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in the Theory of Fibrations

by

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Abstract

The results contained in this thesis appeared in Annals of Mathematics in 1963 in a paper by A. Dold. His paper was targeted to professional mathematicians; hence, his proofs are brief and many "easy" parts are omitted. In this thesis, I will elaborate the first four chapters of his paper so that these results become accessible to non-specialists and graduate students.

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Table of Contents

Table of Notation

$(\omega \cdot \omega')(t)$: The product path is defined by

$$(\omega \cdot \omega')(t) = \begin{cases} \omega(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ \omega'(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

$\mathbb{B}^n$: The n-ball is defined by $\mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\| \le 1\}$.

$\mathbb{S}^{n-1}$: The $(n-1)$-sphere is defined by $\mathbb{S}^{n-1} = \{x \in \mathbb{B}^n : \|x\| = 1\}$.

$\mathbb{R}^+$: The positive real numbers.

$\mathcal{C}_B$: The category consisting of spaces over B and maps over B.

$\overline{\mathcal{C}}_B$: The category consisting of spaces over B and vertical homotopy classes of maps over B.

${}_\tau\omega$: The path ${}_\tau\omega$ is defined by ${}_\tau\omega(t) = \omega(\tau t)$.

${}^\tau\omega$: The path ${}^\tau\omega$ is defined by ${}^\tau\omega(t) = \omega(1 - \tau + \tau t) = \omega(1 - \tau(1-t))$.

Δ_q : The standard q-simplex.

Δ_X : The diagonal function of $X \to X \times X$.

Φ_e : The characteristic map for a cell e of a CW-complex.

e^j : The j-th face of the standard simplex.

∇_X : The folding function of $X \vee X \to X$.

ω^- : The reverse path is defined by $\omega^-(t) = \omega(1-t)$.

∂_q : The q-th boundary operator.

$A \subset B$: A is a (not necessarily proper) subset of B.

φ_e : The attaching map for a cell e of a CW-complex.

B_q : The barycentre of Δ_q is defined by $B_q = \sum_{i=0}^{q} \frac{e^i}{q+1}$.

$S_q(X)$: The free abelian group generated by the singular q-simplices of a space X.

X^n : The n-skeleton of a CW-complex X.

c_x : The constant path at $x \in X$.

$dimX$: The dimension of a CW-complex X.

Introduction

One of our main problems in algebraic topology is the **lifting problem**, which is a dual of the extension problem: given maps $p : E \to B$ and $f : X \to B$, does there exist a map $g : X \to E$ with $pg = f$? A variant of the lifting problem is the homotopy lifting problem, which can be stated as follows:
Given maps $p : E \to B$, $f : X \to E$ and a homotopy $H : X \times I \to B$ with $pf(x) = H(x, 0)$ for all $x \in X$, does there exist a homotopy $G : X \times I \to E$ with $pG = H$ and $G(x, 0) = f(x)$ for all $x \in X$?

We say that the map $p : E \to B$ has the **homotopy lifting property**, abbreviated as **HLP** in the future, **with respect to X**, if such a G exists for all f and H with $pf(x) = H(x, 0)$. Note that if $p : E \to B$ has the HLP with respect to X and $f \simeq f' : X \to B$, then f can be lifted to E if and only if f' can be lifted to E. It turns out that HLP is a crucial property in homotopy theory, and we have the following definitions.

A map $p : E \to B$ is said to be a **Hurewicz fibration** if it has the HLP with respect to every space X, and it is said to be a **Serre fibration** if it has the HLP with respect to every I^n, $n \geq 0$ where I is the unit interval $[0, 1]$ and $I^0 = \{0\}$. Clearly, any Hurewicz fibration is a Serre fibration. For example, let $f : X \to Y$ and define $E_f = \{(x, \omega) \in X \times Y^I : \omega(0) = f(x)\} \subset X \times Y^I$. Then $p : E_f \to Y$ defined by $p(x, \omega) = \omega(1)$ is a Hurewicz fibration (cf. [5], p.86). An example of a Serre fibration which is not a Hurewicz fibration is the map $p : E \to B$ defined by $p(x, y) = x$, where $L = \{(x, x-1) \in \mathbb{R}^2 : x \in I\}$, $E = \cup_{n \in \mathbb{N}} \{(x, \frac{1}{n}) : x \in I\} \cup L$ and $B = I$ (cf. [8], p.360).

The study of fibrations is important, since any continuous function is a Hurewicz fibration up to homotopy. (cf. [5], p.86) We also have the following results:

1. Let $p : E \to B$ be a Hurewicz fibration and b_0, b_1 be elements of a path connected space B. Then the fibres $p^{-1}(b_0)$ and $p^{-1}(b_1)$ have the same homotopy type. For a Serre fibration, we must also require that the fibres $p^{-1}(b_0)$ and $p^{-1}(b_1)$ have the homotopy type of a finite polyhedron.

2. Let $p : E \to B$ be a Serre fibration with fibre F. Then there is an exact sequence of homotopy groups $\cdots \to \pi_2(E, x_0) \xrightarrow{p_*} \pi_2(B, b_0) \to \pi_1(F, x_0) \to \pi_1(E, x_0) \xrightarrow{p_*} \pi_1(B, b_0) \to \pi_0(F, x_0) \to \pi_0(E, x_0) \xrightarrow{p_*} \pi_0(B, b_0)$, where x_0 is the base point of E and of F, and b_0 is the base point of B.

The second theorem is very useful to compute homotopy groups. For example, we can find the homotopy groups of complex projective space $\mathbb{C}P^\infty$ from the fibration $S^1 \hookrightarrow S^\infty \xrightarrow{p} \mathbb{C}P^\infty$.

Many examples of Serre fibrations are provided by locally trivial bundles. A **locally trivial bundle with fibre F** is a map $p : E \to B$ for which there is an open cover $\mathcal{V}$ of B and homeomorphisms $\varphi_V : V \times F \to p^{-1}(V)$ for all V in $\mathcal{V}$ such that $p\varphi_V(v, x) = v$ for all $(x, v) \in V \times F$. Here are a couple of examples.

1. Let $E = B \times F$. Then the projection $p : E \to B$ defined by $p(b, x) = b$ is a locally trivial bundle with fibre F. In fact, this is called the **trivial fibration.**

2. The map $exp : \mathbb{R} \to S^1$ defined by $exp(t) = e^{2\pi i t}$ is a locally trivial bundle with fibre $\mathbb{Z}$. The map exp is a standard example of a covering space, and in fact, we have: every covering space $p : \tilde{X} \to X$ is a locally trivial bundle with discrete fibre.

A further important example can be found in the theory of fibre bundles (cf. [6], p.210).

The important concepts in this thesis are fibre homotopy equivalence, the section extension property (SEP), and the covering homotopy property (CHP). For definitions of these concepts, see pages p. 8, p. 23, and p. 67.

In 1963, A. Dold wrote an important paper [3] in the Annals of Mathematics. In this paper, he proved that fibre homotopy equivalence, the SEP, and the CHP are local properties in the following sense. Let $\{V_\lambda\}_{\lambda\in\Lambda}$ be a covering of B which admits a refinement by a locally finite partition of unity (see p. 18 for definition). If $p_\lambda : p^{-1}(V_\lambda) \to V_\lambda$ has these properties for each λ, then so does p.

Dold uses the local finiteness of the partitions of unity cleverly; however, his proofs are condensed and many parts are omitted. To make these concepts accessible to non-specialists and graduate students, I have added some examples to clarify the definitions and explicitly verified the statements where he omitted proofs. I have also included standard definitions and properties referred to in his paper. My major contribution in this thesis is to fill in all the details that Dold omitted.

The idea of fibre homotopy equivalence is crucial to many further developments in homotopy theory and bundle theory. In particular, it has led to the development of stable fibre homotopy equivalence and J-homomorphism (cf. [2], [6]), which in turn was the basis of much of the development of the subject in the 70's and 80's.

Chapter 1

Spaces Over B

Definition 1.1 *A continuous map $p : E \to B$ is called a* **space over B**. *If $p : E \to B$ and $p' : E' \to B$ are spaces over B, then a continuous map $f : E \to E'$ is called a* **map over B** *if $p'f = p$.*

Definition 1.2 *The category $\mathcal{C}_B$ has spaces over B as its objects, maps over B as its morphisms, and composition is the usual composition of functions (cf. Appendix, p. 80, for definition of a category).*

Remark 1.3

1. *$\mathcal{C}_B$ is indeed a category. The usual composition is associative, hence maps over B are also associative. Also, it has the identity map 1_E for all $f \in hom(p,p)$, where $(p : E \to B) \in obj\mathcal{C}_B$ which satisfies the given axioms. Thus, $\mathcal{C}_B$ is a category.*

2. *Any category $\mathcal{C}_B$ has a privileged object, the identity map 1_B, and every space over B admits a unique map over B into 1_B; namely, $p : E \to B$.*

Definition 1.4 *For every topological space Y, define a space over B by $E = B \times Y$, $p(b,y) = b$, for all $b \in B$, for all $y \in Y$. A space over B is called* **trivial** *if it is equivalent in $\mathcal{C}_B$ to a space of this form, that is, let $p' : E' \to B$ and $p : E = B \times Y \to B$ be spaces over B where Y is any topological space. The space p' over B is trivial* *if there exist maps $f : E' \to E$, and $g : E \to E'$ over B such that $fg = 1_E$ and $gf = 1_{E'}$.*

$$\begin{tikzcd} E \arrow[r, shift left, "f"] \arrow[d, "p'"'] & B \times Y \arrow[l, shift left, "g"] \arrow[dl, "p"] \\ B & \end{tikzcd}$$

A homotopy $\theta : E \times I \to E'$ is called a **homotopy over B** *or a* **vertical homotopy** *if $\theta_t : E \to E'$, $\theta_t(e) = \theta(e,t)$ is a map over B for all $t \in I = [0,1]$.*

Two maps $f_0, f_1 : E \to E'$ are **vertically homotopic**, *written as $f_0 \simeq_B f_1$, if there exists a vertical homotopy θ with $\theta_0 = f_0$, and $\theta_1 = f_1$. We write $\theta : f_0 \simeq_B f_1$ and read "θ is a vertical homotopy from f_0 to f_1".*

Example 1.5 A homotopy but not a vertical homotopy

Let $E = \{(x, x-1) \in \mathbb{R}^2 : 0 \leq x \leq 1\} \cup \{(x, 1-x) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$ and $B = I = [0,1]$. Let $p : E \to B$ be a space over B defined by $p = \pi|_E$ where π is the natural projection from $\mathbb{R}^2$ to the x-coordinate. Define $f : E \to E$ by $f(x,y) = (x, |y|)$. Let $1_E : E \to E$ be the identity map. Note that f and 1_E are both maps over B. Then, since E is contractible, there exists a homotopy $\theta : 1_E \simeq f$ from 1_E to f; however, there is no vertical homotopy. Notice that the fibre $p^{-1}(0)$ has two path components and $(0,-1)$, $f(0,-1) = (0,1)$ are in distinct path components. If there were a vertical homotopy φ_t of 1_E to f, then $\varphi((0,-1),t) = \varphi_t(0,-1)$ will be a path in $p^{-1}(0)$ joining $(0,-1)$ to $(0,1)$, leading to a contradiction.

The next lemma is useful to construct a new continuous function from existing ones.

Lemma 1.6 Gluing Lemma

1. *Let X be any topological space and X_i, $i = 1, \cdots, n$ closed subsets of X with $X = \cup_{i=1}^n X_i$. Suppose that there are continuous functions $f_i : X_i \to Y$ such that for any $1 \leq i, j \leq n$, $f_i\big|_{X_i \cap X_j} = f_j\big|_{X_i \cap X_j}$, that is, the functions agree on overlaps. Then the function $f : X \to Y$ defined by $f(x) = f_i(x)$ if $x \in X_i$ is well defined and continuous.*

2. *Assume that X is any topological space and $\{X_\alpha\}_{\alpha \in J}$ is any open cover of X. Suppose that, for each $\alpha \in J$, there is a continuous function $f_\alpha : X_\alpha \to Y$ such that $f_\alpha\big|_{X_\alpha \cap X_{\alpha'}} = f_{\alpha'}\big|_{X_\alpha \cap X_{\alpha'}}$, that is, the functions agree on overlaps. Then the function $f : X \to Y$ defined by $f(x) = f_\alpha(x)$ if $x \in X_\alpha$ is well defined and continuous.*

Proof: We first deal with Case 1.. Clearly, the function f is well defined because of the overlaps hypothesis. To see the continuity, let C be a closed subset of Y. Then, $f^{-1}(C) = (\cup_{i=1}^n X_i) \cap f^{-1}(C) = \cup_{i=1}^n (X_i \cap f^{-1}(C)) = \cup_{i=1}^n (X_i \cap f_i^{-1}(C))$. Since $f_i^{-1}(C)$ is closed in X_i and X_i closed in X, $X_i \cap f_i^{-1}(C)$ is closed in X for each $i = 1, \cdots, n$. Since a finite union of closed sets is closed, $f^{-1}(C)$ is closed in X. Hence, f so defined is continuous.

For Case 2., again, well definedness is obvious. For continuity, let O be an open subset of Y. Then, $f^{-1}(O) = (\cup_{\alpha \in J} X_\alpha) \cap f^{-1}(O) = \cup_{\alpha \in J}(X_\alpha \cap f^{-1}(O) = \cup_{\alpha \in J}(X_\alpha \cap f_\alpha^{-1}(O))$. Since $f_\alpha^{-1}(O)$ is open in X_α and X_α open in X, $X_\alpha \cap f_\alpha^{-1}(O)$ is open in X. Since a union of open subsets is open, $f^{-1}(O)$ is open in X. Hence, f so defined is continuous. $\square$

Proposition 1.7 *The relation $\simeq_B$ is an equivalence relation between maps over B which is compatible with composition.*

Proof: Let $f_0, f_1, f_2 : E \to E'$ be maps over B.

reflexivity : The trivial vertical homotopy $\theta : f_0 \simeq_B f_0$ defined by $\theta_t(e) = f_0(e)$ for all $t \in I$ shows the reflexivity.

symmetry : Suppose there exists a vertical homotopy $\theta : f_0 \simeq_B f_1$. Then, $\psi : E \times I \to E'$ is defined by $\psi(e,t) = \theta(e, 1-t)$, and therefore symmetry is shown. Note that $\psi_t = \theta_{1-t}$ is a map over B for all $t \in I$.

transitivity : Suppose there exist vertical homotopies $\theta : f_0 \simeq_B f_1$ and $\varphi : f_1 \simeq_B f_2$. Then, define $\psi : E \times I \to E'$ by

$$\psi(e,t) = \begin{cases} \theta(e, 2t) & \text{if } t \le \frac{1}{2}, \\ \varphi(e, 2t-1) & \text{if } t \ge \frac{1}{2}. \end{cases}$$

By the Gluing Lemma (Lemma 1.6), ψ is well defined and continuous on $E \times I$. Note, again, that $\psi_t = \theta_{2t}$, if $t \le \frac{1}{2}$ and $\psi_t = \varphi_{2t-1}$ if $t \ge \frac{1}{2}$ are maps over B since both θ and φ are vertical homotopies. Thus, ψ is a vertical homotopy from f_0 to f_2.

Thus, the claim holds. □

Definition 1.8 *The category $\overline{\mathcal{C}_B}$ has spaces over B as its objects and vertical homotopy classes of maps over B as its morphisms.*

Definition 1.9 *We say that $p : E \to B$ is* **dominated by $\mathbf{p'}$**: *$E' \to B$ if there exist maps $f : E \to E'$ and $g : E' \to E$ over B such that $gf \simeq_B 1_E$, i.e. p is a "retract" of p' in $\overline{\mathcal{C}_B}$.*

Definition 1.10 *A map $f : E \to E'$ over B whose class in $\overline{\mathcal{C}_B}$ is an equivalence, that is, it has left and right homotopy inverses, is called a* **fibre homotopy equivalence**. *Furthermore, $p : E \to B$ is called* **fibre-homotopically trivial** *if it is fibre homotopy equivalent to a trivial space $B \times Y \to B$.*

Remark 1.11 *If $p : E \to B$ is a space over B and $h : X \to E$ a continuous map, then $ph : X \to B$ is a space over B and h becomes a map over B. More precisely, if $q = ph : X \to B$, then q is a space over B and*

$$\begin{array}{ccc} X & \xrightarrow{h} & E \\ \Big\downarrow{\scriptstyle q=ph} & & \Big\downarrow{\scriptstyle p} \\ B & \overset{1_B}{=\!=\!=} & B \end{array}$$

is commutative. Hence, h is a map over B of q into p.

Now, suppose that $h_0, h_1 : X \to E$ are continuous maps satisfying $ph_0 = ph_1$, then h_0 and h_1 are maps of q into p over B, where $q = ph_0 = ph_1$. In particular, we can talk of h_0, h_1 being vertically homotopic; namely, h_0, h_1 are vertically homotopic if there exists a homotopy $\Theta : X \times I \to E$ such that $\Theta_0 = h_0, \Theta_1 = h_1$ and

$$\begin{array}{ccc} X & \xrightarrow{\Theta_t} & E \\ \Big\downarrow{\scriptstyle q} & & \Big\downarrow{\scriptstyle p} \\ B & \overset{1_B}{=\!=\!=} & B \end{array}$$

is commutative for every $t \in I$. This is equivalent to saying that $p\Theta(x,t) = ph_0(x)$ $= ph_1(x)$ for all $t \in I$ and for all $x \in X$. Thus, two continuous maps h_0, $h_1 : X \to E$ satisfying $ph_0 = ph_1$ are vertically homotopic if there exists a continuous map $\Theta :$ $X \times I \to E$ such that $p\Theta_t = ph_0$ for all $t \in I$ with $\Theta_0 = h_0$ and $\Theta_1 = h_1$. On the other hand, if $p : E \to B$ is a space over B and $h_0, h_1 : X \to E$ with $ph_0 = ph_1$, *in general, h_0 and h_1 are not necessarily vertically homotopic. Let $E = I \times \{0,1\}$, $B = I$ and $p : E \to B$ the space over B defined by $p(x,0) = x$ and $p(x,1) = x$. If h_0,$h_1 : X = I \to E$ are defined by $h_0(x) = (x,0)$ and $h_1(x) = (x,1)$, then $ph_0 = ph_1$. But, there are no homotopies from h_0 to h_1 since $h_0(x)$ and $h_1(x)$ are in different path components of E. In particular, h_0, h_1 are not vertically homotopic.*

An example of $h_0 \simeq h_1$ but $h_0 \not\simeq_B h_1$ can be obtained as follows. Let $E' = \{(x, x-1) \in \mathbb{R}^2 : 0 \leq x \leq 1\} \cup \{(x, 1-x) \in \mathbb{R}^2 : 0 \leq x \leq 1\}$ and $B = [0,1]$. Let $p : E' \to B$ be a space over B defined by $p(x,y) = x$ and define $h_0 : I \to E'$ by $h_0(x) = (x, 1-x)$ and $h_1 : I \to E'$ by $h_1(x) = (x, x-1)$ (here, $X = I$ and $q = 1_I$). Then, as we saw in Example 1.5, $h_0 \simeq h_1$ but $h_0 \not\simeq_B h_1$.

Proposition 1.12 *Let $p : E \to B$ be a space over B. The followings are equivalent:*

1. *The space p is a fibre homotopy equivalence viewed as a map over B into 1_B,*
2. *The space p is dominated by 1_B, and*
3. *There exists a section $s : B \to E$, that is $ps = 1_B$, and a vertical homotopy $\theta : sp \simeq_B 1_E$.*

Proof:

Proof of 1. $\Rightarrow$ 2. Suppose that p is a fibre homotopy equivalence viewed as a map over B into 1_B. Then there exist maps f of p into 1_B and g of 1_B into p in C_B such that $gf \simeq_B 1_E$ and $fg \simeq_B 1_B$. Notice that the only map of p into 1_B is p. Hence $f = p$ and we get $pg \simeq_B 1_B$. This proves that p is dominated by 1_B.

Proof of 2. $\Rightarrow$ 3. Suppose that p is dominated by 1_B, then there exist maps $f : E \to B$ and $g : B \to E$ over B and a vertical homotopy $\theta : gf \simeq_B 1_B$. By setting $s = g$, we have $ps = pg = 1_B$. Also $f = p$ since p is the only map of p to 1_B over B, cf. Remark 1.3. Hence, $sp = gp = gf \simeq_B 1_B$. Thus, 3. holds.

Proof of 3. $\Rightarrow$ 1. Assume that there exists a section $s : B \to E$ such that $sp \simeq_B 1_E$. Since $ps = 1_B$, there exists $\psi : ps \simeq_B 1_B$; namely, $\psi(b,t) = ps(b)$ for all $t \in I$. Thus, p is an equivalence in $\overline{\mathcal{C}_B}$ and hence p is a fibre homotopy equivalence.□

Definition 1.13 *A space $p : E \to B$ over B is called* **shrinkable** *if p is a fibre homotopy equivalence viewed as map over B into 1_B.*

Examples 1.14 Shrinkable Spaces

1. Let $E = B^I = \{\omega : I \to B\}$ with the compact-open topology (cf. [7], p.286) and $p : E \to B$ be the space over B defined by $p(\omega) = \omega(0)$. For any $b \in B$, let e_b denote the constant path at b in B; namely, $e_b(t) = b$ for all $t \in I$. Define $s : B \to E$ by $s(b) = e_b$. Then, clearly, $ps = 1_B$. If we define $\Theta : E \times I \to E$ by $\Theta(\omega, t) = \omega_t$, where $\omega_t(\tau) = \omega(t\tau)$, for all $\tau \in I$, from $p\Theta(\omega,t) = \omega_t(0) = \omega(0) = p(\omega)$, we see that each Θ_t is a map over B where $\Theta_t(\omega) = \Theta(\omega,t)$. Note also that $\Theta_0(\omega) = e_{\omega(0)} = sp(\omega)$ and $\Theta_1(\omega) = \omega$. Thus, Θ is a vertical homotopy between sp and 1_E. By Proposition 1.12, i.e. 3. $\Rightarrow$ 1., p is shrinkable.

2. A fibre-homotopically trivial space over B is shrinkable $\Leftrightarrow$ Y is contractible. Before proving this, we prove the following lemma.

   Lemma 1.15 *Suppose $E_1 \xrightarrow{p_1} B$ is fibre homotopy equivalent to $E_2 \xrightarrow{p_2} B$. Then, p_1 is shrinkable $\Leftrightarrow$ p_2 is shrinkable.*

Proof: Notice that it suffices to show the implication $\Rightarrow$. Since p_1 is fibre homotopy equivalent to p_2, there exist maps $f : E_1 \to E_2$ and $g : E_2 \to E_1$ over B such that $fg \simeq_B 1_{E_2}$ and $gf \simeq_B 1_{E_1}$. Since p_1 is shrinkable, there exists a section $s : B \to E_1$ with $sp_1 \simeq_B 1_{E_1}$. Then $\sigma = fs : B \to E_2$ is easily seen to be a section for $p_2 : E_2 \to B$. Moreover, $\sigma p_2 = fsp_2 = fsp_1 g$ because

$$\begin{array}{ccc} E_2 & \xrightarrow{g} & E_1 \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_1} \\ B & \overset{1_B}{=\!=\!=} & B \end{array}$$

is commutative. Since $sp_1 \simeq_B 1_{E_1}$, $\sigma p_2 \simeq_B fg \simeq_B 1_{E_2}$. □

Now, we deal with the second example.

Proof: Because of Lemma 1.15, we need to show that $B \times Y \xrightarrow{p_1} B$ where $p_1(b, y) = b$ is shrinkable $\Leftrightarrow$ Y is contractible.

First, suppose Y is contractible. Then there exists a homotopy $H : Y \times I \to Y$ such that $H(y, 0) = y$, for all $y \in Y$ and $H(y, 1) = y_0$, a fixed element in Y. Let $s : B \to B \times Y$ be given by $s(b) = (b, y_0)$. Then $\Theta : B \times Y \times I \to B \times Y$ given by $\Theta(b, y, t) = (b, H(y, t))$ is a vertical homotopy between $1_{B\times Y}$ and sp_1.

Conversely, assume that $B \times Y \xrightarrow{p_1} B$ is shrinkable. Then, there exists $s : B \to B \times Y$ with $p_1 s = 1_B$ and $sp_1 \simeq_B 1_{B\times Y}$. Choose some $b_0 \in B$. From $p_1 s = 1_B$, we get $s(b_0) = (b_0, y_0)$ for some $y_0 \in Y$. Let $\Theta : B \times Y \times I \to B \times Y$ yield a vertical homotopy between $1_{B\times Y}$ and sp_1. Then, $\Theta(b_0, y, t) = (b_0, H(y, t))$ for some continuous map $H : Y \times I \to Y$ satisfying $H(y, 0) = y$ and $H(y, 1) = y_0$. Hence, Y is contractible. □

Definition 1.16 *Let $p : E \to B$ be a space over B and $\alpha : A \to B$ a continuous map. Then, we define the* **induced space** p_α: $E_\alpha \to B_\alpha = A$ *by* $p_\alpha(e, a) = a$, *where* $E_\alpha = \{(e, a) \in E \times A : p(e) = \alpha(a)\}$.

Definition 1.17 *A commutative diagram of continuous maps,*

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & E \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{\theta} & B \end{array}$$

is called a **pull-back diagram** *(or said to have the* **pull back property***) if given any pair of continuous maps $\beta_1 : X \to E, \beta_2 : X \to A$ satisfying $p\beta_1 = \theta\beta_2$, there exists a unique continuous map $\beta : X \to F$ with $\varphi\beta = \beta_1$ and $q\beta = \beta_2$.*

Remark 1.18 *Let $p : E \to B$ be a space over B and $\alpha : A \to B$ a continuous map. Let $p_\alpha : E_\alpha \to A$ be the induced space over A, as given in Definition 1.16. Let $\hat{\alpha} : E_\alpha \to E$ be defined by $\hat{\alpha}(e, a) = e$ for every $(e, a) \in E_\alpha$. Then*

$$\begin{array}{ccc} E_\alpha & \xrightarrow{\hat{\alpha}} & E \\ \downarrow{\scriptstyle p_\alpha} & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{\alpha} & B \end{array}$$

is a pull-back diagram in the sense of Definition 1.17. Given continuous maps $\beta_1 : X \to E, \beta_2 : X \to A$ with $p\beta_1 = \alpha\beta_2$, then $\beta(x) = (\beta_1(x), \beta_2(x))$ is in E_α for any $x \in X$. Thus, $\beta : X \to E_\alpha$ defined by $\beta(x) = (\beta_1(x), \beta_2(x))$ is a continuous map (note that E_α is a subspace of the product space $E \times A$); it clearly satisfies $\hat{\alpha}\beta = \beta_1$ and $p_\alpha\beta = \beta_2$. Moreover, the equations $\hat{\alpha}\beta(x) = \beta_1(x)$ and $p_\alpha\beta(x) = \beta_2(x)$ imply that $\beta(x)$ has to be $(\beta_1(x), \beta_2(x))$ for any $x \in X$. Thus, β is unique. This proves the *pull-back property for diagram*

$$\begin{array}{ccc} E_\alpha & \xrightarrow{\tilde{\alpha}} & E \\ \downarrow{\scriptstyle p_\alpha} & & \downarrow{\scriptstyle p} \\ A & \xrightarrow{\alpha} & B. \end{array}$$

Since the pull-back property is a universal property, the diagram above is essentially determined by this pull-back property.

