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C*-ALGEBRAS AND INFINITE QUANTUM SYSTEMS

BY

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled, "C*-ALGEBRAS AND INFINITE QUANTUM SYSTEMS" submitted by David Henry Petiot in partial fulfillment of the requirements for the degree of Master of Science.

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ABSTRACT

This thesis is an examination of the Algebraic Approach to Quantum Mechanics and its use in the description of infinite Quantum systems. This approach is based on the postulate that it is possible to set up a correspondence between Quantum systems and the mathematical theory of C*-algebras.

In Chapter 1 the theory of C*-algebras and their representations is discussed. The main result is the G.N.S. construction.

In Chapter 2 the methods developed in Chapter 1 are applied to Quantum spin sytems. The C*-algebra corresponding to a single spin is constructed and analyzed, and the G.N.S. representation for the state corresponding to a canonical ensemble with definite temperature is constructed. The discussion is then generalized to a two spin system, a multi spin system, and finally to a system consisting of an infinite number of spins. It is shown that the G.N.S. representations corresponding to different temperatures are unitarily equivalent for the finite system, while for the infinite system these representations become unitarily inequivalent. This example illustrates that classical variables, in this case the temperature, arise in quantum systems as labels that distinguish between different inequivalent representations of the Quantum algebra in the case of inifinitely many degrees of freedom.

In Chapter 3 the algebraic description of a non-interacting Bose gas is examined. The C.C.R. algebra is constructed. Methods developed by Araki and Woods are used to construct the state of the infinite Bose gas that has density ρ and in which all particles have zero momentum. The representation constructed by Araki and

Woods is shown to be the G.N.S. representation for this state. It is shown that for the infinite gas the G.N.S. representations corresponding to different densities are unitarily inequivalent. Again a classical variable, the density in this case, arises as a label that distinguishes the different inequivalent representations.

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INTRODUCTION

Traditionally Quantum Mechanics is based on the postulate that it is possible to set up a correspondence between physical systems and the mathematical theory of Hilbert spaces. To a given physical system we can associate a Hilbert space in such a manner that the states of the system are represented by density operators on the Hilbert space (self adjoint, positive, bounded operators with unit trace), while the observables of the system are represented by linear self adjoint operators on the Hilbert space. It is possible to deal instead with the set of linear bounded operators, which are not self adjoint in general. The idea behind the Algebraic Approach to Quantum Mechanics is to consider the algebraic structure of the set of all bounded linear operators on the Hilbert space as the fundamental mathematical object, with the Hilbert space being secondary. We postulate that to a given physical system we can associate a C*-algebra in such a way that the states of the system correspond to (normalized) positive linear functionals over the C*-algebra, while the observables of the system can be expressed in terms of the elements of the C*-algebra. The Hilbert space with its linear operators then corresponds to a concrete representation of the C*-algebra (see [Haag] and [Haag1] for a general discuccion of the Algebraic Approach).

It may happen that the C*-algebra associated with a particular physical system admits only one irreducible representation (up to unitary equivalence), as is usually the case when one deals with systems that have a finite number of degrees of freedom. When we consider systems that possess an infinite number of degrees of freedom we often find that the corresponding C*-algebra has a number of unitarily inequivalent irreducible representations (such systems are encountered in Quantum Field Theory and Quantum Statistical mechanics, see [Haag2]). We shall see an example of this in Chapter 2 when we discuss the canonical commutation relations. The usual motivation for the Algebraic Approach is given by the second possibility (many inequivalent representations). In the first case we could consider the unique representation and return to the Hilbert space formalism. In the second case we have to consider many inequivalent representations side by side and it is the C*-algebra that provides an underlying connection between them.

CHAPTER 1

C*-ALGEBRAS

1.1 DEFINITION OF A C*-ALGEBRA

We begin by defining the structure of a C*-algebra (some good references for this material are [Kadi], [Dixm1], [Brat1] and [Brat2]). We start with a vector space $\mathfrak{A}$ over the field of complex numbers $\mathbb{C}$ and define a C*-algebra by adding more structure to $\mathfrak{A}$.

Definition 1.1 A product AB over a vector space $\mathfrak{A}$ is a rule that associates to each pair $A, B \in \mathfrak{A}$ the product AB such that

i) $AB \in \mathfrak{A}$ (closure),

ii) $A(BC)=(AB)C$ (associative),

iii) $A(\beta B+\gamma C)=\beta AB+\gamma AC$ and $(\beta B+\gamma C)A=\beta BA+\gamma CA$ (distributive),

for all $A, B, C \in \mathfrak{A}$ and $\beta, \gamma \in \mathbb{C}$.

Definition 1.2 A vector space $\mathfrak{A}$ equipped with a product is an algebra.

Definition 1.3 An involution of an algebra $\mathfrak{A}$ is a mapping $A \in \mathfrak{A} \to A^* \in \mathfrak{A}$ that satisfies

i) $(A^*)^*=A$,

ii) $(AB)^*=B^*A^*$,

iii) $(\alpha A+\beta B)^*=\overline{\alpha}A^*+\overline{\beta}B^*$,

for all $A,B \in \mathfrak{A}$ and $\alpha,\beta \in \mathbb{C}$, where $\overline{\alpha}$ is the complex conjugate of α. The element A^* is referred to as the adjoint of A. An algebra $\mathfrak{A}$ that possesses an involution is called a *-algebra.

Definition 1.4 A $*$-norm of a $*$-algebra $\mathfrak{A}$ is a rule that associates a real number $\|A\|$ (the norm of A) with every element $A\in\mathfrak{A}$ such that

i) $\|A\|\geq 0$, $\|A\|=0$ iff $A=0$,

ii) $\|\alpha A\|=|\alpha|\,\|A\|$,

iii) $\|A+B\|\leq\|A\|+\|B\|$ (triangle inequality),

iv) $\|AB\|\leq\|A\|\,\|B\|$ (product inequality),

v) $\|A^*\|=\|A\|$ ($*$-norm property),

for all $A,B\in\mathfrak{A}$ and $\alpha,\beta\in\mathbb{C}$. A $*$-algebra $\mathfrak{A}$ that possesses a $*$-norm is called a normed $*$-algebra.

Since a normed $*$-algebra $\mathfrak{A}$ is, among other things, a set of elements it is possible to equip $\mathfrak{A}$ with various topological structures. One such topology is the uniform topology which uses the $*$-norm to define the neighborhoods of an element $A\in\mathfrak{A}$, $N(A;\varepsilon)=\{B\in\mathfrak{A}:\|A-B\|\leq\varepsilon\}$.

Definition 1.5 A normed $*$-algebra $\mathfrak{A}$ that is complete (in the Cauchy sense) in the uniform topology defined by its $*$-norm is called a Banach $*$-algebra.

Given a $*$-algebra $\mathfrak{A}$ we can attempt to construct a Banach $*$-algebra by defining a $*$-norm for $\mathfrak{A}$ and then completing $\mathfrak{A}$ in the uniform topology defined by this $*$-norm. Since it may be possible to define various inequivalent $*$-norms for $\mathfrak{A}$ it may also be possible to construct different Banach $*$-algebras from one $*$-algebra. This being the case one might wonder whether or not it is possible to fix a unique $*$-norm by adding more conditions to our definition of a $*$-norm. The following theorem shows that this is indeed the case.

Theorem 1.6 Let $\mathfrak{A}$ be a *-algebra. If $\mathfrak{A}$ possesses a *-norm that satisfies $\|A^*A\|=\|A\|^2$ for all $A\in\mathfrak{A}$ and if $\mathfrak{A}$ is complete with respect to the uniform topology defined by this *-norm then this *-norm is unique (i.e. it is the only *-norm that satisfies $\|A^*A\|=\|A\|^2$).

