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Constructions of Galois Categories

by

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A THESIS

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Abstract

Alexandre Grothendieck, through his groundbreaking work in the 1960s, helped bring algebraic geometry into the modern era. Among his vast work from this period, there is the notion of a Galois category - an example of one being the category of finite etale covers of a scheme. In this thesis we shall investigate Galois categories from a purely categorical perspective and construct examples. In particular, we investigate what it means to be a covering of an object and when such objects form a Galois category.

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Introduction

Classical Galois Theory is the study between the interplay of field theory and group theory. Originally, Evariste Galois was interested in studying the symmetries found in the roots of polynomials. His work was not understood, let alone published, until many years after his death. Today, classical Galois theory is standard in any first course algebra book, but modern Galois theory has evolved into something far larger than what Galois had discovered.

In classical Galois theory, one considers special field extensions L/K, or $K \longrightarrow L$ (remember that in this notation $K \subset L$), called *Galois extensions* and their automorphism group, $G := Aut_K(L)$. That is, G is the group of field homomorphisms $L \longrightarrow L$ that fix K. Then an extension is called *Galois* if it is algebraic and there exists a subgroup $H \subset G$ such that $L^H = K$, where L^H is the fixed field of L,

$$L^H := \{x \in L \mid h(x) = x \text{ for all } h \in H\}$$

Given a Galois extension L/K, the Fundamental Theorem of Galois Theory asserts that there is a bijective correspondence between the set of intermediate fields $L/E/K$ (so here, $K \subset E \subset L$) of L/K and the set of subgroups of G. In some sense, this correspondence is a bridge between field theory and group theory. Under a Galois extension, one can take a problem in field theory and using this correspondence convert it to a problem in group theory, and vice versa. A classic example of this is the problem of whether a degree 5 polynomial is solvable by radicals - the answer is no! For details, see for instance Theorem VII.7.8 in [1].

What is interesting is that this is not the only place in mathematics where such a correspondence exists. In algebraic topology we can consider a simplicial complex X and the connected covering spaces of X, say $Y \longrightarrow X$. Then there is a similar correspondence between the connected covering spaces of X and the subgroups of the fundamental group $\pi_1(X)$ (for details, see for instance [12]). In algebraic geometry, if one considers a connected scheme X and the $finite$ *étale* coverings of X, say $Y \longrightarrow X$, then there is a similar result. In fact, if $X = Spec(K)$ where K is a field, we can recover classical Galois theory (see for instance [17]).

Generally speaking, each of the above examples are instances of Galois categories. Galois categories were first defined by Alexander Grothendieck in [10]. Although the definition of a Galois category is first given as a result of studying the algebraic geometry behind it, Grothendieck provides a purely categorical definition in the form of six axioms. If your category satisfies these six axioms then it is a Galois category and conversely.

The goal of this thesis is to study the categorical definition of Galois categories. We shall show how certain constructions lead to Galois categories and the consequences of said constructions. Notice that in the examples given above, each is a "covering" of some kind: a map $Y \longrightarrow X$ over some "base" object X. Or, in the classical case, an injective map $K \longrightarrow L$ where L lives "under" some base object K. We shall bring this notion of a "covering" into a more general setting where it can be utilized to build a Galois category abstractly. We hope that this thesis sheds some light on the categorical aspect of modern Galois theory.

Almost every single page in this thesis has a categorical diagram, so the importance of Chapter 1 cannot be stressed enough. In Chapter 1, we cover the basics of category theory. We will provide here all necessary definitions, theorems, proofs, and notation needed for the rest of the thesis.

In Chapter 2, we will introduce Grothendieck's Galois categories. From here, we will define an important categorical construction called the *family category* which we will see appear throughout the thesis. The remainder of Chapter 2 will be a very general approach to building a Galois category, based on Grothendieck's axioms.

In Chapter 3, we will narrow our focus to a category which attempts to utilize an abstract notion of a covering. Coverings in this chapter will be objects over some base S, $X \longrightarrow S$, that under base change are "trivialized" by some T in the following way:

$$X \times_S T \cong \coprod T$$

We show that under certain conditions, the category of connected coverings is a Galois category. Then we observe that the category of connected strongly separable algebras is an example of such a category. This example from algebra was demonstrated by Michael Barr in [3].

In Chapter 4, we attempt to bring the notion of a covering closer to that of the more familiar Galois theory, in particular the *Kaplansky* conditon, which in the fields case is just that $L^G = K$. This chapter also seeks to bring these notions to a more categorical setting using the pushout, product, and fixed point limit functors.

Notational Warning. In this thesis, whenever we need to denote the composition of two maps, say $f : A \longrightarrow B$ and $g : B \longrightarrow C$ we will write the composition in **diagrammatic order**. That is, the composition will be written in the way that it appears in a diagram. For example,

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in this case, the composition of f and g will be written fg. We will never write maps in compositional order, other than on the rare instance where we actually are performing a computation, in which case we shall write $g(f(-))$.

Chapter 1

Category Theory

Introduction

The goal of this chapter is to provide the necessary category theory notions in order to understand this thesis. The material of this thesis is categorical by nature, since we are explicitly constructing categories, and as such, the reader will require a thorough understanding of category theory. All of the work below is standard and can be found in for instance [2] or [7], with the exception of Section 1.4 which can be found in [6]. We will make explicit where other references were used.

In Section 1.1 we will define categories and provide numerous examples. In Section 1.2 we will define functors, natural transformations, and what it means for two functors to be adjoint. In Section 1.3 we will define limits and colimits and provide all necessary examples, including the lesser known fixed point and quotient objects. Finally, in Section 1.4 we will define extensive categories and what it means for an object to be connected in the categorical sense.

1.1 Categories and Maps

Definition 1.1.1. A *category* $\mathbb{X}$ consists of the following data:

- Objects: A class $\mathbb{X}_0$: $X, Y, Z, \ldots$
- Maps: A class $\mathbb{X}_1$: $f, g, h, \ldots$ where each map f is equipped with two objects $dom(f)$ and $cod(f)$, called the *domain* and *codomain* of f respectively. We

shall write:

$$f : X \longrightarrow Y$$

where $X = dom(f)$ and $Y = cod(f)$.

such that the following conditions hold:

(i) Composition: Given maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ there exists a map $fg : X \longrightarrow Z$ called the *composition* of f and g.

(ii) Identities: For every $X \in \mathbb{X}_0$ there exists a map $1_X : X \longrightarrow X$ called the *identity* of X.

(iii) Unit Law: For all maps $f : X \longrightarrow Y$,

$$1_X f = f = f 1_Y$$

(iv) Associativity Law: For all maps $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$, $h : Z \longrightarrow W$,

$$f(gh) = (fg)h$$

Remark. We will write $X \in \mathbb{X}$ instead of $X \in \mathbb{X}_0$ for objects X and similarly we will write $f \in \mathbb{X}$ instead of $f \in \mathbb{X}_1$ for maps f. On occasion, we will write $f : X \longrightarrow Y \in \mathbb{X}$ by which it is meant that $f \in \mathbb{X}$, not $Y \in \mathbb{X}$. Categories will be defined by describing the class of objects and the class of maps. The composition rule, identity maps, and the unit and associative laws will then be described as needed.

Example 1.1.1. Let **Set** denote the following data:

- Objects: Sets A, B, C, etc.
- Maps: Set functions

$$\begin{aligned} f : A &\longrightarrow B \\ a &\longmapsto f(a) \end{aligned}$$

A composition of two maps is ordinary function composition. The identity map for any set A is defined usual.

To show that **Set** is a category, we will need to prove the unit and associative laws. Given any map $f : A \longrightarrow B \in$ **Set** we have for all $a \in A$,

$$(1_A f)(a) = f(1_A(a)) = f(a) = 1_B(f(a)) = (f 1_B)(a)$$

Hence the unit law is satisfied. For the associative law, let $f : A \longrightarrow B$, $g : B \longrightarrow C$, $h : C \longrightarrow D$. Then for all $a \in A$,

$$f(gh)(a) = (gh)(f(a)) = h(g(f(a))) = h((fg)(a)) = (fg)h(a)$$

Hence the associative law is satisfied, so **Set** is a category.

With a slight modification to the class of objects, we could define the category of finite sets, **fset**:

- Objects: Finite sets A, B, C, ...
- Maps: Set functions $f : A \longrightarrow B$.

In a similar argument for **Set**, it is readily seen that **fset** is also an example of a category.

The category **Set** provides a solid foundation for constructing numerous other categories where the objects look like sets, but have some additional structure. We introduce some of these categories below.

Example 1.1.2. Let **cRing** denote the following data:

- Objects: Commutative rings $(R, +, \times)$, which we will write as R.
- Maps: Ring homomorphisms $\varphi : R \longrightarrow S$.

Given a commutative ring R the identity map $1_R : R \longrightarrow R$ is defined as it is in **Set** and is readily seen to be a ring homomorphism, hence $1_R \in$ **cRing**. Composition is also defined the same way as it is in **Set** and the proofs of the unit and associative laws are similar. Thus **cRing** is a category.

Naturally, we could have done something similar to the above to define the category of rings, **Ring**, for which the objects are rings which are not necessarily commutative. However throughout this thesis, when we discuss examples pertaining to rings, we shall only be concerned with commutative rings (with identity).

Example 1.1.3. Let **Grp** denote the following data:

- Objects: Groups $(G, \cdot)$, which we will write as G.
- Maps: Group homomorphisms $\varphi : G \longrightarrow H$.

Given a group G the identity map $1_G : G \longrightarrow G$ is defined the same way as it is in **Set** and 1_G is also a group homomorphism, hence $1_G \in$ **Grp**. Composition is also defined the same way as it is in **Set**. The proofs of the unit and associative laws for **Grp** are similar to those in **Set**, thus **Grp** is a category.

A more interesting case is to consider a single group $G \in$ **Grp**. Then G itself can be described as a category as follows:

Example 1.1.4. Let $G \in$ **Grp** be a group. Then G is equipped with the following data:

- Objects: A single object G (i.e. the group itself is the only object).
- Maps: Elements $g \in G$ of the group, thought of as maps $g : G \longrightarrow G$.

Since there is only one object, we only need one identity map. G is a group, so it has an identity element e which is the identity map 1_G.

Now, given $g, h \in G$ we define composition of $g : G \longrightarrow G$ and $h : G \longrightarrow G$ by multiplication. That is, $gh = (g \cdot h)$.

From this, we see that the unit law follows naturally and the associative law follows from the associative law within G, that is, the group associativity law. Hence every group $G \in \mathbf{Grp}$ is a category.

Example 1.1.5. ([1] II.9.2, II.9.3). Let $G \in \mathbf{Grp}$ be any group. Then an *action* of G on a set A is a set map:

$$\rho : G \times A \longrightarrow A$$
$$(g, a) \longmapsto \rho(g, a)$$

(where we denote $\rho(g, a)$ by $g \cdot a$) such that

1. $e \cdot a = a$.
2. For all $g, h \in G$ and $a \in A$, $(gh) \cdot a = g \cdot (h \cdot a)$.

We then say that A equipped with an action ρ is a G-set, denoted (A, ρ). Let G-**Set** denote the following data:

- Objects: G-sets (A, ρ).
- Maps: $(A, \rho) \longrightarrow (A', \rho')$ is a map $\varphi : A \longrightarrow A' \in \mathbf{Set}$ such that for all $g \in G$ and $a \in A$,
$$\varphi(g \cdot a) = g \cdot \varphi(a)$$

Then G-**Set** is a category.

Example 1.1.6. *Finite Categories.* Finite categories are categories with a finite number of objects and maps. For instance, the category containing no objects and no arrows is vacuously a category (it automatically satisfies all of the conditions). We will call this category the *initial category*. It is displayed below:

We could also form a category with only one object, say A, and only one map which by the definition of a category must be the identity map 1_A. We will call this category the *terminal category* and denote it by $\mathbf{1}$, displayed below:

$$1_A \circlearrowright A$$

From here, we can continue to add maps and objects, so long as we have an identity map for every object and a composition rule on each of the maps that satisfies the unit and associative laws. For example,

$$1_A \circlearrowright A \longrightarrow B \circlearrowleft 1_B$$

Another example would be to consider any finite group as a category, in the sense of Example 1.1.4. Then any finite group is also a finite category.

Notice that in the definition of a category, we described $\mathbb{X}_0$ and $\mathbb{X}_1$ as classes and not as sets.

Definition 1.1.2. ([2] Definition 1.11). A category $\mathbb{X}$ is *small* if $\mathbb{X}_0$ and $\mathbb{X}_1$ are sets.

Example 1.1.7. Let $G \in \mathbf{Grp}$ be a finite group. Then G, viewed as a category, is a small category. There is only one object, G, so the class of objects is a set with a single element. Second, the collection of all elements of a group forms a set (by the definition of groups - there is an underlying set). Hence G is a small category.

However consider the category $\mathbf{Set}$. The collection of all sets would then be $\mathbf{Set}_0$, but this being a set would lead to Russell's paradox. Hence $\mathbf{Set}$ is not small.

Given any two objects X, Y in a category $\mathbb{X}$ there can be more than one map between them. The class of maps in $\mathbb{X}$ between two objects X and Y will be denoted by $\mathbb{X}(X, Y)$ and is called the *hom-set* of X and Y. However the name is misleading - every hom-set need not be a set.

Definition 1.1.3. A category is *locally small* if for every $X, Y \in \mathbb{X}$ the hom-set $\mathbb{X}(X, Y)$ is a set.

Example 1.1.8. **Set** is locally small. Given any two sets A, B the collection of maps between them, $\mathbf{Set}(A, B)$, is a set.

Unless otherwise stated, any categories listed henceforth are assumed to be locally small.

Definition 1.1.4. Given a category $\mathbb{X}$ we define the *opposite* category of $\mathbb{X}$, denoted $\mathbb{X}^{op}$, as follows:

- Objects: Objects in $\mathbb{X}$.
- Maps: A map $f^{op} : Y \longrightarrow X$ in $\mathbb{X}^{op}$ is the same as a map $f : X \longrightarrow Y \in \mathbb{X}$.

Given two maps $f^{op} : Y \longrightarrow X$ and $g^{op} : Z \longrightarrow Y$, which are the same as maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ respectively in $\mathbb{X}$, their composition $g^{op}f^{op}$ is defined as fg. The identity map $1_X^{op} = 1_X$ by deifnition.

Thus the opposite of a category $\mathbb{X}$ is the category $\mathbb{X}^{op}$ with the same objects, but with the domains and codomains of all the maps exchanged. Given some property of a map $f : X \longrightarrow Y$ in $\mathbb{X}$, intuitively one would think that there should be a "dual" property for the map $f^{op} : Y \longrightarrow X$ in $\mathbb{X}^{op}$, since f^{op} is defined by f. This is the notion of *duality* and it is best seen through an example.

Definition 1.1.5. A map $f : X \longrightarrow Y$ in a category $\mathbb{X}$ is called a *monomorphism* if, given any two maps $g : A \longrightarrow X$ and $h : A \longrightarrow X$ where $g, h \in \mathbb{X}$,

$$A \underset{h}{\overset{g}{\rightrightarrows}} X \xrightarrow{f} Y$$

such that $gf = hf$ then $g = h$.

We shall often write that f is *monic*, instead of monomorphism.

The dual notion of monic would then be the same conditions, but with all of the domains and codomains exchanged. Thus it would be a property for the map $f : Y \longrightarrow X$ defined as follows:

Definition 1.1.6. A map $f : Y \longrightarrow X$ in a category $\mathbb{X}$ is called an *epimorphism* if, given any two maps $g : X \longrightarrow A$ and $h : X \longrightarrow A$ where $g, h \in \mathbb{X}$,

$$Y \xrightarrow{f} X \underset{h}{\overset{g}{\rightrightarrows}} A$$

such that $fg = fh$ then $g = h$.

We shall often write that f is *epic*, instead of epimorphism.

So by these two definitions, we see that $f : X \longrightarrow Y$ is monic in $\mathbb{X}$ if and only if f is epic in $\mathbb{X}^{op}$ (since $f \in \mathbb{X}^{op}$ would be the map $f^{op} : Y \longrightarrow X$). This is a direct application of duality. Naturally, one should ask how monics and epics behave in different categories, since the definitions are entirely general.

Example 1.1.9. Consider the category of sets, **Set**, and suppose we have a map $f : A \longrightarrow B \in$ **Set** that is monic. Further, suppose that $f(a) = f(a')$ for some $a, a' \in A$. Then consider the maps g and h defined on the set $\{a, a'\}$ as follows:

$$\{a, a'\} \underset{h}{\overset{g}{\rightrightarrows}} A \xrightarrow{f} B$$

where $g(a) = a = g(a')$ and $h(a) = a$, $h(a') = a'$. Then,

$$f(g(a)) = f(a) = f(a') = f(h(a'))$$

But since f is monic, this implies that $g(a) = h(a')$, but then $a = a'$. Hence f is injective.

Conversely, suppose that $f : A \longrightarrow B$ is injective. Then suppose we have two maps $g, h : C \longrightarrow B$ such that $gf = hf$, i.e.

$$C \underset{h}{\overset{g}{\rightrightarrows}} A \xrightarrow{f} B$$

Then for all $c \in C$ we have $f(g(c)) = f(h(c))$. But since f is injective, this implies that $g(c) = h(c)$ for all $c \in C$. Hence $g = h$, so f is monic.

One can also show that a map f in **Set** is an epimorphism if and only if f is surjective.

It is important to note that monic and epic do not correspond precisely with the familiar notions of injective and surjective in every category. In fact, one does not need to tread too far from the category **Set** to find an example of where these notions do not coincide.

Example 1.1.10. Consider the ring homomorphism $\iota : \mathbb{Z} \longrightarrow \mathbb{Q}$ in **cRing**, which is the inlusion. This map is not surjective by definition, but it is an epimorphism. To see this, let $R \in$ **cRing** and suppose we have two ring homomorphisms $g, h : \mathbb{Q} \longrightarrow R$ such that:

$$\mathbb{Z} \xrightarrow{\iota} \mathbb{Q} \underset{h}{\overset{g}{\rightrightarrows}} R$$

where $\iota g = \iota h$. We must show that $g = h$. For all $z \in \mathbb{Z}$ we have, by definition of the inclusion ι:

$$g(z) = g(\iota(z)) = h(\iota(z)) = h(z)$$

Hence g and h agree on every integer $z \in \mathbb{Z}$. However recall that both g and h are ring homomorphisms, so in particular we have for any rational $p/q \in \mathbb{Q} \setminus \mathbb{Z}$,

$$g(\frac{p}{q}) = g(p) * g(\frac{1}{q}) = h(p) * g(\frac{1}{q}) = h(q(\frac{p}{q})) * g(\frac{1}{q}) = h(q) * h(\frac{p}{q}) * g(\frac{1}{q})$$
$$= h(\frac{p}{q}) * g(q) * g(\frac{1}{q}) = h(\frac{p}{q}) * g((q)(\frac{1}{q})) = h(\frac{p}{q}) * g(1) = h(\frac{p}{q})$$

Hence $g = h$, so ι is an epimorphism.

Notice that the notions of injective and surjective are inherently dependent on being able to work with elements of any given object in your category. This is not always possible and is in fact rarely useful. Observe that the definitions of monic and epic make no mention of the elements of a given object. We must do the same abstraction for other familiar notions and one of these is what it means to be an isomorphism. To do this, we introduce another pair of dual notions.

Definition 1.1.7. A map $f : X \longrightarrow Y$ in a category $\mathbb{X}$ is a *section* if there exists a map $f' : Y \longrightarrow X$ such that $ff' = 1_X$. That is, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "1_X"'] & Y \arrow[d, "f'"] \\ & X \end{tikzcd}$$

Definition 1.1.8. A map $g : X \longrightarrow Y$ in a category $\mathbb{X}$ is a *retraction* if there exists a map $g' : Y \longrightarrow X$ such that $g'g = 1_Y$. That is, the following diagram commutes:

$$\begin{tikzcd} Y & X \arrow[l, "g"'] \\ & Y \arrow[ul, "1_Y"] \arrow[u, "g'"'] \end{tikzcd}$$

Definition 1.1.9. A map which as both a section and a retraction is called an *isomorphism*.

Example 1.1.11. Let $G \in \mathbf{Grp}$ and consider G as a category, as in Example 1.1.4. Recall that the maps in G viewed as a category are the elements $g \in G$ where $g : G \longrightarrow G$.

Since G is a group, every $g \in G$ has an inverse, $g^{-1} \in G$ which is the map $g^{-1} : G \longrightarrow G$. Observe that g is both a section and a retraction with g^{-1}. That is, $gg^{-1} = 1_G$ and $g^{-1}g = 1_G$. Hence every map in the category G is an isomorphism.

So we have seen that given some definition we can write the dual of the definition by reversing all the arrows (see: section, retraction and monic, epic). We can apply the same idea to proofs. That is, if we provide the proof of a property and the exact same proof, but with all arrows reversed, is the proof of the dual property, then one would say that the proof of the dual property is a "proof by duality". This is best seen through an example.

Proposition 1.1.1. *Every section is monic.*

Proof. Suppose $f : X \longrightarrow Y$ is a section. Then there exists a map $f' : Y \longrightarrow X$ such that $ff' = 1_X$. Suppose that there exists maps $g, h : Z \longrightarrow X$ such that $gf = hf$, i.e.

$$Z \overset{g}{\underset{h}{\rightrightarrows}} X \xrightarrow{f} Y$$

Then,

$$g = g1_X = g(ff') = (gf)f' = (hf)f' = h(ff') = h1_X = h$$

Hence f is monic. $\square$

The dual of Proposition 1.1.1 is the following:

Proposition 1.1.2. *Every retraction is epic.*

The proof of this dual statement is the dual of the proof of the original statement. That is, it is a proof by duality.

Proof. Suppose $g : X \longrightarrow Y$ is a retraction. Then there exists a map $g' : Y \longrightarrow X$ such that $g'g = 1_Y$. Suppose that there exists maps $h, k : Y \longrightarrow Z$ such that $gh = gk$, i.e.

$$X \xrightarrow{g} Y \overset{h}{\underset{k}{\rightrightarrows}} Z$$

Then,

$$h = 1_Y h = (g'g)h = g'(gh) = g'(gk) = (g'g)k = 1_Y k = k$$

Hence g is epic. $\square$

Remark. Not every monic map is a section. Observe that the map $f : \emptyset \longrightarrow A \in \mathbf{Set}$ for any set A is vacuously monic: there are no maps into $\emptyset$ so the monic condition is vacuously true. In order to be a section, we would need a map $f' : A \longrightarrow \emptyset$, but no such map in **Set** exists. Hence f is not a section.

Definition 1.1.10. Let $\mathbb{X}$ be a category and $X \in \mathbb{X}$. Then the *slice category* is the category $\mathbb{X}/X$ which consists of the following data:

- Objects: Maps $y : Y \longrightarrow X \in \mathbb{X}$ with codomain X.
- Maps: $f : Y \longrightarrow Y'$ is a map in $\mathbb{X}$ such that $fy' = y$, that is, the following triangle commutes:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "y"'] & & Y' \arrow[dl, "y'"] \\ & X & \end{tikzcd}$$

Generally speaking, one can think of the objects in the slice category $\mathbb{X}/X$ as objects "over" X. Naturally, if the slice category is the category of objects "over" some specified object, then we can consider the category of objects "under" some specified object as well.

Definition 1.1.11. Let $\mathbb{X}$ be a category and $X \in \mathbb{X}$. Then the *coslice category* is the category $X/\mathbb{X}$ which consists of the following data:

- Objects : Maps $y : X \longrightarrow Y \in \mathbb{X}$ with domain X.
- Maps: $f : Y \longrightarrow Y'$ is a map in $\mathbb{X}$ such that $yf = y'$, that is, the following triangle commutes:

$$\begin{tikzcd} Y \arrow[rr, "f"] & & Y' \\ & X \arrow[ul, "y"] \arrow[ur, "y'"'] & \end{tikzcd}$$

Convention. When it is clear, we shall often refer to the object $Y \longrightarrow X$ in the slice category $\mathbb{X}/X$ as simply Y and similarly in the coslice category. Additionally we will often do the same for the maps in a slice/coslice category. That is, we shall denote the map

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "y"'] & & Y' \arrow[dl, "y'"] \\ & X & \end{tikzcd}$$

in $\mathbb{X}/X$ as simply $f : Y \longrightarrow Y'$.

Remark. It is important to note that an object in a (co)slice category is a map in the original category. Thus, it will have all of the properties that maps in the original category have.

Example 1.1.12. Consider the category of topological spaces **Top** and let $S \in$ **Top**. Then let $Cov(S)$ be the subcategory of the slice category **Top**$/S$ with the following data:

- Objects: Covering maps $p : X \longrightarrow S$. That is, for every $s \in S$ there exists an open neighborhood U of s such that $p^{-1}(U) \cong \coprod_{\alpha} V_\alpha$ where each $V_\alpha \cong U$ and this homeomorphism is via p.
- Maps: Commuting triangles:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "p"'] & & X' \arrow[dl, "p'"] \\ & S & \end{tikzcd}$$

Example 1.1.13. Consider the category of commutative rings **cRing** and let $R \in$ **cRing**. Then the category of R-algebras, denoted **R-Alg**, is the coslice category $R/$**cRing** which consists of the following data:

- Objects: Ring homomorphisms $\varphi : R \longrightarrow A$.
- Maps: Commuting triangles:

$$\begin{tikzcd} A \arrow[rr] & & A' \\ & R \arrow[ul, "\varphi"] \arrow[ur, "\varphi'"'] & \end{tikzcd}$$

Remark. One must be careful when dealing with the opposite category of a slice or coslice category. Notice that for any category $\mathbb{X}$, $(X/\mathbb{X})^{op}$ is *not* the same as $\mathbb{X}/X$. This is because taking the opposite of a category only reverses the arrows of the maps and would not reverse the arrows defining the objects. On the other hand, $X/(\mathbb{X}^{op})$ is the same as $\mathbb{X}/X$ since a map with domain X in $\mathbb{X}^{op}$ is the same as a map with codomain X in $\mathbb{X}$.

1.2 Functors, Natural Transformations, and Adjoints

All of the properties we have defined so far have been strictly referring to maps in a category: epic, monic, sections, retractions, and isomorphisms. Generally speaking, studying the maps between the objects of a category is more important than studying the objects themselves. Thus one could ask: are there maps between categories? The answer is thankfully yes.

Definition 1.2.1. A *functor* $F : \mathbb{X} \longrightarrow \mathbb{Y}$ between two categories consists of a map between the classes of objects, $F_0 : \mathbb{X}_0 \longrightarrow \mathbb{Y}_0$, and a map between the classes the maps, $F_1 : \mathbb{X}_1 \longrightarrow \mathbb{Y}_1$ such that:

(i) $F_1(f : X \longrightarrow Y) = F_1(f) : F_0(X) \longrightarrow F_0(Y)$.

(ii) $F_1(1_X) = 1_{F_0(X)}$.

(iii) $F_1(fg) = F_1(f)F_1(g)$.

Much like we dropped $\mathbb{X}_0$ and $\mathbb{X}_1$ as notation for the classes of objects and maps of a category $\mathbb{X}$, we will drop the notation F_0 and F_1 and simply use F. When defining a new functor, we will make clear how it acts on both the objects and the maps of the category.

Example 1.2.1. Define the functor $U : \mathbf{Grp} \longrightarrow \mathbf{Set}$ as follows:

$$\begin{aligned} U : \mathbf{Grp} &\longrightarrow \mathbf{Set} \\ G &\longmapsto G \text{ with no group structure.} \\ \varphi : G \longrightarrow H &\longmapsto \varphi : G \longrightarrow H \text{ as a map of sets - no group homomorphism structure.} \end{aligned}$$

Then U is indeed a functor and is an example of a *forgetful functor*. One can always define a forgetful functor so long as it is possible to "remove structure" from one categroy to arrive at another.

Example 1.2.2. Let $\mathbb{X}$ be a locally small category and let $A \in \mathbb{X}$. Then consider the following map, called the $hom\text{-}functor$:

$$\begin{aligned} \mathbb{X}(A,-) : \mathbb{X} &\longrightarrow \mathbf{Set} \\ X &\longmapsto \mathbb{X}(A,X) \\ f : X \longrightarrow Y &\longmapsto \mathbb{X}(A,f) : \mathbb{X}(A,X) \longrightarrow \mathbb{X}(A,Y) \end{aligned}$$

where $\mathbb{X}(A,f)$ is defined by composition with f. That is, $\mathbb{X}(A,f)$ sends a map $A \longrightarrow X$ to $A \longrightarrow X \longrightarrow Y$. Then $\mathbb{X}(A,-)$ is a functor.

Example 1.2.3. Let $f : X \longrightarrow Y$ be a map in $\mathbb{X}$. Then the *composition functor*, denoted f_*, is defined as follows:

$$f_* : \mathbb{X}/X \longrightarrow \mathbb{X}/Y$$

$$\begin{tikzcd} & & Z \arrow[d] \arrow[dd, bend left] \\ Z \arrow[d] & \longmapsto & X \arrow[d, "f"'] \\ X & & Y \end{tikzcd}$$

$$\begin{tikzcd} Z \arrow[rr, "\varphi"] \arrow[dr] & & Z' \arrow[dl] \\ & X & \end{tikzcd} \longmapsto \begin{tikzcd} Z \arrow[rr, "f_*(\varphi)"] \arrow[dr] \arrow[ddr, bend right] & & Z' \arrow[dl] \arrow[ddl, bend left] \\ & X \arrow[d, "f"] & \\ & Y & \end{tikzcd}$$

So the composition functor sends Z, as an object over X, to Z, but now as an object over Y. Similarly, the map $\varphi : Z \longrightarrow Z'$ over X is sent to a map $f_*(\varphi) : Z \longrightarrow Z'$ but now with a different base, Y. These objects and maps need not be the same. For instance, given a map $f : S \longrightarrow S'$ in **Top**, a covering space $p : X \longrightarrow S$ in $Cov(S)$ is not the same as the covering space $X \longrightarrow S \longrightarrow S'$.

Definition 1.2.2. A functor $F : \mathbb{X} \longrightarrow \mathbb{Y}$ is called *contravariant* if for all $f : X \longrightarrow Y \in \mathbb{X}$, $F(f) : F(Y) \longrightarrow F(X)$. Otherwise the functor is called *covariant*.

Intuitively, a contravariant functor is one that reverses a map when applied.

Example 1.2.4. Let $\mathbb{X}$ be a locally small category and let $A \in \mathbb{X}$. Then consider the following:

$$\begin{aligned} \mathbb{X}(-, A) : \mathbb{X} &\longrightarrow \mathbf{Set} \\ X &\longmapsto \mathbb{X}(X, A) \\ f : X \longrightarrow Y &\longmapsto \mathbb{X}(f, A) : \mathbb{X}(Y, A) \longrightarrow \mathbb{X}(X, A) \end{aligned}$$

where $\mathbb{X}(f, A)$ is defined by pre-composition with f. That is $\mathbb{X}(f, A)$ sends each $Y \longrightarrow A$ to $X \longrightarrow Y \longrightarrow A$. Then $\mathbb{X}(-, A)$ is a contravariant functor.

