Searching faster using self-loops in quantum walks

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1 Background

Searching is one the most important tasks in computer science. Searching is used everywhere by everyone at all times, and it is one of the main drivers of research in quantum computing. Quantum walks, which are a generalization of random walks, can be used to build faster quantum searching algorithms. A random walk is analogous to walking the streets in Calgary at random until eventually reaching our destination. A quantum walk is similar but conducted on a quantum computer. Quantum walks enable us to solve searching problems quadratically faster than classical random walks [1].

Quantum walks are probabilistic, which means that the walk reaches the destination with a probability that depends on the number of steps taken. The challenge is to develop a quantum walk that reaches the destination with high probability after few number of steps. In a recent paper[2], Wong shows that adding self-loops to a square grid can speed up the walk and boost the success probability to 1 in some specific cases. A self-loop is analogous to pausing at an intersection before continuing walking down a next street. This idea is known to be insufficient in general. The main problem is that the walk only works for a unique destination; when there are multiple possible destinations, the success probability of Wong's walk may drop to almost 0. My main objective is to design a more general quantum walk that can simulate Wong's quantum walk in case of a unique marked node, and that also works for multiple marked nodes.

2 Introduction

The graph that we apply the quantum walk on, is a $\sqrt{N} \times \sqrt{N}$ torus grid with self-loops everywhere, for which there is only one unique marked node (or destination). The weight of each self-loop is $l = \frac{4}{N}$ and weight of other edges is 1. Each node represents a intersection and each edge represents a path between two intersections. The directions of edges are $\{|\uparrow\rangle, |\downarrow\rangle, |\leftarrow\rangle, |\rightarrow\rangle, |\circlearrowleft\rangle\}$. Thus the edges can be represented by states in the Hilbert space $\mathbb{C}^N \otimes \mathbb{C}^5$. For example, the state $|x, \leftarrow\rangle$ denotes the edge at node x pointing to the left.

In Wong's paper, the quantum walk is defined as $A_5 = W \cdot G_5$ where

$$W = SWAP \cdot (2\sum_x |x, P_x\rangle \langle x, P_x| - \mathbb{1}).$$

$$G_5 = \mathbb{1} - 2 \sum_{v \in \{\uparrow,\downarrow,\leftarrow,\rightarrow,\circlearrowleft\}} |m, v\rangle \langle m, v| .$$

with

$$P_x = \frac{1}{\sqrt{4+l}} \left(|\uparrow\rangle + |\downarrow\rangle + |\leftarrow\rangle + |\rightarrow\rangle + \sqrt{l}\, |\circlearrowleft\rangle \right).$$

$SWAP$ is the operator that changes the direction of every edge and m denotes the marked node.

The quantum walk algorithm that we designed in this summer is defined as $A_2 = W \cdot G_2$ where

$$G_2 = \mathbb{1} - 2\Big(|m, \circlearrowleft\rangle \langle m, \circlearrowleft| + |m, \uparrow\downarrow\leftarrow\rightarrow\rangle \langle m, \uparrow\downarrow\leftarrow\rightarrow| \Big).$$

Here $|\uparrow\downarrow\leftarrow\rightarrow\rangle = |\uparrow\rangle + |\downarrow\rangle + |\leftarrow\rangle + |\rightarrow\rangle$, which is the superposition of four edges.

Instead of having a 5-dimensional reflection G_5, the new quantum walk A_2 use a 2-dimensional reflection G_2. Our hypothesis is that A_2 acts the same as A_5 on the square grid with a unique marked node. This simplification helps us to analyze why the quantum walk can find faster with high success probability on the grid. We will also test A_2 on grid with multiple marked nodes to see if it works in this case.

