UNIVERSITY OF CALGARY

Ball-Polyhedra, Spindle-Convexity and Their Properties

by

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Abstract

Traditional results in convexity are extended to a new type of convexity called spindle convexity. In this context line segments are replaced by spindles. A spindle joining two points is the union of all circular arcs where the radii of the circles are larger than a fixed radius but the arcs are at most a semicircle. Standard results in convexity are examined in this new context. This topic leads to natural extensions of polyhedra called ball-polyhedra, where the region bounded by a family of planes is replaced by the region bounded by a family of spheres. The topics are examined in euclidean, hyperbolic and spherical spaces to provide full generality.

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Dedication

For my mother, my daughter and my wife.

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Chapter 1

Introduction

This thesis presents a variety of ideas and results that are based on the concepts traditionally covered in convexity, but we examine them here in a new context. This context is a generalization of convexity called spindle convexity. We have further broadened the discussion by looking at these results not just in euclidean space, but in hyperbolic space and spherical space as well. We introduce the concept of ball-polyhedra and carefully examine the two-dimensional case.

1.1 Thesis Overview

Traditional results in convexity are extended to a new type of convexity called spindle convexity. In this context line segments are replaced by spindles. A spindle joining two points is the union of all circular arcs where the radii of the circles are larger than a fixed radius but the arcs are at most a semicircle. Standard results in convexity are examined in this new context. This topic leads to natural extensions of polyhedra called ball-polyhedra, where the region bounded by a family of planes is replaced by the region bounded by a family of spheres. The topics are examined in euclidean, hyperbolic and spherical spaces to provide full generality.

1.2 Thesis Layout

Chapter 2 lays the ground work by providing definitions, notation and preliminary results. In Chapter 3 we begin to explore some of the critical issues. The key results deal with the monotonicity of the perimeter of certain spindle polygons. Many of the results broadly generalize to the three spaces. We conclude our study in Chapter 4 where we restrict our attention in some cases to a particular space. The chapter concludes with results demonstrating the, somewhat surprising, failure of certain Helly type theorems in hyperbolic space.

Chapter 2

Spindle-Convexity and Ball-Polyhedra: Notations and Basic Facts

2.1 Introduction

The study of convexity has a rich and deep history in mathematics. Various attempts at generalizing the notion of convexity or developing an analogous concept have been made. In [43], Mayer examines one such generalization of convexity called Überkonvexität and Soltan, in [48], attempts an axiomatic study of convexity. In this section we examine the idea of spindle-convexity which is similar to Mayer's Überkonvexität, but does not fall within Soltan's discussion. The definition we use is broader than Mayer's as we attempt to generalize across euclidean, hyperbolic and spherical geometries. This chapter is a meticulous development of basic facts and notations related to spindle-convexity and ball-polyhedra in the three spaces $\mathbb{R}^n, \mathbb{H}^n$, and $\mathbb{S}^n$. We conclude this chapter with an explanation of the condensed notation that is used throughout the remainder of this thesis.

2.2 Preliminary Notation

We begin with a development of some preliminary notation. At the end of this chapter we condense the notation considerably. However, in certain situations it is necessary and advantageous to have this robust notation. For example, there are several occasions when we utilize the fact that hyperbolic and spherical spaces can be embedded in a euclidean space. Once embedded in a euclidean space, objects in hyperbolic and spherical space have euclidean counterparts. In these situations it is necessary to identify the ambient space, dimension and other important characteristics.

Let $\mathbb{R}^n$, $n \geq 2$, denote an n-dimensional euclidean space. We denote the origin by $o_{\mathbb{R}^n}$. The euclidean distance between $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ is $d^n_{\mathbb{R}}(a,b)$. The closed line segment between two points is denoted by $[a,b]^n_{\mathbb{R}}$, the open line segment is denoted by $(a,b)^n_{\mathbb{R}}$. For the closed, n-dimensional ball with center $a \in \mathbb{R}^n$ and of radius $r > 0$, we use the notation $\mathbf{B}^n_{\mathbb{R}}[a,r] = \{x \in \mathbb{R}^n : d^n_{\mathbb{R}}(a,x) \leq r\}$. For the open n-dimensional ball with center $a \in \mathbb{R}^n$ and of radius $r > 0$, we use the notation $\mathbf{B}^n_{\mathbb{R}}(a,r) = \{x \in \mathbb{R}^n : d^n_{\mathbb{R}}(a,x) < r\}$. The $(n-1)$-dimensional sphere with center $a \in \mathbb{R}^n$ and of radius $r > 0$ is denoted by $\mathbb{S}^{n-1}_{\mathbb{R}}(a,r) = \{x \in \mathbb{R}^n : d^n_{\mathbb{R}}(a,x) = r\}$.

Let $\mathbb{H}^n$, $n \geq 2$, denote an n-dimensional hyperbolic space. Unless otherwise specified, we use the Poincaré Ball Model of hyperbolic space. In this model, $\mathbb{H}^n$ is embedded in $\mathbb{R}^n$ as a ball, where the ball, which is called the Poincaré Ball, has radius one. We denote the center of the ball by $o_{\mathbb{H}^n}$ and assume that it coincides with $o_{\mathbb{R}^n}$. The hyperbolic distance between $a \in \mathbb{H}^n$ and $b \in \mathbb{H}^n$ is $d^n_{\mathbb{H}}(a,b)$. The closed line segment between two points is denoted by $[a,b]^n_{\mathbb{H}}$, the open line segment is denoted by $(a,b)^n_{\mathbb{H}}$. For the closed, n-dimensional ball with center $a \in \mathbb{H}^n$ and of radius $r > 0$, we use the notation $\mathbf{B}^n_{\mathbb{H}}[a,r] = \{x \in \mathbb{H}^n : d^n_{\mathbb{H}}(a,x) \leq r\}$. For the open n-dimensional ball with center $a \in \mathbb{H}^n$ and of radius $r > 0$, we use the notation $\mathbf{B}^n_{\mathbb{H}}(a,r) = \{x \in \mathbb{H}^n : d^n_{\mathbb{H}}(a,x) < r\}$. The $(n-1)$-dimensional sphere with center $a \in \mathbb{H}^n$ and of radius $r > 0$ is denoted by $\mathbb{S}^{n-1}_{\mathbb{H}}(a,r) = \{x \in \mathbb{H}^n : d^n_{\mathbb{H}}(a,x) = r\}$.

Let $\mathbb{S}^n$, $n \geq 2$, denote an n-dimensional spherical space. Unless otherwise specified, we use the Spherical Model of spherical space. In this model, $\mathbb{S}^n$ is embedded in $\mathbb{R}^{n+1}$ as a sphere, where the sphere has radius one. We assume the center of the sphere is $o_{\mathbb{R}^{n+1}}$ and let $o_{\mathbb{S}^n}$ be some fixed point on the sphere which we call the *origin* of $\mathbb{S}^n$. The spherical distance between $a \in \mathbb{S}^n$ and $b \in \mathbb{S}^n$ is $d^n_{\mathbb{S}}(a,b)$. The closed line segment between two points is denoted by $[a,b]^n_{\mathbb{S}}$, the open line segment is denoted by $(a,b)^n_{\mathbb{S}}$. For the closed, n-dimensional ball with center $a \in \mathbb{S}^n$ and of radius $r > 0$, we use the notation $\mathbf{B}^n_{\mathbb{S}}[a,r] = \{x \in \mathbb{S}^n : d^n_{\mathbb{S}}(a,x) \leq r\}$. For the open n-dimensional ball with center $a \in \mathbb{S}^n$ and of radius $r > 0$, we use the notation $\mathbf{B}^n_{\mathbb{S}}(a,r) = \{x \in \mathbb{S}^n : d^n_{\mathbb{S}}(a,x) < r\}$. The $(n-1)$-dimensional sphere with center $a \in \mathbb{S}^n$ and of radius $r > 0$ is denoted by $\mathbb{S}^{n-1}_{\mathbb{S}}(a,r) = \{x \in \mathbb{S}^n : d^n_{\mathbb{S}}(a,x) = r\}$.

Any sphere or ball in this thesis is of positive, possibly infinite, radius r. We use the notation $(0,\infty]$ to be the collection of all positive real numbers and the symbol ∞. Furthermore, we make the convention that planes in euclidean and hyperbolic space are spheres where $r = \infty$. In euclidean and hyperbolic space, $r \in (0,\infty]$. In spherical space, $r \in (0,\pi/2]$ and lines in spherical space are spheres with $r = \pi/2$. Finally, we note that a 0-dimensional sphere is a pair of distinct points.

We conclude this section by introducing an operation on sets in the spaces $\mathbb{R}^n$, $\mathbb{H}^n$ and $\mathbb{S}^n$. This operation is an important one that appears throughout the thesis.

For a set $X \subset \mathbb{R}^n$ and $r \in (0,\infty)$, let

$$\mathbf{B}^n_{\mathbb{R}}[X,r] = \bigcap_{x\in X} \mathbf{B}^n_{\mathbb{R}}[x,r] \quad\text{and}\quad \mathbf{B}^n_{\mathbb{R}}(X,r) = \bigcap_{x\in X} \mathbf{B}^n_{\mathbb{R}}(x,r). \tag{2.2.1}$$

For a set $X \subset \mathbb{H}^n$ and $r \in (0,\infty)$, let

$$\mathbf{B}^n_{\mathbb{H}}[X,r] = \bigcap_{x\in X} \mathbf{B}^n_{\mathbb{H}}[x,r] \quad\text{and}\quad \mathbf{B}^n_{\mathbb{H}}(X,r) = \bigcap_{x\in X} \mathbf{B}^n_{\mathbb{H}}(x,r). \tag{2.2.2}$$

For a set $X \subset \mathbb{S}^n$ and $r \in (0,\pi/2)$, let

$$\mathbf{B}^n_{\mathbb{S}}[X,r] = \bigcap_{x\in X} \mathbf{B}^n_{\mathbb{S}}[x,r] \quad\text{and}\quad \mathbf{B}^n_{\mathbb{S}}(X,r) = \bigcap_{x\in X} \mathbf{B}^n_{\mathbb{S}}(x,r). \tag{2.2.3}$$

2.3 Euclidean Spindle, Spindle-Geodesic and Spindle-Distance

In this section we develop the concept of a spindle in euclidean space and the associated notation. In the theory of spindle-convexity, the spindle plays the same role that line segments play in the theory of convexity. The goal here is to develop a theoretical construct, in this case spindle-convexity, that has features analogous to classical convexity. Just as two points are joined by a line segment we find that, under the right conditions, two points are joined by a spindle. If the dimension of the space is n, then a spindle joining two distinct points is an n-dimensional body. In the classical theory, the line segment between two points a and b has the property that it is the shortest path between these two points. In the current setting, if two points are joined by a bounded spindle, then a spindle-geodesic is the shortest path on the boundary of the spindle between these points. The spindle-distance between two points is just the length of the spindle-geodesic joining them, just like the distance between two points is the length of the line segment joining them.

Definition 2.3.1 *Let $r \in (0, \infty]$. Let a and b be two points in $\mathbb{R}^n$. If $d^n_{\mathbb{R}}(a, b) < 2r$, then the* closed euclidean r-spindle *of a and b, denoted by* $\text{spin}[a, b, r]^n_{\mathbb{R}}$, *is defined as the union of all circular arcs, with end points a and b, that are of radii at least r and shorter than πr. If $d^n_{\mathbb{R}}(a, b) = 2r$ and m is the midpoint of $[a, b]^n_{\mathbb{R}}$, then* $\text{spin}[a, b, r]^n_{\mathbb{R}} = \mathbf{B}^n_{\mathbb{R}}[m, r]$. *If $d^n_{\mathbb{R}}(a, b) > 2r$, then we define* $\text{spin}[a, b, r]^n_{\mathbb{R}}$ *to be $\mathbb{R}^n$. In all cases, the* open euclidean r-spindle, *denoted by* $\text{spin}(a, b, r)^n_{\mathbb{R}}$, *is the interior of the closed one.*

Definition 2.3.2 *Let $r \in (0, \infty]$. Let a and b be two points in $\mathbb{R}^n$ with $d^n_{\mathbb{R}}(a, b) \leq 2r$. A* closed euclidean r-spindle-geodesic *between a and b, denoted by* $\text{geo}[a, b, r]_{\mathbb{R}}$, *is a circular arc of radius r, with end points a and b, and of length at most πr. An* open euclidean r-spindle geodesic, *denoted* $\text{geo}(a, b, r)_{\mathbb{R}}$, *is the relative interior of*

$\text{geo}[a, b, r]_{\mathbb{R}}$.

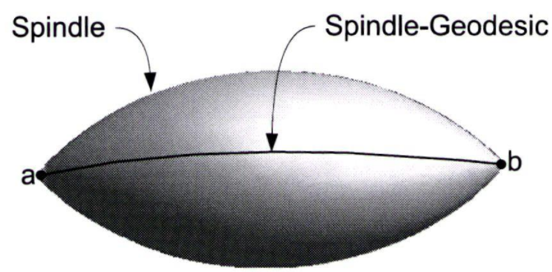

Figure 2.1: Euclidean r-Spindle and Euclidean r-Spindle-Geodesic in $\mathbb{R}^3$ where r is finite.

The points a and b in Figure 2.1 are joined by a euclidean r-spindle, where r is finite. The euclidean r-spindle is the union of all circular arcs of radius at least r joining a and b with length at most πr. The non-unique spindle-geodesic is a circular arc of radius r and of length πr joining a and b. In the event that it is clear what are the radius and the ambient space, we simplify the terminology and call $\text{spin}[a, b, r]^n_{\mathbb{R}}$ a spindle and $\text{geo}[a, b, r]_{\mathbb{R}}$ a spindle-geodesic.

The next two remarks provide useful characterizations of spindles and spindle-geodesics. In Section 2.8, we examine Remark 2.3.3 (and Remarks 2.4.4, 2.5.3) in detail and simply state it here for reference. Remark 2.3.4 follows easily from Definition 2.3.2 and the conventions established earlier.

Remark 2.3.3 *Let $r \in \mathbb{R}$ such that $r > 0$. If $d^n_{\mathbb{R}}(a, b) \leq 2r$, then* $\text{spin}[a, b, r]^n_{\mathbb{R}} = \mathbf{B}^n_{\mathbb{R}}[\mathbf{B}^n_{\mathbb{R}}[\{a, b\}, r], r]$, *and* $\text{spin}(a, b, r)^n_{\mathbb{R}} = \mathbf{B}^n_{\mathbb{R}}(\mathbf{B}^n_{\mathbb{R}}[\{a, b\}, r], r)$.

Remark 2.3.4 *Let a and b be two points in $\mathbb{R}^n$. The non-unique closed or open euclidean r-spindle-geodesic is a 2-dimensional curve. It lies on the boundary of* $\text{spin}[a, b, r]^n_{\mathbb{R}}$ *and is a circular arc of radius r and length at most πr. The line segment* *through a and b is the euclidean r-spindle where r is infinite. In the case $r = \infty$ we do not distinguish between the closed euclidean r-spindle and the open euclidean r-spindle. The closed (respectively open) euclidean line segment between a and b is the closed (respectively open) euclidean r-spindle-geodesic where r is infinite.*

All spindle-geodesics joining two points have the same length and this common value is called the spindle-distance. The spindle-distance is a useful measurement and has interesting properties. The next definition formalizes the concept and the following remark provides an explicit formula for the spindle-distance between two points.

Definition 2.3.5 *Let $r \in (0, \infty]$. If a and b are two points in $\mathbb{R}^n$, $n \geq 2$, such that $d^n_{\mathbb{R}}(a, b) \leq 2r$, then the* euclidean r-spindle-distance *between a and b, denoted $\rho^n_{\mathbb{R}}[r](a, b)$, is the euclidean arc-length of any euclidean r-spindle-geodesics joining a and b. If $d^n_{\mathbb{R}}(a, b) > 2r$, then the euclidean r-spindle-distance is undefined. By letting r be infinity, the euclidean r-spindle-distance becomes the euclidean distance.*

Let $r \in \mathbb{R}$ such that $r > 0$. If a and b are two points in $\mathbb{R}^n$, $n \geq 2$, such that $d^n_{\mathbb{R}}(a, b) \leq 2r$, then there is some spindle-geodesic $\mathrm{geo}[a, b, r]_{\mathbb{R}}$ joining a and b. Recall that $\mathrm{geo}[a, b, r]_{\mathbb{R}}$ is a circular arc in $\mathbb{R}^n$. Let $\mathbb{S}^1_{\mathbb{R}}(c, r)$ be the circle containing $\mathrm{geo}[a, b, r]_{\mathbb{R}}$, see Figure 2.2.

By elementary trigonometry, the euclidean r-spindle-distance between a and b is simply $\rho^n_{\mathbb{R}}[r](a, b) = 2r\theta$, where

$$\sin\theta = \frac{d^n_{\mathbb{R}}(a, b)}{2r}.$$

This discussion verifies the following remark.

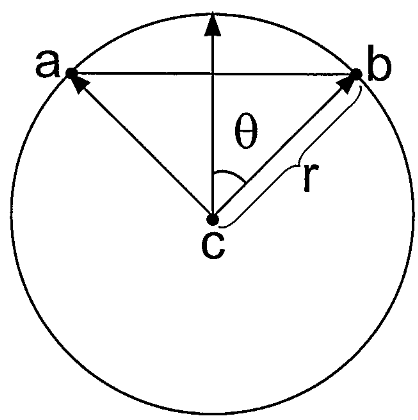

Figure 2.2: Computing the Euclidean and Hyperbolic Spindle-Distance between a and b.

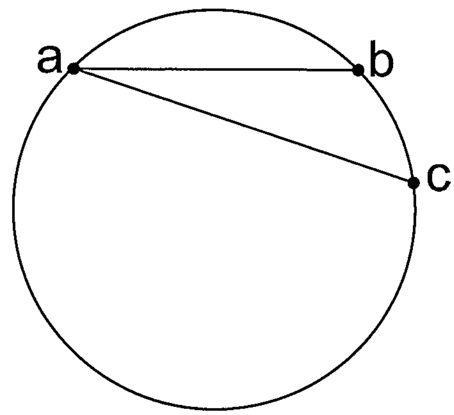

Figure 2.3: Monotonicity of the Euclidean and Hyperbolic Spindle-Distance.