Remark 1.19 *If $\alpha : A \hookrightarrow B$ is the inclusion of a subspace $A \subset B$, we write* $\boldsymbol{p_A}$*:$E_A \to A$ for the induced space which can be identified with $p\,|_{p^{-1}(A)} : p^{-1}(A) \to A$. This is because $E_\alpha = \{(e,a) \in E \times A : p(e) = a\}$ and the map $\theta : E_\alpha \to p^{-1}(A)$ given by $\theta(e,a) = e$ is a homeomorphism with*

$$\begin{array}{ccc} E_\alpha & \xrightarrow{\theta} & p^{-1}(A) \\ \downarrow{\scriptstyle p_\alpha} & & \downarrow{\scriptstyle p|_{p^{-1}(A)}} \\ A & \overset{1_A}{=\!=\!=} & A \end{array}$$

commutative.

Definition 1.20 *If $f : E \to E'$ is a map over B, the* **induced map** $\boldsymbol{f_\alpha}$*: $E_\alpha \to E'_\alpha$ is defined by $f_\alpha(e,a) = (f(e),a)$ and we write* $\boldsymbol{f_A}$*: $E_A \to E'_A$ if $\alpha : A \hookrightarrow B$.*

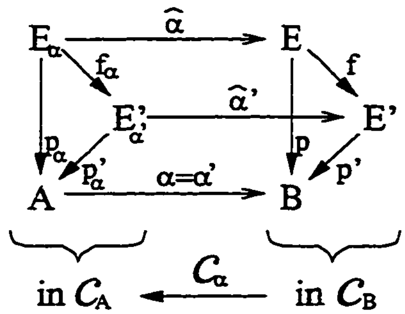
Remark 1.21 *The induced spaces and maps form a covariant functor $\mathcal{C}_\alpha : \mathcal{C}_B \to \mathcal{C}_A$(cf. Appendix, p. 80, for definition of a category). Since $\mathcal{C}_\alpha$ preserves vertical homotopies, it induces a functor $\overline{\mathcal{C}_\alpha} : \overline{\mathcal{C}_B} \to \overline{\mathcal{C}_A}$.*

Let $p, p', p'' \in obj\mathcal{C}_B, f \in hom(p,p'), f' \in hom(p',p''), \alpha : A \to B$. Then, clearly, p_α is a map over A; it is continuous since it is a projection. We also have $(\mathcal{C}_\alpha(ff'))(e,a) = (ff')_\alpha(e,a) = (ff'(e),a) = f_\alpha(f'(e),a) = f_\alpha f'_\alpha(e,a) =$

$(\mathcal{C}_\alpha(f)\mathcal{C}_\alpha(f'))(e,a)$ whenever ff' is defined. Hence, $\mathcal{C}_\alpha : \mathcal{C}_B \longrightarrow \mathcal{C}_A$ is a covariant functor.

Let $f,g : E \longrightarrow E'$ be maps over B, $\alpha : A \longrightarrow B$ and $D : f \simeq_B g$. We want to show that there exists a vertical homotopy over A from f_α to g_α. Define $D_\alpha : E_\alpha \times I \longrightarrow E_\alpha$ by $D_\alpha((e,a),t) = (D(e,t),a)$. Then, $p'D(e,t) = p'D_t(e) = p(e) = \alpha(a)$ and therefore D_α is well defined and clearly, it is continuous. Moreover, $D_\alpha((e,a),0) = (D(e,0),a) = (f(e),a) = f_\alpha(e,a)$, $D_\alpha((e,a),1) = (D(e,1),a) = (g(e),a) = g_\alpha(e,a)$, and $p'_\alpha D_\alpha((e,a),t) = p'_\alpha(D(e,t),a) = a = p_\alpha(e,a)$ for all $t \in I$. Hence, $D_\alpha : f_\alpha \simeq_A g_\alpha$.

A similar argument shows that $\overline{\mathcal{C}_\alpha}$ is a functor, that is, for $p,p' \in obj\mathcal{C}_B$, $f \in hom(p,p')$, $\mathcal{C}_\alpha(p) = p_\alpha : E_\alpha \longrightarrow A$; $\mathcal{C}_\alpha(f : E \longrightarrow E') = f_\alpha : E_\alpha \longrightarrow E'_\alpha$, and $\overline{\mathcal{C}_\alpha}(p) = p_\alpha$; $\overline{\mathcal{C}_\alpha}[f] = [f_\alpha] = \{g : E_\alpha \longrightarrow E'_\alpha : g \simeq_A f_\alpha\}$.



Definition 1.22 *If P is a property of continuous maps, then we say that $p : E \longrightarrow B$ (respectively $f : E \longrightarrow E'$)* **has the property P over $A \subset B$** *if p_A (respectively f_A) has the property P. In this sense, we use, for example, "p is trivial over A", "f is a fibre homotopy equivalence over A", and so on. We say that f* **has the property P locally** *if every $b \in B$ has a neighbourhood V such that f has the property P over V.*

For example, one can combine this definition with Definition 1.4 and thereby obtain the definition of a locally trivial space over B.

Chapter 2

The Section Extension Property (SEP)

Definition 2.1 *A* **halo around $A \subset B$** *is a subset V of B such that there exists a continuous function $\tau : B \to I$ with $A \subset \tau^{-1}(1)$ and $V^c \subset \tau^{-1}(0)$ where $V^c = B \backslash V$.*

Definition 2.2 *A topological space X is called* **normal** *if any disjoint closed subsets A and B can be separated by disjoint open neighbourhoods; that is, there exist open subsets U and V of X such that $A \subset U$, $B \subset V$ and $U \cap V = \phi$.*

Example 2.3 Halo

1. Every subset $V \subset B$ is a halo around ϕ.

   Proof: Take $\tau : B \to I$ by $\tau(b) = 0$ for all $b \in B$. Then $\phi \subset \tau^{-1}(1)$ and $V^c \subset B = \tau^{-1}(0)$. □

2. Let B be normal, A closed in B. Then, any neighbourhood V of A is a halo around A.

   Proof: Since V is a neighbourhood of A, there is an open set U of B such that $A \subset U \subset V$. Then $C = B \backslash U$ is closed in B and $A \cap C = \phi$. Hence, by Urysohn's Lemma, there exists a continuous function $\tau : B \to I$ such that $\tau(a) = 1$ for all $a \in A$ and $\tau(c) = 0$ for all $c \in C$. For this τ, we have $A \subset \tau^{-1}(1)$ and $B \backslash V \subset B \backslash U \subset \tau^{-1}(0)$. □

3. Let $\tau : B \to I$ be continuous. Then, $V = \tau^{-1}(0,1]$ is a halo around $\tau^{-1}[\epsilon, 1]$ for every $\epsilon > 0$.

   Proof: Let $A = \tau^{-1}[\epsilon, 1]$. Define $\tau' : B \to I$ by $\tau'(b) = min(1, \frac{1}{\epsilon}\tau(b))$. Then, $b \in A \Rightarrow \frac{1}{\epsilon}\tau(b) \geq 1$, hence $\tau'(b) = min(1, \frac{1}{\epsilon}\tau(b)) = 1$ and $b \in V^c \Rightarrow \tau(b) = 0 \Rightarrow \tau'(b) = min(1, 0) = 0$. Since the minimum of two continuous real valued functions is a continuous function, τ' is continuous. Also $A \subset \tau'^{-1}(1)$ and $V^c \subset \tau'^{-1}(0)$. □

Definition 2.4 *A family $\{A_\alpha\}_{\alpha \in J}$ of subsets of a topological space X is said to be* **locally finite** *if for any $x \in X$, there exists a neighbourhood U_x of x in X with the property that $I(U_x)$ is finite, where $I(U_x) = \{\alpha \in J : A_\alpha \cap U_x \neq \phi\}$.*

Definition 2.5 *Let $\{A_\alpha\}_{\alpha \in J}$ be a covering of X. A covering $\{B_\lambda\}_{\lambda \in \Lambda}$ is said to be a* **refinement of $\{A_\alpha\}_{\alpha \in J}$** *if for each $\lambda \in \Lambda$, there exists some $\alpha \in J$, depending on λ, with $B_\lambda \subset A_\alpha$; equivalently, if there exists a map $\theta : \Lambda \to J$ with $B_\lambda \subset A_{\theta(\lambda)}$.*

Definition 2.6 *A Hausdorff space X is said to be* **paracompact** *if every open covering of X admits an open locally finite refinement.*

Definition 2.7 *Let X be a topological space and $f : X \to \mathbb{R}$ a real-valued function defined on X. The* **support of f**, *denoted by* $\boldsymbol{supp(f)}$, *is the closure of the set $\{x \in X : f(x) \neq 0\}$ in X. Thus, $x \notin supp(f)$ implies that there exists an open neighbourhood U_x of x with $f(U_x) = 0$.*

Definition 2.8 *Let $\mathcal{A} = \{A_\alpha\}_{\alpha \in j}$ be a covering of a topological space X. A* **partition of unity** *is a family $\mathcal{F} = \{f_\lambda : X \to I\}_{\lambda \in \Lambda}$ of continuous functions defined on X with values in I satisfying*

1. *$\{supp(f_\lambda)\}_{\lambda \in \Lambda}$ is a locally finite family of (closed sets in X), and*
2. *$\Sigma_{\lambda \in \Lambda} f_\lambda(x) = 1$ for all $x \in X$.*

Note that the above sum makes sense since it is a finite sum.

If further, $\{supp(f_\lambda)\}_{\lambda \in \Lambda}$ is a locally finite refinement of $\mathcal{A}$, then $\mathcal{F}$ is said to be a **partition of unity subordinate to $\mathcal{A}$**. *A partition $\mathcal{G} = \{g_\alpha\}_{\alpha \in J}$ of unity is* **subordinate to the cover $\mathcal{A}$ of X with the same index set** *if $supp(g_\alpha) \subset A_\alpha$ for all $\alpha \in J$.*

Definition 2.9 *A (not necessarily open) covering $\{V_\lambda\}_{\lambda \in \Lambda}$ of B is called* **numerable** *if it admits a refinement by a locally finite partition of unity, that is, if there exists a locally finite partition of unity $\{\pi_\gamma : B \to I\}_{\gamma \in \Gamma}$ such that every set $\pi_\gamma^{-1}(0,1]$ is contained in some V_λ (Dold calls such a partition of unity a* **numeration of $\{V_\lambda\}_{\lambda \in A}$** *).*

Example 2.10 Numerable Covering

1. Paracompact (respectively normal) spaces are characterized by the fact that every open covering (respectively every locally finite open covering) is numerable.

   Before proving this, we need some preliminaries.

Lemma 2.11 Shrinking Lemma

Let $\{A_\alpha\}_{\alpha\in J}$ be a locally finite open covering of a normal space X. Then, there is an open covering $\{B_\alpha\}_{\alpha\in J}$ of X such that $\bar{B}_\alpha \subset A_\alpha$ for each $\alpha \in J$.

Proof: Well order J. We will define a family $\{B_\alpha\}_{\alpha\in J}$ of open sets in X, by transfinite induction, such that

i) $\bar{B}_\alpha \subset A_\alpha$ for all $\alpha \in J$

ii) for each $\alpha \in J$, the family formed by the B_λ for $\lambda \leq \alpha$ and by the A_λ for $\lambda > \alpha$ is an open covering of X.

Let $\gamma \in J$ be fixed. Suppose that we have defined the B_α for $\alpha < \gamma$ so that the both conditions are satisfied for all $\alpha < \gamma$. We want to construct B_γ that satisfies the conditions above.

First, we will show that the $B_\alpha, \alpha < \gamma$ and $A_\alpha, \alpha \geq \gamma$ form a covering of X. Since $\{A_\alpha\}_{\alpha\in J}$ is a locally finite cover of X, for each $x \in X$, there are only a finite number of indices $\lambda \in J$ such that $x \in A_\lambda$, say $\lambda_1 < \lambda_2 < \ldots < \lambda_n$. Let λ_h be the largest of the λ_i such that $\lambda_i < \gamma$. If $h < n$ then, we have $x \in A_{\lambda_n}$ and $\lambda_n \geq \gamma$, and if $h = n$, then $\lambda_n < \gamma$ and therefore the inductive hypothesis $X = (\cup_{\lambda\leq\lambda_n} B_\lambda) \cup (\cup_{\lambda>\lambda_n} A_\lambda)$ shows that x belongs to B_λ for some $\lambda \leq \lambda_n < \gamma$ since $x \notin A_\lambda$ for any $\lambda > \lambda_n$. Hence $X = (\cup_{\alpha<\gamma} B_\alpha) \cup (\cup_{\alpha\geq\gamma} A_\alpha)$.

Let $C = (\cup_{\alpha<\gamma} B_\alpha) \cup (\cup_{\alpha>\gamma} A_\alpha)$. Then, C is open, and $A_\gamma^c \subset C$ since $C \cup A_\gamma$ is an open cover for X. Hence, there exists an open set V such that $A_\gamma^c \subset V \subset \bar{V} \subset C$. If we put $B_\gamma = (\bar{V})^c$, we have $\bar{B}_\gamma \subset V^c \subset A_\gamma$ and $B_\gamma^c = \bar{V} \subset C \Rightarrow B_\gamma^c \cap C^c = \phi \Rightarrow B_\gamma \cup C = X$, so that the B_α such that $\alpha \leq \gamma$ and the A_α such that $\alpha > \gamma$ cover X.

Thus, there exist open sets $\{\beta_\alpha\}_{\alpha\in J}$ of X satisfying the above conditions i) and ii). To prove the Shrinking Lemma, we have to show that $\cup_{\alpha\in J}B_\alpha = X$. Given any x, there exist only finitely many elements $\lambda_1 < \cdots < \lambda_r$ in J with $x \in A_{\lambda_i}$. Thus, $x \notin A_\lambda$ whenever $\lambda > \lambda_r$. From $(\cup_{\alpha\leq\lambda_r}B_\alpha) \cup (\cup_{\lambda>\lambda_r}A_\lambda) = X$, we see that $x \in \cup_{\alpha\leq\lambda_r}B_\alpha$. Hence, the Shrinking Lemma holds. □

Proposition 2.12 Existence of a Partition of Unity

Given any locally finite open covering $\{A_\alpha\}_{\alpha\in J}$ of a normal space X, there exists a partition of unity $\{f_\alpha\}_{\alpha\in J}$ on X subordinate to the covering $\{A_\alpha\}_{\alpha\in J}$.

Proof: By the Shrinking Lemma above, there exists an open covering $\{B_\alpha\}_{\alpha\in J}$ of X such that $\bar{B}_\alpha \subset A_\alpha$ for each $\alpha \in J$. It is clear that the covering $\{B_\alpha\}_{\alpha\in J}$ is locally finite (since $\{B_\alpha\}_{\alpha\in J}$ is a refinement of $\{A_\alpha\}_{\alpha\in J}$ and $\{A_\alpha\}_{\alpha\in J}$ is locally finite). For each $\alpha \in J$, let C_α be an open set of X such that $\bar{B}_\alpha \subset C_\alpha \subset \bar{C}_\alpha \subset A_\alpha$. By Urysohn's Lemma, there exists a continuous function $g_\alpha : X \to I$ with $g_\alpha(\bar{B}_\alpha) \subset \{1\}$ and $g_\alpha(X\backslash C_\alpha) \subset \{0\}$, that is, $g_\alpha^{-1}(0,1] \subset C_\alpha$. Thus, $supp(g_\alpha) \subset \bar{C}_\alpha$ for all $\alpha \in J$. Since $\{B_\alpha\}_{\alpha\in J}$ is a covering of X, we have $\Sigma_{\alpha\in J}g_\alpha(x) \geq 1$ for each $x \in X$. This sum makes sense since $\{B_\alpha\}_{\alpha\in J}$ is locally finite. Let $f_\alpha(x) = \frac{g_\alpha(x)}{\Sigma_{\beta\in J}g_\beta(x)}$, for all $\alpha \in J$ and for all $x \in X$. The function f_α is well-defined since $\Sigma_{\alpha\in J}g_\alpha(x) \neq 0$ and it is continuous since each $\alpha \in J, g_\alpha$ is continuous. Then, the f_α form a partition of unity subordinate to the covering $\{A_\alpha\}_{\alpha\in J}$. (Note that by construction, $supp(f_\alpha) \subset \bar{C}_\alpha$ for all $\alpha \in J$ is locally finite, $f_\alpha(x) \geq 0$ for any x in X, and $\Sigma_{\alpha\in J}f_\alpha(x) = 1$.) □

Theorem 2.13 *Every paracompact space is normal.*

Proof: First, we prove the regularity. Let a be a point of a paracompact space X and B be a closed subset of X disjoint from a. Then, for each $b \in B$, there exists an open neighbourhood U_b of b such that $a \notin \bar{U}_b$ since X is Hausdorff. So, $\{U_b\}_{b\in B} \cup X\backslash B$ is an open cover for X. Let $\mathcal{C}$ be a locally finite open refinement and $\mathcal{D} = \{U \in \mathcal{C} : U \cap B \neq \phi\}$ the sub-collection of $\mathcal{C}$ consisting of all elements of $\mathcal{C}$ which meet B. Thus, $\mathcal{D}$ is a family of open sets in X and $D \in \mathcal{D} \Rightarrow a \notin \bar{D}$, since D intersects with B, D lies in some U_b whose closure is disjoint from a. Let $V = \cup_{D\in\mathcal{D}} D$, then V is an open set in X containing B. Since $\mathcal{D}$ is locally finite, $\bar{V} = \cup_{D\in\mathcal{D}} \bar{D}$ is disjoint from a. Thus, X is regular, that is, V, $X\backslash\bar{V}$ are disjoint open sets of X containing B, a, respectively.

To prove the normality, repeat the same argument replacing a with a closed subset A and the Hausdorff condition with the regularity. □

Corollary 2.14 *Given any open covering $\{A_\alpha\}_{\alpha\in J}$ of a paracompact space X, there exists a partition of unity $\{f_\alpha\}_{\alpha\in J}$ on X subordinate to the covering $\{A_\alpha\}_{\alpha\in J}$ (with the same index set).*

Proof: Since X is paracompact, there exists a locally finite open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of X which refines $\{A_\alpha\}_{\alpha\in J}$. We choose a set theoretic map $\theta : \Lambda \to J$ satisfying $U_\lambda \subset A_{\theta(\lambda)}$ for all $\lambda \in \Lambda$. From the Proposition 2.12, there exists a partition of unity $\{g_\lambda : X \to I\}$ with *supp*$(g_\lambda) \subset U_\lambda$, for all $\lambda \in \Lambda$. Let $f_\alpha = \Sigma_{\lambda\in\theta^{-1}(\alpha)} g_\lambda = \Sigma_{\theta(\lambda)=\alpha} g_\lambda$. First of all, observe that f_α is continuous from X into I. To see this, notice that any $x \in X$ admits an open set V_x with

$\Delta(x) = \{\lambda \in \Lambda : V_x \cap U_\lambda \neq \phi\}$ finite. Then clearly $f_\alpha|_{V_x} = \Sigma_{\lambda\in\Delta(x)\cap\theta^{-1}(\alpha)} g_\lambda$ is a finite sum. Hence, $f_\alpha|_{V_x}$ is continuous. From $g_\lambda \geq 0$ and $\Sigma_{\lambda\in\Lambda} g_\lambda = 1$, we see that $0 \leq f_\alpha \leq 1$.

It is clear that the set $\{x \in X : f_\alpha(x) > 0\}$ is the same set as $\cup_{\lambda\in\theta^{-1}(\alpha)}\{x : g_\lambda(x) > 0\}$. Since the family $\{x : g_\lambda(x) > 0\}$ is a locally finite family of sets in X, we have $Cl(\cup_{\lambda\in\theta^{-1}(\alpha)}\{x : g_\lambda(x) > 0\}) = \cup_{\lambda\in\theta^{-1}(\alpha)} Cl\{x : g_\lambda(x) > 0\}$. Thus, $supp(f_\alpha) = \cup_{\lambda\in\theta^{-1}(\alpha)} supp(g_\lambda) \subset \cup_{\lambda\in\theta^{-1}(\alpha)} U_\lambda \subset A_\alpha$. Hence, $\{f_\alpha : X \to I\}_{\alpha\in J}$ are continuous functions with $supp(f_\alpha) \subset A_\alpha$ for all $\alpha \in J$. Moreover, $\alpha \neq \alpha' \Rightarrow \theta^{-1}(\alpha) \cap \theta^{-1}(\alpha') = \phi$ and therefore $\Sigma_{\alpha\in J} f_\alpha = \Sigma_{\lambda\in\Lambda} g_\lambda = 1$. This shows that $\{f_\alpha : X \to I\}_{\alpha\in J}$ is a partition of unity subordinate to $\{A_\alpha\}_{\alpha\in J}$ (with the same indexing set).

This completes the proof of 1. in Example 2.10 since in both cases, the converses are obvious. □

2. If $\{V_\lambda\}_{\lambda\in\Lambda}$ is a numerable covering of B and if $\alpha : X \to B$ is continuous, then $\{\alpha^{-1}(V_\lambda)\}$ is a numerable covering of X.

   Proof: Let $x \in X$, then $\alpha(x) \in V_\lambda$ for some $\lambda \in \Lambda$, and therefore $x \in \alpha^{-1}(V_\lambda)$. Hence $\{\alpha^{-1}(V_\lambda)\}_{\lambda\in\Lambda}$ covers X. To see if $\{\alpha^{-1}(V_\lambda)\}_{\lambda\in\Lambda}$ admits a refinement by a locally finite partition of unity, define $\pi'_\gamma : X \to I$ by $\pi'_\gamma = \pi_\gamma\alpha$ for each $\gamma \in \Gamma$ where $\{\pi_\gamma\}_{\gamma\in\Gamma}$ is a partition of unity subordinate to $\{V_\lambda\}_{\lambda\in\Lambda}$. First note that $supp(\pi'_\gamma) \subset \alpha^{-1}(supp(\pi_\gamma))$. For $\alpha(x) \in B$, there exists an open neighbourhood $U_x \subset B$ of $\alpha(x)$ which intersects with only finitely many $supp(\pi_\gamma)$, say, $\gamma_1, \ldots, \gamma_k$. Hence the open neighbourhood $U'_x = \alpha^{-1}(U_x) \subset X$ of x intersects at most k of $\alpha^{-1}(supp(\pi_\gamma))$; namely, $\alpha^{-1}(supp(\pi_{\gamma_1})), \ldots, \alpha^{-1}(supp(\pi_{\gamma_k}))$.

Clearly, we have $\Sigma_{\gamma\in\Gamma}\pi'_\gamma(x) = \Sigma_{\gamma\in\Gamma}\pi_\gamma\alpha(x) = 1$ since $\alpha(x) \in B$. Thus, $\{\pi'_\gamma\}_{\gamma\in\Gamma}$ is a partition of unity on X. Since $\{\pi_\gamma\}_{\gamma\in\Gamma}$ is a refinement of $\{V_\lambda\}_{\lambda\in\Lambda}$, there exists $\theta : \Lambda \to \Gamma$ with $supp(\pi_\gamma) \subset V_{\theta(\lambda)}$ for all $\gamma \in \Gamma$. For the same θ, we get $supp(\pi'_\gamma) \subset \alpha^{-1}(supp(\pi_\gamma)) \subset \alpha^{-1}(V_{\theta(\lambda)})$. This proves that $\{\alpha^{-1}(V_\lambda)\}_{\lambda\in\Lambda}$ is a numerable covering of X. □

Definition 2.15 *A space $p : E \to B$ over B has the* **Section Extension Property (SEP)** *if the following holds. For every $A \subset B$ and every section s over A which admits an extension as a section to a halo V around A, there exists a section $S : B \to E$ over B with $S\,|_A = s$. We refer to such an S an extension over B of s.*

Remark 2.16 *In particular, if $p : E \to B$ has the SEP then p always has a section by taking $A = \phi = V$.*

Proposition 2.17 *Suppose $p : E \to B$ is dominated by $p' : E' \to B$. Assume p' has the SEP. Then so does p. (In particular, every shrinkable space has the SEP since 1_B has the SEP.)*

Proof: Since p' dominates p, there are maps $f : E \to E'$, $g : E' \to E$ over B and a vertical homotopy $\Theta : gf \simeq_B 1_E$. Let $A \subset B$ and $s : A \to E$ a section over A which admits an extension to a halo V. Let $\tau : B \to I$ be a haloing function and $s' : V \to E$ the extension of s. We ought to find a section $S : B \to E$ with $S\,|_A = s\,|_A$. By above Example 2.3(halo), $\tau^{-1}(0,1]$ is a halo around $\tau^{-1}[\frac{1}{2},1]$ and $\tau^{-1}(0,1] \subset V$. Note that $fs' : V \to E'$ is a section. Since $V \supset \tau^{-1}(0,1]$, V is also a halo around $\tau^{-1}[\frac{1}{2},1]$.