3 Procedure

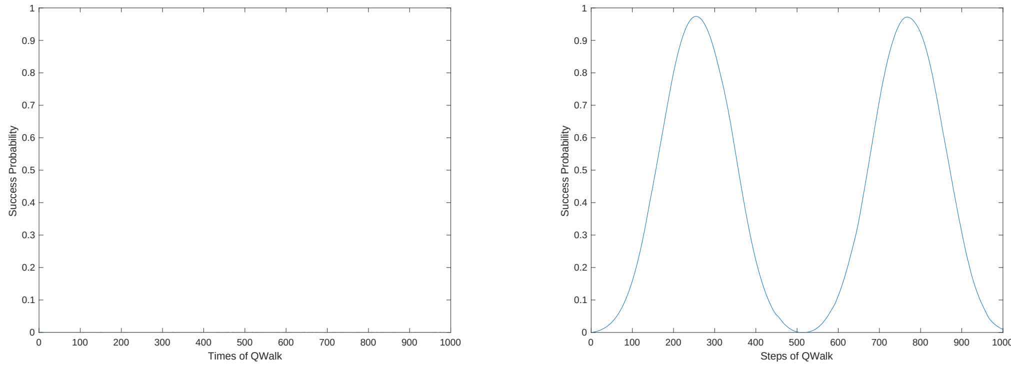
We implemented Wong's quantum walk A_5 and our quantum walk A_2 in MATLAB code. Using MATLAB, we simulated A_2 and A_5 on the same grid graph with a unique marked node, starting from the same initial quantum state. Comparing the quantum states and success probabilities after each step of A_2 and A_5, we found that A_2 acts the same as A_5. Based on this experimental result, we developed and gave mathematical proofs to show A_2 and A_5 act identically in this specific case. We also tested the case of multiple marked nodes. Our MATLAB simulations show that A_2 also finds fast with high success probability when there are multiple marked nodes. In contrast, the known A_5 walk fails in finding for some configurations with multiple marked nodes.

4 Results

The main result is the construction of the 2-dimensional quantum walk A_2. We find that A_2 can simulate A_5 in case of unique marked node and A_2 can find multiple marked nodes with high success probability (as showed in Figure 1).

We gave the mathematical proof that showed A_2 and A_5 act identically on the square grid with a unique marked node.

Assume there is a state $|\psi\rangle$ for which the amplitudes of edges are rotational symmetric about the marked node m, and reflectional symmetric about horizontal and vertical line across



(a) quantum walk A_5 (b) quantum walk A_2

Figure 1: Apply quantum walk A_2 and A_5 on the 2D grid of N vertices with two adjacent marked nodes. $N = 100 \times 100 = 10000$.

the marked node m:

1. $|m,\uparrow\rangle, |m,\downarrow\rangle, |m,\leftarrow\rangle, |m,\rightarrow\rangle$ have the same amplitude in $|\psi\rangle$. G_2 and G_5 only change the state of marked node m, we can write marked state as

$$|\psi_m\rangle = \alpha\, |m, \circlearrowleft\rangle + \beta\, |m, \uparrow\downarrow\leftarrow\rightarrow\rangle .$$

Applying G_5 on $|\psi_m\rangle$ yields

$$\begin{aligned} G_5 \cdot |\psi_m\rangle &= \left(\mathbb{1} - 2 \sum_{v \in \{\uparrow,\downarrow,\leftarrow,\rightarrow,\circlearrowleft\}} |m, v\rangle \langle m, v| \right) \cdot \big(\alpha\, |m, \circlearrowleft\rangle + \beta\, |m, \uparrow\downarrow\leftarrow\rightarrow\rangle\big) \\ &= -\alpha\, |m, \circlearrowleft\rangle - \beta\, |m, \uparrow\downarrow\leftarrow\rightarrow\rangle \\ &= -\, |\psi_m\rangle . \end{aligned}$$

Applying G_2 on $|\psi_m\rangle$ yields

$$\begin{aligned} G_2 \cdot |\psi_m\rangle &= \Big[\mathbb{1} - 2\big(|m, \circlearrowleft\rangle \langle m, \circlearrowleft| + |m, \uparrow\downarrow\leftarrow\rightarrow\rangle \langle m, \uparrow\downarrow\leftarrow\rightarrow| \big)\Big] \cdot \big(\alpha\, |m, \circlearrowleft\rangle + \beta\, |m, \uparrow\downarrow\leftarrow\rightarrow\rangle\big) \\ &= -\alpha\, |m, \circlearrowleft\rangle - \beta\, |m, \uparrow\downarrow\leftarrow\rightarrow\rangle \\ &= -\, |\psi_m\rangle . \end{aligned}$$

We find that G_2 and G_5 act as the same on $|\psi_m\rangle$. Both of them reverse the state $|\psi_m\rangle$ to $-|\psi_m\rangle$, which means G_2 and G_5 preserve the rotational and reflectional symmetry.