Suppose we have a *-algebra $\mathfrak{A}$ that possesses two inequivalent *-norms. If we complete $\mathfrak{A}$ with respect to each of these norms then Theorem 1.6 implies that the resulting *-algebras are different.

Since $\|A^*A\|=\|A\|^2$ implies that $\|A^*\|=\|A\|$ we can replace condition (v) in our definition of a *-norm with the more stringent condition:

(v') $\|A^*A\|=\|A\|^2$ (C*-norm property).

Definition 1.7 A *-norm that satisfies condition (v') is called a C*-norm, and a *-algebra $\mathfrak{A}$ that is complete with respect to the uniform topology defined by a C*-norm is called a C*-algebra. The foregoing discussion shows that it is possible to construct at most one C*-algebra from a given *-algebra.

The most basic example of a C*-algebra is the complex numbers, with involution defined as complex conjugation and with the usual norm $\|\alpha\|\equiv|\alpha\,\overline{\alpha}|^{1/2}$. A more relevant example is the set $\mathcal{L}(\mathfrak{H})$ of all bounded linear operators on a Hilbert space $\mathfrak{H}$, equipped with the usual algebraic structure. The operator adjoint defines an involution of $\mathcal{L}(\mathfrak{H})$, and the operator norm

$$\|A\| = \mathrm{Sup}\left\{\frac{\|A\psi\|}{\|\psi\|} : \psi \in \mathfrak{H}\,;\, \psi \neq 0\right\}$$

defines a C*-norm on $\mathcal{L}(\mathfrak{H})$ (that $\mathcal{L}(\mathfrak{H})$ is complete follows from the fact that a Cauchy sequence of bounded linear operators converges to a bounded linear operator).

Definition 1.8 An identity of a C*-algebra $\mathfrak{A}$ (or of any algebra) is an element $\mathbb{1} \in \mathfrak{A}$ that satisfies $A\mathbb{1} = \mathbb{1}A = A$ for all $A \in \mathfrak{A}$.

If an identity exists it is unique. Furthermore it is easy to see that $\mathbb{1}^*$ is an identity if $\mathbb{1}$ is, so that $\mathbb{1}^* = \mathbb{1}$. Since $\|\mathbb{1}\| = \|\mathbb{1}^*\mathbb{1}\| = \|\mathbb{1}\|^2$ we must have $\|\mathbb{1}\| = 0$ or $\|\mathbb{1}\| = 1$. From the product inequality we have that $\|A\| = \|\mathbb{1}A\| \leq \|\mathbb{1}\|\,\|A\|$, so that $\|\mathbb{1}\| = 0$ implies that $\|A\| = 0$ for all $A \in \mathfrak{A}$. This last statement means that $A = 0$ for all $A \in \mathfrak{A}$ and the algebra is identically zero. This being the case we shall assume that $\|\mathbb{1}\| = 1$. If a C*-algebra does not contain an identity (there is no guarantee that it does) we can proceed in one of two ways. First we can adjoin an identity to the C*-algebra (i.e. embed $\mathfrak{A}$ in a larger C*-algebra that contains an identity), or we can construct an approximate identity (see, for example, [Brat1] sections 2.1.1 and 2.2.3). We will ignore these technical difficulties and assume that the C*-algebras with which we deal contain an identity.

1.2 STRUCTURE OF A C*-ALGEBRA

The structure of a C*-algebra is completely determined by the assumptions previously laid out, i.e. a C*-algebra is an algebra equipped with an involution and complete with respect to the uniform topology defined by a C*-norm. Although this structure is quite simple, we shall see that these structural assumptions lead to a great wealth of properties (it is, of course, always pleasing to get a lot from a little). We begin by classifying the elements of a C*-algebra.

Definition 1.9 Let $\mathfrak{A}$ be a C*-algebra, then an element $A\in\mathfrak{A}$ is

i) normal if $AA^*=A^*A$,

ii) self adjoint or real if $A^*=A$,

iii) isometric if $A^*A=\mathbb{1}$,

iv) unitary if $A^*A=AA^*=\mathbb{1}$.

An arbitrary element $A\in\mathfrak{A}$ can be uniquely decomposed in terms of self adjoint elements A_1 and A_2 as $A=A_1+iA_2$, where the real and imaginary parts of A are given, respectively, by $A_1=(A+A^*)/2$ and $A_2=(A-A^*)/2i$. A general self adjoint element $A\in\mathfrak{A}$ can be decomposed in terms of unitary elements as $A=(U_++U_-)/2$ with $U_\pm=(A\pm i\sqrt{\|A\|^2I-A^2})$, (we shall soon see that the square root operation in the previous expression is well defined). Using the above decompositions we then see that an arbitrary element $A\in\mathfrak{A}$ can be decomposed as $A=\alpha_1U_1+\alpha_2U_2+\alpha_3U_3+\alpha_4U_4$ where the U_i are unitary elements of $\mathfrak{A}$ and the $\alpha_i\in\mathbb{C}$ are such that $|\alpha_i|\le\|A\|/2$.

Definition 1.10 An element A of a C*-algebra $\mathfrak{A}$ (with identity) is said to be invertible if there exists an element $A^{-1}\in\mathfrak{A}$ (called the inverse of A) such that $AA^{-1}=A^{-1}A=\mathbb{1}$.

All the usual results hold, i.e., an inverse is unique if it exists and AB is invertible if and only if A and B are invertible. Then $(AB)^{-1}=B^{-1}A^{-1}$. Furthermore, if an element $A\in\mathfrak{A}$ is invertible, then A^* is invertible with $(A^*)^{-1}=(A^{-1})^*$. If A is not invertible then it is said to be singular. The notion of an inverse allows us to define the spectrum.

Definition 1.11 Let $\mathfrak{A}$ be a C*-algebra (with identity). The resolvent set $r_{\mathfrak{A}}(A)$ of an element $A\in\mathfrak{A}$ is defined to be

$$r_{\mathfrak{A}}(A)\equiv\{\lambda\in\mathbb{C}:\lambda\mathbb{1}-A \text{ is invertible}\}.$$

The spectrum $\sigma_{\mathfrak{A}}(A)$ of an element $A\in\mathfrak{A}$ is then defined to be the complement of the resolvent set, i.e.,

$$\sigma_{\mathfrak{A}}(A)\equiv\{\lambda\in\mathbb{C}:\lambda\mathbb{1}-A \text{ is singular}\}.$$

It will be noticed that we have attached the symbol $\mathfrak{A}$ to $r_{\mathfrak{A}}(A)$ and $\sigma_{\mathfrak{A}}$. The reason for this is that if we consider a subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ then there are two possible spectra, $\sigma_{\mathfrak{A}}(A)$ and $\sigma_{\mathfrak{B}}(A)$ (i.e., an element may be singular in $\mathfrak{B}$ but invertible in $\mathfrak{A}$; this would be the case if $A\in\mathfrak{B}$ and $A^{-1}\in\mathfrak{A}$ but $A^{-1}\notin\mathfrak{B}$). As it turns out C*-algebras have the property that $\sigma_{\mathfrak{A}}(A)=\sigma_{\mathfrak{B}}(A)$ for all $A\in\mathfrak{B}$. The reason is that if A^{-1} exists it is contained in the C*-subalgebra generated by $\mathbb{1}$, A, and A^* ([Brat1] Proposition 2.2.7). With this in mind we shall simply write $\sigma(A)$ for the spectrum of an element $A\in\mathfrak{A}$.