Remark 2.3.6 *Let $r \in (0, \infty]$. Let a and b be two points in $\mathbb{R}^n$, $n \geq 2$, such that $d^n_{\mathbb{R}}(a, b) \leq 2r$. If $r \in \mathbb{R}$ such that $r > 0$, then an explicit formula for the euclidean r-spindle-distance is*

$$\rho^n_{\mathbb{R}}[r](a, b) = 2r \sin^{-1}\left(\frac{d^n_{\mathbb{R}}(a, b)}{2r}\right).$$

If $r = \infty$, then

$$\rho^n_{\mathbb{R}}[r](a, b) = d^n_{\mathbb{R}}(a, b).$$

The final remark in this section simply says that if the euclidean distance between two points, say a and c, is greater than the euclidean distance between two points, say a and b, then spindle-distance between a and c is greater than the spindle-distance between a and b. Let $r \in (0, \infty]$ and $a, b, c \in \mathbb{R}^n$ be points such that $d^n_{\mathbb{R}}(a, b) < d^n_{\mathbb{R}}(a, c) \leq 2r$. If $r = \infty$, then there is nothing to prove. Thus, we assume $r < \infty$. Since $d^n_{\mathbb{R}}(a, b) < d^n_{\mathbb{R}}(a, c) \leq 2r$, there is a map of the line segments $[a, b]^n_{\mathbb{R}}$ and $[a, c]^n_{\mathbb{R}}$ so that the points a, b, c all lie on a circle of radius r, see Figure 2.3. The longer chord, joining a and c, subtends a longer arc on the circle than does the chord joining a and b.

Remark 2.3.7 *Let $r \in (0, \infty]$. If $a, b, c \in \mathbb{R}^n$ are points such that $d^n_{\mathbb{R}}(a, b) < d^n_{\mathbb{R}}(a, c) \leq 2r$, then $\rho^n_{\mathbb{R}}[r](a, b) < \rho^n_{\mathbb{R}}[r](a, c)$.*

2.4 Hyperbolic Spindle, Spindle-Geodesic and Spindle-Distance

This section repeats the discussion presented in Section 2.3, but the setting is now hyperbolic space. We include the information primarily for reference. We highlight the subtle differences which vary from the euclidean case.

In the current setting we are working strictly in hyperbolic space. Any circle or circular arc is a hyperbolic circle or hyperbolic circular arc, its radius is the hyperbolic radius and its center is the hyperbolic center. Furthermore, lines and line segments are hyperbolic lines and hyperbolic line segments. Finally any lengths and distances are measured using hyperbolic distance.

Similar to Section 2.3, we define a hyperbolic r-spindle and hyperbolic r-spindle-geodesic. Figure 2.4 provides examples of both objects in $\mathbb{H}^2$. We obtain characterizations of hyperbolic r-spindles and hyperbolic r-spindle-geodesics similar to Section 2.3 and these are collected in Remark 2.4.4 (which is examined in Section 2.8) and Remark 2.4.5, which follows from the definitions. Finally, Definition 2.4.6 provides an analogous definition for spindle-distance in hyperbolic space.

The key issue that arises in this context has to do with the fact that in some cases not all of the circles required to form the spindle reside in the given hyperbolic space. In particular there are instances when the spindle requires the inclusion of circular arcs which are on hypercycles. Nonetheless, the portion of the hypercycles contained in the Poincaré Ball have familiar properties which we exploit here.

Definition 2.4.1 *Let $r \in (0, \infty]$. Let $\bar{r}$ be the euclidean radius of the hyperbolic circle with hyperbolic radius r. Let a and b be two points in $\mathbb{H}^n$. If the euclidean spindle-geodesic* $\text{geo}[a, b, \bar{r}]_{\mathbb{R}}$ *is contained in Poincaré Ball then we call it a geodesic with end points a and b and radius $\bar{r}$. If $d^n_{\mathbb{H}}(a, b) < 2r$, then the* closed hyperbolic r-spindle *of a and b, denoted by* $\text{spin}[a, b, r]^n_{\mathbb{H}}$, *is defined as the union of all geodesics with end points a and b and radii at least $\bar{r}$. If $d^n_{\mathbb{H}}(a, b) = 2r$ and m is the midpoint of $[a, b]^n_{\mathbb{H}}$, then* $\text{spin}[a, b, r]^n_{\mathbb{H}} = \mathbf{B}^n_{\mathbb{H}}[m, r]$. *If $d^n_{\mathbb{H}}(a, b) > 2r$, then we define* $\text{spin}[a, b, r]^n_{\mathbb{H}}$ *to be $\mathbb{H}^n$. In all cases, the* open hyperbolic r-spindle, *denoted by* $\text{spin}(a, b, r)^n_{\mathbb{H}}$, *is the interior of the closed one.*

Remark 2.4.2 *Let a and b be two points in $\mathbb{H}^n$. The open (resp. closed) euclidean line segment joining a and b is contained in the open (resp. closed) hyperbolic spindle of a and b.*

Definition 2.4.3 *Let $r \in (0, \infty]$. Let a and b be two points in $\mathbb{H}^n$ with $d^n_{\mathbb{H}}(a, b) \leq 2r$. A* closed hyperbolic r-spindle-geodesic *between a and b, denoted* $\text{geo}[a, b, r]_{\mathbb{H}}$, *is a circular arc of radius r, with end points a and b, and of length at most $\pi \sinh(r)$. An* open hyperbolic r-spindle-geodesic, *denoted* $\text{geo}(a, b, r)_{\mathbb{H}}$, *is the relative interior of* $\text{geo}[a, b, r]_{\mathbb{H}}$.

The points a and b in Figure 2.4 lie on the dashed hyperbolic line and are joined by the solid hyperbolic line segment. The two dashed circles, passing through a and b, are congruent in $\mathbb{H}^2$. The intersection of their closures is the spindle joining a and b. Either circular arc, joining a and b, lying on the boundary of the spindle is a spindle-geodesic.

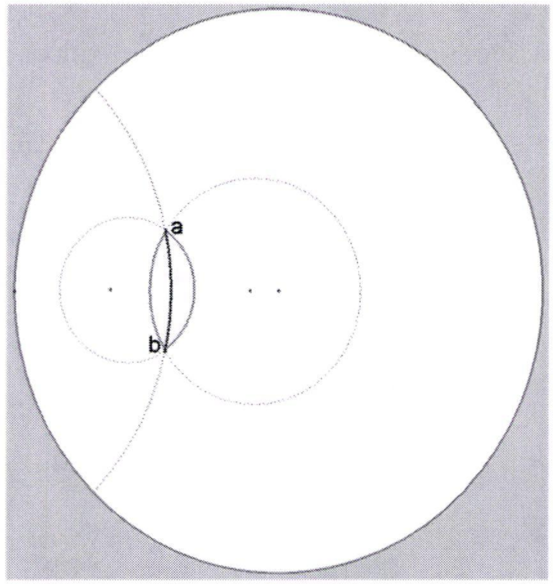

Figure 2.4: Hyperbolic r-Spindle and Hyperbolic r-Spindle-Geodesic in $\mathbb{H}^2$.

Remark 2.4.4 *Let $r \in \mathbb{R}$ such that $r > 0$. If $d^n_{\mathbb{H}}(a,b) \leq 2r$, then* $\text{spin}[a,b,r]^n_{\mathbb{H}} = \mathbf{B}^n_{\mathbb{H}}[\mathbf{B}^n_{\mathbb{H}}[\{a,b\},r],r]$, *and* $\text{spin}(a,b,r)^n_{\mathbb{H}} = \mathbf{B}^n_{\mathbb{H}}(\mathbf{B}^n_{\mathbb{H}}[\{a,b\},r],r)$.

Remark 2.4.5 *Let a and b be two points in $\mathbb{H}^n$. The non-unique closed or open hyperbolic r-spindle geodesic is a 2-dimensional curve in hyperbolic space lying on the boundary of* $\text{spin}[a,b,r]^n_{\mathbb{H}}$. *It is a hyperbolic circular arc of hyperbolic radius r. The hyperbolic length of this curve is at most $\pi\sinh(r)$. The hyperbolic line through a and b is the hyperbolic r-spindle where r is infinite. In the case $r = \infty$, we do not distinguish between the closed hyperbolic r-spindle and the open hyperbolic r-spindle. The closed (respectively open) hyperbolic line segment between a and b is the closed (respectively open) hyperbolic r-spindle-geodesic where r is infinite.*

Definition 2.4.6 *Let $r \in (0,\infty]$. If a and b are two points in $\mathbb{H}^n$, $n \geq 2$, such that $d^n_{\mathbb{H}}(a,b) \leq 2r$, then the* hyperbolic r-spindle-distance *between a and b, denoted $\rho^n_{\mathbb{H}}[r](a,b)$, is the hyperbolic arc-length of any hyperbolic r-spindle-geodesics joining a*

and b. If $d^n_{\mathbb{H}}(a,b) > 2r$, then the hyperbolic r-spindle-distance is undefined. By letting r be infinity, the hyperbolic r-spindle-distance becomes the hyperbolic distance.

Let $r \in \mathbb{R}$ such that $r > 0$. If a and b are two points in $\mathbb{H}^n$, $n \geq 2$, such that $d^n_{\mathbb{H}}(a,b) \leq 2r$, then there is some spindle-geodesic $\text{geo}[a,b,r]_{\mathbb{H}}$ joining a and b. Recall that $\text{geo}[a,b,r]_{\mathbb{H}}$ is a circular arc in $\mathbb{H}^n$. Let $\mathbb{S}^1_{\mathbb{H}}(c,r)$ be the circle containing $\text{geo}[a,b,r]_{\mathbb{H}}$. After translating the center c of $\mathbb{S}^1_{\mathbb{H}}(c,r)$ to the origin $o_{\mathbb{H}^n}$, Figure 2.2 accurately represents the configuration. The length of the circular arc joining a and b is $2\theta \sinh r$. By the hyperbolic law of cosines,

$$\cosh[d^n_{\mathbb{H}}(a,b)] = \cosh^2 r - [\cos(2\theta)] \sinh^2 r.$$

Thus, we easily obtain the following remark.

Remark 2.4.7 *Let $r \in (0,\infty]$. Let a and b be two points in $\mathbb{H}^n$, $n \geq 2$, such that $d^n_{\mathbb{H}}(a,b) \leq 2r$. If $r \in \mathbb{R}$ such that $r > 0$, then an explicit formula for the hyperbolic r-spindle-distance is*

$$\rho^n_{\mathbb{H}}[r](a,b) = \sinh(r)\cos^{-1}\left(\frac{\cosh^2(r) - \cosh[d^n_{\mathbb{H}}(a,b)]}{\sinh^2(r)}\right).$$

If $r = \infty$, then

$$\rho^n_{\mathbb{H}}[r](a,b) = d^n_{\mathbb{H}}(a,b).$$

Let $r \in (0,\infty]$. If $a,b,c \in \mathbb{H}^n$ are points such that $d^n_{\mathbb{H}}(a,b) < d^n_{\mathbb{H}}(a,c) \leq 2r$, then there is a map of the line segments $[a,b]^n_{\mathbb{H}}$ and $[a,c]^n_{\mathbb{H}}$ so that the points a,b,c all lie on a circle of radius r. Furthermore, if we translate the point a to the origin then Figure 2.3 accurately reflects the configuration. The next remark follows from the comments preceding Remark 2.3.7.

Remark 2.4.8 *Let $r \in (0,\infty]$. If $a,b,c \in \mathbb{H}^n$ are points such that $d^n_{\mathbb{H}}(a,b) < d^n_{\mathbb{H}}(a,c) \leq 2r$, then $\rho^n_{\mathbb{H}}[r](a,b) < \rho^n_{\mathbb{H}}[r](a,c)$.*

2.5 Spherical Spindle, Spindle-Geodesic and Spindle-Distance

As with Section 2.4, this section repeats the discussion presented in Section 2.3, but the setting is spherical space. Again, we collect facts and notation relevant to the discussion of spindle, spindle-geodesic and spindle-distance for spherical space. Note that all generic terms, such as line, circle, distance etc., refer to objects in spherical space.

Similar to Section 2.3 and Section 2.4, we define a spherical r-spindle, r-spindle-geodesic and spindle distance. We obtain characterizations of these objects which are collected in the following figures and remarks. As before, Remark 2.4.4 is examined in Section 2.8.

Definition 2.5.1 *Let $r \in (0, \pi/2]$. Let a and b be two points in $\mathbb{S}^n$. If $d^n_{\mathbb{S}}(a,b) < 2r$, then the* closed spherical r-spindle *of a and b, denoted by* $\operatorname{spin}[a,b,r]^n_{\mathbb{S}}$, *is defined as the union of all circular arcs, with end points a and b, that are of radii at least r and shorter than $\pi\sin(r)$. If $d^n_{\mathbb{S}}(a,b) = 2r$ and m is the midpoint of $[a,b]^n_{\mathbb{S}}$, then* $\operatorname{spin}[a,b,r]^n_{\mathbb{S}} = \mathbf{B}^n_{\mathbb{S}}[m,r]$. *If $d^n_{\mathbb{S}}(a,b) > 2r$, then we define* $\operatorname{spin}[a,b,r]^n_{\mathbb{S}}$ *to be $\mathbb{S}^n$. In all cases, the* open spherical r-spindle, *denoted by* $\operatorname{spin}(a,b,r)^n_{\mathbb{S}}$, *is the interior of the closed one. The closed (respectively open) spherical line segment between a and b is the closed (respectively open) spherical r-spindle where r is $\pi/2$.*

Definition 2.5.2 *Let $r \in (0, \pi/2]$. Let a and b be two points in $\mathbb{S}^n$ with $d^n_{\mathbb{S}}(a,b) \leq 2r$. A* closed spherical r-spindle-geodesic *between a and b, denoted* $\operatorname{geo}[a,b,r]_{\mathbb{S}}$, *is a circular arc of radius r, with end points a and b, and of length at most $\pi\sin(r)$. An* open spherical r-spindle geodesic, *denoted* $\operatorname{geo}(a,b,r)_{\mathbb{S}}$, *is the relative interior of* $\operatorname{geo}[a,b,r]_{\mathbb{S}}$.

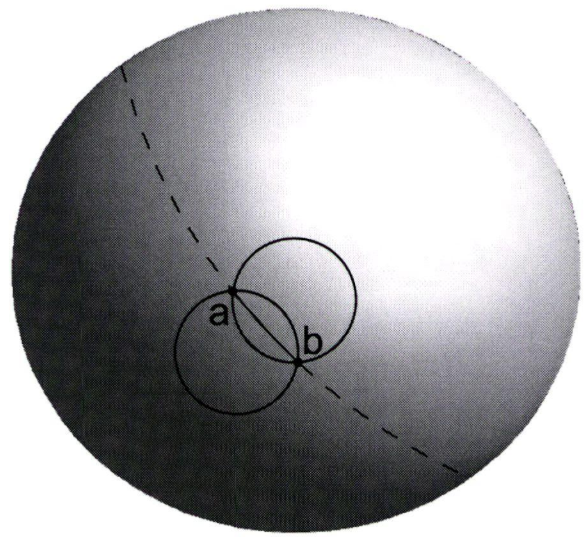

Figure 2.5: Spherical r-Spindle and Spherical r-Spindle-Geodesic in $\mathbb{S}^2$.

The dashed great circle passing through a and b in Figure 2.5 is a line in spherical space and the solid arc on the great circle joining a and b is a spherical line segment. The two circles passing through a and b have the same radius, namely r. The intersection of the disks bounded by these two circles is $\text{spin}[a, b, r]^2_{\mathbb{S}}$. Either of the two arcs joining a and b on the boundary of the spindle are spindle-geodesics.

Remark 2.5.3 *Let* $r \in (0, \pi/2)$. *If* $d^n_{\mathbb{S}}(a, b) \leq 2r$, *then* $\text{spin}[a, b, r]^n_{\mathbb{S}} = \mathbf{B}^n_{\mathbb{S}}[\mathbf{B}^n_{\mathbb{S}}[\{a, b\}, r], r]$, *and* $\text{spin}(a, b, r)^n_{\mathbb{S}} = \mathbf{B}^n_{\mathbb{S}}(\mathbf{B}^n_{\mathbb{S}}[\{a, b\}, r], r)$.

Remark 2.5.4 *Let a and b be two points in $\mathbb{S}^n$. The non-unique closed or open spherical r-spindle-geodesic is a 2-dimensional curve in spherical space lying on the boundary of* $\text{spin}[a, b, r]^n_{\mathbb{S}}$. *It is a spherical circular arc of spherical radius r. The spherical length of this curve is at most $\pi \sin(r)$. The spherical line through a and b is the spherical r-spindle where r is $\pi/2$. In the case $r = \pi/2$, we do not distinguish between the closed spherical r-spindle and the open spherical r-spindle. The closed* *(respectively open) spherical line segment between* a *and* b *is the closed (respectively open) spherical* r*-spindle-geodesic where* r *is* $\pi/2$.