Example 1.2.5. Putting examples 1.2.2 and 1.2.4 together, we can construct another functor on the hom-sets. Let $\mathbb{X}$ be a locally small category. Then,

$$\mathbb{X}(-,-) : \mathbb{X}^{op} \times \mathbb{X} \longrightarrow \mathbf{Set}$$

$$(X, Y) \longmapsto \mathbb{X}(X, Y)$$

$$(f^{op} : X \longrightarrow Y, g : X' \longrightarrow Y') \longmapsto \mathbb{X}(f^{op}, g) : \mathbb{X}(X, X') \longrightarrow \mathbb{X}(Y, Y')$$

where $f^{op} : X \longrightarrow Y \in \mathbb{X}^{op}$ hence it is $Y \longrightarrow X \in \mathbb{X}$. Then $\mathbb{X}(f^{op}, g)$ is defined as follows:

$$\mathbb{X}(f^{op}, g) : \mathbb{X}(X, X') \longrightarrow \mathbb{X}(Y, Y')$$

$$(X \longrightarrow X') \longmapsto (Y \xrightarrow{f} X \longrightarrow X' \xrightarrow{g} Y')$$

Now we have a collection of objects, categories, and maps between them, functors. Let **Cat** denote the following data:

- Objects: Categories $\mathbb{X}$, $\mathbb{Y}$, ...

- Maps: Functors $F : \mathbb{X} \longrightarrow \mathbb{Y}$.

Then we get the following natural result.

Proposition 1.2.1. Cat *is a category.*

Proof. Given a category $\mathbb{X}$, define the identity functor $\mathbb{1}_{\mathbb{X}}$ as follows:

$$\mathbb{1}_{\mathbb{X}} : \mathbb{X} \longrightarrow \mathbb{X}$$

$$X \longmapsto X$$

$$f : X \longrightarrow Y \longmapsto f : X \longrightarrow Y$$

Let $F : \mathbb{X} \longrightarrow \mathbb{Y}$ and $G : \mathbb{Y} \longrightarrow \mathbb{Z}$ be two functors. The composition functor FG is defined as follows:

$$FG : \mathbb{X} \longrightarrow \mathbb{Z}$$

$$X \longmapsto G(F(X))$$

$$f : X \longrightarrow Y \longmapsto G(F(f)) : G(F(X)) \longrightarrow G(F(Y))$$

The unit and associative laws are clear from the above definitions. Thus **Cat** is a category.

□

Consider for a moment an arbitrary functor $F : \mathbb{X} \longrightarrow \mathbb{Y}$. Given any two objects $X, Y \in \mathbb{X}$ there are objects $F(X), F(Y) \in \mathbb{Y}$. Thus there are associated hom-sets for each pair of objects: $\mathbb{X}(X, Y)$ is the set of maps between X and Y in $\mathbb{X}$ and $\mathbb{Y}(F(X), F(Y))$ is the set of maps between $F(X)$ and $F(Y)$ in $\mathbb{Y}$. So for every pair of objects $X, Y \in \mathbb{X}$ we can build a natural map of sets between these two hom-sets using the functor F:

$$F_{X,Y} : \mathbb{X}(X, Y) \longrightarrow \mathbb{Y}(F(X), F(Y))$$
$$f : X \longrightarrow Y \longmapsto F(f) : F(X) \longrightarrow F(Y)$$

Since $F_{X,Y}$ is a map in **Set**, it is natural to think about when this map is injective and when it is surjective. This leads to the following definitions.

Definition 1.2.3. A functor $F : \mathbb{X} \longrightarrow \mathbb{Y}$ is *faithful* if for all $X, Y \in \mathbb{X}$, $F_{X,Y}$ is injective.

Definition 1.2.4. A functor $F : \mathbb{X} \longrightarrow \mathbb{Y}$ is *full* if for all $X, Y \in \mathbb{X}$, $F_{X,Y}$ is surjective.

Example 1.2.6. Define the functor $F : \mathbf{fset} \longrightarrow \mathbf{Set}$ as follows:

$$F : \mathbf{fset} \longrightarrow \mathbf{Set}$$
$$A \longmapsto A$$
$$f : A \longrightarrow B \longmapsto f : A \longrightarrow B$$

In other words, F is an inclusion of categories. Then it is not difficult to see that for any two finite sets $A, B \in \mathbf{fset}$, $F_{A,B}$ is both injective and surjective. Hence F is full and faithful.

Example 1.2.7. Recall the forgetful functor $U : \mathbf{Grp} \longrightarrow \mathbf{Set}$ from Example 1.2.1. Let $G, H \in \mathbf{Grp}$ be any two groups and consider the map:

$$U_{G,H} : \mathbf{Grp}(G, H) \longrightarrow \mathbf{Set}(U(G), U(H))$$

Suppose that $U_{G,H}(\varphi) = U_{G,H}(\sigma)$. As maps between sets, $\varphi(g) = \sigma(g)$ for all $g \in G$. Hence $U_{G,H}$ is injective so U is faithful.

However, notice that one could construct a map in **Set** $G \longrightarrow H$ that is not a group homomorphism (eg. pick $e \neq h \in H$ and define the map $G \longrightarrow H$ by $g \longmapsto h$ for all $g \in G$). Hence $U_{G,H}$ is not surjective, so U is not full.

The notion of an isomorphism was introduced in the previous section to establish what it means for two objects in a category to be essentially the same. We would like to do the same for categories. At first glance, it would be tempting to think that a full and faithful functor would provide such a notion, but Example 1.2.6 shows us otherwise since intuitively we would not want to think of **fset** and **Set** as the same. So we need an additional property.

Definition 1.2.5. A functor $F : \mathbb{X} \longrightarrow \mathbb{Y}$ is *essentially surjective* if for every object $Y \in \mathbb{Y}$ there exists an object $X \in \mathbb{X}$ such that $F(X) \cong Y$.

Example 1.2.8. Recall the inclusion functor F : **fset** $\longrightarrow$ **Set** from Example 1.2.6. Let $X \in$ **Set** be an infinite set. Then there is no finite set $A \in$ **fset** such that $F(A) = X$ hence F is not essentially surjective.

Putting all of the above notions together, we get a notion that tells us when two categories are "essentially the same".

Definition 1.2.6. A functor $F : \mathbb{X} \longrightarrow \mathbb{Y}$ is an *equivalence of categories* if F is full, faithful, and essentially surjective.

If $F : \mathbb{X} \longrightarrow \mathbb{Y}$ is an equivalence of categories, we write $\mathbb{X} \simeq \mathbb{Y}$.

Example 1.2.9. Let $\mathbb{X}$ denote the category of finite dimensional $\mathbb{R}$-vector spaces (the objects are such vector spaces and the maps are $\mathbb{R}$-linear maps) and let $\mathbb{Y}$ be the category of matrices with entries in $\mathbb{R}$, $\mathbf{Mat}(\mathbb{R})$, which consists of the following data:

- Objects: $n \in \mathbb{N}$ (including 0).
- Maps: $m \longrightarrow n$ is given by an $m \times n$ matrix A.

Then the identity $n \longrightarrow n$ is the $n \times n$ identity matrix I and composition of maps is given by matrix multiplication. We claim that $\mathbf{Mat}(\mathbb{R}) \simeq \mathbb{X}$. To see this, consider the following functor:

$$F : \mathbf{Mat}(\mathbb{R}) \longrightarrow \mathbb{X}$$

$$n \longrightarrow \mathbb{R}^n$$

$$A : n \longrightarrow m \longmapsto A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

Given any finite dimensional $\mathbb{R}$-vector space V, we know that $V \cong \mathbb{R}^n$ (see for instance Proposition VI.1.7 in [1]) so F is essentially surjective. Every vector space map $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ has a corresponding matrix representation which is unique, so F is both full and faithful as well. Hence F is an equivalence of categories.

We have seen various examples of properties on maps in a category, for instance monic, epic, sections, retractions, etc. One can ask when such properties hold after applying a functor.

Definition 1.2.7. Let P be some property in a category $\mathbb{X}$. Then a functor $F : \mathbb{X} \longrightarrow \mathbb{Y}$ *preserves* property P if whenever P holds for some collection of maps (f_i) and objects (X_i) then P holds for the maps $(F(f_i))$ and the objects $(F(X_i))$.

Example 1.2.10. Let $F : \mathbb{X} \longrightarrow \mathbb{Y}$ be any functor. Then F preserves isomorphisms. To see this, suppose that $f : X \longrightarrow Y$ is an isomorphism. Then in particular f is both a section and a retraction so there exists maps $f' : Y \longrightarrow X$ and $f'' : Y \longrightarrow X$ such that $ff' = 1_X$

and $f''f = 1_Y$. Since F is a functor, we have:

$$1_{F(X)} = F(1_X) = F(ff') = F(f)F(f')$$
$$1_{F(Y)} = F(1_Y) = F(f''f) = F(f'')F(f')$$

Hence $F(f)$ is both a section and a retraction in $\mathbb{Y}$, so $F(f)$ is an isomorphism.

Definition 1.2.8. Let P be some property in a category $\mathbb{Y}$. Then a functor $F : \mathbb{X} \longrightarrow \mathbb{Y}$ *reflects* property P if whenever there exists maps $(f_i) \in \mathbb{X}$ and objects $(X_i) \in \mathbb{X}$ such that P holds for the maps $(F(f_i))$ and objects $(F(X_i))$ then P holds for the maps (f_i) and the objects (X_i).

Remark. Let us consider what it means for a functor $F : \mathbb{X} \longrightarrow \mathbb{Y}$ to reflect isomorphisms. Given a map $f : X \longrightarrow Y$ such that $F(f) : F(X) \longrightarrow F(Y)$ is an isomorphism in $\mathbb{Y}$, for F to reflect isomorphisms means that then f is an isomorphism in $\mathbb{X}$. It does not mean that if I have an isomorphism between two objects in $F(X')$ and $F(Y')$ in $\mathbb{Y}$ that X' and Y' are isomorphic in $\mathbb{X}$. The fact that you must start with a map f in $\mathbb{X}$ is important, since a map between X' and Y' may not even exist in $\mathbb{X}$.

Definition 1.2.9. A functor F is *conservative* if F reflects isomorphisms.

Keeping with the theme of establishing what it means to be a map between certain objects, be they objects in a category or categories themselves, we shall now define what it means to be a map between two functors.

Definition 1.2.10. Let $F, G : \mathbb{X} \longrightarrow \mathbb{Y}$ be two functors. A *natural transformation* $\eta : F \Rightarrow G$ is a collection of maps $\eta_X : F(X) \longrightarrow G(X)$, one for every $X \in \mathbb{X}$, such that for every map $f : X \longrightarrow Y$ we have $F(f)\eta_Y = \eta_X G(f)$. That is, the following diagram commutes:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\eta_X"'] & F(Y) \arrow[d, "\eta_Y"] \\ G(X) \arrow[r, "G(f)"'] & G(Y) \end{tikzcd}$$

Convention. We will refer to the condition of a natural transformation requiring the above square to commute as the naturality condition.

Example 1.2.11. Consider the category of commutative rings, **cRing**, and the category of groups, **Grp**. We shall form two different functors **cRing** $\longrightarrow$ **Grp** as follows.

$$GL_n(-) : \mathbf{cRing} \longrightarrow \mathbf{Grp}$$
$$R \longmapsto GL_n(R)$$
$$\varphi : R \longrightarrow S \longmapsto GL_n(\varphi) : GL_n(R) \longrightarrow GL_n(S)$$

where $GL_n(R)$ is the general linear group - the set of all $n \times n$ invertice matrices with entries in R. Also for $A \in GL_n(R)$, $GL_n(\varphi)(A) = \varphi(A)$ where by $\varphi(A)$ we mean φ applied to every entry of A.

Next consider the following functor.

$$(-)^\times : \mathbf{cRing} \longrightarrow \mathbf{Grp}$$
$$R \longmapsto R^\times$$
$$\varphi : R \longrightarrow S \longmapsto \varphi^\times : R^\times \longrightarrow S^\times$$

where given a ring R, $R^\times$ is the group of all units of R (a unit in R is an element $u \in R$ such that there exists a $v \in R$ with $uv = 1_R = vu$). For every $u \in R^\times$, $\varphi^\times(u) = \varphi(u)$ where $\varphi(u) \in S^\times$ since,

$$1_S = \varphi(1_R) = \varphi(uv) = \varphi(u)\varphi(v) = \varphi(v)\varphi(u)$$

It is not difficult to show that $GL_n(-)$ and $(-)^\times$ are both functors. One can construct a natural transformation between these two functors by considering the determinant as a natural transformation. That is,

$$det(-) : GL_n(-) \Rightarrow (-)^\times$$

To explicitly define $det(-)$ as a natural transformation we need a map $det(R) \in \mathbf{Grp}$ for every $R \in \mathbf{cRing}$. The map $det(R)$ is defined as follows.

$$\begin{aligned} det(R) : GL_n(R) &\longrightarrow R^\times \\ A &\longmapsto det(A) \end{aligned}$$

where $det(A)$ is indeed a unit, since $A \in GL_n(R)$ if and only if $det(A)$ is a unit in R (see for instance, Proposition VI.3.3 in [1]).

Now suppose we have a map $\varphi : R \longrightarrow S \in \mathbf{cRing}$. We must show that the following diagram commutes.

$$\begin{tikzcd} GL_n(R) \arrow[r, "GL_n(\varphi)"] \arrow[d, "det(R)"'] & GL_n(S) \arrow[d, "det(S)"] \\ R^\times \arrow[r, "\varphi^\times"'] & S^\times \end{tikzcd}$$

For $A \in GL_n(R)$ consider:

$$det(S)(GL_n(\varphi)(A)) = det(S)(\varphi(A)) = det(\varphi(A))$$

$$\varphi^\times(det(R)(A)) = \varphi^\times(det(A)) = \varphi(det(A))$$

and $det(\varphi(A)) = \varphi(det(A))$ since φ is a ring homomorphism. Hence the diagram commutes and so $det(-)$ is a natural transformation.

Definition 1.2.11. A *natural isomorphism* $\eta : F \Rightarrow G$ between two functors $F, G : \mathbb{X} \longrightarrow \mathbb{Y}$ is a natural transformation such that for all $X \in \mathbb{X}$, $\eta_X : F(X) \longrightarrow G(X)$ is an isomorphism in $\mathbb{Y}$.

Proposition 1.2.2. ([2], Proposition 7.26). *$F : \mathbb{X} \longrightarrow \mathbb{Y}$ is an equivalence of categories if and only if there exists a functor $G : \mathbb{Y} \longrightarrow \mathbb{X}$ and a pair of natural isomorphisms:*

$$\alpha : \mathbb{1}_{\mathbb{X}} \Rightarrow FG$$

$$\beta : \mathbb{1}_{\mathbb{Y}} \Rightarrow GF$$

Proposition 1.2.3. *For any categories $\mathbb{X}$ and $\mathbb{Y}$,* $\mathrm{Fun}(\mathbb{X}, \mathbb{Y})$ *called the functor category is a category with the following data:*

- *Objects: Functors $F : \mathbb{X} \longrightarrow \mathbb{Y}$.*
- *Maps: Natural transformations $F \Rightarrow G$.*

Proof. Given a functor $F : \mathbb{X} \longrightarrow \mathbb{Y}$, there is an identity natural transformation $1_F : F \Rightarrow F$ defined by taking the identity map on $F(X)$ for every $X \in \mathbb{X}$. That is, $(1_F)_X : F(X) \longrightarrow F(X)$.

For composition, suppose we have two natural transformations $\alpha : F \Rightarrow G$ and $\beta : G \Rightarrow H$ where $F, G, H : \mathbb{X} \longrightarrow \mathbb{Y}$. We need to define a natural transformation which will act as the composition of α and β, i.e. $\alpha\beta : F \Rightarrow G \Rightarrow H$. For an object $X \in \mathbb{X}$ define the composition $(\alpha\beta)_X := \alpha_X \beta_X$. We must show that $\alpha\beta$ defined above satisfies the naturality condition. But for any map $f : X \longrightarrow Y \in \mathbb{X}$, concatenating the naturality conditions for both α and β we get the following commutative diagram:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "\alpha_X"] \arrow[dd, bend right, "\alpha_X\beta_X =: (\alpha\beta)_X"'] & F(Y) \arrow[d, "\alpha_Y"'] \arrow[dd, bend left, "\alpha_Y\beta_Y =: (\alpha\beta)_Y"] \\
G(X) \arrow[r, "G(f)"'] \arrow[d, "\beta_X"] & G(Y) \arrow[d, "\beta_Y"'] \\
H(X) \arrow[r, "H(f)"'] & H(Y)
\end{tikzcd}$$

The outer square is exactly the naturality condition for $\alpha\beta$, thus $\alpha\beta$ is a natural transformation. The unit and associative laws then follow by definition. $\square$

Example 1.2.12. Recall that sets equipped with an action by a group G formed a category, denoted G-**Set** (see Example 1.1.5). We want to generalize this by thinking about a group acting on objects of any category, not just sets. Fix a group $G \in$ **Grp** and consider G as a category as in Example 1.1.4 and let $\mathbb{X}$ be any category. Then consider the functor category $\mathrm{Fun}(G, \mathbb{X})$ from Proposition 1.2.3 which will consist of the following data:

- Objects: Functors $\rho_X : G \longrightarrow \mathbb{X}$, one for every object $X \in \mathbb{X}$.
- Maps: Natural transformations $\eta : \rho_X \Rightarrow \rho_Y$.

where ρ_X denotes the functor that sends G to the object $X \in \mathbb{X}$. For all $X \in \mathbb{X}$ we define the functors ρ_X below:

$$\rho_X : G \longrightarrow \mathbb{X}$$

$$G \longmapsto X$$

$$g : G \longrightarrow G \longmapsto \rho_X(g) : X \longrightarrow X$$

Since there is only one object, G, a natural transformation $\eta : \rho_X \Rightarrow \rho_Y$ is a map η_G : $\rho_X(G) \longrightarrow \rho_Y(G)$. But by definition of ρ_X and ρ_Y, this is just a map $\eta : X \longrightarrow Y$ (we denote this map by η for convenience) such that for all maps $g : G \longrightarrow G$ the following diagram commutes, which is the naturality condition for η:

$$\begin{tikzcd} X \arrow[r, "\rho_X(g)"] \arrow[d, "\eta"'] & X \arrow[d, "\eta"] \\ Y \arrow[r, "\rho_Y(g)"'] & Y \end{tikzcd}$$

The commutativity of the above diagram tells us that for all $g \in G$,

$$\eta\rho_Y(g) = \rho_X(g)\eta$$

This is much like the condition required for a map between G-sets in Example 1.1.5. Thus, we will say that an object $X \in \mathbb{X}$ has an *action* by a group G if there exists a functor $\rho_X \in \mathit{Fun}(G, \mathbb{X})$.

Definition 1.2.12. An *adjunction* consists of two functors

$$\begin{tikzcd} \mathbb{X} \arrow[r, bend left, "F"] & \mathbb{Y} \arrow[l, bend left, "G"] \end{tikzcd}$$

and a natural isomorphism

$$\alpha : \mathbb{Y}(F(-), -) \longrightarrow \mathbb{X}(-, G(-))$$

That is, for all $X \in \mathbb{X}$ and $Y \in \mathbb{Y}$, α is a family of bijections

$$\alpha_{X,Y} : \mathbb{Y}(F(X), Y) \longrightarrow \mathbb{X}(X, G(Y))$$

We say that F is *left adjoint* to G and G is *right adjoint* to F and denote such an adjunction by $F \dashv G$.

Example 1.2.13. (For full details on this example, please refer to [1] V.3.3). Let R and S be commutative rings and denote by **R-Mod** and **S-Mod** the categories of R-modules and S-modules respectively. An S-module N is an abelian group equipped with an action by S, that is, a ring homomorphism:

$$\sigma : S \longrightarrow End(N)$$

where $End(N) := \mathbf{Ab}(N, N)$, the hom-set of N in the category of abelian groups **Ab**, which is a ring through composition.

Now let $f : R \longrightarrow S$ be a ring homomorphism. Define the precomposition functor f_* as follows:

$$f_* : \textbf{S-Mod} \longrightarrow \textbf{R-Mod}$$

$$S \xrightarrow{\sigma} End(N) \longmapsto R \xrightarrow{f} S \xrightarrow{\sigma} End(N)$$

We can define a second functor f^* by taking the tensor product over $f : R \longrightarrow S$. That is,

$$f^* : \textbf{R-Mod} \longrightarrow \textbf{S-Mod}$$

$$M \longmapsto M \otimes_R S$$

Finally, we define a third functor $f^!$ as follows:

$$\begin{aligned} f^! : \mathbf{R\text{-}Mod} &\longrightarrow \mathbf{S\text{-}Mod} \\ M &\longmapsto \mathbf{R\text{-}Mod}(S, M) \end{aligned}$$

where the hom-set $\mathbf{R\text{-}Mod}(S, M)$ has a natural S-module structure by setting $s \cdot \alpha(s') = \alpha(ss')$ for every $s, s' \in S$ and $\alpha \in \mathbf{R\text{-}Mod}(S, M)$.

Then, f_* is right adjoint to f^* and left adjoint to $f^!$. Notationally, $f^* \dashv f_* \dashv f^!$. A proof of this can be found in Proposition VIII.3.6 of [1].

Remark. The notation we used above for the precomposition functor is the same we used for the composition functor of Example 1.2.3. Whenever these functors appear, the context with which they appear will make it clear which functor we are referring to.

Theorem 1.2.1. ([19] Theorem 1 p. 93). *If a functor F is an equivalence of categories then F is both a left and right adjoint.*

1.3 Limits and Colimits

Definition 1.3.1. An object 0 in a category $\mathbb{X}$ is called an *initial object* of $\mathbb{X}$ if for every $X \in \mathbb{X}$ there is exactly one map $0 \longrightarrow X \in \mathbb{X}$.

Definition 1.3.2. An object 1 in a category $\mathbb{X}$ is called a *terminal object* of $\mathbb{X}$ if for every $X \in \mathbb{X}$ there is exactly one map $X \longrightarrow 1 \in \mathbb{X}$.

Example 1.3.1. In **Set** there is always exactly one map $\emptyset \longrightarrow A$ for every $A \in$ **Set** (the map is the empty map, vacuously defined). Thus $\emptyset$ is an initial object of **Set**.

On the other hand there is also always exactly one map $A \longrightarrow \{*\}$ for every $A \in$ **Set** by sending every $a \in A$ to the single point $\{*\}$. Thus $\{*\}$ is a terminal object of **Set**.

Example 1.3.2. Let $\mathbb{X}$ be any category, $X \in \mathbb{X}$, and consider the slice category $\mathbb{X}/X$. Then for any object $f : Y \longrightarrow X \in \mathbb{X}/X$ there is always the following map in $\mathbb{X}/X$:

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "f"'] & & X \arrow[dl, "1_X"] \\ & X & \end{tikzcd}$$

Thus $1_X : X \longrightarrow X$ is a terminal object in $\mathbb{X}/X$.

On the other hand, consider the coslice category $X/\mathbb{X}$. Then for any object $f : X \longrightarrow Y \in X/\mathbb{X}$ there is always the following map in $\mathbb{X}/X$:

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ & X \arrow[ul, "1_X"] \arrow[ur, "f"'] & \end{tikzcd}$$

Thus $1_X : X \longrightarrow X$ is an initial object in $X/\mathbb{X}$.

Proposition 1.3.1. *If $\mathbb{X}$ has a terminal object 1 then it is unique up to isomorphism. Dually, if $\mathbb{X}$ has an inital object 0 then it is also unique up to isomorphism.*

Proof. Suppose that 1 and A were both terminal objects in $\mathbb{X}$. Then for all $X \in \mathbb{X}$ there is exactly one map of the form:

$$X \longrightarrow 1$$

$$X \longrightarrow A$$

In particular, since 1 is terminal there is exactly one map $\varphi : A \longrightarrow 1$ and since A is terminal there is exactly one map $\psi : 1 \longrightarrow A$. Hence $\varphi\psi : A \longrightarrow 1 \longrightarrow A$ is the only map from $A \longrightarrow A$ so $\varphi\psi = 1_A$. So φ is a section.

On the other hand, $\psi\varphi : 1 \longrightarrow A \longrightarrow 1$ is the only map from $1 \longrightarrow 1$ so $\psi\varphi = 1_1$. So φ is also a retraction. Hence φ is an isomorphism.

The proof that the initial object is unique up to isomorphism is dual. $\square$

Throughout the previous sections, we have come accross various "diagrams" - graphs with objects and arrows between the objects. For example, the naturality condition for a natural transformation required that a certain square diagram commute. This idea of a diagram in a category can be generalized and will provide the foundation for defining limits and colimits of a category.

Definition 1.3.3. Let $\mathbb{I}$ and $\mathbb{X}$ be any two categories. Then a *diagram of type* $\mathbb{I}$ in $\mathbb{X}$ is a functor:

$$D : \mathbb{I} \longrightarrow \mathbb{X}$$

The idea behind a diagram is that the category $\mathbb{I}$ is an indexing for the objects and maps of the diagram you wish to describe in $\mathbb{X}$. For example, a diagram in the shape of a square in $\mathbb{X}$ would be the functor:

$$\begin{tikzcd} i \arrow[r, "a"] \arrow[d, "b"'] & j \arrow[d, "c"] \\ k \arrow[r, "d"'] & l \end{tikzcd} \xrightarrow{D} \begin{tikzcd} D(i) \arrow[r, "D(a)"] \arrow[d, "D(b)"'] & D(j) \arrow[d, "D(c)"] \\ D(k) \arrow[r, "D(d)"'] & D(l) \end{tikzcd}$$

Definition 1.3.4. A *cone* of a diagram $D : \mathbb{I} \longrightarrow \mathbb{X}$ consists of an object $C \in \mathbb{X}$ and a family of maps $(c_i : C \longrightarrow D(i))_{i \in \mathbb{I}}$ such that for every map $a : i \longrightarrow j \in \mathbb{I}$ the following triangle commutes:

$$\begin{tikzcd} & C \arrow[dl, bend right, "c_i"'] \arrow[dr, bend left, "c_j"] & \\ D(i) \arrow[rr, "D(a)"'] & & D(j) \end{tikzcd}$$

Let $D : \mathbb{I} \longrightarrow \mathbb{X}$ be a diagram and $(C, c_i)_{i \in \mathbb{I}}$, $(C', c'_i)_{i \in \mathbb{I}}$ be two cones of D. Then we say that a map between the cones $(C, c_i)_{i \in \mathbb{I}} \longrightarrow (C', c'_i)_{i \in \mathbb{I}}$ is a map $\varphi : C \longrightarrow C' \in \mathbb{X}$ such that for all $i \in \mathbb{I}$ the following triangle commutes:

$$\begin{tikzcd} C \arrow[rr, "\varphi"] \arrow[dr, "c_j"'] & & C' \arrow[dl, "c'_i"] \\ & D(i) & \end{tikzcd}$$

A map between the same cone, $(C, c_i)_{i\in\mathbb{I}} \longrightarrow (C, c_i)_{i\in\mathbb{I}}$ is just the identity map $1_C : C \longrightarrow C \in \mathbb{X}$ and there is a natural composition of maps between cones:

$$\begin{tikzcd}
C \arrow[r] \arrow[dr, "c_i"'] & C' \arrow[d, "c_i'"] \arrow[r] & C'' \arrow[dl, "c_i''"] \\
& D(i) &
\end{tikzcd}$$

Thus we have formed a "category of cones of the diagram D", denoted $\mathbf{Cone}(D)$.

Definition 1.3.5. Let $D : \mathbb{I} \longrightarrow \mathbb{X}$ be a diagram. Then a *limit* of D is the terminal object in $\mathbf{Cone}(D)$.

The limit of a diagram will be denoted $\lim D$. That $\lim D$ is the terminal object in the category $\mathbf{Cone}(D)$ means that $\lim D$ is in particular a cone, so it comes equipped with maps $p_i : \lim D \longrightarrow D(i)$ for every $i \in \mathbb{I}$. Additionally given any other cone C, there must exist a unique map $\alpha : C \longrightarrow \lim D$ such that all triangles between this map and the maps c_i and p_i commute for all $i \in \mathbb{I}$. This idea is illustrated in the following example:

$$\begin{tikzcd}
& C \arrow[d, dashed, "\alpha"] \arrow[ddl, "c_i"'] \arrow[ddr, "c_j"] & \\
& \lim D \arrow[dl, "p_i"] \arrow[dr, "p_j"'] & \\
D(i) \arrow[rr] & & D(j)
\end{tikzcd}$$

Where in the above diagram, we have indicated that the map α is unique by denoting it with a dashed arrow. This is standard notation which we will use for the remainder of the thesis. The above commutes at every such triangle, that is, $\alpha p_i = c_i$ for all $i \in \mathbb{I}$.

Generally speaking, we will mostly be concerned with $finite$ limits, which are limits over a diagram $D : \mathbb{I} \longrightarrow \mathbb{X}$ such that $\mathbb{I}$ is a finite category (see Example 1.1.6). We provide numerous examples below, each of which will be used throughout this thesis.

Examples of Limits.

Terminal Object. Let $\mathbb{I}$ be the initial category (no objects and no maps). Then any cone $(C, c_i)_{i \in \mathbb{I}}$ is simply the object $C \in \mathbb{X}$ (there are no maps c_i). Thus for every $C \in \mathbb{X}$ there is a unique map to the limit $\lim D$:

$$C \dashrightarrow \lim D$$

But this is precisely the definition of the terminal object. So $\lim D$ is the terminal object of $\mathbb{X}$.

Product. Let $\mathbb{I}$ be the category with exactly two objects i, j and only the identity maps for each object. For simplicity, denote $D(i) = A$ and $D(j) = B$. Then we have the following diagram for the limit $\lim D$ for every cone (C, c_i, c_j):

$$\begin{tikzcd}
 & & A \\
C \arrow[r, dashed, "\alpha"] \arrow[urr, bend left] \arrow[drr, bend right] & \lim D =: A \times B \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\
 & & B
\end{tikzcd}$$

$\lim D =: A \times B$ is called the *product* of objects A and B and the maps π_1 and π_2 are called the *projection maps*. This diagram represents the following statement: $A \times B$ is called the *product* of A and B if there exists maps $\pi_1 : A \times B \longrightarrow A$ and $\pi_2 : A \times B \longrightarrow B$ such that given an object C with maps $c_1 : C \longrightarrow A$ and $c_2 : C \longrightarrow B$ there exists a unique map $\alpha : C \longrightarrow A \times B$ such that $\alpha\pi_1 = c_1$ and $\alpha\pi_2 = c_2$.