Definition 1.12 Let $\mathfrak{A}$ be a C*-algebra. The spectral radius $\rho(A)$ of an element $A\in\mathfrak{A}$ is defined to be $\rho(A)\equiv\mathrm{Sup}\{|\lambda|:\lambda\in\sigma(A)\}$.

The spectrum of an element A of a C*-algebra $\mathfrak{A}$ is related to its norm through the spectral radius $\rho(A)$. In particular it is found that $\rho(A) \leq \|A\|$. A natural question is: when does equality hold? Is it possible to define a subset $\mathfrak{B}$ of $\mathfrak{A}$ such that equality holds for all elements of $\mathfrak{B}$? The answer is contained in the following theorem ([Brat1] Theorem 2.25).

Theorem 1.13 Let $\mathfrak{A}$ be a C*-algebra (with identity),

i) if $A \in \mathfrak{A}$ is normal, self adjoint or unitary then $\rho(A)=\|A\|$,

ii) if $A \in \mathfrak{A}$ is unitary then $\sigma(A) \subseteq \{\lambda : \lambda \in \mathbb{C}, |\lambda|=1\}$ and $\rho(A)=1$,

iii) if $A \in \mathfrak{A}$ is self adjoint, the spectrum is real and $\sigma(A) \subseteq [-\|A\|, \|A\|]$,

iv) for general $A \in \mathfrak{A}$ and polynomial P, $\sigma(P(A))=P(\sigma(A))$. In particular $\sigma(\lambda \mathbb{1}-A)=\lambda-\sigma(A)$. Also $\sigma(A^*)=\overline{\sigma(A)}$.

In Theorem 1.6 we claimed that the C*-norm property served to fix a unique norm and we now offer a proof. The spectrum $\sigma(A)$, and therefore the spectral radius $\rho(A)$, depend solely on the algebraic structure of a C*-algebra $\mathfrak{A}$. Given a general $A \in \mathfrak{A}$, A^*A is self adjoint, so that (using the C*-norm property and Theorem 1.13 (i)) $\|A\|=\|A^*A\|^{1/2}=\rho(A^*A)^{1/2}$. This shows that the C*-norm $\|A\|$ is unique.

We now wish to set up an order relation between elements of a C*-algebra. This is made possible by the identification of positive elements.

Definition 1.14 An element A of a C*-algebra $\mathfrak{A}$ is said to be positive if it is self adjoint and its spectrum is contained in the positive half line. The set of all positive elements of $\mathfrak{A}$ is denoted by $\mathfrak{A}_+$.

The reason that we can use the definition of positivity to set up an order relation is that $\mathfrak{A}_+ \cap \{-\mathfrak{A}_+\} = \{0\}$, i.e. if A is positive and -A is positive then A is necessarily zero. Thus we can make the following

Definition 1.15 Two elements A,B of a C*-algebra $\mathfrak{A}$ are said to be in the relation $A \geq B$ whenever $A-B \in \mathfrak{A}_+$.

This relation satisfies i) $A \geq 0$ and $A \leq 0 \Rightarrow A=0$,

ii) $A \leq B$ and $B \leq C \Rightarrow A \leq C$.

So we see that we have indeed defined an order relation on $\mathfrak{A}$.

The positive elements of a C*-algebra admit the notion of a square root. Corresponding to every positive $A \in \mathfrak{A}$ is a unique positive $B \in \mathfrak{A}$, called the positive square root of A, such that $A=B^2$. It should be noted that B^*B is positive for all $B \in \mathfrak{A}$, and furthermore it is a fact that we can obtain all positive elements in this manner. Combining these two results leads to the notion of the modulus of an element $A \in \mathfrak{A}$,

Definition 1.16 The modulus of an element A of a C*-algebra $\mathfrak{A}$ is the element of $\mathfrak{A}$ defined by $|A| \equiv (A^*A)^{1/2}$.

Completing our discussion on the algebraic structure of a C*-algebra we note that we have the following decompositions. If $A \in \mathfrak{A}$ is self adjoint and we define $A_{\pm} = \frac{(|A| \pm A)}{2}$ then

i) $A_{\pm} \in \mathfrak{A}_+$,

ii) $A = A_+ - A_-$,

iii) $A_+ A_- = 0$.

It also follows that $A_\pm$ are the unique elements with these properties. We also have the following " polar decomposition " for invertible elements. An invertible $A\in\mathfrak{A}$ can be uniquely written as $A=U|A|$, where $U\equiv A|A|^{-1}$ is unitary.

1.3 STATES AND REPRESENTATIONS: THE G.N.S. CONSTRUCTION.

We now wish to discuss the representation theory of C*-algebras. Since the states over a C*-algebra play a major role in representation theory we shall begin our discussion with them. The dual $\mathfrak{A}^*$ of a C*-algebra is defined to be the space of continuous linear functionals over $\mathfrak{A}$. We can define the norm of an element $f\in\mathfrak{A}^*$ as

$$\|f\|=\mathrm{Sup}\left\{\frac{|f(A)|}{\|A\|}: A\in\mathfrak{A};A\neq 0\right\}.$$

An important subset of $\mathfrak{A}^*$ is the set of states.

Definition 1.17 A linear functional f over a C*-algebra $\mathfrak{A}$ is said to be positive if $f(A^*A)\geq 0$ for all $A\in\mathfrak{A}$, i.e. f takes on positive values for positive elements of $\mathfrak{A}$ (recall that $A^*A\geq 0$). A state ω over $\mathfrak{A}$ is a positive linear functional over $\mathfrak{A}$ with unit norm, i.e. $\|\omega\|=1$.

It turns out that that a linear functional f is positive if and only if f is continuous and satisfies $\|f\|=f(\mathbb{1})$. So a linear functional ω is a state if and only if ω is continuous and $\|\omega\|=\omega(\mathbb{1})=1$.

Definition 1.18 A mapping between two $*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$, $\pi: A \in \mathfrak{A} \to \pi(A) \in \mathfrak{B}$, that is defined for all $A \in \mathfrak{A}$ and preserves the algebraic structure of $\mathfrak{A}$, i.e.

i) $\pi(\alpha A+\beta B)=\alpha\pi(A)+\beta\pi(B)$,

ii) $\pi(AB)=\pi(A)\,\pi(B)$,

iii) $\pi(A^*)=\pi(A)^*$,

for all $A,B \in \mathfrak{A}$ and $\alpha,\beta \in \mathbb{C}$, is a $*$-morphism between $\mathfrak{A}$ and $\mathfrak{B}$. A $*$-morphism that is one to one is a $*$-isomorphism.

When $\mathfrak{A}$ and $\mathfrak{B}$ are C*-algebras we find that all $*$-morphisms between $\mathfrak{A}$ and $\mathfrak{B}$ are positivity preserving ($A \geq 0 \Rightarrow \pi(A) \geq 0$), continuous, and satisfy $\|\pi(A)\| \leq \|A\|$. This last result implies the the set $\pi(\mathfrak{A})$ is itself a C*-algebra, for it is obviously a $*$-algebra and the condition $\|\pi(A)\| \leq \|A\|$ implies that it is complete with respect to the uniform topology defined by the C*-norm of $\mathfrak{B}$. The kernel of a $*$-morphism is defined to be $\mathrm{Ker}\,\pi \equiv \{A \in \mathfrak{A} : \pi(A)=0\}$. A $*$-morphism π is one to one and onto (i.e. it is a $*$-isomorphism) if and only if $\mathrm{Ker}\pi=\{0\}$. We now define a representation of a C*-algebra.