2. Applying $(2\,|x, P_x\rangle \langle x, P_x| - \mathbb{1})$ on every node x preserves the rotational and reflectional symmetry since the states $|P_x\rangle$ are the same for each node x.

3. *SWAP* operator does not break this symmetry because it just swap the amplitudes of two edges with opposite direction.

So for the state $|\psi\rangle$ with rotational and reflectional symmetry, we have

$$A_5 \cdot |\psi\rangle = W \cdot G_5 \cdot |\psi\rangle = W \cdot G_2 \cdot |\psi\rangle = A_2 \cdot |\psi\rangle \, .$$

And the result state $|\psi'\rangle = A_5 \cdot |\psi\rangle = A_2 \cdot |\psi\rangle$ is still rotational and reflectional symmetric. Hence we can conclude that

$$A_5^i \cdot |\psi\rangle = A_2^i \cdot |\psi\rangle \, , \qquad i = 1, 2, 3... \, .$$

The quantum walk starts from the initial state

$$|init\rangle = \frac{1}{\sqrt{N}} \sum_{x=1}^{N} |x, P_x\rangle \, .$$

For the initial state $|init\rangle$, amplitudes of edges are rotational symmetric about the marked node m, and reflectional symmetric about horizontal and vertical line across the marked node m. Then we have

$$A_5^i \cdot |init\rangle = A_2^i \cdot |init\rangle \, , \qquad i = 1, 2, 3... \, .$$

This above reasoning is a key part in our proof that A_2 and A_5 act identically on the square grid with a unique marked node.

5 Experience Gained

A useful skill I have learned during this summer was using MATLAB to implement and simulate quantum algorithms. Before this summer, I learned quantum algorithms by reading papers. Some methods and ideas in those papers are complicated and not intuitive, so that sometimes I cannot get enough insight from reading them. Learning how to simulate quantum walk in MATLAB helps me better understand technical details and gain more insight of quantum computing. This skill helped me make great progress in the summer research and I believe it will continue to benefit me in my future research activities.

Although doing experiment by MATLAB simulation is important and helpful, mathematical proof is the core of this theoretical research project. In past courses, I learned some proof techniques. This research project provided me an opportunity to practice what I have learned before. Using those techniques to solve real problems and generate the final result gave me a better understanding of mathematical proofs.

During the summer research, I learned many useful research skill like reading papers and writing in LaTeX. By meeting with my supervisor regularly, I developed communication skills such as how to prepare for meetings, how to present my findings, how to write research reports, and how to ask questions. Also our research group had weekly group meetings, I did presentations to introduce my research to our group. After my presentations, I received many useful feedbacks and suggestions from the group members, which helped me to improve my skills.

6 Acknowledgements

First, deep thanks goes to my supervisor Dr. Peter Høyer. I greatly appreciate his encouragement and guidance in my research, as well as the numerous explanations of quantum walk. And I would like to thank PURE and the University of Calgary for supporting me to conduct research in quantum computing during this summer. The research project exposed me to some open challenges in quantum computing and gave me an opportunity to solve one. This research experience allowed me to not only apply my knowledge but also to expand it and get something new. My research and presentations also benefit our research group to become stronger. The research and communication skills I learned will be a treasure for my future research career. I believe this summer research experience is a great preparation for my graduate study.

References

[1] Mario Szegedy. Quantum speed-up of Markov chain based algorithms. In *45th IEEE Symposium on Foundations of Computer Science*, FOCS'04, pages 32–41, 2004.

[2] Thomas G. Wong. Faster search by lackadaisical quantum walk, March 2018.