Since p' has the SEP, there is a section $S' : B \to E'$ with $S'\big|_{\tau^{-1}[\frac{1}{2},1]} = fs'\big|_{\tau^{-1}[\frac{1}{2},1]}$. Now define $S : B \to E$ by

$$S(b) = \begin{cases} gS'(b) & \text{if } \tau(b) \le \frac{1}{2}, \\ \Theta(s'(b), 2\tau(b) - 1) & \text{if } \tau(b) \ge \frac{1}{2}. \end{cases}$$

Then, for $\tau(b) = \frac{1}{2}$, we have $gS'(b) = gfs'(b)$, and $\Theta(s'(b), 2\tau(b) - 1) = \Theta(s'(b), 0) = gfs'(b)$. Hence, S is well defined and continuous by the Gluing Lemma. Using the fact that $pgf = p'f = p$, we see that

$$pS(b) = \begin{cases} pS'(b) = pgfs'(b) = ps'(b) = b & \text{if } \tau(b) \le \frac{1}{2}, \\ p\Theta(s'(b), 2\tau(b) - 1) = ps'(b) = b & \text{if } \tau(b) \ge \frac{1}{2}. \end{cases}$$

Thus, S is a section; hence p has the SEP, that is, $S|_A = s$. □

Proposition 2.18 *Let $p : E \to B$ be a space over B, $B \subset B'$ and $r : B' \to B$ a retraction. If the induced map $p_r : E_r \to B'$ has the SEP, so does p.*

<u>Proof</u>: Let $A \subset B$, V a halo around A, $\tau : B \to I$ a haloing function, $s : A \to E$ a section and $\tilde{s} : V \to E$ a section extending s. We want to find an extension $S : B \to E$ with $S|_A = s$. Let $A' = r^{-1}(A)$, $V' = r^{-1}(V)$ and $\tau' = \tau r$ where $\tau' : B' \to I$. Then V' is a halo around A' in B' with $\tau' : B' \to [0,1]$ as the haloing function.

$$\begin{tikzcd}
 & & E_r \arrow[d, "p_r"] & \\
A \arrow[r, hook] \arrow[urr, "s"] & V \arrow[r, hook] \arrow[ur, "\tilde{s}"'] & B & E \arrow[ul] \arrow[d, "p"] \\
 & A' \arrow[ul] \arrow[r, hook] & V' \arrow[ul] \arrow[r, hook] & B' \arrow[ul, "r"]
\end{tikzcd}$$

Note that $E_r = \{(e, b') \in E \times B' : p(e) = r(b')\}$. Also note that $p(\tilde{s}r(v')) = r(v')$. Hence $(\tilde{s}r(v'), v') \in E \times B'$ satisfies $p(\tilde{s}r(v')) = r(v')$ and therefore $(\tilde{s}r(v'), v') \in E_r$.

Also, $p_r(e,b') = b'$ for any $(e,b') \in E_r$. Hence $p_r(\tilde{s}r(v'),v') = v'$. It follows that $s' : V' \to E_r$ defined by $s'(v') = (\tilde{s}r(v'),v')$ is a section of p_r over V'.

Since p_r has the SEP, there is a section $S' : B' \to E_r$ with $S'|_{A'} = s'|_{A'}$. We know that the diagram

$$\begin{array}{ccc} E_r & \xrightarrow{\hat{r}} & E \\ \downarrow{\scriptstyle p_r} & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{r} & B \end{array}$$

is commutative, where $\hat{r}(e,b') = e$ for any $(e,b') \in E_r$. Let $S(b) = \hat{r}S'(b)$ (note that B is a subspace of B'). Then $pS(b) = p\hat{r}S'(b) = (rp_r)S'(b) = r(b) = b$. Hence $S : B \to E$ is a section of p. Also for any $a \in A$, since $A \subset A'$, we have $S'(a) = s'(a) = (\tilde{s}(a),a) = (s(a),a)$ and therefore $S(a) = \hat{r}S'(a) = s(a)$ for all $a \in A$. This proves Proposition 2.18. $\square$

Example 2.19 SEP

If $p : E \to B$ is dominated by a trivial space $B \times Y$ with $\pi_i(Y) = 0$ for $i < n$ ($\pi_0(Y) = 0$ means that Y is pathwise connected), and if B is a retract of a CW-complex of dimension $\leq n$, then p has the SEP. (Here, n is either an integer ≥ 0 or $n = \infty$.)

Before proving this, we need some basic definitions and properties.

Definition 2.20 *Let $\mathbb{B}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ and $\mathbb{S}^{n-1} = \{x \in \mathbb{B}^n : \|x\| = 1\}$. Then $\mathbb{B}^n$ is called an* **n-ball** *and $\mathbb{S}^{n-1}$ is called an* **(n-1)-sphere**.

Definition 2.21 *Let X be a Hausdorff space. A* **CW-decomposition of X** *is a set $\mathcal{E}$ of subspaces of X with the following properties:*

1. $X = \cup_{e \in \mathcal{E}} e, e \neq e' \Rightarrow e \cap e' = \phi$, *that is,* $\mathcal{E}$ *is a covering of* X *by pointwise disjoint sets.*

2. *Every* $e \in \mathcal{E}$ *is homeomorphic to some Euclidean space* $\mathbb{R}^{|e|}$, *where the number* $|e|$ *is called the* **dimension of e**. *The sets* $e \in \mathcal{E}$ *which are homeomorphic with* $\mathbb{R}^n$ *are the* **n-cells**, *and the union* $X^n = \cup_{|e| \leq n} e$ *is the* **n-skeleton** *of the CW-decomposition.*

3. *For every* n*-cell* $e \in \mathcal{E}$, *there exists a relative homeomorphism* $\Phi_e : (\mathbb{B}^n, \mathbb{S}^{n-1}) \rightarrow (X^{n-1} \cup e, X^{n-1})$.

   Remark 2.22 *Condition 3. refines Condition 2 in the following sense: not only is* e *homeomorphic with* $\mathbb{R}^n \approx \mathbb{B}^n - \mathbb{S}^{n-1}$, *but also a homeomorphism can be chosen which extends to the boundary* $\mathbb{S}^{n-1}$, *that is, on* $\mathbb{S}^{n-1}$, Φ_e *need not be homeomorphic but* $\Phi_e(\mathbb{S}^{n-1}) \subset X^{n-1}$. *The function* Φ_e *is called a* **characteristic map** *for* e, *and* $\varphi_e = \Phi_e |_{\mathbb{S}^{n-1}} : \mathbb{S}^{n-1} \rightarrow X^{n-1}$ *an* **attaching map** *for* e.

4. *(Closure finiteness)*

   The closure $\bar{e}$ *of every cell is contained in a finite union of cells.*

5. *(Weak topology)*

   A subset $A \subset X$ *is closed in* X *if and only if* $A \cap \bar{e}$ *is closed in* $\bar{e}$ *for every cell* $e \in \mathcal{E}$. *Equivalently, a map* $f : X \rightarrow Y$ *is continuous if every* $f|_{\bar{e}}$ *is continuous.*

A Hausdorff space X *together with a CW-decomposition* $\mathcal{E}$ *is called a* **CW-complex** *or* **CW-space** . *The* **dimension** *of a CW-complex,* **dimX**, *is the least integer* n *such that* $X^n = X$; *if no such* n *exists, then* $dimX = \infty$.

Definition 2.23 *The* **standard q-simplex Δ_q** *consists of all points* $x \in \mathbb{R}^{q+1}$ *such that* $0 \le x_i \le 1$, $i = 0, 1, \cdots, q$, *and* $\sum_{i=0}^{q} x_i = 1$, *where* $\mathbb{R}^{q+1}$ *denotes Euclidean space and* $\{x_i\}$ *are the coordinates of* $x = (x_0, x_1, \cdots, x_q) \in \mathbb{R}^{q+1}$. *The unit points* $e^j = (0, \cdots, 0, 1, 0, \cdots, 0)$ *of* $\mathbb{R}^{q+1}$ *are called the* **vertices of Δ_q**.

Definition 2.24 *A continuous function* f *of* Δ_q *into* $\mathbb{R}^n$ *is called* **affine linear** *if there exists a linear map* $F : \mathbb{R}^{q+1} \to \mathbb{R}^n$ *in the usual sense such that* $F\big|_{\Delta_q} = f$. *For a given set of points* $P_0, P_1, \cdots, P_q \in \mathbb{R}^n$, *the unique affine linear map* $g : \Delta_q \to \mathbb{R}^n$ *with* $g(e^i) = P_i$, *namely,* $g(x) = \sum_{i=1}^{q} x_i P_i$, *is called an* **affine linear simplex with vertices $P_0, P_1, \cdots, P_q$**.

Definition 2.25 *The* **j-th face** *of* Δ_q *is the (affine) linear map* $\epsilon^j = \epsilon_q^j : \Delta_{q-1} \to \Delta_q$ *defined by*

$$\epsilon^j(e^i) = \begin{cases} e^i & \text{if } i < j, \\ e^{i+1} & \text{if } i \ge j, \end{cases}$$

where $j = 0, 1, \cdots, q$.

Definition 2.26 *Let* X *be a topological space. A* **singular q-simplex** *of* X *is a continuous map* $\sigma = \sigma_q : \Delta_q \to X, q \ge 0$. *We consider the free abelian group* $\boldsymbol{S_q(X)}$ *which is generated by the set of all singular q-simplices.*

Define $\partial_q : S_q(X) \to S_{q-1}(X)$ *by* $\partial_q(\sigma) = \sum_{j=0}^{q}(-1)^j(\sigma\epsilon_q^j)$ *for each singular q-simplex* σ, *and then extending linearly to all of* $S_q(X)$. *The homomorphism* ∂_q *is called a* **boundary operator** .

Remark 2.27 *Any continuous map* $f : X \to Y$ *induces homomorphisms* $f_* : S_q(X) \to S_q(Y)$ *given by* $f_*(\sigma) = f\sigma$ *for every singular q-simplex* $\sigma : \Delta_q \to X$.

Definition 2.28 *Let X be a non-empty convex subspace of Euclidean space $\mathbb{R}^n$. Let $P \in X$. For every $\sigma_q : \Delta_q \to X, q \geq 0$, define $(P \cdot \sigma_q) : \Delta_{q+1} \to X$ by*

$$(P \cdot \sigma_q)(x_0, x_1, \cdots, x_{q+1}) = \begin{cases} P & \text{if } x_0 = 1, \\ x_0 P + (1 - x_0)\sigma_q(\frac{x_1}{1-x_0}, \cdots, \frac{x_{q+1}}{1-x_0}) & \text{if } x_0 \neq 1. \end{cases}$$

This defines homomorphisms $P = P_q : S_q(X) \to S_{q+1}(X), P_q(\sigma) = P \cdot \sigma$, called the **cone construction.**

Remark 2.29 *Intuitively speaking, $P \cdot \sigma$ is obtained by erecting the cone with vertex P over σ.*

Definition 2.30 *For every space X, we define homomorphisms $\beta_q : S_q(X) \to S_q(X), q \geq 0$, called the* **barycentric subdivision,** *as follows:*

1. *$\beta_0 = 1_{S_0 X}$, and for $q > 0$,*
2. *$\beta_q i_q = B_q \cdot \beta_{q-1}(\partial i_q)$, and*
3. *$\beta_q(\sigma_q) = (\sigma_q)_*(\beta_q i_q)$,*

where $i_q \in S_q \Delta_q$ denotes the **identity map** *of Δ_q, $(\sigma_q)_* : S_q(\Delta_q) \to S_q(X)$ is induced by $\sigma_q : \Delta_q \to X$, $B_q = (\frac{1}{q+1}, \frac{1}{q+1}, \cdots, \frac{1}{q+1}) = \sum_{i=0}^{q} \frac{e^i}{q+1}$ is the* **barycentre** *of Δ_q, $B_q \cdot$ is the cone construction, and $\sigma_q : \Delta_q \to X$ is an arbitrary singular simplex.*

Remark 2.31 *Loosely speaking, the barycentric subdivision of σ_q is obtained by projecting the barycentric subdivision of $\partial \sigma_q$ from the centre of σ_q.*

Definition 2.32 *The* **diameter of $\tau : \Delta_q \to \mathbb{R}^k$** *is defined by*

$$\|\tau\| = max\{\|\tau(x) - \tau(y)\| : x, y \in \Delta_q\}.$$

Lemma 2.33 *If $\sigma : \Delta_q \to \mathbb{R}^k$ is a linear simplex with vertices $P_0, P_1, \cdots, P_q$, then*

1. $\|P - P'\| \leq max_{0\leq i\leq q} \|P - P_i\|$ *for all $P, P' \in \sigma(\Delta_q)$, and*
2. $\|\sigma\| = max_{0\leq i,j\leq q} \|P_i - P_j\|$.

Proof:

1. Let $P' = \Sigma_{i=0}^q x_i' P_i$ with $x_i' \geq 0$ and $\sum_{i=0}^q x_i' = 1$. Then,

$$\begin{aligned} \|P - P'\| &= \|P - \textstyle\sum_{i=0}^q x_i' P_i\| = \|\sum_{i=0}^q x_i'(P - P_i)\| \\ &\leq \textstyle\sum_{i=0}^q x_i' \|P - P_i\| \leq \sum_{i=0}^q x_i'(max_{0\leq i\leq q}\{\|P - P_i\|\}) \\ &= max_{0\leq i\leq q}\{\|P - P_i\|\}. \end{aligned}$$

2. It is clear that $\|\sigma\| = max_{P,P'\in\sigma(\Delta_q)}\{\|P - P'\|\} \geq max_{0\leq i,j\leq q}\{\|P_i - P_j\|\}$. For the other inequality, by applying the property above,

$$\begin{aligned} \|\sigma\| &= max_{P,P'\in\sigma(\Delta_q)}\{\|P - P'\|\} \leq max_{0\leq i\leq q}\{\|P - P_i\|\} \\ &\leq max_{0\leq i,j\leq q}\{\|P_i - P_j\|\}. \end{aligned}$$

Hence, the equality holds. □

Lemma 2.34 *Let $\sigma : \Delta_q \to \mathbb{R}^k$ be a linear simplex. Then $\beta(\sigma)$ contains only linear simplices of diameter less than or equal to $\frac{q}{q+1} \|\sigma\|$. In particular, $\beta^n(i_q)$ contains only simplices of diameter less than or equal to $(\frac{q}{q+1})^n \|i_q\|$.*

Proof:

1. Given $\tau : \Delta_r \to \mathbb{R}^l$, $P \in \mathbb{R}^l$, and a linear map $f : \mathbb{R}^l \to \mathbb{R}^k$, we have $f(P \cdot \tau) = (fP) \cdot (f\tau)$.

Recall that $P \cdot \tau : \Delta_{r+1} \to \mathbb{R}^l$ by

$$P \cdot \tau(x_0, \cdots, x_{r+1}) = \begin{cases} P & \text{if } x_0 = 1, \\ x_0 P + (1 - x_0)\tau(\frac{x_1}{1-x_0}, \cdots, \frac{x_{r+1}}{1-x_0}) & \text{if } x_0 \neq 1. \end{cases}$$

So, we get $f(P \cdot \tau)(x_0, \cdots, x_{r+1}) = f(P) = ((fP) \cdot (f\tau))(x_0, \cdots, x_{r+1})$ if $x_0 = 1$, and for $x_0 \neq 1$, we have

$$\begin{aligned} f(P \cdot \tau)(x_0, \cdots, x_{r+1}) &= f(x_0 P) + f((1 - x_0)\tau(\frac{x_1}{1 - x_0}, \cdots, \frac{x_{r+1}}{1 - x_0})) \\ &= x_0 f(P) + (1 - x_0) f(\tau(\frac{x_1}{1 - x_0}, \cdots, \frac{x_{r+1}}{1 - x_0})) \\ &= ((fP) \cdot (f\tau))(x_0, \cdots, x_{r+1}). \end{aligned}$$

2. If $\tau : \Delta_r \to \mathbb{R}^l$ is a linear map with vertices $Q_0, \cdots, Q_r$, then $P \cdot \tau : \Delta_{r+1} \to \mathbb{R}^l$ is linear map with vertices $P, Q_0, \cdots, Q_r$.

   Let $(x_0, \cdots, x_{r+1}) \in \Delta_r$ with $x_i = 0$ or $1, 0 \leq i \leq r + 1$. Then, we have $P \cdot \tau(x_0, \cdots, x_{r+1}) = P$ if $x_0 = 1$ and for $x_i = 1$ and $1 \leq i \leq r + 1$, we have $P \cdot \tau(x_0, \cdots, x_{r+1}) = \tau(0, \cdots, 0, 1, 0, \cdots, 0) = Q_i$.

3. Using 1. above, $\beta_q \sigma_q = \sigma_q(\beta_q i_q) = \sigma_q(B_q \cdot \beta_{q-1}(\partial i_q)) = (\sigma_q B_q) \cdot (\sigma_q \beta_{q-1} \partial i_q)$ $= (\sigma_q B_q) \cdot (\beta_{q-1} \sigma_q \partial i_q) = (\sigma_q B_q) \cdot (\beta_{q-1} \sigma_q \sum_{j=0}^{q} (-1)^j \epsilon^j) = \sum_{j=0}^{q} (-1)^j (\sigma_q B_q) \cdot (\beta_{q-1} \sigma_q \epsilon^j)$. Thus, $\beta_q \sigma_q$ contains only simplices of the form $\sigma' = (\sigma_q B_q) \cdot \tau$ where τ is contained in some $\beta_{q-1}(\sigma_q \epsilon^j)$.

   By the Lemma 2.33 above, $\|\sigma'\| = \|P - Q\|$ where P, Q are either vertices of τ, or $\sigma_q B_q$. If P, Q are both vertices of τ, then, using Lemma 2.34,

$$\|\sigma'\| = \|P - Q\| \leq \|\tau\| \leq \frac{q-1}{q} \|\sigma\epsilon^j\| \leq \frac{q-1}{q} \|\sigma\| \leq \frac{q}{q+1} \|\sigma\| .$$

If one of them, say $P = \sigma_q B_q$, then $\| \sigma' \| = \| P - Q \| \leq \| P - P_i \|$ for some i. Hence, we have,

$$\begin{aligned} \|\sigma'\| &= \|P - Q\| \leq \|\sigma_q B_q - P_i\| = \|(\Sigma_{\mu=0}^{q} \frac{1}{q+1} P_\mu) - P_i\| \\ &= \|\Sigma_{\mu=0}^{q} \frac{1}{q+1}(P_\mu - P_i)\| \leq \frac{1}{q+1} \Sigma_{\mu=0}^{q} \|P_\mu - P_i\| \leq \frac{q}{q+1} \|\sigma\| . \end{aligned}$$

□

Now we will deal with Example 2.19.

By Proposition 2.17, we can assume $E = B \times Y$, and by Proposition 2.18 assume that B itself is a CW-complex of dimension less than or equal to n.

Let $A \subset B$, V a halo around A in B with a section $s : V \to E$, i.e. $s|_A : A \to E$ is a section which admits an extension s to a halo V, and $\tau : B \to I$ a haloing function. Composing the projection $B \times Y \to Y$ with s, we get a map $\sigma : V \to Y$ with $s(v) = (v, \sigma(v))$, and we want to find $\Sigma : B \to Y$ with $\Sigma|_A = \sigma|_A$.

Let B^i denote the i-skeleton of B, and $T^i = B^i \cup \tau^{-1}[\frac{i+1}{i+2}, 1], i = 0, 1, \cdots$. Clearly, $A \subset \tau^{-1}(1) \subset T^i, \forall i$. By induction on i, we will construct $\Sigma^i : T^i \to Y$ such that $\Sigma^i|_{B^{i-1}} = \Sigma^{i-1}|_{B^{i-1}}, i = 1, 2, 3, \cdots$, and $\Sigma^i|_A = \sigma|_A, i = 0, 1, 2, \cdots$. The first equation shows that $\Sigma = lim_{i\to\infty}\Sigma^i$ is well defined and continuous, and the second gives $\Sigma|_A = \sigma|_A$.

To start the induction, define Σ^0 to be σ on $\tau^{-1}[\frac{1}{2}, 1]$ and let Σ^0 have arbitrary values on the remaining vertices of B. Now, assume that Σ^{i-1} has already been found for $i > 0$. Pick an i-cell e^i_λ, and let $\Phi = \Phi^i_\lambda : \Delta^i \to B$ be its characteristic map, where Δ^i is the standard i-simplex which is homeomorphic to $\mathbb{B}^i$. Note that we can make N so large that every simplex which meets $(\tau\Phi)^{-1}[\frac{i+1}{i+2}, 1]$ lies in $(\tau\Phi)^{-1}[\frac{i}{i+1}, 1]$

by Lemma 2.34. Thus, for large N, the N-fold barycentric subdivision of Δ^i contains a subcomplex K with $(\tau\Phi)^{-1}[\frac{i+1}{i+2}, 1] \subset K \subset (\tau\Phi)^{-1}[\frac{i}{i+1}, 1]$.

Notice that the map $\Sigma^{i-1}\Phi$ is defined on K since $\Phi(K) \subset \tau^{-1}[\frac{i}{i+1}, 1]$, and it is defined on the boundary $\dot{\Delta}^i$ of Δ^i since $\Phi(\dot{\Delta}^i) \subset B^{i-1}$. Thus, $\Sigma^{i-1}\Phi$ is defined on the subcomplex $K \cup \dot{\Delta}^i$ of Δ^i. Since $\pi_\mu(Y) = 0$ for $\mu < i$, $\Sigma^{i-1}\Phi$ can be extended by the usual skeleton-after-skeleton construction to $\Sigma^i_\lambda : \Delta^i \to Y$. Apply the same process to all i-cells e^i_λ, and define

$$\Sigma^i(x) = \begin{cases} \Sigma^{i-1}(x) & \text{for } x \in B^{i-1} \cup \tau^{-1}[\frac{i+1}{i+2}, 1] \supset B^{i-1} \cup A, \\ \Sigma^i_\lambda(\Phi^{-1}(x)) & \text{for } x \in e^i_\lambda. \end{cases}$$

The function Σ^i is clearly well defined by the construction, that is, $e^i_\lambda \cap (B^{i-1} \cup \tau^{-1}[\frac{i+1}{i+2}, 1]) \subset \dot{e}^i_\lambda \cup K$. It is continuous on closed i-cells and therefore on B^i. Moreover, it is continuous on the closed subset $\tau^{-1}[\frac{i+1}{i+2}, 1]$, hence on the union $T^i = B^i \cup \tau^{-1}[\frac{i+1}{i+2}, 1]$. It is clear that Σ^i satisfies the conditions $\Sigma^i|_{B^{i-1}} = \Sigma^{i-1}|_{B^{i-1}}, i = 1, 2, 3, \cdots$ and $\Sigma^i|_A = \sigma|_A, i = 0, 1, 2, \cdots$. Thus, as we mentioned above, $S : B \to B \times Y$ defined by $S(b) = (b, \Sigma(b))$ is a section of p over B. This completes the proof of Example 2.19. $\square$

Our next goal is to show that the SEP passes from a space B to any "nice" open subset $W \subset B$. First, a technical lemma on patching continuous functions is needed.

Lemma 2.35 *Let B be any topological space, $\rho : B \to I$ be continuous and $W = \rho^{-1}(0, 1]$. Let $\theta : W \to I$ be continuous and suppose that there exists a δ, $0 < \delta < 1$ such that for any $0 < \epsilon < \delta$, there exists an $\eta(\epsilon) > 0$ with the property $\theta^{-1}[\epsilon, 1] \subset \rho^{-1}[\eta(\epsilon), 1]$; equivalently, $\rho(b) < \eta(\epsilon)$ implies $\theta(b) < \epsilon$. Then, the function $\varphi : B \to I$ given by $\varphi|_W = \theta$ and $\varphi(b) = 0$ for all $b \in B \backslash W$ is continuous.*

Proof: We know that W is open and $\varphi|_W = \theta$ is continuous, so $\varphi : B \to I$ is continuous at every point in W. To show the continuity of φ on B, we have only to check the continuity of $\varphi : B \to I$ at every point of $B\backslash W$. Let $b \in B\backslash W$. Then, $\varphi(b) = 0$. We need to check that for any $\epsilon > 0$, $\varphi^{-1}[0, \epsilon)$ is a neighbourhood of b in B. Without loss of generality, we can assume $\epsilon < \delta$. We show that $\varphi^{-1}[\epsilon, 1]$ is closed in B.

Since $\varphi(b) = 0$, for $b \in B\backslash W$, $\varphi^{-1}[\epsilon, 1] \cap B\backslash W = \phi$; in other words, $\varphi^{-1}[\epsilon, 1] \subset W$. Since $\varphi = \theta$ on W, we see that $\varphi^{-1}[\epsilon, 1] \subset \theta^{-1}[\epsilon, 1]$. Hence, $\varphi^{-1}[\epsilon, 1] \subset \rho^{-1}[\eta(\epsilon), 1]$ for some $\eta(\epsilon) > 0$. Since $\varphi|_W$ is continuous, so is $\varphi\big|_{\rho^{-1}[\eta(\epsilon),1]}$. Hence $\varphi^{-1}[\epsilon, 1]$ is closed in $\rho^{-1}[\eta(\epsilon), 1]$ and therefore closed in B. Thus, φ, so defined, is continuous on B. □

Proposition 2.36 *If $p : E \to B$ has the SEP, and if $W \subset B$ is an open set such that $W = \rho^{-1}(0, 1]$ for some continuous function $\rho : B \to I$, then $p_W : E_W \to W$ has the SEP.*

Proof: We have the following commutative diagram:

$$\begin{array}{ccc} E_W & \xrightarrow{\hat{i}} & E \\ \downarrow{\scriptstyle p_W} & & \downarrow{\scriptstyle p} \\ W & \xrightarrow{i} & B \end{array}$$

where $E_W = \{(e, w) \in E \times W : p(e) = w\}$ and $W = \rho^{-1}(0, 1]$. Let α be a section of p_W over $X \subset W$ Then $\beta = \hat{i}\alpha : X \to E$ is a section of p over X since we have $p\beta = p\hat{i}\alpha = ip_W\alpha = i1_X = 1_X$. Thus, we can restate the proposition as follows:

(*) Let $p : E \to B$ have the SEP, $\rho : B \to I$ a continuous map, and $W = \rho^{-1}(0, 1]$. Let $\tau : W \to I$ be any continuous map and $s : \tau^{-1}(0, 1] \to E$ any section of p over $\tau^{-1}(0, 1]$. Then, there exists a section $S : W \to E$ of p such that $S\big|_{\tau^{-1}(1)} = s\big|_{\tau^{-1}(1)}$.

In the sense of Definition 2.15, $\tau^{-1}(1) = A$ and $\tau^{-1}(0,1] = V$. Note also that τ is NOT defined on B but on W and the section $S_W : W \to E_W$ is defined by $S_W(w) = (S(w), w)$.

To show (*), we will construct sections $S_n : B \to E$, $n = 2, 3, \ldots$, such that

i) $S_n(b) = S_{n-1}(b)$ for $\rho(b) > \frac{1}{n-1}$, $n = 3, 4, \ldots$

ii) $S_n(b) = s(b)$ for $\{\tau(b) > 1 - \frac{1}{n}$ and $\rho(b) > \frac{1}{n+1}\}$, $n = 2, 3, \ldots$. Note that this condition is well defined since $\rho(b) > \frac{1}{n+1}$ implies that $b \in W$, and therefore $\tau(b)$ and $s(b)$ are defined.

We impose these conditions since the first implies that $S = \lim_{n\to\infty} S_n$ is a well-defined section over $W = \rho^{-1}(0,1]$, and the second implies that $S(b) = s(b)$ for $\tau(b) = 1$, $b \in W$. Now we start the induction on n.