Definition 1.19 Let $\mathfrak{A}$ be a C*-algebra. A representation of $\mathfrak{A}$ is a pair $\{\mathfrak{H},\pi\}$, where $\mathfrak{H}$ is a Hilbert space and π is a $*$-morphism from $\mathfrak{A}$ into $\mathcal{L}(\mathfrak{H})$ (the set of all bounded linear operators on $\mathfrak{H}$). A representation $\{\mathfrak{H},\pi\}$ is faithful if $\mathrm{Ker}\pi=\{0\}$, i.e., if π is a $*$-isomorphism from $\mathfrak{A}$ into $\mathcal{L}(\mathfrak{H})$.

A set of operators $\pi(\mathfrak{A})$ on a Hilbert space $\mathfrak{H}$ representing a C*-algebra $\mathfrak{A}$ is itself a C*-algebra, which we refer to as a concrete C*-algebra. We should mention at this point that a representation is faithful if and only if it is norm preserving, i.e., $\|A\|=\|\pi(A)\|$ for all $A \in \mathfrak{A}$. Following the well known procedure from group theory we can always construct a faithful representation from a non-faithful representation. If $\pi:\mathfrak{A}\rightarrow\mathcal{L}(\mathfrak{H})$ is not a faithful representation, we can construct a faithful representation $\hat{\pi}$ of the quotient algebra $\mathfrak{A}_\pi\equiv\mathfrak{A}/\mathrm{Ker}\pi$.

Two important classes of representations are the irreducible ones and the cyclic ones.

Definition 1.20 A set of operators $\mathfrak{M}$ acting on a Hilbert space $\mathfrak{H}$ is said to be irreducible whenever the only closed subspaces of $\mathfrak{H}$ that are invariant under the action of $\mathfrak{M}$ are the trivial ones $\{0\}$ and $\mathfrak{H}$. A representation $\{\mathfrak{H},\pi\}$ of a C*-algebra $\mathfrak{A}$ is then said to be irreducible whenever the set of operators $\pi(\mathfrak{A})$ is irreducible.

Definition 1.21 A vector Ω in a Hilbert space $\mathfrak{H}$ is said to be cyclic for a set $\mathfrak{M}$ of bounded linear operators on $\mathfrak{H}$ whenever the set $\{A\Omega:A\in\mathfrak{M}\}$ is dense in $\mathfrak{H}$. A cyclic representation of a C*-algebra $\mathfrak{A}$ is then defined to be a triple $\{\mathfrak{H},\pi,\Omega\}$, where $\{\mathfrak{H},\pi\}$ is a representation of $\mathfrak{A}$ and Ω is a cyclic vector for the set of operators $\pi(\mathfrak{A})$ on $\mathfrak{H}$.

The irreducible and cyclic representations are connected as follows ([Emch] page 84).

Lemma 1.22 A nonzero representation $\{\mathfrak{H},\pi\}$ of a C*-algebra $\mathfrak{A}$ is irreducible if and only if every nonzero vector $\Psi\in\mathfrak{H}$ is cyclic for $\pi(\mathfrak{A})$.

Definition 1.23 Let $\mathfrak{M}$ be a set of bounded linear operators acting on a Hilbert space $\mathfrak{H}$. The commutant $\mathfrak{M}'$ of $\mathfrak{M}$ is defined to be

$\mathfrak{M}'\equiv\{A\in\mathfrak{L}(\mathfrak{H})$: AM=MA for all $M\in\mathfrak{M}\}$. The bicommutant $\mathfrak{M}''$ of $\mathfrak{M}$ is defined to be the commutant of $\mathfrak{M}'$.

Lemma 1.24 (Schur's Lemma) A representation $\{\mathfrak{H},\pi\}$ of a C*-algebra $\mathfrak{A}$ is irreducible if and only if $\pi(\mathfrak{A})'=\{\lambda\mathbb{1}:\lambda\in\mathbb{C}\}$.

Suppose that we have a representation $\{\mathfrak{H},\pi\}$ of a C*-algebra $\mathfrak{A}$. For any $\Omega\in\mathfrak{H}$ with $\|\Omega\|=1$ we can use this representation to define a state ω_Ω over $\mathfrak{A}$, $\omega_\Omega(A)\equiv(\Omega,\pi(A)\Omega)$, which we refer to as a vector state. A fundamental result in representation theory is that the converse to this is also true: every state over a C*-algebra is a vector state in some representation ([Brat1] Theorem 2.3.16).

Theorem 1.25 To every state ω over a C*-algebra $\mathfrak{A}$ corresponds a cyclic representation, $\{\mathfrak{H}_\omega,\pi_\omega,\Omega_\omega\}$, such that $\omega(A)=(\Omega_\omega,\pi_\omega(A)\Omega_\omega)$.

Proof: The theorem is proved by constructing a representation with the desired properties. This construction is known as the G.N.S. construction, named for Gelfand, Naimark, and Segal (the origins of the G.N.S. construction may be found in [Gelf] and [Sega]). It is based on the observation that the elements of a C*-algebra $\mathfrak{A}$ may be viewed in two ways, first as vectors in a complex vector space and second as linear transformations on this vector space. Consider the set E consisting of elements of $\mathfrak{A}$, $E=\{\Psi_A=A:A\in\mathfrak{A}\}$, equipped with the additive structure of $\mathfrak{A}$ through the definitions $\Psi_A+\Psi_B\equiv\Psi_{A+B}$ and $\alpha\Psi_A\equiv\Psi_{\alpha A}$. The set E is a complex vector space. We can attempt to define a representation of $\mathfrak{A}$ on this vector space by using the algebraic structure of $\mathfrak{A}$ to define a

*-homomorphism π from $\mathfrak{A}$ into the set of linear transformations on E. For all $A \in \mathfrak{A}$ we define $\pi(A)\Psi_B \equiv \Psi_{AB}$ for all $\Psi_A \in E$. Note that

$$\pi(A)[\beta\Psi_B+\gamma\Psi_C]=\pi(A)\Psi_{\beta B+\gamma C}=\Psi_{\beta AB+\gamma AC}=\beta\pi(A)\Psi_B+\gamma\pi(A)\Psi_C,$$

so that the $\pi(A)$ are linear transformations on E. It is possible to show at this point that π is a homomorphism, i.e., it preserves the algebraic structure. We have, for arbitrary A, B, and $C \in \mathfrak{A}$ and $\alpha,\beta \in \mathbb{C}$,

i) $\pi(\alpha A+\beta B)\Psi_C=\Psi_{\alpha AC+\beta BC}=\alpha\Psi_{AC}+\beta\Psi_{AB}=[\alpha\pi(A)+\beta\pi(B)]\Psi_C$,
so $\pi(\alpha A+\beta B)=\alpha\pi(A)+\beta\pi(B)$,

ii) $\pi(AB)\Psi_C=\Psi_{ABC}=\pi(A)\Psi_{BC}=\pi(A)\pi(B)\Psi_C$,
so $\pi(AB)=\pi(A)\pi(B)$.