One could also consider the product over more than two objects. If $(A_i)_{i \in I}$ is a family of objects in $\mathbb{X}$ then the product of the A_i's is denoted $\prod\limits_{i \in I} A_i$ and is equipped with projection maps $\pi_i : \prod\limits_{i \in I} A_i \longrightarrow A_i$. If I is finite, then we say $\prod\limits_{i \in I} A_i$ is a *finite* product.

Example 1.3.3. Let $A, B \in \mathbf{Set}$. Then the product of A and B is the following set:

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}$$

Equalizer. Let $\mathbb{I}$ be the category with two objects and exactly two maps between them. Then $D : \mathbb{I} \longrightarrow \mathbb{X}$ is a diagram of the following form in $\mathbb{X}$:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

Then the limit of this diagram $\lim D =: Eq(f,g)$, called the *equalizer* of f and g, is the object in $\mathbb{X}$ such that for any cone (C, c_1, c_2) the following diagram commutes:

$$\begin{tikzcd} C \arrow[d, dashed, "\alpha"'] \arrow[dr, "c_1"] & & \\ Eq(f,g) \arrow[r, "e_1"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

Observe that by the definition of cones, $e_1 f = e_2 = e_1 g$ and $c_1 f = c_2 = c_1 g$. This diagram represents the following statement: $Eq(f,g)$ is the *equalizer* of f and g if there exists a map $e_1 : Eq(f,g) \longrightarrow A$ such that $e_1 f = e_1 g$ such that given an object C with a map $c_1 : C \longrightarrow A$ such that $c_1 f = c_1 g$ there exists a unique map $\alpha : C \longrightarrow Eq(f,g)$ such that $\alpha e_1 = c_1$.

Example 1.3.4. Let $f, g : A \longrightarrow B$ be two maps in $\mathbf{Set}$. Then the equalizer of f and g is the set:

$$Eq(f,g) = \{a \in A \mid f(a) = g(a)\}$$

equipped with the inclusion map $Eq(f,g) \longrightarrow A$.

One could also ask for the limit of a diagram with an arbitrary number of maps $A \longrightarrow B$. We will also call such limits equalizers (or *finite* equalizers if there are a finite number of maps), keeping the following specific example in mind.

Example 1.3.5. Let $\mathbb{X}$ be a category with equalizers, $X \in \mathbb{X}$, and define $G := Aut(X)$, the group of automorphisms (i.e. isomorphisms) of X in $\mathbb{X}$. Then the *fixed point* of X by G, denoted X^G, is the limit of the diagram consisting of all maps $g \in G$, i.e.

$$\begin{tikzcd} X \arrow[r, shift left=2, "g"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & X \end{tikzcd}$$

That is, the fixed point object is the equalizer of the above diagram. So for all maps $\varphi : C \longrightarrow X$ such that $\varphi g = \varphi$ for all $g \in G$, there exists a unique map $\alpha : C \longrightarrow X^G$ such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[d, dashed, "\alpha"'] \arrow[dr, "\varphi"] & & \\ X^G \arrow[r, "f"'] & X \arrow[r, shift left=2, "g"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & X \end{tikzcd}$$

If $\mathbb{X} = \mathbf{Set}$ then the fixed point object $X^G = \{x \in \mathbb{X} \mid g(x) = x \text{ for all } g \in G\}$.

Pullback. Let $D : \mathbb{I} \longrightarrow \mathbb{X}$ be the diagram of the following form in $\mathbb{X}$:

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Then the limit of this diagram $limD =: X \times_Y Z$, called the *pullback* of f and g over Y, is the object in $\mathbb{X}$ such that for any cone (C, c_1, c_2) the following diagram commutes:

$$\begin{tikzcd} C \arrow[dr, dashed, "\alpha"] \arrow[drr, bend left, "c_2"] \arrow[ddr, bend right, "c_1"'] & & \\ & X \times_Y Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Z \arrow[d, "g"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

This diagram represents the following statement: $X \times_Y Z$ is the pullback of f and g over Y if there exists maps $\pi_1 : X \times_Y Z \longrightarrow X$ and $\pi_2 : X \times_Y Z \longrightarrow Z$ such that $\pi_1 f = \pi_2 g$ and given any other object $C \in \mathbb{X}$ with maps $c_1 : C \longrightarrow X$ and $c_2 : C \longrightarrow Z$ such that $c_1 f = c_2 g$ there exists a unique map $\alpha : C \longrightarrow X \times_Y Z$ such that $\alpha\pi_1 = c_1$ and $\alpha\pi_2 = c_2$.

Example 1.3.6. Let $f : X \longrightarrow Y$ and $g : Z \longrightarrow Y$ be maps in **Set**. Then the pullback of f and g over Y is the following set:

$$X \times_Y Z := \{(x, z) \in X \times Z \mid f(x) = g(z)\}$$

Example 1.3.7. Let $\mathbb{X}$ be a category with pullbacks and fix an object $S \in \mathbb{X}$. Then the pullback of $X \longrightarrow S$ and $Y \longrightarrow S$ is given by the following commuting square:

$$\begin{tikzcd} X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

But now consider the slice category $\mathbb{X}/S$. The object $X \times_S Y$ is now just the product in $\mathbb{X}/S$, i.e.

$$\begin{tikzcd} & X \\ X \times_S Y \arrow[ur] \arrow[dr] & \\ & Y \end{tikzcd}$$

This is because in $\mathbb{X}/S$ the objects *are maps* $X \longrightarrow S$. A pullback in $\mathbb{X}/S$ would be something of the form:

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

where $X \longrightarrow S, Y \longrightarrow S, Z \longrightarrow S$ are all objects in $\mathbb{X}/S$.

As we see from the above examples, all of the limits have the following "universal property": given any other object C that satisfies the original diagram in the required way, there exists a unique map from the object C to the limit. So for instance in the pullback example above, we will say that the map α exists by the "universal property of the pullback".

Before we continue, we need to discuss a few important properties of the pullback.

Lemma 1.3.1. *(Pullback Square Lemma,* [2] p. 95*). Let $\mathbb{X}$ be a category with pullbacks. Suppose we have the following diagram in $\mathbb{X}$:*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & E \arrow[r] & F \end{tikzcd}$$

(i) If the left and right squares are pullback diagrams, then so is the outer square. That is,

$$D \times_F C \cong D \times_E (E \times_F C)$$

(ii) If the rightmost square and the outer square are pullback diagrams, then so is the leftmost square.

In particular, (i) of Lemma 1.3.1 gives us a sort of “cancellation” property for pullbacks, if we have such a diagram where both inner squares are pullbacks.

Definition 1.3.6. Let $\mathbb{X}$ be a category with pullbacks and fix a map $f : X \longrightarrow Y$. Then the *pullback functor* is a functor $f^* : \mathbb{X}/Y \longrightarrow \mathbb{X}/X$ defined as follows:

$$f^* : \mathbb{X}/Y \longrightarrow \mathbb{X}/X$$

$$\begin{tikzcd} Z \arrow[d] & & f^*(Z) := X \times_Y Z \arrow[r] \arrow[d] & Z \arrow[d] \\ Y & \longmapsto & X \arrow[r, "f"'] & Y \end{tikzcd}$$

$$\begin{tikzcd} & & & X \times_Y Z \arrow[rr] \arrow[dr, dashed, "f^*(\varphi)"] \arrow[ddr, bend right] & & Z \arrow[d, "\varphi"] \\ Z \arrow[rr, "\varphi"] \arrow[dr] & & Z' \arrow[dl] & \longmapsto & X \times_Y Z' \arrow[r] \arrow[d] & Z' \arrow[d] \\ & Y & & & X \arrow[r, "f"'] & Y \end{tikzcd}$$

For any map f in a category with pullbacks, f^* is indeed a functor (see for instance Propositon 5.10 in [2]). We will frequently use this functor and the * notation will only be used for such a functor. This functor is also important since it is part of an adjunction.

Proposition 1.3.2. *Let $f : X \longrightarrow Y \in \mathbb{X}$ where $\mathbb{X}$ is a category with pullbacks. Then $f^* : \mathbb{X}/Y \longrightarrow \mathbb{X}/X$ is right adjoint to the composition functor f_* of Example 1.2.3.*

Proof. We want to show that the following pair of functors:

$$\begin{tikzcd} \mathbb{X}/Y \arrow[r, bend left, "f^*"] & \mathbb{X}/X \arrow[l, bend left, "f_*"] \end{tikzcd}$$

is an adjunction. So by definition, for all $a : A \longrightarrow X \in \mathbb{X}/X$ and $b : B \longrightarrow Y \in \mathbb{X}/Y$ we need to show that there is a bijection between the following hom-sets:

$$\mathbb{X}/Y(f_*(A), B) \longrightarrow \mathbb{X}/X(A, f^*(B))$$

We will construct two maps which are inverse to each other, α and β. First we define α which will be the map which sends a map in $\mathbb{X}/Y(f_*(A), B)$ to the map given by the universal property of the pullback $X \times_Y B$. That is,

$$\alpha : \mathbb{X}/Y(f_*(A), B) \longrightarrow \mathbb{X}/X(A, f^*(B))$$

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "af"'] & & B \arrow[dl, "b"] \\ & Y & \end{tikzcd} \longmapsto \begin{tikzcd} A \arrow[dr, dashed, "\alpha(\varphi)"] \arrow[drr, bend left, "\varphi"] \arrow[ddr, bend right, "a"'] & & \\ & X \times_Y B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "b"] \\ & X \arrow[r, "f"'] & Y \end{tikzcd}$$

Next we need to define β. The map β will be post composition with the projection map $\pi_2 : X \times_Y B \longrightarrow B$. That is,

$$\beta : \mathbb{X}/X(A, f^*(B)) \longrightarrow XX/Y(f_*(A), B)$$

$$\begin{tikzcd} A \arrow[rr, "\psi"] \arrow[dr, "a"'] & & X \times_Y B \arrow[dl, "\pi_1"] \\ & X & \end{tikzcd} \longmapsto \begin{tikzcd} A \arrow[r, "\psi"] \arrow[rr, bend left, "\beta(\psi)"] \arrow[drr, "af"'] & X \times_Y B \arrow[r, "\pi_2"] & B \arrow[d, "b"] \\ & & Y \end{tikzcd}$$

So it remains to show that $\alpha(\beta(\psi)) = \psi$ and $\beta(\alpha(\varphi)) = \varphi$. By definition of α, $\alpha(\beta(\psi))$ is defined by the universal property of the pullback $X \times_Y B$:

$$\begin{tikzcd}
A \arrow[rr, "\psi"] \arrow[dr, dashed, "\alpha(\beta(\psi))"] \arrow[ddr, bend right, "a"'] & & X \times_Y B \arrow[d, "\pi_2"] \\
& X \times_Y B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "b"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

But ψ is also valid for the unique map $A \longrightarrow X \times_Y B$ above since $\psi\pi_1 = a$ by definition of ψ. So by the uniqueness, $\alpha(\beta(\psi)) = \psi$.

On the other hand, consider $\beta(\alpha(\varphi))$. By definition of β, $\beta(\alpha(\varphi))$ is $\alpha(\varphi)\pi_2$, but by the pullback diagram defining $\alpha(\varphi)$, $\alpha(\varphi)\pi_2 = \varphi$. Hence $\beta(\alpha(\varphi)) = \varphi$.

Thus we have shown the required bijection. $\square$

Definition 1.3.7. Let $\mathbb{X}$ be a category with pullbacks. Then given a map $f : X \longrightarrow Y \in \mathbb{X}$, a property P of a map $\varphi : Z \longrightarrow Y$ is *stable under pullback by f* if the map $\pi_1 : X \times_Y Z \longrightarrow X$ has property P. That is,

$$\begin{tikzcd}
X \times_Y Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Z \arrow[d, "\varphi"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Example 1.3.8. A monic map is always stable under pullback. To see this, suppose $m : Z \longrightarrow Y$ is monic. Then consider the map $\pi_1 : X \times_Y Z \longrightarrow X$, the projection map from the pullback of m along f. Suppose that we had maps $p, q : A \longrightarrow X \times_Y Z$ such that $p\pi_1 = q\pi_1$. We need to show that $p = q$.

To show that $p = q$, we will show that both p and q satisfy the same universal property of the pullback. First observe that by the commutativity of the pullback square,

$$p\pi_2\varphi = p\pi_1 f = q\pi_1 f = q\pi_2\varphi$$

Since φ was assumed to be monic, we have: $p\pi_2 = q\pi_2$. This, along with the assumption $p\pi_1 = q\pi_1$ gives us the following by the universal property of the pullback:

$$\begin{tikzcd}
A \arrow[dr, dashed, "{p,q}"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Y Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Z \arrow[d, "\varphi"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

But by the above, both p and q will make the above diagram commute (where the maps $A \longrightarrow X$ and $A \longrightarrow Z$ are $p\pi_1 = q\pi_1$ and $p\pi_2 = q\pi_2$ respectively). Hence by the uniqueness, $p = q$.

There is a natural dual to the notion of a limit which we call a colimit ("co" is often used to refer to a dual notion). Colimits are built up from the category of *cocones*, denoted $\mathbf{coCone}(D)$ which consists of the following data for some diagram $D : \mathbb{I} \longrightarrow \mathbb{X}$:

- Objects: Cocones - $(C, c_i)_{i\in\mathbb{I}}$ where $C \in \mathbb{X}$ and $c_i : D(i) \longrightarrow C$ such for all $a : i \longrightarrow j \in \mathbb{I}$ the following triangle commutes:

$$\begin{tikzcd}
D(i) \arrow[rr, "D(a)"] \arrow[dr, bend right, "c_i"'] & & D(j) \arrow[dl, bend left, "c_j"] \\
& C &
\end{tikzcd}$$

- Maps: $\varphi : (C, c_i)_{i\in\mathbb{I}} \longrightarrow (C', c'_i)_{i\in\mathbb{I}}$ is a map $\varphi : C \longrightarrow C' \in \mathbb{X}$ such that for all $i \in \mathbb{I}$ the following triangle commutes:

$$\begin{tikzcd}
& D(i) \arrow[dl, "c_i"'] \arrow[dr, "c'_i"] & \\
C \arrow[rr, "\varphi"'] & & C'
\end{tikzcd}$$

Definition 1.3.8. Let $D : \mathbb{I} \longrightarrow \mathbb{X}$ be a diagram. Then a *colimit* of D is the initial object in $\mathbf{coCone}(D)$.

The colimit of a diagram D will be denoted colim$D$. The idea of the colimit is illustrated in the below diagram (think of the diagram for the limit below Definition 1.3.5, but with all the maps reversed).

$$\begin{tikzcd}
D(i) \arrow[rr] \arrow[dr, "c_i"] \arrow[ddr, "p_i"'] & & D(j) \arrow[dl, "c_j"'] \arrow[ddl, "p_j"] \\
& C \arrow[d, dashed, "\alpha"] & \\
& \mathrm{colim}D &
\end{tikzcd}$$

A colimit is called $finite$ if the category $\mathbb{I}$ of the diagram D is a finite category. We provide some common examples below.

Examples of Colimits.

Initial Object. Take $\mathbb{I}$ to be the initial category and consider $D : \mathbb{I} \longrightarrow \mathbb{X}$. Then the colimit of D, colim$D$, is the object such that for all cones $C \in \mathbb{X}$ (there are no maps c_i since there are no objects $D(i)$) there exists a unique map:

$$C \dashleftarrow \mathrm{colim}D$$

Thus colim$D$ is the initial object of $\mathbb{X}$.

Coproduct. The colimit of the diagram D of two objects A, B with no maps between them is called the *coproduct* of A and B, denoted $A \coprod B$. It is defined in the same way as the product, but with all maps reversed. That is, for all cocones (C, c_1, c_2) there exists a unique map α such that the following diagram commutes:

$$\begin{tikzcd}
& & A \arrow[dl, "\iota_1"] \arrow[dll, bend right, "c_1"'] \\
C & A \coprod B \arrow[l, dashed, "\alpha"'] & \\
& & B \arrow[ul, "\iota_2"'] \arrow[ull, bend left, "c_2"]
\end{tikzcd}$$

Example 1.3.9. Let $A, B \in \mathbf{Set}$. Then the coproduct of A and B is the disjoint union of the sets.

Just like we did with products, we can consider arbitrary coproducts over families of objects $(X_i)_{i\in I}$. The coproduct of this family would then be denoted $\coprod\limits_{i\in I} X_i$ with inclusion maps $X_i \longrightarrow \coprod\limits_{i\in I} X_i$. If I is finite, then we say that $\coprod\limits_{i\in I} X_i$ is a $finite$ coproduct.

Coequalizer. The colimit of the diagram D of two maps $f, g : A \longrightarrow B$ between two objects $A, B \in \mathbb{X}$ is called the coequalizer of f and g, denoted $Coeq(f, g)$. Again, it is defined in the same way as the equalizer, but with all maps reversed. So for all cocones (C, c_1) there exists a unique map α such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "c_1"'] & Coeq(f,g) \arrow[d, dashed, "\alpha"] \\
& & C
\end{tikzcd}$$

Example 1.3.10. Recall Example 1.3.5 where we defined a special equalizer called the fixed point. Here we do the dual. Let $\mathbb{X}$ be a category with coequalizers, let $X \in \mathbb{X}$, and denote $G := Aut(X)$, the group of automorphisms of X in $\mathbb{X}$. Then the *quotient* of X by G, denoted X_G (or alternatively X/G), is the coequalizer of the diagram consisting of all maps $g : X \longrightarrow X \in G$, i.e.

$$\begin{tikzcd}
X \arrow[r, shift left=2, "g"] \arrow[r, shift right=2] \arrow[r, phantom, "\vdots"] & X
\end{tikzcd}$$

That is, X_G is the object equipped with a map $q : X \longrightarrow X_G$ such that given any other object C with a map $\varphi : X \longrightarrow C$ such that $g\varphi = g$ for all $g \in G$ there exists a unique map

$\alpha : X_G \longrightarrow C$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, shift left, "g"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right] & X \arrow[r, "q"] \arrow[dr, "\varphi"'] & X_G \arrow[d, dashed, "\alpha"] \\ & & C \end{tikzcd}$$

Proposition 1.3.3. *Let $\mathbb{X}$ be a category with coequalizers. Then any coequalizer map is epic.*

Proof. Suppose we have the following coequalizer $Coeq(f, g)$ of maps $f, g : A \longrightarrow B$:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] & Coeq(f,g) \end{tikzcd}$$

We want to show that q is epic. So suppose we have two maps $r, s : Coeq(f, g) \longrightarrow C$ such that $qr = qs$. We need to show that $r = s$.

Notice that $f(qr) = (fq)r = (gq)r = g(qr)$. So by the universal property of the coequalizer, we have the following diagram:

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \arrow[r, "q"] \arrow[dr, "qr=qs"'] & Coeq(f,g) \arrow[d, dashed] \\ & & C \end{tikzcd}$$

Both r and s make this diagram commute, so by uniqueness $r = s$. $\square$

Pushout. The pushout is, according to our nomenclature, the "copullback", but the name pushout is used. It is the colimit of the following diagram for objects $X, Y, Z \in \mathbb{X}$ and maps $f, g \in \mathbb{X}$:

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "g"'] & Z \\ X & \end{tikzcd}$$

The pushout will be denoted $X \otimes_Y Z$. It is a colimit, so for all cocones (C, c_1, c_2) there exists a unique map $\alpha : X \otimes_Y Z \longrightarrow C$ such that the following diagram commutes:

$$\begin{tikzcd}
Y \arrow[r, "f"] \arrow[d, "g"'] & Z \arrow[d] \arrow[ddr, bend left, "c_1"] & \\
X \arrow[r] \arrow[drr, bend right, "c_2"'] & X \otimes_Y Z \arrow[dr, dashed, "\alpha"] & \\
& & C
\end{tikzcd}$$

Example 1.3.11. In **cRing** the pushout of two ring homomorphisms $f : B \longrightarrow A$ and $g : B \longrightarrow C$ is the tensor product. That is, the following is a pushout square:

$$\begin{tikzcd}
B \arrow[r, "g"] \arrow[d, "f"'] & C \arrow[d] \\
A \arrow[r] & A \otimes_B C
\end{tikzcd}$$

Proposition 1.3.4. ([2] p.100). *A category has finite products and equalizers if and only if it has pullbacks and a terminal object.*

Proposition 1.3.5. ([2] p. 104). *A category has all finite limits if and only if it has pullbacks and a terminal object.*

Each of the above propositions can be dualized. For instance a category has all finite colimits if and only if it has pushouts and an initial object.

Definition 1.3.9. A functor $F : \mathbb{X} \longrightarrow \mathbb{Y}$ *preserves limits* if given a diagram $D : \mathbb{I} \longrightarrow \mathbb{X}$ and the limit $(\lim D, \pi_i)_{i \in \mathbb{I}}$, the cone $(F(\lim D), F(\pi_i))_{i \in \mathbb{I}}$ is the limit for the diagram $DF : \mathbb{I} \longrightarrow \mathbb{X} \longrightarrow \mathbb{Y}$.

Definition 1.3.10. A functor $F : \mathbb{X} \longrightarrow \mathbb{Y}$ *reflects limits* if given a diagram $D : \mathbb{I} \longrightarrow \mathbb{X}$ and a cone $(C, \pi_i)_{i \in \mathbb{I}}$ in $\mathbb{X}$ such that $(F(C), F(\pi_i))_{i \in \mathbb{I}}$ is a limit of the diagram $DF : \mathbb{I} \longrightarrow \mathbb{X} \longrightarrow \mathbb{Y}$ then the original cone $(C, \pi_i)_{i \in \mathbb{I}}$ is a limit of the diagram $D : \mathbb{I} \longrightarrow \mathbb{X}$.

Theorem 1.3.1. ([2] p. 225). *Right adjoints preserve limits and left adjoints preserve colimits.*

Example 1.3.12. By Proposition 1.3.2 given a map $f : X \longrightarrow Y$ in $\mathbb{X}$ a category with pullbacks, the pullback functor f^* is a right adjoint. Hence f^* preserves limits.

1.4 Extensive Categories and Connected Objects

In this section, we will define what it means for a category to be extensive and what connected means in the categorical sense. For brevity, most proofs have been omitted, but the work below on extensive categories can all be found in [6], unless otherwise stated.

Much like we used pullbacks to define the pullback functor in Definition 1.3.6 we can also use coproducts to form a coproduct functor.

Definition 1.4.1. ([5] p.194). Let $\mathbb{X}$ be a category with coproducts and let $(X_i)_{i\in I}$ be a family of objects in $\mathbb{X}$. Then the *coproduct functor* on $(X_i)_{i\in I}$, denoted $\coprod(-)$ is defined as follows:

$$\coprod(-) : \prod_I(\mathbb{X}/X_i) \longrightarrow \mathbb{X}\Big/\coprod_I X_i$$

$$\left(\begin{array}{c} Y_i \\ \big\downarrow f_i \\ X_i \end{array}\right)_{i\in I} \longmapsto \begin{array}{c} \coprod_I Y_i \\ \big\downarrow \coprod f_i \\ \coprod_I X_i \end{array}$$

$$\left(\begin{tikzcd} Y_i \arrow[rr, "\varphi_i"] \arrow[dr, "f_i"'] & & Y_i' \arrow[dl, "f_i'"] \\ & X_i & \end{tikzcd}\right)_{i\in I} \longmapsto \begin{tikzcd} Y_i \arrow[rrr, "\varphi_i"] \arrow[dr] & & & Y_i' \arrow[dl] \\ & \coprod_I Y_i \arrow[r, dashed, "\coprod \varphi_i"] \arrow[dr, "\coprod f_i"'] & \coprod_I Y_i' \arrow[d, "\coprod f_i'"] & \\ & & \coprod_I X_i & \end{tikzcd}$$

The map $\coprod \varphi_i$ above is defined via the universal property of the coproduct $\coprod_I Y_i$, since for every Y_i we have a map:

$$Y_i \xrightarrow{\varphi_i} Y_i' \longrightarrow \coprod_I Y_i'$$

Definition 1.4.2. A category $\mathbb{X}$ with coproducts is called *extensive* if for all families of objects $(X_i)_{i\in I}$ the coproduct functor $\coprod(-)$ of Definition 1.4.1 is an equivalence of categories.

Note that the index I is not necessarily finite. If $\mathbb{X}$ is a category with finite coproducts and I is finite, we say that $\mathbb{X}$ is *finitely extensive.* If $\mathbb{X}$ has all coproducts (I is any set) then we say that $\mathbb{X}$ is *infinitary extensive.* When necessary, the distinction will be made clear.

Example 1.4.1. The category of topological spaces, **Top**, is extensive ([5] 6.4).

Proposition 1.4.1. ([16] Proposition 1.1). *Let $\mathbb{X}$ be a category with coproducts. Then,*

(1) $\mathbb{X}$ is finitely extensive if and only if the pullbacks along inclusions exist and for any commutative diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \arrow[d] & Y \arrow[l] \arrow[d] \\ A \arrow[r] & A \coprod B & B \arrow[l] \end{tikzcd}$$

the top row is a coproduct if and only if both inner squares are pullbacks.

(2) $\mathbb{X}$ is extensive if and only if the following two conditions are satisfied:

(i) For any pair of maps $f : X \longrightarrow A$ and $g : Y \longrightarrow B$,

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X \coprod Y \arrow[d, "f \coprod g"] \\ A \arrow[r] & A \coprod B \end{tikzcd}$$

is a pullback (i.e. $X \cong A \times_{A \coprod B} X \coprod Y$).

(ii) For every map $f : Z \longrightarrow X \coprod Y$ there exists maps $Z_X \longrightarrow X$ and $Z_Y \longrightarrow Y$ and an isomorphism $\alpha : Z_X \coprod Z_Y \longrightarrow Z$ such that the following diagram commutes:

$$\begin{tikzcd}
Z_X \arrow[r] \arrow[dd] & Z_X \coprod Z_Y \arrow[d, "\alpha", "\cong"'] & Z_Y \arrow[l] \arrow[dd] \\
& Z \arrow[d, "f"] & \\
X \arrow[r] & X \coprod Y & Y \arrow[l]
\end{tikzcd}$$

Definition 1.4.3. In a category $\mathbb{X}$ with coproducts, a coproduct $X \coprod Y$ is called *disjoint* if

(i) The inclusions $X \longrightarrow X \coprod Y$ and $Y \longrightarrow X \coprod Y$ are monic.

(ii) The pullback along the inclusions is the initial object.

An arbitrary coproduct $\coprod_I X_i$ is called *disjoint* if all inclusions $X_i \longrightarrow \coprod_I X_i$ are monic and for any two inclusions $X_i \longrightarrow \coprod_I X_i$ and $X_j \longrightarrow \coprod_I X_i$ the pullback is the initial object.

Example 1.4.2. Take the coproduct of two objects $A, B \in \mathbf{Set}$ for instance. Then the inclusion maps $\iota_A : A \longrightarrow A \coprod B$ and $\iota_B : B \longrightarrow A \coprod B$ are certainly injective, hence also monic. By definition of the pullback in **Set**,

$$\begin{aligned} A \times_{A \sqcup B} B &:= \{(a, b) \in A \times B \mid \iota_A(a) = \iota_B(b)\} \\ &= \emptyset \end{aligned}$$

Hence coproducts in **Set** are disjoint.

Proposition 1.4.2. ([6] Proposition 2.6). *In an extensive category, all coproducts are disjoint.*

So in an extensive category, given a coproduct $X \coprod Y$ the following are each pullback diagrams:

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d, "1_X"'] & X \arrow[d] \\ X \arrow[r] & X \coprod Y \end{tikzcd} \quad \begin{tikzcd} Y \arrow[r, "1_Y"] \arrow[d, "1_Y"'] & Y \arrow[d] \\ Y \arrow[r] & X \coprod Y \end{tikzcd} \quad \begin{tikzcd} 0 \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X \coprod Y \end{tikzcd}$$

where the first two pullbacks illustrate that the inclusions are monic and the third illustrates that the pullback along each inclusion is the initial object.

Definition 1.4.4. Let $\mathbb{X}$ be any locally small category. Then $C \in \mathbb{X}$ is *connected* if the functor:

$$\mathbb{X}(C, -) : \mathbb{X} \longrightarrow \mathbf{Set}$$

preserves coproducts.

Theorem 1.4.1. ([9] Theorem 2.1, p. 141). *Let $\mathbb{X}$ be an infinitary extensive category. Then the following are equivalent:*

(i) C is connected.

(ii) C is not the initial object and if $C \cong A \coprod B$ then either A or B is isomorphic to C.

(iii) C is not the initial object and if $C \cong A \coprod B$ then either A or B is the initial object.

(iv) Any map $C \longrightarrow \coprod A_i$ factors through one of the coproduct inclusions $A_i \longrightarrow \coprod A_i$.

Example 1.4.3. In **Top**, a topological space X is connected in the usual sense if and only if it is a connected object in **Top** ([5] Proposition 6.1.4).

Example 1.4.4. Let G-**Set** be the category of sets equipped with an action by a group G. An action by G on a set X is called *transitive* if $X \neq \emptyset$ and for all $x, y \in X$ there exists a $g \in G$ such that $g \cdot x = y$. Then $(A, \rho) \in G$-**Set** is a connected object if and only if the action of G on A is transitive ([17]).

The notion of a connected object in a category should not be confused with what it means for a *category* to be connected.

Definition 1.4.5. A *connected category* $\mathbb{X}$ is a category that is not the initial category and for every two objects $X, Y \in \mathbb{X}$ there exists a zigzag of maps between them:

$$\begin{tikzcd} & A_1 \arrow[dl] \arrow[dr] & & \arrow[dl] & \cdots & \arrow[dr] & \\ A & & A_2 & & & & B \end{tikzcd}$$

Intuitively, in a connected category one can always draw a "line" between any two objects by following the maps between the objects (regardless of the direction of the maps).

Example 1.4.5. The following finite category is connected:

$$\begin{tikzcd} A \arrow[loop left, "1_A"] \arrow[r] & B \arrow[loop right, "1_B"] \end{tikzcd}$$

Definition 1.4.6. A *connected limit* (resp. *connected colimit*) is a limit (resp. colimit) over a diagram $D : \mathbb{I} \longrightarrow \mathbb{X}$ such that $\mathbb{I}$ is a connected category.

Example 1.4.6. Pullbacks are connected limits, since the pullback in $\mathbb{X}$ of maps $f : X \longrightarrow Y$ and $g : Z \longrightarrow Y$ is the limit of the following diagram $D : \mathbb{I} \longrightarrow \mathbb{X}$:

$$\begin{tikzcd} & i \arrow[d] & \xrightarrow{D} & & Z := D(i) \arrow[d, "g"] \\ j \arrow[r] & k & & D(j) =: X \arrow[r, "f"'] & Y := D(k) \end{tikzcd}$$

where $\mathbb{I}$ above is certainly connected as a category.