For $n = 2$, define $\tau' : B \to I$ by

$$\tau'(b) = \begin{cases} \tau(b)\rho(b) & \text{if } b \in W, \\ 0 & \text{otherwise.} \end{cases}$$

By setting $\theta(b) = \tau(b)\rho(b)$, $b \in W$, and $\eta(\epsilon) = \epsilon$, we see that τ' is continuous on B by Lemma 2.35 above. Notice that the set

$$\{b \in B : \tau(b) > 1 - \frac{1}{n} \text{ and } \rho(b) > \frac{1}{n+1}\}_{n=2} = \{b \in B : \tau(b) > \frac{1}{2} \text{ and } \rho(b) > \frac{1}{3}\}$$

is an open subset of B contained in $\tau'^{-1}[\frac{1}{6}, 1]$. Thus, if we find a section $S_2 : B \to E$ such that $S_2 \big|_{\tau'^{-1}[\frac{1}{6},1]} = s \big|_{\tau'^{-1}[\frac{1}{6},1]}$, we are done the case $n = 2$. Let $V' = \{b \in B : \tau'(b) > 0\}$, then $V' \subset W$. Note that $s\,|_{V'}$ is a section of p over V' and V' is a halo around $\tau'[\frac{1}{6}, 1]$. Since p has the SEP, by assumption, there exists a section S_2 such that $S_2 \big|_{\tau'^{-1}[\frac{1}{6},1]} = s \big|_{\tau'^{-1}[\frac{1}{6},1]}$, hence the case $n = 2$ holds.

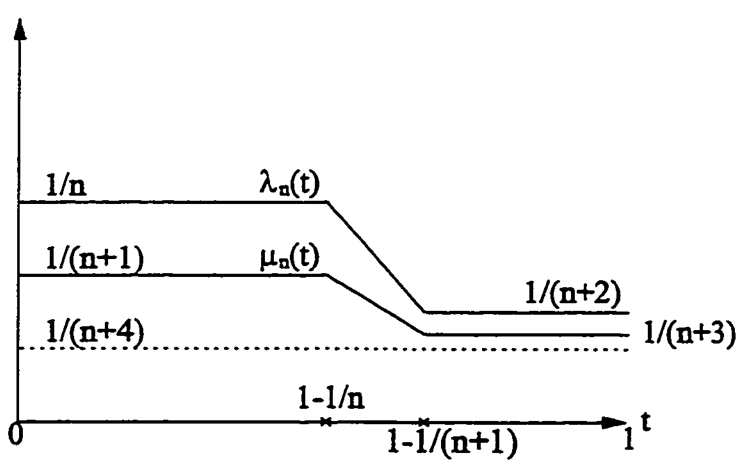
Suppose that sections $S_2, \ldots, S_n$ have been constructed satisfying the two conditions i) and ii). We will define $A_{n+1} \subset B$ and V_{n+1}, a halo around A_{n+1} in B so that we can conclude, by the SEP for p, that there exists a section $S_{n+1} : B \to E$ with $S_{n+1}\big|_{A_{n+1}} = s_{n+1}\big|_{A_{n+1}}$ where $s_{n+1} : V_{n+1} \to E$ is a suitably constructed section over V_{n+1} of p. Thus, $s_{n+1}\big|_{A_{n+1}}$ is a section of p over A_{n+1} which has an extension to a halo V_{n+1} around A_{n+1} in B. Of course, we still need to verify the two conditions i) and ii) above for S_{n+1}.

For this, we consider the functions $\lambda_n, \mu_n : I \to I$ defined by

$$\lambda_n(t) = \begin{cases} \frac{1}{n+1} & \text{if } t \leq 1 - \frac{1}{n}, \\ \frac{1}{n+3} & \text{if } t \geq 1 - \frac{1}{n+1}, \\ \frac{1}{n+1} + \frac{t+\frac{1}{n}-1}{\frac{1}{n}-\frac{1}{n+1}}\left(\frac{1}{n+3} - \frac{1}{n+1}\right) & \text{otherwise,} \end{cases}$$

$$\mu_n(t) = \begin{cases} \frac{1}{n} & \text{if } t \leq 1 - \frac{1}{n}, \\ \frac{1}{n+2} & \text{if } t \geq 1 - \frac{1}{n+1}, \\ \frac{1}{n+1} + \frac{t+\frac{1}{n}-1}{\frac{1}{n}-\frac{1}{n+1}}\left(\frac{1}{n+2} - \frac{1}{n}\right) & \text{otherwise.} \end{cases}$$

The graphs below show the relation between the functions λ_n and μ_n.



It is easy to check that both λ_n and μ_n are well defined and continuous. Then, we have $0 < \frac{1}{n+4} < \lambda_n(t) < \mu_n(t)$, $\mu_n(t) \leq \frac{1}{n}$ and $\lambda_n(t) \leq \frac{1}{n+1}$, for all $t \in I$. Now, define $V_{n+1} = \{b \in W : \rho(b) > \lambda_n(\tau(b))\}$. Observe that $b \in V_{n+1} \Rightarrow \lambda_n(\tau(b)) > 1 - \frac{1}{n}$ or $\rho(b) > \frac{1}{n+1}$. Note that $\tau(b) \leq 1 - \frac{1}{n} \Rightarrow \lambda_n(\tau(b)) = \frac{1}{n+1}$ by definition of $\lambda_n \Rightarrow \rho(b) > \frac{1}{n+1}$ Thus, $V_{n+1} \subset \{b \in W : \tau(b) > 1 - \frac{1}{n}\} \cup \{b \in W : \rho(b) > \frac{1}{n+1}\}$. Define a section $s_{n+1} : V_{n+1} \longrightarrow E$ over V_{n+1} of p by

$$s_{n+1}(b) = \begin{cases} S_n(b) & \text{if } b \in V_{n+1} \text{ and } \rho(b) > \frac{1}{n+1}, \\ s(b) & \text{if } b \in V_{n+1} \text{ and } \tau(b) > 1 - \frac{1}{n}. \end{cases}$$

Recall that $s : X = \tau^{-1}(0,1] \longrightarrow E$ is a section of p over X and $S_n : B \longrightarrow E$ is a section of p over B satisfying the two conditions $i)$ and $ii)$. Note that the condition $ii)$ of S_n guarantees that s_{n+1} is well defined. Let $A_{n+1} = \{b \in W : \rho(b) > \mu_n(\tau(b))\}$, then $A_{n+1} \subset V_{n+1}$ since $\mu_n > \lambda_n$. As we have mentioned earlier, we want to show that V_{n+1} is a halo around A_{n+1} in B. Thus, we have to define a haloing function $h : B \longrightarrow I$ with $h\big|_{A_{n+1}} = 1$ and $h\big|_{V^c_{n+1}} = 0$. We construct such an h in three steps.

1. Define $\beta : W \longrightarrow \mathbb{R}$ by $\beta(b) = \frac{\rho(b)-\lambda_n(\tau(b))}{\mu_n(\tau(b))-\lambda_n(\tau(b))}$. Clearly β is continuous on W since ρ, τ, λ_n and μ_n are all continuous on W, and $\mu_n > \lambda_n$. Then, the function $f : W \longrightarrow \mathbb{R}$ defined by $f(b) = min(1, \beta(b))$ is continuous on W. A couple of facts need to be mentioned. First,

$$\begin{aligned} b \in A_{n+1} &\Leftrightarrow \rho(b) > \mu_n(\tau(b)) > 0 \\ &\Leftrightarrow \rho(b) - \lambda_n(\tau(b)) > \mu_n(\tau(b)) - \lambda_n(\tau(b)) > 0 \\ &\Leftrightarrow \beta(b) = \frac{\rho(b) - \lambda_n(\tau(b))}{\mu_n(\tau(b)) - \lambda_n(\tau(b))} > 1 \\ &\Leftrightarrow f(b) = min(1, \beta(b)) = 1, \end{aligned}$$

and

$$\begin{aligned}
b \in W\backslash A_{n+1} &\Leftrightarrow \rho(b) \leq \mu_n(\tau(b)) \\
&\Leftrightarrow \rho(b) - \lambda_n(\tau(b)) \leq \mu_n(\tau(b)) - \lambda_n(\tau(b)) \\
&\Leftrightarrow \frac{\rho(b) - \lambda_n(\tau(b))}{\mu_n(\tau(b)) - \lambda_n(\tau(b))} \leq 1 \\
&\Leftrightarrow f(b) = \beta(b).
\end{aligned}$$

Thus, we can rewrite f as

$$f(b) = \begin{cases} 1 & \text{if } b \in A_{n+1}, \\ \beta(b) & \text{if } b \in W\backslash A_{n+1}. \end{cases}$$

Secondly, $b \in V_{n+1} \Rightarrow \rho(b) > \lambda_n(\tau(b)) \Rightarrow \rho(b) - \lambda_n(\tau(b)) > 0 \Rightarrow f(b) > 0$ since $\mu_n(\tau(b)) - \lambda_n(\tau(b)) > 0$. Thus, $f(b) > 0$, for all $b \in V_{n+1}$. Also, $b \notin V_{n+1} \Rightarrow \rho(b) - \lambda_n(\tau(b)) \leq 0 \Rightarrow f(b) \leq 0$. Note that $f(b) \leq 1$, for all $b \in W$.

2. Since f is continuous on W, $g(b) = max(0, f(b))$ is continuous on W. Since $f(b) \leq 1$, we have $0 \leq g(b) \leq 1$. Explicitly,

$$g(b) = \begin{cases} 1 & \text{if } b \in A_{n+1}, \\ 0 & \text{if } b \in W\backslash V_{n+1} \text{ (since } f(b) \leq 0 \text{ on } W\backslash V_{n+1}), \\ \beta(b) & otherwise. \end{cases}$$

3. Observe that for $b \in V_{n+1}$, we have $\rho(b) > \lambda(\tau(b)) > \frac{1}{n+4}$. Define $h : B \to I$ by $h|_W = g$ and $h\big|_{B\backslash W} = 0$. Notice that $h^{-1}[\epsilon, 1] \subset V_{n+1} \subset \rho[\frac{1}{n+4}, 1]$ whenever $0 < \epsilon < \frac{1}{n+4}$. Hence, h is continuous on B by Lemma 2.35 above (take $\delta = \frac{1}{n+4} = \eta(\epsilon)$).

Thus, V_{n+1} is a halo around A_{n+1} in B with the haloing function h. By the SEP for p, there is a section $S_{n+1} : B \to E$ with $S_{n+1}\big|_{A_{n+1}} = s_{n+1}\big|_{A_{n+1}}$. We now check the conditions i) and ii).

i) $\rho(b) > \frac{1}{n} \Rightarrow \rho(b) > \mu_n(\tau(b))$(because $\frac{1}{n} \geq \mu_n(t)$, for all $t \in I$) $\Rightarrow b \in A_{n+1} \Rightarrow S_{n+1}(b) = s_{n+1}(b)$ (by the very construction of S_{n+1}). Also, from $A_{n+1} \subset V_{n+1}$ and $\frac{1}{n} > \frac{1}{n+1}$, any b with $\rho(b) > \frac{1}{n}$ automatically satisfies $b \in V_{n+1}, \rho(b) > \frac{1}{n+1}$. By definition, $s_{n+1}(b) = S_n(b)$ then.

ii) $\tau(b) > 1 - \frac{1}{n+1}$ and $\rho(b) > \frac{1}{n+2} \Rightarrow \rho(b) > \frac{1}{n+2} = \mu_n(\tau(b)) \Rightarrow b \in A_{n+1} \Rightarrow S_{n+1}(b) = s_{n+1}(b)$ by the construction of S_{n+1}. Also, $1 - \frac{1}{n+1} > 1 - \frac{1}{n}$. Hence, $\tau(b) > 1 - \frac{1}{n+1} \Rightarrow b \in A_{n+1}, \tau(b) > 1 - \frac{1}{n} \Rightarrow b \in V_{n+1}, \tau(b) > 1 - \frac{1}{n} \Rightarrow s_{n+1}(b) = s_n(b)$. Hence, $S_{n+1}(b) = s(b)$.

This completes the inductive step.

Now, by setting $S = \lim_{n\to\infty} S_n$, we get a section S from B to E such that $S\big|_{\tau^{-1}(1)} = s\big|_{\tau^{-1}(1)}$ and therefore the claim holds. □

The last proposition may be thought of as passing from a global property to a local property. We now turn to the main theorem of this chapter, which may be thought of as passing from a local property to a global one.

Theorem 2.37 Section Extension Theorem

Let $p : E \to B$ be a space over B. If there exists a numerable covering $\{V_\lambda\}_{\lambda\in\Lambda}$ of B such that p has the SEP over each V_λ, then p has the SEP.

Proof: Note that $\{V_\lambda\}_{\lambda\in\Lambda}$ numerable implies that there exists a locally finite partition of unity $\{\pi_a : B \to I\}_{a\in A}$ such that each $\pi_a^{-1}(0,1]$ is contained in some V_λ, $\lambda = \lambda(a)$.

Since $p\big|_{p^{-1}(V_\lambda)} : p^{-1}(V_\lambda) \to V_\lambda$ has the SEP, Proposition 2.36 implies that the induced space, $p_{\pi_a^{-1}(0,1]} : E_{\pi_a^{-1}(0,1]} \to \pi_a^{-1}(0,1]$, has the SEP. So, without loss of generality, we can assume that the covering $\{V_\lambda\}_{\lambda\in\Lambda}$ itself corresponds to a locally finite partition of unity $\{\pi_\lambda : B \to I\}_{\lambda\in\Lambda}$, that is, $V_\lambda = \pi_\lambda^{-1}(0,1]$.

Let A be a subset of B, V a halo around A with a haloing function $\tau : B \to I$, and $s : V \to E$ a section of p over V. We want to find a section $S : B \to E$ such that $S|_A = s|_A$. Define $\pi'_\lambda : B \to I$ by $\pi'_\lambda(b) = (1-\tau(b))\pi_\lambda(b)$, and let $\pi'_0 = \tau$ and $\Lambda' = \Lambda\dot{\cup}\{0\}$, the disjoint union of Λ and $\{0\}$. Then, $\pi'^{-1}_\lambda(0,1] = \{b \in B : \pi_\lambda(b) \neq 0 \text{ and } \tau(b) \neq 1\} = \{b \in B : \tau(b) \neq 1\} \cap \pi_\lambda^{-1}(0,1]$. Note also that $\sum_{\lambda\in\Lambda'} \pi'_\lambda(b) = \sum_{\lambda\in\Lambda} \pi'_\lambda(b) + \pi'_0(b) = \sum_{\lambda\in\Lambda}(1-\tau(b))\pi_\lambda(b) + \tau(b) = (1-\tau(b))\sum_{\lambda\in\Lambda}\pi_\lambda(b) + \tau(b) = (1-\tau(b))\cdot 1 + \tau(b) = 1$ for all $b \in B$ and therefore $\{\pi'_\lambda\}_{\lambda\in\Lambda'}$ is a locally finite partition of unity. (The local finiteness comes from $\{\pi_\lambda\}_{\lambda\in\Lambda}$ being locally finite.) For any $\lambda \in \Lambda$, $(1-\tau)\big|_{\pi_\lambda^{-1}(0,1]} : \pi_\lambda^{-1}(0,1] \to I$ is continuous. Writing θ_λ for $(1-\tau)\big|_{\pi_\lambda^{-1}(0,1]}$, we see that $\theta_\lambda^{-1}(0,1] = \pi'^{-1}_\lambda(0,1]$. By Proposition 2.36, $E_{\theta_\lambda^{-1}(0,1]} \to \theta_\lambda^{-1}(0,1]$ has the SEP; equivalently, $E_{\pi'^{-1}_\lambda(0,1]} \to \pi'^{-1}_\lambda(0,1]$ has the SEP.

For every $\Gamma \subset \Lambda'$, define $\pi_\Gamma : B \to I$ by $\pi_\Gamma = \sum_{\gamma\in\Gamma}\pi'_\gamma$. Consider the family $\mathcal{F}$ of all pairs (Γ, S_Γ) where $0 \in \Gamma \subset \Lambda'$ and $S_\Gamma : \pi_\Gamma^{-1}(0,1] \to E$ is a section with $S_\Gamma\big|_{\tau^{-1}(1)} = s\big|_{\tau^{-1}(1)}$. Note that $(\{0\}, s)$ is such a pair for $s = S_{\{0\}} : \pi'^{-1}_0(0,1] = \tau^{-1}(0,1] \to E$ is a section since $V \supset \tau^{-1}(0,1]$. Hence the collection of pairs (Γ, S_Γ) is not empty. Now we construct a section $S : B \to E$ with $S|_A = s|_A$ using the following steps.

Step 1: Define a partial order on $\mathcal{F}$ and show that any chain of this collection has an upper bound in $\mathcal{F}$.

Step 2: By Zorn's Lemma, there exists a maximal element (M, S_M) in $\mathcal{F}$. We will show that $M = \Lambda'$.

Proof of step 1:

We define $(\Gamma, S_\Gamma) \leq (\Gamma', S_{\Gamma'}) \Leftrightarrow \Gamma \subset \Gamma'$ and $S_\Gamma(b) = S_{\Gamma'}(b)$ at all $b \in B$ satisfying $\pi_\Gamma(b) = \pi_{\Gamma'}(b) > 0$. Observe the following: if $(\Gamma, S_\Gamma) \leq (\Gamma', S_{\Gamma'})$ in $\mathcal{F}$, then

1. $S_\Gamma(b)$ is defined $\Leftrightarrow b \in \pi_\Gamma^{-1}(0,1] \Leftrightarrow \pi_\Gamma(b) \neq 0$.

2. $\pi_\Gamma(b) = \pi_{\Gamma'}(b) > 0 \Rightarrow \sum_{\gamma\in\Gamma} \pi'_\gamma(b) = \sum_{\gamma\in\Gamma'} \pi'_\gamma(b)$
$= \sum_{\gamma\in\Gamma} \pi'_\gamma(b) + \sum_{\gamma\in\Gamma'\setminus\Gamma} \pi'_\gamma(b) \Rightarrow \sum_{\gamma\in\Gamma'\setminus\Gamma} \pi'_\gamma(b) = 0 \Rightarrow \pi'_\gamma(b) = 0$ for all $\gamma \in \Gamma'\setminus\Gamma$.

3. $S_\Gamma(b) \neq S_{\Gamma'}(b) \Rightarrow \pi_\Gamma(b) \neq \pi_{\Gamma'}(b)$ and both positive by 1. above imply that there exists $\nu \in \Gamma'\setminus\Gamma$ such that $\pi'_\nu(b) \neq 0$, that is, $\pi_{\Gamma'}(b) > \pi_\Gamma(b)$.

Let $(\Gamma^\sigma, S_{\Gamma^\sigma})_{\sigma\in\Sigma}$ be a strictly ordered system of such pairs. Let $\Delta = \cup_{\sigma\in\Sigma}\Gamma^\sigma$. We want to show that there is a section $S_\Delta : \pi_\Delta^{-1}(0,1] \to E$ such that $(\Delta, S_\Delta) \geq (\Gamma^\sigma, S_{\Gamma^\sigma})$ for any $\sigma \in \Sigma$. We construct S_Δ as follows: Let $b \in \pi_\Delta^{-1}(0,1]$. Since $\{\pi'^{-1}_\lambda(0,1]\}_{\lambda\in\Lambda'}$ is locally finite, there exists an open neighbourhood W_b of b which intersects only finitely many of the sets $\{\pi'^{-1}_\lambda(0,1]\}_{\lambda\in\Lambda'}$, hence only finitely many of the sets $\{\pi'^{-1}_\lambda(0,1]\}_{\lambda\in\Delta}$. Without loss of generality, we may assume $W_b \subset \pi_\Delta^{-1}(0,1]$. Let $\lambda_1, \ldots, \lambda_r$ be all the elements of Δ satisfying $\pi'^{-1}_\lambda(0,1] \cap W_b \neq \phi$. Since $\Delta = \cup_{\sigma\in\Sigma}\Gamma^\sigma$, we can find elements $\sigma_i \in \Sigma$ for $1 \leq i \leq r$, with $\lambda_i \in \Gamma^{\sigma_i}$. Since $(\Gamma^\sigma, S_{\Gamma^\sigma})_{\sigma\in\Sigma}$ is totally ordered, one of these Γ^{σ_i} will satisfy $\Gamma^{\sigma_j} \subset \Gamma^{\sigma_i}$ for $1 \leq j \leq r$. Then, $\lambda_j \in \Gamma^{\sigma_i}$ for $1 \leq j \leq r$. Writing ρ for σ_i, we see that $\lambda_j \in \Gamma^\rho$ for $1 \leq j \leq r$ and $\rho \in \Sigma$. If $\sigma \in \Sigma$ satisfies $(\Gamma^\sigma, S_{\Gamma^\sigma}) \geq (\Gamma^\rho, S_{\Gamma^\rho})$, then for any $\mu \in \Gamma^\sigma\setminus\Gamma^\rho$, we have $\pi'^{-1}_\mu(0,1] \cap W_b = \phi$, since $(\Gamma^\rho, S_{\Gamma^\rho}) \geq (\Gamma^{\sigma_j}, S_{\Gamma^{\sigma_j}})$ for $1 \leq j \leq r$. Hence π'_μ vanishes on W_b, that is, $\pi'_\mu(\beta) = 0$ for all $\beta \in W_b$. It follows from observation 3 mentioned above that $S_{\Gamma^\rho}|_{W_b} = S_{\Gamma^\sigma}|_{W_b}$. Thus, for any $b \in \pi_\Delta^{-1}(0,1]$, we have an open set W_b of $\pi_\Delta^{-1}(0,1]$ with $b \in W_b$ satisfying the following condition:

(*) There exists a $\rho \in \Sigma$ with $S_{\Gamma^\sigma}|_{W_b} = S_{\Gamma^\rho}|_{W_b}$ for any $(\Gamma^\sigma, S_{\Gamma^\sigma}) \geq (\Gamma^\rho, S_{\Gamma^\rho})$.

We define $S_\Delta|_{W_b} = S_{\Gamma^\rho}|_{W_b}$ for any choice of such a ρ. To see S_Δ is well defined, let b, b' be any two elements of $\pi_\Delta^{-1}(0,1]$; $W_b, W_{b'}$ are open sets in $\pi_\Delta^{-1}(0,1]$ with $b \in W_b, b' \in W_{b'}$ and ρ, ρ' are elements in Σ satisfying $S_{\Gamma^\sigma}|_{W_b} = S_{\Gamma^\rho}|_{W_b}$ for all $(\Gamma^\sigma, S_{\Gamma^\sigma}) \geq (\Gamma^\rho, S_{\Gamma^\rho})$; $S_{\Gamma^\sigma}\Big|_{W_{b'}} = S_{\Gamma^\rho}\Big|_{W_{b'}}$ for all $(\Gamma^\sigma, S_{\Gamma^\sigma}) \geq (\Gamma^{\rho'}, S_{\Gamma^{\rho'}})$. Since either one of $(\Gamma^\rho, S_{\Gamma^\rho}) \geq (\Gamma^{\rho'}, S_{\Gamma^{\rho'}})$ or $(\Gamma^{\rho'}, S_{\Gamma^{\rho'}}) \geq (\Gamma^\rho, S_{\Gamma^\rho})$ is valid, we see that $S_{\Gamma^\sigma} = S_{\Gamma^\rho} = S_{\Gamma^{\rho'}}$, on $W_b \cap W_{b'}$, for every σ with $(\Gamma^\sigma, S_{\Gamma^\sigma}) \geq max((\Gamma^\rho, S_{\Gamma^\rho}), (\Gamma^{\rho'}, S_{\Gamma^{\rho'}}))$. Thus, $S_\Delta : \pi_\Delta^{-1}(0,1] \to E$ is a well-defined section.

We claim that $(\Delta, S_\Delta) \geq (\Gamma^\sigma, S_{\Gamma^\sigma})$ for any $\sigma \in \Sigma$. Clearly, $\Delta \supset \Gamma^\sigma$ for any $\sigma \in \Sigma$. To prove the claim, we have only to show that $b \in \pi_\Delta^{-1}(0,1] \cap \pi_{\Gamma^\sigma}^{-1}(0,1]$, and $S_\Delta(b) \neq S_{\Gamma^\sigma}(b)$ imply that there exists some $\mu \in \Delta \backslash \Gamma^\sigma$ with $\pi_\mu(b) > 0$. Since $b \in \pi_\Delta^{-1}(0,1]$, our earlier discussion shows that there exists a $\rho \in \Sigma$ satisfying $S_\Delta(b) = S_{\Gamma^\delta}(b) = S_{\Gamma^\rho}(b)$ for all $(\Gamma^\delta, S_{\Gamma^\delta}) \geq (\Gamma^\rho, S_{\Gamma^\rho})$. Since $S_\Delta(b) \neq S_{\Gamma^\sigma}(b)$, it follows that $(\Gamma^\rho, S_{\Gamma^\rho}) > (\Gamma^\sigma, S_{\Gamma^\sigma})$ and that $\pi_\mu(b) \neq 0$ for some $\mu \in \Gamma^\rho \backslash \Gamma^\sigma$. This μ is clearly in $\Delta \backslash \Gamma^\sigma$. This completes the proof of Step 1.

Proof of step 2:

By Zorn's lemma, there exists a maximal element (M, S_M) in $\mathcal{F}$. Suppose, if possible, that $M \neq \Lambda'$. Let $\mu \in \Lambda' \backslash M$ and $M' = M \cup \{\mu\}$. Notice that the sets $D_1 = \{b \in \pi_{M'}^{-1}(0,1] : \pi_\mu(b) \leq \pi_M(b)\}, D_2 = \{b \in \pi_{M'}^{-1}(0,1] : \pi_\mu(b) \geq \pi_M(b)\}$ are closed subsets of $\pi_{M'}^{-1}(0,1]$ with $D_1 \cup D_2 = \pi_{M'}^{-1}(0,1]$ and $D_1 \cap D_2 = \{b \in \pi_{M'}^{-1}(0,1] : \pi_\mu(b) = \pi_{M'}(b)\}$.