For π to be a *-homomorphism we also require $\pi(A^*)=\pi(A)^*$. We must first define the action of $\pi(A)^*$ on E. A natural candidate for $\pi(A)^*$ would be the adjoint of $\pi(A)$, but to define the adjoint we require E to possess a scalar product. If we could construct a scalar product we would have a representation since we could use this scalar product to define a norm on E, making E a pre-Hilbert space, and then complete E in the uniform topology arising from this norm, making E a Hilbert space. It is in the definition of a scalar product where we make a connection with the states over $\mathfrak{A}$. Let ω be a state over $\mathfrak{A}$ and define $(\Psi_A ,\Psi_B)\equiv\omega(A^*B)$ for all A, $B \in \mathfrak{A}$. We must show that (Ψ_A ,Ψ_B) is a scalar product for E. Explicitly, (Ψ_A ,Ψ_B) must satisfy

i) $(\alpha\Psi_A +\beta\Psi_B, \gamma\Psi_C)=\overline{\alpha}\gamma(\Psi_A ,\Psi_C)+\overline{\beta}\gamma(\Psi_B ,\Psi_C)$,

ii) $(\Psi_A, \Psi_B) = \overline{(\Psi_B, \Psi_A)}$ (the bar denotes complex conjugation),

iii) $(\Psi_A, \Psi_A) \geq 0$.

and iv) $(\Psi_A, \Psi_A) = 0$ if and only if $\Psi_A = 0$,

for all A, B, C$\in \mathfrak{A}$ and $\alpha, \beta \in \mathbb{C}$. Using the fact that ω is a state it is easy to see that the first three conditions conditions are satisfied.

$$
\begin{aligned}
\text{i)}\ (\alpha\Psi_A + \beta\Psi_B, \gamma\Psi_C) &= \omega([\alpha A + \beta B]^*\gamma C) \\
&= \omega(\overline{\alpha}\gamma A^*C + \overline{\beta}\gamma B^*C) \\
&= \overline{\alpha}\gamma\omega(A^*C) + \overline{\beta}\gamma\omega(B^*C) \\
&= \overline{\alpha}\gamma(\Psi_A, \Psi_C) + \overline{\beta}\gamma(\Psi_B, \Psi_C),
\end{aligned}
$$

ii) $(\Psi_A, \Psi_B) = \omega(A^*B) = \overline{\omega(B^*A)} = \overline{(\Psi_B, \Psi_A)}$,

iii) $(\Psi_A, \Psi_A) = \omega(A^*A) \geq 0$, since ω is positive.

It may happen, however, that $\omega(A^*A)=0$ for some $A \neq 0$. This means that condition iv) might fail. To salvage the construction we must redefine our vector space E in such a way that $(\Psi_A, \Psi_A)=0$ if and only if $\Psi_A = 0$. Fortunately this can be done in a straight forward way.

We construct the representation space as follows. Given a state ω over $\mathfrak{A}$, consider the set $\mathfrak{I}_\omega \equiv \{A \in \mathfrak{A} : \omega(A^*A)=0\}$. $\mathfrak{I}_\omega$ is a left ideal of $\mathfrak{A}$, i.e., $\mathfrak{I}_\omega$ is a subset of $\mathfrak{A}$ such that $I \in \mathfrak{I}_\omega$ and $A \in \mathfrak{A}$ imply that $AI \in \mathfrak{I}_\omega$. The fact that that the ideal is left is not important as we could go through the construction using the right ideal $\mathfrak{I}'_\omega \equiv \{A \in \mathfrak{A} : \omega(AA^*)=0\}$. What is important is that $\mathfrak{I}_\omega$ is an ideal, so that the quotient algebra $\mathfrak{A}/\mathfrak{I}_\omega$, with equivalence classes $\Psi_A \equiv \{\hat{A} : \hat{A}=A+I,\ I \in \mathfrak{I}_\omega\}$, is well defined. These equivalence classes are now used, rather than the elements of $\mathfrak{A}$, to define our vector space. We define $E_\omega = \{\Psi_A : A \in \mathfrak{A}\}$ and equip E_ω with the structure $\Psi_A + \Psi_B \equiv \Psi_{A+B}$ and $\alpha\Psi_A \equiv \Psi_{\alpha A}$. At this point E_ω is a complex vector space. We can define a scalar product on the vector space E_ω as $(\Psi_A, \Psi_B) \equiv \omega(A^*B)$, and in turn define a norm as $\|\Psi_A\| \equiv \omega(A^*A)^{1/2}$. Since we are dealing with the quotient algebra $\mathfrak{A}/\mathfrak{I}_\omega$ we have that $\|\Psi_A\|=0$ if and only if $A \in \mathfrak{I}_\omega$, i.e. if and only if $\Psi_A=0$. This, of course, is why we deal with the quotient algebra $\mathfrak{A}/\mathfrak{I}_\omega$ and not the original C* -algebra $\mathfrak{A}$. Now, with respect to this norm, E_ω is a pre-Hilbert space which we denote by H_ω, and completion of H_ω then gives the representation space $\mathfrak{H}_\omega$. Note that we should verify that our scalar product is independent of the particular representatives used in its definition. We have, for all $I_1, I_2 \in \mathfrak{I}_\omega$ and $A, B \in \mathfrak{A}$,

$$\omega((A+I_1)^*(B+I_2)) = \omega(A^*B) + \overline{\omega(B^*I_1)} + \omega(A^*I_2) + \omega(I_1^*I_2)$$

$$= \omega(A^*B),$$

where the last three terms vanish because $\mathfrak{I}_\omega$ is a left ideal.

Now consider the second role of the elements of $\mathfrak{A}$, namely that of linear operators over $\mathfrak{H}_\omega$. We fist define the action of the $\pi_\omega(A)$ on H_ω. For $\Psi_B \in H_\omega$ and any $A \in \mathfrak{A}$ we define $\pi_\omega(A)\Psi_B \equiv \Psi_{AB}$. As before it is possible to show that the $\pi_\omega(A)$ are linear transformations on H_ω that π_ω is a morphism from A into the set of linear tansformations on H_ω. Since the $\pi_\omega(A)$ are bounded on H_ω we can easily extend them to all of $\mathfrak{H}_\omega$. If we define $\pi_\omega(A^*) \equiv \pi_\omega(A)^\dagger$ (= the adjoint of $\pi_\omega(A)$), then π_ω is a *-morphism from $\mathfrak{A}$ into the set of bounded linear operators on the Hilbert space $\mathfrak{H}_\omega$. The pair $\{\mathfrak{H}_\omega, \pi_\omega\}$ is therefore a representation of $\mathfrak{A}$. Finally we define the vector $\Omega_\omega \equiv \Psi_{\mathbb{1}}$. Since $\Psi_A = \pi_\omega(A)\Psi_{\mathbb{1}}$, Ω_ω is a cyclic vector and the representation $\{\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega\}$ is therefore cyclic. Finally we note that

$$(\Omega_\omega, \pi_\omega(A)\Omega_\omega) = (\Psi_{\mathbb{1}}, \pi_\omega(A)\Psi_{\mathbb{1}}) = (\Psi_{\mathbb{1}}, \Psi_A) = \omega(\mathbb{1}^* A) = \omega(A),$$

so the cyclic representation $\{\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega\}$ has the desired property. This completes the proof.

We now discuss the notion of unitary equivalence for representations of a C*-algebra.

Definition 1.26 Two representations of a C*-algebra, $\{\mathfrak{H}, \pi\}$ and $\{\mathfrak{H}', \pi'\}$, are unitarily equivalent if there exists a unitary transformation U of $\mathfrak{H}$ onto $\mathfrak{H}'$ such that $\pi'(A)U = U\pi(A)$ for all $A \in \mathfrak{A}$. Two cyclic representations $\{\mathfrak{H}, \pi, \Omega\}$ and $\{\mathfrak{H}', \pi', \Omega'\}$ are unitarily equivalent if $\{\mathfrak{H}, \pi\}$ and $\{\mathfrak{H}', \pi'\}$ are unitarily equivalent and using the same unitary transformation, $\Omega' = U\Omega$.