Definition 2.5.5 *Let* $r \in (0, \pi/2]$. *If* a *and* b *are two points in* $\mathbb{S}^n$, $n \geq 2$, *such that* $d^n_{\mathbb{S}}(a,b) \leq 2r$, *then the* spherical r-spindle-distance *between* a *and* b, *denoted* $\rho^n_{\mathbb{S}}[r](a,b)$, *is the spherical arc-length of any spherical* r*-spindle-geodesics joining* a *and* b. *If* $d^n_{\mathbb{S}}(a,b) > 2r$, *then the spherical* r*-spindle-distance is undefined. By letting* r *be* $\pi/2$, *the spherical* r*-spindle-distance becomes the spherical distance.*

Let $r \in (0, \pi/2]$. If a and b are two points in $\mathbb{S}^n$, $n \geq 2$, such that $d^n_{\mathbb{S}}(a,b) \leq 2r$, then there is some spindle-geodesic $\mathrm{geo}[a,b,r]_{\mathbb{S}}$ joining a and b. Note that $\mathrm{geo}[a,b,r]_{\mathbb{S}}$ is a circular arc in $\mathbb{S}^n$. Let $\mathbb{S}^1_{\mathbb{S}}(c,r)$ be the circle containing $\mathrm{geo}[a,b,r]_{\mathbb{S}}$. Recall that the spherical model is embedded in $\mathbb{R}^{n+1}$ and the spherical circle $\mathbb{S}^1_{\mathbb{S}}(c,r)$ is simply a euclidean circle, say $\mathbb{S}^1_{\mathbb{R}}(c',r')$, in $\mathbb{R}^n$. To obtain c', we project the point c onto the plane in $\mathbb{R}^n$ containing $\mathbb{S}^1_{\mathbb{S}}(c,r)$, see Figure 2.6.

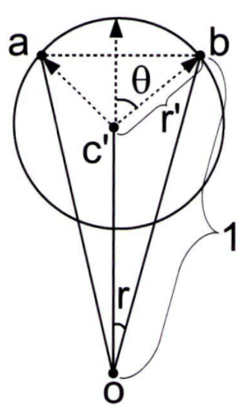

Figure 2.6: Computing Spindle-Distance in Spherical Space.

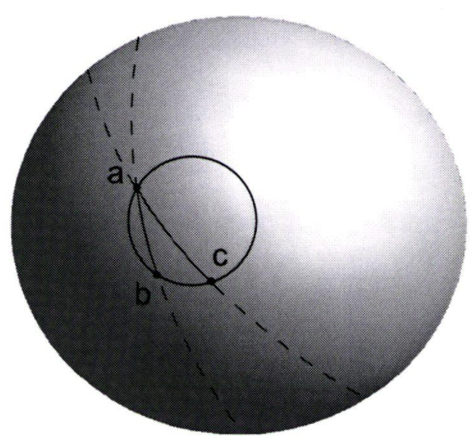

Figure 2.7: Monotonicity of Spherical Spindle-Distance.

In Figure 2.6, the point o is the origin, which is the same for both euclidean spherical space, the circle is $\mathbb{S}^1_{\mathbb{R}}(c',r')$ and the dashed segments lie in the same euclidean plane as the circle. The dashed segments are radii of $\mathbb{S}^1_{\mathbb{R}}(c', r')$. Now, b lies on the circle $\mathbb{S}^1_{\mathbb{S}}(c, r)$, so the spherical distance between c and b is r. The spherical distance is the measure of the angle $\angle cob$. In particular, $\angle cob = r$. Since $[o, c']^{n+1}_{\mathbb{R}} \subset [o, c]^{n+1}_{\mathbb{R}}$, we have that $\angle c'ob = r$. Next, let $2\theta = \angle ac'b$, where $\angle ac'b$ is the angle between the segments $[a, c']^{n+1}_{\mathbb{R}}$ and $[b, c']^{n+1}_{\mathbb{R}}$ in the euclidean plane containing $\mathbb{S}^1_{\mathbb{R}}(c', r')$. Since b is in spherical space and lies on a sphere which is of radius one and centered at o, it follows that the euclidean distance between o and b is one.

From Figure 2.6, we readily compute $\rho^n_{\mathbb{S}}[r](a, b) = 2\theta r' = 2\theta \sin r$ where,

$$\sin\theta = \frac{d^{n+1}_{\mathbb{R}}(a, b)}{2r'} = \frac{d^{n+1}_{\mathbb{R}}(a, b)}{2\sin(r)}.$$

Thus, we easily obtain the following remark.

Remark 2.5.6 *Let $r \in (0, \pi/2]$. If a and b are two points in $\mathbb{S}^n$, $n \geq 2$, such that $d^n_{\mathbb{S}}(a, b) \leq 2r$, then an explicit formula for the spherical r-spindle-distance is*

$$\rho^n_{\mathbb{S}}[r](a, b) = 2\sin(r)\sin^{-1}\left(\frac{d^{n+1}_{\mathbb{R}}(a, b)}{2\sin(r)}\right)$$

Let $r \in (0, \pi/2]$. If $a, b, c \in \mathbb{S}^2$ are points such that $d^2_{\mathbb{S}}(a, b) < d^2_{\mathbb{S}}(a, c) \leq 2r$, then there is a map of the line segments $[a, b]^n_{\mathbb{S}}$ and $[a, c]^n_{\mathbb{S}}$ so that the points a, b, c all lie on a circle of radius r, see Figure 2.3. The longer chord, joining a and c, subtends a longer arc on the circle than does the chord joining a and b. This clarifies the following remark.

Remark 2.5.7 *Let $r \in (0, \pi/2]$. If $a, b, c \in \mathbb{S}^2$ are points such that $d^2_{\mathbb{S}}(a, b) < d^2_{\mathbb{S}}(a, c) \leq 2r$, then $\rho^2_{\mathbb{S}}[r](a, b) < \rho^2_{\mathbb{S}}[r](a, c)$.*

2.6 Spindle-Convexity

We now begin a discussion of the topic of spindle-convexity and what it means for a set to be spindle-convex.

Definition 2.6.1 *Let $r \in (0, \infty]$. A set $C \subset \mathbb{R}^n$ is* r-spindle-convex in $\mathbb{R}^n$ *if, and only if, for every pair of points $a, b \in C$, there exists a closed euclidean r-spindle-geodesic joining them and every closed euclidean r-spindle-geodesic joining a and b is contained in C.*

Definition 2.6.2 *Let $r \in (0, \infty]$. A set $C \subset \mathbb{H}^n$ is* r-spindle-convex in $\mathbb{H}^n$ *if, and only if, for every pair of points $a, b \in C$, there exists a closed hyperbolic r-spindle-geodesic joining them and every closed hyperbolic r-spindle-geodesic joining a and b is contained in C.*

Definition 2.6.3 *Let $r \in (0, \pi/2]$. A set $C \subset \mathbb{S}^n$ is* r-spindle-convex in $\mathbb{S}^n$ *if, and only if, for every pair of points $a, b \in C$, there exists a closed spherical r-spindle-geodesic joining them and every closed spherical r-spindle-geodesic joining a and b is contained in C.*

When $r = \infty$ (resp. $r = \infty$, resp. $r = \pi/2$) then there is a unique closed euclidean (resp. hyperbolic, resp. spherical) r-spindle-geodesic joining any two points. In fact, the euclidean (resp. hyperbolic, resp. spherical) r-spindle-geodesic joining any two points is just the line segment between them. Thus, when $r = \infty$ (resp. $r = \infty$, resp. $r = \pi/2$), r-spindle-convexity in $\mathbb{R}^n$ (resp. $\mathbb{H}^n$, resp. $\mathbb{S}^n$) is equivalent to convexity in $\mathbb{R}^n$ (resp. $\mathbb{H}^n$, resp. $\mathbb{S}^n$).

Let $r \in \mathbb{R}$ such that $r > 0$ (resp. let $r \in \mathbb{R}$ such that $r > 0$, resp. let $r \in (0, \pi/2)$). Suppose $C \subset \mathbb{R}^n$ (resp. $C \subset \mathbb{H}^n$, resp. $C \subset \mathbb{S}^n$) is an r-spindle-convex set. If C is a single point then it is clearly convex and 0-dimensional. Let a and b be distinct elements of C. Now, C contains all euclidean (resp. hyperbolic, resp. spherical) r-spindle-geodesic joining a and b. The union of all such curves is the boundary of the closed euclidean (resp. hyperbolic, resp. spherical) r-spindle joining a and b. The closed euclidean (resp. hyperbolic, resp. spherical) r-spindle joining the points a and b contains, by definition, the euclidean (resp. hyperbolic, resp. spherical) line segment joining a and b. In the euclidean case, $[a,b]^n_{\mathbb{R}} \subset \mathrm{spin}[a,b,r]^n_{\mathbb{R}} \subseteq C$ implies C is convex, in the euclidean sense. In the hyperbolic case, $[a,b]^n_{\mathbb{H}} \subset \mathrm{spin}[a,b,r]^n_{\mathbb{H}} \subseteq C$ implies C is convex, in the hyperbolic sense. In the spherical case, $[a,b]^n_{\mathbb{S}} \subset \mathrm{spin}[a,b,r]^n_{\mathbb{S}} \subseteq C$ implies C is convex, in the spherical sense.

Remark 2.3.3 (resp. Remark 2.4.4, resp. Remark 2.5.3) shows that a closed euclidean (resp. hyperbolic, resp. spherical) r-spindle is the intersection of closed balls, and it is either an empty set, or a point, or homeomorphic to a closed ball. The latter is true in this setting. Since C contains the aforementioned spindle, it has a non-empty interior and is full-dimensional.

Let $C_i \subset \mathbb{R}^n$ (resp. $C_i \subset \mathbb{H}^n$, resp. $C_i \subset \mathbb{S}^n$) be an r-spindle-convex set for each $i \in I$. If the points a and b are contained in

$$\bigcap_{i\in I} C_i,$$

then a and b are contained in C_i for each $i \in I$. Since C_i is an r-spindle-convex set for each $i \in I$, it follows that $\mathrm{spin}[a,b,r]^n_{\mathbb{R}}$ (resp.$\mathrm{spin}[a,b,r]^n_{\mathbb{H}}$, resp.$\mathrm{spin}[a,b,r]^n_{\mathbb{S}}$) is contained in C_i for each $i \in I$. Finally,

$$\mathrm{spin}[a,b,r]^n_{\mathbb{R}} \subseteq \bigcap_{i\in I} C_i, (\text{resp. } \mathrm{spin}[a,b,r]^n_{\mathbb{H}} \subseteq \bigcap_{i\in I} C_i, \text{resp. } \mathrm{spin}[a,b,r]^n_{\mathbb{S}} \subseteq \bigcap_{i\in I} C_i)$$

yields that the intersection of r-spindle-convex sets is an r-spindle-convex set. We collect these facts in the following remarks.

Remark 2.6.4 *Let $r \in (0, \infty]$ and $C \subset \mathbb{R}^n$ be a non-empty r-spindle-convex set in $\mathbb{R}^n$. Then C is convex. Furthermore, if $r \neq \infty$, then C is either* 0*-dimensional (a point) or n-dimensional. Finally, the intersection of r-spindle-convex sets is an r-spindle-convex set.*

Remark 2.6.5 *Let $r \in (0, \infty]$ and $C \subset \mathbb{H}^n$ be a non-empty r-spindle-convex set in $\mathbb{H}^n$. Then C is convex. Furthermore, if $r \neq \infty$, then C is either* 0*-dimensional (a point) or n-dimensional. Finally, the intersection of r-spindle-convex sets is an r-spindle-convex set.*

Remark 2.6.6 *Let $r \in (0, \pi/2]$ and $C \subset \mathbb{S}^n$ be a non-empty r-spindle-convex set in $\mathbb{S}^n$. Then C is convex. Furthermore, if $r \neq \pi/2$, then C is either* 0*-dimensional (a point) or n-dimensional. Finally, the intersection of r-spindle-convex sets is an r-spindle-convex set.*

In the standard theory convexity, any two points are joined by a line segment, which is a bounded set. In the theory of spindle-convexity, it may be the case that points are too far apart for them to be joined by a spindle. We are primarily interested in bounded sets where the points are not too far apart. The following definitions clarify this.

Definition 2.6.7 *The* circumradius $\mathrm{cr}_{\mathbb{R}}(X)$ *of a bounded set $X \subseteq \mathbb{R}^n$ is the radius of the unique smallest ball that contains X (also known as the circumball of X); that is,*

$$\mathrm{cr}_{\mathbb{R}}(X) = \inf\{r > 0 : X \subseteq \mathbf{B}^n_{\mathbb{R}}[q, r] \textit{ for some } q \in \mathbb{R}^n\}.$$

If X is unbounded, then $\mathrm{cr}_{\mathbb{R}}(X) = \infty$.

Definition 2.6.8 *The* circumradius $\mathrm{cr}_{\mathbb{H}}(X)$ *of a bounded set* $X \subseteq \mathbb{H}^n$ *is the radius of the unique smallest ball that contains* X *(also known as the circumball of* X*); that is,*

$$\mathrm{cr}_{\mathbb{H}}(X) = \inf\{r > 0 : X \subseteq \mathbf{B}^n_{\mathbb{H}}[q, r] \textit{ for some } q \in \mathbb{H}^n\}.$$

If X *is unbounded, then* $\mathrm{cr}_{\mathbb{H}}(X) = \infty$.

Definition 2.6.9 *The* circumradius $\mathrm{cr}_{\mathbb{S}}(X)$ *of a bounded set* $X \subseteq \mathbb{S}^n$ *is the radius of the unique smallest ball that contains* X *(also known as the circumball of* X*); that is,*

$$\mathrm{cr}_{\mathbb{S}}(X) = \inf\{r > 0 : X \subseteq \mathbf{B}^n_{\mathbb{S}}[q, r] \textit{ for some } q \in \mathbb{S}^n\}.$$

If X *is unbounded, then* $\mathrm{cr}_{\mathbb{S}}(X) = \infty$.

Again, in the standard theory of convexity, the convex hull of a set S is the intersection of all convex sets containing S. We may also identify S as being in a convex position provided that no point in S is contained in the convex hull of the remaining points in S. The following extends these ideas to the current setting.

Definition 2.6.10 *Let* $r \in (0, \infty]$ *and* $X \subset \mathbb{R}^n$. *Then the* euclidean r-spindle-convex hull *of* X *is*

$$\mathbb{R}\,\mathrm{conv_r}\, X = \bigcap\{C \subseteq \mathbb{R}^n : X \subseteq C \textit{ and } C \textit{ is } r\textit{-spindle-convex in } \mathbb{R}^n\}.$$

Furthermore, X *is in an* r-spindle-convex position *in* $\mathbb{R}^n$ *if, and only if,*

$$x \notin \mathbb{R}\,\mathrm{conv_r}(X \setminus x), \textit{ for any } x \in X.$$

Definition 2.6.11 *Let* $r \in (0, \infty]$ *and* $X \subset \mathbb{H}^n$. *Then the* hyperbolic r-spindle-convex hull *of* X *is*

$$\mathbb{H}\,\mathrm{conv_r}\, X = \bigcap\{C \subseteq \mathbb{H}^n : X \subseteq C \textit{ and } C \textit{ is } r\textit{-spindle-convex in } \mathbb{H}^n\}.$$

Furthermore, X is in an r-spindle-convex position *in* $\mathbb{H}^n$ *if, and only if,*

$$x \notin \mathbb{H}\operatorname{conv}_{\mathrm{r}}(X \setminus x), \text{ for any } x \in X.$$

Definition 2.6.12 *Let $r \in (0, \pi/2]$ and $X \subset \mathbb{S}^n$. Then the* spherical r-spindle-convex hull *of X is*

$$\mathbb{S}\operatorname{conv}_{\mathrm{r}} X = \bigcap \{C \subseteq \mathbb{S}^n : X \subseteq C \text{ and } C \text{ is } r\text{-spindle-convex in } \mathbb{S}^n\}.$$

Furthermore, X is in an r-spindle-convex position *in* $\mathbb{S}^n$ *if, and only if,*

$$x \notin \mathbb{S}\operatorname{conv}_{\mathrm{r}}(X \setminus x), \text{ for any } x \in X.$$

2.7 Ball-Polyhedra

Recall that a convex polytope is the convex hull of a finite set of points, and that a convex polyhedron is the intersection of a finite set of half-spaces. Convex polyhedra may be unbounded, but when bounded, they are equivalent to convex polytopes. We define ball-polyhedra, the main subject of study throughout this thesis.

Definition 2.7.1 *Let $r \in \mathbb{R}$ such that $r > 0$ and $X \subset \mathbb{R}^n$ be a finite set such that $\mathrm{cr}_{\mathbb{R}}(X) \leq r$. We call $P = \mathbf{B}^n_{\mathbb{R}}[X, r] \neq \emptyset$, a* euclidean r-ball-polyhedron. *For any $x \in X$, we call $\mathbf{B}^n_{\mathbb{R}}[x, r]$, a* generating ball *of P, and $\mathbb{S}^{n-1}_{\mathbb{R}}(x, r)$, a* generating sphere *of P. If $n = 2$, then we say that a euclidean r-ball-polyhedron is a* euclidean r-disk-polygon.

Figure 2.8 shows how a euclidean ball-polyhedron is constructed as the intersection of four congruent balls.

Definition 2.7.2 *Let $r \in \mathbb{R}$ such that $r > 0$ and $X \subset \mathbb{H}^n$ be a finite set such that $\mathrm{cr}_{\mathbb{H}}(X) \leq r$. We call $P = \mathbf{B}^n_{\mathbb{H}}[X, r] \neq \emptyset$, a* hyperbolic r-ball-polyhedron. *For any*

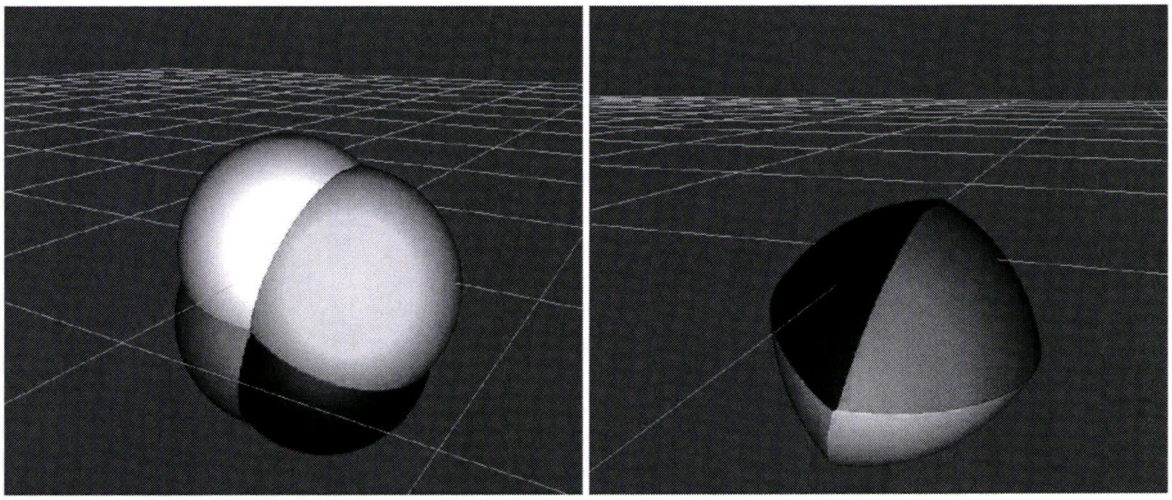

Figure 2.8: Euclidean Ball-Polyhedron in $\mathbb{R}^3$.