In a similar fashion, equalizers are also connected limits. However notice that a product is not a connected limit, since a product of two objects X, Y is the limit the diagram with two objects and no maps between them.

Dually, pushouts and coequalizers are connected colimits, but the coproduct is not a connected colimit.

Proposition 1.4.3. *Let $\mathbb{I}$ be a finite connected category, $D : \mathbb{I} \longrightarrow \mathbb{X}$ a diagram, and suppose that every object in $\mathbb{X}$ is connected. Then the finite colimit of D is itself a connected object in $\mathbb{X}$.*

Chapter 2

Galois Categories and the Family Category

Introduction

This chapter is devoted to an investigation of Galois categories. In [10] Grothendieck gave a purely axiomatic approach to the definition of a Galois category. The goal of this chapter is to study the following question: given any category $\mathbb{X}$, what requirements are necessary on $\mathbb{X}$ in order to construct a Galois category? Of course, our previous sentence has just answered this question - Grothendieck has already described this. However, we wish to take a slightly more general approach invoking a notion called the "family category".

In Section 2.1 we introduce Grothendieck's Galois categories. The material presented in that section is all due to Grothendieck in [10], but we use [17] as a main reference. In Section 2.2 we introduce the family category, which will be our main tool for constructing a Galois category. Then the next two Sections, 2.3 and 2.4 are a verification of the six axioms for the category $Fam(\mathbb{X}/S)$ which will be defined at the beginning of Section 2.3.

We note here that there will be a distinct lack of examples throughout this chapter. This is because the focus of this chapter is to study the axioms that make a Galois category, not the examples. In the end, we will have constructed an example: an example of a Galois category.

2.1 Galois Categories

Definition 2.1.1. ([17] Definition 3.1). Let $\mathbb{X}$ be a category and $F : \mathbb{X} \longrightarrow \mathbf{fset}$ a functor. Then $\mathbb{X}$ is a *Galois category* with F its *fundamental functor* if the following six axioms hold:

(G1) $\mathbb{X}$ has a terminal object and pullbacks.

(G2) $\mathbb{X}$ has coproducts, an initial object, and quotients by a finite group of automorphisms.

(G3) Any map $f \in \mathbb{X}$ can be written as $f = em$ where e is epic and m is monic. Furthermore, any monic map $m : X \longrightarrow Y$ is an isomorphism of X with a direct summand of Y.

(G4) F preserves terminal objects and pullbacks.

(G5) F preserves coproducts, epimorphisms, and quotients by a finite group of automorphisms.

(G6) F reflects isomorphisms.

Discussion.

(G1) By Proposition 1.3.5 if **G1** is satisfied for $\mathbb{X}$ then $\mathbb{X}$ has all finite limits.

(G2) Recall from Example 1.3.10 that quotients are special types of coequalizers. Let $H \subset Aut(X)$ be a subgroup of the automorphism group of the object $X \in \mathbb{X}$. Then the quotient of X by H, denoted X_H (or X/H) is the coequalizer of the maps $h : X \longrightarrow X \in H \subset Aut(X) \subset \mathbb{X}_1$. That is, we have the following coequalizer diagram:

$$\begin{tikzcd} X \arrow[r, shift left, "h"] \arrow[r, shift right, "\vdots"'] & X \arrow[r, "q"] \arrow[dr] & X_H \arrow[d, dashed] \\ & & Y \end{tikzcd}$$

(G3) The first property of this axiom is straightforward. For the second, let $m : X \longrightarrow Y$ be a monic map in $\mathbb{X}$. Then for m to be an isomorphism with a direct summand of Y means that there exists some map $m' : Z \longrightarrow Y$ such that m, m' together with Y is the coproduct of X and Z. In particular, we have the following coproduct:

$$\begin{tikzcd} & X \arrow[dl, "m"'] \\ Y \cong X \coprod Z & \\ & Z \arrow[ul, "m'"] \end{tikzcd}$$

The other axioms regarding the fundamental functor F are simply requiring that F preserve certain limits and colimts and epics and reflect isomorphisms. (see Definitions 1.2.7 and 1.2.8 for preservation and reflection properties of a functor).

Example 2.1.1. **fset**, the category of finite sets, is certainly a Galois category, with the identity functor **fset** $\longrightarrow$ **fset** as the fundamental functor.

Example 2.1.2. Let $S \in$ **Top** be connected and choose a base point $s \in S$. Let $fCov(S)$ be the category of finite covering spaces of S - that is, covering spaces $p : X \longrightarrow S$ (i.e. as in Example 1.1.12) such that $p^{-1}(s)$ as a set is finite. Let F_s be the following functor:

$$F_s : fCov(S) \longrightarrow \mathbf{fset}$$

$$\begin{tikzcd} X \arrow[d, "p"] \\ S \end{tikzcd} \longmapsto p^{-1}(s)$$

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "p"'] & & Y \arrow[dl, "q"] \\ & S & \end{tikzcd} \longmapsto p^{-1}(s) \longrightarrow q^{-1}(s)$$

Then $fCov(S)$ is a Galois category with fundamental functor F_s given above. For details, see [17] 3.7.

Galois categories have an equivalent characterization which is often taken to be the definition, rather than the six axioms listed above. A Galois category is a category $\mathbb{X}$ which is equivalent to the category π-**fset** - finite sets equipped with a continuous action by a profinite group π. We shall describe how given a Galois category $\mathbb{X}$ as in Definition 2.1.1 we can construct an appropriate profinite group π so that $\mathbb{X}$ will be equivalent to π-**fset**. Because this material does not appear in the rest of this thesis, we shall provide only an outline. This material can be found in full in [17] and [18] and of course by the original author Grothendieck in [10].

For the remainder of this section, let $\mathbb{X}$ be a Galois category with $F : \mathbb{X} \longrightarrow$ **fset** its fundamental functor.

Definition 2.1.2. An *automorphism* of F is a natural transformation $\sigma : F \Rightarrow F$ such that for all $X \in \mathbb{X}$ each map $\sigma_X : F(X) \longrightarrow F(X)$ is a bijection.

Since an automorphism σ of F is a natural transformation, for any map $f : X \longrightarrow Y \in \mathbb{X}$ the following square (the naturality condition of σ) commutes:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle \sigma_Y} \\ F(X) & \xrightarrow[F(f)]{} & F(Y) \end{array}$$

We can then form the *automorphism group* of F, $Aut(F)$, by taking the set of all automorphisms of F. Then we want to show that $Aut(F)$ is a profinite group, so we will need a few definitions. This is the bare minimum of profinite groups, for details see [18] or [21].

Definition 2.1.3. A *topological group* is a group G, with a given topology, such that the multiplication map:

$$G \times G \longrightarrow G$$
$$(g, h) \longmapsto gh$$

and the inversion map

$$G \longrightarrow G$$
$$g \longmapsto g^{-1}$$

are continuous, with respect to the given topology on G.

Definition 2.1.4. A topological group G is *profinite* if G is

(i) Hausdorff

(ii) Compact

(iii) Totally disconected - the largest connected subsets are single points.

Example 2.1.3. A simple example: any finite group is profinite, when given the discrete topology.

One can show that $Aut(F)$ is in fact a profinite group (see for instance Section 3.4 in [17] or for a more detailed analysis, see Section 2.1.6 in [11]).

Next we need to show that we have an action of $Aut(F)$ on $F(X)$ for any $X \in \mathbb{X}$ and that this action is continuous. Define the action on $F(X)$ as follows:

$$\rho_X : Aut(F) \times F(X) \longrightarrow F(X)$$
$$(\sigma, y) \longmapsto \sigma_X(y) =: \sigma \cdot y$$

Then one can show that this action is indeed continuous [17]. The main part of this is that the action is continuous if and only if the kernel of the action is open as a subset of $Aut(F)$ (see [17] 1.19).

With this, we can define a functor:

$$H : \mathbb{X} \longrightarrow Aut(F)\text{-}\mathbf{fset}$$

$$X \longmapsto (F(X), \rho_X)$$

$$f \longmapsto F(f)$$

The functor H will be our equivalence. That is,

Theorem 2.1.1. ([17] Theorem 3.5). *Let $\mathbb{X}$ be a Galois category with fundamental functor F. Then,*

*(i) The functor $H : \mathbb{X} \longrightarrow Aut(F)$-**fset** is an equivalence of categories.*

*(ii) If π is a profinite group such that $\mathbb{X} \simeq \pi$-**fset** then π is canonically isomorphic to $Aut(F)$.*

(iii) If F' is a second fundamental functor on $\mathbb{X}$, then F and F' are isomorphic.

*(iv) If π is a profinite group such that $\mathbb{X} \simeq \pi$-**fset** then there is an isomorphism of profinite groups $\pi \cong Aut(F)$ that is determined up to an inner automorphism of $Aut(F)$.*

2.2 The Family Category

Definition 2.2.1. ([5], 6.1). Let $\mathbb{X}$ be any category. Then define the *family category*, denoted $Fam(\mathbb{X})$ to be the category consisting of the following data:

- Objects: $(X_i)_{i \in I}$ for some indexing set $I \in$ **fset**.
- Maps: $(\Lambda, \lambda) : (X_i)_{i \in I} \longrightarrow (Y_j)_{j \in J}$ where

$\Lambda : I \longrightarrow J$ is a map in **fset**.

$\lambda_i : X_i \longrightarrow Y_{\Lambda(i)}$ is a map in $\mathbb{X}$ for every $i \in I$.

Remark. We see that $Fam(\mathbb{X})$ is indeed a category, we need an identity map and a composition rule. The identity map on an object $(X_i)_{i\in I} \in Fam(\mathbb{X})$ is given by the identity maps in **fset** and $\mathbb{X}$ as follows:

$$(X_i)_{i\in I} \longrightarrow (X_i)_{i\in I}$$
$$1_I : I \longrightarrow I$$
$$1_{X_i} : X_i \longrightarrow X_i$$

Composition between two maps $(\Lambda, \lambda) : (X_i)_{i\in I} \longrightarrow (Y_j)_{j\in J}$ and $(\Phi, \varphi) : (Y_j)_{j\in J} \longrightarrow (Z_k)_{k\in K}$ is given by composition at each step in **fset** and $\mathbb{X}$ i.e.

$$(X_i)_{i\in I} \longrightarrow (Y_j)_{j\in J} \longrightarrow (Z_k)_{k\in K}$$
$$I \longrightarrow J \longrightarrow K$$
$$X_i \longrightarrow Y_{\Lambda(i)} \longrightarrow Z_{\Phi(\Lambda(i))}$$

Example 2.2.1. $Fam(\textbf{fset})$ is equivalent to **fset** under the following two functors:

$$F : Fam(\textbf{fset}) \longrightarrow \textbf{fset}$$
$$(X_i)_{i\in I} \longmapsto \bigsqcup_{i\in I} X_i$$

$$G : \textbf{fset} \longrightarrow Fam(\textbf{fset})$$
$$A \longmapsto (\{*\})_A$$

This example provides the intuition that the family category is very similar to a category of coproducts. Indeed, the family category of $\mathbb{X}$ is often referred to as the "coproduct completion" of $\mathbb{X}$ (for instance, in [5] Remark 6.2.2). Below, we will see how the family category interacts with connected objects. But first, we need a lemma.

Lemma 2.2.1. *Let $\mathbb{X}$ be a category with coproducts. Then for any $\coprod_I A_i \in \mathbb{X}$ and $X \in \mathbb{X}$,*

$$\mathbb{X}(\coprod_I A_i, X) \cong \prod_I \mathbb{X}(A_i, X)$$

Proof. The bijection will be constructed using the inclusion maps $a_i : A_i \longrightarrow \coprod_I A_i$ and the universal property of the coproduct. Note that we will denote an object or a map in the product as a tuple, but this is not to be confused with the family category notation. Consider:

$$\mathbb{X}(\coprod_I A_i, X) \longrightarrow \prod_I \mathbb{X}(A_i, X)$$

$$\coprod_I A_i \xrightarrow{f} X \longmapsto (A_i \xrightarrow{a_i} \coprod_I A_i \xrightarrow{f} X)_{i \in I}$$

So this map is just precomposing f with a_i for all $i \in I$. And for the other direction,

$$\prod_I \mathbb{X}(A_i, X) \longrightarrow \mathbb{X}(\coprod_I A_i, X)$$

$$\begin{tikzcd}
& & & A_i \arrow[dl, "a_i"] \arrow[dll, bend right, "\varphi_i"'] \\
(\varphi_i : A_i \longrightarrow X)_I \longmapsto & X & \coprod_I A_i \arrow[l, dashed] & \\
& & & A_j \arrow[ul, "a_j"'] \arrow[ull, bend left, "\varphi_j"]
\end{tikzcd}$$

Then it is straightforward to check that these maps are inverse to each other. Hence we get our desired bijection. □

Definition 2.2.2. ([9] Definition 2.2 (c), p. 142). A category $\mathbb{X}$ is called *locally connected* if for any object $X \in \mathbb{X}$,

$$X = \coprod_I X_i$$

where each X_i is connected in $\mathbb{X}$.

Definition 2.2.3. Let $\mathbb{X}$ be a category with connected objects. Then $Conn(\mathbb{X})$ is the category consisting of the following data:

- Objects: $X \in \mathbb{X}$ such that X is connected as an object in $\mathbb{X}$.
- Maps: $f : X \longrightarrow Y \in \mathbb{X}$ such that X and Y are connected as objects in $\mathbb{X}$.

With this, we come to an important Proposition.

Proposition 2.2.1. *Let $\mathbb{X}$ be a locally connected category. Then there is an equivalence of categories,*

$$\mathbb{X} \simeq Fam(Conn(\mathbb{X}))$$

Proof. $\mathbb{X}$ being locally connected means that for any object $X \in \mathbb{X}$ we can write X as a coproduct of connected objects, called the connected components of X, i.e.

$$X = \coprod_I X_i$$

Now we need to define a functor between $\mathbb{X}$ and $Fam(Conn(\mathbb{X}))$. Consider:

$$F : \mathbb{X} \longrightarrow Fam(Conn(\mathbb{X}))$$
$$X = \coprod_I X_i \longmapsto (X_i)_{i \in I}$$

where $(X_i)_{i\in I} \in Fam(Conn(\mathbb{X}))$ since each X_i is connected by definition. To define a functor, we need to see how it acts on the maps. So let $f : X \longrightarrow Y$ be a map in $\mathbb{X}$. Then f is of the following form:

$$f : \coprod_I X_i \longrightarrow \coprod_I Y_j$$

By Lemma 2.2.1 the map f is in one-to-one corresondence with a tuple $(X_i \longrightarrow \coprod_J Y_j)_I$. But then, since each Y_j is connected, each map $X_i \longrightarrow \coprod_J Y_j$ is in one-to-one correspondence with a map $\varphi_i : X_i \longrightarrow Y_j$, where this corespondence is given by the definition of connected. This choice of $\varphi_i : X_i \longrightarrow Y_j$ is a choice of a unique j, given an i (it is unique by the bijection in the definition of connected). This will act as our function on sets, $I \longrightarrow J$. So we define F on maps as follows:

$$F : \mathbb{X} \longrightarrow Fam(Conn(\mathbb{X}))$$

$$f : X \longrightarrow Y \longmapsto (\Phi, \varphi) : (X_i)_{i \in I} \longrightarrow (Y_j)_{j \in J}$$

Where $(\Phi, \varphi) =: F(f)$ is defined by the discussion above. That is, Φ is the choice of j, given an $i \in I$ described above by the definition of connected and $\varphi_i : X_i \longrightarrow Y_j$ is the map given above. We must show that F is faithful, full, and essentially surjective.

F *faithful*: Notice that the map (Φ, φ) is uniquely defined, given a map $f : X \longrightarrow Y$, so F is faithful.

F *full*: Suppose $(\Lambda, \lambda) : F(X) \longrightarrow F(Y)$ is a map in $Fam(Conn(\mathbb{X}))$. Then,

$$(\Lambda, \lambda) : (X_i)_{i \in I} \longrightarrow (Y_j)_{j \in J}$$

$$\Lambda : I \longrightarrow J$$

$$\lambda_i : X_i \longrightarrow Y_{\Lambda(i)}$$

But then the collection of maps $(\lambda_i : X_i \longrightarrow Y_{\Lambda(i)})_I$ corresponds uniquely to the collection $(X_i \longrightarrow \coprod_I Y_{\Lambda(i)})_I$ by the connectedness of each $Y_{\Lambda(i)}$. But by Lemma 2.2.1 this collection corresponds uniquely to a map $f : \coprod_I X_i \longrightarrow \coprod_I Y_{\Lambda(i)}$. Hence $F(f) = (\Lambda, \lambda)$ so F is full.

F *essentially surjective*: Let $(A_k)_{k \in K}$ be an arbitrary object in $Fam(Conn(\mathbb{X}))$. Then $F(\coprod_K A_k) = (A_k)_{k \in K}$ so F is essentially surjective. □

Example 2.2.2. ([5] Proposition 6.1.1). Let $Conn(\mathbf{Top})$ be the full subcategory of $\mathbf{Top}$ of connected topological spaces. Then $Fam(Conn(\mathbf{Top}))$ is equivalent to the subcategory of $\mathbf{Top}$ of topological spaces with open connected components.

The next Lemma is Lemma 6.2.3 in [5], but where a sketch of a proof is provided. Here we provide the full details.

Lemma 2.2.2. *Let $\mathbb{X}$ be any category with limits. Then $Fam(\mathbb{X})$ has limits.*

Proof. Suppose that $\mathbb{X}$ is a category with limits and consider a diagram in $Fam(\mathbb{X})$:

$$D : \mathbb{Y} \longrightarrow Fam(\mathbb{X})$$

We wish to show that this diagram has a limit L. That is, L is an object in $Fam(\mathbb{X})$ along with projection maps π such that for any map $y : Y \longrightarrow Y' \in \mathbb{Y}$ the following diagram commutes:

$$\begin{tikzcd} & & L \arrow[dll, bend right, "\pi_Y"'] \arrow[dl, bend left, "\pi_{Y'}"] \\ D(Y) \arrow[r, "D(y)"] & D(Y') & \end{tikzcd}$$

and such an L is universal with respect to this property. For notational purposes, since $D(Y) \in Fam(\mathbb{X})$ for every $Y \in \mathbb{Y}$ it is a family object with an index $I_Y \in$ **fset**, so write $D(Y) := D(Y)_{I_Y} = (D(Y)_i)_{i \in I_Y}$.

Since **fset** is complete, consider $I := \prod_{Y \in \mathbb{Y}} I_Y$ which comes equipped with projection maps ρ_Y:

$$\begin{tikzcd} & I_Y \\ I = \prod_{Y \in \mathbb{Y}} I_Y \arrow[ur, "\rho_Y"] \arrow[dr, "\rho_{Y'}"'] & \\ & I_{Y'} \end{tikzcd}$$

Consider $\iota \in I$, so $\iota = (\dots, i_Y, \dots)$ where $i_Y \in I_Y$. In particular, $i_Y = \rho_Y(\iota)$ for every $Y \in \mathbb{Y}$. Using this, we obtain a new diagram:

$$D_\iota : \mathbb{Y} \longrightarrow \mathbb{X}$$

$$D_\iota(Y) := D(Y)_{i_Y}$$

Then let L_ι denote the limit of this diagram, i.e. $L_\iota := \lim D_\iota$, which exists since we assumed $\mathbb{X}$ to have limits. So for every $\iota \in I$ we obtain such a limit L_ι.

Now consider $L := (L_\iota)_{\iota \in I}$. We claim that this is indeed the limit of the original diagram D. To see this, we must first construct projection maps $\pi_Y : L \longrightarrow D(Y)$. We define these maps as follows:

$$\pi_Y : (L_\iota)_{\iota \in I} \longrightarrow (D(Y)_i)_{i \in I_Y}$$

$$\rho_Y : I \longrightarrow I_Y$$

$$(p_Y)_\iota : L_\iota \longrightarrow D(Y)_{\rho_Y(\iota)} = D(Y)_{i_Y}$$

where ρ_Y is the projection given by the product $\prod_{Y \in \mathbb{Y}} I_Y = I$ and for every $\iota \in I$, the maps $(p_Y)_\iota$ are the projection maps for the limit $L_\iota \in \mathbb{X}$.

It remains to show that the object L is universal. So suppose we have an object $(A_j)_{j \in J} \in Fam(\mathbb{X})$ along with maps α_Y such that:

$$\begin{tikzcd}
 & (A_j)_{j \in J} \arrow[d, dashed, "?"] \arrow[ddl, bend right, "\alpha_Y"'] \arrow[ddr, bend left, "\alpha_{Y'}"] & \\
 & (L_\iota)_{\iota \in I} \arrow[dl, "\pi_Y"'] \arrow[dr, "\pi_{Y'}"] & \\
D(Y) \arrow[rr, "D(y)"'] & & D(Y')
\end{tikzcd}$$

Lets examine one of the maps α_Y, which is a map in the family category. So $\alpha_Y := (A_Y, a_Y)$ where:

$$\alpha_Y : (A_j)_{j \in J} \longrightarrow D(Y)_{I_Y}$$

$$A_Y : J \longrightarrow I_Y$$

$$(a_Y)_j : A_j \longrightarrow D(Y)_{A_Y(j)}$$

Using the maps A_Y above we have a unique map $\Lambda : J \longrightarrow I$ by the universal property of the product $I = \prod_{Y \in \mathbb{Y}} I_Y$ in **fset** as follows:

$$\begin{tikzcd}
 & & I_Y \\
J \arrow[r, dashed, "\Lambda"] \arrow[urr, bend left, "A_Y"] \arrow[drr, bend right, "A_{Y'}"'] & I \arrow[ur, "\rho_Y"] \arrow[dr, "\rho_{Y'}"'] & \\
 & & I_{Y'}
\end{tikzcd}$$

And using the maps $(a_Y)_j$ above we have a unique map $\lambda_j : A_j \longrightarrow L_\iota$ for each $j \in J$ given by the definition of L_ι being a limit in $\mathbb{X}$. That is,

$$\begin{tikzcd}
 & A_j \arrow[d, dashed, "\lambda_j"] \arrow[ddl, bend right, "(a_Y)_j"'] \arrow[ddr, bend left, "(a_{Y'})_j"] & \\
 & L_\iota \arrow[dl, "(p_Y)_\iota"'] \arrow[dr, "(p_{Y'})_\iota"] & \\
D(Y)_{A_Y(j)} \arrow[rr] & & D(Y')_{A_{Y'}(j)}
\end{tikzcd}$$

Then (Λ, λ) is our desired map $(A_j)_{j \in J} \longrightarrow (L_\iota)_{\iota \in I}$ and is unique by construction. $\square$

There is a very natural dual to this Proposition which we will also need.

Proposition 2.2.2. *Let $\mathbb{X}$ be any category with colimits. Then $Fam(\mathbb{X})$ has colimits.*

Epic and monic maps in the family category are precisely what one would expect, which immediately follows from the definition of composition.

Lemma 2.2.3. *Consider a map in an arbitrary family category $Fam(\mathbb{X})$:*

$$(\Lambda, \lambda) : (X_i)_{i \in I} \longrightarrow (Y_j)_{j \in J}$$

$$\Lambda : I \longrightarrow J$$

$$\lambda_i : X_i \longrightarrow Y_{\Lambda(i)}$$

*Then (Λ, λ) is epic (resp. monic) in $Fam(\mathbb{X})$ if Λ is epic (resp. monic) in **fset** and for each $i \in I$, λ_i is epic (resp. monic) in $\mathbb{X}$.*

Proposition 2.2.3. *For any category $\mathbb{X}$,*

(i) $Fam(\mathbb{X})$ *has an initial object.*

(ii) If $\mathbb{X}$ *has a terminal object, then* $Fam(\mathbb{X})$ *has a terminal object.*

(iii) If every map $f \in \mathbb{X}$ *can be written as* $f = em$ *where* e *is epic and* m *is monic, then so can any map in* $Fam(\mathbb{X})$.

(iv) $Fam(\mathbb{X})$ *has coproducts.*

Proof. (i) Consider the empty collection of objects over the empty set $()_\emptyset$. We want to show that for any object $(X_i)_{i \in I} \in Fam(\mathbb{X})$ there exists a unique map $()_\emptyset \longrightarrow (X_i)_{i\in I}$. Recall that maps in $Fam(\mathbb{X})$ have both a map between the indexing sets, in this case it is the unique map $\emptyset \longrightarrow I$ and for every element in the first indexing set, a map between the objects. Since the empty set has no elements, there is no map between the objects so $()_\emptyset \longrightarrow (X_i)_{i\in I}$ is defined exactly by $\emptyset \longrightarrow I$ which is unique.

(ii) Suppose $\mathbb{X}$ has a terminal object and denote it by 1. Then consider the object $(1)_{\{*\}} \in Fam(\mathbb{X})$, which is the collection of the same object, 1, over a one-point set. Now given any object $(X_i)_{i\in I} \in Fam(\mathbb{X})$, consider the map:

$$(X_i)_{i\in I} \longrightarrow (1)_{\{*\}}$$

$$I \longrightarrow \{*\}$$

$$X_i \longrightarrow 1$$

The map $I \longrightarrow \{*\}$ in **Set** is unique, since $\{*\}$ is terminal in **Set** and for each $i \in I$ the map $X_i \longrightarrow 1$ is also unique, since 1 is terminal in $\mathbb{X}$. Hence $(1)_{\{*\}}$ is terminal in $Fam(\mathbb{X})$.

(iii) Let (Λ, λ) be a map in $Fam(\mathbb{X})$ given as follows:

$$(\Lambda, \lambda) : (X_i)_{i \in I} \longrightarrow (Y_j)_{j \in J}$$

$$\Lambda : I \longrightarrow J$$

$$\lambda_i : X_i \longrightarrow Y_{\Lambda(i)}$$

We need to show that $(\Lambda, \lambda) = (E, e)(M, m)$ where (E, e) is epic and (M, m) is monic in $Fam(\mathbb{X})$. By Lemma 2.2.3 this means that we need E and M epic and monic respectively in **fset** and for each $i \in I$, e_i and m_i epic and monic respectively in $\mathbb{X}$. First, factor Λ in **fset** through its image:

$$\begin{tikzcd} I \arrow[rr, "\Lambda"] \arrow[dr, "E"'] & & J \\ & Im(\Lambda) \arrow[ur, "M"'] & \end{tikzcd}$$

where E is surjective by construction, hence epic and the inclusion map M is injective, hence monic.

Next, by assumption we know that since $\lambda_i \in \mathbb{X}$, $\lambda_i = e_i m_i$ where e_i is epic and m_i is monic. Hence we have a factorization in $Fam(\mathbb{X})$.

(iv) For this, let us consider the binary coproduct case as the more general case is similar. So let $(X_i)_{i \in I}, (Y_j)_{j \in J} \in Fam(\mathbb{X})$. We want to show that the coproduct of these two exists, i.e. $(X_i)_{i \in I} \coprod (Y_j)_{j \in J}$. Since the coprouct must be some object in $Fam(\mathbb{X})$, write $(X_i)_{i \in I} \coprod (Y_j)_{j \in J} = (Z_k)_{k \in K}$ where $(Z_k)_{k \in K}$ is defined as follows:

$$K := I \sqcup J$$

$$Z_k := \begin{cases} X_k & : k \in I \\ Y_k & : k \in J \end{cases}$$

Notice that $(Z_k)_{k \in K}$ is indeed an object in $Fam(\mathbb{X})$ since it is a collection of objects in $\mathbb{X}$.

Then the map $(X_i)_{i\in I} \longrightarrow (Z_k)_{k\in K}$ is:

$$I \hookrightarrow K = I \sqcup J \text{ (inclusion in } \mathbf{fset}).$$

$$id_{X_i} : X_i \to X_i$$

where the map $(Y_j)_{j\in J} \longrightarrow (Z_k)_{k\in K}$ is defined similarily. We denote these maps by (X, x) and (Y, y) respectively.

It remains to check that $(Z_k)_{k\in K}$ has the universal property of the coproduct. So suppose that we have maps (P, p) and (Q, q) and an object $(A_l)_{l\in L}$ with:

$$\begin{tikzcd}
 & & (X_i)_{i\in I} \arrow[dl, "{(X,x)}"] \arrow[dll, bend right, "{(P,p)}"'] \\
(A_l)_{l\in L} & (Z_k)_{k\in K} \arrow[l, dashed, "{(D,d)}"'] & \\
 & & (Y_j)_{j\in J} \arrow[ul, "{(Y,y)}"'] \arrow[ull, bend left, "{(Q,q)}"]
\end{tikzcd}$$

Define the map (D, d) as follows:

$$(D, d) : (Z_k)_{k\in K} \longrightarrow (A_l)_{l\in L}$$

$$D : I \sqcup J \longrightarrow L$$

$$d_k : Z_k \longrightarrow A_{D(k)}$$

$$d_k := \begin{cases} p_k & : k \in I \\ q_k & : k \in J \end{cases}$$

where D is the unique map defined by the universal property of the coproduct in **fset**. Since x_i and y_j for every $i \in I$ and $j \in J$ are the identity maps, the commutativity follows. Uniqueness of (D, d) also follows from this and so $Fam(\mathbb{X})$ has coproducts. $\square$

2.3 $Fam(\mathbb{X}/S)$ is a Galois Category: $(G1) - (G3)$

Given a category $\mathbb{X}$ and an object $S \in \mathbb{X}$ recall that we had defined the slice category, denoted $\mathbb{X}/S$ (see Definition 1.1.10) in which the objects were maps of the form $X \longrightarrow S$ for $X \in \mathbb{X}$. We claim that $Fam(\mathbb{X}/S)$ where $S \in \mathbb{X}$ and $\mathbb{X}$ is a category with certain restrictions, along with a special functor $Fam(\mathbb{X}/S) \longrightarrow$ **fset** is a Galois category. In this section we will examine the category itself, $Fam(\mathbb{X}/S)$ and show that it satisfies the first three axioms which characterize Galois categories $(G1) - (G3)$.

Assumption. Our category $\mathbb{X}$ will require a number of special properties. We list them below and provide a discussion afterwards:

1. $\mathbb{X}$ is locally connected.

2. $\mathbb{X}$ has pullbacks and quotients by finite subgroups of automorphism groups.

3. For any $f : X \longrightarrow Y \in \mathbb{X}$ we can uniquely (up to isomorphism) factor f as follows:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q"'] & & Y \\ & X_H \arrow[ur, "m"'] & \end{tikzcd}$$

   where $H = Aut_Y(X)$, $q : X \longrightarrow X_H$ is the quotient map (which is epic, as it is a coequalizer - see Proposition 1.3.3), and $m : X_H \longrightarrow Y$ is given by the universal property of X_H and is assumed to be monic.

4. Given any monic map $m : X \longrightarrow Y$ in $\mathbb{X}$, there exists an object Z and a map $n : Z \longrightarrow Y$ such that $Y = X \coprod Z$ with inclusion maps m and n.