Unitary equivalence of two cyclic representations $\{\mathfrak{H}, \pi, \Omega\}$ and $\{\mathfrak{H}', \pi', \Omega'\}$ implies unitary equivalence of the representations $\{\mathfrak{H}, \pi\}$ and

$\{\mathfrak{H}',\pi'\}$, but not the converse. In connection with Theorem 1.25 we have the following ([Emch] page 81).

Theorem 1.27 Let ω be a state over a C*-algebra $\mathfrak{A}$ and $\{\mathfrak{H}_\omega,\pi_\omega,\Omega_\omega\}$ the corresponding G.N.S. representation, then any cyclic representation $\{\mathfrak{H},\pi,\Omega\}$ of $\mathfrak{A}$ with the property $(\Omega,\pi(A)\Omega)=\omega(A)$ for all $A\in\mathfrak{A}$ is unitarily equivalent to the cyclic representation $\{\mathfrak{H}_\omega,\pi_\omega,\Omega_\omega\}$. In particular, the representations $\{\mathfrak{H}_\omega,\pi_\omega\}$ and $\{\mathfrak{H},\pi\}$ are unitarily equivalent.

Proof: Define a transformation U from the pre-Hilbert space H_ω into $\mathfrak{H}$ as

$$U\pi_\omega(A)\Omega_\omega\equiv \pi(A)\Omega,$$

for all $A\in\mathfrak{A}$. We have

$$(U\pi_\omega(A)\Omega_\omega,U\pi_\omega(B)\Omega_\omega)=(\pi(A)\Omega,\ \pi(B)\Omega)=(\Omega,\ \pi(A^*B)\Omega)$$

$$=\omega(A^*B)=(\pi_\omega(A)\Omega_\omega,\pi_\omega(B)\Omega_\omega),$$

so that U preserves scalar products. U is also bounded on H_ω so that we may extend by continuity the definition of U to all of $\mathfrak{H}_\omega$. Since Ω and Ω_ω are cyclic we obtain in this manner a unitary transformation from $\mathfrak{H}_\omega$ into $\mathfrak{H}$. This unitary transformation is such that

i) $U\Omega_\omega=\Omega$, by definition,

ii) $\omega(A)=(\Omega,\pi(A)\Omega)=(U\Omega_\omega,\pi(A)U\Omega_\omega)=(\Omega_\omega,U^{-1}\pi(A)U\Omega_\omega)$, and $\omega(A)=(\Omega_\omega,\pi_\omega(A)\Omega_\omega)$, together with the cyclicity of Ω_ω, imply that $U^{-1}\pi(A)U=\pi_\omega(A)$ for all $A\in\mathfrak{A}$, i.e., $\pi(A)U=U\pi_\omega(A)$.

So the cyclic representations $\{\mathfrak{H},\pi,\Omega\}$ and $\{\mathfrak{H}_\omega,\pi_\omega,\Omega_\omega\}$ are unitarily equivalent. This proves Theorem 1.27 by construction.

Quite often it is too complicated to identify the G.N.S. representation corresponding to a state ω with a known Hilbert space and set of bounded operators. In practice we often invent a cyclic representation in which the state ω is the vector state corresponding to the cyclic vector, and use Theorem 1.27 to conclude that this representation is unitarily equivalent to the G.N.S. representation.

An important class of states are the pure ones.

Definition 1.28 A state ω over a C*-algebra $\mathfrak{A}$ is said to be pure if it is not possible to decompose ω into $\omega=\lambda\omega_1+(1-\lambda)\omega_2$, where $\omega_1 \neq \omega_2$ are states over $\mathfrak{A}$ and $0<\lambda<1$. A state that is not pure is mixed.

The following theorem establishes a connection between the pure states over a C*-algebra $\mathfrak{A}$ and the irreducible G.N.S. representations of $\mathfrak{A}$ ([Emch] page 87).

Theorem 1.29 Consider a state ω over a C*-algebra $\mathfrak{A}$ and the corresponding G.N.S. representation $\{\mathfrak{H}_\omega,\pi_\omega,\Omega_\omega\}$. Then $\pi_\omega(\mathfrak{A})$ is irreducible if and only if ω is pure.

Corollary 1.30 Consider an irreducible representation $\{\mathfrak{H},\pi\}$ of a C*-algebra $\mathfrak{A}$. Every vector state in this representation is pure.

Proof: Let $\omega(A)=(\Omega,\pi(A)\Omega)$ be a vector state in an irreducible representation $\{\mathfrak{H},\pi\}$. Since the representation is irreducible every vector in $\mathfrak{H}$ is cyclic. The representation $\{\mathfrak{H},\pi,\Omega\}$ is therefore cyclic and hence unitarily equivalent to the G.N.S. representation $\{\mathfrak{H}_\omega,\pi_\omega,\Omega_\omega\}$, by Theorem 1.27. The G.N.S. representation is then irreducible because the representation $\{\mathfrak{H},\pi\}$ is irreducible. The state ω is therefore pure by Theorem 1.29.

1.4 Von NEUMANN ALGEBRAS AND TYPES OF EQUIVALENCE

The physical states of a system correspond to linear functionals over the appropriate C*-algebra, which in turn correspond (via the G.N.S. construction) to representations of the C*-algebra as bounded operators acting on a Hilbert space. We now review those aspects of the theory of operator algebras that appear to be most relevant to Quantum Mechanics (this material is covered in [Emch]). In addition we will discuss two types of equivalence between representations of a C*-algebra, quasi and physical equivalence.

Consider the set $\mathcal{L}(\mathfrak{H})$ of bounded linear operators on a Hilbert space $\mathfrak{H}$. A variety of topologies can be defined on $\mathcal{L}(\mathfrak{H})$, but we will confine our attention to the uniform, strong, and weak topologies. We will characterize a topology on $\mathcal{L}(\mathfrak{H})$ by specifying when a sequence of operators converges in that topology.

Definition 1.31 A sequence of operators $\{A_n\} \in \mathcal{L}(h)$ converges to an operator $A \in \mathcal{L}(\mathfrak{H})$

i) uniformly if $\lim_{n\to\infty} \|A-A_n\|=0$, where $\|A\|$ is the operator norm on $\mathcal{L}(\mathfrak{H})$,

ii) strongly if $\lim_{n\to\infty} \|(A-A_n)\Psi\|=0$, for all $\Psi \in \mathfrak{H}$,

iii)weakly if $\lim_{n\to\infty} (A_n\Psi,\phi)=(A\psi,\phi)$ for all $\Psi,\phi \in \mathfrak{H}$.

It is easy to demonstrate that uniform convergence implies strong convergence, which in turn implies weak convergence. When a set of operators contains its uniform (strong, weak) limits we say that the set is closed in the uniform (strong, weak) topology. If a set of operators is not closed in a particular topology we can close it by adding to it its limit points in that topology. The set $\mathcal{L}(\mathfrak{H})$ is closed in the uniform topology which is the topology generated by the operator norm on $\mathcal{L}(\mathfrak{H})$. Since this norm is a C*-norm the set $\mathcal{L}(\mathfrak{H})$ is a C*-algebra.

We now consider some special subalgebras of $\mathcal{L}(\mathfrak{H})$, the von Neumann algebras. A set of bounded linear operators $\mathfrak{M}$ on $\mathfrak{H}$ that is an algebra under the usual operations of $\mathcal{L}(\mathfrak{H})$ is a subalgebra of $\mathcal{L}(\mathfrak{H})$. If $\mathfrak{M}$ is closed under the involution of $\mathcal{L}(\mathfrak{H})$ (i.e., the adjoint operation), then it is a *-subalgebra. Recalling the definitions of the commutant $\mathfrak{M}'$

and bi-commutant $\mathfrak{M}''$ of $\mathfrak{M}$ (Definition 1.23) we now define a special class of *-subalgebras of $\mathcal{L}(\mathfrak{H})$, the von Neumann algebras.