$x \in X$, we call $\mathbf{B}^n_{\mathbb{H}}[x, r]$, a generating ball *of P, and $\mathbb{S}^{n-1}_{\mathbb{H}}(x, r)$, a* generating sphere *of P. If $n = 2$, then we say that a hyperbolic r-ball-polyhedron is a* hyperbolic r-disk-polygon.

Definition 2.7.3 *Let $r \in (0, \pi/2]$ and $X \subset \mathbb{S}^n$ be a finite set such that $\mathrm{cr}_{\mathbb{S}}(X) \leq r$. We call $P = \mathbf{B}^n_{\mathbb{S}}[X, r] \neq \emptyset$, a* spherical r-ball-polyhedron. *For any $x \in X$, we call $\mathbf{B}^n_{\mathbb{S}}[x, r]$, a* generating ball *of P, and $\mathbb{S}^{n-1}_{\mathbb{S}}(x, r)$, a* generating sphere *of P. If $n = 2$, then we say that a spherical r-ball-polyhedron is a* spherical r-disk-polygon.

The dotted circles in Figure 2.9 are congruent in their respective space. Taking the intersections of the closures of these circles produces a disk-polygon in either $\mathbb{H}^2$ or $\mathbb{S}^2$.

2.8 Notation Simplified

It is evident from the preceding that the definitions, results and ensuing arguments vary little, if at all, with respect to the spaces $\mathbb{R}^n, \mathbb{H}^n$ and $\mathbb{S}^n$. This remains true of the

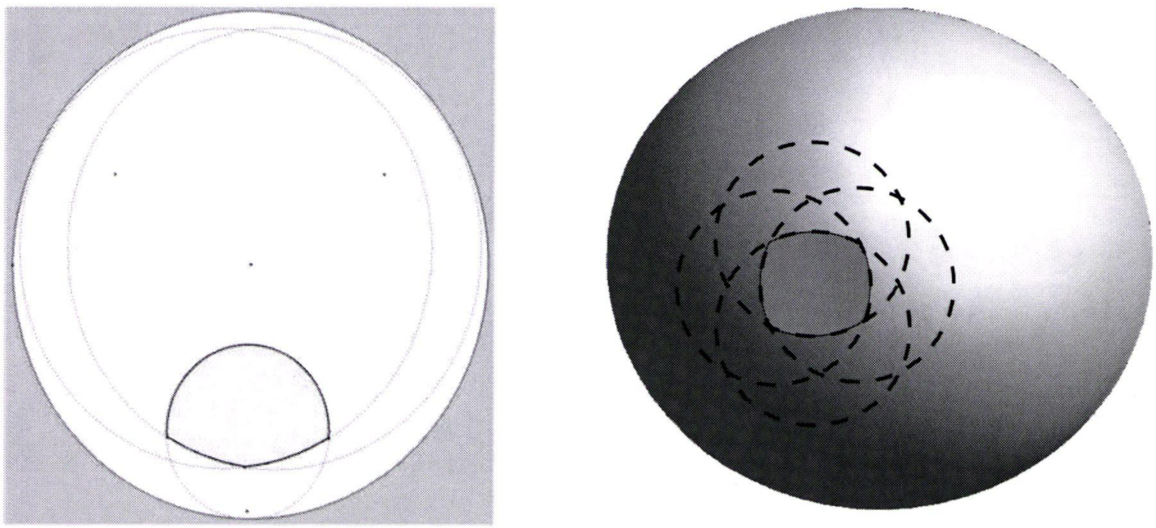

Figure 2.9: Example of a Hyperbolic Ball-Polyhedron in $\mathbb{H}^2$ and Spherical Ball-Polyhedron in $\mathbb{S}^2$.

results examined in the subsequent chapters of this thesis. Accordingly, we simplify the notation. Furthermore, with the new notation, we present only one statement of definitions and results which are valid in all three spaces. In general, only one argument or proof is necessary in these situations. When warranted, we distinguish results or arguments that vary between the three spaces. In certain situations, when it is necessary to distinguish space, dimension or radii, we utilize the more cumbersome notation developed in the preceding sections. Otherwise we use the following conventions.

We now introduce the *generic space* $Y \in \{\mathbb{R}^n, \mathbb{H}^n, \mathbb{S}^n : n \geq 1\}$. Many of the objects discussed in this chapter arose from spheres, balls and circular arcs. Each of these objects depends on some radius. In general for each space Y, there is some fixed radius, say r_Y; furthermore, r_Y lies in some interval, called the *radial domain of the space* Y, denoted D_Y. If $Y \in \{\mathbb{R}^n, \mathbb{H}^n : n \geq 1\}$, then $D_Y = (0, \infty]$ and $r_Y \in D_Y$. If $Y = \mathbb{S}^n, n \geq 1$, then $D_Y = (0, \pi/2]$ and $r_Y \in D_Y$. Finally, there arise situations where we do not allow $r_Y = \infty$, for $Y \in \{\mathbb{R}^n, \mathbb{H}^n\}$, or $r_Y = \pi/2$, for $Y = \mathbb{S}^n$. If $Y \in \{\mathbb{R}^n, \mathbb{H}^n : n \geq 1\}$, then let $\overline{D}_Y = (0, \infty)$. If $Y = \mathbb{S}^n, n \geq 1$, then let $\overline{D}_Y = (0, \pi/2)$. We call $\overline{D}_Y$ the *restricted radial domain of the space* Y.

Using this fixed radius, we construct the previously described objects. We frequently omit r_Y, with the understanding that it is in the correct interval. Also, in the case of objects, such as k-dimensional balls and spheres, which depend on a dimension argument we frequently omit the dimension when it is clear from the context. Table 2.8 summarizes the notational conventions used in the remainder of the thesis.

Object	Notation	Notes
Origin of Y.	o	Depends on Y.
Distance between $a, b \in Y$.	$d_Y(a, b)$	Depends on Y.
Open Line Segment between $a, b \in Y$. Closed Line Segment between $a, b \in Y$.	$(a, b)_Y$ $[a, b]_Y$	Depends on Y.
k-dimensional Open Ball centered at $a \in Y$. k-dimensional Closed Ball centered at $a \in Y$. k-dimensional Sphere centered at $a \in Y$. k-dimensional Open Ball Operator on $X \subset Y$. k-dimensional Closed Ball Operator on $X \subset Y$.	$\mathbf{B}^k_Y(a)$ or $\mathbf{B}_Y(a)$ $\mathbf{B}^k_Y[a]$ or $\mathbf{B}_Y[a]$ $\mathbb{S}^k_Y(a)$ or $\mathbb{S}_Y(a)$ $\mathbf{B}^k_Y(X)$ or $\mathbf{B}_Y(X)$ $\mathbf{B}^k_Y[X]$ or $\mathbf{B}_Y[X]$	Depends on Y. Depends on r_Y. Depends on k, whenever k is omitted.
Open Spindle between $a, b \in Y$. Closed Spindle between $a, b \in Y$.	$\mathrm{spin}(a, b)_Y$ $\mathrm{spin}[a, b]_Y$	Depends on Y. Depends on r_Y.
Open Spindle-Geodesic between $a, b \in Y$. Closed Spindle-Geodesic between $a, b \in Y$.	$\mathrm{geo}(a, b)_Y$ $\mathrm{geo}[a, b]_Y$	Depends on Y. Depends on r_Y.
Spindle-Distance in Y between $a, b \in Y$.	$\rho_Y(a, b)$	Depends on r_Y.
Circumradius of $X \subset Y$.	cr_Y	Depends on r_Y.
r-Spindle-Convex Hull of $X \subset Y$.	$\mathrm{conv}_{(\mathrm{r},\mathrm{Y})}\, X$	Depends on r,Y.

Table 2.1: Summary of Simplified Notation.

To demonstrate the new notation we restate Remarks 2.3.3, 2.4.4 and 2.5.3 and examine the proof in detail here.

Remark 2.8.1 *Let* $r_Y \in \overline{D}_Y$. *If* $d_Y(a, b) \leq 2r_Y$, *then* $\mathrm{spin}[a, b, r_Y]_Y = \mathbf{B}^n_Y[\mathbf{B}^n_Y[\{a, b\}, r_Y], r_Y]$, *and* $\mathrm{spin}(a, b, r_Y)_Y = \mathbf{B}^n_Y(\mathbf{B}^n_Y[\{a, b\}, r_Y], r_Y)$.

Proof. We prove $\operatorname{spin}[a, b, r_Y]_Y = \mathbf{B}_Y^n[\mathbf{B}_Y^n[\{a, b\}, r_Y], r_Y]$; the other equality is similar. The set $\mathbf{B}_Y^n[\mathbf{B}_Y^n[\{a, b\}, r_Y], r_Y]$ is the intersection of spindle-convex sets and is therefore spindle-convex. Since it contains a and b, it contains the spindle $\operatorname{spin}[a, b, r_Y]_Y$.

Suppose that $x \in \mathbf{B}_Y^n[\mathbf{B}_Y^n[\{a, b\}]]$, then for every $c \in \mathbf{B}_Y^n[\{a, b\}, r_Y]$ we note that $x \in \mathbf{B}_Y^n[c, r_Y]$. If $Y \in \{\mathbb{R}^n, \mathbb{S}^n\}$ and there is $\operatorname{geo}[a, b, r]_Y$, with $r \geq r_Y$ such that $x \in \operatorname{geo}[a, b, r]_Y$ then $\operatorname{spin}[a, b, r_Y]_Y \supseteq \mathbf{B}_Y^n[\mathbf{B}_Y^n[\{a, b\}, r_Y], r_Y]$ and there is nothing to prove. If $Y = \mathbb{H}^n$ and there is $\operatorname{geo}[a, b, r]_{\mathbb{R}^n}$, with $r \geq \bar{r}$ where $\bar{r}$ is the euclidean radius of the circle in the Poincaré Ball with hyperbolic radius r_Y such that $x \in \operatorname{geo}[a, b, r]_{\mathbb{R}^n}$ then $\operatorname{spin}[a, b, r_Y]_Y \supseteq \mathbf{B}_Y^n[\mathbf{B}_Y^n[\{a, b\}, r_Y], r_Y]$ and again there is nothing to prove.

Suppose, for a contradiction, that the points a, b and x determine a circle with radius $r < r_Y$ and the plane, H, determined by the points a, b and x contains a point c such that $x \notin \mathbf{B}_Y^2[c, r_Y]$ but $\{a, b\} \in \operatorname{bd} \mathbf{B}_Y^2[c, r_Y]$. There are two circles of radius r_Y in H passing through a and b. One circle has its center in the same half plane in H determined by the line through a and b containing x and the other has its center in the half plane in H determined by the line through a and b not containing x. The latter center is the point c. Now, $\{a, b\} \in \mathbf{B}_Y^n[c, r_Y]$ which implies $c \in \mathbf{B}_Y^n[\{a, b\}, r_Y]$ but $x \notin \mathbf{B}_Y^n[c, r_Y]$ and this contradicts our earlier observation. ∎

Chapter 3

Planar Results

3.1 Introduction

In this chapter, we explore several fascinating planar results pertaining to spindle-convexity and related objects. Since the setting is the plane, $Y \in \{\mathbb{R}^2, \mathbb{H}^2, \mathbb{S}^2\}$. Unless otherwise specified, we assume $r_Y \in D_Y$.

The euclidean version of Lemma 3.2.2 and a special case in euclidean space of Corollary 3.4.2 are found in [10]. Furthermore, the definition of spindle-polygons and the corresponding proofs presented here expand on the ideas of [10], [28], [24] and [12]. These results are applied in Lemma 3.5.2.

3.2 Spindle-distance is Not a Metric.

Thus far we have referred to the length of a spindle-geodesic as spindle-distance, and this reference might casually suggest that it is a metric. We now develop a series of results which we combine to demonstrate that the spindle-distance is not a metric. This idea is a very important consideration in many of the results examined later.

Claim 3.2.1 *[Arm Lemma] Let a, c, b and b' be points in Y such that*

(1) b and b' lie in the same open half plane bounded by the line through a and c,

(2) $d_Y(b, c) = d_Y(b', c)$ and $\angle acb < \angle acb'$.

Then, $d_Y(a, b) < d_Y(a, b')$.

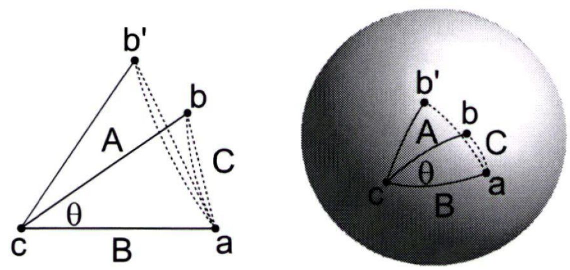

Figure 3.1: A Demonstration of the Arm Lemma.

The left diagram in Figure 3.1 demonstrates the configuration of Claim 3.2.1 in the euclidean plane, where the sides opposite c are the dashed line segments. In the Poincaré Disk Model, this figure accurately demonstrates the hyperbolic case after c is mapped to the origin of the hyperbolic plane. In this case the sides opposite c are the dashed circular arcs. The diagram on the right in Figure 3.1 demonstrates the configuration of Claim 3.2.1 in the spherical plane. The lengths, in Y, of the sides opposite a, b and c are denoted by A, B and C, respectively. We show that if the angle θ increases, which corresponds with b moving to b', then the length C increases.

Proof. For the triangle determined by the points a, b and c in Y let A (resp. B, resp. C) be the length of the side opposite the vertex a (resp. b, resp. c). Thus, $A = d_Y(b,c)$, $B = d_Y(a,c)$ and $C = d_Y(a,b)$. Furthermore, let θ be the angle $\angle acb$. Since the models we are using are conformal, we see that $\theta \in (0,\pi)$ in all three spaces. See Figure 3.1.

The euclidean law of cosines states that $C^2 = A^2+B^2-2AB\cos\theta$. The hyperbolic law of cosines states that $\cosh C = \cosh A \cosh B - \sinh A \sinh B \cos\theta$. And the spherical law of cosines states that $\cos C = \cos A \cos B + \sin A \sin B \cos\theta$. Thus, in all three spaces, we may express C as a function of θ and readily obtain that $\frac{dC}{d\theta} > 0$.

Since C is an increasing function of θ it follows that $\angle acb < \angle acb'$ implies that $d_Y(a, b) < d_Y(a, b')$. ■

The spindle-distance is not a metric in general. However, it is a metric in the cases $Y \in \{\mathbb{R}^2, \mathbb{H}^2\}$ and $r_Y = \infty$ or $Y = \mathbb{S}^n$ and $r_Y = \pi/2$. In these instances, spindle-distance corresponds to the straight line distance. The remaining possibilities require that r_Y be in the restricted radial domain and are examined in the next lemma.

Lemma 3.2.2 *Let* $r_Y \in \overline{D}_Y$ *and* $a, b, c \in Y$ *be points such that* $d_Y(a, b) \leq 2r_Y, d_Y(a, c) \leq 2r_Y$, *and* $d_Y(b, c) \leq 2r_Y$. *Then,*

(i) $\rho_Y(a, b) + \rho_Y(b, c) > \rho_Y(a, c) \iff b \notin \operatorname{spin}[a, c]_Y$;

(ii) $\rho_Y(a, b) + \rho_Y(b, c) = \rho_Y(a, c) \iff b \in \operatorname{bd}\operatorname{spin}[a, c]_Y$;

(iii) $\rho_Y(a, b) + \rho_Y(b, c) < \rho_Y(a, c) \iff b \in \operatorname{spin}(a, c)_Y$.

Proof. We begin by examining the case $b \in \operatorname{bd}\operatorname{spin}[a, c]_Y$. The boundary of $\operatorname{spin}[a, c]_Y$ is the union of two closed spindle-geodesics connecting a to c. Thus, b is an element of one of these spindle-geodesics, say $\operatorname{geo}[a, c]_Y$. Furthermore, $\operatorname{geo}[a, c]_Y$ may be expressed as the union of two closed spindle-geodesics. One of these spindle-geodesics connects a to b, denoted $\operatorname{geo}[a, b]_Y$, and the other connects b to c, denoted $\operatorname{geo}[b, c]_Y$. Since $\operatorname{geo}[a, b]_Y$ and $\operatorname{geo}[b, c]_Y$ have only the point b in common, it readily follows that

$$\rho_Y(a, b) + \rho_Y(b, c) = \rho_Y(a, c).$$

Next, we examine the case $b \notin \operatorname{spin}[a, c]_Y$. In this setting there are two possibilities, either

$$d_Y(a, c) \leq d_Y(a, b) \text{ or } d_Y(a, c) > d_Y(a, b).$$

Figure 3.2 demonstrates the case where $b \notin \operatorname{spin}[a, c]_Y$ and $d_Y(a, c) > d_Y(a, b)$. On the left, is the euclidean case. In the Poincaré Disk Model, the figure on the

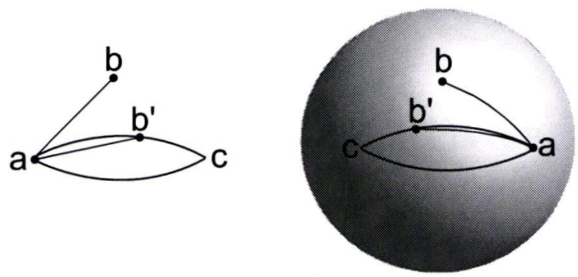

Figure 3.2: A Demonstration for Lemma 3.2.2.

left accurately demonstrates the hyperbolic case after c is mapped to the origin of the hyperbolic plane. The figure on the right demonstrates the case in the spherical plane.