Since $b \in D_1 \Rightarrow \pi_M(b) + \pi_\mu(b) = \pi_{M'}(b) > 0$ with $\pi_M(b) \geq \pi_\mu(b)$, we get $\pi_M(b) > 0$ on D_1. Similarly, $\pi_\mu(b) > 0$ on D_2. Define $\varphi : \pi_{M'}^{-1}(0,1] \to I$ by

$$\varphi(b) = \begin{cases} 1 & \text{if } b \in D_1, \\ \frac{\pi_M(b)}{\pi_\mu(b)} & \text{if } b \in D_2. \end{cases}$$

The function φ is continuous on D_1 and D_2 separately. Also, on $D_1 \cap D_2, \pi_M(b) = \pi_\mu(b)$, hence φ is well defined. It follows that φ is a continuous function $\pi_{M'}^{-1}(0,1] \to I$. Note that $\varphi(b) > 0 \Rightarrow$ either $b \in D_1$ or $\pi_M(b) > 0$ if $b \in D_2$. But as already seen, $\pi_M(b) > 0$ if $b \in D_1$. Hence, $\varphi(b) > 0 \Rightarrow \pi_M(b) > 0$. Thus, $\varphi^{-1}(0,1] \subset \pi_M^{-1}(0,1]$. Hence $S_M \big|_{\varphi^{-1}(0,1]}$ is a section over $\varphi^{-1}(0,1]$. Moreover, $D_1 \subset \varphi^{-1}(0,1]$. Notice that $\varphi \big|_{\pi_\mu^{-1}(0,1]} : \pi_\mu^{-1}(0,1] \to I$ is a continuous function and $\varphi^{-1}(0,1] \cap \pi_\mu^{-1}(0,1]$ is a halo around $\varphi^{-1}(1) \cap \pi_\mu^{-1}(0,1]$ in $\pi_\mu^{-1}(0,1]$ with a haloing function $\tau = \varphi \big|_{\pi_\mu^{-1}(0,1]}$. Since p has the SEP over $\pi_\mu^{-1}(0,1]$, we see that there exists a section $S_\mu : \pi_\mu^{-1}(0,1] \to E$ with $S_\mu \big|_{\varphi^{-1}(1) \cap \pi_\mu^{-1}(0,1]} = S_M \big|_{\varphi^{-1}(1) \cap \pi_\mu^{-1}(0,1]}$. Define $S_{M'} : \pi_{M'}^{-1}(0,1] \to E$ by

$$S_{M'}(b) = \begin{cases} S_M(b) & \text{if } b \in D_1, \\ S_\mu(b) & \text{if } b \in D_2. \end{cases}$$

On $D_1 \cap D_2$, we have $\varphi(b) = 1$; hence $S_M = S_\mu$ on $D_1 \cap D_2$. It follows that $S_{M'}$ is a section over $\pi_{M'}^{-1}(0,1]$. Also, for $b \in \pi_{M'}^{-1}(0,1], S_M(b) \neq S_{M'}(b) \Rightarrow b \notin \varphi^{-1}(1) \Rightarrow \varphi(b) < 1 \Rightarrow 0 \leq \pi_M(b) < \pi_\mu(b)$, that is, $\mu \in M' \backslash M$ with $\pi_\mu(b) > 0$. Hence $(M', S_{M'}) > (M, S_M)$, contradicting with the maximality of (M, S_M).

Hence $M = \Lambda'$. This yields $B = \pi_{\Lambda'}^{-1}(0,1]$ and hence S_M is a section over B with $S_M \big|_{\tau^{-1}(1)} = s \big|_{\tau^{-1}(1)}$. □

Theorem 2.38 *Let $p : E \to B$ be a space over B, $A \subset B$, and s a section over A which admits an extension s' to a halo V around A. Assume there exists a numerable* *covering $\{V_\lambda\}_{\lambda\in\Lambda}$ of $B\backslash A$ such that*

1. *p_{V_λ} is shrinkable for each λ, that is, fibre homotopy equivalent to a trivial space $V_\lambda \times Y$ with contractible Y, or*

2. *p_{V_λ} is dominated by a trivial space $V_\lambda \times Y_\lambda$ with $\pi_i(Y_\lambda) = 0$ for $i < n_\lambda$, and V_λ is a retract of a CW-complex of dimension $\leq n_\lambda$, $n_\lambda \leq \infty$.*

Then, there exists a section $S : B \to E$ with $S|_A = s$.

If S, S' are two sections of p with $S|_V = S'|_V$, then $S \simeq_B S' rel A$, provided that, in case 2, we further have $\pi_i(Y_\lambda) = 0$ for $i < n_\lambda + 1$.

Proof: From the previous results, p has the SEP over each V_λ, and by the Section Extension Theorem, p has the SEP over $B' = B\backslash A$. Let $\tau : B \to [0,1]$ be a haloing function for V around A. Let $A' = B' \cap \tau^{-1}[\frac{1}{2}, 1]$ and $\tau' = min(1, 2\tau)$. Then V is a halo around A' with a haloing function τ'. Since $A' \subset V$, $s'|_{A'}$ is a section over A'. By the SEP for B', there exists a section $S' : B' \to E$ such that $S'|_{A'} = s'|_{A'}$. Define $S : B \to E$ by

$$S(b) = \begin{cases} s'(b) & \text{if } \tau(b) > \frac{1}{2}, \\ S'(b) & \text{if } \tau(b) < 1. \end{cases}$$

Note that $\tau(b) > \frac{1}{2} \Rightarrow b \in A \cup A'$ and $\tau(b) < 1 \Rightarrow b \notin A \Rightarrow b \in B'$. Note also that $\{b \in B : \tau(b) > \frac{1}{2}\} \cap \{b \in B : \tau(b) < 1\} \subset A'$. Let $a' \in A'$, then $s'(a') = S'(a')$ by the property of S'. Thus, S is a well defined section from B to E and continuous by the Gluing Lemma. Now, if $a \in A$, we get $S(a) = s'(a) = s(a)$ since s' is an extension of $s : A \to E$, hence $S|_A = s$. Thus, p has the SEP over B.

For the second assertion, let $\tilde{A} = B \times \{0,1\} \cup A \times I$, $\tilde{V} = B \times I\backslash\{\frac{1}{2}\} \cup V \times I$, then $\tilde{A} \subset \tilde{V}$. Further, $\tilde{V}$ is a halo around $\tilde{A}$ with a haloing function $\tilde{\tau} : B \times I \to I$

by $\tilde{\tau}(b,t) = min(\tau(b) + |2t-1|, 1)$. To see this, let $\tilde{a} \in \tilde{A}$ then $\tilde{a} = (a,t)$, $(b,0)$ or $(b,1)$, for some $a \in A$, $b \in B$, $t \in I$. Since we have

1. $\tilde{\tau}(a,t) = min(\tau(a) + |2t-1|, 1) = min(1+$ non-negative $, 1) = 1$,

2. $\tilde{\tau}(b,0) = min(\tau(b) + |2 \cdot 0 - 1|, 1) = min(\tau(b) + 1, 1) = 1$, and

3. $\tilde{\tau}(b,1) = min(\tau(b) + |2 \cdot 1 - 1|, 1) = min(\tau(b) + 1, 1) = 1$,

$\tilde{\tau}(\tilde{a}) = 1$. For $\tilde{b} \in \tilde{V}^c$, we have $\tilde{b} = (\tilde{v}, \frac{1}{2})$, for some $\tilde{v} \in V^c$. Hence, $\tilde{\tau}(\tilde{v}, \frac{1}{2})$ $= min(\tau(\tilde{v}) + |2 \cdot \frac{1}{2} - 1|, 1) = min(0+0, 1) = 0$. Define $\tilde{S} : \tilde{V} \to E \times I$ by

$$\tilde{S}(b,t) = \begin{cases} (S(b),t) & \text{if } (b,t) \in B \times [0, \frac{1}{2}), \\ (S'(b),t) & \text{if } (b,t) \in B \times (\frac{1}{2}, 1], \\ (S(b),t) & \text{if } (v,t) \in V \times I. \end{cases}$$

Since $S|_V = S'|_V$, $\tilde{S}$ is well defined. Since $B \times [0, \frac{1}{2})$ and $B \times (\frac{1}{2}, 1]$ are both open, and $(S(b),t)$ and $(S'(b),t)$ are both continuous on these open sets, we only have to show the continuity of $\tilde{S}$ on $V \times I$. Since $\tilde{S}(v,t) = (S(v),t)$, we have to show that $\tilde{S}^{-1}(O)$ is open in $\tilde{V}$ where O is an open set containing the point $(S(v),t)$. But, $(S, 1_I)$ is continuous on $B \times I$, so $O' = (S, 1_I)^{-1}(O)$ is open in $B \times I$ containing (v,t) and therefore $O' = O \cap V$ is open in $\tilde{V}$. Thus, $\tilde{S}$ is continuous on $\tilde{V}$. Hence $\tilde{S}$ is a section over $\tilde{V}$. Note that in case 1., $(p \times 1_I)_{V_\lambda \times I}$ is shrinkable. In case 2., $(p \times 1_I)_{V_\lambda \times I}$ is dominated by $V_\lambda \times I \times Y_\lambda$. Now $V_\lambda \times I$ is a retract of a CW-complex of dimension $n_\lambda + 1$. By assumption, $\pi_i(Y_\lambda) = 0$ for $0 \leq i \leq n_\lambda + 1$. Hence, the result proved already yields a section $\bar{S} : B \times I \to E \times I$ with $\bar{S}|_{\tilde{A}} = \tilde{S}|_{\tilde{A}}$. Define $\Theta : B \times I \to E$ by $\Theta(b,t) = \hat{\alpha}\bar{S}(b,t)$ where $\hat{\alpha} : E \times I \to E$ is a projection by $\hat{\alpha}(e,t) = e$. Then, we get the following:

1. $\Theta(b,0) = \hat{\alpha}\bar{S}(b,0) = \hat{\alpha}\tilde{S}(b,0) = \hat{\alpha}(S(b),0) = S(b)$,

2. $\Theta(b,1) = \hat{\alpha}\bar{S}(b,1) = \hat{\alpha}\tilde{S}(b,1) = \hat{\alpha}(S'(b),1) = S'(b)$, and

3. $\Theta(a,t) = \hat{\alpha}\bar{S}(a,t) = \Theta(a,t) = \hat{\alpha}\tilde{S}(a,t) = \hat{\alpha}(S(a),t) = S(a)$ for all $t \in I$ and for all $a \in A$,

since $(b,0)$, $(b,1)$ and $(a,t) \in \tilde{A}$. Thus, $\Theta : S \simeq_B S'$ *rel* A since $\tilde{S}_t(b) = \tilde{S}(b,t)$ is a map over B. □

Remark 2.39 *There exist spaces over B which are not locally trivial (cf. Definition 1.22) for which the corollary above applies. Let $E = \{(x,y) \in \mathbb{R}^2 : |y| \leq |x|\}$, $B = \mathbb{R}$ and $p : E \to B$, $p(x,y) = x$. Clearly, p is shrinkable. If $s(x) = (x,0) \in E$, then $ps = 1_B$ and $sp \simeq_B 1_E$. However, $p : E \to B$ itself is not locally trivial; the fibre of p over $\{0\}$ is a single point $\{0\}$ and for any $t \neq 0$ in $\mathbb{R}$, the fibre of p over $\{t\}$ is the closed interval $[-t,t]$ of $\mathbb{R}$.*

Chapter 3

Hereditary SEP and Fibre Homotopy Equivalence

In this chapter, we will show that fibre homotopy equivalence is a local property, that is, for a numerable covering $\{V_\lambda\}_{\lambda\in\Lambda}$ of B, if $f_\lambda : p^{-1}(V_\lambda) \to p'^{-1}(V_\lambda)$ is a fibre homotopy equivalence for each $\lambda \in \Lambda$, then so is f.

Proposition 3.1 *Let $p : E \to B$ be a space over B. The following are equivalent:*

1. *Every induced space p_α has the SEP.*
2. *Given $G : X \to B$, a halo V around $A \subset X$, and $f : V \to E$ with $pf = G|_V$, there exists $F : X \to E$ with $pF = G$ and $F|_A = f$.*
3. *Given $G : X \to B$, $A \subset X$, and $f : A \to E$ with $pf = G|_A$, there exists a lift $F : X \to E$ of G with $F|_A \simeq_B f$.*
4. *The space p over B is shrinkable.*
5. *The space p over B is fibre homotopy equivalent to a trivial space $B \times Y \to B$ with Y contractible.*
6. *The space p over B is dominated by a trivial space $B \times Y \to B$ with Y contractible.*

<u>Proof</u>: Recall 2 of Example 1.14. This shows the equivalence of 4. $\Leftrightarrow$ 5.. The implication 5. $\Rightarrow$ 6. follows directly from Definition 1.9. Moreover, a trivial space

$B \times Y \to B$ is homotopic to B when Y is contractible, the hypothesis 6. is equivalent to saying that the space p over B is dominated by 1_B. It follows from Proposition 1.12 that 6. $\Rightarrow$ 4.. Hence we need only to verify the equivalence of 1. $-$ 4..

Proof of 1. $\Rightarrow$ 2. Recall $E_G = \{(e,x) \in E \times X : p(e) = G(x)\}$, $p_G(e,x) = x$. Then $p_G : E_G \to X$ is the induced space over X. If $\hat{G}(e,x) = e$, then

$$\begin{tikzcd} E_G \arrow[r, "\hat{G}"] \arrow[d, "p_G"] & E \arrow[d, "p"] \\ X \arrow[r, "G"] & B \end{tikzcd}$$

is commutative. From the commutative diagram (and the hypotheses),

$$\begin{tikzcd} & E_G \arrow[r, "\hat{G}"] \arrow[d] & E \arrow[d, "p"] \\ V \arrow[r, hook] \arrow[urr, "f"] & X \arrow[r, "G"'] & B \end{tikzcd}$$

we can express $s : V \to E_G$ by $s(v) = (f(v), v)$. Since $pf(v) = G(v)$, $(f(v), v) \in E_G$. The function s is continuous, since f and 1_V are both continuous. Further, $p_G s(v) = p_G(f(v), v) = v \Rightarrow s$ is a section of p_G over V. Since p_G has the SEP, there exists a section $S : X \to E_G$ with $S|_A = s|_A$ (since $s|_A$ is a section and has an extension $s : V \to E_G$). Define $F : X \to E$ by $F = \hat{G}S$, then $pF = p\hat{G}S = Gp_GS = G1_X = G$ and $F(a) = \hat{G}S(a) = \hat{G}S|_A(a) = \hat{G}s|_A(a) = \hat{G}(f(a), a) = f(a)$, for all $a \in A$. Thus, 2. holds.

Proof of 2. $\Rightarrow$ 1. Let $\alpha : X \to B$ be continuous, $p_\alpha : E_\alpha \to X$ the induced space, V a halo around $A \subset X$, and $s : V \to E_\alpha$ a section of p_α over V. We want to find a section $S : X \to E_\alpha$ with $S|_A = s|_A$.

$$\begin{tikzcd}
 & & E_\alpha \arrow[r, "\hat{\alpha}"] \arrow[d, shift left, "p_\alpha"] & E \arrow[d, "p"] \\
A \arrow[r, hook] \arrow[urr, "s_A"] & V \arrow[r, hook] \arrow[ur, "s"] & X \arrow[r, "\alpha"'] \arrow[u, shift left, "S"] & B
\end{tikzcd}$$

Let $f : V \to E$ be defined by $s(v) = (f(v), v)$. The continuity of f is clear since s is continuous. Since $(f(v), v) \in E_\alpha$, $pf(v) = \alpha(v)$. Thus, $pf = \alpha|_V$. By hypothesis 2., there exists $F : X \to E$ with $F|_A = f|_A$ and $pF = \alpha$. Define $S : X \to E_\alpha$ by $S(x) = (F(x), x)$. Since $pF(x) = \alpha(x)$, $(F(x), x) \in E_\alpha$. Further, F and 1_X are both continuous on X, so S is continuous. Note also that S is a section since $p_\alpha S(x) = p_\alpha(F(x), x) = x$. Finally, $S(a) = (F(a), a) = (f(a), a) = s|_A(a)$, for all $a \in A$. Thus, p_α has the SEP.

Proof of 2. $\Rightarrow$ 4. Recall Example 2.3. Then, by hypothesis 2., there exists a map $F' : B \to E$ such that $pF' = 1_B$, hence F' is a section of p over B.

$$\begin{tikzcd}
 & & E \arrow[d, shift right, "p"'] \\
\phi \arrow[r, hook] & \phi \arrow[r, hook] \arrow[ur, "f"] & B \arrow[u, dashed, shift right, "F'"']
\end{tikzcd}$$

Define $G : E \times I \to B$ by $G(e, t) = p(e)$ and let $V = E \times I \backslash \{\frac{1}{2}\}$, $A = E \times \{0, 1\} = E \times \{0\} \cup E \times \{1\}$. Then, V is a halo around A with a haloing function $\tau : E \times I \to I$ by $\tau(e, t) = |2t - 1|$. Note that $\tau(e, 0) = |2 \cdot 0 - 1| = |-1| = 1 = |2 \cdot 1 - 1| = \tau(e, 1)$ and $\tau(e, \frac{1}{2}) = |2 \cdot \frac{1}{2} - 1| = 0$. Thus, $A \subset \tau^{-1}(1)$ and $V^c \subset \tau^{-1}(0)$. Define $f : V \to E$ by

$$f(e, t) = \begin{cases} e & \text{if } t < \frac{1}{2}, \\ F'p(e) & \text{if } t > \frac{1}{2}. \end{cases}$$

Then, we have $pf = G|_V$.

$$\begin{tikzcd}
 & & & E \arrow[d, "p"] \\
E\times\dot{I} \arrow[r, hook] & E\times I\backslash\{\frac{1}{2}\} \arrow[r, hook] \arrow[urr, "f"] & E\times I \arrow[r] \arrow[ur, dashed, "F"] & B
\end{tikzcd}$$

By hypothesis 2., there exists a lift F of G with $F|_A = f|_A$. Then we have $F(e,0) = f(e,0) = e = 1_E(e)$ and $F(e,1) = f(e,1) = F'p(e)$. Note that $pF = G = p$ is a map over B and therefore $F : 1_E \simeq_B F'p$. Thus, p is shrinkable.

Proof of 4. $\Rightarrow$ 3. Since p is shrinkable, there exists a section $S : B \to E$ and a vertical homotopy $\Theta : 1_E \simeq_B Sp$. Given $G : X \to B$, $f : A \to E$ with $pf = G|_A$, define $F : X \to E$ by $F = SG$. Then $\Theta(f(a),0) = f(a)$ and $\Theta(f(a),1) = Spf(a) = SG|_A(a) = F|_A(a)$. Since Θ is a vertical homotopy, $\Theta|_{A\times I} : f \simeq_B F|_A$ as required.

$$\begin{tikzcd}
 & & E \arrow[d, shift left, "p"] \\
A \arrow[r, hook] \arrow[urr] & X \arrow[r] \arrow[ur, dashed, "F"] & B \arrow[u, shift left, "S"]
\end{tikzcd}$$

Proof of 3. $\Rightarrow$ 2. Suppose that for given $G : X \to B$, $A \subset X$, and $f : A \to E$ with $pf = G|_A$, there exists a lift $F' : X \to E$ with $F'|_A \simeq_B f$.

$$\begin{tikzcd}
 & & E \arrow[d, "p"] \\
A \arrow[r, hook] \arrow[urr] & X \arrow[r, "F"'] \arrow[ur, dashed, "F'"] & B
\end{tikzcd}$$

Let $G : X \to B$, V a halo around $A \subset X$ with a haloing function $\tau : X \to I$, and $f : V \to E$ with $pf = G|_V$. We want to find $F : X \to E$ such that

$pF = G$ and $F|_A = f|_A$. By assumption, there exists a lift $F' : X \to E$ with $F'|_V \simeq_V f$. Let D be a vertical homotopy from $F'|_V$ to f, and define $F : X \to E$ by

$$F(x) = \begin{cases} F'(x) & \text{if } \tau(x) \leq \frac{1}{2}, \\ D(x, 2\tau(x) - 1) & \text{if } \tau(x) \geq \frac{1}{2}. \end{cases}$$

Note that $\tau(x) \geq \frac{1}{2} \Rightarrow x \notin V^c \Rightarrow x \in V$ and when $\tau(x) = \frac{1}{2}$, we get $F(x) = D(x, 2\tau(x) - 1) = D(x, 0) = F'(x)$, so F is well defined. Hence, by the Gluing Lemma, it is continuous on X. If $a \in A \subset X$, then $F(a) = D(a, 2\tau(a) - 1) = D(x, 1) = f(a)$ since $A \subset \tau^{-1}(1)$. Recall that F' and D_t are maps over B where $D_t(x) = D(x, t)$.

$$\begin{tikzcd}
 & & & E \arrow[d, "p"] \\
A \arrow[r, hook] \arrow[urrr, "f|_A"] & V \arrow[r, hook] \arrow[urr, "f"] & X \arrow[r] \arrow[ur, "F'"] \arrow[ur, dashed, "F"'] & B
\end{tikzcd}$$

Hence $pF' = F$ and $pD_t = G$. Thus, $pf = G$ and therefore the claim holds. □

Lemma 3.2 *If $p : E \to B$ is shrinkable, then, for any $\alpha : X \to B$, the induced space $p_\alpha : E_\alpha \to X$ is shrinkable.*

Proof: The space p over B shrinkable implies that there exists a section $s : B \to E$ and a vertical homotopy $\varphi : sp \simeq_B 1_E$. Define $S : X \to E_\alpha = \{(e, x) \in E \times X : p(e) = \alpha(x)\}$ by $S(x) = (s\alpha(x), x)$ and $\psi : E_\alpha \times I \to E_\alpha$ by $\psi((e, x), t) = (\varphi(e, t), x)$. Since $p(s\alpha(x)) = \alpha(x)$, S is well defined, and $p_\alpha S(x) = p_\alpha(s\alpha(x), x) = x = 1_X(x) \Rightarrow S$ is a section of p_α over X. Note that the continuity of S is clear, since each coordinate function is continuous.

Since $p(\varphi(e,t)) = p\varphi_t(e) = p(e) = \alpha(x)$, ψ is well defined. The continuity of ψ is obvious. Since $p_\alpha\psi_t(e,x) = p_\alpha\psi((e,x),t) = p_\alpha(\varphi(e,t),x) = x = p_\alpha(e,x)$, ψ_t is a map over X for all $t \in I$. Moreover, $\psi((e,x),0) = (\varphi(e,0),x) = (sp(e),x) = (s\alpha(x),x) = S(x) = Sp_\alpha(e,x)$, and $\psi((e,x),1) = (\varphi(e,1),x) = (e,x) = 1_{E_\alpha}(e,x)$. Hence, ψ is a vertical homotopy from Sp_α to 1_{E_α}, and therefore p_α is shrinkable by definition. □

Corollary 3.3 *If $p : E \to B$ is shrinkable over each set V_λ of a numerable covering $\{V_\lambda\}$ of B, then so is p.*

Proof: Let $\alpha : X \to B$ be continuous. Then, by Lemma 3.2, p_α is shrinkable over $\alpha^{-1}(V_\lambda)$, and p_α has the SEP over $\alpha^{-1}(V_\lambda)$ by Proposition 2.17 for each $\lambda \in \Lambda$ (note that p_α should be written as $p_\alpha\big|_{p_\alpha^{-1}(\alpha^{-1}(V_\lambda))}$ to be precise). But, $\{\alpha^{-1}(V_\lambda)\}_{\lambda\in\Lambda}$ is a numerable covering of X by 2. of Example 2.10. Hence, p_α has the SEP over X by the Section Extension Theorem (Theorem 2.37). Since α is chosen arbitrary, every induced space p_α has the SEP, and therefore, by 1. $\Leftrightarrow$ 4.. in Proposition 3.1, p is shrinkable. □

We now turn to the proof of a technical result (Lemma 3.4), after which the main theorem of this chapter (Theorem 3.6) will follow comparatively easily. Although the proof of the lemma occupies over ten pages, most of it consists of tedious verification of many details and is not conceptually difficult.

Let $E^I = \{\omega : I \to E\}$ with the compact-open topology. Define $R = \{(y,\omega) \in E' \times E^I : p'(y) = p\omega(t) \text{ for all } t \in I \text{ and } \omega(1) = f(y)\}$. Then, a point $(y,\omega) \in R$ is a pair consisting of a point $y \in E'$ and a path ω completely contained in $p^{-1}(p'(y))$. Now we state the lemma.

Lemma 3.4 *If $f : E' \to E$ is a fibre homotopy equivalence over B, then $q : R \to E$ defined by $q(y, \omega) = \omega(0)$ is shrinkable with R as defined above.*

Proof: We will show this by defining a section $\sigma : E \to R$ ($q\sigma = 1_E$) such that there exists a vertical homotopy $D : 1_R \simeq_E \sigma q$. Throughout this proof, we will adapt the following notations: let $\omega, \omega' : I \to X$ be paths in X, and $\tau \in I$ fixed.

1. The product path $\omega \cdot \omega'$ is defined by

$$(\omega \cdot \omega')(t) = \begin{cases} \omega(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ \omega'(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

   For simplicity, we take $[\frac{i-1}{n}, \frac{i}{n}]$ to be the domain of the ith path of a product of n paths where $i = 1, \ldots, n$.

2. ${}_\tau\omega : I \to X$ is defined by ${}_\tau\omega(t) = \omega(\tau t)$.

3. ${}^\tau\omega : I \to X$ is defined by ${}^\tau\omega(t) = \omega(1 - \tau + \tau t) = \omega(1 - \tau(1-t))$.

4. $\omega^- : I \to X$ is defined by $\omega^-(t) = \omega(1-t)$, the inverse path (the reverse path).

5. $c_x : I \to X$ is the constant path at $x \in X$.

For example, $\omega = {}_1\omega = {}^1\omega$, ${}_0\omega = c_{\omega(0)}$ and ${}^0\omega = c_{\omega(1)}$.

Let $f' : E \to E'$ be a fibre homotopy inverse of f over B, that is, $ff' \simeq_B 1_E$ and $f'f \simeq_B 1_{E'}$, and let $\varphi : 1_{E'} \simeq_B f'f$ and $\psi : 1_E \simeq_B ff'$ be vertical homotopies. Let $t \in I$ be fixed, then $\varphi_t : E' \to E'$ and $\psi_t : E \to E$ are maps over B, and therefore $p'\varphi_t = p'$, $p\varphi_t = p$. Similarly, for fixed $y \in E'$ and $z \in E$, $\varphi_y : I \to E'$ and $\psi_z : I \to E$ are paths in E' and E, respectively. Note that for any $(y, \omega) \in R$ and for any $t \in I$, we have

$$(*)\quad \begin{cases} p\psi_{\omega(0)}(t) = p\psi(\omega(0), t) = p\psi_t(\omega(0)) = p(\omega(0)) = p'(y) \text{ and,} \\ p'\varphi_y(t) = p'\varphi(y, t) = p'\varphi_t(y) = p'(y). \end{cases}$$

We now return to the proof. First, define $\sigma : E \to R$ by $\sigma(z) = (f'(z), \psi_z)$. Note that $(f'(z), \psi_z) \in R$ since $p\psi_z(t) = p\psi(z,t) = p\psi_t(z) = p(z) = p'f'(z) = p'(f'(z))$, for all $t \in I$, and $\psi_z(1) = \psi(z, 1) = ff'(z) = f(f'(z))$. Since both f' and ψ_z are continuous on E, so is σ. Furthermore, since $q\sigma(z) = q(f'(z), \psi_z) = \psi_z(0) = \psi(z, 0) = 1_E(z) = z$, σ is a section of q over E.