Definition 1.32 A *-subalgebra $\mathfrak{M}$ of $\mathcal{L}(\mathfrak{H})$ that has the property $\mathfrak{M}''=\mathfrak{M}$ is a von Neumann algebra.

Since $\mathfrak{M}'=\mathfrak{M}'''$ and $\mathfrak{M}''=\mathfrak{M}''''$, the commutant and bi-commutant of an arbitrary *-algebra $\mathfrak{M}$ are von Neumann algebras.

The set $\mathcal{L}(\mathfrak{H})$ is an example of a von Neumann algebra. As was mentioned above it is also a C*-algebra. A natural question to ask is whether or not all von Neumann algebras are C*-algebras. The answer is contained in the following theorem (see [Emch] page 116 or [Brat1] page 72).

Theorem 1.33 For a *-subalgebra $\mathfrak{M}$ of $\mathcal{L}(\mathfrak{H})$ that contains the identity the following conditions are equivalent.

i) $\mathfrak{M}''=\mathfrak{M}$, i.e., $\mathfrak{M}$ is a von Neumann algebra,

ii) $\mathfrak{M}$ is weakly closed,

iii) $\mathfrak{M}$ is strongly closed.

Furthermore any of the above conditions imply that $\mathfrak{M}$ is uniformly closed.

Every von Neumann algebra is closed in the uniform topology and is therefore a concrete C*-algebra (i.e. a C*-algebra of operators on a Hilbert space). The converse is not true, a concrete C*-algebra is not necessarily a von Neumann algebra. This means that the set of operators $\pi(\mathfrak{A})$ in a representation $\{\mathfrak{H},\pi\}$ of a C*-algebra $\mathfrak{A}$ is not a von Neumann algebra in general. We can, however, always generate a von Neumann algebra from $\pi(\mathfrak{A})$ by forming the bi-commutant $\pi(\mathfrak{A})''$, or equivalently by completing $\pi(\mathfrak{A})$ in the weak or strong topologies.

We have mentioned that in the Traditional Approach to Quantum Mechanics one is always dealing with the set of all self adjoint operators on a Hilbert space, and that the states are represented by density operators. A state ω over a von Neumann algebra $\mathfrak{M}$ is said to be normal if there exists a density operator $\rho\in\mathfrak{M}$ (i.e., a self adjoint, positive, bounded operator with finite trace) such that $\omega(A)=\frac{\mathrm{Tr}\rho A}{\mathrm{Tr}\rho}$ for all $A\in\mathfrak{M}$. Given a representation $\{\mathfrak{H},\pi\}$ of a C*-algebra $\mathfrak{A}$, a state ω over $\mathfrak{A}$ is said to be π-normal if there exists a density operator ρ in the von Neumann algebra $\pi(\mathfrak{A})''$ such that $\omega(A)=\frac{\mathrm{Tr}\rho\pi(A)}{\mathrm{Tr}\rho}$ for all $A\in\mathfrak{A}$. We then define two representations $\{\mathfrak{H}_1,\pi_1\}$ and $\{\mathfrak{H}_2,\pi_2\}$ of a C*-algebra $\mathfrak{A}$ to be quasi-equivalent if every π_1-normal state of $\mathfrak{A}$ is a π_2-normal state. We denote this equivalence by $\pi_1\approx\pi_2$. The following is Theorem 2.4.26 in [Brat1].

Theorem 1.34 Two representations are quasi-equivalent if and only if there exists a *-isomorphism α from $\pi_1(\mathfrak{A})''$ to $\pi_2(\mathfrak{A})''$ such that $\pi_2(A)=\alpha\pi_1(A)$ for all $A\in\mathfrak{A}$.

Some authors choose this latter property as the definition of quasi-equivalence.

We now discuss another type of equivalence between representations, that of physical equivalence ([Emch] page 97). Given a representation $\{\mathfrak{H},\pi\}$ of a C*-algebra $\mathfrak{A}$, we know that the set of operators $\pi(\mathfrak{A})$ form a C*-algebra. Denote by S the set of all states over $\mathfrak{A}$ and by S_π the set of all states over $\pi(\mathfrak{A})$. For an arbitrary linear functional f_π on $\pi(\mathfrak{A})$ we can define a linear functional f on $\mathfrak{A}$ by $f(A)\equiv f_\pi(\pi(A))$ for all $A\in\mathfrak{A}$. Since a representation is positivity preserving, f is positive whenever f_π is. We also note that f vanishes on Ker π. Going the other way we see that any linear functional f over $\mathfrak{A}$ that vanishes on Ker π gives rise to a linear functional f_π over $\pi(\mathfrak{A})$ by the definition $f_\pi(\pi(A))\equiv f(A)$ for all $A\in\mathfrak{A}$. Using the above definitions we may then consider S_π to be a subset of S. Two representations $\{\mathfrak{H}_1,\pi_1\}$ and $\{\mathfrak{H}_2,\pi_2\}$ of $\mathfrak{A}$ are then said to be physically equivalent if their sets of states S_{π_1} and S_{π_2} are identical when considered as subsets of S. This is the case if and only if Ker π_1=Ker π_2. In particular we note that all faithful representations are physically equivalent.

We now have three types of equivalence between representations of C*-algebras; unitary, quasi, and physical. Two representations $\{\mathfrak{H}_1,\pi_1\}$ and $\{\mathfrak{H}_2,\pi_2\}$ that are unitarily equivalent have the same kernel and are therefore physically equivalent. The unitary equivalence between $\pi_1(\mathfrak{A})$ and $\pi_2(\mathfrak{A})$ can be extended by continuity to a unitary equivalence between $\pi_1(\mathfrak{A})''$ and $\pi_2(\mathfrak{A})''$. So each density operator in $\pi_1(A)''$ is unitarily equivalent to a density operator in $\pi_2(A)''$. The two representations are therefore quasi-equivalent. Now assume that the two representations $\{\mathfrak{H}_1,\pi_1\}$ and $\{\mathfrak{H}_2,\pi_2\}$ are quasi-equivalent. The von Neumann algebras $\pi_1(\mathfrak{A})''$ and $\pi_2(\mathfrak{A})''$ are then *-isomorphic, where the

*-isomorphism α is such that $\alpha\pi_2(A)=\pi_1(A)$ for all $A\in\mathfrak{A}$. This implies that the two representations have the same kernel and are therefore physically equivalent. We now have that unitary equivalence implies quasi-equivalence, which in turn implies physical equivalence. It is also possible to show that unitary and quasi-equivalence coincide for irreducible representations.

1.5 QUASI-LOCAL ALGEBRAS

As we have mentioned, the Algebraic Approach to Quantum Mechanics is based on the postulate that it is possible to construct a C*-algebra for a given physical system in such a manner that the bounded observables of the system are represented by the self adjoint elements of the C*-algebra and the states of the system are represented by linear functionals over the C*-algebra. We now wish to extend this formalism to infinite systems. This is made possible by exploiting the fact that all physical measurements performed on a system are limited in space and time. This means that we usually understand the local structure of the system, and the structure of the infinite system is built up from this knowledge. The C*-algebras that we associate with observables that can be measured in a finite space-time region are referred to as the local algebras and the C*-algebra that we build from them is said to be a quasi-local algebra.