If $d_Y(a,c) \leq d_Y(a,b)$ then, depending on which space Y is, one of the Remarks 2.3.7, 2.4.8 or 2.5.7 may be used to show that $\rho_Y(a,b) \geq \rho_Y(a,c)$. Consequently, the desired inequality,

$$\rho_Y(a,b) + \rho_Y(b,c) > \rho_Y(a,c),$$

follows immediately. If $d_Y(a,c) > d_Y(a,b)$, then we proceed in the following manner. Let $\mathrm{geo}(a,c)_Y$ be the open spindle-geodesic which lies in the half plane containing b and bounded by the line through a and c. Now, rotate the point b about a, keeping $d_Y(a,b)$ constant, until it intersects $\mathrm{geo}(a,c)_Y$ in a new point which we label b', see Figure 3.2. Note that

$$\rho_Y(a,b) = \rho_Y(a,b'). \tag{3.2.1}$$

Using the Arm Lemma (Claim 3.2.1) we obtain $d_Y(b,c) > d_Y(b',c)$. Then, depending on which space Y is, one of the Remarks 2.3.7, 2.4.8, 2.5.7 may be applied to obtain

$$\rho_Y(b,c) > \rho_Y(b',c). \tag{3.2.2}$$

Now, $b' \in \operatorname{bd}\operatorname{spin}[a, c]_Y$ is a case we have already examined and established that,

$$\rho_Y(a, b') + \rho_Y(b', c) = \rho_Y(a, c). \tag{3.2.3}$$

Finally, by combining 3.2.1, 3.2.2, 3.2.3 we obtain

$$\rho_Y(a, b) + \rho_Y(b, c) > \rho_Y(a, b') + \rho_Y(b', c) = \rho_Y(a, c).$$

The proof that $b \in \operatorname{spin}(a, c)_Y$ implies $\rho_Y(a, b) + \rho_Y(b, c) < \rho_Y(a, c)$ is a similar application of the Arm Lemma.

For a contradiction, suppose $\rho_Y(a, b) + \rho_Y(b, c) > \rho_Y(a, c)$ and $b \in \operatorname{spin}[a, c]_Y$. Then, $b \in \operatorname{spin}[a, c]_Y$ implies

$$\rho_Y(a, b) + \rho_Y(b, c) \leq \rho_Y(a, c)$$

by the first part of the proof, which provides an immediate contradiction. The remaining two implications are analogous. ∎

3.3 Spindle-Polygons

Let V be the vertex set and E the edge set of a graph $G = (V, E)$. We assume that the graph G is drawn in Y, meaning that $V \subset Y$. Thus, the vertex set is a collection of points, say $V = \{x_0, x_1, \ldots, x_m\}$, where x_i is a point in the plane Y for each i. The collection of edges satisfies

$$E \subseteq \{\{x_i, x_j\} : x_i, x_j \in V \text{ and } i \neq j\}$$

and any two vertices lying on an edge are joined by a continuous curve in Y.

Suppose that the graph G satisfies the following properties. (1) G is a Hamiltonian circuit. (2) $d_Y(x_i, x_j) \leq 2r_Y$ for each edge $\{x_i, x_j\} \in E$. Property (2) ensures that each pair of vertices lying on an edge of G may be joined by a spindle-geodesic. In fact, there are two spindle geodesics joining such vertices. Suppose that for each edge $\{x_i, x_j\} \in E$, we select one closed spindle-geodesic, say $\text{geo}[x_i, x_j]_Y$. Let $\widehat{G}$ be the union of these spindle-geodesics. In particular,

$$\widehat{G} = \bigcup_{\{x_i, x_j\} \in E} \text{geo}[x_i, x_j]_Y.$$

Definition 3.3.1 *If $Y = \mathbb{R}^2$ (resp. $Y = \mathbb{H}^2$, resp. $Y = \mathbb{S}^2$), then $\widehat{G}$ is called a euclidean (resp. hyperbolic, resp. spherical) r_Y-spindle-polygon corresponding to the graph $G(V, E)$.*

The object $\widehat{G}$ depends on the space Y, the radius r_Y and the graph $G(V, E)$. In this context, where we are using the generic space Y and understand these dependencies, we simply call $\widehat{G}$ a *spindle-polygon corresponding to the graph G* or even more simply a *spindle-polygon.* The points $x_0, x_1, \ldots, x_m \in V$ are the *vertices* of the spindle-polygon. The spindle-polygon $\widehat{G}$ is a union of spindle-geodesics. The collection of these spindle-geodesics,

$$\widehat{E} = \{\text{geo}[x_i, x_j]_Y : \{x_i, x_j\} \in E\},$$

is called the *side set* of the spindle-polygon. The spindle-geodesics in $\widehat{E}$ are the *sides*, of the spindle-polygon. Clearly, the elements in the collection of sides $\widehat{E}$ of the spindle-polygon $\widehat{G}$ is in one-to-one correspondence with the elements in the collection of edges E of the graph G.

Definition 3.3.2 *The graph $G = (V, E)$ is the* underlying graph *of $\widehat{G}$. The* underlying polygon *of $\widehat{G}$, which we denote by $\overline{G}$, is the union of all line segments between each pair of vertices joined by an edge of G. In particular,*

$$\overline{G} = \bigcup_{\{x_i, x_j\} \in E} [x_i, x_j]_Y.$$

A spindle-polygon is called regular *if the underlying polygon is regular.*

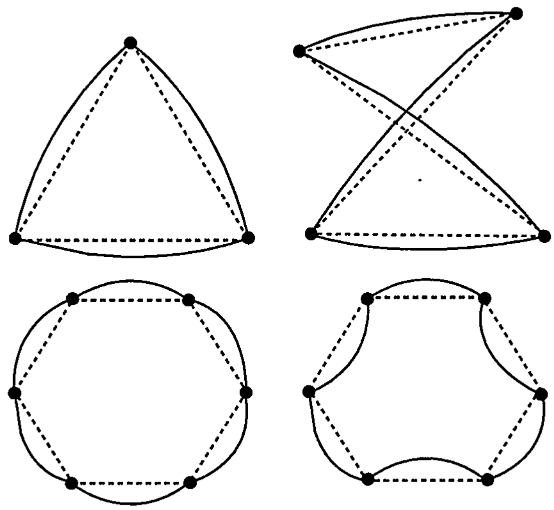

Figure 3.3: Euclidean Spindle-Polygons.

This collection of spindle-polygons in Figure 3.3 is located in the euclidean plane. The underlying polygons are demonstrated using dashed lines. Notice self intersections are possible. The two six sided spindle-polygons demonstrate how the same graph is the underlying graph of two distinct spindle-polygons.

Let $r_Y \in \overline{D}_Y$ and suppose that the graph $G = (V, E)$ is the underlying graph of the r_Y-spindle-polygon $\widehat{G}$. Now, for any edge in E, say $\{a, b\}$, there are two spindle-geodesics joining a and b, say g_1 and g_2. One of these spindle-geodesics is a side of $\widehat{G}$, say $g_1 \subset \widehat{G}$. Replacing g_1 by g_2 produces a new spindle-polygon, distinct from $\widehat{G}$. If $r_Y = \infty$ (resp. $r_Y = \infty$, resp. $r_Y = \pi/2$) when $Y = \mathbb{R}^2$ (resp. $Y = \mathbb{H}^2$, resp. $Y = \mathbb{S}^2$), then this does not occur because there exists a unique spindle-geodesic joining the vertices of any edge in E. We collect these observations in the following remark.

Remark 3.3.3 *If $r_Y = \infty$ (resp. $r_Y = \infty$, resp. $r_Y = \pi/2$) when $Y = \mathbb{R}^2$ (resp. $Y = \mathbb{H}^2$, resp. $Y = \mathbb{S}^2$) then $\widehat{G}$ is a polygon in $\mathbb{R}^2$ (resp. $\mathbb{H}^2$, resp. $\mathbb{S}^2$). In this* *case, there exists a unique spindle-polygon corresponding to the graph $G = (V, E)$. If $r_Y \in \overline{D}_Y$, then there exist more than one distinct spindle-polygons corresponding to the graph $G = (V, E)$.*

Recall that a set of points $X \subset Y$ is in spindle-convex position in Y if $x \notin \mathrm{conv}_{(\mathrm{r},\mathrm{Y})}(X \setminus x)$ for all $x \in X$.

Definition 3.3.4 *Let $\widehat{G}$ be a spindle-polygon in Y, where $G = (V, E)$ is the underlying graph. Suppose $V = \{x_0, x_1, \ldots, x_m\}$ is in spindle-convex position in Y. Then, we call $\widehat{G}$ a* weakly spindle-convex spindle-polygon. *Suppose $\widehat{G}$ is a weakly spindle-convex spindle-polygon and the underlying polygon $\overline{G}$ has no self-intersections. Then, we call $\widehat{G}$ a* cyclic spindle-polygon. *Finally, if $\widehat{G}$ is a cyclic spindle-polygon such that it is the boundary of the spindle-convex hull of $\widehat{G}$, then $\widehat{G}$ is called a* spindle-convex spindle-polygon.

A weakly spindle-convex spindle-polygon need not be a spindle-convex spindle-polygon, only the vertices are in spindle-convex position. Furthermore, a cyclic spindle-polygon $\widehat{G}$ may have sides which intersect. It is not necessary for $\widehat{G}$ to be devoid of self-intersections, only that the underlying polygon $\overline{G}$ have no self-intersections. See Figure 3.4 for examples.

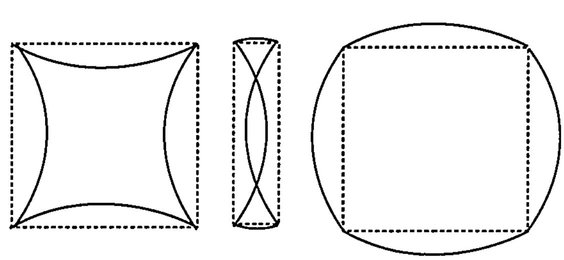

Figure 3.4: Weakly Spindle-Convex Spindle-Polygons.

The solid arcs in Figure 3.4 represent the spindle polygon and the underlying polygons are denoted by the dashed lines. All three of these four-sided spindle-polygons are weakly spindle-convex, because the vertices are in spindle convex position. Even though the sides of the second one intersect the underlying polygon has no self-intersection and therefore all three are cyclic. However, only the third one is spindle-convex.

Suppose $V = \{x_0, x_1, \ldots, x_m\}$ is the vertex set of an arbitrary weakly spindle-convex spindle-polygon $\widehat{G}$. Without loss of generality, we assume that the vertices of $\widehat{G}$ are labeled based on their cyclic order of appearance on the boundary of the spindle-convex hull of V. Thus, the the edge set of a cyclic spindle-polygon is simply

$$E = \{\{x_0, x_1\}, \{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{m-1}, x_m\}, \{x_m, x_0\}\}.$$

3.4 A Result on Spindle-Quadrilaterals

For simplicity, we call a weakly spindle-convex spindle-polygon with four vertices a *spindle-quadrilateral.* Recall that a convex quadrilateral in the plane Y is a polygon with four vertices such that no vertex is contained in the Y convex hull of the other. three. In the case that $r_y = \infty$ when $Y = \mathbb{R}^2$, $r_y = \infty$ when $Y = \mathbb{H}^2$ or $r_y = \pi/2$ when $Y = \mathbb{S}^2$, a spindle-quadrilateral is just a convex quadrilateral and vice versa.

Lemma 3.4.1 states that in a convex quadrilateral the total length of the diagonals is greater than the total length of an opposite pair of sides. The generalization to spindle-quadrilaterals, Corollary 3.4.2, requires the additional constraint that the circumradius of the vertices be at most r_Y. This ensures that there is a spindle-geodesic joining the diagonal vertices. For example, consider a square in the euclidean plane with euclidean side length 2. Now, join each pair of points connected by a side of the square with a 1-spindle-geodesic. In this way we obtain a euclidean 1-spindle-polygon, see Figure 3.5. However, the euclidean distance between a pair of diagonal vertices is greater than 2.82. The longest 1-spindle-geodesic joining two points in $\mathbb{R}^2$ is π and the maximum straight line distance between these two points is 2. Consequently, there is no spindle-geodesic joining vertices lying on a diagonal. The vertices, of the spindle-quadrilateral in Figure 3.5, lying along a diagonal cannot be connected by a spindle-geodesic, because they are too far apart.

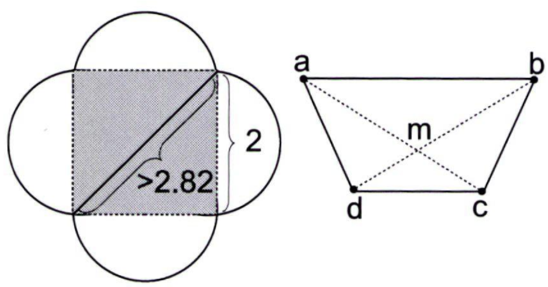

Figure 3.5: A Quadrilateral and a Spindle-Quadrilateral.

The points $a, b, c, d \in Y$ are the vertices of an arbitrary spindle-quadrilateral and they are given *in this cyclic order*; which means that, without loss of generality, they are labeled in their clockwise order of appearance on $\operatorname{bd}\operatorname{conv}_{(\mathrm{r},\mathrm{Y})}\{a, b, c, d\}$.

Lemma 3.4.1 *Let $a, b, c, d \in Y$ be vertices of a convex quadrilateral, in this cyclic order. Then,*

$$d_Y(a,c) + d_Y(b,d) > d_Y(a,b) + d_Y(c,d) \text{ and}$$
$$d_Y(a,c) + d_Y(b,d) > d_Y(a,d) + d_Y(b,c).$$

That is to say, the total length of the diagonals is greater than the total length of an opposite pair of sides.

Proof. Let m be the intersection of the diagonal line segments, namely $m = [a,c]_Y \cap [b,d]_Y$, see Figure 3.5. By the triangle inequality $d_Y(a,m) + d_Y(m,b) > d_Y(a,b)$

and $d_Y(d,m) + d_Y(m,c) > d_Y(c,d)$ which implies

$$d_Y(a,c) + d_Y(b,d) = d_Y(a,m) + d_Y(m,c) + d_Y(m,b) + d_Y(d,m) > d_Y(a,b) + d_Y(c,d).$$

The other inequality is proved similarly. ■

Corollary 3.4.2 *Let $a, b, c, d \in Y$ be vertices of a spindle-quadrilateral, in this cyclic order, where* $\mathrm{cr}_Y\{a,b,c,d\} \leq r_Y$. *Then,*

$$\rho_Y(a,c) + \rho_Y(b,d) > \rho_Y(a,b) + \rho_Y(c,d) \text{ and}$$
$$\rho_Y(a,c) + \rho_Y(b,d) > \rho_Y(a,d) + \rho_Y(b,c).$$

That is to say, the total spindle-distance between diagonal points is greater than the total spindle-distance between points joined by an opposite pair of sides.

Proof. We show the first inequality. The second is obtained from the first by permuting the labeling on the vertices and repeating the following argument. Suppose that $a, b, c, d \in Y$ are vertices of a spindle-quadrilateral in this cyclic order where $\mathrm{cr}_Y\{a,b,c,d\} \leq r_Y$. Since the vertices are in spindle-convex position, they are also in convex position. Thus, the underlying polygon is a convex quadrilateral.

Figure 3.6 provides a schematic representation of the configuration. It accurately reflects one possible configuration when $Y = \mathbb{R}^2$. The diagram can be used to infer the result in the remaining two spaces.

As noted earlier a, b, c, d are the vertices of a convex quadrilateral. By Lemma 3.4.1 one of the sides, $[a,b]_Y$ or $[c,d]_Y$, of the quadrilateral is shorter than one of the diagonals, $[a,c]_Y$ or $[b,d]_Y$, of the quadrilateral. Without loss of generality, we assume that $d_Y(a,b) < d_Y(b,d)$.

Since no three vertices are collinear, there exists a closed half-plane, bounded by the line through b and d, that contains a in its interior. Call this closed half-plane H.

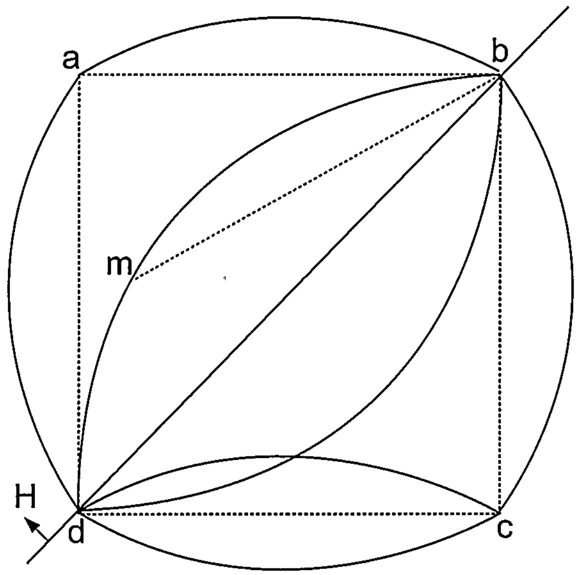

Figure 3.6: Schematic Spindle-Quadrilateral.