It remains to construct a vertical homotopy $D : R \times I \to R$ over E from 1_R to σq.

$$\begin{tikzcd}
R\times I \arrow[r, "D"] & R \arrow[r, "u"] \arrow[d, shift right, "q"'] & (E')^I & \\
E\times I \arrow[r, "\Psi"] & E \arrow[u, shift right, "\sigma"'] \arrow[r, shift left, "f'"] \arrow[dr, "p"'] & E' \arrow[l, shift left, "f"] \arrow[d, "p'"] & E'\times I \arrow[l, "\Phi"'] \\
& & B &
\end{tikzcd}$$

Consider $u : R \to (E')^I$ by $u(y, \omega) = \varphi_y \cdot f' \circ (\omega^- \cdot \psi_{\omega(0)} \cdot ff'\omega \cdot f\varphi_y^- \cdot \omega^-)$. Here φ_y^- means $(\varphi_y)^-$. The following steps verify that u is well defined.

1. $\varphi_y(1) = \varphi(y, 1) = f'f(y) = f'\omega(1) = f'\omega^-(0) = f'(\omega^- \cdot \psi_{\omega(0)} \cdot ff'\omega \cdot f\psi_y^- \cdot \omega(0))$.

2. $\omega^-(1) = \omega(1-1) = \omega(0) = 1_E(\omega(0)) = \psi(\omega(0), 0) = \psi_{\omega(0)}(0)$.

3. $\psi_{\omega(0)}(1) = \psi(\omega(0), 1) = ff'(\omega(0)) = ff'\omega(0)$.

4. Since $(y, \omega) \in R$, $ff'\omega(1) = ff'f(y) = f\varphi(y, 1) = f\varphi_y(1-0) = f\varphi_y^-(0)$.

5. $f\varphi_y^-(1) = f\varphi_y(1-1) = f\varphi(y, 0) = f1_{E'}(y) = f(y) = \omega(1) = \omega(1-0) = \omega^-(0)$.

The continuity of u is clear by the definition of product path. Since a composition of continuous functions and any product path is again continuous, we will not show the continuity in this proof if a function is so defined. Similarly, we do not prove the continuity of a function if each coordinate function is continuous as well as those satisfying the assumptions of the Gluing Lemma.

We will construct a vertical homotopy $D : R \times I \to R$ of 1_R to σq in six stages, namely,

$$D((y,\omega),\tau) = \begin{cases} d_1((y,\omega),6\tau) & \text{if } 0 \le \tau \le \frac{1}{6}, \\ d_2((y,\omega),6\tau-1) & \text{if } \frac{1}{6} \le \tau \le \frac{2}{6}, \\ d_3((y,\omega),6\tau-2) & \text{if } \frac{2}{6} \le \tau \le \frac{3}{6}, \\ d_4((y,\omega),6\tau-3) & \text{if } \frac{3}{6} \le \tau \le \frac{4}{6}, \\ d_5((y,\omega),6\tau-4) & \text{if } \frac{4}{6} \le \tau \le \frac{5}{6}, \\ d_6((y,\omega),6\tau-5) & \text{if } \frac{5}{6} \le \tau \le 1, \end{cases}$$

where the vertical homotopies d_i and v_y are defined below. For suitable maps $K_i : R \to R$, the method to be used can be outlined as $d_1 : 1_R \simeq_E K_1$, $d_2 : K_1 \simeq_E K_2$, $d_3 : K_2 \simeq_E K_3K_2$, $d_4 : K_3K_2 \simeq_E K_4$, $d_5 : K_4 \simeq_E K_5$, $d_6 : K_5 \simeq_E K_6 = \sigma q$.

Let $K_0 = 1_R$. For each $i = 1, \ldots, 6$, we will

1. define K_i and show that the path used in the definition by $K_i : R \to R$ is well defined,
2. show that $imK_i \subset R$ and K_i is a map over E,
3. define d_i and show that the path defined by $(d_i)_\tau$, $\tau \in I$ is well defined,
4. show that $imd_i \subset R$ and $(d_i)_\tau$ is a map over E, and

5. show that d_i is a vertical homotopy from K_{i-1} to K_i.

$d_1 : R \times I \to R$:

1. Define $K_1 : R \to R$ by

$$K_1(y, \omega) = (y, \omega \cdot c_{\omega(1)}),$$

where $c_{\omega(1)}$ is the constant path at $\omega(1)$. The product path $\omega \cdot c_{\omega(1)}$ is clearly well defined.

2. Since $(y, \omega) \in R$, $K_1(y, \omega) = (y, \omega \cdot c_{\omega(1)}) \in R$. Clearly, K_1 is a map over E since $qK_1(y, \omega) = q(y, \omega \cdot c_{\omega(1)}) = \omega \cdot c_{\omega(1)}(0) = \omega(0) = q(y, \omega)$.

3. Define $d_1 : R \times I \to R$ by

$$d_1((y, \omega), \tau) = (y, \omega g_\tau),$$

where $g_\tau : I \to I$ by $g_\tau(t) = min((1+\tau)t, 1)$. Since g_τ is continuous and $0 \leq g_\tau(t) \leq 1$ for any t, $\tau \in I$, the path $\omega g_\tau \in E^I$ is well defined.

4. Clearly $p(\omega g_\tau(t)) = p'(y)$, for all $t \in I$ and $\omega g_\tau(1) = \omega(1) = p'(y)$. Hence d_1 is well defined. Since $q(y, \omega g_\tau) = \omega g_\tau(0) = \omega(min((1+\tau) \cdot 0, 1)) = \omega(0)$ $= q(y, \omega)$, $(d_1)_\tau$ is a map over E.

5. Note that $\omega g_0(t) = \omega((min((1+0)t, 1)) = \omega(min(t, 1)) = \omega(t)$ and

$$\begin{aligned} \omega g_1(t) &= \omega(min((1+1)t, 1)) \\ &= \omega(min(2t, 1)) \\ &= \begin{cases} \omega(2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ c_{\omega(1)}(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases} \\ &= (\omega \cdot c_{\omega(1)})(t). \end{aligned}$$

Thus, d_1 is a vertical homotopy from K_0 to K_1.

$d_2 : R \times I \to R$:

1. Define $K_2 : R \to R$ by

$$K_2(y, \omega) = (f'\omega(0), \omega \cdot f \circ u(y, \omega)),$$

where $u \circ (y, \omega) = \varphi_y \cdot f'(\omega^- \cdot \psi_{\omega(0)} \cdot ff'\omega \cdot f\varphi_y^- \cdot \omega^-)$ (recall that $u : R \to (E')^I$). Since $(y, \omega) \in R$, we have

$$\begin{aligned} f \circ u(y,\omega)(0) &= f(\varphi_y \cdot f'(\omega^- \cdot \psi_{\omega(0)} \cdot ff'\omega \cdot f\varphi_y^- \cdot \omega^-)(0)) \\ &= f\varphi_y(0) = f\varphi(y,0) = f1_{E'}(y) = f(y) = \omega(1), \end{aligned}$$

and therefore $\omega \cdot f \circ u(y, \omega)$ is well defined.

2. Note that $p'f'\omega(0) = p\omega(0) = p'(y)$ and

$$p(\omega \cdot f \circ u(y,\omega))(t) = \begin{cases} p\omega(t) = p'(y) & 0 \le t \le \frac{1}{2}, \\ p'u(y,\omega)(t) & \frac{1}{2} \le t \le 1. \end{cases}$$

Thus, by $(*)$, we see that $p'u(y,\omega)(t) = p'(y)$, for all $t \in I$, and therefore $p'f'\omega(0) = p(\omega \cdot f \circ u(y,\omega))$, for all $t \in I$. Note also that $\omega \cdot f \circ u(y,\omega)(1)$ $= f(\varphi_y \cdot f' \circ (\omega^- \cdot \psi_{\omega(0)} \cdot ff'\omega \cdot f\varphi_y^- \cdot \omega^-)(1)) = ff'\omega^-(1) = f(f'\omega(0))$, and therefore $K_2(y,\omega) \in R$. Moreover, $qK_2(y,\omega) = q(f'\omega(0), \omega \cdot f \circ u(y,\omega))$ $= \omega \cdot f \circ u(y,\omega)(0) = \omega(0) = q(y,\omega)$, hence K_2 is a map over E.

3. Define $d_2 : R \times I \to R$ by

$$d_2((y,\omega),\tau) = (u(y,\omega)(\tau), \omega \cdot f \circ ({}_\tau u(y,\omega))).$$

The product path $\omega \cdot f \circ ({}_\tau u(y,\omega))$ is well defined since $f({}_\tau u(y,\omega)(0))$ $= f(u(y,\omega)(\tau \cdot 0)) = f(u(y,\omega)(0)) = f(\psi_y(0)) = f\psi(y,0) = f1_E(y) = f(y)$ $= \omega(1)$.

4. We have just seen that $p(\omega \cdot f(u(y,\omega)(t))) = p'(y)$ for all $t \in I$, and since $\omega \cdot f({}_\tau u(y,\omega)(1)) = f({}_\tau u(y,\omega)(1)) = f(u(y,\omega)(\tau))$, d_2 is well defined. Note that $q(d_2)_\tau(y,\omega) = q(u(y,\omega)(\tau), \omega \cdot f \circ ({}_\tau u(y,\omega))) = \omega \cdot f({}_\tau u(y,\omega)(0))$ $= \omega(0) = q(y,\omega)$ shows that $(d_2)_\tau$ is a map over E.

5. Notice that $d_2((y,\omega),0) = (u(y,\omega)(0), \omega \cdot f \circ ({}_0u(y,\omega))) = (\psi_y(0), \omega \cdot f \circ ({}_0u(y,\omega))) = (\psi(y,0), \omega \cdot f \circ ({}_0u(y,\omega))) = (y, \omega \cdot f \circ ({}_0u(y,\omega)))$. Notice also that, for $0 \le t \le \frac{1}{2}$, we have $\omega \cdot f \circ ({}_0u(y,\omega))(t) = \omega(2t)$, and for $\frac{1}{2} \le t \le 1$, $\omega \cdot f \circ ({}_0u(y,\omega))(t) = f(u(y,\omega)(0 \cdot (2t-1))) = f(\psi_y(0)) = f(\psi(y,0)) = f1_E(y) = f(y) = \omega(1) = c_{\omega(1)}(2t-1)$. Thus, $(\omega \cdot f)({}_0u(y,\omega))$ $= \omega \cdot c_{\omega(1)}$, hence $d_2((y,\omega),0) = K_1(y,\omega)$. Since $d_2((y,\omega),1) = (u(y,\omega)(1), \omega \cdot f \circ ({}_1u(y,\omega))) = (f'\omega^-(1), \omega \cdot f \circ ({}_1u(y,\omega))) = (f'\omega(0), \omega \cdot f \circ (u(y,\omega)))$ $= K_2(y,\omega)$, d_2 is a vertical homotopy from K_1 to K_2.

Remark 3.5 *Notice the following properties:*

1. $(g\omega)^-(t) = (g\omega)(1-t) = g\omega(1-t) = g\omega^-(t)$.
2. $f(\omega \cdot \omega'(t)) = \left\{ \begin{array}{ll} f\omega(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ f\omega'(2t-1) & \text{if } \frac{1}{2} \le t \le 1 \end{array} \right\} = (f\omega \cdot f\omega')(t)$.

3.

$$
\begin{aligned}
(\omega\cdot\omega')^-(t) &= (\omega\cdot\omega')(1-t)\\
&= \begin{cases} \omega(2-2t) & \text{if } 0\le 1-t\le \frac{1}{2},\\ \omega'(2-2t-1) & \text{if } \frac{1}{2}\le 1-t\le 1,\end{cases}\\
&= \begin{cases} \omega(1-(2t-1)) & \text{if } -1\le -t\le -\frac{1}{2},\\ \omega'(1-2t) & \text{if } -\frac{1}{2}\le -t\le 0,\end{cases}\\
&= \begin{cases} \omega^-(2t-1) & \text{if } \frac{1}{2}\le t\le 1,\\ \omega'^-(2t) & \text{if } 0\le t\le \frac{1}{2},\end{cases}\\
&= (\omega'^-\cdot\omega^-)(t).
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\omega\cdot f\circ u(y,\omega) &= \omega\cdot(f(\phi_y\cdot f'(\omega^-\cdot\psi_{\omega(0)}\cdot ff'\omega\cdot f\phi_y^-\cdot\omega^-)))\\
&= \omega\cdot f\phi_y\cdot ff'\omega^-\cdot ff'\psi_{\omega(0)}\cdot ff'ff'\omega\cdot ff'f\phi_y^-\cdot ff'\omega^-\\
&= \omega\cdot f\phi_y\cdot ff'\omega^-\cdot ff'\psi_{\omega(0)}\cdot ff'\circ(\omega\cdot f\phi_y\cdot ff'\omega^-)^-\\
&= v_y\cdot ff'\psi_{\omega(0)}\cdot ff'v_y^-\\
&\qquad \textit{where } v_y=\omega\cdot f\varphi_y\cdot ff'\omega^-\\
&= v_y\cdot ff'\circ(\psi_{\omega(0)}\cdot v_y^-).
\end{aligned}
$$

Hence, we can write $K_2: R\to R$ *by* $K_2(y,\omega)=(f'\omega(0), v_y\cdot ff'\circ(\psi_{\omega(0)}\cdot v_y^-))$. *Note that by* $(*)$, *we have* $pv_y(I)=p'(y)$.

$d_3 : K_2(R)\times I\to R$:

1. Define $K_3 : K_2(R)\to R$ by

$$
\begin{aligned}
K_3K_2(y,\omega) &= K_3(f'\omega(0), v_y\cdot ff'\circ(\psi_{\omega(0)}\cdot v_y^-))\\
&= (f'\omega(0), v_y\cdot c_1\cdot ff'(\psi_{\omega(0)}\cdot v_y^-)\cdot c_2),
\end{aligned}
$$

where $c_1 = c_2 = c_{ff'\omega(0)}$, the constant path at $ff'\omega(0)$. To see the product path is well defined, we check the following: $v_y(1) = \omega \cdot f\varphi_y^- \cdot ff'(\omega^-(1))$ $= ff'(\omega(1-1)) = c_1(0)$, $ff'((\psi_{\omega(0)} \cdot v_y^-)(0)) = ff'(\psi(\omega(0), 0)) = ff'(\omega(0))$ $= c_1(1)$, and $ff'((\psi_{\omega(0)} \cdot v_y^-)(1)) = ff'(v_y(1-1)) = ff'((\omega \cdot f\varphi_y \cdot ff'\omega^-)(0))$ $= ff'(\omega(0)) = c_2(0)$. Hence, $v_y \cdot c_1 \cdot ff' \circ (\psi_{\omega(0)} \cdot v_y^-) \cdot c_2$ is well defined.

2. Clearly, $(f'\omega(0), v_y \cdot ff'(\psi_{\omega(0)} \cdot v_y^-)) \in R$ implies that $K_3(K_2(R)) \subset R$, hence K_3 is well defined. We see that K_3 is a map over E from $K_2(R)$ to R since $qK_3(f'\omega(0), (v_y \cdot ff')(\psi_{\omega(0)} \cdot v_y^-)) = v_y \cdot c_1 \cdot ff'(\psi_{\omega(0)} \cdot v_y^-) \cdot c_2(0)$ $= v_y(0) = v_y \cdot ff'(\psi_{\omega(0)} \cdot v_y^-)(0) = q\,\big|_{K_2(R)}\, (f'\omega(0), v_y \cdot ff'(\psi_{\omega(0)} \cdot v_y^-))$.

3. Define $d_3 : R \times I \to R$ by

$$d_3((y, \omega), \tau) = (f'\omega(0), \beta_\tau \cdot \beta'_\tau),$$

where $\beta_\tau = v_y g_\tau$, $\beta'_\tau = ff'(\psi_{\omega(0)} \cdot v_y^-)g_\tau$, and $g_\tau : I \to I$ is defined by $g_\tau(t) = min((1+\tau)t, 1)$ as before. Since $\beta_\tau(1) = v_y g_\tau(1) = v_y(1) =$ $\omega \cdot f\phi_y \cdot ff'\omega^-(1) = ff'\omega(0)$ and $\beta'_\tau(0) = ff'(\psi_{\omega(0)} \cdot v_y^-)g_\tau(0) = ff'\psi_{\omega(0)}(0)$ $= ff'\omega(0)$, $\beta_\tau \cdot \beta'_\tau$ is well defined.

4. Since $\beta_\tau \cdot \beta'_\tau = v_y g_\tau \cdot ff' \circ (\psi_{\omega(0)} \cdot v_y^-)g_\tau = (v_y \cdot ff' \circ (\psi_{\omega(0)} \cdot v_y^-))g_\tau$ and $(f'\omega(0), v_y \cdot ff' \circ (\psi_{\omega(0)} \cdot v_y^-)) \in R$, d_3 is well defined. Moreover, $q(d_3)_\tau(f'\omega(0), v_y \cdot ff' \circ (\psi_{\omega(0)} \cdot v_y^-)) = q(f'\omega(0), \beta_\tau \cdot \beta'_\tau) = \beta_\tau(0) = v_y(0)$ $= v_y \cdot ff' \circ (\psi_{\omega(0)} \cdot v_y^-)(0) = q(f'\omega(0), v_y \cdot ff' \circ (\psi_{\omega(0)} \cdot v_y^-))$ shows that $(d_3)_\tau$ is a map over E.

5. The following show that d_3 is a vertical homotopy from K_2 to K_3K_2.

$$\begin{aligned}
\beta_0 \cdot \beta_0'(t) &= \begin{cases} \beta_0(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ \beta_0'(2t-1) & \text{if } \frac{1}{2} \le t \le 1, \end{cases} \\
&= \begin{cases} v_y(min(2t,1)) & \text{if } 0 \le t \le \frac{1}{2}, \\ ff'(\psi_{\omega(0)} \cdot v_y^-)(min((2t-1),1)) & \text{if } \frac{1}{2} \le t \le 1, \end{cases} \\
&= \begin{cases} v_y(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ ff'(\psi_{\omega(0)} \cdot v_y^-)(2t-1) & \text{if } \frac{1}{2} \le t \le 1, \end{cases} \\
&= v_y \cdot ff'(\psi_{\omega(0)} \cdot v_y^-)(t), \text{ and}
\end{aligned}$$

$$\begin{aligned}
\beta_1 \cdot \beta_1'(t) &= \begin{cases} \beta_1(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ \beta_1'(2t-1) & \text{if } \frac{1}{2} \le t \le 1, \end{cases} \\
&= \begin{cases} v_y(min(2 \cdot 2t,1)) & \text{if } 0 \le t \le \frac{1}{2}, \\ ff'(\psi_{\omega(0)} \cdot v_y^-)(min(2(2t-1),1)) & \text{if } \frac{1}{2} \le t \le 1, \end{cases} \\
&= \begin{cases} v_y(4t) & \text{if } 0 \le t \le \frac{1}{4}, \\ c_{v_y(1)}(t) = c_{ff'\omega(0)}(4t-1) & \text{if } \frac{1}{4} \le t \le \frac{1}{2}, \\ ff'(\psi_{\omega(0)} \cdot v_y^-)(2(2t-1)) & \text{if } \frac{1}{2} \le t \le \frac{3}{4}, \\ c_{ff'(\psi_{\omega(0)} \cdot v_y^-)(1)}(t) = c_{ff'\omega(0)}(4t-3) & \text{if } \frac{3}{4} \le t \le 1, \end{cases} \\
&= v_y \cdot c_1 \cdot ff'(\psi_{\omega(0)} \cdot v_y^-) \cdot c_2(t).
\end{aligned}$$

$d_4 : R \times I \to R :$

1. Define $K_4 : R \to R$ by

$$K_4(y,\omega) = (f'\omega(0), v_y \cdot \psi_{\omega(0)}^- \cdot \psi_{\omega(0)} \cdot v_y^- \cdot \psi_{\omega(0)}).$$

The following verifies the product path $v_y \cdot \psi_{\omega(0)}^- \cdot \psi_{\omega(0)} \cdot v_y^- \cdot \psi_{\omega(0)}$ is well defined:

(a) $v_y(1) = \omega \cdot f\phi_y \cdot ff'\omega^-(1) = ff'\omega(0) = \psi(\omega(0), 1) = \psi^-_{\omega(0)}(0)$.

(b) $\psi^-_{\omega(0)}(1) = \psi(\omega(0), 0) = \psi_{\omega(0)}(0)$.

(c) $\psi_{\omega(0)}(1) = \psi(\omega(0), 1) = ff'\omega(0) = \omega \cdot f\phi_y \cdot ff'\omega^-(1) = v_y(1) = v_y^-(0)$.

(d) $v_y^-(1) = v_y(0) = \omega \cdot f\phi_y \cdot ff'\omega^-(0) = \omega(0) = \psi(\omega(0), 0) = \psi_{\omega(0)}(0)$.

2. By $(*)$, $p(v_y \cdot \psi^-_{\omega(0)} \cdot \psi_{\omega(0)} \cdot v_y^- \cdot \psi_{\omega(0)})(t) = p'(y)$, for all $t \in I$. Also, $v_y \cdot \psi^-_{\omega(0)} \cdot \psi_{\omega(0)} \cdot v_y^- \cdot \psi_{\omega(0)}(1) = \psi_{\omega(0)}(1) = ff'\omega(0)$, and hence $K_4(y, \omega) \in R$. Furthermore, $qK_4(y, \omega) = q(f'\omega(0), v_y \cdot \psi^-_{\omega(0)} \cdot \psi_{\omega(0)} \cdot v_y^- \cdot \psi_{\omega(0)})$ $= (v_y \cdot \psi^-_{\omega(0)} \cdot \psi_{\omega(0)} \cdot v_y^- \cdot \psi_{\omega(0)})(0) = v_y(0) = \omega(0) = q(y, \omega)$, and therefore K_4 is a map over E.

3. Define $d_4 : R \times I \to R$ by

$$d_4((y, \omega), \tau) = (f'\omega(0), v_y \cdot {}_\tau\psi^-_{\omega(0)} \cdot \psi_{1-\tau}(\psi_{\omega(0)} \cdot v_y^-) \cdot {}^\tau\psi_{\omega(0)}).$$

Notice that $v_y(1) = ff'\omega(0) = \psi(\omega(0), 1) = \psi_{\omega(0)}(1) = \psi^-_{\omega(0)}(0) = {}_\tau\psi^-_{\omega(0)}(0)$, ${}_\tau\psi^-_{\omega(0)}(1) = \psi^-_{\omega(0)}(\tau) = \psi^{\omega(0)}(1 - \tau) = \psi(\omega(0), 1 - \tau) = \psi_{1-\tau}(\omega(0)) =$ $\psi_{1-\tau}(\psi(\omega(0), 0)) = \psi_{1-\tau}(\psi_{\omega(0)}(0)) = \psi_{1-\tau}(\psi_{\omega(0)} \cdot v_y^-)(0)$, and $\psi_{1-\tau}(\psi_{\omega(0)} \cdot v_y^-)(1) = \psi_{1-\tau}v_y^-(1) = \psi_{1-\tau}v_y(0) = \psi_{1-\tau}(\omega \cdot f\phi_y \cdot ff'\omega^-)(0)$ $= \psi_{1-\tau}(\omega(0)) = \psi(\omega(0), 1 - \tau) = \psi_{\omega(0)}(1 - \tau + 0\tau) = {}^\tau\psi_{\omega(0)}(0)$. Hence, the product path $v_y \cdot {}_\tau\psi^-_{\omega(0)} \cdot \psi_{1-\tau}(\psi_{\omega(0)} \cdot v_y^-) \cdot {}^\tau\psi_{\omega(0)}$ is well defined.

4. Again by $(*)$, we have $p(v_y \cdot {}_\tau(\psi^-_{\omega(0)} \cdot \psi_{1-\tau}(\psi_{\omega(0)} \cdot v_y^-) \cdot {}^\tau\psi_{\omega(0)}))(t) = p'(y)$, for all $t \in I$. Since $v_y \cdot {}_\tau(\psi^-_{\omega(0)} \cdot \psi_{1-\tau}(\psi_{\omega(0)} \cdot v_y^-) \cdot {}^\tau\psi_{\omega(0)})(1) = {}^\tau\psi_{\omega(0)}(1)$ $= \psi(\omega(0), 1 - \tau + \tau) = \psi(\omega(0), 1) = ff'\omega(0)$, $d_4((y, \omega), \tau) \in R$. Hence, d_4 is well defined. The map $(d_4)_\tau$ is a map over E, since $q(d_4)_\tau(y, \omega)$ $= q(f'\omega(0), v_y \cdot {}_\tau\psi^-_{\omega(0)} \cdot \psi_{1-\tau}(\psi_{\omega(0)} \cdot v_y^-) \cdot {}^\tau\psi_{\omega(0)}) = v_y(0) = \omega(0) = q(y, \omega)$.

5. Finally, we show that d_4 is a vertical homotopy from K_3K_2 to K_4. For $d_4((y,\omega),0) = (f'\omega(0), v_y \cdot {}_0(\psi^-_{\omega(0)}) \cdot \psi_{1-0}(\psi_{\omega(0)} \cdot v^-_y) \cdot {}^0\psi_{\omega(0)}) = (f'\omega(0), v_y \cdot c' \cdot ff'(\psi_{\omega(0)} \cdot v^-_y) \cdot c'') = K_3K_2(y,\omega)$, where $c' = c_{ff'\omega(0)} = c''$, and $d_4((y,\omega),1) = (f'\omega(0), v_y \cdot {}_1(\psi^-_{\omega(0)}) \cdot \psi_{1-1}(\psi_{\omega(0)} \cdot v^-_y) \cdot {}^1\psi_{\omega(0)}) = (f'\omega(0), v_y \cdot \psi^-_{\omega(0)} \cdot (\psi_{\omega(0)} \cdot v^-_y) \cdot \psi_{\omega(0)})$, we have $d_4 : K_3K_2 \simeq_E K_4$.

$d_5 : R \times I \to R$:

1. Define $K_5 : R \to R$ by

$$K_5(y,\omega) = (f'\omega(0), c_3 \cdot \psi_{\omega(0)}),$$

where $c_3 = c_{\omega(0)}$, the constant path at $\omega(0)$, and clearly the path is well defined.

2. Since $\psi_{\omega(0)}(1) = ff'\omega(0)$, by $(*)$, $K_5(y,\omega) \in R$, and $qK_5(y,\omega) = q(f'\omega(0), c_3 \cdot \psi_{\omega(0)}) = c_3(0) = \omega(0) = q(y,\omega)$ implies that K_5 is a map over E.