Discussion. The idea is that we want a category $\mathbb{X}$ so that $Fam(\mathbb{X}/S)$ is a Galois category. These requirements on $\mathbb{X}$ were chosen as necessities for this to hold. Note that given the above requirements on $\mathbb{X}$, they are also true in $\mathbb{X}/S$.

1. Recall that for $\mathbb{X}$ to be locally connected means that for any $X \in \mathbb{X}$,

$$X = \coprod_I X_i$$

   where each $X_i \in \mathbb{X}$ is connected and are called the connected components of $\mathbb{X}$. As we shall see, this is necessary for the second part of axiom $(G3)$.

2. Given a category without pullbacks or quotients, say $\mathbb{C}$, it is not necessarily true that $Fam(\mathbb{C})$ will have pullbacks or quotients. However as we have seen, this does not apply for all limits and colimits. For instance by Proposition 2.2.3 $Fam(\mathbb{C})$ has coproducts for any category $\mathbb{C}$, regardless of whether $\mathbb{C}$ has coproducts or not.

3. This specific factorization through a particular quotient is required for an issue regarding the preservation of epic maps by the fundamental functor. This will be discussed more in the next section, but the issue is that in general epics are not stable under pullback (see Definition 1.3.7). This is true in general in a "regular category" (see for instance [4]), but asking our category to be regular is stronger than what we require.

4. This requirement is exactly the second part of axiom $(G3)$. That is, we require any monic map to be a direct summand for some coproduct.

Proposition 2.3.1. *Let $\mathbb{X}$ be a category with assumptions 1.— 4. Then $Fam(\mathbb{X}/S)$ satisfies $(G1) - (G3)$.*

We divide the proof up into three parts, one for each axiom.

Proof. **Proof of (G1).** *Terminal Object* : Since slice categories always have a terminal object, in this case $1_S : S \longrightarrow S$, $\mathbb{X}/S$ has a terminal object. Then by (ii) of Proposition 2.2.3 it follows that $Fam(\mathbb{X}/S)$ has a terminal object (in our case, the object is the singleton collection $(S \longrightarrow S)_{\{*\}}$).

Pullbacks : By Proposition 2.2.2 this follows immediately. However, for completeness, we will explicitly construct the pullbacks in $Fam(\mathbb{X}/S)$. This construction is based off the work in [5] section 6.2. Suppose we have the following diagram in $Fam(\mathbb{X}/S)$:

$$\begin{tikzcd} & (Y_j)_{j\in J} \arrow[d, "(V,v)"] \\ (X_i)_{i\in I} \arrow[r, "(U,u)"'] & (Z_k)_{k\in K} \end{tikzcd}$$

Since **fset** has pullbacks, we can form the pullback of the indexing sets:

$$\begin{tikzcd} I \times_K J \arrow[r] \arrow[d] & J \arrow[d, "V"] \\ I \arrow[r, "U"'] & K \end{tikzcd}$$

where by definition of pullback in **fset**, $I \times_K J := \{(s,t) \in I \times J \mid U(s) = V(t)\}$. Now by assumption $\mathbb{X}$ has pullbacks and hence so does $\mathbb{X}/S$. So for every $(s,t) \in I \times_K J$ we have the following pullback in $\mathbb{X}/S$:

$$\begin{tikzcd} P_{(s,t)} \arrow[r] \arrow[d] & Y_t \arrow[d] \\ X_s \arrow[r] & Z_{U(s)} = Z_{V(t)} \end{tikzcd}$$

Then the family $(P_{(s,t)} \longrightarrow S)_{(s,t)\in I\times_K J}$ is the pullback of the original diagram in $Fam(\mathbb{X}/S)$.

$\square$

Proof. **Proof of (G2).** *Initial Object* : This was shown in (i) of Proposition 2.2.3.

Coproducts : This was shown in (iv) of Proposition 2.2.3.

Quotients : By Proposition 2.2.2 this follows immediately since $\mathbb{X}$ has quotients by assumption. But, for completeness, we will explicitly construct the quotients in $Fam(\mathbb{X}/S)$.

We need to show that the quotient of an object in $Fam(\mathbb{X}/S)$ by a finite subgroup of its automorphism group exists. So let $(X_i)_{i\in I} \in Fam(\mathbb{X}/S)$, $G := Aut((X_i)_{i\in I})$, and H be a finite subgroup of G.

Consider for a moment a map $(\Sigma, \sigma) \in H$ defined as follows:

$$(\Sigma, \sigma) : (X_i)_{i\in I} \longrightarrow (X_i)_{i\in I}$$

$$\Sigma : I \longrightarrow I$$

$$\sigma_i : X_i \longrightarrow X_{\Sigma(i)}$$

To be an isomorphism in $Fam(\mathbb{X}/S)$ means that both Σ and σ_i for every $i \in I$ must also be isomorphisms in their respective categories. But since every σ_i is an isomorphism, $X_{\Sigma(i)} \cong X_i$ for all i. So either $X_i \cong X_k \cong X$ for all $i, k \in I$ or Σ is the identity map, in which case σ_i is an automorphism of X_i. For the former case, the following argument would apply by removing the index i.

For every (Σ, σ) in H, there is an automorphism σ_i of X_i. Denote the group of automorphisms for X_i by H_i. Notice that H_i is finite since H is finite.

With this in mind define the quotient of $(X_i)_{i\in I}$ by H to be:

$$(X_i)_{i\in I}/H := (X_i/H_i)_{i\in I}$$

with the quotient map (Q, q) defined as follows:

$$(Q, q) : (X_i)_{i\in I} \longrightarrow (X_i/H_i)_{i\in I}$$

$$Q := id_I : I \longrightarrow I$$

$$q_i : X_i \longrightarrow X_i/H_i$$

where for every $i \in I$, q_i is the quotient map for X_i/H_i and exists since we assumed quotients exist in $\mathbb{X}/S$. $\square$

In order to prove $(G3)$ we will need a lemma.

Lemma 2.3.1. *Let $\mathbb{X}$ be a locally connected category with coproducts. Then for any two families with the same index I, $(X_i)_{i\in I}, (Z_i)_{i\in I}$,*

$$(X_i)_{i\in I} \coprod (Z_i)_{i\in I} \cong (X_i \coprod Z_i)_{i\in I}$$

In other words, the family of coproducts is isomorphic to the coproduct of the families.

Proof. Since $\mathbb{X}$ is locally connected, by Proposition 2.2.1 we have:

$$\mathbb{X} \simeq Fam(Conn(\mathbb{X}))$$

where the equivalence on the objects is given by sending $\coprod_I A_i$ to $(A_i)_{i\in I}$. Then for all $i \in I$ by the equivalence we have:

$$X_i \coprod Z_i \simeq (X_i, Z_i)$$

where (X_i, Z_i) is the family consisting of two objects, X_i and Z_i. Notice that here we implicitly assumed that the objects X_i, Z_i were connected. If they were not connected, we could write each as a coproduct of its connected components, then apply the equivalence. Thus without loss of generality, we can assume that X_i, Z_i are connected for all $i \in I$. It is also important to note that this is *not* an isomorphism: the objects above are not in the same category.

In order to proceed, we recall that an equivalence of categories is a pair of functors $F : \mathbb{X} \longrightarrow Fam(Conn(\mathbb{X}))$ and $G : Fam(Conn(\mathbb{X})) \longrightarrow \mathbb{X}$ which are naturally isomorphic

(Proposition 1.2.2). This equivalence was also seen in Proposition 2.2.1. Applying G to both $(X_i)_{i\in I}\coprod(Z_i)_{i\in I}$ and $(X_i\coprod Z_i)_{i\in I}$ we get the following in $\mathbb{X}$:

$$G\Big((X_i)_{i\in I}\coprod(Z_i)_{i\in I}\Big)=\Big(\coprod_I X_i\Big)\coprod\Big(\coprod_I Z_i\Big)$$
$$G\Big((X_i\coprod Z_i)_{i\in I}\Big)=\coprod_I\Big(X_i\coprod Z_i\Big)$$

But by expanding these coproducts, we see that they are in fact the same object. Hence in $\mathbb{X}$ we have the following equality:

$$G\Big((X_i)_{i\in I}\coprod(Z_i)_{i\in I}\Big)=G\Big((X_i\coprod Z_i)_{i\in I}\Big)$$

But now we apply F to each of these objects. Since $GF\cong\mathbb{1}_{Fam(Conn(\mathbb{X}))}$ and F preserves isomorphisms we have the following in $Fam(Conn(\mathbb{X}))$:

$$\begin{aligned}(X_i)_{i\in I}\coprod(Z_i)_{i\in I}&=F\Bigg(G\Big((X_i)_{i\in I}\coprod(Z_i)_{i\in I}\Big)\Bigg)\\&=F\Bigg(G\Big((X_i\coprod Z_i)_{i\in I}\Big)\Bigg)\\&\cong(X_i\coprod Z_i)_{i\in I}\end{aligned}$$

which was the required result. □

Proof. **Proof of (G3).** *Factorization* : By Assumption 3, given any map $f\in\mathbb{X}$ we can factor f as qm where q is epic and m is monic. Thus we can do the same in $\mathbb{X}/S$. Hence by Proposition 2.2.3 (*iii*) $Fam(\mathbb{X}/S)$ has such a factorization.

Monomorphisms are Direct Summands: Suppose we have a monomorphism $(M,m):(X_i)_{i\in I}\longrightarrow(Y_j)_{j\in J}$ in $Fam(\mathbb{X}/S)$. We need to show that there exists an object $(U_l)_{l\in L}\in Fam(\mathbb{X}/S)$ along with a map $(N,n):(U_l)_{l\in L}\longrightarrow(Y_j)_{j\in J}$ such that:

$$(Y_j)_{j\in J}\cong(X_i)_{i\in I}\coprod(U_l)_{l\in L}$$

First observe that for every $i \in I$, $m_i : X_i \longrightarrow Y_{M(i)}$ is monic in $\mathbb{X}/S$ and thus is a direct summand since we have assumed that $\mathbb{X}/S$ has this property. So for every $i \in I$ there exists an object Z_i and a map $z_i : Z_i \longrightarrow Y_{M(i)}$ such that $Y_{M(i)} = X_i \coprod Z_i$. So we have the following inclusions for every $i \in I$:

$$\begin{tikzcd} X_i \arrow[dr, "m_i"] & \\ & Y_{M(i)} = X_i \coprod Z_i \\ Z_i \arrow[ur, "z_i"'] & \end{tikzcd}$$

Hence we can write $(Y_j)_{j\in J}$ as follows:

$$\begin{aligned} (Y_j)_{j\in J} &= (Y_{M(i)})_{i\in I} \coprod (Y_j)_{j\in J\setminus M(I)} \text{ by definition of coproduct in } Fam(\mathbb{X}/S). \\ &= (X_i \coprod Z_i)_{i\in I} \coprod (Y_j)_{j\in J\setminus M(I)} \\ &\cong (X_i)_{i\in I} \coprod (Z_i)_{i\in I} \coprod (Y_j)_{j\in J\setminus M(I)} \text{ by Lemma 2.3.1.} \end{aligned}$$

Then define the required family $(U_l)_{l\in L}$ as follows:

$$\begin{aligned} (U_l)_{l\in L} &:= (Z_i)_{i\in I} \coprod (Y_j)_{j\in J\setminus M(I)} \\ &= (B_k)_{k\in K} \end{aligned}$$

Where, by definition of the coproduct in $Fam(\mathbb{X}/S)$ ((iv) in Proposition 2.2.3):

$$B_k := \begin{cases} Z_k : k \in I \\ Y_k : k \in J \setminus M(I) \end{cases}$$

Note that $K = I \sqcup J \setminus M(I)$, but M is monic so $M(I) \cong I$. Hence $K \cong J$. Then $(U_j)_{j\in J} := (B_j)_{j\in J}$ is the required family for $(G3)$.

Hence $Fam(\mathbb{X}/S)$ satisfies $(G3)$. $\square$

2.4 $Fam(\mathbb{X}/S)$ is a Galois Category: $(G4) - (G6)$

Recall from Definition 2.1.1 that the characterization of a Galois category includes the existence of the fundamental functor. In our case, we are seeking a functor $Fam(\mathbb{X}/S) \longrightarrow \mathbf{fset}$, i.e. from our category in question to the category of finite sets, that satisfies axioms $(G4) - (G6)$. In addition to the assumptions of Section 2.3 we require a critical assumption on our category $\mathbb{X}$ and that is the existence of a very special object $p : s \longrightarrow S \in \mathbb{X}/S$.

Assumption: In addition to the assumptions from Section 2.2, we assume the existence of an object $p : s \longrightarrow S \in \mathbb{X}/S$ which we will call a *geometric point*. This object must satisfy three conditions:

1. p^* reflects isomorphisms.
2. s is connected as an object in $\mathbb{X}/S$.
3. $\mathbb{X}/s \simeq \mathbf{fset}$.

Discussion.

1. This condition was imposed specifically for axiom $(G6)$, which requires that the fundamental functor reflects isomorphisms.
2. Intuitively, we can think of $s \longrightarrow S$ as a "point", so requiring this point to be connected is not too far from the intuition.
3. This condition is very strong and indeed we will make frequent use of this assumption for the remainder of the section. This assumption originates from the example of finite étale covers where the geometric point is a map $Spec(K) \longrightarrow S$ where K is an algebraically closed field. One can show that

the category of finite etale covers over $Spec(K)$ is equivalent to **fset**. For details on this, see 5.23 of [17].

With the third assumption in mind, there are a few important observations with regard to the category $\mathbb{X}/s$ that we must address. Observe that we are assuming the existence of two functors M, N to form the equivalence $\mathbb{X}/s \simeq \mathbf{fset}$.

In general we do not know what these functors are, but we can still utilize them in examining the category $\mathbb{X}/s$. Recall that any finite set A can always be written as the disjoint union of all of its elements and this is isomorphic (in **fset**) to the disjoint union of the one-pointed set. That is,

$$A = \bigsqcup_{a \in A} a \cong \bigsqcup_{A} \{*\}$$

Now suppose $X \longrightarrow s \in \mathbb{X}/s$ (which we will denote as simply X). Then $M(X) \in \mathbf{fset}$ so we can write $M(X) \cong \bigsqcup_{M(X)} \{*\}$. Since a functor that is part of an equivalence preserves limits and colimits (by Theorem 1.2.1 and Theorem 1.3.1), N preserves the terminal object and coproducts. So we have:

$$X = id_{\mathbb{X}/s}(X) \cong N(M(X)) = N(\bigsqcup_{M(X)} \{*\}) = \coprod_{M(X)} N(\{*\}) = \coprod_{M(X)} s$$

where by s above we mean the object $1_s : s \longrightarrow s \in \mathbb{X}/s$. So each object $X \longrightarrow s \in \mathbb{X}/s$ is "trivial" in the sense that $X \cong \coprod_{M(X)} s$.

Lemma 2.4.1. *For all $X \in \mathbb{X}/S$, $\mathbb{X}/S(s, X) \cong M(X \times_S s)$.*

Proof. From the work above, we know that $X \times_S s \cong \coprod_{M(X \times_S s)} s$. Suppose we have a map $\alpha : s \longrightarrow X$ (so $\alpha \in \mathbb{X}/S(s, X)$). Then α determines a map $\alpha' : s \longrightarrow X \times_S s$ by the universal property of the pullback $X \times_S s$ as follows:

$$\begin{tikzcd}
s \arrow[dr, dashed, "\alpha'"] \arrow[drr, bend left, "\alpha"] \arrow[ddr, bend right, "1_s"'] & & \\
& X \times_S s \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X \arrow[d] \\
& s \arrow[r] & S
\end{tikzcd}$$

Then since $X \times_S s \cong \coprod_{M(X \times_S s)} s$, α' is really a map $s \longrightarrow \coprod_{M(X \times_S s)} s$. Hence $\alpha' \in \mathbb{X}/s(s, \coprod_{M(X \times_S s)} s)$. So we have constructed a map $\mathbb{X}/S(s, X) \longrightarrow \mathbb{X}/s(s, \coprod_{M(X \times_S s)} s)$, which we will call φ. That is, $\varphi(\alpha) = \alpha'$ as defined above. But then, consider the following:

$$\begin{aligned}
\varphi : \mathbb{X}/S(s, X) \longrightarrow & \mathbb{X}/s(s, \coprod_{M(X \times_S s)} s) \\
& \cong \coprod_{M(X \times_S s)} \mathbb{X}/s(s, s) \text{ since } s \text{ is connected by assumption.} \\
& \cong \coprod_{M(X \times_S s)} \{*\} \text{ since } \mathbb{X}/s(s, s) \cong \{*\} \text{ as } 1_s : s \longrightarrow s \text{ is terminal.} \\
& \cong M(X \times_S s)
\end{aligned}$$

Thus it remains to show that φ is a bijection. First, suppose that $\varphi(\alpha) = \varphi(\beta)$ for some maps $\alpha, \beta \in \mathbb{X}/S(s, X)$. Then each of $\varphi(\alpha)$ and $\varphi(\beta)$ are defined by the universal property of the pullback $X \times_S s$. By the top triangle of the universal property diagram (see above when we defined α') we have:

$$\alpha = \varphi(\alpha)\pi_1 = \varphi(\beta)\pi_1 = \beta$$

So $\alpha = \beta$ hence φ is injective.

Second, suppose we have a map $\gamma : s \longrightarrow X \times_S s$. Then $\varphi(\gamma\pi_1) = \gamma$ so φ is surjective. $\square$

We will now utilize the geometric point in order to construct the fundamental functor for $Fam(\mathbb{X}/S)$. To do this, we will first consider the pullback functor along $s \longrightarrow S$ from

$\mathbb{X}/S$ to $\mathbb{X}/s$ defined as follows:

$$F_s : \mathbb{X}/S \longrightarrow \mathbb{X}/s$$

$$\begin{tikzcd} X \arrow[d] & & s \times_S X \arrow[r] \arrow[d] & X \arrow[d] \\ S & \mapsto & s \arrow[r] & S \end{tikzcd}$$

$$\begin{tikzcd} & & & s \times_S X \arrow[rr] \arrow[dr, dashed, "F_s(\varphi)"] \arrow[ddr, bend right] & & X \arrow[d, "\varphi"] \\ X \arrow[rr, "\varphi"] \arrow[dr] & & Y \arrow[dl] & \longmapsto & s \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\ & S & & & s \arrow[r] & S \end{tikzcd}$$

This is clearly not the functor we need to show $Fam(\mathbb{X}/S)$ is a Galois category since it does not have $Fam(\mathbb{X}/S)$ in its domain, but we will use F_s in order to define our fundamental functor. As such it is important that we understand how the functor F_s behaves. But first, we need a lemma.

Lemma 2.4.2. *Let $\mathbb{X}$ be a category with finite coproducts and a connected terminal object T. Then a map between finite coproducts*

$$\coprod_I T \longrightarrow \coprod_J T$$

is uniquely determined by a map between the indexes

$$I \longrightarrow J$$

Proof. Consider the following:

$$\begin{aligned} \mathbb{X}(\coprod_I T, \coprod_I T) &\cong \prod_I \mathbb{X}(T, \coprod_I T) \text{ by Lemma 2.2.1.} \\ &\cong \prod_I \Big(\coprod_J \mathbb{X}(T,T) \Big) \text{ since } T \text{ is connected.} \\ &\cong \prod_I \Big(\coprod_J \{*\} \Big) \text{ since } T \text{ is terminal, } \mathbb{X}(T,T) \text{ has only one map.} \end{aligned}$$

We claim that a map $I \longrightarrow J$ uniquely determines an object in the product $\prod_I \left(\coprod_J \{*\} \right)$. First, notice that we have already seen in **fset** that $\coprod_J \{*\} \cong J$. Then by the definition of the product $\prod_I J$ we have projection maps $\pi_i : \prod_I J \longrightarrow J$, one for each $i \in I$. So an element $(x)_{i \in I} \in \prod_I J$ uniquely determines a map $f : I \longrightarrow J$ defined by $f(i) := \pi_i((x)_I)$. □

This lemma will actually be surprisingly useful, not only for the Proposition below, but also in Chapter 3. We shall call back to this Lemma on numerous occasions.

Proposition 2.4.1. *$F_s : \mathbb{X}/S \longrightarrow \mathbb{X}/s$ preserves limits, preserves quotients and epimorphisms, and reflects isomorphisms.*

Proof. Limits: From Proposition 1.3.2, we know that F_s is a right adjoint. Thus by Theorem 1.3.1 F_s preserves all limits.

Quotients: Consider the quotient $X_H \in \mathbb{X}/S$ of the object $X \longrightarrow S \in \mathbb{X}/S$ by a subgroup of the automorphism group $H \subseteq Aut_S(X)$:

$$X \underset{\longrightarrow}{\overset{h}{\longrightarrow}} \!\!\!\vdots\;\; X \xrightarrow{q} X_H$$

We need to show that $F_s(X_H)$ is also a quotient, but in the category $\mathbb{X}/s$. So apply the functor F_s to the above quotient diagram to get:

$$X \times_S s \underset{\longrightarrow}{\overset{F_s(h)}{\longrightarrow}} \!\!\!\vdots\;\; X \times_S s \xrightarrow{F_s(q)} X_H \times_S s$$

Recall that $\mathbb{X}/s \simeq$ **fset**, so there is a pair of functors M, N forming the equivalence. In particular, we know that every object in $\mathbb{X}/s$ can be written as a coproduct over the terminal object, namely s. So we can rewrite the diagram above as follows:

$$\coprod_{M(X \times_S s)} s \underset{\longrightarrow}{\overset{F_s(h)}{\longrightarrow}} \!\!\!\vdots\;\; \coprod_{M(X \times_S s)} s \xrightarrow{F_s(q)} \coprod_{M(X_H \times_S s)} s$$

But notice that each of the maps $F_s(h)$ are determined by the map between the indexing sets $M(X \times_S s) \longrightarrow M(X \times_S s)$ by Lemma 2.4.2. Thus, it suffices to consider the following diagram in **fset**, which is the quotient diagram on the indexing sets of the coproducts:

$$M(X \times_S s) \underset{\longrightarrow}{\overset{h}{\longrightarrow}} M(X \times_S s) \longrightarrow M(X \times_S s)_H$$

where we have called the maps above h for convenience, since they correspond to the maps $F_s(h)$.

Since **fset** is cocomplete the maps h above have a colimit, in particular it is the quotient $M(X \times_S s)_H$. Since we are in finite sets, this is the set of all orbits of H.

However we can simplify this diagram even further. By lemma 2.4.1, for all $X \in \mathbb{X}/S$, $M(X \times_S s) \cong \mathbb{X}/S(s, X)$. The diagram then becomes:

$$\mathbb{X}/S(s,X) \underset{\longrightarrow}{\overset{h}{\longrightarrow}} \mathbb{X}/S(s,X) \longrightarrow \mathbb{X}/S(s,X)_H$$

where $\mathbb{X}/S(s,X)_H$ is just notation for the quotient (the set of orbits). We must show that $\mathbb{X}/S(s,X)_H \cong \mathbb{X}/S(s,X_H)$ since this would give us that $M(X \times_S s)_H \cong M(X_H \times_S s)$. Then since this is the quotient in the diagram in **fset** on the index of the coproducts, it implies that the quotient for the diagram in $\mathbb{X}/s$ is the coproduct $\coprod\limits_{M(X_H \times_S s)} s$ which is what we needed to show. So define $\varphi : \mathbb{X}(s,X)_H \longrightarrow \mathbb{X}(s, X_H)$ as follows:

$$\varphi : \mathbb{X}/S(s,X)_H \longrightarrow \mathbb{X}/S(s,X_H)$$
$$[\alpha : s \longrightarrow X] \longmapsto \alpha q : s \longrightarrow X \longrightarrow X_H$$

where $[\alpha : s \longrightarrow X]$ is the equivalence class of α in $\mathbb{X}/S(s,X)_H$ and recall $q : X \longrightarrow X_H$ is the map equipped with the quotient object X_H. Notice that φ is independent of choice of representative since $\alpha h q = \alpha q$ for all $h \in H$, so φ is well-defined.

φ injective : Suppose that $\alpha q = \beta q$. Then since q is epic it follows immediately that $\alpha = \beta$ so φ is injective.

φ *surjective* : Now suppose we have a map $f : s \longrightarrow X_H$ (so $f \in \mathbb{X}/S(s, X_H)$). We want to show that there exists a map $d : s \longrightarrow X$ such that $f = dq$. Consider the pullback diagram of $q : X \longrightarrow X_H$ and $f : s \longrightarrow X_H$ in $\mathbb{X}/S$:

$$\begin{tikzcd} X \times_{X_H} s \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & s \arrow[d, "f"] \\ X \arrow[r, "q"'] & X_H \end{tikzcd}$$

Notice that the top map $\pi_2 : X \times_{X_H} s \longrightarrow s$ is also a map in $\mathbb{X}/s$, so we can apply the functor M:

$$M(\pi_2) : M(X \times_{X_H} s) \longrightarrow M(s)$$

But M must preserve the terminal object, so $M(s) = \{*\}$ i.e. the one-point set and additionally $M(X \times_{X_H} s) \cong \coprod_{M(X \times_{X_H} s)} M(s)$ since $M(X \times_{X_H} s)$ is a finite set. So for every element of $M(X \times_{X_H} s)$ there are maps into the coproduct:

$$\begin{tikzcd} M(s) \arrow[dr, "\gamma"] & \\ & \coprod_{M(X \times_{X_H} s)} M(s) \\ M(s) \arrow[ur] & \end{tikzcd}$$

Choose any one of these maps, say γ. Then,

$$N(\gamma) : N(M(s)) \longrightarrow N(M(X \times_{X_H} s))$$

$$N(\gamma) : s \longrightarrow X \times_{X_H} s$$

We claim that $N(\gamma)\pi_1$ is our desired map d. To see this, construct the codiagonal map which is the following map defined by the universal property of the coproduct:

$$\begin{tikzcd}
M(s) \arrow[dr, "\gamma"] \arrow[drr, bend left, "1_{M(s)}"] & & \\
& \coprod_{M(X\times_{X_H} s)} M(s) \arrow[r, dashed, "\nabla"] & M(s) \\
M(s) \arrow[ur] \arrow[urr, bend right, "1_{M(s)}"'] & &
\end{tikzcd}$$

Observe that $M(\pi_2) \cong \nabla$. Indeed, the following square commutes:

$$\begin{tikzcd}
M(X \times_{X_H} s) \arrow[r, "\cong"] \arrow[d, "M(\pi_2)"'] & \coprod_{M(X\times_{X_H} s)} M(s) \arrow[d, "\nabla"] \\
M(s) \arrow[r, "1_{M(s)}"] & M(s)
\end{tikzcd}$$

Now recall our original pullback square for $X \times_{X_H} s$. We re-produce this diagram below:

$$\begin{tikzcd}
X \times_{X_H} s \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & s \arrow[d, "f"] \\
X \arrow[r, "q"'] & X_H
\end{tikzcd}$$

By commutativity, $\pi_2 f = \pi_1 q$, but precomposing with $N(\gamma)$ we get $N(\gamma)\pi_2 f = N(\gamma)\pi_1 q$. Thus is remains to show that $N(\gamma)\pi_2 = 1_s$. But,

$1_{M(s)} = \gamma\nabla \cong \gamma M(\pi_2)$ by definition of the codiagonal and since $\nabla \cong M(\pi_2)$.

$N(1_{M(s)}) \cong N(\gamma)N(M(\pi_2))$ by applying N to the previous line.

$1_s \cong N(\gamma)\pi_2$

Hence $f = N(\gamma)\pi_1 q$ so our desired map is $d := N(\gamma)\pi_1$.

This implies that φ is a bijection (so an isomorphism in **fset**). Hence $\coprod_{M(X_H \times_S s)} s = F_s(X_H)$ is the quotient of the original diagram in $\mathbb{X}/s$. Hence F_s preserves quotients.

Epimorphisms: In order to show F_s preserves epics we need to recall a few facts about our factorization in $\mathbb{X}/S$.

First, we know that we can factor any map $f : X \longrightarrow Y \in \mathbb{X}/S$ as follows:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q"'] & & Y \\ & X_{Aut_Y(X)} \arrow[ur, "m"'] & \end{tikzcd}$$

where m is given by the universal property of the quotient $X_{Aut_Y(X)}$ and q is the quotient map.

Second, recall that by Assumption 3. in Section 2.3, for any factorization as above that m is monic. Since q is always epic, this gives us a factorization of any map as an epic followed by a monic.

Now apply F_s to this factorization:

$$\begin{tikzcd} X \times_S s \arrow[rr, "F_s(f)"] \arrow[dr, "F_s(q)"'] & & Y \times_S s \\ & X_{Aut_Y(X)} \times_S s \arrow[ur, "F_s(m)"'] & \end{tikzcd}$$

Since F_s preserves quotients, $F_s(q) : X \times_S s \longrightarrow X_{Aut_Y(X)} \times_S s$ is a quotient map and hence epic in $\mathbb{X}/s$. Also since monics are stable under pullback (Example 1.3.8) $F_s(m)$ is monic. Hence $F_s(f)$ can be factored as an epic map $F_s(q)$ followed by a monic map $F_s(m)$.

Now suppose that f is epic. Since factorizations are assumed to be unique the following two factorizations must be the same, up to isomorphism:

$$\begin{tikzcd} & Y \arrow[dr, bend left, "\cong"] & \\ X \arrow[ur, bend left, "f"] \arrow[rr, "f"] \arrow[dr, bend right, "q"'] & & Y \\ & X_{Aut_Y(X)} \arrow[ur, bend right, "m"'] & \end{tikzcd}$$

That is, $m : X_{Aut_Y(X)} \longrightarrow Y$ is an isomorphism . But then since F_s preserves isomorphisms (since it is a functor), $F(m)$ is also an isomorphism. Applying F_s to the factorization of f

we have $F_s(f) = F_s(q)F_s(m)$, but $F_s(q)$ is epic and $F_s(m)$ is an isomorphism so that implies $F_s(f)$ is also epic. Hence F_s preserves epics.

Reflects Isomorphisms: This is immediate since we have assumed that F_s reflects isomorphisms (it was part of the assumption of $s \longrightarrow S$). □

We now define our canditate for the fundamental functor of $Fam(\mathbb{X}/S)$.

$$\mathcal{F}_s : Fam(\mathbb{X}/S) \longrightarrow Fam(\mathbb{X}/s)$$

$$(X_i \longrightarrow S)_{i \in I} \longmapsto (F_s(X_i) \longrightarrow s)_{i \in I} = (X \times_S s \longrightarrow s)_{i \in I}$$

That is, $\mathcal{F}_s$ is base change at each object of the family. Now consider a map in $Fam(\mathbb{X}/S)$:

$$(\Lambda, \lambda) : (X_i)_{i \in I} \longrightarrow (Y_j)_{j \in J}$$

$$\Lambda : I \longrightarrow J$$

$$\lambda_i : X_i \longrightarrow Y_{\Lambda(i)}$$

Then $\mathcal{F}_s((\Lambda, \lambda)) = (\Lambda, F_s(\lambda))$ where by $F_s(\lambda)$ we mean that for all $i \in I$, $F_s(\lambda)_i := F_s(\lambda_i)$. So again, $\mathcal{F}_s$ on a map is just F_s applied to each indexed object in the family.