Let $\text{geo}[b, d]_Y$ be the spindle-geodesic joining b to d which is contained in H. Since $d_Y(a, b) < d_Y(b, d)$, there is a point $m \in \text{geo}(b, d)_Y$ such that

$$\rho_Y(b, m) = \rho_Y(a, b). \tag{3.4.1}$$

Because the points a, b, c, d are in spindle-convex position, a is not in the closed spindle $spin[b, d]_Y$. However, $m \in \text{geo}(b, d)_Y \subset spin[b, d]_Y$. So, a and m are distinct points. Thus, the acute angles $\angle mbd$ and $\angle abd$ are also distinct. Since a and m are in the interior of H and $a \notin spin[b, d]_Y$ we see that $\angle mbd < \angle abd$. Thus,

$$\angle mbc = \angle mbd + \angle dbc < \angle abd + \angle dbc = \angle abc.$$

Since our models of Y are conformal, the inequality $\angle mbc < \angle abc$ holds in all spaces. Finally, applying the arm-lemma gives $d_Y(m, c) < d_Y(a, c)$ which yields

$$\rho_Y(m, c) < \rho_Y(a, c). \tag{3.4.2}$$

If $m \notin \mathrm{spin}[c,d]_Y$, then by Lemma 3.2.2 $\rho_Y(c,d) < \rho_Y(d,m) + \rho_Y(m,c)$. Now, using 3.4.1, 3.4.2 and this last inequality we carry out the following computation,

$$\begin{aligned}\rho_Y(c,d) &< \rho_Y(d,m) + \rho_Y(m,c) \\ &< \rho_Y(d,m) + \rho_Y(a,c) = \rho_Y(b,d) - \rho_Y(m,b) + \rho_Y(a,c) \\ &\qquad\qquad\qquad\qquad\;\; = \rho_Y(b,d) - \rho_Y(a,b) + \rho_Y(a,c),\end{aligned}$$

and rearranging this expression gives the desired result

$$\rho_Y(a,b) + \rho_Y(c,d) < \rho_Y(a,c) + \rho_Y(b,d).$$

So, to complete the proof we need only to verify that $m \notin \mathrm{spin}[c,d]_Y$.

Suppose that x and y are vertices of the spindle-quadrilateral joined by a side and u and v are the remaining two vertices. Since the convex hull of the vertices forms a convex quadrilateral, x and y lie in the interior of a closed half plane bounded by the line through u and v. Denote this closed half plane by $H(u,v)$.

If $\mathrm{geo}[c,d]_Y$ is the spindle-geodesic contained in $H(c,d)$, then let $\mathbb{S}^1_Y(x,r_Y)$ be the circle containing $\mathrm{geo}[c,d]_Y$. By Remark 2.8.1, $\mathrm{spin}[c,d]_Y \subset \mathrm{cl}\,\mathbb{S}^1_Y(x,r_Y)$. Since we have already established that $m \in \mathrm{geo}(b,d)_Y$, to show that $m \notin \mathrm{spin}[c,d]_Y$ we need only demonstrate that $\mathrm{geo}(b,d)_Y \subset Y \setminus \mathrm{cl}\,\mathbb{S}^1_Y(x,r_Y)$.

The intersection of the disk $\mathrm{cl}\,\mathbb{S}^1_Y(x,r_Y)$ and the half plane $H(c,d)$ is a subset of $\mathrm{spin}[c,d]_Y$. Since $b \in H(c,d)$, if b was also in the disk $\mathrm{cl}\,\mathbb{S}^1_Y(x,r_Y)$, then

$$b \in \mathrm{cl}\,\mathbb{S}^1_Y(x,r_Y) \cap H(c,d) \subset \mathrm{spin}[c,d]_Y,$$

which contradicts the spindle-convexity of the vertices. Thus, $b \notin \mathrm{cl}\,\mathbb{S}^1_Y(x,r_Y)$ and if $\mathbb{S}^1_Y(y,r_Y)$ denotes the circle containing $\mathrm{geo}(b,d)_Y$, then $\mathbb{S}^1_Y(x,r_Y) \neq \mathbb{S}^1_Y(y,r_Y)$.

If $Y = \mathbb{R}^2$ then $\mathbb{S}^1_Y(x,r_Y)$ and $\mathbb{S}^1_Y(y,r_Y)$ are congruent circles. Thus, $\mathrm{geo}(b,d)_Y$ and $\mathrm{geo}[c,d]_Y$ have the same curvature. Since they are no longer than a semi-circle, it is readily apparent that they do not intersect. Hence, $\mathrm{geo}(b,d)_Y \subset Y \setminus \mathrm{cl}\,\mathbb{S}^1_Y(x,r_Y)$.

If $Y = \mathbb{H}^2$ then we may assume, without loss of generality, that d is at the origin, $o_{\mathbb{H}^n}$. By embedding the Poincaré Disc in the euclidean plane we see $\mathbb{S}^1_Y(x, r_Y)$ and $\mathbb{S}^1_Y(y, r_Y)$ are congruent as *euclidean* circles. Let the euclidean centers of $\mathbb{S}^1_Y(x, r_Y)$ and $\mathbb{S}^1_Y(y, r_Y)$ be $\bar{x}$ and $\bar{y}$, respectively, and let the common euclidean radius be $\bar{r}$. The preceding argument applied to the circles $\mathbb{S}^1_{\mathbb{R}}(\bar{x}, \bar{r})$ and $\mathbb{S}^1_{\mathbb{R}}(\bar{y}, \bar{r})$ in the euclidean plane yields the desired result.

If $Y = \mathbb{S}^2$ then consider the stereographic projection $\mathrm{proj} : \mathbb{S}^2 \to \mathbb{R}^2$ where $\mathrm{proj}(d) = o_{\mathbb{R}^n}$. This map takes the circles $\mathbb{S}^1_Y(x, r_Y)$ and $\mathbb{S}^1_Y(y, r_Y)$ to the circles $\mathrm{proj}(\mathbb{S}^1_Y(x, r_Y))$ and $\mathrm{proj}(\mathbb{S}^1_Y(y, r_Y))$, respectively, which are congruent as *euclidean* circles. Let the euclidean centers of $\mathrm{proj}(\mathbb{S}^1_Y(x, r_Y))$ and $\mathrm{proj}(\mathbb{S}^1_Y(y, r_Y))$ be $\bar{x}$ and $\bar{y}$, respectively, and let $\bar{r}$ be the common euclidean radius. The preceding argument applied to the circles $\mathrm{proj}(\mathbb{S}^1_Y(x, r_Y)) = \mathbb{S}^1_{\mathbb{R}}(\bar{x}, \bar{r})$ and $\mathrm{proj}(\mathbb{S}^1_Y(y, r_Y)) = \mathbb{S}^1_{\mathbb{R}}(\bar{y}, \bar{r})$ in the euclidean plane yields the desired result. ■

The main result obtained in this section is Corollary 3.4.2. It is used several times in this thesis and we have isolated it here in this section. Contrast the proof of Corollary 3.4.2 and the proof of the classical result from which it follows, Lemma 3.4.1.

3.5 Isoperimetric Inequalities

The following proposition, based on the work in [10], is a generalization of Corollary 3.4.2. The result says that, amongst all weakly spindle-convex spindle-polygons with the same vertex set the one with the least perimeter is a cyclic spindle-polygon.

Proposition 3.5.1 *Let $\widehat{G}$ be a cyclic spindle-polygon, where $G = (V, E)$ is the underlying graph. Furthermore, suppose that $\mathrm{cr}_Y V \leq r_Y$. Then, for all weakly spindle-*

polygons $\widehat{G}'$, with underlying graph $G' = (V, E')$,

$$\text{Perimeter}(\widehat{G}') \geq \text{Perimeter}(\widehat{G}).$$

Equality is achieved if, and only if, $\widehat{G}'$ is a cyclic spindle-polygon.

Proof. Let the collection of vertices be $V = \{x_0, x_1, \ldots, x_m\}$ and suppose that $\text{cr}_Y V \leq r_Y$. If $m = 3$, then the result is trivial. If $m = 4$, then the result follows from Corollary 3.4.2. Thus, we may assume $m \geq 5$.

Recall that the underlying graph of a spindle-polygon with vertices in V is a Hamiltonian cycle on the vertices. Since $\text{cr}_Y V \leq r_Y$, every Hamiltonian cycle on V corresponds to a spindle-polygon. Furthermore, there are only finitely many such Hamiltonian cycles, because V is finite. By examining each one, we find a spindle-polygon $\widehat{G}$ such that $\text{Perimeter}(\widehat{G})$ is minimal.

Let $G = (V, E)$ be the underlying graph of $\widehat{G}$. After possibly relabeling the vertices, we may assume the graph G is directed with edge set,

$$E = \{(x_0, x_1), (x_1, x_2), (x_2, x_3), \ldots, (x_{m-1}, x_m), (x_m, x_0)\}.$$

Given an edge (x_i, x_{i+1}) it is understood that the indices are taken modulo m.

Either there exist vertices x_i and x_{i+1} such that the line through them strictly separates two points in $V \setminus \{x_i, x_{i+1}\}$, or not. If this condition is not satisfied, then for each edge (x_i, x_{i+1}) there exists a closed half plane bounded by the line through x_i and x_{i+1} containing $V \setminus \{x_i, x_{i+1}\}$. Now, the vertices are in spindle-convex position and therefore they are in convex position. Consequently, no three vertices are collinear which implies that for each edge (x_i, x_{i+1}) the points in $V \setminus \{x_i, x_{i+1}\}$ lie in the open half plane, denoted $H(x_i, x_{i+1})$, bounded by the line through x_i and x_{i+1}. The intersection of all such half planes is a convex polyhedron,

$$I = H(x_0, x_1) \cap H(x_1, x_2) \cap \ldots \cap H(x_{m-1}, x_m) \cap H(x_m, x_0).$$

Since the vertices are in convex position, I is also the convex hull of the vertices. Thus, $\operatorname{bd} I$ is a polygon with vertex set V and from the construction of I,

$$\operatorname{bd} I = (x_0, x_1) \cup (x_1, x_2) \cup (x_2, x_3) \cup \ldots, (x_{m-1}, x_m) \cup (x_m, x_0).$$

In particular, $\operatorname{bd} I$ is the underlying polygon of $\widehat{G}$. Since the underlying polygon has no self intersections, $\widehat{G}$ is a cyclic spindle-polygon, and there is nothing left to prove.

Suppose the line through x_i and x_{i+1} strictly separates two points of $V \setminus \{x_i, x_{i+1}\}$, say x_j and x_k. Let H be the closed half plane bounded by the line through x_i and x_{i+1} containing x_j. Let $\operatorname{path}(x_a, x_b)$ denote the directed path joining x_a to x_b. In particular,

$$\operatorname{path}(x_a, x_b) = (x_a, x_{a+1}) \cup (x_{a+1}, x_{a+2}) \cup \ldots \cup (x_{b-1}, x_b).$$

If $\operatorname{path}(x_{i+1}, x_i) \subset H$ then the the Hamiltonian cycle on the vertices does pass through x_k. Hence, there exist vertices x_l and x_{l+1} in $V \setminus \{x_i, x_{i+1}\}$ such that either $x_l \in H$ and $x_{l+1} \in Y \setminus H$ or $x_l \in Y \setminus H$ and $x_{l+1} \in H$. Without loss of generality, assume that it is the former.

Starting at x_i we move to x_{i+1} along an edge, see Figure 3.7. Now, following a sequence of edges, $\operatorname{path}(x_{i+1}, x_l)$, we reach x_l. From x_l we go to x_{l+1} along an edge. Finally, x_{l+1} is joined to x_i by $\operatorname{path}(x_{l+1}, x_i)$, passing through the remaining vertices. Thus, the underlying graph, which is a (directed) Hamiltonian cycle, is given by

$$\{(x_i, x_{i+1})\} \cup \operatorname{path}(x_{i+1}, x_l) \cup \{(x_l, x_{l+1})\} \cup \operatorname{path}(x_{l+1}, x_i).$$

Since x_i and x_l are not already joined by an edge and neither are x_{i+1} and x_{l+1}, we replace the edges (x_i, x_{i+1}) and (x_l, x_{l+1}) by (x_i, x_l) and (x_{i+1}, x_{l+1}), respectively. If we permute the labeling of the vertices so that $0 \leq i+1 < l \leq m$, then the inequality $i+1 \leq p \leq l$ is unambiguous. We replace the directed edges (x_p, x_{p+1}),

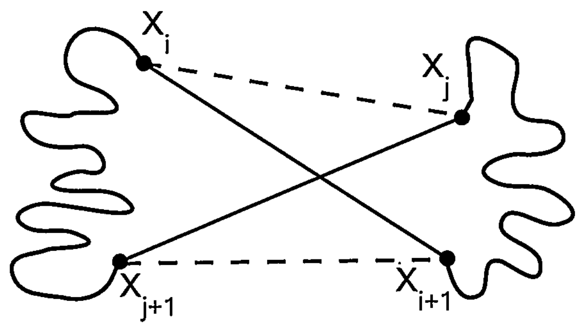

Figure 3.7: Schematic of the edge replacement described in Proposition 3.5.1.

where $i+1 < p < l$, by (x_{p+1}, x_p). Informally, we reverse the direction of the path $\text{path}(x_{i+1}, x_l)$ to get the new path $\text{path}(x_l, x_{i+1})$. This leads to a new (once again directed) Hamiltonian cycle

$$\overline{G}' = (x_i, x_l) \cup \text{path}(x_l, x_{i+1}) \cup (x_{i+1}, x_{l+1}) \cup \text{path}(x_{l+1}, x_i).$$

We denote the edge set of the graph corresponding to this new Hamiltonian cycle by E'. The new graph is denoted $G' = (V, E')$. As we noted earlier, every Hamiltonian cycle on V corresponds to a spindle-polygon. In this case we obtain $\widehat{G}'$ a spindle-polygon with underlying graph $G' = (V, E')$.

Finally, if we view x_i, x_{i+1}, x_j and x_{j+1} as the vertices of a spindle-quadrilateral, then the procedure described above amounts to replacing the diagonals, (x_i, x_{i+1}) and (x_l, x_{l+1}), by a pair of opposite sides, (x_i, x_l) and (x_{i+1}, x_{l+1}). Corollary 3.4.2 ensures

$$\text{Perimeter}(\widehat{G}) > \text{Perimeter}(\widehat{G}').$$

This contradicts the minimality of Perimeter$(\widehat{G})$. ■

Next, we examine theorems concerning cyclic spindle-polygons that are analogous to those studied by Dowker in [24] and L. Fejes Tóth in [28] for polygons. The arguments extend the ideas of ([28], pp.162-170), by using [10] and [21].

Let $C = \mathbb{S}^1_Y(o_{Y^2}, r)$ be a circle of radius $r < r_Y$. A spindle-polygon $\widehat{G}$, with underlying graph $G = (V, E)$, is *inscribed in* C if, and only if, $V \subset C$. Notice that the condition $r < r_Y$ implies $\widehat{G} \subset \operatorname{conv} C$, which is just the disk $\mathbf{B}^2_Y(o_{Y^2}, r)$. For each positive integer n, a standard compactness argument ensures the existence of an n-sided cyclic spindle-polygon of largest perimeter inscribed in C.

Lemma 3.5.2 *Let C be a circle of radius $r < 1$. Let P_n, where n is a positive integer, be an n-sided cyclic spindle-polygon of largest perimeter inscribed in C. Then,*

$$\operatorname{Perimeter}(P_{n-1}) + \operatorname{Perimeter}(P_{n+1}) < 2\operatorname{Perimeter}(P_n), \textit{ for all } n \geq 4. \tag{3.5.1}$$

Proof. Let $\widehat{Q}$ be an $(n-1)$-sided cyclic spindle-polygon and $\widehat{R}$ be an $(n+1)$-sided cyclic spindle-polygon, both inscribed in C. To prove the theorem we need only construct two n-sided cyclic spindle-polygons $\widehat{S}$ and $\widehat{T}$ such that

$$\operatorname{Perimeter}(\widehat{Q}) + \operatorname{Perimeter}(\widehat{R}) \leq \operatorname{Perimeter}(\widehat{S}) + \operatorname{Perimeter}(\widehat{T}). \tag{3.5.2}$$

Without loss of generality, inscribe $\widehat{Q}$ and $\widehat{R}$ into C so that their respective vertices do not coincide. Any arc of C with length at least πr contains a vertex from each of $\widehat{Q}$ and $\widehat{R}$. Otherwise, there exists an $(n-1)$-sided cyclic spindle-polygon (resp. $(n+1)$-sided cyclic spindle-polygon) with larger perimeter than $\widehat{Q}$ (resp. $\widehat{R}$).

Suppose that a, b, l and m are points on C such that a is joined to b by a spindle-geodesic and l is joined to m by a spindle-geodesic. Furthermore, suppose that the closed r-spindle-geodesics $\operatorname{geo}[a,b,r]_Y \subset C$ and $\operatorname{geo}[l,m,r]_Y \subset C$ satisfy $\operatorname{geo}[l,m,r]_Y \subset \operatorname{geo}[a,b,r]_Y$, see Figure 3.8. Let $\operatorname{seg}(a,b) = \operatorname{conv}([a,b]_Y \cup \operatorname{geo}[a,b,r]_Y)$ and $\operatorname{seg}(l,m) = \operatorname{conv}([l,m]_Y \cup \operatorname{geo}[l,m,r]_Y)$, both of which are just segments of the disc $\operatorname{conv} C$. Then,

$$\operatorname{seg}(l,m) \subset \operatorname{seg}(a,b).$$

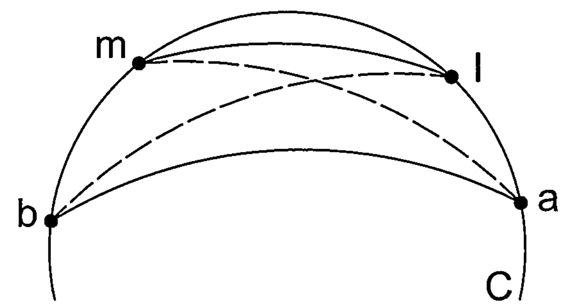

Figure 3.8: Schematic configuration for Lemma 3.5.2.