3. Define $\bar{K}_5(y,\omega) = (f'\omega(0), v_y \cdot \psi^-_{\omega(0)} \cdot (v_y \cdot \psi^-_{\omega(0)})^- \cdot \psi_{\omega(0)})$ and $\tilde{K}_5(y,\omega) = (f'\omega(0), c_{\omega(0)} \cdot c_{\omega(0)} \cdot \psi_{\omega(0)})$. Then, clearly, there exist vertical homotopies $\bar{d}_5 : K_4 \simeq_E \bar{K}_5$ and $\tilde{d}_5 : \tilde{K}_5 \simeq_E K_5$. Define $d'_5 : R \times I \to R$ by

$$d'_5((y,\omega),\tau) = (f'\omega(0), {}_{1-\tau}(v_y \cdot \psi^-_{\omega(0)}) \cdot {}^{1-\tau}((v_y \cdot \psi^-_{\omega(0)})^-) \cdot \psi_{\omega(0)}).$$

Since $v_y(1) = ff'\omega^-(1) = \psi(\omega(0),0) = \psi^-_{\omega(0)}(0)$, the product path $v_y \cdot \psi^-_{\omega(0)}$ is well defined, and ${}_{1-\tau}(v_y \cdot \psi^-_{\omega(0)})(1) = v_y \cdot \psi^-_{\omega(0)}(1-\tau) = (v_y \cdot \psi^-_{\omega(0)})^-(1-(1-\tau)) = {}^{1-\tau}((v_y \cdot \psi^-_{\omega(0)})^-)(0)$ and ${}^{1-\tau}((v_y \cdot \psi^-_{\omega(0)})^-)(1) = (v_y \cdot \psi^-_{\omega(0)})^-(1-(1-\tau)+(1-\tau)) = (v_y \cdot \psi^-_{\omega(0)})(1) = v_y(0) = \omega(0) = \psi_{\omega(0)}(0)$ show that the entire path is well defined.

4. By $(*)$, we have $p({}_{1-\tau}(v_y \cdot \psi^-_{\omega(0)}) \cdot {}^{1-\tau}((v_y \cdot \psi_{\omega(0)})^-) \cdot \psi_{\omega(0)})(t) = p'(y)$ for all $t \in I$. Since ${}_{1-\tau}(v_y \cdot \psi^-_{\omega(0)}) \cdot {}^{1-\tau}((v_y \cdot \psi_{\omega(0)})^-) \cdot \psi_{\omega(0)}(1) = ff'\omega(0)$, d_5' is well defined. Also

$$\begin{aligned} q(d_5')_\tau(y,\omega) &= q(f'\omega(0), {}_{1-\tau}(v_y \cdot \psi^-_{\omega(0)}) \cdot {}^{1-\tau}((v_y \cdot \psi_{\omega(0)})^-) \cdot \psi_{\omega(0)}) \\ &= {}_{1-\tau}(v_y \cdot \psi^-_{\omega(0)}) \cdot {}^{1-\tau}((v_y \cdot \psi_{\omega(0)})^-) \cdot \psi_{\omega(0)}(0) \\ &= {}_{1-\tau}(v_y \cdot \psi^-_{\omega(0)})(0) = v_y(0) = \omega(0) = q(y,\omega), \end{aligned}$$

and therefore $(d_5')_\tau$ is a map over E.

5. Since $d_5'((y,\omega),0) = (f'\omega(0), {}_{1-0}(v_y \cdot \psi^-_{\omega(0)}) \cdot {}^{1-0}((v_y \cdot \psi^-_{\omega(0)})^-) \cdot \psi_{\omega(0)}) = (f'\omega(0), {}_0(v_y \cdot \psi^-_{\omega(0)}) \cdot {}^0((v_y \cdot \psi^-_{\omega(0)})^-) \cdot \psi_{\omega(0)}) = (f'\omega(0), c_{\omega(0)} \cdot c_{\omega(0)} \cdot \psi_{\omega(0)})$, d_5' is a vertical homotopy from $\bar{K}_5$ to $\tilde{K}_5$. Putting these vertical homotopies $\bar{d}_5 : K_4 \simeq_E \bar{K}_5$, $d_5' : \bar{K}_5 \simeq_E \tilde{K}_5$, and $\tilde{d}_5 : \tilde{K}_5 \simeq_E K_5$ together gives a vertical homotopy $d_5 : K_4 \simeq_E K_5$ where d_5 is the vertical homotopy obtained by concatenation of $\bar{d}_5$, d_5', and $\tilde{d}_5$.

$d_6 : R \times I \to R :$

1. Define $K_6 : R \to R$ by

$$K_6(y,\omega) = (f'\omega(0), \psi_{\omega(0)}),$$

then, the path is clearly well defined.

2. There is nothing to show.

3. Define $d_6 : R \times I \to R$ by

$$d_6((y,\omega),\tau) = (f'\omega(0), \gamma_\tau),$$

where $\gamma_\tau(t) = \psi_{\omega(0)}(max(0, \frac{2t+\tau-1}{1+\tau}))$. It is also obvious that the path is well defined since $0 \le max(0, \frac{2t+\tau-1}{1+\tau}) \le 1$ for all $t \in I$.

4. By $(*)$ together with $\frac{2t+\tau-1}{1+\tau} = 1$ when $t = 1$ we see that d_6 is well defined. Clearly, $d_6(R \times I) \subset R$. Moreover, $q(d_6)_\tau(y,\omega) = q(f'\omega(0), \gamma_\tau) = \gamma_\tau(0)$ $= \psi_{\omega(0)} max(0, \frac{\tau-1}{1+\tau}) = \psi_{\omega(0)}(0) = \omega(0) = q(y,\omega)$. Hence, $(d_6)_\tau$ is a map over E.

5. $d_6((y,\omega),0)$ and $= (f'\omega(0), \gamma_0)$ $\gamma_0(t) = \psi_{\omega(0)} max(0, \frac{2t-1}{1})$
$$= \left\{ \begin{array}{ll} \psi_{\omega(0)}(0) & \text{if } 2t-1 \le 0 \\ \psi_{\omega(0)}(2t-1) & \text{if } 2t-1 \ge 0 \end{array} \right\} = c_3 \cdot \psi_{\omega(0)}(t).$$
Finally, $d_6((y,\omega),1) = (f'\omega(0), \gamma_1)$, and $\gamma_1(t) = \psi_{\omega(0)} max(0, \frac{2t}{2}) = \psi_{\omega(0)}(t)$. Hence, d_6 is a vertical homotopy from K_5 to K_6 over E.

Notice that $\sigma q(y,\omega) = \sigma(\omega(0)) = (f'\omega(0), \psi_{\omega(0)}) = K_6(y,\omega)$. Note also that $d_1 : 1_R \simeq_E K_1$, $d_2 : K_1 \simeq_E K_2$, $d_3 : K_2 \simeq_E K_3K_2$, $d_4 : K_3K_2 \simeq_E K_4$, $d_5 : K_4 \simeq_E K_5$, $d_6 : K_5 \simeq_E K_6 = \sigma q$, and therefore D is well defined. Hence, D is continuous by the Gluing Lemma. Thus, we have $D : 1_R \simeq_E \sigma q$. This completes the proof. □

Theorem 3.6 *Let $f : E' \to E$ be a map over B. If f is a fibre homotopy equivalence over each set V_λ of a numerable covering $\{V_\lambda\}$ of B, then f is a fibre homotopy equivalence.*

More generally, if under the same assumptions on f, a partial homotopy inverse $f_V^- : p^{-1}(V) \to p'^{-1}(V)$ of f, and a vertical homotopy $D_V : 1_{p^{-1}(V)} \simeq_V f_V f_V^-$ are given over a halo V around $A \subset B$, then f_A^-, D_A can be extended to B.

Proof: For each $\lambda \in \Lambda$, let $f_\lambda : p'^{-1}(V_\lambda) \to p^{-1}(V_\lambda)$ be the part of f over V_λ, and $f_\lambda^- : p^{-1}(V_\lambda) \to p'^{-1}(V_\lambda)$ be a homotopy inverse of f_λ over V_λ, and write, for each λ,

$p_\lambda = p\big|_{p^{-1}(V_\lambda)} : p^{-1}(V_\lambda) \to V_\lambda$ and $p'_\lambda = p'\big|_{p'^{-1}(V_\lambda)} : p^{-1}(V_\lambda) \to V_\lambda$. By Lemma 3.4, for each $\lambda \in \Lambda$, $q_\lambda : R_\lambda \to E'_\lambda$ is shrinkable where $R_\lambda = \{(y,\omega) \in p'^{-1}_\lambda(V_\lambda) \times (p_\lambda^{-1}(V_\lambda))^I : p'_\lambda(y) = p_\lambda\omega(t)$ for all $t \in I$ and $\omega(1) = f_\lambda(y)\}$, and $E'_\lambda = p'^{-1}(V_\lambda)$. Thus, by Corollary 3.3, $q : R \to E$ itself is shrinkable where $R = \{(y,\omega) \in E' \times E^I : p'(y) = p\omega(t)$ for all $t \in I$ and $\omega(1) = f(y)\}$ and therefore by Proposition 2.17, it has the SEP. A section of q is a pair $S = (f', \theta) : E \to R$ by $S(z) = (f'(z), \theta(z))$ where $\theta(z) = \theta_z : I \to E$. Note that $f' : E \to E'$ is a map over B. By the SEP, we can choose this f' to be an extension of the given map $f_{\bar{V}}^-$ with $f'|_A = f_{\bar{V}}^-|_A$. Now, define $\Theta : E \times I \to E$ by $\Theta(z,t) = \theta(z)(t) = \theta_z(t)$. Then, since $p\theta_z(t) = p'(f'(z))$ for all $t \in I$, we have $p\Theta_t(z) = p\Theta(z,t) = p\theta_z(t) = p'(f'(z)) = p(z)$. Hence, Θ is a vertical homotopy over B. From $\Theta(z,0) = \theta_z(0) = q(f'(z),\theta_z) = qS(z) = z$ and $\Theta(z,1) = \theta_z(1) = f(f'(z))$, we see that $\Theta : 1_E \simeq_B ff'$. Recall how we defined the ψ_z in Lemma 3.4; it is defined by the vertical homotopy $\psi_\lambda : 1_{p^{-1}(V_\lambda)} \simeq_{V_\lambda} f_\lambda f_\lambda^-$. Thus, again, by the SEP, we can choose Θ to be an extension of the given vertical homotopy D_V with $\Theta|_A = D_V|_A$, i.e. ψ_λ is a restriction of D_V over V_λ.

Now, it remains to show that $1_{E'} \simeq_B f'f$. By applying the analogous process to this f' as above, we see that there exist a map $f'' : E' \to E$ over B and a vertical homotopy $\Theta' : 1_{E'} \simeq_B f'f''$. Here, R_λ is replaced by $R'_\lambda = \{(z,\omega') \in p_\lambda^{-1}(V_\lambda) \times (p_\lambda^{-1}(V_\lambda))^I : p_\lambda(z) = p'_\lambda\omega'(I)$ and $\omega'(1) = f_\lambda^-(z)\}$. But, $f'f'' = f'1_Ef'' \simeq_B f'ff'f'' \simeq_B f'f1_{E'} = f'f$; hence $1_{E'} \simeq_B f'f$, as desired. □

Remark 3.7 *One cannot, in addition to f_A^-, D_A, prescribe $D'_A : 1_{p'^{-1}(A)} \simeq_A f_A^- f_A$, as the following example shows. Let $B = [0,1], E = B \times S^1 = E', p : E \to B$ by $p(b,z) = b, q : E \to B$ by $q(b,z) = b, f = 1_E : E \to E(= E')$, and $A =$*

$\{0\} \cup \{1\}$, where $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Prescribe $f_A^- = 1_{E'}, D : E \times I \to E$ by $D(0, z, t) = (0, z), D(1, z, t) = (1, e^{2\pi i t} z)$, and $D' : E' \times I \to E'$ by $D'(0, z, t) = (0, z), D'(1, z, t) = (1, z)$. Since D and D' are vertical homotopies, we have $pD_1 = p = qf = q = qD'_1$. But, D rotates $\{1\} \times S^1$ while D' fixes $\{1\} \times S^1$, a contradiction.

Chapter 4

The Covering Homotopy Property (CHP)

Definition 4.1 *Let $p : E \to B$ be a space over B and $\bar{H} : X \times I \to B$ a homotopy. We say that p has* **the covering homotopy property** *(abbreviated henceforth as CHP)* **for** $\bar{H}$ *if the following holds:*

Given $h : X \to E$ with $ph(x) = \bar{H}(x,0)$ for all $x \in X$, any $\tau : X \to I$ and any $H' : \tau^{-1}(0,1] \times I \to E$ satisfying $pH'(x,t) = \bar{H}(x,t)$ and $H'(x,0) = h(x)$ for all $x \in \tau^{-1}(0,1]$ and $t \in I$, there exists an $H : X \times I \to E$ with

$$\left.\begin{array}{l} pH = \bar{H}, H(x,0) = h(x) \text{ for all } x \in X \text{ and} \\ H\big|_{\tau^{-1}(1)\times I} = H'\big|_{\tau^{-1}(1)\times I}. \end{array}\right\} (*)$$

In the above definition, I can be replaced by an arbitrary interval $[a,b]$ of $\mathbb{R}$ where $a < b$. We replace 0 by a and 1 by b in the above postulated definition.

We say that p has the **CHP for** X *if it has the CHP for all homotopies $\bar{H}$ with domain $X \times I$. If it has the CHP for all spaces X, we say that it has the* **CHP**.

In the special case when $\tau = 0 : X \to I$, we have $\tau^{-1}(0,1] = \phi$ and the condition postulated in the Definition 4.1 takes on the following form:

Given a homotopy $\bar{H} : X \times I \to B$ and a map $h : X \to E$ satisfying $ph(x) = \bar{H}(x,0)$, then there exists $H : X \times I \to E$ satisfying $H(x,0) = h(x)$ for all $x \in X$ and $pH = \bar{H}$. In fact, this apparently weaker condition (which is the "classical" CHP, cf. p.1) implies the conditions of the Definition 4.1. Namely, we have the following.

Proposition 4.2 *Suppose for every $\bar{G} : X \times I \to B$ and $g : X \to E$ with $pg(x) = \bar{G}(x,0)$ for all $x \in X$, we can find a $G : X \times I \to E$ satisfying $pG = \bar{G}$ and $G(x,0) = g(x)$, then p has the CHP for X.*

Proof: Let $\bar{H} : X \times I \to B$, $h : X \to E$, $\tau : X \to I$ and $H' : \tau^{-1}(0,1] \times I \to E$ satisfy

$$ph(x) = \bar{H}(x,0) \text{ for all } x \in X,$$

$$\left.\begin{array}{l} pH'(x,t) = \bar{H}(x,t), \text{ and} \\ H'(x,0) = h(x), \end{array}\right\} \text{ for all } x \in \tau^{-1}(0,1] \text{ and for all } t \in I.$$

Let $\tau' : X \to I$ be given by $\tau' = max(0, 2\tau - 1)$. Then τ' is continuous. Let $D = \{(x,t) \in X \times I : t \leq \tau'(x)\}$. Note that if $(x,t) \in D$ with $t > 0$, then $0 < 2\tau(x) - 1$, hence $\tau(x) > \frac{1}{2}$ or $x \in \tau^{-1}(\frac{1}{2}, 1] \subset \tau^{-1}(0,1]$. Also, since $\tau'(x) \geq 0$ for all $x \in X$, we have $X \times \{0\} \subset D$. As already observed, $(x,t) \in D, \tau(x) \leq \frac{1}{2} \Rightarrow t = 0$.

Define $H'' : D \to E$ by $H''(x,0) = h(x)$ for all $x \in X$ and $H''(x,t) = H'(x,t)$ if $\tau(x) \geq \frac{1}{2}$. The sets $\{(x,t) \in D : \tau(x) \leq \frac{1}{2}\} = X \times \{0\}$ and $\{(x,t) \in D : \tau(x) \geq \frac{1}{2}\}$ are closed subsets of D. When $\tau(x) = \frac{1}{2}$, we have $(x,t) \in D \Rightarrow t = 0 \Rightarrow H'(x,0) = h(x)$. Hence, H'' is well defined. Since H'' is separately continuous on the above closed sets of D, we see that $H'' : D \to E$ is continuous by the Gluing Lemma.

Let $\bar{G} : X \times I \to B$ and $g : X \to E$ be defined by

$$\bar{G}(x,t) = \bar{H}(x, min(1, \tau'(x) + t))$$

and

$$g(x) = H''(x, \tau'(x)).$$

Then $\bar{G}(x,0) = \bar{H}(x,\tau'(x))$. Also

$$g(x) = \begin{cases} h(x) & \text{if } \tau'(x) = 0, \\ H'(x,\tau'(x)) & \text{if } \tau'(x) > 0. \end{cases}$$

Hence,

$$\begin{aligned} pg(x) &= \begin{cases} ph(x) & \text{if } \tau'(x) = 0, \\ pH'(x,\tau'(x)) & \text{if } \tau'(x) > 0, \end{cases} \\ &= \begin{cases} \bar{H}(x,0) & \text{if } \tau'(x) = 0, \\ \bar{H}(x,\tau'(x)) & \text{if } \tau'(x) > 0, \end{cases} \\ &= \bar{H}(x,\tau'(x)) \text{ in each case.} \end{aligned}$$

Thus, $\bar{G}(x,0) = pg(x)$ for all $x \in X$.

By our assumption, there exists a $G : X \times I \to E$ satisfying $pG = \bar{G}$ and $G(x,0) = g(x)$. Define $H : X \times I \to E$ by

$$H(x,t) = \begin{cases} H''(x,t) & \text{if } t \leq \tau'(x), \\ G(x,t-\tau'(x)) & \text{if } t \geq \tau'(x). \end{cases}$$

Note that when $t = \tau'(x)$, $H''(x,\tau'(x)) = g(x) = G(x,0)$. Thus, the two definitions agree when $t = \tau'(x)$. It follows that $H : X \times I \to E$ is a well defined continuous map.

We claim that H satisfies all the requirements in $(*)$ of Definition 4.1. Since $\tau(x) \geq 0$ for all $x \in X$, we get $H(x,0) = H''(x,0) = h(x)$. Also,

$$pH(x,t) = \begin{cases} ph(x) = \bar{H}(x,0) & \text{if } t = 0, \\ pH'(x,t) = \bar{H}(x,t) & \text{if } 0 < t \leq \tau'(x), \\ pG(x,t-\tau'(x)) = \bar{G}(x,t-\tau'(x)) & \text{if } t \geq \tau'(x), \end{cases}$$

$$= \begin{cases} \bar{H}(x,t) & \text{if } 0 \leq t \leq \tau'(x), \\ \bar{H}(x, min(1, \tau'(x) + t - \tau'(x))) & \text{if } t \geq \tau'(x), \end{cases}$$
$$= \bar{H}(x,t) \text{ for all } t \in I.$$

Also when $\tau(x) = 1$, we have $\tau'(x) = 1$. Hence, $H(x,t) = H''(x,1) = H'(x,t)$. Thus, $H\big|_{\tau^{-1}(1)\times I} = H'\big|_{\tau^{-1}(1)\times I}$. This completes the proof of Proposition 4.2. □

Proposition 4.3 *If $p : E \to B$ has the CHP for X, then so does every induced space $p_\alpha : E_\alpha \to B_\alpha$ where $\alpha : B_\alpha \to B, E_\alpha = \{(e,a) \in E \times B_\alpha : p(e) = \alpha(a)\}$, and $p_\alpha(e,a) = a$.*

Proof: Let $p : E \to B$ have the CHP for X, $\alpha : B_\alpha \to B, g : X \to E_\alpha$ and $\bar{G} : X \times I \to B_\alpha$ with $p_\alpha g = \bar{G}$. Then, $\alpha\bar{G} = \alpha p_\alpha g = p\hat{\alpha}g$ where $\hat{\alpha} : E_\alpha \to E$ is given by $\hat{\alpha}(e,a) = e$. Since p has the CHP for X, there exists $G' : X \times I \to E$ with $G'(x,0) = \hat{\alpha}g(x), pG'(x,t) = \alpha\bar{G}(x,t)$. Define $G : X \times I \to E_\alpha$ by $G = (G', \bar{G})$. Then, we have $G(x,0) = (G'(x,0), \bar{G}(x,0)) = (\hat{\alpha}g(x), p_\alpha g(x)) = g(x)$ and $p_\alpha G(x,t) = p_\alpha(G'(x,t), \bar{G}(x,t)) = \bar{G}(x,t)$. Hence, p_α has the CHP for X. □

Example 4.4 *Every trivial space $B \times Y \to B$ has the CHP.*

Proof: Consider $p : B \times Y \to B, g : X \to B \times Y$ and $\bar{G} : X \times I \to B$ with $pg(x) = \bar{G}(x,0)$ for all $x \in X$. Define $G : X \times I \to B \times Y$ by $G(x,t) = (\bar{G}(x,t), p'g(x))$, where $p' : B \times Y \to Y$ is the projection to the second coordinate. Then, we get $G(x,0) = (\bar{G}(x,0), p'g(x)) = (pg(x), p'g(x)) = g(x)$ for all $x \in X$ and $pG(x,t) = p(\bar{G}(x,t), p'g(x)) = \bar{G}(x,t)$ for all $x \in X$ and for all $t \in I$. □

Definition 4.5 *Given $\hat{H} : X \times I \to B$, for any $h : X \to E$ satisfying $ph(x) = \hat{H}(x, 0)$ for all $x \in X$, we define a space $q_h : R \to X$ over X as follows:*

$$R = \{(x, \omega) \in X \times E^I : h(x) = \omega(0), p\omega(t) = \hat{H}(x, t)\}, q_h(x, \omega) = x.$$

The following lemma will enable us to use the results of previous chapters (on SEP) to obtain new results on CHP.

Lemma 4.6 *The space p over B has the CHP for $\bar{H} : X \times I \to B \Leftrightarrow q = q_h$ has the SEP for all $h : X \to E$ with $ph(x) = \bar{H}(x, 0)$.*

Proof: First, suppose that p has the CHP for $\bar{H}$. Let V be a halo around $A \subset X$ with a haloing function $\tau : X \to I$, $s : A \to R$ a section of q over A, $s' : V \to R$ an extension of s, and $h : X \to E$ with $ph(x) = \bar{H}(x, 0)$ for all $x \in X$. Then $s'(v) = (v, \omega_v)$ where $\omega_v : I \to E$ with $\omega_v(0) = h(v)$ and $p\omega_v(t) = \bar{H}(v, t)$, for all $v \in V$ for all $t \in I$.

Define $H' : \tau^{-1}(0, 1] \times I \to E$ by $H'(v, t) = \omega_v(t)$ (note that $V \supset \tau^{-1}(0, 1]$ since $V^c \subset \tau^{-1}(0)$). Then, we have $pH'(v, t) = p\omega_v(t) = \bar{H}(v, t)$ and $H'(v, 0) = \omega_v(0) = h(v)$ for all $v \in \tau^{-1}(0, 1]$ for all $t \in I$. Since p has the CHP for $\bar{H}$, there exists $H : X \times I \to E$ with $H(x, 0) = h(0), H\left|_{\tau^{-1}(1)\times I}\right. = H'\left|_{\tau^{-1}(1)\times I}\right.$, and $pH = \bar{H}$. Write $H_x(t)$ for $H(x, t)$, and define $S : X \to R$ by $S(x) = (x, H_x)$. Note that $H_x(0) = H(x, 0) = h(x)$ and $pH_x(t) = pH(x, t) = \bar{H}(x, t)$ imply that S is well defined. The continuity of S is clear. Also, $qS(x) = q(x, H_x) = x$ shows that S is a section of q over X. Also from $A \subset \tau^{-1}(1)$, we see that $S(a) = (a, H_a) = (a, H'_a) = (a, \omega_a) = s(a)$. Since $S\,|_A = s$, q has the SEP for any $h : X \to E$ with $ph = \bar{H}$.

Conversely, assume that q has the SEP for all $h : X \to E$ with $ph(x) = \bar{H}(x, 0)$. Let $\tau : X \to I, H' : \tau^{-1}(0, 1] \times I \to E$ with $pH' = \bar{H}$ and $H'(x, 0) = h(x)$ for all $x \in$

$\tau^{-1}(0,1]$ for all $t \in I$. Let $A = \tau^{-1}(1)$ and $V = \tau^{-1}(0,1]$, then V is a halo around $A \subset X$. Define $s : V \times I \to R$ by $s(v) = (v, H'_v)$ where $H'_v(t) = H'(v,t)$. Notice that $H'_v(0) = H'(v,0) = h(v)$ and $pH'_v(t) = pH'(v,t) = \bar{H}(V,t)$ for all $v \in V$ and for all $t \in I$ imply $s(v) \in R$. Clearly, s is a section of q over V, that is, $qs(v) = q(v, H'_v) = v$. Since q has the SEP for any $h : X \to E$ with $ph(x) = \bar{H}(x,0)$, there exists a section $S : X \to R$ of q over X with $S\,|_A = s\,|_A$. Then $S(x) = (x, \omega_x)$ where $\omega_x : I \to E$ with $\omega_x(0) = h(x)$ and $p\omega_x(t) = \bar{H}(x,t)$ for all $x \in X$ and for all $t \in I$.

Define $H : X \times I \to E$ by $H(x,t) = \omega_x(t)$. Then, $pH(x,t) = p\omega_x(t) = \bar{H}(x,t)$ and $H(x,0) = \omega_x(0) = h(0)$, for all $x \in X$ for all $t \in I$. Since $S\Big|_{\tau^{-1}(1)} = s\Big|_{\tau^{-1}(1)}$, we see that $H\Big|_{\tau^{-1}(1)\times I} = H'\Big|_{\tau^{-1}(1)\times I}$. $\square$

Lemma 4.7 *Let $a < b < c \in \mathbb{R}$ and $\bar{H} : X \times [a,c] \to B$. If $p : E \to B$ has the CHP for $\bar{H}\Big|_{X\times[a,b]}$ and $\bar{H}\Big|_{X\times[b,c]}$, then p has the CHP for $\bar{H}$ itself.*

Proof: Let $h : X \to E$ with $ph(x) = \bar{H}(x,a)$ for all $x \in X, \tau : X \to [a,c]$ continuous, $H' : \tau^{-1}(a,c] \times [a,c] \to E$ with $H'(x,a) = h(x)$ and $pH'(x,t) = \bar{H}(x,t)$ for all $x \in \tau^{-1}(a,c]$ and for all $t \in [a,c]$.