To be a fundamental functor, we need the codomain to be **fset**. But since $\mathbb{X}/s \simeq$ **fset** by assumption, Example 2.2.1 implies that $Fam(\mathbb{X}/s) \simeq$ **fset**.

Lemma 2.4.3. *Let $\mathbb{X}/S$ be a cateogry with limits. If F_s preserves limits then $\mathcal{F}_s$ preserves limits.*

Proof. Let $H : \mathbb{Y} \longrightarrow Fam(\mathbb{X}/S)$ be a diagram and suppose that there exists a limit of this diagram in $Fam(\mathbb{X}/S)$, say $M = (M_\iota)_{\iota \in I}$. From Proposition 2.2.2 we know that each M_ι is a limit in $\mathbb{X}/S$. For M to be a limit means that for any map $y : Y \longrightarrow Y'$ the following diagram commutes and is universal:

$$\begin{tikzcd}
 & & M = (M_\iota)_{\iota \in I} \arrow[dll, bend right, "\pi_Y"'] \arrow[dl, bend left, "\pi_{Y'}"] \\
H(Y)_{I_Y} \arrow[r, "H(y)"'] & H(Y')_{I_{Y'}} &
\end{tikzcd}$$

where each $H(Y)_{I_Y}$ is some family object in $Fam(\mathbb{X}/S)$ indexed by I_Y.

We want to show that $(\mathcal{F}_s(M), \mathcal{F}_s(\pi_Y))_{Y \in \mathbb{Y}}$ is a limit in $Fam(\mathbb{X}/S)$. That is, we want to show that for all maps $y : Y \longrightarrow Y' \in \mathbb{Y}$ applying the functor $\mathcal{F}_s$ to all of the possible above diagrams yields a commuting diagram which is universal:

$$\begin{tikzcd}
 & & \mathcal{F}_s((M_\iota)_{\iota \in I}) \arrow[dll, bend right, "\mathcal{F}_s(\pi_Y)"'] \arrow[dl, bend left, "\mathcal{F}_s(\pi_{Y'})"] \\
\mathcal{F}_s(H(Y)_{I_Y}) \arrow[r, "\mathcal{F}_s(H(y))"'] & \mathcal{F}_s(H(Y')_{I_{Y'}}) &
\end{tikzcd}$$

By definition of $\mathcal{F}_s$, $\mathcal{F}_s(M) = (F_s(M_\iota))_{\iota \in I}$. Then since F_s preserves limits by assumption and each M_ι is a limit, $F_s(M_\iota)$ is the limit of the diagram:

$$H_\iota F_s : \mathbb{Y} \longrightarrow \mathbb{X}/S \longrightarrow \mathbb{X}/s$$

To continue, we need to investigate the maps $\mathcal{F}_s(\pi_Y)$ which are defined as follows:

$$\mathcal{F}_s(\pi_Y) : \mathcal{F}_s(M) = (F_s(M_\iota))_I \longrightarrow (F_s(H(Y)_{\iota_Y}))_{I_Y} = \mathcal{F}_s(H(Y)_{I_Y})$$

$$I \longrightarrow I_Y$$

$$F_s(M_\iota) \longrightarrow F_s(H(Y)_{\iota_Y})$$

where $I \longrightarrow I_Y$ is the projection map $I = \prod_{Y \in \mathbb{Y}} I_Y \longrightarrow I_Y$ from Proposition 2.2.2 and each map $F_s(M_\iota) \longrightarrow F_s(H(Y)_{\iota_Y})$ is the map from the limit diagram of $F_s(M_\iota)$. Thus the universal property of the limit $\mathcal{F}_s(M)$ follows from the universal property of each of the limits $F_s(M_\iota)$. $\square$

Dually,

Lemma 2.4.4. *If F_s preserves colimits then $\mathcal{F}_s$ preserves colimits.*

Proposition 2.4.2. *$\mathcal{F}_s : Fam(\mathbb{X}/S) \longrightarrow Fam(\mathbb{X}/s)$ satisfies axioms $(G4) - (G6)$.*

Proof. **(G4).** We must show that $\mathcal{F}_s$ preserves the terminal object and pullbacks.

$Terminal\ Object$: Recall that $(S \longrightarrow S)_{\{*\}}$ is the terminal object in $Fam(\mathbb{X}/S)$. Then,

$$\mathcal{F}_s((S \longrightarrow S)_{\{*\}}) = (S \times_S s \longrightarrow s)_{\{*\}} = (s \longrightarrow s)_{\{*\}}$$

where $(s \longrightarrow s)_{\{*\}}$ is terminal in $Fam(\mathbb{X}/s)$.

$Pullbacks$: By 2.4.1, F_s preserves pullbacks. Hence by Lemma 2.4.3, $\mathcal{F}_s$ preserves pullbacks.

Hence $\mathcal{F}_s$ satisfies $(G4)$.

(G5). We need to show that $\mathcal{F}_s$ preserves epimorphisms, quotients, and coproducts.

$Epimorphisms$: Suppose $(\Lambda, \lambda) : (X_i)_{i \in I} \longrightarrow (Y_j)_{j \in J}$ is an epimorphism in $Fam(\mathbb{X}/S)$. Then Λ is an epimorphism in **fset** (hence surjective) and for each $i \in I$, $\lambda_i : X_i \longrightarrow Y_{\Lambda(i)}$ is an epimorphism in $\mathbb{X}/S$.

Then $\mathcal{F}_s((\Lambda, \lambda)) = (\Lambda, F_s(\lambda))$ where for all $i \in I$, $F_s(\lambda)_i := F_s(\lambda_i)$. But since λ_i is an epimorphism by assumption and by Proposition 2.4.1 F_s preserves epimorphisms, $F_s(\lambda_i)$ is also an epimorphism. Hence $\mathcal{F}_s$ preserves epimorphisms.

$Quotients$: Follows from Proposition 2.4.1 and Lemma 2.4.4.

Coproducts : Follows from Proposition 2.4.1 and Lemma 2.4.4.

(G6). Suppose $\mathcal{F}_s((\Lambda, \lambda))$ is an isomorphism. Then $\mathcal{F}_s((\Lambda, \lambda)) = (\Lambda, F_s(\lambda))$ is an isomorphism. So Λ is an isomorphism in **fset** and for all $i \in I$, $F_s(\lambda)_i = F_s(\lambda_i)$ is an isomorphism. But since F_s reflects isomorphisms, λ_i is an isomorphism for all $i \in I$. Hence (Λ, λ) is an isomorphism in $Fam(\mathbb{X}/S)$, so $\mathcal{F}_s$ reflects isomorphisms.

□

Concluding Remarks.

The goal of this chapter was, starting with an arbitrary category $\mathbb{X}$, construct a Galois category under certain restrictions on $\mathbb{X}$. This goal has been reached as is seen above, but at a great cost. This procedure has lead to requiring a large list of assumptions on $\mathbb{X}$. We required a total of 7 assumptions on $\mathbb{X}$, which includes the geometric point $s \longrightarrow S$, which is larger than the list of axioms $(G1) - (G6)$. As was mentioned in the introduction, we refrained from providing examples throughout this chapter because it is difficult to find such examples.

A particular difficulty is the point $s \longrightarrow S$ and the fact that $\mathbb{X}/s \simeq$ **fset**. This is a very restrictive requirement on the category. Recall that the fundamental functor $\mathcal{F}_s$ was constructed using the pullback functor along $s \longrightarrow S$, F_s. But the equivalence $M, N : \mathbb{X}/s \simeq$ **fset** tells us that these pullbacks were "trivial" in the following sense:

$$F_s(X) = X \times_S s \cong \coprod_{M(X \times_S s)} s$$

That is, the pullbacks of our "covers", $X \longrightarrow S$ are trivial in that the pullbacks are just copies of the "point" $s \longrightarrow S$. The existence of such a point is too strong and the next chapter seeks to address this issue.

Chapter 3

Coverings and Trivializations

Introduction

The objective of this Chapter is to define a new category, denoted $\mathcal{A}$, so that taking the family category of this category yields a Galois category. We did a similar thing in Chapter 2 by putting assumptions on a category $\mathbb{X}/S$ so that $Fam(\mathbb{X}/S)$ was a Galois category, but as we saw, these assumptions were very restrictive. In this chapter we will construct a category, $\mathcal{A}$, using a generalization of a property found in the example of strongly separable algebras. Then, we will show that this category satisfies certain axioms so that $Fam(\mathcal{A})$ is a Galois category. This approach is made possible through the work done by Michael Barr in [3] where he provides an equivalent characterization of Galois categories.

Section 3.1 is devoted to introducing Barr's characterization. In Section 3.2, we introduce a new notion called a covering which will be the foundation for our category $\mathcal{A}$ which we also define in this section. In Section 3.3, we show that $\mathcal{A}$ satisfies the five axioms of Barr, thus making $Fam(\mathcal{A})$ a Galois category. In Section 3.4, we finally see a non-trivial example appear - strongly separable algebras.

3.1 Barr's Characterization of Galois Categories

Here we provide Barr's characterization of Galois categories from [3]. First we have Barr's 5 axioms for a category $\mathcal{A}$.

(REC). *Regular Epimorphism Condition.* Every map is a regular epimorphism. That is, for all maps $f : X \longrightarrow Y$ there exists a parallel pair of maps $g, h : A \longrightarrow X$

such that Y equipped with f is the coequalizer of g and h.

$$\begin{tikzcd} A \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

(AP). *Amalgamation Property.* For every pair of maps

$$\begin{tikzcd} & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

there exists an object C and maps $X \longleftarrow C \longrightarrow Z$ such that we have a commuting square:

$$\begin{tikzcd} C \arrow[r] \arrow[d] & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

(UBM). *Uniformly Bounded Multisums.* Let X be any object. Then for all Z, there exists a natural number $R(X)$ (to be defined later), objects $C_1, \ldots, C_n$ and maps:

$$\begin{tikzcd} C_i \arrow[r] \arrow[d] & Z \\ X & \end{tikzcd}$$

such that $n \leq R(X)$ and whenever there exists an object C and maps

$$\begin{tikzcd} C \arrow[r] \arrow[d] & Z \\ X & \end{tikzcd}$$

there is a unique i and a unique map $C \longrightarrow C_i$ such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & C_i \arrow[r] \arrow[d] & Z \\ & X & \end{tikzcd}$$

(TO). *Terminal Object.* There is a terminal object.

(EC). *Exactness Condition.* Coequalizers exist and products in $Fam(\mathcal{A})$ preserve coequalizers.

Proposition 3.1.1. ([3] Proposition 2.4). *Suppose $\mathcal{A}$ satisfies REC and has coequalizers. Then for every $A \in Fam(\mathcal{A})$, the functor $A \times -$ commutes with finite colimits.*

Let $\mathcal{A}$ be a category which satisfies the above five conditions.

Definition 3.1.1. An object $A \in \mathcal{A}$ is called *normal* if for any pair of maps $f, g : B \longrightarrow A$ there exists an automorphism σ of A such that $f\sigma = g$. Furthermore, a map $f : X \longrightarrow Y$ is called a *normal envelope* of Y if X is normal and given any other normal object N such that $N \longrightarrow Y$, the following diagram commutes:

$$\begin{tikzcd} & N \arrow[dl] \arrow[ddl] \\ X \arrow[d] & \\ Y & \end{tikzcd}$$

Proposition 3.1.2. ([3] Proposition 4.6). *Let N be normal. Then $N \times N \cong \coprod N$.*

Barr's two main Theorems from [3] are as follows.

Theorem 3.1.1. *Suppose $\mathcal{A}$ satisfies AP, UBM, TO, and every map in $\mathcal{A}$ is an epimorphism. Then every object of $\mathcal{A}$ has a normal envelope.*

Theorem 3.1.2. *Suppose $\mathcal{A}$ satisfies, in addition, RMC and EC. Then $\mathcal{A}$ is equivalent to the category of transitive π-sets for a uniquely determined profinite group π and conversely, such a category satisfies the above conditions.*

From Example 1.4.4 we know that the transitive π-sets are the connected objects of the category π-**fset**. Then by Proposition 2.2.1 we have:

$$\pi\text{-}\mathbf{fset} \simeq Fam(Conn(\pi\text{-}\mathbf{fset})) \simeq Fam(\mathcal{A})$$

where the second equivalence is by Theorem 3.1.2. Hence this theorem implies that if $\mathcal{A}$ satisfies the above five axioms, then $Fam(\mathcal{A})$ is a Galois category.

3.2 Coverings and Trivializations

In this section we will construct a new category $\mathcal{A}$. Much like how $Fam(\mathbb{X}/S)$ was built from an already established category $\mathbb{X}$, the category $\mathcal{A}$ will be constructed from an already existing category, $\mathbb{Y}$. In addition, we will utilize a certain sub-class of maps in $\mathbb{Y}$ which we will denote by $\mathcal{C}$. We list our assumptions on $\mathbb{Y}$ and $\mathcal{C}$ below:

Assumptions.

1. Let $\mathbb{Y}$ be a category such that:
    - $\mathbb{Y}$ is finitely extensive (Definition 1.4.2).
    - $\mathbb{Y}$ has pullbacks.
    - $\mathbb{Y}$ has coequalizers.
    - $\mathbb{Y}$ is locally connected (Definition 2.2.2).
2. There exists a connected object $S \in \mathbb{Y}$.
3. Let $\mathcal{C} \subseteq Mor(\mathbb{Y})$ be the class of morphisms in $\mathbb{Y}$ such that:
    - For all $f : X \longrightarrow Y \in \mathcal{C}$ the pullback functor f^* preserves and reflects regular epimorphisms and preserves coequalizers.

- $\mathcal{C}$ is closed under composition and stable under pullback (Definition 1.3.7).

Discussion.

1. Most of the assumptions we have made on $\mathbb{Y}$ are self explanatory. $\mathbb{Y}$ being locally connected implies that we can write any object in $\mathbb{Y}$ as a coproduct of connected objects, i.e. for any $Y \in \mathbb{Y}$ we have:
$$Y = \coprod_{i \in I} C_i$$
We shall call the C_i the *connected components* of Y.

2. The object S will act as our "base space" with which we will work over. This will of course take various meanings in different categories, but we only require S to be connected.

3. Generally speaking, this is a rather strong list of requirements on the class $\mathcal{C}$. However, we will see that we do not necessarily need all of these conditions. This will be explained at the end of Section 3.

Our first definition will play a crucial role in defining our category.

Definition 3.2.1. A map $f : X \longrightarrow Y$ in $\mathbb{Y}$ is called a *covering* if there exists a map $p : T_X \longrightarrow Y \in \mathcal{C}$ such that $X \times_Y T_X \cong \coprod_{[X:T_X]} T_X$ as objects over T_X. We call such a map $T_X \longrightarrow Y$ a *trivialization* of f and $[X : T_X]$ the *index* of X over T_X.

In other words, if $f : X \longrightarrow Y$ is a covering with $p : T_X \longrightarrow Y$ its trivialization, then $p^*(X) \cong \coprod_{[X:T_X]} T_X$ where p^* is the pullback functor along p.

Remark. The index need not be finite. If the index is finite for all possible trivializations, we say that f is a *finite covering.* If it is clear or unneccesary to make mention of the index, then it will be omitted. If $T \longrightarrow Y$ is *not* a trivialization of $X \longrightarrow Y$, then $[X : T] = 0$.

The next few lemmas describe a few important properties of coverings.

Lemma 3.2.1. *Let $\mathbb{Y}$ be given as in the assumptions and suppose $Y \in \mathbb{Y}$. Let $\mathcal{D}$ be a finite collection of maps with codomain Y that are coverings. Then there exists an object $T_{\mathcal{D}}$ and a map $T_{\mathcal{D}} \longrightarrow Y$ such that for all $X \longrightarrow Y \in \mathcal{D}$, $X \times_Y T_{\mathcal{D}} \cong \coprod T_{\mathcal{D}}$.*

Proof. Let $\mathcal{D}$ be a finite collection of maps in $\mathbb{Y}$ with codomain $Y \in \mathbb{Y}$ that are coverings. Then for any $X \longrightarrow Y \in \mathcal{D}$ there exists a map $T_X \longrightarrow Y \in \mathbb{Y}$ such that $X \times_Y T_X \cong \coprod T_X$. For any two objects $X \longrightarrow Y, Z \longrightarrow Y \in \mathcal{D}$ consider the pullback of their trivializations:

$$\begin{tikzcd}
T_X \times_Y T_Z \arrow[r] \arrow[d] & T_X \arrow[d] \\
T_Z \arrow[r] & Y
\end{tikzcd}$$

This pullback diagram is in $\mathbb{Y}$ so it exists by assumption. Then observe that the induced map $T_X \times_Y T_Z \longrightarrow Y$ obtained by going around the above square in either direction is in $\mathcal{C}$ since $\mathcal{C}$ is closed under composition and pullback. Then since $\mathbb{Y}$ is assumed to be extensive, we have:

$$X \times_Y (T_X \times_Y T_Z) \cong (\coprod_{[X:T_X]} T_X) \times_Y T_Z \cong \coprod_{[X:T_X]} (T_X \times_Y T_Z)$$

$$Z \times_Y (T_X \times_Y T_Z) \cong Z \times_Y (T_Z \times_Y T_X) \cong (\coprod_{[Z:T_Z]} T_Z) \times_Y T_X \cong \coprod_{[Z:T_Z]} (T_X \times_Y T_Z)$$

Hence both $X \longrightarrow Y$ and $Z \longrightarrow Y$ have $T_X \times_Y T_Z$ as a common trivialization. We can generalize this argument: consider the pullback of all of the trivializations of the maps in $\mathcal{D}$, $T_{\mathcal{D}} := T_X \times_Y \dots$. Then,

$$X \times_Y T_{\mathcal{D}} \cong X \times_Y (T_X \times_Y \dots) \cong (\coprod T_X) \times_Y \dots \cong \coprod (T_X \times_Y \dots) \cong \coprod T_{\mathcal{D}}.$$

Hence every $X \longrightarrow Y \in \mathcal{D}$ has $T_{\mathcal{D}} \longrightarrow Y$ as its common trivialization. $\square$

Lemma 3.2.2. *Suppose we have the following diagram in $\mathbb{Y}$:*

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "fg"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

If f and g are (finite) coverings then fg is a (finite) covering.

Proof. Suppose f and g are coverings. Then there exists trivializations of both; that is, there exists $T_f \longrightarrow Y$ and $T_g \longrightarrow Z$ such that:

$$X \times_Y T_f \cong \coprod T_f$$

$$Y \times_Z T_g \cong \coprod T_g$$

Using the map g we can consider $T_f \longrightarrow Y$ as an object over Z, i.e. $T_f \longrightarrow Y \xrightarrow{g} Z$. Then consider the pullback of $T_f \longrightarrow Z$ and $T_g \longrightarrow Z$:

$$\begin{tikzcd}
T := T_f \times_Z T_g \arrow[r] \arrow[dd] & T_f \arrow[d] \\
& Y \arrow[d, "g"] \\
T_g \arrow[r] & Z
\end{tikzcd}$$

We want to show that there exists a $T_{fg} \longrightarrow Z$ such that $X \times_Z T_{fg} \cong \coprod T_{fg}$. We claim that T defined above is the desired trivialization. To see this, consider the following diagram where in each inner square we have taken the pullback:

$$\begin{tikzcd}
X \times_Y (Y \times_Z T) \arrow[r] \arrow[d] & Y \times_Z T \arrow[r] \arrow[d] & T \arrow[d] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

Since each inner square is a pullback, the pullback square lemma tells us that $X \times_Y (Y \times_Z T) \cong X \times_Z T$. Considering T as an object with a map to Z, then as an object with a map to Y

we have:

$$Y \times_Z T \cong Y \times_Z (T_f \times_Z T_g) \cong (\coprod T_g) \times_Z T_f \cong \coprod T$$
$$X \times_Y T \cong X \times_Y (T_f \times_Z T_g) \cong (X \times_Y T_f) \times_Z T_g \cong (\coprod T_f) \times_Z T_g \cong \coprod T$$

That is, both f and g are trivialized by $T \longrightarrow Y$ and $T \longrightarrow Y \longrightarrow Z$ respectively. But then substituting the above into our pullback $X \times_Y (Y \times_Z T)$ we get:

$$\begin{aligned} X \times_Z T &\cong X \times_Y (Y \times_Z T) \\ &\cong X \times_Y (\coprod T) \\ &\cong \coprod (X \times_Y T) \\ &\cong \coprod (\coprod T) \\ &\cong \coprod T \end{aligned}$$

So $T \longrightarrow Y \longrightarrow Z$ is a trivialization of fg, hence fg is a covering. $\square$

Lemma 3.2.3. *Suppose we have the following pullback diagram in $\mathbb{Y}$:*

$$\begin{tikzcd} X \times_Y Z \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

If f is a (finite) covering then π_2 is a (finite) covering.

Proof. Suppose we have a pullback diagram as in the Lemma and that $f : X \longrightarrow Y$ is a covering. Then there exists a map $T_X \longrightarrow Y$ such that

$$X \times_Y T_X \cong \coprod_{[X:T_X]} T_X$$

We need to show that $\pi_2 : X \times_Y Z \longrightarrow Z$ is a covering. This means that we need some map $T \longrightarrow Z$ that is a trivialization for π_2. Notice that the trivialization we need must be a map over Z. Taking the pullback of T_X and Z over Y we get:

$$\begin{tikzcd} Z \times_Y T_X \arrow[r] \arrow[d] & T_X \arrow[d] \\ Z \arrow[r, "g"'] & Y \end{tikzcd}$$

Then $Z \times_Y T_X \longrightarrow Z$ from the above diagram will be our candidtate for the trivialization of π_2. Observe:

$$\begin{aligned} (X \times_Y Z) \times_Z (Z \times_Y T_X) &\cong X \times_Y (Z \times_Y T_X) \\ &\cong Z \times_Y (X \times_Y T_X) \text{ by associativity of pullbacks.} \\ &\cong Z \times_Y (\coprod_{[X:T_X]} T_X) \\ &\cong \coprod_{[X:T_X]} (Z \times_Y T_X) \text{ by commutativity of pullbacks and coproducts.} \end{aligned}$$

Hence $\pi_2 : X \times_Y Z \longrightarrow Z$ is a covering. If $[X : T_X]$ is finite, then π_2 is a finite covering. $\square$

Throughout these lemmas, the index $[X : T_X]$ has not played a major role. However, we need to be able to keep track of the indexes and we do this through the following notion.

Definition 3.2.2. Let $f : X \longrightarrow Y$ be a covering in $\mathbb{Y}$. Then the *rank of* f (or the rank of X), denoted $R(X)$, is the maximum over all indexes:

$$R(X) = \max_{T \in \mathcal{C}} [X : T]$$

Remark. If $f : X \longrightarrow Y$ is not a finite covering, then there exists some $T \longrightarrow Y$ such that $[X : T]$ is not finite, so $R(X) = \infty$. Thus the rank will be more important when dealing with finite coverings.

Before we discuss some of the various properties of the rank which we will need for the next section, we need to know more about how connected objects and coproducts interact. This is accomplished through the following Proposition.

Proposition 3.2.1. *Let $\{X_i\}_{i\in I}$ and $\{Y_j\}_{j\in J}$ be collections of connected objects in $\mathbb{Y}$. Then if $\coprod_I X_i \cong \coprod_J Y_j$,*

(i) $|I| \cong |J|$.

(ii) For each X_i, there exists a Y_j such that $X_i \cong Y_j$.

Proof. Let $H : \coprod_I X_i \longrightarrow \coprod_J Y_J$ be an isomorphism, with $\{X_i\}_{i\in I}$ and $\{Y_j\}_{j\in J}$ all connected in $\mathbb{Y}$. Then,

$$\begin{aligned}\mathbb{Y}(\coprod_I X_i, \coprod_J Y_j) &\cong \prod_I \mathbb{Y}(X_i, \coprod_J Y_j) \text{ by Lemma 2.2.1.}\\ &\cong \prod_I \Big(\coprod_J \mathbb{Y}(X_i, Y_j)\Big) \text{ since } X_i \text{ is connected.}\end{aligned}$$

So H corresponds to some collection of maps:

$$\Big(H_{i,h(i)} : X_i \longrightarrow Y_{h(i)}\Big)_{i\in I}$$

where h is the unique function on the indexes, given by the unique choice of j made to form $H_{i,j}$:

$$h : I \longrightarrow J$$

Since H is an isomorphism, we have a map $H^{-1} : \coprod_J Y_j \longrightarrow \coprod_I X_i$ which then corresponds to a collection of maps:

$$\Big((H^{-1})_{j,g(j)} : Y_j \longrightarrow X_{g(j)}\Big)_{j\in J}$$

where g is the function on the indexes:

$$g : J \longrightarrow I$$

By the isomorphism on the hom-sets, note that for some i, j:

$$H_{i,h(i)} = H\restriction_{X_i} \text{ and } (H^{-1})_{j,g(j)} = H^{-1}\restriction_{Y_j}$$

Now $H^{-1}H = 1_Y$ so in particular $(H^{-1}H)\restriction_{Y_j} = 1_{Y_j}$. Then,

$$\begin{aligned} 1_{Y_j} = (H^{-1}H)\restriction_{Y_j} &= H^{-1}\restriction_{Y_j} H\restriction_{H^{-1}(Y_j)} \\ &= H^{-1}\restriction_{Y_j} H\restriction_{X_{g(j)}} \end{aligned}$$

Similarly for the other inverse,

$$\begin{aligned} 1_{X_i} = (HH^{-1})\restriction_{X_i} &= H\restriction_{X_i} H^{-1}\restriction_{H(X_i)} \\ &= H\restriction_{X_i} H^{-1}\restriction_{Y_{h(i)}} \end{aligned}$$

Then $X_i \cong Y_{h(i)}$ and $g = h^{-1}$. □

Since the category $\mathbb{Y}$ is assumed to be locally connected, we have that $X \cong \coprod_{i\in I} X_i$ for some collection of connected objects $\{X_i\}_{i\in I}$ in $\mathbb{Y}$. So by the above proposition we know that this decomposition is unique, up to isomorphism.

Our next result shows us that if we decompose a covering into its connected components, each of the connected components are themselves coverings.

Lemma 3.2.4. *Suppose that $X \longrightarrow Y \in \mathbb{Y}$ is a (finite) covering. Then given $X = \coprod_I C_i$ where each C_i is connected, for all $i \in I$ $C_i \longrightarrow Y$ is a (finite) covering.*

Proof. Let $T \longrightarrow Y$ be a trivialization for $X \longrightarrow Y$. Then,

$$X \times_Y T \cong \coprod_{[X:T]} T$$

Now $X = \coprod_I C_i$ so in particular we have the following map for every C_i.

$$C_i \longrightarrow \coprod_I C_i = X \longrightarrow Y$$

Thus we can take the pullback of each $C_i \longrightarrow Y$ over $T \longrightarrow Y$ to get $C_i \times_Y T$, which is also an object in $\mathbb{Y}$. Hence it can also be written as a coproduct of its connected components, i.e. $C_i \times_Y T \cong \coprod_{K_i} D_{k_i}$. We then have the following:

$$\begin{aligned}\coprod_{[X:T]} T &\cong X \times_Y T \\ &\cong (\coprod_I C_i) \times_Y T \\ &\cong \coprod_I (C_i \times_Y T) \\ &\cong \coprod_I (\coprod_{K_i} D_{k_i})\end{aligned}$$

From here, there are two cases to consider. First suppose that T is connected. Then since each D_{k_i} is connected by Proposition 3.2.1 each $D_{k_i} \cong T$. Hence $K_i = [C_i : T]$ and in particular,

$$C_i \times_Y T \cong \coprod_{[C_i:T]} T$$

so $C_i \longrightarrow Y$ is a covering. Furthermore, by Proposition 3.2.1

$$[X : T] = \sum_{i \in I} K_i = \sum_{i \in I} [C_i : T]$$

So $[X : T] \geq [C_i : T]$ for each $i \in I$ and if $[X : T]$ is finite then so is $[C_i : T]$.

For the second case, suppose that T is not connected. Then since $T \in \mathbb{Y}$ we can write T as a coproduct of its connected components, i.e.

$$T \cong \coprod_{n \in N} T_n$$

Now we claim that each $T_n \longrightarrow Y$ is also a trivialization for $X \longrightarrow Y$. To see this, consider the following for every $n \in N$:

$$\begin{aligned}
X \times_Y T_n &\cong X \times_Y (T \times_T T_n) \\
&\cong (X \times_Y T) \times_T T_n \\
&\cong (\coprod_{[X:T]} T) \times_T T_n \\
&\cong \coprod_{[X:T]} (T \times_T T_n) \\
&\cong \coprod_{[X:T]} T_n
\end{aligned}$$

Now using the trivialization $T_n \longrightarrow Y$ where T_n is a connected component of T, we can proceed exactly as we did in the first case to get the desired result. □

We can now prove a series of lemmas regarding the rank.

Lemma 3.2.5. *Suppose that $X \longrightarrow Y \in \mathbb{Y}$ is a finite covering. Then given $X = \coprod_I C_i$ where each C_i is a connected component of X, $I \leq R(X)$.*

Proof. We can use the exact same argument as in Lemma 3.2.4 to get that:

$$[X : T] = \sum_{i \in I} [C_i : T]$$

for some trivialization $T \longrightarrow Y$ of $X \longrightarrow Y$.

$X \longrightarrow Y$ is a finite covering by assumption, so each $[C_i : T]$ must also be finite. Thus there is some minimum of the $[C_i : T]$, say K_{min}. Then,

$$R(X) \geq [X : T] = \sum_{i \in I} [C_i : T] \geq \sum_I K_{min} = I \cdot K_{min} \geq I$$

Hence $R(X) \geq I$ as desired. □

Lemma 3.2.6. *Let $X \longrightarrow Y$, $Z \longrightarrow Y$ be finite coverings and $X \times_Y Z \cong \coprod_I C_i$ be the decomposition of $X \times_Y Z$ into its connected components. Then,*

$$R(X \times_Y Z) \geq \sum_{i \in I} R(C_i)$$

Proof. We have the same situation as we did in Lemma 3.2.5, but with $X \times_Y Z$ instead of X, since $X \times_Y Z \longrightarrow Y$ is a covering by Lemmas 3.2.3 and 3.2.2. So we have:

$$[X \times_Y Z : T] = \sum_{i \in I} [C_i : T]$$

But then this immediately implies that

$$R(X \times_Y Z) \geq \sum_{i \in I} R(C_i)$$

as desired. $\square$

We are now ready to define our category of interest, which will be denoted $\mathcal{A}$. For a fixed connected object $S \in \mathbb{Y}$, the category $\mathcal{A}$ will be the subcategory of connected objects of the following category:

Definition 3.2.3. Let $Cov(\mathbb{Y}/S)$ be the full subcategory of $\mathbb{Y}/S$ with the following data:

- Objects: $X \longrightarrow S$ in $\mathbb{Y}/S$ such that $X \longrightarrow S$ is a covering.
- Maps: $\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$ maps in $\mathbb{Y}/S$ where $X \longrightarrow S$, $Y \longrightarrow S \in Cov(\mathbb{Y}/S)$.