This configuration, where a segment is contained in another segment, may arise when two spindle-polygons, say $\widehat{G}_1$ and $\widehat{G}_2$, are inscribed in a single circle, or when a single self-intersecting spindle-polygon, say $\widehat{F}$, is inscribed in a circle. Suppose that we start with the former and, as in the figure, assume that the cyclic ordering of the vertices is a, l, m, b.

Let path(a, b) (resp. path(l, m)) be the collection of edges in the underlying graph of $\widehat{G}_1$ (resp. $\widehat{G}_2$) joining a to b (resp. l to m) not including the edge $\{a, b\}$ (resp. $\{l, m\}$). Let $\widehat{G}_3$ be the spindle-polygon with underlying graph

$$\text{path}(a, b) \cup [b, l]_Y \cup \text{path}(l, m) \cup [m, a]_Y.$$

Now, by Corollary 3.4.2,

$$\rho_Y(a, b) + \rho_Y(l, m) < \rho_Y(a, m) + \rho_Y(b, l).$$

Thus, replacing geo$[a, b]$ and geo$[l, m]$ with geo$[a, m]$ and geo$[b, l]$ results in a single, possibly self-intersecting, spindle-polygon. Furthermore, the total perimeter of $\widehat{G}_3$ is strictly larger than the total perimeter of $\widehat{G}_1$ and $\widehat{G}_2$. Finally, had we started with $\widehat{F}$ the same argument applied to $\widehat{F}$ results in the construction of two spindle-polygons, say $\widehat{F}_1$ and $\widehat{F}_2$, where the total perimeter of $\widehat{F}_1$ and $\widehat{F}_2$ is strictly larger than the total perimeter of $\widehat{F}$. This completes the description of an algorithm we use to carry out the proof.

Suppose $\widehat{F}$ and $\widehat{G}$ are cyclic spindle-polygons inscribed in a circle. Now, suppose we can apply the preceding algorithm at least twice to $\widehat{F}$ and $\widehat{G}$. In other words there are at least two instances where a segment is contained in another segment. On the even iterations we obtain two spindle-polygons and we claim that their underlying polygons have no self-intersections. Let $\overline{F} = (V_F, E_F)$ and $\overline{G} = (V_G, E_G)$ be the underlying graphs of $\widehat{F}$ and $\widehat{G}$, respectively. Suppose that the vertex sets $V_F = \{x_0, x_1, \ldots, x_\alpha\}$ and $V_G = \{y_0, y_1, \ldots, y_\beta\}$ are labeled based on the, say clockwise, order of appearance of the vertices on the circle. There are at least two instances of a segment contained in another. So, by a permutation of the vertices if necessary, one of

$$\operatorname{seg}(x_0, x_1) \subset \operatorname{seg}(y_0, y_1) \text{ or } \operatorname{seg}(y_0, y_1) \subset \operatorname{seg}(x_0, x_1)$$

holds. Furthermore, there exist integers i, j, l, m where $1 < i < j \leq \alpha$ and $1 < l < m \leq \beta$ such that one of

$$\operatorname{seg}(x_i, x_j) \subset \operatorname{seg}(y_l, y_m) \text{ or } \operatorname{seg}(y_l, y_m) \subset \operatorname{seg}(x_i, x_j)$$

holds. Regardless of which case we consider, after two applications of the algorithm we obtain two spindle-polygons with vertex sets

$$V_1 = \{x_0, y_1, y_2, \ldots, y_k, x_j, x_{j+1}, \ldots x_\alpha\} \text{ and}$$
$$V_2 = \{y_0, x_1, x_2, \ldots, x_i, y_l, y_{l+1}, \ldots y_\beta\}$$

In both vertex sets, the vertices appear in ascending order based on the clockwise ordering imposed earlier. Thus, neither underlying polygon has any self-intersections. Finally, suppose that after 2γ applications of the algorithm, where γ is a positive integer, we obtain two spindle-polygons where at least one of the underlying polygons has self-intersections. Then, on the $2\gamma-2$-th iteration we start with two cyclic spindle-polygons, but after two more applications of the algorithm they are no longer cyclic. This contradicts the preceding discussion. Hence, the two spindle-polygons obtained on every even iteration are cyclic.

Starting from the cyclic spindle-polygons $\widehat{Q}$ and $\widehat{R}$, we carry out the algorithm for each segment contained in another segment. After every odd numbered iteration of the algorithm we obtain a single, self-intersecting spindle-polygon and after every even numbered iteration we obtain two cyclic spindle-polygons. Since we have finitely many vertices and the perimeter increases strictly with each step, the algorithm terminates. A simple counting argument shows that the process terminates with two spindle-polygons. Since the algorithm terminates when there is no segment contained within another segment, the only possibility is that each of the the two spindle-polygons obtained is n-sided. Since the perimeter increased at each step of the process these n-sided cyclic spindle-polygons are the desired $\widehat{S}$ and $\widehat{T}$. ■

Theorem 3.5.3 *Let C be a circle of radius $r < 1$. Let P be an n-sided cyclic spindle-polygon of largest perimeter that can be inscribed in C. Then P is regular.*

Proof. Suppose that P is not regular. Starting with P and a suitable rotation of P we modify the argument in the proof of the preceding lemma to construct two n-sided cyclic spindle-polygons Q and R inscribed in C. By construction, $\text{Perimeter}(Q) + \text{Perimeter}(R) > 2\,\text{Perimeter}(P)$. Hence, one of Q or R has larger perimeter than P. ■

Theorem 3.5.4 *Let C be a circle of radius $r < 1$. Let P be an n-sided cyclic spindle-polygon of largest area that can be inscribed in C. Then P is regular.*

Proof. Suppose P is not regular. Let P_0 be the regular n-sided cyclic spindle-polygon with the same perimeter as P. By the discrete isoperimetric inequality for spindle-polygons proved in [21], $\mathrm{Area}(P) < \mathrm{Area}(P_0)$. Furthermore, by the preceding theorem, P_0 is inscribed in a circle C_0 with radius $r_0 < r$. Thus, P_1, the regular n-sided cyclic spindle-polygon inscribed in C, clearly satisfies $\mathrm{Area}(P_0) < \mathrm{Area}(P_1)$ which completes the proof. ■

Chapter 4

Generalizing Results in Convexity to Spindle Convexity

Inspired by classical results in the study of convex sets, this chapter attempts to generalize these ideas to spindle-convex sets.

4.1 Separation

The idea of supporting a convex set by a hyperplane or separating two convex sets by a hyperplane is of fundamental importance in the classical theory. Motivated by the following example, we would like to extend the idea of separation and support. Suppose that we have a sufficiently small convex body in $\mathbb{R}^3$ supported by a plane. We now replace our surface of zero curvature, the plane, with a surface that has non-zero curvature. We might wonder under which circumstances is this always possible? As first approximation we might try surfaces of constant curvature, spheres. In the current framework, where our spindle convex sets arise from intersections of balls this idea is very natural.

This section describes results dealing with the separation and support of spindle convex sets by spheres motivated by the basic facts about separation and support of convex sets by hyperplanes as they are introduced in standard textbooks; e.g., [17]. In this context $Y \in \{\mathbb{R}^n, \mathbb{H}^n, \mathbb{S}^n\}$, r_Y is fixed and in the appropriate radial domain. The spheres that we consider as potential candidates for support and separation have radius r_Y.

Lemma 4.1.1 *Let a spindle convex set $C \subset Y$ be supported by the hyperplane H in*

Y at $x \in \operatorname{bd} C$. Then, the closed ball of radius r_Y supported by H at x and lying in the same half-space as C contains C.

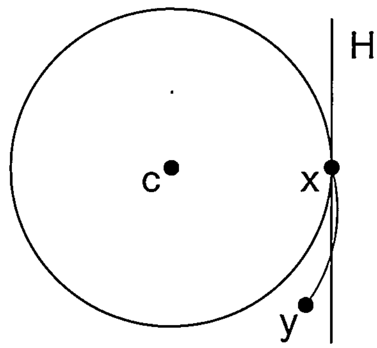

Figure 4.1: Support by a ball.

Proof. We use the following initial setup in all spaces Y. Let $\mathbf{B}^n_Y[c]$ be the ball of radius r_Y that is supported by H at x and is in the same closed half-space bounded by H as C. We show that $\mathbf{B}^n_Y[c]$ is the desired ball. Assume that C is not contained in $\mathbf{B}^n_Y[c]$. So, there is a point $y \in C$, $\;y \notin \mathbf{B}^n_Y[c]$.

Consider the euclidean case $Y = \mathbb{R}^n$. By taking the intersection of the configuration with the plane that contains x, y and c, call this plane P, we reduce the general problem to a planar problem. The intersection of P and $\mathbf{B}^n_Y[c]$ is a disk, $\mathbf{B}^2_Y[c, r_Y]$, which contains x. Furthermore, $P \cap H$ is a line in P that supports $\mathbf{B}^2_Y[c, r_Y]$ at x (Figure 4.1). Let P^+ be the open half-plane contained in P bounded by the line $P \cap H$ that does not contain $\mathbf{B}^2_Y[c, r_Y]$. Now, the spindle-geodesic connecting x and y that does not intersect $\mathbf{B}^2_Y(c, r_Y)$ intersects P^+. Hence, H cannot be a supporting hyperplane of C at x, a contradiction.

Next, consider the hyperbolic case $Y = \mathbb{H}^n$. We exploit the embedding of the Poincaré Ball Model in euclidean space as follows. By translating x to the origin and taking the intersection of the configuration with the euclidean plane that contains x, y and c, call this plane P, the current problem reduces to a problem in the euclidean plane already examined. As a result of translating x to the origin there are two key observations that we now make. First, after the translation, the hyperbolic hyperplane H is embedded in a euclidean hyperplane, say $\bar{H}$. Thus, the intersection of $\bar{H}$ and the euclidean plane P is a line, call it l, in P. Second, the spindle-geodesic connecting x and y in P is part of a hyperbolic circle with radius r_Y. Hence, the boundary of the disk $\mathbf{B}^2_Y[c, r_Y]$ and the circle containing the spindle-geodesic connecting x and y are congruent as euclidean circles. Thus, if $\mathbf{B}^2_Y[c, r_Y]$ is identified with a euclidean disk, embedded in the euclidean plane P where the euclidean line l supports the disk at x, then the problem reduces to the one in the euclidean plane examined in the preceding paragraph.

Finally, consider the spherical case $Y = \mathbb{S}^n$. Using stereographic projection, where the point x is taken to the origin, reduces the current problem to one in euclidean space. The stereographic projection transforms the spherical hyperplane H into a euclidean hyperplane. This hyperplane supports the image, under the projection, of $\mathbf{B}^n_Y[c, r_Y]$ at the origin. Furthermore, the projection of the spherical spindle connecting x and y is the euclidean spindle connecting the projections of x and y. Since the stereographic projection preserves containments, we may apply the result in the euclidean case, proved earlier, to obtain the desired result. ■

The following definition clarifies the notion of a supporting sphere in this context. Accordingly, the ball obtained in Lemma 4.1.1 supports C at x and it's boundary also supports C at x.

Definition 4.1.2 *If a ball $\mathbf{B}^n_Y[c, r]$ contains a set $C \subset Y$ and a point $x \in \operatorname{bd} C$ is on $\mathbb{S}^{n-1}_Y(c, r)$, then we say that $\mathbb{S}^{n-1}_Y(c, r)$* supports *C at x. If $\mathbb{S}^{n-1}_Y(c, r)$ supports C at x,* *then we say* $\mathbf{B}^n_Y[c,r]$ supports C *at* x.

Casually, we may call such a ball a supporting ball and it's boundary a supporting sphere. However, it should be noted that there are in fact two kinds of supporting spheres that arise in the course of this discussion. The ones that have been identified in Definition 4.1.2 can be distinguished as *generic* supporting spheres. While those that arise in the context of ball-polyhedra, and in particular standard ball-polyhedra, can be distinguished as *standard* supporting spheres. We have not discussed standard ball-polyhedra, but the idea of standard supporting spheres has a highly specialized meaning as we shall see. The generating spheres of a ball-polyhedron are also generic supporting spheres according to Definition 4.1.2. However, not all of the standard supporting spheres of a standard ball-polyhedron are generic supporting spheres. We explore these relations in subsequent chapters.

The euclidean version of the following corollary appears in [37] without proof.

Corollary 4.1.3 *Let* $A \subset Y$ *be a closed convex set. Then the following are equivalent.*

(i) *A is spindle-convex.*

(ii) *A is the intersection of all balls with radius r_Y containing A; that is,* $A = \mathbf{B}_Y[\mathbf{B}_Y[A]]$.

(iii) *For every boundary point of A, there is a ball with radius r_Y that supports A at that point.*

Proof. The implication (i) $\Longrightarrow$ (iii) is a restatement of Lemma 4.1.1. Since the intersection of spindle-convex sets are themselves spindle-convex, the implication (ii) $\Longrightarrow$ (i) is trivial. So, all that remains to be checked is (iii) $\Longrightarrow$ (ii).

Since $A \subseteq \mathbf{B}_Y[\mathbf{B}_Y[A]]$, we suppose that there exists $x \in \mathbf{B}_Y[\mathbf{B}_Y[A]]$, but $x \notin A$. Since A and x are closed and convex they can be strictly separated a hyperplane H. By applying an appropriate translation to H we may assume that it intersects the boundary of A at a point, say p. By (iii) there is a ball $B^n_Y[c, r_Y]$ which supports A at p. Clearly $c \in \mathbf{B}_Y[A]$ so $\mathbf{B}_Y[\mathbf{B}_Y[A]] \subseteq B^n_Y[c, r_Y]$ but H strictly separates $B^n_Y[c, r_Y]$ and x which implies $x \notin \mathbf{B}_Y[\mathbf{B}_Y[A]]$ for a contradiction. Thus, $A \supseteq \mathbf{B}_Y[\mathbf{B}_Y[A]]$. ∎

Corollary 4.1.4 *Let $C \subset Y$ be a spindle convex set. If* $\operatorname{cr}(C) = r_Y$ *then* $C = \mathbf{B}^n_Y[q, r_Y]$ *for some* $q \in Y$.

Proof. Observe that if C has two distinct supporting balls of radius r_Y then $\operatorname{cr}(C) < r_Y$ and if C is not contained in a ball of radius r_Y, that is $\operatorname{cr}(C) > r_Y$ then $C = Y$. Thus, the assertion follows. ∎

Theorem 4.1.5 *Let $C, D \subset Y$ be spindle convex sets. Suppose C and D have disjoint relative interiors. Then there is a closed ball $\mathbf{B}^n_Y[c, r_Y]$ such that $C \subseteq \mathbf{B}^n_Y[c, r_Y]$ and $D \subset Y \backslash \mathbf{B}^n_Y(c, r_Y)$.*

Furthermore, if C and D have disjoint closures and one, say C, is not a ball of radius r_Y, then there is a closed ball $\mathbf{B}^n_Y[c, r_Y]$ such that $C \subset \mathbf{B}^n_Y(c, r_Y)$ and $D \subset Y \backslash \mathbf{B}^n_Y[c, r_Y]$.

Proof. Since C and D are spindle convex, they are convex, bounded sets with disjoint relative interiors. So, their closures are convex, compact sets with disjoint relative interiors. Hence, they can be separated by a hyperplane H that supports C at a point, say x. The closed ball, say $\mathbf{B}$, obtained from Lemma 4.1.1 satisfies the conditions of the first statement.

For the second statement, we assume that C and D have disjoint closures, so the ball $\mathbf{B}$, as constructed in the preceding paragraph, is disjoint from the closure of D and remains so even after a sufficiently small translation. Furthermore, C is a spindle convex set that is different from a unit ball, so there is a sufficiently small translation of $\mathbf{B}$ that satisfies the second statement. ∎

Definition 4.1.6 *Let $C, D \subset Y, c \in Y, r > 0$. We say that $\mathbb{S}^{n-1}(c,r)$* separates *C from D if $C \subseteq \mathbf{B}^n_Y[c,r]$ and $D \subseteq Y \backslash \mathbf{B}^n_Y(c,r)$, or $D \subseteq \mathbf{B}^n_Y[c,r]$ and $C \subseteq Y \backslash \mathbf{B}^n_Y(c,r)$. If $C \subseteq \mathbf{B}^n_Y(c,r)$ and $D \subseteq Y \backslash \mathbf{B}^n_Y[c,r]$, or $D \subseteq \mathbf{B}^n_Y(c,r)$ and $C \subseteq Y \backslash \mathbf{B}^n_Y[c,r]$, then we say that C and D are* strictly separated *by $\mathbb{S}^{n-1}(c,r)$.*

4.2 Radon's Theorem

The following theorem is a generalization of Radon's Theorem.

Theorem 4.2.1 *Let X be a collection of $n+2$ points in $Y \in \{\mathbb{R}^n, \mathbb{H}^n, \mathbb{S}^n\}$ such that $\operatorname{cr}_Y(X) < r_Y$. Then, the points can be partitioned into two sets such that the spindle-convex hulls of these two sets have non-empty intersection.*

Proof. If $Y = \mathbb{R}^n$, then by Radon's Theorem the points can be partitioned into two sets, say A and B, where the convex hulls of these sets intersect. Since the spindle-convex hulls of A and B contain the respective convex hulls of A and B the result is immediate.

Next, if $Y = \mathbb{H}^n$, then using the embedding of $\mathbb{H}^n$ in $\mathbb{R}^n$ we apply Radon's Theorem to the points in euclidean space. Thus, the points can be partitioned into two sets, say A and B, where the convex hulls of these sets intersect. In general, the hyperbolic spindle joining any two points in $\mathbb{H}^n$ contains the euclidean line segment joining the two points . In particular, the hyperbolic spindle-convex hull of a collection of points in $\mathbb{H}^n$ contains the euclidean convex hull of the same collection of points under the embedding of $\mathbb{H}^n$ in $\mathbb{R}^n$. Thus, the spindle-convex hulls of A and B intersect.