Define $\bar{H}_1 = \bar{H}\Big|_{X\times[a,b]}, h_1 = h, \tau_1 : X \to [a,b]$ by $\tau_1(x) = min(b, \tau(x))$ and $H'_1 = H'\Big|_{\tau_1^{-1}(a,b]\times[a,b]}$. Note that $x \in \tau_1^{-1}(a,b] \Leftrightarrow \tau_1(x) = min(b,\tau(x)) > a \Leftrightarrow \tau(x) > a \Leftrightarrow x \in \tau^{-1}(a,c]$ and therefore

$$H'_1 = H'\Big|_{\tau^{-1}(a,c]\times[a,b]}. \tag{4.1}$$

Note also that $ph_1(x) = ph(x) = \bar{H}(x,a) = \bar{H}_1(x,a)$ and $H'_1(x,a) = H'(x,a) = h(x) = h_1(x)$ for all $x \in \tau_1^{-1}(a,b]$. Moreover, $pH'_1(x,t) = pH'(x,t) = \bar{H}(x,t) =$

$\bar{H}_1(x,t)$ for all $x \in \tau_1^{-1}(a,b]$ and for all $t \in [a,b]$. Since p has the CHP for $\bar{H}\,\big|_{X\times[a,b]}$ (with τ_1), there exists $H_1 : X \times [a,b] \to E$ with $pH_1(x,t) = \bar{H}_1(x,t)$ and $H_1(x,a) = h_1(x)$ for all $x \in X$ and for all $t \in [a,b]$ and $H_1\,\big|_{\tau_1^{-1}(b)\times[a,b]} = H_1'\,\big|_{\tau_1^{-1}(b)\times[a,b]}$. But, $x \in \tau_1^{-1}(b) \Leftrightarrow \tau_1(x) = min(b,\tau(x)) = b \Leftrightarrow \tau(x) \geq b \Leftrightarrow x \in \tau^{-1}[b,c]$. Thus, we have

$$H_1\,\big|_{\tau^{-1}[b,c]\times[a,b]} = H_1'\,\big|_{\tau^{-1}[b,c]\times[a,b]}\,. \tag{4.2}$$

Define $\bar{H}_2 = \bar{H}\,\big|_{X\times[b,c]}$, $h_2(x) = H_1(x,b)$, $\tau_2 : X \to [b,c]$ by $\tau_2(x) = max(b,\tau(x))$, and $H_2' = H'\,\big|_{\tau_2^{-1}(b,c]\times[b,c]}$. Then, we get $ph_2(x) = pH_1(x,b) = pH_1'(x,b) = \bar{H}_1(x,b) = \bar{H}_2(x,b)$ for all $x \in X$. Note that $x \in \tau_2^{-1}(b,c] \Leftrightarrow \tau_2(x) = max(b,\tau(x)) > b \Leftrightarrow \tau(x) > b \Leftrightarrow x \in \tau^{-1}(b,c]$ and therefore $H_2' = H'\,\big|_{\tau^{-1}(b,c]\times[b,c]}$. Thus, for $x \in \tau_2^{-1}(b,c] = \tau^{-1}(b,c] \subset \tau^{-1}[b,c] \subset \tau^{-1}(a,c]$, $H_2'(x,b) = H'(x,b) = H_1'(x,b) = H_1(x,b) = h_2(x)$. Furthermore, $pH_2'(x,t) = pH'(x,t) = \bar{H}(x,t) = \bar{H}_2(x,t)$ for all $x \in \tau_2^{-1}(b,c]$ for all $t \in [b,c]$. By the CHP for $\bar{H}\,\big|_{X\times[b,c]}$ (with τ_2), we get $H_2 : X \times [b,c] \to E$ with $pH_2(x,t) = \bar{H}_2(x,t)$ for all $x \in X$ and for all $t \in [b,c]$, $H_2(x,b) = h_2(x)$ for all $x \in X$ and

$$H_2\,\big|_{\tau_2^{-1}(c)\times[b,c]} = H_2'\,\big|_{\tau_2^{-1}(c)\times[b,c]}\,. \tag{4.3}$$

Since $x \in \tau_2^{-1}(c) \Leftrightarrow \tau_2(x) = max(b,\tau(x)) = c \Leftrightarrow \tau(x) = c \Leftrightarrow x \in \tau^{-1}(c)$,

$$H_2\,\big|_{\tau^{-1}(c)\times[b,c]} = H'\,\big|_{\tau_2^{-1}(c)\times[b,c]}\,. \tag{4.4}$$

Now define $H : X \times [a,c] \to E$ by

$$H(x,t) = \begin{cases} H_1(x,t) & \text{if } t \in [a,b], \\ H_2(x,t) & \text{if } t \in [b,c]. \end{cases}$$

Notice that $t = b \Rightarrow H_2(x,b) = h_2(x) = H_1(x,b)$ for all $x \in X$, hence H is well defined and its continuity is obvious from the Gluing Lemma. For any $x \in X$, we have

$$pH(x,t) = \left\{ \begin{array}{ll} pH_1(x,t) = \bar{H}_1(x,t) & \text{if } t \in [a,b] \\ pH_2(x,t) = \bar{H}_2(x,t) & \text{if } t \in [b,c] \end{array} \right\} = \bar{H}(x,t),$$

and $H(x,a) = H_1(x,a) = h_1(x) = h(x)$. Moreover,

$$\begin{aligned} H\Big|_{\tau^{-1}(c)\times[a,c]} &= \begin{cases} H_1\Big|_{\tau^{-1}(c)\times[a,b]} & \text{if } t \in [a,b], \\ H_2\Big|_{\tau^{-1}(c)\times[b,c]} & \text{if } t \in [b,c], \end{cases} \\ &= \begin{cases} H_1'\Big|_{\tau^{-1}(c)\times[a,b]} & \text{if } t \in [a,b] \text{ by (4.2)}, \\ H_2'\Big|_{\tau^{-1}(c)\times[b,c]} & \text{if } t \in [b,c] \text{ by (4.3)}, \end{cases} \\ &= \begin{cases} H'\Big|_{\tau^{-1}(c)\times[a,b]} & \text{if } t \in [a,b] \text{ by (4.1)}, \\ H'\Big|_{\tau^{-1}(c)\times[b,c]} & \text{if } t \in [b,c] \text{ by (4.4)}, \end{cases} \\ &= H'\Big|_{\tau^{-1}(c)\times[a,c]} . \end{aligned}$$

Thus, p has the CHP for $\bar{H}$. □

We are now ready to obtain the main theorems on the local nature of the CHP.

Theorem 4.8 *Let $p : E \to B$ be a space over B, and $\bar{H} : X \times I \to B$ a homotopy. If there exists a numerable covering $\{V_\lambda\}_{\lambda\in\Lambda}$ of X, and for every $\lambda \in \Lambda$ real numbers $0 = t_0^\lambda < t_1^\lambda < \cdots < t_{r_\lambda}^\lambda = 1$ such that p has the CHP for $\bar{H}\Big|_{V_\lambda\times[t_i^\lambda,t_{i+1}^\lambda]}$, $\forall\lambda, i$, then p has the CHP for $\bar{H}$.*

Proof: Lemma 4.7 shows that p has the CHP for $\bar{H}\,|_{V_\lambda\times I}$ for all $\lambda \in \Lambda$. Let $q = q_h$ where $h : X \to E$ is any map with $ph(x) = \hat{H}(x,0)$. Then, by Lemma 4.6, $q : R \to X$ has the SEP over each V_λ, hence q itself has the SEP X, by the Section Extension Theorem. By Lemma 4.6, p has the CHP for $\bar{H}$. □

Theorem 4.9 *If $p : E \to B$ has the CHP over every set V_λ where $\{V_\lambda\}_{\lambda\in\Lambda}$ is either* *i) a numerable covering, or*

ii) an open covering

of B, then p has the CHP for all spaces X in case i), respectively for all paracompact spaces X in case ii).

Proof: We first deal with with case i).

Let $\bar{H} : X \times I \to B$ be a homotopy. We can assume that $\{V_\lambda\}_{\lambda \in \Lambda}$ is given by a locally finite partition of unity, say $\{\pi_\lambda : B \to I\}_{\lambda\in\Lambda}$ (cf. see the first paragraph of the proof of the Section Extension Theorem, 2.37). So, $V_\lambda = \pi_\lambda^{-1}(0,1]$, for all $\lambda \in \Lambda$. For every ordered r-tuple $\lambda_1, \lambda_2, \cdots, \lambda_r \in \Lambda$, define $\pi_{\lambda_1\cdots\lambda_r} : X \to I$ by $\pi_{\lambda_1\cdots\lambda_r}(x) = \Pi_{i=1}^r inf\{\pi_{\lambda_i}\bar{H}(x,t) : t \in [\frac{i-1}{r}, \frac{i}{r}]\}$. Note that $\pi_{\lambda_i}\bar{H} : X \times [\frac{i-1}{r}, \frac{i}{r}] \to I$ is continuous. Since $[\frac{i-1}{r}, \frac{i}{r}]$ is compact Hausdorff, $\theta_{\lambda_i} : X \to I$ given by $\theta_{\lambda_i}(x) = inf_{t\in[\frac{i-1}{r},\frac{i}{r}]}\pi_{\lambda_i}\bar{H}(x,t)$ is continuous. Now, $\pi_{\lambda_1\cdots\lambda_r}$ is the pointwise product of $\theta_{\lambda_1}\cdots\theta_{\lambda_r}$, hence continuous. We claim that $\{W_\gamma = \pi_\gamma^{-1}(0,1]\}_{\gamma\in\Gamma}$ is a numerable covering of X. Here Γ is the set of all finite tuples of elements from Λ; namely, $\gamma \in \Gamma \Leftrightarrow \gamma = \lambda_1\lambda_2\cdots\lambda_r$ for some $r \in \mathbb{N}, \lambda_i \in \Lambda, 1 \le i \le r$. Note that for any $\lambda_1\cdots\lambda_r$ in Γ, we will have $\pi_{\lambda_1\cdots\lambda_r}(x) \neq 0 \Leftrightarrow \bar{H}(x \times [\frac{i-1}{r}, \frac{i}{r}]) \subset V_{\lambda_i}$ for $1 \le i \le r$.

Let $(x,t) \in X \times I$, then there exists an open neighbourhood $V_{(x,t)} \subset B$ of $\bar{H}(x,t)$ such that $V_{(x,t)} \subset V_\lambda$ for some $\lambda \in \Lambda$ and $V_{(x,t)}$ intersects only finitely many V_λ. Thus, every pair $(x,t) \in X \times I$ has an open neighbourhood which is contained in one of the sets $\{\bar{H}^{-1}(V_\lambda)\}_{\lambda\in\Lambda}$ and which meets only a finite number of $\{\bar{H}^{-1}(V_\lambda)\}_{\lambda\in\Lambda}$. Hence, by compactness of I, for any $x \in X$, there exists an open neighbourhood $U_x \subset X$ of x and a natural number r_x depending on x, with the property

1. $U_x \times [\frac{i-1}{r_x}, \frac{i}{r_x}] \subset \bar{H}^{-1}(V_{\lambda_i})$ for some $\lambda_i \in \Lambda, 1 \le i \le r_x$, and

2. $U_x \times I$ meets only finitely many $\bar{H}^{-1}(V_\lambda)$.

From 1., we see that $\pi_{\lambda_1 \cdots \lambda_{r_x}}(u) \neq 0$ for every $u \in U_x$. In particular, $W_{\lambda_1 \cdots \lambda_{r_x}} \supset U_x$. It follows that $\cup_{\gamma \in \Gamma} W_\gamma = X$. For any given natural number r and an r-tuple $\lambda_1, \cdots, \lambda_r$ of elements from Λ, we have observed already that $x \in W_{\lambda_1 \cdots \lambda_r}$ $\Leftrightarrow x \times [\frac{i-1}{r}, \frac{i}{r}] \subset \bar{H}^{-1}(V_{\lambda_i})$ for $1 \leq i \leq r$. Condition 2. guarantees that for any given natural number r, the family $\{W_\gamma\}_{\gamma \in \Gamma_r}$ is locally finite where Γ_r is the set of r-tuples of elements from Λ. However, $\Gamma = \cup_{\gamma \geq 1} \Gamma_r$ and the family $\{W_\gamma\}_{\gamma \in \Gamma}$ may fail to be locally finite. We will show, however, that $\{W_\gamma\}_{\gamma \in \Gamma}$ is numerable. For any natural number k and $\gamma = (\lambda_1, \cdots, \lambda_k) \in \Gamma_k$, let us write π_γ for the function $\pi_{\lambda_1 \cdots \lambda_k} : X \to I$.

Let $\mathbb{R}^+ = \{t \in \mathbb{R} : t > 0\}$. For any $r \in \mathbb{N}$, we define a continuous function $q_r : X \to \mathbb{R}^+ \cup \{0\}$ as follows: $q_1(x) = 0$ for all $x \in X$. If $r > 1$, let $\Delta_r = \cup_{k<r} \Gamma_k$ and $q_r(x) = \sum_{\gamma \in \Delta_r} \pi_\gamma(x)$.

For any $(\lambda_1, \cdots, \lambda_r) \in \Gamma_r$, define $\pi'_{\lambda_1 \cdots \lambda_r} : X \to I$ by $\pi'_{\lambda_1 \cdots \lambda_r}(x)$ $= max(0, \pi_{\lambda_1 \cdots \lambda_r}(x) - r q_r(x))$ for any $x \in X$ and let $W'_\gamma = \{x \in X : \pi'_\gamma(x) > 0\}$ for $\gamma \in \Gamma_r$. We will show that $\{W'_\gamma\}_{\gamma \in \Gamma}$ is a locally finite open covering of X. Clearly, $W'_\gamma \subset W_\gamma = \{x \in X : \pi_\gamma(x) > 0\}$. Thus, $\{W'_\gamma\}_{\gamma \in \Gamma}$ is a refinement of $\{W_\gamma\}_{\gamma \in \Gamma}$.

To see that $\{W'_\gamma\}_{\gamma \in \Gamma}$ cover X, let $x \in X$. We know that $X = \cup_{\gamma \in \Gamma} W_\gamma$. Choose $k \in \mathbb{N}$ minimal with respect to the property that $x \in W_\mu$ for some $\mu \in \Gamma_k$. This means $x \in W_\mu$ for some $\mu \in \Gamma_k$ and $x \notin W_\delta$ for any $\delta \in \Delta_k = \cup_{l<k} \Gamma_l$. Then $q_k(x) = 0$ and hence $\pi'_\mu(x) = \pi_\mu(x) > 0$ showing that $x \in W'_\mu$. Hence $\{W'_\gamma\}_{\gamma \in \Gamma}$ cover X.

Next, we prove that $\{W'_\gamma\}_{\gamma \in \Gamma}$ is locally finite. With k having the same meaning as above, let $x \in W_\mu$ with $\mu \in \Gamma_k$. Then $\pi_\mu(x) > 0$. Choose N a sufficiently large integer greater than k to satisfy $\pi_\mu(x) > \frac{1}{N}$. Since $\pi_\mu(x)$ is one of the terms occurring in the expression $q_N(x) = \sum_{\gamma\in\Delta_N} \pi_\gamma(x)$, we see that $q_N(x) > \frac{1}{N}$. Since q_N is continuous, there exists an open set V in X with $x \in V$ and $q_N(y) > \frac{1}{N}$ for all $y \in V$. It follows that $lq_l(y) > 1$ for all $y \in V$ whenever $l \geq N$, hence $\pi'_\gamma(y) = 0$ for all $y \in V$ and $\gamma \in \Gamma_l$ whenever $l \geq N$. Hence, $V \cap W'_\gamma = \phi$ for all $\gamma \in \Gamma_l$ with $l \geq N$. Since $W'_\gamma \subset W_\gamma$ and $\{W_\gamma\}_{\gamma\in\Delta_N} = \{W_\gamma\}_{\gamma\in\Gamma_1\cup\cdots\cup\Gamma_{N-1}}$ is locally finite, we see that $\{W'_\gamma\}_{\gamma\in\Gamma}$ is locally finite. Clearly $\pi''_\gamma = \frac{\pi'_\gamma}{\sum_{\nu\in\Gamma}\pi'_\nu}$ is a partition of unity with $W'_\gamma = \{x \in X : \pi''_\gamma(x) > 0\}$ for every $\gamma \in \Gamma$.

Thus, $\{W_\gamma\}_{\gamma\in\Gamma}$ is numerable. Now p has the CHP for $\bar{H}\Big|_{W_{\lambda_1\cdots\lambda_r}\times[\frac{i-1}{r},\frac{i}{r}]}$ for any $(\lambda_1, \cdots, \lambda_r) \in \Gamma_r$ and $1 \leq i \leq r$. From Theorem 4.8, it follows that p has the CHP for $\bar{H}$. This completes the proof of case i).

In case ii), $\{V_\lambda\}_{\lambda\in\Lambda}$ is an open covering of B and $\bar{H} : X \times I \to B$ is a homotopy with X paracompact. Using the compactness of I, for any $x \in X$, we can find an open set U_x of X with $x \in U_x$ and a natural number r_x satisfying the condition that $U_x \times [\frac{i-1}{r_x}, \frac{i}{r_x}] \subset \bar{H}^{-1}(V_{\lambda_i})$ for some $\lambda_i \in \Lambda$, $1 \leq i \leq r_x$. Then automatically $\{U_x\}_{x\in X}$ is a covering of X which is numerable, because every open covering of a paracompact space is numerable. Now p has the CHP for $\bar{H}\Big|_{U_x\times[\frac{i-1}{r_x},\frac{i}{r_x}]}$ for $1 \leq i \leq r_x$. Theorem 4.8 implies that p has the CHP for $\bar{H}$. □

Definition 4.10 *We say that $p : E \to B$* **has the local CHP for CW-complexes of dimension $\leq m$** *, if given any homotopy $\bar{H} : X \times I \to B$ with X a CW-complex of dimension $\leq m$, we can find an open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ of X such that p has the CHP for $\bar{H}\,|_{U_\lambda\times I} : U_\lambda \times I \to B$ for each $\lambda \in \Lambda$.*

Definition 4.11 *A space X is called a* **local CW-complex of dimension $\leq m$,** *if for every point x of X, we can find a neighbourhood N_x of x in X with N_x a*

CW-complex of dimension $\leq m$.

Note that N_x itself need not be open in X. All we require is that $x \in int(N_x)$ where $int(N_x)$ is the interior of N_x in X as a topological space.

With these definitions, we have the following result which is actually a modification of Theorem 4.9 *ii*).

Theorem 4.12 *Let $p : E \to B$ have the local CHP for CW-complexes of dimension $\leq m$. Then p has the CHP for all paracompact spaces X which are locally CW-complexes of dimension $\leq m$.*

Proof: Let X be a paracompact space which is locally a CW-complex of dimension $\leq m$ and $\bar{H} : X \times I \to B$ be any homotopy. For any $x \in X$, we have a neighbourhood N_x of x in X with N_x a CW-complex of dimension $\leq m$. By assumption, $p : E \to B$ has the local CHP for CW-complexes of dimension $\leq m$. Hence, we can find an open set U_x in N_x with $x \in U_x$ satisfying the condition that p has the CHP for $\bar{H}\,|_{U_x \times I} : U_x \times I \to B$. Since N_x is a neighbourhood of x in X, we can choose U_x to be open in X.

Thus for each $x \in X$, there exists an open set U_x in X satisfying the condition that p has the CHP for $\bar{H}\,|_{U_x \times I}$. Since X is paracompact, $\{U_x\}_{x \in X}$ is a numerable covering of X. Hence, by Theorem 4.9, p has the CHP for $\bar{H} : X \times I \to B$. □

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Appendix A

Categories and Functors

Definition A.1 *A* **category** $\mathcal{C}$ *consists of*

1. *a class of* **objects**, $\boldsymbol{obj\mathcal{C}}$,

2. *sets of* **morphisms**, $\boldsymbol{hom(X,Y)}$$= hom_{\mathcal{C}}(X,Y)$, *for each pair* $X, Y \in obj\mathcal{C}$, *and*

3. *a map* $hom(X,Y) \times hom(Y,Z) \to hom(X,Z)$, $(f,g) \mapsto gf$, *called* **composition**, *for every triple* $X, Y, Z \in obj\mathcal{C}$,

satisfying the following axioms:

1. *the family of* $hom(X,Y)$*'s is pairwise disjoint,*

2. *composition is associative whenever defined, and*

3. *for each* $X \in obj\mathcal{C}$, *there exists an* **identity** $1_X \in hom(X,X)$ *with* $1_X f = f$ *and* $g 1_X = g$ *for all* $f \in hom(Y,X)$, *for all* $g \in hom(X,Z)$, *where* $Y, Z \in obj\mathcal{C}$.

Examples A.2 1. $\mathcal{C} = \mathcal{S}ets$ *with* $obj\mathcal{C}$ *consisting of all sets,*

$$hom(X,Y) = \{\text{all functions } X \to Y\},$$

and composition is the usual composition of functions.

2. $\mathcal{C} = \mathcal{T}op_*$ *with* $obj\mathcal{C}$ *consisting of all pointed topological spaces,*

$$hom(X, Y) = \{\text{all continuous functions } X \to Y\},$$

and composition is the usual composition.

3. $\mathcal{C} = \mathcal{G}rp$ *with* $obj\mathcal{C}$ *consisting of all groups,*

$$hom(X, Y) = \{\text{all group homomorphisms } X \to Y\},$$

and composition is the usual composition.

4. $\mathcal{C} = \mathcal{A}b$ *with* $obj\mathcal{C}$ *consists of all abelian groups,*

$$hom(X, Y) = \{\text{all group homomorphisms } X \to Y\},$$

and composition is the usual composition. Notice that $\mathcal{A}b$ *is a subcategory of* $\mathcal{G}rp$.

5. $obj\mathcal{C}$ *is the elements in a quasi-ordered set, and*

$$hom(x, y) = \begin{cases} \phi & \text{if } x \not\leq y, \\ \{i_y^x\} & \text{if } x \leq y. \end{cases}$$

The composition is defined by $i_z^y i_y^x = i_z^x$. *This is an example of an abstract category in contrast with concrete categories (meaning subcategories of the category of sets and functions) in the previous examples.*

Definition A.3 *Let* $\mathcal{C}, \mathcal{D}$ *be categories. A* **covariant functor** $T : \mathcal{C} \to \mathcal{D}$ *is a function on the objects and morphisms of* $\mathcal{C}$ *into (respectively) objects and morphisms of* $\mathcal{D}$ *as follows.*

1. $X \in obj\mathcal{C} \Rightarrow TX \in obj\mathcal{D}$, *and*

2. $f \in hom_{\mathcal{C}}(X,Y) \Rightarrow Tf \in hom_{\mathcal{D}}(TX,TY)$

such that $T(gf) = T(g)T(f)$ *for any morphisms of* $\mathcal{C}$ *whenever* gf *is defined, and* $T(1_X) = 1_{TX}$, *for all* $X \in obj\mathcal{C}$.

Definition A.4 *Let* $\mathcal{C}, \mathcal{D}$ *be categories. A* **contravariant functor** $T : \mathcal{C} \to \mathcal{D}$ *is a function on the objects and morphisms of* $\mathcal{C}$ *into (respectively) objects and morphisms of* $\mathcal{D}$ *as follows.*

1. $X \in obj\mathcal{C} \Rightarrow TX \in obj\mathcal{D}$, *and*

2. $f \in hom_{\mathcal{C}}(X,Y) \Rightarrow Tf \in hom_{\mathcal{D}}(TY,TX)$

such that $T(gf) = T(f)T(g)$ *for any morphisms of* $\mathcal{C}$ *whenever* gf *is defined, and* $T(1_X) = 1_{TX}$, *for all* $X \in obj\mathcal{C}$.

Examples A.5

1. *The forgetful functor* $F : \mathcal{T}op_* \to \mathcal{S}ets$ *assigns to each pointed topological space its underlying set and to each continuous function itself, i.e. "forgetting" the base point and continuity.*

2. *Let* X *be fixed in a category* $\mathcal{C}$, *then* $hom(X, \ ) : \mathcal{C} \to \mathcal{S}ets$ *is a covariant functor assigning to each object* $Y \in obj\mathcal{C}$ *the set* $hom(X,Y)$ *and to each morphism* $f \in hom(Y,Y')$ *the induced map* $f_* = hom(X,f) : hom(X,Y) \to hom(X,Y')$ *defined by* $h \mapsto fh$.

3. *Similarly, for* $Y \in \mathcal{C}$ *fixed,* $hom(\ ,Y) : \mathcal{C} \to \mathcal{S}ets$ *is a contravariant functor assigning to each object* $X \in obj\mathcal{C}$ *the set* $hom(X,Y)$ *and to each morphism* $g \in hom(X,X')$ *the induced map* $g^* = hom(g,Y) : hom(X',Y) \to hom(X,Y)$ *defined by* $h \mapsto hg$.

4. Let n be an integer greater than 0. The covariant functor $\pi_n : \mathcal{T}op_* \to \mathcal{G}rp$ is defined by $\pi_n(X, x_0) = [(S^n, s^n), (X, x_0)]$ for each $X \in obj\mathcal{T}op_*$ and $\pi_n(f) = f_* : [(S^n, s^n), (X, x_0)] \to [(S^n, s^n), (Y, y_0)]$ for $f \in hom(X, Y)$, where x_0 is the base point of X, y_0 is the base point of $Y \in obj\mathcal{T}op_*$, $s^n = (1, 0, \cdots, 0) \in \mathbb{R}^{n+1}$ is the base point of S^n, and $[(S^n, s^n), (X, x_0)]$ denotes the family of homotopy classes of all base point preserving maps from S^n to X. For the composition of the equivalence classes, we need the following definition.

Definition A.6 *Let (X, x_0) be a pointed space and define $\boldsymbol{\Delta}_x : (X, x_0) \to (X, x_0) \times (X, x_0)$ by $\Delta_x(x) = (x, x)$. Let $i : (X, x_0) \vee (X, x_0) \hookrightarrow (X, x_0) \times (X, x_0)$ be the inclusion, where $(X, x_0) \vee (X, x_0) = X \times \{x_0\} \cup \{x_0\} \times X$. A pointed space (X, x_0) is called a* **co-Hspace** *if there exists a continuous function, called a* **co-multiplication** $c : (X, x_0) \to (X, x_0) \vee (X, x_0)$ *with $ic \simeq \Delta_x$.*

Since S^n is a co-Hspace for $n \geq 1$ (cf. [8], p.331), we define the composition by $[f] * [g] = [\nabla_X(f \vee g)c]$, where $f, g \in \pi_n(X, x_0)$, $\boldsymbol{\nabla}_X : (X, x_0) \vee (X, x_0) \to (X, x_0)$ is the **folding map** *defined by $\nabla_X(x, x_0) = x$ and $\nabla_X(x_0, x) = x$.*

Index

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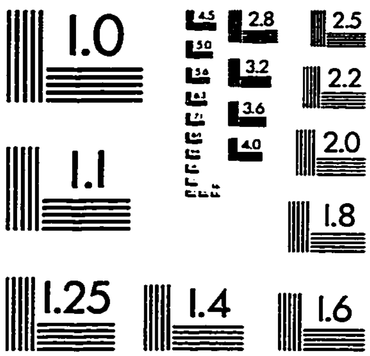

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