Notice that we have shifted our view of coverings slightly when considering $Cov(\mathbb{Y}/S)$. Coverings are still maps in $\mathbb{Y}$, but we are now only interested in coverings that have codomain S - thus making them objects in $Cov(\mathbb{Y}/S)$.

We now arrive at the main category of interest for this chapter.

Definition 3.2.4. Let $\mathcal{A}$ be the full subcategory of $Cov(\mathbb{Y}/S)$ with the following data:

- Objects: finite, connected coverings $X \longrightarrow S$.
- Maps: $\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$ maps in $\mathbb{Y}/S$ where $X \longrightarrow S$, $Y \longrightarrow S \in \mathcal{A}$.

Recall that $X \longrightarrow S$ being a finite covering means that given any $T_X \longrightarrow S \in \mathcal{C}$ ($\mathcal{C}$ being the class of maps defined at the beginning of this section) such that $X \times_S T_X \cong \coprod_{[X:T_X]} T_X$ the index $[X : T_X]$ is finite. By connected we mean that $X \longrightarrow S$ is connected as an object in the category $\mathbb{Y}/S$.

On the classes of objects, each of these categories is contained in the one before it. That is,

$$\mathcal{A} \subset Cov(\mathbb{Y}/S) \subset \mathbb{Y}/S$$

Since each is a full subcategory, the maps are simply maps in $\mathbb{Y}/S$ with the domain and codomain in the category.

3.3 The Category $\mathcal{A}$

The goal of this section is to prove that $Fam(\mathcal{A})$ is a Galois category where $\mathcal{A}$ is the category of finite connected covers of S defined in the last section. Coverings in $Fam(\mathcal{A})$ are dealt with extensively in Chapter 6 of [5] by Borceux and Janelidze, where a cover is defined by imposing two different conditions. First, Borceux and Janelidze require that the trivializations $T \longrightarrow S$ in $\mathcal{C}$ be effective descent maps. Informally, this means that the trivializations $t : T \longrightarrow S$ have the property that the pullback functor t^* is conservative and reflects certain types of coequalizers. Second, they require that the covers themselves, $p : X \longrightarrow S$, are split by an effective descent map $t : T \longrightarrow S$. Again, informally this means that there exists an isomorphism $X \times_S T \cong \coprod T$ in the category of covers over T, which is the definition of cover that we have incorporated.

One of our goals in constructing the category $\mathcal{A}$ as we have done was to see if it was possible to replace the condition of effective descent with something else. We were able to replace this condition, provided we have the following additional assumptions on $\mathcal{A}$:

Assumptions.

1. Given a map $f : X \longrightarrow Y \in \mathcal{A}$ and a common trivialization of both X and Y, say $T \longrightarrow S$, the induced map:

$$\coprod_{[X:T]} T \longrightarrow \coprod_{[Y:T]} T$$

   is uniquely determined by the map on the indices

$$[X : T] \longrightarrow [Y : T]$$

2. Given a map $f : X \longrightarrow Y \in \mathcal{A}$ and a common trivialization as in assumption 1, the induced map on the indices:

$$[X : T] \longrightarrow [Y : T]$$

   is surjective.

Discussion.

1. The first assumption is inspired by the definition of a covering space of topological spaces. Lemma 2.4.2 ensures that if T is connected, then the map $\coprod_{[X:T]} T \longrightarrow \coprod_{[Y:T]} T$ is uniquely determined by the associated map $[X : T] \longrightarrow [Y : T]$. But if T is not connected, then this need not be the case. However, a covering space of topological spaces is locally a trivial covering over a connected open subset of the base space. Proposition 6.5.2 of [5] shows that the first assumption holds for these covers (since it shows that the covering maps are sums of maps to connected components of T), and this assumption is encoded in the notion of splitting incorporated in Definition 5.1.7 of [5].

2. The second assumption is satisfied by covering maps of topological spaces, where any map between covers must be a surjection. In fact, this assumption is meant to generalize another condition satisfied by covers of topological spaces: that for any point x in the base space S, there exists a connected component T_i (i.e. connected open subset of S) of the trivialization $t : T \longrightarrow S$ such that $x \in t(T_i)$. We have identified assumption 2. as the condition that we need to verify to show that we have a Galois category; in future work we intend to show that covers with the property that the "union" of the connected components of the trivialization T is equal to S automatically satisfy 2., but at the time of writing it remains unclear what "union" means in this context.

With these assumptions in hand, we now show that $Fam(\mathcal{A})$ is a Galois category by verifying Barr's five axioms for the category $\mathcal{A}$ of Definition 3.2.4.

(REC). Let $f : X \longrightarrow Y$ be a map in $\mathcal{A}$. So f is of the form:

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

We want to show that f is regular epic. Suppose that $p : T \longrightarrow S$ is the common trivialization for $X \longrightarrow S$ and $Y \longrightarrow S$. Now apply the functor p^* to the triangle above:

$$\coprod_{[X:T]} T \xrightarrow{p^*(f)} \coprod_{[Y:T]} T$$

By assumption 1 at the beginning of this section, the map $p^*(f)$ is unquely determined by the map on the indexes:

$$[X : T] \longrightarrow [Y : T]$$

Now by assumption 2, this map is surjective. A map in **fset** is epic if and only if it is regular epic ([4] p. 42) hence $[X : T] \longrightarrow [Y : T]$ is regular epic. This means that the map is the coequalizer of a pair of maps $g, h : D \longrightarrow [X : T]$ for some $D \in \mathbf{fset}$.

$$D \overset{g}{\underset{h}{\rightrightarrows}} [X : T] \longrightarrow [Y : T]$$

Each map $g, h : D \longrightarrow [X : T]$ corresponds uniquely to maps $\coprod_D T \longrightarrow \coprod_{[X:T]} T$ so the diagram,

$$\coprod_D T \rightrightarrows \coprod_{[X:T]} T \xrightarrow{p^*(f)} \coprod_{[Y:T]} T$$

is also a coequalizer. Hence $p^*(f)$ is also regular epic. Now, $p \in \mathcal{C}$ and by assumption, p^* reflects regular epics. Thus, f is also regular epic.

In order to show **(AP)**, we will need a lemma.

Lemma 3.3.1. *Suppose $X \longrightarrow S$, $Y \longrightarrow S$, and $Z \longrightarrow S$ are all in $\mathcal{A}$. Then the pullback $X \times_Y Z \longrightarrow S$ is a finite covering.*

Proof. We have the following pullback diagram in $\mathbb{Y}/S$:

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Each of X, Y, Z are in $\mathcal{A}$ so we can find a common trivialization of all three, say $p : T \longrightarrow S$. Now apply the functor p^* to the above pullback diagram to get:

$$\begin{tikzcd} (X \times_Y Z) \times_S T \arrow[r] \arrow[d] & \coprod_{[Z:T]} T \arrow[d] \\ \coprod_{[X:T]} T \arrow[r] & \coprod_{[Y:T]} T \end{tikzcd}$$

Recall that p^* is a right adjoint, hence preserves limits by Theorem 1.3.1 . So the above diagram is a pullback diagram which implies that:

$$(X \times_Y Z) \times_S T \cong (\coprod_{[X:T]} T) \times_{\coprod_{[Y:T]} T} (\coprod_{[Z:T]} T)$$

Then by commutativity of coproduct and pullback, we have:

$$\begin{aligned}(\coprod_{[X:T]} T) \times_{\coprod_{[Y:T]} T} (\coprod_{[Z:T]} T) &\cong \coprod_{[X:T]} \Big(T \times_{\coprod_{[Y:T]} T} (\coprod_{[Z:T]} T)\Big) \\ &\cong \coprod_{[X:T]} \Big(\coprod_{[Z:T]} (T \times_{\coprod_{[Y:T]} T} T)\Big)\end{aligned}$$

where $T \times_{\coprod_{[Y:T]} T} T$ is the following pullback:

$$\begin{tikzcd} T \times_{\coprod_{[Y:T]} T} T \arrow[r] \arrow[d] & T \arrow[d] \\ T \arrow[r] & \coprod_{[Y:T]} T \end{tikzcd}$$

By the disjointness of the coproduct, the inclusion maps $T \longrightarrow \coprod_{[Z:T]} T$ are monic and the pullback along two different inclusion maps is the initial object (see Definition 1.4.3). Hence this pullback diagram is the same as exactly one of the following two pullbacks, depending on if the inclusions are onto the same component.

$$\text{Either: } \begin{tikzcd} 0 \arrow[r] \arrow[d] & T \arrow[d] \\ T \arrow[r] & \coprod_{[Z:T]} T \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} T \arrow[r] \arrow[d] & T \arrow[d] \\ T \arrow[r] & \coprod_{[Z:T]} T \end{tikzcd}$$

In either case, we see that $(\coprod_{[X:T]} T) \times_{\coprod_{[Y:T]} T} (\coprod_{[Z:T]} T)$ is a coproduct of some index of T (the index being either 0 or $[X:T][Z:T]$). Hence $X \times_Y Z \longrightarrow S$ is a finite covering. $\square$

(AP). Suppose we have the following diagram in $\mathcal{A}$:

$$\begin{tikzcd} & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

These maps are also in $\mathbb{Y}/S$ so we can form the pullback to get the following diagram in $\mathbb{Y}/S$:

$$\begin{tikzcd} X \times_Y Z \arrow[r] \arrow[d] & Z \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

By Lemma 3.3.1 above, $X \times_Y Z \longrightarrow S$ is a finite covering. However it may not be true that $X \times_Y Z$ is connected, thus it cannot be used for our original diagram, which is in $\mathcal{A}$. To work around this, consider the connected components:

$$X \times_Y Z \cong \coprod_I C_i$$

By the above, $X \times_Y Z \longrightarrow S$ is a finite covering hence by Lemma 3.2.4, each $C_i \longrightarrow S$ is also a finite covering. Choose any connected component, say $C_i \longrightarrow S$. Then $C_i \longrightarrow S$ is a finite, connected covering of S, hence is in $\mathcal{A}$. Using the inclusion map, we can construct maps from C_i to Z and to X as follows:

$$C_i \longrightarrow \coprod_I C_i \xrightarrow{\cong} X \times_Y Z \longrightarrow Z$$

$$C_i \longrightarrow \coprod_I C_i \xrightarrow{\cong} X \times_Y Z \longrightarrow X$$

$C_i \longrightarrow S \in \mathcal{A}$ and $X \longrightarrow S$, $Z \longrightarrow S \in \mathcal{A}$ so the above maps are in $\mathcal{A}$. It remains to show that the following square (in $\mathcal{A}$) commutes:

$$\begin{tikzcd} C_i \arrow[r] \arrow[d] & Z \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

First, the map $C_i \longrightarrow Z$ factors through the coproduct $\coprod_I C_i$ via the inclusion map $C_i \longrightarrow \coprod_I C_i$ and then through the isomorphism with $X \times_Y Z$ on the other side. Thus the map,

$$C_i \longrightarrow Z \longrightarrow Y$$

is the same as the map,

$$C_i \longrightarrow \coprod_I C_i \xrightarrow{\cong} X \times_Y Z \longrightarrow Z \longrightarrow Y$$

Now, since the pullback diagram commutes the map,

$$X \times_Y Z \longrightarrow Z \longrightarrow Y$$

is equal to the map,

$$X \times_Y Z \longrightarrow X \longrightarrow Y$$

Then, notice that the map $C_i \longrightarrow X$ factors through the coproduct $\coprod_I C_i$ in exactly the same way as the map $C_i \longrightarrow Z$. Hence the map,

$$C_i \longrightarrow \coprod_I C_i \xrightarrow{\cong} X \times_Y Z \longrightarrow Z \longrightarrow Y$$

is the same as the map,

$$C_i \longrightarrow X \longrightarrow Y$$

Hence $\mathcal{A}$ satisfies **(AP)**.

(UBM). Let $X \longrightarrow S \in \mathcal{A}$. Given any other object $Z \longrightarrow S$ we can form the pullback over S:

$$\begin{tikzcd} X \times_S Z \arrow[r] \arrow[d] & Z \arrow[d] \\ X \arrow[r] & S \end{tikzcd}$$

By Lemma 3.2.3 $X \times_S Z \longrightarrow Z$ is a finite covering. Then since $Z \longrightarrow S$ is also a finite covering, by Lemma 3.2.2 $X \times_S Z \longrightarrow Z \longrightarrow S$ is a finite covering. Hence $X \times_S Z \longrightarrow S$ is a finite covering. So writing $X \times_S Z$ in terms of its connected components,

$$X \times_S Z \cong \coprod_I C_i$$

each $C_i \longrightarrow S$ is also a finite covering by Lemma 3.2.4. We claim that the connected components C_i of $X \times_S Z$ are precisely the required $C_i's$ for UBM.

First we need maps $C_i \longrightarrow Z$ and $C_i \longrightarrow X$. These maps will be the inclusion map into the coproduct followed by the projection down from the pullback, i.e. for all $i \in I$ we use the composition of maps given from the following diagram:

$$\begin{tikzcd}
C_i \arrow[r] \arrow[rrr, bend left] \arrow[drr, bend right] & \coprod_I C_i \arrow[r, "\cong"] & X \times_S Z \arrow[r] \arrow[d] & Z \arrow[d] \\
& & X \arrow[r] & S
\end{tikzcd}$$

Next we need to show that $I \leq R(X)$, but this is from Lemma 3.2.5.

Now suppose we had an object $C \in \mathcal{A}$ with maps in $\mathbb{Y}/S$:

$$\begin{tikzcd}
C \arrow[r] \arrow[d] & Z \\
X &
\end{tikzcd}$$

Then by the universal property of the pullback $X \times_S Y$ in $\mathbb{Y}$ we have:

$$\begin{tikzcd}
C \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_S Z \arrow[r] \arrow[d] & Z \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

So we have the following map in $\mathbb{Y}/S$:

$$C \longrightarrow X \times_S Z \xrightarrow{\cong} \coprod_I C_i$$

Since $C \in \mathcal{A}$, C is in particular connected in $\mathbb{Y}/S$. Hence the above map $C \longrightarrow \coprod_I C_i$ corresponds uniquely to a map:

$$\gamma : C \longrightarrow C_i$$

for some C_i. Thus γ is our candidate for the required unique map for UBM. It remains to show that the following diagram commutes:

$$\begin{tikzcd}
C \arrow[dr, dashed, "\gamma"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& C_i \arrow[r] \arrow[d] & Z \\
& X &
\end{tikzcd}$$

By expanding the diagram above, i.e. including the definitions of $C_i \longrightarrow Z$ and $C_i \longrightarrow X$ we get the following commuting diagram:

$$\begin{tikzcd}
C \arrow[dr, dashed, "\gamma"] \arrow[dddr, dashed] \arrow[dddrr, bend left] \arrow[ddddr, bend right] & & \\
& C_i \arrow[d] & \\
& \coprod_I C_i \arrow[d, "\cong"] & \\
& X \times_S Z \arrow[r] \arrow[d] & Z \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}$$

which gives us the desired commutativity. Hence $\mathcal{A}$ satisfies UBM.

(TO). We need to show that $\mathcal{A}$ has a terminal object. Consider the arrow $1_S : S \longrightarrow S$ in $\mathbb{Y}/S$. Given any other $f : X \longrightarrow S \in \mathbb{Y}/S$ there is always a map:

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "f"'] & & S \arrow[dl, "1_S"] \\
& S &
\end{tikzcd}$$

So 1_S is the terminal object in $\mathbb{Y}/S$. Since S is connected by assumption, it remains to show that $1_S : S \longrightarrow S$ is a covering. It is straightforward to show that $1_S : S \longrightarrow S$ is in $\mathcal{C}$ and so by:

$$S \times_S S \cong S \cong \coprod_{\{*\}} S$$

$1_S : S \longrightarrow S$ is a covering (by the above, 1_S is the trivialization of 1_S). So $1_S \in \mathcal{A}$ which means $\mathcal{A}$ has a terminal object.

(EC). We need to show that $\mathcal{A}$ has coequalizers. So suppose we have a parallal pair $g, h : X \longrightarrow Y$ in $\mathcal{A}$

$$X \underset{h}{\overset{g}{\rightrightarrows}} Y$$

Since $\mathbb{Y}/S$ has coequalizers, there exists an object $C \in \mathbb{Y}/S$ and a map $q : Y \longrightarrow C$ such that

$$X \underset{h}{\overset{g}{\rightrightarrows}} Y \xrightarrow{q} C$$

is a coequalizer diagram. Both $X, Y \in \mathcal{A}$ so we can find a common trivialization $p : T \longrightarrow S \in \mathcal{C}$. Applying $p^* : \mathbb{Y}/S \longrightarrow \mathbb{Y}/T$ to the above diagram we get:

$$\begin{tikzcd}
X \times_S T \arrow[r, shift left, "p^*(g)"] \arrow[r, shift right, "p^*(h)"'] \arrow[d, "\cong"'] & Y \times_S T \arrow[r, "p^*(q)"] \arrow[d, "\cong"] & C \times_S T \\
\coprod_{[X:T]} T \arrow[r, shift left, "p^*(g)"] \arrow[r, shift right, "p^*(h)"'] & \coprod_{[Y:T]} T &
\end{tikzcd}$$

where we use $p^*(g)$, $p^*(h)$ in two ways for convenience. Taking the bottom row with the map $p^*(q)$ we get the following diagram:

$$\coprod_{[X:T]} T \underset{p^*(h)}{\overset{p^*(g)}{\rightrightarrows}} \coprod_{[Y:T]} T \xrightarrow{p^*(q)} C \times_S T$$

Since $p \in \mathcal{C}$, p^* in particular preserves coequalizers. Hence $C \times_S T$ along with the map $p^*(q)$ is the coequalizer of the parallel pair

$$\coprod_{[X:T]} T \underset{p^*(h)}{\overset{p^*(g)}{\rightrightarrows}} \coprod_{[Y:T]} T$$

Then by Lemma 2.4.2 the maps $p^*(g)$ and $p^*(h)$ are determined by two maps between the indexes in **Set**:

$$[X : T] \rightrightarrows [Y : T]$$

The coequalizer of this parallel pair in **Set** will uniquely determine the coequalizer of the parallel pair $p^*(g)$ and $p^*(h)$. That is, in **Set** we have the coequalizer diagram:

$$[X : T] \rightrightarrows [Y : T] \longrightarrow D$$

which determines the coequalizer diagram in $\mathbb{Y}/S$ of $p^*(g)$ and $p^*(h)$ as follows:

$$\coprod_{[X:T]} T \underset{p^*(h)}{\overset{p^*(g)}{\rightrightarrows}} \coprod_{[Y:T]} T \longrightarrow \coprod_{D} T$$

Hence $C \times_S T \cong \coprod_D T$, so $T \longrightarrow S \in \mathcal{C}$ is a trivialization for $C \longrightarrow S$. Hence $C \longrightarrow S$ is a finite covering. Lastly, C is a coequalizer of connected objects, hence is a connected colimit so C itself is connected by Proposition 1.4.3. Hence $C \in \mathcal{A}$ so $\mathcal{A}$ has coequalizers.

The last step is to show that products in $Fam(\mathcal{A})$ preserve coequalizers. But, since $\mathcal{A}$ satisfies REC and has coequalizers, Proposition 3.1.1 tells us that this is true.

3.4 An Example: Strongly Separable Algebras

Let S be a ring with no idempotents except 0 and 1. In other words, S is *connected*. An S-algebra is called *strongly separable* if A is both S-projective and $A \otimes_S A$-projective. For a classical reference on strongly separable algebras, see [13]. Then consider the following result due to Barr in [3]:

Theorem 3.4.1. *A is strongly separable if and only if there exists a faithfully flat S-algebra B such that $A \otimes_S B \cong B^n$ as B-algebras.*

Notice the similarity with our definition of $\mathcal{A}$. The objects in $\mathcal{A}$ are maps $X \longrightarrow S$, coverings, such that there exists a special object in $\mathcal{C}$, $T \longrightarrow S$, with $X \times_S T \cong \coprod T$. The trivialization $X \times_S T \cong \coprod T$ is exactly the opposite of the condition in the above Theorem. So it is reasonable to expect to be able to construct the category of strongly separable algebras as a category of coverings.

Take the opposite of the category $\mathbb{Y}$ in our assumptions above, $\mathbb{Y}^{op}$. Then $\mathbb{Y}^{op}/S = S/\mathbb{Y}$ since by definition of the opposite of a category, all maps in $\mathbb{Y}^{op}$ are reversed. So let $\mathbb{Y} =$ **cRing** the category of commutative rings. Then $\mathbb{Y}^{op}/S = S/\mathbb{Y}$ is the category of S-algebras. All of our operations have been dualized: pullback is now pushout, coproduct is now product, coequalizers are now equalizers, epic is now monic, etc. In particular, a covering over S in this sense is an S-algebra $S \longrightarrow X$ such that there exists a map $S \longrightarrow T \in \mathcal{C}$ with:

$$X \otimes_S T \cong \prod T$$

In order to invoke Theorem 3.4.1 we need to know that all maps of the form $S \longrightarrow T \in \mathcal{C}$ are faithfully flat S-algebras. But this follows from the assumptions listed on $\mathcal{C}$. Hence $\mathcal{A}$, the category of finite connected coverings, is now precisely the category of connected strongly separable S-algebras by Theorem 3.4.1. Then $Fam(\mathcal{A})$ is the category of strongly separable algebras, which was proven to be a Galois category by Barr in [3].

Chapter 4

Normal Objects

Introduction

In the last chapter, we saw the notion of a covering which was an object $X \longrightarrow S$ in $\mathbb{Y}/S$ with a so-called trivialization $T \longrightarrow S$ such that:

$$X \times_S T \cong \coprod T$$

In this chapter, we investigate coverings which are "self-trivializable". This condition is called *normal* and we have seen this before, in particular in Barr's characterization of a Galois category in the previous chapter. This chapter seeks to build a connection between more classical notions, like Kaplansky and descent, to normality.

In Section 4.1 we examine three functors, each defined by a particular limit. However here we will show how to abstractly incorporate a G-action on the objects in the image of each of the functors. We have seen abstract G-actions on objects in a category before, in particular in Example 1.2.12. In Section 4.2 we first write down the definitions of Kaplansky, descent, and normality using the functors defined in Section 4.1. Then, we show that these three properties are all equivalent under certain hypothesis on the functors. In particular, the notion of a functor being conservative (Definition 1.2.9) returns yet again as a hypothesis.

4.1 Limit Functors and G-Actions

For the remainder of this chapter, let $\mathbb{X}$ be a category with pushouts, products, and fixed point limits and let R be some object in $\mathbb{X}$. We will be working in the coslice category $R/\mathbb{X}$, where objects are maps $R \longrightarrow X$ for $X \in \mathbb{X}$. The reason for the change in perspective from slice to coslice categories is that we wish to focus on the category of R-algebras, **R-Alg**, as our motivating example and this category is exactly $R/\mathbf{cRing}$.

Now, fix an object $f : R \longrightarrow A \in R/\mathbb{X}$ and let G be the automorphism group of f in $R/\mathbb{X}$, that is, $G := Aut_R(A)$. This section is devoted entirely to the study of three limit functors E_A, H_A, and F_A which all depend on the chosen fixed object $f : R \longrightarrow A$ (hence the subscript notation). We have seen functors which are defined by limits before, for example the pullback functor f^* from Definition 1.3.6. However these functors will carry additional structure, namely the G-action.

The Pushout Functor

Recall that we can consider the group $G = Aut_R(A)$ as a category (Example 1.1.4). Then the category $Fun(G, R/\mathbb{X})$ is the category whose objects are functors between the categories G and $R/\mathbb{X}$ and whose maps are natural transformations between the functors (Proposition 1.2.3).

With this, define the pushout functor along $f : R \longrightarrow A$ with G-action, denoted E_A, as follows:

$$E_A : R/\mathbb{X} \longrightarrow R/\mathbb{X} \longrightarrow Fun(G, R/\mathbb{X})$$

$$\begin{tikzcd} R \arrow[d] & R \arrow[r, "f"] \arrow[d] & A \arrow[d] & \\ X \arrow[r, phantom, "\longmapsto" near start, shift left=4ex] & X \arrow[r] & X \otimes_R A =: X_A \arrow[r, phantom, "\longmapsto \rho_{X_A}", shift left=4ex] & {} \end{tikzcd}$$

The first part of the composition above sends an object $R \longrightarrow X \in R/\mathbb{X}$ to the pushout along f; the object $A \longrightarrow X_A$. Notice that $A \longrightarrow X_A$ is an object in $A/\mathbb{X}$, but by pre-composing with f it is also an object in $R/\mathbb{X}$. So we have the object $R \longrightarrow A \longrightarrow X_A$ which is now an object of $R/\mathbb{X}$ and we will denote it by X_A. The second part of the composition will then send X_A to a functor $\rho_{X_A} \in Fun(G, R/\mathbb{X})$. Recall from Example 1.2.12 the functor category $Fun(G, R/\mathbb{X})$ is the category of $R/\mathbb{X}$ objects X equipped with a G-action, where a G-action on X is regarded as a functor $\rho_X \in Fun(G, R/\mathbb{X})$. For the pushout X_A, the action ρ_{X_A} is defined by the universal property of the pushout X_A, given a $\sigma \in G$:

$$\rho_{X_A} : G \longrightarrow R/\mathbb{X}$$

$$G \longmapsto R \xrightarrow{f} A \longrightarrow X_A$$

$$\begin{tikzcd}
 & R \arrow[r, "f"] \arrow[d] & A \arrow[r, "\sigma"] \arrow[d] & A \arrow[dd, bend left] \\
\sigma : G \longrightarrow G \longmapsto & X \arrow[r] \arrow[drr, bend right] & X_A \arrow[dr, dashed, "\rho_{X_A}(\sigma)"] & \\
 & & & X_A
\end{tikzcd}$$

In order for such a map ρ_{X_A} to exist, we would need to check that the outer square in the above diagram commutes. Since $G = Aut_R(A)$, $f\sigma = f$ since f has domain R (σ fixes R). The outer square then commutes by the commutativity of the pushout square for X_A.

Next we need to describe how E_A acts on maps in $R/\mathbb{X}$. So suppose we have a map $\varphi \in R/\mathbb{X}$, shown below:

$$\begin{tikzcd}
 & R \arrow[dl, "f_X"'] \arrow[dr, "f_Y"] & \\
X \arrow[rr, "\varphi"] & & Y
\end{tikzcd}$$

For the first part of E_A, we need to construct a map between the pushouts of $f_X : R \longrightarrow X$ and $f_Y : R \longrightarrow Y$. Consider the pushout diagrams for X_A and Y_A respectively:

$$\begin{tikzcd}
R \arrow[r, "f"] \arrow[d, "f_X"'] & A \arrow[d, "\iota_A^X"] & R \arrow[r, "f"] \arrow[d, "f_Y"'] & A \arrow[d, "\iota_A^Y"] \\
X \arrow[r, "\iota_X"'] & X_A & Y \arrow[r, "\iota_Y"'] & Y_A
\end{tikzcd}$$

Where we have labeled all of the above maps X_A and Y_A for use later. We need to construct a map from X_A to Y_A. For this, we use the universal property of the pushout X_A along with the map φ in the following way:

$$\begin{tikzcd}
R \arrow[r, "f"] \arrow[d, "f_X"'] & A \arrow[d, "\iota_A^X"] \arrow[ddr, bend left, "\iota_A^Y"] & \\
X \arrow[r, "\iota_X"'] \arrow[d, "\varphi"'] & X_A \arrow[dr, dashed, "\eta"] & \\
Y \arrow[rr, bend right, "\iota_Y"'] & & Y_A
\end{tikzcd}$$

For the above map η to exist, we need to see that the outer square commutes. Consider the following:

$$\begin{aligned} f\iota_A^Y &= f_Y\iota_Y \text{ by the commutativity of the pushout square for } Y_A. \\ &= f_X\varphi\iota_Y \text{ by the commutativity of the triangle for } \varphi. \end{aligned}$$

Hence the outer square commutes, so η exists. Thus we will define $E_A(\varphi) := \eta$.

For the second part of E_A, we need to send η to a map between the functors ρ_{X_A} and ρ_{Y_A} i.e. a natural transformation $\rho_{X_A} \Longrightarrow \rho_{Y_A}$. We claim that η is in fact the desired natural transformation. To see this, note that our functors ρ_{X_A} and ρ_{Y_A} begin in a category with only one object, G. So for every $g \in G$ the naturality condition on η,

$$\begin{tikzcd}
\rho_{X_A}(G) \arrow[r, "\rho_{X_A}(g)"] \arrow[d, "\eta"'] & \rho_{X_A}(G) \arrow[d, "\eta"] \\
\rho_{Y_A}(G) \arrow[r, "\rho_{Y_A}(g)"'] & \rho_{Y_A}(G)
\end{tikzcd}
\quad \text{becomes} \quad
\begin{tikzcd}
X_A \arrow[r, "\rho_{X_A}(g)"] \arrow[d, "\eta"'] & X_A \arrow[d, "\eta"] \\
Y_A \arrow[r, "\rho_{Y_A}(g)"'] & Y_A
\end{tikzcd}$$

So defining a natural transformation $\rho_{X_A} \Longrightarrow \rho_{Y_A}$ amounts to defining a map $X_A \longrightarrow Y_A$ such that the above second diagram commutes for every $g \in G$. The map η defined earlier is certainly a map from X_A to Y_A so it remains to show that η is natural, that is, we must show $\rho_{X_A}(g)\eta = \eta\rho_{Y_A}(g)$ for all $g \in G$. To show this, consider the following new pushout diagram which in particular involves a map $g : A \longrightarrow A$:

$$\begin{tikzcd}
R \arrow[r, "f"] \arrow[d, "f_X"'] & A \arrow[r, "g"] \arrow[d, "\iota_A^X"] & A \arrow[dd, "\iota_A^Y"] \\
X \arrow[r, "\iota_X"'] \arrow[d, "\varphi"'] & X_A \arrow[dr, dashed, "?"] & \\
Y \arrow[rr, "\iota_Y"'] & & Y_A
\end{tikzcd}$$

where every map has been labeled according to our earlier labeling when we first introduced the pushout diagrams for X and Y. First we will show that the unique dashed map above exists by showing that the outer square commutes:

$$\begin{aligned}
f_X\varphi\iota_Y &= f_Y\iota_Y \text{ since } f_X\varphi = f_Y \text{ by the deifnition of } \varphi. \\
&= f\iota_A^Y \text{ by the commutativity of the pushout square for } Y_A. \\
&= fg\iota_A^Y \text{ since } g \in Aut_R(A), \text{ so } f \text{ is invariant under } g.
\end{aligned}$$

Hence there is a unique map that makes the above diagram commute. In fact, both $\eta\rho_{Y_A}(g)$ and $\rho_{X_A}(g)\eta$ satisfy this unique universal property. To show this, we need to show that the pushout diagram commutes for both of these maps. This follows from the definitions of $\rho_{X_A}(g)$, $\rho_{Y_A}(g)$, and η:

$$\iota_X(\eta\rho_{Y_A}(g)) = (\iota_X\eta)\rho_{Y_A}(g) = \varphi\iota_Y\rho_{Y_A}(g) = \varphi\iota_Y$$
$$\iota_A^X(\eta\rho_{Y_A}(g)) = (\iota_A^X\eta)\rho_{Y_A}(g) = \iota_A^Y\rho_{Y_A}(g) = g\iota_A^Y$$

and

$$\iota_X(\rho_{X_A}(g)\eta) = (\iota_X\rho_{X_A}(g))\eta = \iota_X\eta = \varphi\iota_Y$$
$$\iota_A^X(\rho_{X_A}(g)\eta) = (\iota_A^X\rho_{X_A}(g))\eta = g\iota_A^X\eta = g\iota_A^Y$$

Hence $\rho_{X_A}(g)\eta = \eta\rho_{Y_A}(g)$ so $\eta := E_A(\varphi)$ is natural.