Finally, if $Y = \mathbb{S}^n$, then apply a central projection. The projection of X, denoted $\operatorname{proj}(X)$, is a collection of $n+2$ points in $\mathbb{R}^n$. By Radon's Theorem $\operatorname{proj}(X)$ can be partitioned into two sets, say A and B, where the convex hulls of these sets intersect. The pre-images of A and B, denoted $\mathrm{proj}^{-1}(A)$ and $\mathrm{proj}^{-1}(B)$ respectively, partition X. The spindle-convex hulls of these sets, namely $\mathrm{conv}_{(\mathrm{r},\mathrm{Y})}[\mathrm{proj}^{-1}(A)]$ and $\mathrm{conv}_{(\mathrm{r},\mathrm{Y})}[\mathrm{proj}^{-1}(B)]$, are spherically convex sets in $\mathbb{S}^n$. In particular, $\mathrm{proj}\,\mathrm{conv}_{(\mathrm{r},\mathrm{Y})}[\mathrm{proj}^{-1}(A)]$ and $\mathrm{proj}\,\mathrm{conv}_{(\mathrm{r},\mathrm{Y})}[\mathrm{proj}^{-1}(B)]$ are convex sets in $\mathbb{R}^n$ that contain A and B respectively. Thus,

$$\mathrm{proj}\,\mathrm{conv}_{(\mathrm{r},\mathrm{Y})}[\mathrm{proj}^{-1}(A)] \cap \mathrm{proj}\,\mathrm{conv}_{(\mathrm{r},\mathrm{Y})}[\mathrm{proj}^{-1}(B)] \neq \emptyset$$

which completes the proof. ■

4.3 Carathéodory's Theorem

The next theorem is a generalization of Carathéodory's Theorem to ball polytopes. Before dealing with the theorem we examine the following lemma.

Lemma 4.3.1 *Let $A \subseteq \mathbb{R}^n$ and contained in a unit ball. Then,* $\mathrm{conv}_{\mathrm{r}} A = \bigcup_{\substack{F \subseteq A \\ |F| \leq \infty}} \mathrm{conv}_{\mathrm{r}} F$.

Proof. Clearly, $\mathrm{conv}_{\mathrm{r}} A \supseteq \bigcup_{\substack{F \subseteq A \\ |F| \leq \infty}} \mathrm{conv}_{\mathrm{r}} F$. Next, we show $B = \bigcup_{\substack{F \subseteq A \\ |F| \leq \infty}} \mathrm{conv}_{\mathrm{r}} F$ is spindle-convex. Let $p \in B$ and $q \in B$, then there exist sets F_1 and F_2 both finite and contained in A such that $p \in \mathrm{conv}_{\mathrm{r}} F_1$ and $q \in \mathrm{conv}_{\mathrm{r}} F_2$. Now, $F_1 \cup F_2$ is finite and contained in A and $\mathrm{conv}_{\mathrm{r}}(F_1 \cup F_2)$ is a spindle-convex set containing p and q. Therefore, $\mathrm{spin}[p, q] \subseteq \mathrm{conv}_{\mathrm{r}}(F_1 \cup F_2) \subseteq B$. Thus, B is a spindle-convex set and $A \subseteq B$, together these two facts imply that $\mathrm{conv}_{\mathrm{r}} A \subseteq B$. ■

Theorem 4.3.2 *Let $A \subseteq \mathbb{R}^2$ and contained in a unit ball. Then,* $\mathrm{conv}_{\mathrm{r}} A = \bigcup_{\substack{F \subseteq A \\ |F| \leq 3}} \mathrm{conv}_{\mathrm{r}} F$.

Proof. We show that $\mathrm{conv}_{\mathrm{r}} F = \bigcup_{\substack{G \subseteq F \subseteq A \\ |G| \leq 3}} \mathrm{conv}_{\mathrm{r}} G$. Let $\bar{F}$ be the set obtained from F by removing all points which do no contribute to $\mathrm{conv}_{\mathrm{r}} F$. In particular, we remove all points of F which are not in spindle-convex position. Observe that $\mathrm{conv}_{\mathrm{r}} F = \mathrm{conv}_{\mathrm{r}} \bar{F}$.

If $|\bar{F}| \leq 3$ then we are done; otherwise we proceed inductively on the cardinality of $\bar{F}$. Suppose $\operatorname{conv}_{\mathrm{r}} \bar{F} = \bigcup_{\substack{G \subseteq F \subseteq A \\ |G| \leq 3}} \operatorname{conv}_{\mathrm{r}} G$ whenever $|\bar{F}| \leq k$. Next, let $|\bar{F}| = k+1$. There is a natural cyclic ordering of the points in $\bar{F}$, so take three consecutive points $a, b, c \in \bar{F}$. Let H_1 be the closed half space bounded by the line through a and c containing b and let H_2 be the closed half space bounded by the line through a and c not containing b. Let $G = \{a, b, c\}$ and $\bar{G} = \bar{F} \setminus \{b\}$. Since G and $\bar{G}$ are contained in $\bar{F}$, they are also contained in $\operatorname{conv}_{\mathrm{r}} \bar{F}$. Thus, $\operatorname{conv}_{\mathrm{r}} \bar{F} \supseteq \operatorname{conv}_{\mathrm{r}} G \cup \operatorname{conv}_{\mathrm{r}} \bar{G}$. Let $x \in \operatorname{conv}_{\mathrm{r}} \bar{F}$. Without loss of generality assume that $x \in H_1$. If $x \notin \operatorname{conv}_{\mathrm{r}} G$ then the point x is not in either of the two circles, one determined by the points a and b and the other determined by the points b and c, both containing $\operatorname{conv}_{\mathrm{r}} G$ in their closures. These two circles also contain $\operatorname{conv}_{\mathrm{r}} \bar{F}$ in their closures, which contradicts the assumption $x \in \operatorname{conv}_{\mathrm{r}} \bar{F}$. Thus, $x \in \operatorname{conv}_{\mathrm{r}} G$ which implies $\operatorname{conv}_{\mathrm{r}} \bar{F} \subseteq \operatorname{conv}_{\mathrm{r}} G \cup \operatorname{conv}_{\mathrm{r}} \bar{G}$.

Thus, $\operatorname{conv}_{\mathrm{r}} \bar{F} = \operatorname{conv}_{\mathrm{r}} G \cup \operatorname{conv}_{\mathrm{r}} \bar{G}$. Now, $|\bar{G}| \leq k$ so $\operatorname{conv}_{\mathrm{r}} \bar{G} = \bigcup_{\substack{G \subseteq F \subseteq A \\ |G| \leq 3}} \operatorname{conv}_{\mathrm{r}} G$ and $|G| = 3$, therefore the result is immediate. ■

A rigorous treatment of the higher dimensional case is found in [12].

4.4 Helly's Theorem

A straightforward application of Helly's Theorem provides a proof of the following result.

Theorem 4.4.1 *Given a finite family of spindle-convex sets in $\mathbb{R}^n$ containing at least $n+1$ members, if every subfamily of at least $n+1$ members has non-empty intersection then the whole family has non-empty intersection.*

In an effort to explore Helly type results we look to the next theorem, due to Maehara [39], as a possible direction for study.

Theorem 4.4.2 *Let $\mathfrak{F}$ be a family of at least $n+3$ distinct $(n-1)$-spheres in $\mathbb{R}^n$. If any $n+1$ of the spheres in $\mathfrak{F}$ have a point in common, then all of the spheres in $\mathfrak{F}$ have a point in common.*

Maehara points out that neither $n+3$ nor $n+1$ can be reduced in Theorem 4.4.2. In particular, there is a planar configuration of four circles such that any three intersect but not all four share a common point of intersection. This configuration of circles in the euclidean plane can be seen in Figure 4.2 and forms part of the proof of our next result. First, we prove a variant of Theorem 4.4.2.

Theorem 4.4.3 *Let $\mathfrak{F}$ be a family of $(n-1)$-spheres in $\mathbb{R}^n$, and k be an integer such that $0 \leq k \leq n-1$. Suppose that $\mathfrak{F}$ has at least $n-k$ members and that any $n-k$ of them intersect in a sphere of dimension at least $k+1$. Then they all intersect in a sphere of dimension at least $k+1$. Furthermore, $k+1$ cannot be reduced to k.*

Proof. Amongst all the intersections of any $n-k$ spheres from the family, let S be such an intersection of minimal dimension. By assumption, S is a sphere of dimension at least $k+1$. Now, one of the $n-k$ spheres is redundant in the sense that S is contained entirely in this sphere. After discarding this redundant sphere, S is now the intersection of only $(n-k)-1$ members of the family, but any $n-k$ members intersect in a sphere of dimension at least $k+1$. So, the remaining members of the family intersect S. Since the dimension of S is minimal, S is contained in these members. In particular, $\bigcap \mathfrak{F} = S$.

Fixing n and k, $0 \leq k \leq n-1$, we show that $k+1$ cannot be reduced to k by considering a regular n-simplex in $\mathbb{R}^n$, with circumradius one, and a family of $n+1$ unit spheres centered at the vertices of this simplex. The intersection of any $n-k$ of them is a sphere of dimension at least k, but the intersection of all of them is a single point which, as we recall, is not a sphere in the current setting. ■

Theorem 4.4.4 *Given a family, $\mathfrak{F}$, of at least four congruent circles in the hyperbolic plane such that any three intersect then they all intersect.*

Proof. Let A, B, C, D be congruent circles in the hyperbolic plane such that any three intersect but not all four do. Consider the conformal ball model of the hyperbolic plane. In this model our four circles are euclidean circles. Observe that any two circles intersect in exactly two points, because a single point of intersection implies all the circles intersect. Also, we may assume that any three circles intersect in exactly one point because if any three intersect in two points then all the circles intersect.

Let u and v be points in the conformal ball model of the hyperbolic plane. These points are also points in the euclidean plane. Thus, there are two distances between these points, the euclidean distance, denoted $|uv|_e$, and the hyperbolic distance, denoted $|uv|_h$.

The circles in our present setting may be interpreted as euclidean circles with a euclidean center and a locus of points that are the same euclidean distance from the euclidean center. Circles are congruent as euclidean circles if and only if the euclidean length of the radii are the same. In this case we say the circles are euclidean congruent. The circles may also be interpreted as hyperbolic circles with a hyperbolic center and a locus of points that are the same hyperbolic distance from the hyperbolic center. Circles are congruent as hyperbolic circles if and only if the hyperbolic length of the radii are the same. In this case we say the circles are hyperbolic congruent.

Let x be the point of intersection of A, B and C. Translate so that x coincides with the center of the circle at infinity. Following this translation all four circles remain congruent as hyperbolic circles. Furthermore, the properties that any two circles intersect in exactly two points and that any three circles intersect in exactly one point continue to hold. Since x lies on the circles A, B and C, the hyperbolic centers of A, B and C are the same hyperbolic distance from x. Now, the symmetry of the configuration of the circles A, B and C ensures that the euclidean centers of these circles is the same euclidean distance from x. Thus, the circles A, B and C are congruent as euclidean circles. Furthermore, using the three circle problem D is euclidean congruent to A, B and C.

Suppose that the center of D coincides with x (see Figure 4.2). In this case we can easily check that the euclidean centers of A, B and C lie on D. Let e denote the euclidean center of B and h the hyperbolic center of B. The hyperbolic radius of D is $|xe|_h$ and the hyperbolic radius of $|xh|_h$. Now, $|xe|_h < |xh|_h$, so the four circles are not congruent as hyperbolic circles, which contradicts the original assumption.

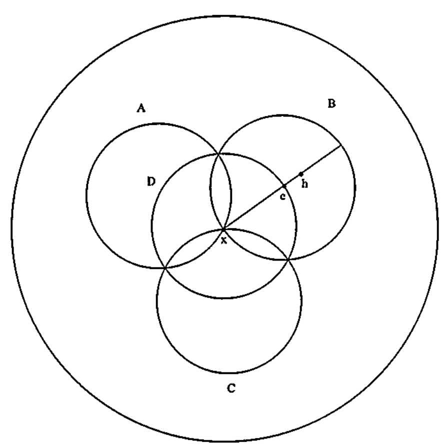

Figure 4.2: Center of D coincides with x.

Thus, the center of D does not coincide with x (see Figure 4.3). Consequently, one of the euclidean centers of A, B or C does not lie in the interior of D. Suppose that the euclidean center of A, denoted by E, does not lie in D. Let H denote the hyperbolic center of A, e denote the euclidean center of D and h denote the hyperbolic center of D. The points x, e and h all lie on a diameter of D with e and h on one side of x. Let p denote the point where the diameter meets D on the side opposite e and h.

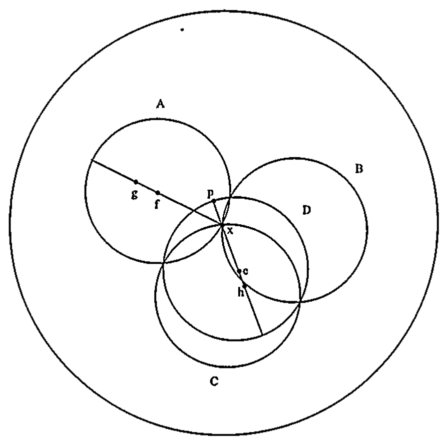

Figure 4.3: Center of D does not coincide with x.

Given two circles U and V in the conformal ball model of the hyperbolic plane such that they are euclidean congruent but the euclidean distance of the euclidean center of U to the center of the circle at infinity is less than the euclidean distance of the euclidean center of V to the center of the circle at infinity. Then, the euclidean distance and hyperbolic distance between the hyperbolic center of U and the euclidean center of U is less than the euclidean distance and hyperbolic distance between the hyperbolic center of V and the euclidean center of V. Thus, $|HE|_h > |he|_h$.

Since, $|Ex|_e = |ep|_e$ and both segments contain x which coincides with the center of the circle at infinity we get $|Ex|_h \geq |ep|_h$.

The hyperbolic radius of A is $|HE|_h + |Ex|_h$ and he hyperbolic radius of D is $|he|_h + |ep|_h$, but $|HE|_h + |Ex|_h > |he|_h + |ep|_h$. Thus the circles are not congruent as hyperbolic circles, a contradiction. ■

Theorem 4.4.5 *Given a family, $\mathfrak{F}$, of at least four horocycles in the hyperbolic plane such that any three intersect then they all intersect.*

Proof. Suppose there are four horocycles, A, B, C, D, such that any three intersect, but not all four. Observe that no two intersect in a single point, ideal or otherwise. Hence, A and B intersect in two points x and y. Take the three horocycles A, B, C. By assumption they intersect in at least a point. Suppose C intersects x and y. Since A, B and D have a point in common, and that point must be either x or y, we have that A, B, C, D all have a point in common. Thus, C can only intersect one of x or y. We may assume it does so at y which means that D intersects x. As noted earlier, the horocycle D intersects each of A and B in points other than x, let a and b be these points respectively. Since any three horocycles have non-empty intersection, C intersects $A \cap D$. However not all four circles intersect, so C passes through a. Similarly, C passes through b.

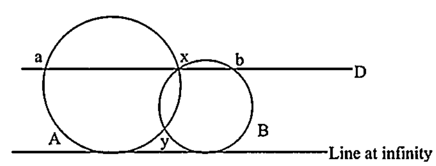

Figure 4.4: The circle through y, a, b is not tangent to the line at infinity.

Consider the half space model of the hyperbolic plane where the horocycle D is one of the horocycles parallel to the line at infinity. In this model we can readily observe (see Figure 4.4) that the unique circle through y, a, b is not tangent to the line at infinity. In fact, it can be strictly separated from the line at infinity by a line parallel to it. However, this circle is C which contradicts the fact that C is a horocycle. ■

Similar proofs to the above can be constructed to show the following theorem.

Theorem 4.4.6 *Given a family, $\mathfrak{F}$, of at least four hypercycles in the hyperbolic plane such that any three intersect then they all intersect.*

Finally, the above results can be extended to higher dimensions as demonstrated in the following series of theorems.

Theorem 4.4.7 *Given a family, $\mathfrak{F}$, of at least $n+2$ congruent spheres in hyperbolic n-space such that any $n+1$ intersect then they all intersect.*

Proof. Suppose there is a family of $n+2$ congruent spheres in hyperbolic n-space such that any $n+1$ intersect, but not all do. In this case, translate so that the point of intersection of $n+1$ spheres is at the center of the circle at infinity in the conformal ball model of hyperbolic space. These $n+1$ spheres are now euclidean congruent but from the proof above the last sphere is a smaller euclidean sphere. Furthermore, it contains the center of the circle at infinity. Thus, these spheres are not congruent as hyperbolic spheres. ■

Theorem 4.4.8 *Given a family, $\mathfrak{F}$, of at least $n+2$ horospheres in hyperbolic space such that any $n+1$ intersect then they all intersect.*

Theorem 4.4.9 *Given a family, $\mathfrak{F}$, of at least $n+2$ hyperspheres in hyperbolic space such that any $n+1$ intersect then they all intersect.*

Chapter 5

Conclusion

This thesis has presented a variety of results that are based on topics encountered in the study of convexity. They have been examined in the more general context of spindle convexity and further broadened by looking beyond euclidean space to hyperbolic and spherical spaces as well. Some results followed easily while others were not nearly as transparent or simply did not hold.

In Chapter 2 definitions, notation and preliminary results were laid out. The definitions provided the necessary foundation for the subsequent discussion. The notation provided for a streamlined discussion and many results that held in all three spaces were easily presented.

The next chapter, Chapter 3, was the starting point of our exploration into this subject. The key results dealt with the monotonicity of the perimeter of spindle polygons. In particular, we showed that the perimeter of spindle polygons is minimal for cyclic spindle polygons and extended Dowker's results to inscribed spindle polygons.

We concluded the journey in Chapter 4 by demonstrating the failure of certain Helly type theorems in hyperbolic space. The failure was the result of a particular configuration of circles in euclidean space. This result is surprising in light of the fact that the planar euclidean equivalent holds.

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