Example 4.1.1. Let $\mathbb{X} = \textbf{cRing}$ the category of commutative rings, so $R/\mathbb{X} = \textbf{R-Alg}$ the category of R-algebras (i.e. the category of ring homomorphisms with domain R) and fix an object in **R-Alg**, say $f : R \longrightarrow A$. Then the functor E_A sends an R-algebra B to the tensor product, $E_A(B) = B \otimes_R A$ which is again an R-algebra.

But as we have seen, this functor does more: it equips the tensor product with a G-action, where $G = Aut_R(A)$. The G-action is a functor, $\rho_{B \otimes_R A}$ which for the purposes of this example we will simply denote ρ, which is defined as follows for $\sigma \in G$:

$$\begin{tikzcd}
R \arrow[r, "f"] \arrow[d] & A \arrow[r, "\sigma"] \arrow[d] & A \arrow[dd, bend left] \\
B \arrow[r] \arrow[drr, bend right] & B \otimes_R A \arrow[dr, dashed, "\rho(\sigma)"] & \\
& & B \otimes_R A
\end{tikzcd}$$

From the commutativity of the above diagram, we can see that G acts on the second coordinate of $B \otimes_R A$. Specifically,

$$\rho(\sigma)(b \otimes a) := b \otimes \sigma(a)$$

Now we can check if this is a G-action via the classical definition. Observe:

$$\rho(1_A)(b \otimes a) = b \otimes 1_A(a) = b \otimes a$$

$$\rho(\sigma\tau)(b \otimes a) = b \otimes (\sigma\tau)(a) = b \otimes (\tau(\sigma(a)) = \rho(\tau)(\rho(\sigma)(b \otimes a))$$

Remark. An R-algebra $\varphi : R \longrightarrow X$ can be seen as an R-module, where the action of R on X is given as:

$$r \cdot x := \varphi(r)x$$

Now consider the functor ρ where $\rho(G) = X \otimes_R A$. The tensor product $X \otimes_R A$ can be seen as both an A-algebra (hence an A-module) and an R-algebra (hence an R-module). In our definition of the functor E_A, we have chosen to consider $X \otimes_R A$ as an R-algebra, with precomposition by $f : R \longrightarrow A$. This was done because the map $\rho(\sigma)$ is, in general, not A-linear. Consider:

$$\rho(\sigma) : X \otimes_R A \longrightarrow X \otimes_R A$$

$$\begin{aligned}\rho(\sigma)(a' \cdot (x \otimes a)) &= \rho(\sigma)(x \otimes a'a) \\ &= x \otimes \sigma(a'a) = x \otimes \sigma(a')\sigma(a) \\ &= \sigma(a') \cdot (x \otimes \sigma(a)) = \sigma(a') \cdot \rho(\sigma)(x \otimes a)\end{aligned}$$

Where $\sigma(a') = a'$ only if $a' \in R$. Hence $\rho(\sigma)$ is R-linear, but not A-linear in general.

The Product Functor

Next define the product functor with G-action, denoted H_A, as follows:

$$H_A : R/\mathbb{X} \longrightarrow R/\mathbb{X} \longrightarrow Fun(G, R/\mathbb{X})$$

$$\begin{array}{ccccc} R & & R & & \\ \downarrow & \longmapsto & \downarrow & \longmapsto & \rho_{\mathcal{A}} \\ X & & \prod_{\sigma \in G} X =: \mathcal{A} & & \end{array}$$

The first part of the composition sends an object $R \longrightarrow X \in R/\mathbb{X}$ to the product of X with itself $|G|$-many times, which is still in $R/\mathbb{X}$ via the map $R \longrightarrow X \longrightarrow \prod_{\sigma \in G} X$. For convenience, denote $\prod_{\sigma \in G} X$ by $\mathcal{A}$. The second part of the composition will send this product to a functor $\rho_{\mathcal{A}} \in Fun(G, R/\mathbb{X})$, i.e. equip $\mathcal{A}$ with an action by G. The action, $\rho_{\mathcal{A}} : G \longrightarrow R/\mathbb{X}$, is the functor defined as follows:

- Objects: $\rho_{\mathcal{A}}(G) = R \longrightarrow \mathcal{A}$.
- Maps: Recall that the object $\mathcal{A}$ comes equipped with projection maps, one for

each X in the product, which we will denote by π_σ for $\sigma \in G$.

$$\begin{tikzcd}
 & X_e \\
\mathcal{A} \arrow[ur, "\pi_e"] \arrow[dr, "\pi_\sigma"'] & \\
 & \vdots \; X_\sigma
\end{tikzcd}$$

where each $X_\sigma = X$ and $e \in G$ denotes the identity element of the group. Now to define $\rho_{\mathcal{A}}(g)$ for $g \in G$, we will use the universal property of products by "twisting" the projection maps by g. The diagram below illustrates this twisting:

$$\begin{tikzcd}
 & & X_e \\
\mathcal{A} \arrow[r, dashed, "\rho_{\mathcal{A}}(g)"] \arrow[urr, bend left, "\pi_{eg}"] \arrow[drr, bend right, "\pi_{\sigma g}"'] & \mathcal{A} \arrow[ur, "\pi_e"] \arrow[dr, "\pi_\sigma"'] & \\
 & & X_\sigma
\end{tikzcd}$$

Next we need to see how H_A acts on maps $\varphi \in R/\mathbb{X}$. Suppose we have a map $\varphi \in R/\mathbb{X}$:

$$\begin{tikzcd}
 & R \arrow[dl] \arrow[dr] & \\
X \arrow[rr, "\varphi"] & & Y
\end{tikzcd}$$

We will denote $\prod X =: \mathcal{A}$ and $\prod Y =: \mathcal{B}$. The first part of H_A will send φ to $\prod \varphi : \mathcal{A} \longrightarrow \mathcal{B}$, where $\prod \varphi$ is defined using the universal property of the product $\mathcal{B}$. That is $\prod \varphi$ is the unqiue map such that the following diagram commutes:

$$\begin{tikzcd}
 & & Y \\
\mathcal{A} \arrow[r, dashed, "\prod \varphi"] \arrow[urr, bend left, "\pi_e \varphi"] \arrow[drr, bend right, "\pi_\sigma \varphi"'] & \mathcal{B} \arrow[ur, "\pi_e"] \arrow[dr, "\pi_\sigma"'] & \\
 & & Y
\end{tikzcd}$$

For the second part of H_A, we will need a natural transformation:

$$\eta : \rho_{\mathcal{A}} \Longrightarrow \rho_{\mathcal{B}}$$

But, as we did in the construction of E_A, we see that the naturality condition shows us that η is simply a map between $\mathcal{A}$ and $\mathcal{B}$ such that the following diagram commutes for each $g \in G$:

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\rho_{\mathcal{A}}(g)"] \arrow[d, "\eta"'] & \mathcal{A} \arrow[d, "\eta"] \\
\mathcal{B} \arrow[r, "\rho_{\mathcal{B}}(g)"'] & \mathcal{B}
\end{tikzcd}$$

There is only one natural choice for η and that is $\prod \varphi$ from above. To see that $\big(\prod \varphi\big)\rho_{\mathcal{B}}(g) = \rho_{\mathcal{A}}(g)\big(\prod \varphi\big)$ one can check that both maps satisfy the same universal property of the product $\mathcal{B}$ shown below:

$$\begin{tikzcd}
 & & Y \\
\mathcal{A} \arrow[r, dashed] \arrow[urr, bend left, "\pi_{eg}\varphi"] \arrow[drr, bend right, "\pi_{\sigma g}\varphi"'] & \mathcal{B} \arrow[ur, "\pi_e"] \arrow[dr, "\pi_\sigma"'] & \\
 & & Y
\end{tikzcd}$$

Example 4.1.2. Let $\mathbb{X} = \mathbf{cRing}$, fix any set $R \in \mathbb{X}$, and consider the object $R \longrightarrow X$ in $R/\mathbf{cRing} = \mathbf{R\text{-}Alg}$. Consider an element $(x_e, \ldots, x_\sigma, \ldots) \in \prod\limits_{\sigma \in G} X$. Notice that in this notation $x_e \in X_e$, $x_\sigma \in X_\sigma$, etc. but remember that $X_\sigma = X$ for all $\sigma \in G$. Let us see what the group action is on $(x_e, \ldots, x_\sigma, \ldots)$ in this case. Recall the diagram which defines the action on $\prod\limits_{\sigma \in G} X =: \mathcal{A}$:

$$\begin{tikzcd}
 & & X_e \\
\mathcal{A} \arrow[r, dashed, "\rho_{\mathcal{A}}(g)"] \arrow[urr, bend left, "\pi_{eg}"] \arrow[drr, bend right, "\pi_{\sigma g}"'] & \mathcal{A} \arrow[ur, "\pi_e"] \arrow[dr, "\pi_\sigma"'] & \\
 & & X_\sigma
\end{tikzcd}$$

Consider a single element $x_{\sigma g}$ of the n-tuple $(x_e, \ldots, x_\sigma, \ldots)$. By the commutativity of the above diagram, we have:

$$x_{\sigma g} = \pi_{\sigma g}(x_e, \ldots, x_\sigma, \ldots) = \pi_\sigma(\rho_{\mathcal{A}}(g)(x_e, \ldots, x_\sigma, \ldots))$$

Notice that this equality implies that in the σ-slot of $\rho_{\mathcal{A}}(g)(x_e, \ldots, x_\sigma, \ldots)$ we must have the element $x_{\sigma g}$. Since σ was arbitrary, we could construct the same equality for any element of $(x_e, \ldots, x_\sigma, \ldots)$. Hence,

$$\rho_{\mathcal{A}}(g)(x_e, \ldots, x_\sigma, \ldots) = (x_{eg}, \ldots, x_{\sigma g}, \ldots).$$

That is, $\rho_{\mathcal{A}}(g)$ is a permutation of the n-tuple by g.

The Fixed Point Functor

Finally we will define the functor F_A which will be the fixed point functor, where we equip the fixed point object with a G-action. Given a functor $\rho_X \in Fun(G, R/\mathbb{X})$, F_A takes the functor to the fixed point object of its image, i.e.

$$F_A : Fun(G, R/\mathbb{X}) \longrightarrow R/\mathbb{X}$$

$$\rho_X \longmapsto \begin{array}{c} R \\ \downarrow \\ X^G \end{array}$$

Recall X^G is the equalizer that is the "dual" version of the quotient object (see Example 1.3.5), which we saw much of in Chapter 2. In our case, starting with the functor ρ_X we construct the fixed point object X^G of the maps $\rho_X(\sigma) : X \longrightarrow X$, which comes equipped with a map $f_X : X^G \longrightarrow X$ such that given any other map $\varphi : Y \longrightarrow X$ with $\varphi\rho_X(\sigma) = \varphi$ there exists a unique map α such that the following diagram commutes:

$$\begin{tikzcd} Y \arrow[d, dashed, "\alpha"'] \arrow[dr, "\varphi"] & & \\ X^G \arrow[r, "f_X"'] & X \arrow[r, shift left, "\rho_X(\sigma)"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right] & X \end{tikzcd}$$

We need to make sure that such a map $R \longrightarrow X^G$ exists, so that the codomain of F_A is really $R/\mathbb{X}$. By the universal property of X^G we have:

$$\begin{tikzcd} R \arrow[d, dashed] \arrow[dr, "\varphi"] & & \\ X^G \arrow[r, "f_X"'] & X \arrow[r, shift left, "\rho_X(\sigma)"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right] & X \end{tikzcd}$$

where $\rho_X(\sigma)(\varphi(r)) = \varphi(r)$ since R is fixed ($\rho_X(\sigma)$ is a map in $R/\mathbb{X}$). Hence we have $R \longrightarrow X^G$.

Remark. Notice that unlike the first two functors E_A and H_A, the domain of F_A is the functor category $Fun(G, R/\mathbb{X})$. However, as we have seen before, these functors are really representing objects in $R/\mathbb{X}$ with a G-action, say X. So we will often denote $F_A(\rho_X)$ as simply $F_A(X)$ where we mean the object X, equipped with an action by G.

Now suppose we have a map in $Fun(G, R/\mathbb{X})$, that is, a natural transformation $\eta : \rho_X \Longrightarrow \rho_Y$. We want to define $F_A(\eta)$. Notice that by the nature of $Fun(G, R/\mathbb{X})$, η is just a map $X \longrightarrow Y$ with the naturality condition (for all $\sigma \in G$):

$$\begin{tikzcd}
X \arrow[r, "\rho_X(\sigma)"] \arrow[d, "\eta"'] & X \arrow[d, "\eta"] \\
Y \arrow[r, "\rho_Y(\sigma)"'] & Y
\end{tikzcd}$$

Using the naturality condition above and the universal property of the fixed point object of Y, (Y^G, f_Y), we can find a map $\alpha : X^G \longrightarrow Y^G$:

$$\begin{tikzcd}
X^G \arrow[dr, "f_X"] \arrow[dd, dashed, "\alpha"'] & & & \\
& X \arrow[dr, "\eta"] & & \\
Y^G \arrow[rr, "f_Y"'] & & Y \arrow[r, shift left=1.5, "\rho_Y(\sigma)"] \arrow[r, shift right=1.5] \arrow[r, phantom, "\vdots"] & Y
\end{tikzcd}$$

In order to show that $\alpha : X^G \longrightarrow Y^G$ exists, we need to check that $f_X\eta\rho_Y(\sigma) = f_X\eta\rho_Y(\sigma)$ for all $\sigma \in G$. However this follows from the naturality of η and the definition of f_X:

$$f_X\eta\rho_Y(\sigma) = f_X\rho_X(\sigma)\eta = f_X\eta$$

So, define $F_A(\eta) := \alpha$. So F_A sends η to the map defined above via the universal property of Y^G.

Example 4.1.3. Once again, consider the category of R-algebras **R-Alg** and let $\rho_X \in \mathrm{Fun}(G, \mathbf{R\text{-}Alg})$. That is, we have an R-algebra $\varphi : R \longrightarrow X$ equipped with a G-action, ρ_X. The R-algebra X^G is then defined as follows:

$$X^G := \{x \in X \mid \rho_X(\sigma)(x) = x \text{ for all } \sigma \in G.\}$$

where $X^G \in \mathbf{cRing}$ since $\rho_X(\sigma) : X \longrightarrow X$ is a map in **cRing**.

4.2 Descends, Kaplansky, and Normal

We begin with an important Lemma.

Lemma 4.2.1. $H_A F_A \cong \mathbb{1}_{R/\mathbb{X}}$.

Proof. First we consider the *diagonal map* of an object X, denoted Δ_X, which is the unique map given by the universal property of the product:

$$\begin{tikzcd} & & X \\ X \arrow[urr, bend left, "1_X"] \arrow[drr, bend right, "1_X"'] \arrow[r, dashed, "\Delta_X"] & \prod_{\sigma\in G} X \arrow[ur, "\pi_e"] \arrow[dr, "\pi_\sigma"] & \\ & & X \end{tikzcd}$$

In order to show that $H_A F_A \cong \mathbb{1}_{R/\mathbb{X}}$ we need to find a natural isomorphism $\eta : H_A F_A \Longrightarrow \mathbb{1}_{R/\mathbb{X}}$. That is, for every map $\varphi : X \longrightarrow Y$ we need maps η_X, η_Y such that the following diagram commutes:

$$\begin{tikzcd} F_A(H_A(X)) \arrow[r, "F_A(H_A(\varphi))"] \arrow[d, "\eta_X"'] & F_A(H_A(Y)) \arrow[d, "\eta_Y"] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

and that η_X and η_Y are isomorphisms. By definition of the functors F_A and H_A:

$$F_A(H_A(X)) = F_A(\prod_{\sigma\in G} X) = \big(\prod_{\sigma\in G} X\big)^G$$

For convenience, denote $\mathcal{A} := \prod_{\sigma \in G} X$ and $\mathcal{B} := \prod_{\sigma \in G} Y$. Consider the following fixed point limit diagram of $\mathcal{A}$:

$$\begin{tikzcd}
X \arrow[d, dashed, "\alpha"'] \arrow[dr, "\Delta_X"] & & \\
\mathcal{A}^G \arrow[r, "f_{\mathcal{A}}"'] & \mathcal{A} \arrow[r, shift left=2, "\rho_{\mathcal{A}}(\sigma)"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & \mathcal{A}
\end{tikzcd}$$

Where $\Delta_X \rho_{\mathcal{A}}(\sigma) = \Delta_X$ since both maps satisfy the same product universal property. That is, the following diagram commutes:

$$\begin{tikzcd}
& & & X \\
X \arrow[urrr, bend left, "1_X"] \arrow[drrr, bend right, "1_X"'] \arrow[r, "\Delta_X"] & \mathcal{A} \arrow[urr, bend left, "\pi_{eg}"] \arrow[drr, bend right, "\pi_{\sigma g}"'] \arrow[r, "\rho_{\mathcal{A}}(g)"] & \mathcal{A} \arrow[ur, "\pi_e"] \arrow[dr, "\pi_\sigma"'] & \\
& & & X
\end{tikzcd}$$

So we know that the map α exists. We want to show that α is an isomorphism, so we need to construct an inverse.

First, we claim that $f_{\mathcal{A}} \pi_\sigma = f_{\mathcal{A}} \pi_g$ for all $\sigma, g \in G$. To see this, notice that the following diagram commutes by definition of $f_{\mathcal{A}}$ and $\rho_{\mathcal{A}}(g)$:

$$\begin{tikzcd}
& \mathcal{A} \arrow[dd, "\rho_{\mathcal{A}}(g)"] \arrow[dr, "\pi_{\sigma g}"] & \\
\mathcal{A}^G \arrow[ur, "f_{\mathcal{A}}"] \arrow[dr, "f_{\mathcal{A}}"'] & & X \\
& \mathcal{A} \arrow[ur, "\pi_\sigma"'] &
\end{tikzcd}$$

So we get $f_{\mathcal{A}} \pi_{\sigma g} = f_{\mathcal{A}} \rho_{\mathcal{A}}(g) \pi_\sigma = f_{\mathcal{A}} \pi_\sigma$. Since this diagram commutes for all $\sigma, g \in G$, take $g = \sigma^{-1} g$ and we get $f_{\mathcal{A}} \pi_\sigma = f_{\mathcal{A}} \pi_g$. So for the rest of the proof, we will denote the projection map simply by π when precomposing with $f_{\mathcal{A}}$.

We will now show that $f_{\mathcal{A}} \pi$ is the inverse of α. One direction is not difficult: $\alpha f_{\mathcal{A}} \pi = \Delta_X \pi = 1_X$. For the other direction we need to show that $f_{\mathcal{A}} \pi \alpha = 1_{\mathcal{A}^G}$. Consider the following diagram for the fixed point object $\mathcal{A}^G$:

$$\begin{tikzcd}
\mathcal{A}^G \arrow[d, dashed, "\hat{f}_{\mathcal{A}}"'] \arrow[dr, "f_{\mathcal{A}}"] & & \\
\mathcal{A}^G \arrow[r, "f_{\mathcal{A}}"'] & \mathcal{A} \arrow[r, shift left, "\rho_{\mathcal{A}}(g)"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right] & \mathcal{A}
\end{tikzcd}$$

By definition of $\mathcal{A}^G$ the lift $\hat{f}_{\mathcal{A}}$ must exist and by observation we see that $\hat{f}_{\mathcal{A}} = 1_{\mathcal{A}^G}$. So now we must show that the map $f_{\mathcal{A}}\pi\alpha$ also satisfies the above diagram. That is, we need to show that $(f_{\mathcal{A}}\pi\alpha)f_{\mathcal{A}} = f_{\mathcal{A}}$.

By definition of α, $(f_{\mathcal{A}}\pi\alpha)f_{\mathcal{A}} = f_{\mathcal{A}}\pi(\alpha f_{\mathcal{A}}) = f_{\mathcal{A}}\pi\Delta_X$. So it remains to show that $f_{\mathcal{A}}\pi\Delta_X = f_{\mathcal{A}}$. To see this, notice that by the universal property of the product $\mathcal{A}$ there must exist a unique map in the following diagram:

$$\begin{tikzcd}
 & & X \\
\mathcal{A}^G \arrow[r, dashed] \arrow[urr, bend left, "f_{\mathcal{A}}\pi_e"] \arrow[drr, bend right, "f_{\mathcal{A}}\pi_\sigma"'] & \mathcal{A} \arrow[ur, "\pi_e"] \arrow[dr, "\pi_\sigma"'] & \\
 & & X
\end{tikzcd}$$

By observation, the unique map must be $f_{\mathcal{A}}$. However we also get:

$$(f_{\mathcal{A}}\pi\Delta_X)\pi_\sigma = f_{\mathcal{A}}\pi(\Delta_X\pi_\sigma) = f_{\mathcal{A}}\pi 1_X = f_{\mathcal{A}}\pi = f_{\mathcal{A}}\pi_\sigma$$

for all $\sigma \in G$. Hence $f_{\mathcal{A}}\pi\Delta_X$ also satisfies this diagram, so $f_{\mathcal{A}}\pi\Delta_X = f_{\mathcal{A}}$ and it follows that α is an isomorphism.

Hence we have shown that η_X is an isomorphism for all X. It remains to show the naturality condition. That is for any $\varphi : X \longrightarrow Y \in R/\mathbb{X}$, we must show that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A}^G \arrow[r, "F_A(H_A(\varphi))"] \arrow[d, "f_{\mathcal{A}}\pi"', "\cong"] & \mathcal{B}^G \arrow[d, "\cong"', "f_{\mathcal{B}}\pi"] \\
X \arrow[r, "\varphi"'] & Y
\end{tikzcd}$$

For this, we need to recall the definitions of H_A and F_A on maps. $H_A(\varphi) = \prod \varphi$ and $F_A(\prod \varphi)$ is defined by the universal property of the fixed point object $\mathcal{B}^G$:

$$
\begin{tikzcd}
\mathcal{A}^G \arrow[dr, "f_{\mathcal{A}}"] \arrow[dd, "F_A(\prod\varphi)"'] & & & \\
& \mathcal{A} \arrow[dr, "\prod\varphi"] & & \\
\mathcal{B}^G \arrow[rr, "f_{\mathcal{B}}"] & & \mathcal{B} \arrow[r, shift left=2, "\rho_{\mathcal{B}}(\sigma)"] \arrow[r, phantom, "\vdots"] \arrow[r, shift right=2] & \mathcal{B}
\end{tikzcd}
$$

Then,

$$F_A\Big(\prod \varphi\Big)(f_{\mathcal{B}}\pi) = (F_A\Big(\prod \varphi\Big)f_{\mathcal{B}})\pi = (f_{\mathcal{A}}\prod \varphi)\pi = f_{\mathcal{A}}(\prod \varphi\pi) = f_{\mathcal{A}}(\pi\varphi) = (f_{\mathcal{A}}\pi)\varphi$$

as required. $\square$

We now wish to introduce some new and some familiar definitions, but in the context of these functors.

Definition 4.2.1. Suppose we have $f : R \longrightarrow A$ denoted by A and let $G = Aut_R(A)$. Then,

- A is *Kaplansky* if $A^G \cong R$.
- A *descends* if $E_A(F_A(A)) \cong A$.
- A is *normal* if $E_A(A) \cong H_A(A)$.

Example 4.2.1. Let L/K be a (possibly infinite) field extension. In other words, L/K is an injective map $f : K \longrightarrow L$. So the category of field extensions can be thought of as the coslice category $K/\mathbb{X}$ where $\mathbb{X}$ is the category of fields. $f : K \longrightarrow L$ is a Galois extension if the extension is algebraic and and $L^{Aut_K(L)} = K$. The second condition is exactly the Kaplansky condition mentioned above.

Example 4.2.2. Consider again the case where A is an R-algebra. Then A descends if we have the following:

$$A^G \otimes_R A \cong A$$

where each is considered as an R-algebra with a G-action. This exactly matches the classical notion of *Galois descent*, see for instance section 2 in [8] or Chapter 17 in [22].

Example 4.2.3. Let L/K be a Galois extension of fields. Then, by [8] Section 5, we have:

$$L \otimes_K L \cong \prod_{\sigma \in G} L$$

which in our notation translates to:

$$E_L(L) \cong H_L(L)$$

Hence if L/K is Galois (in particular, Kaplansky) then L/K is normal.

In the above example, we have seen a relationship between Kaplansky and normal in the case of fields. That is, if L/K is Kaplansky then L/K is normal. In fact, we shall show that the three notions in Definition 4.2.1 are equivalent, under a special hypothesis. We must start with the assumption that E_A is conservative (E_A reflects isomorphisms). We have seen this condition before: in Chapter 2 the geometric point $s \longrightarrow S$ was assumed to be conservative and in Chapter 3 our class $\mathcal{C}$ contained, amoung other conditions, maps such that the pullback functor was conservative.

Theorem 4.2.1. *Let $\mathbb{X}$ be a category with pushouts, products, and fixed points and let $f : R \longrightarrow A$ (denoted A) be an object in the category $R/\mathbb{X}$. Suppose that E_A is conservative and that the functor E_A preserves fixed points. Then the following are equivalent:*

(i) A is Kaplansky.

(ii) A descends.

(iii) A is normal.

Proof. $(i) \Rightarrow (ii)$: Suppose A is Kaplansky. Then $A^G \cong R$, that is, R is the fixed point object of A (by R we mean the object $1_R : R \longrightarrow R$). We want to show that A descends, i.e. that $(A^G)_A \cong A$. But since A is Kaplansky we have:

$$(A^G)_A \cong (R)_A = E_A(R)$$

where $E_A(R)$ is the pushout of $f : R \longrightarrow A$ and $1_R : R \longrightarrow R$ which is just A, i.e.

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "1_R"'] & A \arrow[d, "1_A"] \\ R \arrow[r, "f"'] & A \end{tikzcd}$$

Thus $(A^G)_A \cong A$ so A descends.

(ii) $\Rightarrow$ *(iii)*: Suppose A descends. Then $E_A(F_A(A)) \cong A$. We want to show that A is normal, i.e. $H_A(A) \cong E_A(A)$. Applying the functor $F_A E_A$ to $H_A(A)$ we get the following:

$$H_A(A) \cong E_A(F_A(H_A(A))) \cong E_A(A)$$

where $F_A(H_A(A)) \cong A$ by Lemma 4.2.1. Hence we have $H_A(A) \cong E_A(A)$ so A is normal.

(iii) $\Rightarrow$ *(i)*: Since A is normal we have that $A_A = E_A(A) \cong H_A(A) = \prod_{g\in G} A$. Now since F_A preserves isomorphisms, we can apply F_A to the above isomorphism to get:

$$F_A(E_A(A)) \cong F_A(H_A(A)) \cong A$$

Where $F_A(H_A(A)) \cong A$ by Lemma 4.2.1. On the other hand, by the definitions of F_A and E_A we have:

$$F_A(E_A(A)) = F_A(A_A) = (A_A)^G$$

Putting these together, we get:

$$A \cong (A_A)^G$$

But now we invoke the additional hypothesis made on the functor E_A - that it preserves fixed points. So in particular, we have:

$$(A^G)_A = E_A(A^G) \cong (E_A(A))^G = (A_A)^G$$

Thus $A \cong (A^G)_A$. But by definition of the functor E_A, this is the same as $E_A(R) \cong E_A(A^G)$. But since E_A reflects isomorphisms, $R \cong A^G$. Hence A is Kaplansky.

$\square$

Concluding Remarks.

Theorem 4.2.1 has shown us, under certain hypothesis for the functor E_A, that the notions of Kaplansky, Descends, and Normal are all equivalent. Thus we see the connection with the notion of normal, which we saw from Barr in Theorem 3.1.1 and Kaplansky, which directly relates to classical Galois theory. Recall the notion of coverings from Chapter 3: objects $X \longrightarrow S$ such that there exists a trivialization $T \longrightarrow S \in \mathcal{C}$ for which we had:

$$X \times_S T \cong \coprod T$$

If for instance $X \longrightarrow S$ itself was a trivialization for $X \longrightarrow S$ then we would have:

$$X \times_S X \cong \coprod X$$

which is precisely the dual of normal as defined earlier in this Chapter. However the entirety of this chapter was designed to be dualizable - Definition 4.2.1 refers only to the three functors E_A, H_A, F_A which are all defined via limits, hence the dual versions would be defined with the corresponding colimits. Thus we could just as easily call the above normal and the connection to our covering maps from Chapter 3 is now clear.

Another connection to Chapter 3 is in the required hypothesis on the functor E_A. In this chapter, E_A was the pushout functor of $f : R \longrightarrow A$, but the dual would be the pullback of $\varphi : A \longrightarrow R$ which then would just be φ^*. Then, Theorem 4.2.1 would require φ^* to be conservative and preserve quotients, a type of coequalizer. This is remarkably similar to the requirements we had on the class of maps $\mathcal{C}$ and also on the effective descent maps mentioned at the end of Chapter 3.

Although we see that there is a connection, we recognize that these are only observations and not results. One further research topic is to investigate effective descent maps. As was mentioned in Chapter 3, there has been recent work on these class of maps and effective descent is very closely related to the ideas we have seen throughout this thesis. Another avenue is to investigate Janelidze's "Galois structures" (see for instance section 5 in [14]). We would like to know how the category $\mathcal{A}$ defined in Chapter 3 relates to these Galois structures. In general, further research into Janelidze's abstract Galois theory is required.

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