In what follows we will assume that the configuration space of the system in question is either $\mathbb{R}^3$ or $\mathfrak{M}^4$ (the four dimensional Minkowski space). The following may be found in [Emch], page 253. To each bounded region Z of the configuration space we assume that we can associate a C*-algebra $\mathfrak{A}_Z$ in such a manner that the self adjoint elements of $\mathfrak{A}_Z$ correspond to the observables of the system that can be measured within the region Z. We order the regions Z by inclusion. This ordering is a partial ordering (for all pairs Z_1 and Z_2 there exists a Z_3 such that $Z_1 \leq Z_3$ and $Z_2 \leq Z_3$) and hence the set Σ of all bounded regions Σ is a directed set. What allows us to construct a C*-algebra that corresponds to the infinite system is the postulate of isotony. We assume that for any pair of regions Z_1 and Z_2 we can construct a *-homomorphism $i_{2,1}$ that takes all of $\mathfrak{A}_{Z_1}$ into $\mathfrak{A}_{Z_2}$ (i.e. $i_{2,1}$ is an injection) that satisfies

i) $i_{2,1}(\mathbb{1}_1)=\mathbb{1}_2$, where $\mathbb{1}_1$ and $\mathbb{1}_2$ are the identities of $\mathfrak{A}_{Z_1}$ and $\mathfrak{A}_{Z_2}$, respectively,

ii) $i_{3,2}\, i_{2,1}=i_{3,2}$ whenever $Z_1 \leq Z_2 \leq Z_3$.

The postulate of isotony is a sufficient condition for a family $\{\mathfrak{A}_Z : Z \in \Sigma\}$ of C*-algebras (with Σ a directed set) to admit a C*-inductive limit. This is a C*-algebra $\mathfrak{A}$ with identity $\mathbb{1}$ that has the property that for every $Z \in \Sigma$ there exists an injective *-homomorphism i_Z from $\mathfrak{A}_Z$ into $\mathfrak{A}$ that satisfies

i) $i_Z(\mathbb{1}_Z)=\mathbb{1}$, where $\mathbb{1}_Z$ is the identity of $\mathfrak{A}_Z$,

ii) $i_{Z_2}(\mathfrak{A}_{Z_2})\supset i_{Z_1}(\mathfrak{A}_{Z_1})$, whenever $Z_2\geq Z_1$,

and iii) $\overline{\bigcup_{Z\in\Sigma} i_Z(\mathfrak{A}_Z)}=\mathfrak{A}$, where the bar denotes the uniform closure.

The C*-algebra $\mathfrak{A}$ is referred to as the quasi-local algebra for the infinite system. As the name would suggest, every element of $\mathfrak{A}$ can be approximated to any degree by elements of the local algebras $\mathfrak{A}_Z$ (this is the content of condition (iii) above).

States of the infinite system correspond to states over the quasi-local algebra for the system. Of particular importance is the Gibbs equilibrium state, and we now discuss how we may attempt to construct this state as a state over the quasi-local algebra. Consider the quasi-local algebra $\mathfrak{A}$ generated by the family of C*-algebras $\{\mathfrak{A}_Z : Z\in\Sigma\}$. Let Σ_o be a subset of Σ which consists of an increasing sequence $\{Z_n\}$. Assume that this sequence has the property that for every region Z in Σ there is an integer N(Z) such that $Z_n\supset Z$ for all $n\geq N(Z)$. Let H_n be the Hamiltonian for the region Z_n and ρ_n the corresponding canonical density matrix $\rho_n=\frac{e^{-\beta H_n}}{Tre^{-\beta H_n}}$. For every $A\in\mathfrak{A}_Z$ and $n>N(Z_n)$ we then define the state ω_n on Z as $\omega_n(A)\equiv TrA\rho_n$. If $\lim_{n\to\infty}\omega_n(A)$ exists and defines a state ω on $\mathfrak{A}$ we refer to it as the canonical (or Gibb's) equilibrium state at natural temperature β. The grand canonical state is defined in in a similar fashion with the Hamiltonian H_n replaced with $H-\mu N$, where N is the number of particles in the region Z_n.

1.6 THE CONNECTION BETWEEN THE TRADITIONAL AND ALGEBRAIC APPROACHES TO QUANTUM MECHANICS

In this section we will attempt to clarify two points. (a) In the Traditional Approach the representation we are working in is irreducible and pure states are vector states. In the Algebraic Approach all states are vector states and pure states correspond to irreducible representations (see Theorems 1.25, 1.29,1.30 Chapter 1). (b) In the Introduction to this thesis we made the statement that the algebra associated with a finite system usually admits only one irreducible representation, and when this is the case we might as well work in this unique irreducible representation, thus returning to the traditional Hilbert space formalism.

We first discuss (a). In the Hilbert space formalism one is always dealing with a concrete Hilbert space $\mathfrak{H}$. The algebra is assumed to be the set $\mathcal{L}(\mathfrak{H})$ of all bounded linear operators on this Hilbert space, and this set is always irreducible (roughly speaking the set is too large to have any nontrivial invariant subspaces). A state is a now positive linear functional over $\mathcal{L}(\mathfrak{H})$. It is possible to show that any such state ω is of the form $\omega(A)=\dfrac{\mathrm{Tr}\rho A}{\mathrm{Tr}\rho}$ (for all $A\in \mathcal{L}(\mathfrak{H})$), where ρ is a bounded, self adjoint , positive operator of finite trace. If we assume that ρ has a discrete spectrum , then according to the spectral theorem ρ can be written as

$\rho=\sum\lambda_\phi P_\phi$, and therefore $\omega(A)=\dfrac{\sum\lambda_\phi(\phi,A\phi)}{\sum\lambda_\phi(\phi,\phi)}$, where the vectors ϕ are the eigenvectors of ρ and the P_ϕ are projection operators onto the one-dimensional subspaces spanned by the ϕ. Pure states then correspond to the special case in which ρ is a projection operator onto a one dimensional subspace. This is the case when the above linear combination contains only one term. In this case $\omega(A)=\dfrac{(\phi,A\phi)}{(\phi,\phi)}$, so the pure states are vector states and one can prove that all vector states are pure. Corollary 1.30 makes it clear that the latter proof depends on the irreducibility of the set of operators $\mathcal{L}(\mathfrak{H})$. In the Algebraic Approach we gain the result that all states are vector states in some representation, with the price being that the representation is irreducible only when the state is pure. We can gain an understanding of what is going on here by considering the case when ρ is a projection operator onto a two dimensional subspace, and has unit trace. If this is the case then $\omega(A)=(\phi_1,A\phi_1)+(\phi_2,A\phi_2)$, with ϕ_1 and ϕ_2 the eigenvectors of ρ. In the Hilbert space $\mathfrak{H}$, ω is not a vector state. However, in $\mathfrak{H}\oplus\mathfrak{H}$ ω is a vector state with $\omega(A)=(\phi_1\oplus\phi_2, (A\oplus A)\phi_1\oplus\phi_2)$. Thus we can express ω as a vector state, but in doing so the algebra $\mathcal{L}(\mathfrak{H})\oplus\mathcal{L}(\mathfrak{H})$ becomes reducible.

We now discuss (b). Assume that the C*-algebra $\mathfrak{A}$ corresponding to some physical system admits only one irreducible representation, $\{\mathfrak{H},\pi\}$. This is usually the case when the system possesses a finite number of degrees of freedom. In Chapter 2 we will show that this is the case for the C*-algebra that corresponds to a finite spin system and in Chapter 3 we will show that this is true for the C*-algebra that corresponds to a finite Bose gas. This representation will be the irreducible representation in which one is working in the Hilbert space formalism. Let ω be a mixed state over $\mathfrak{A}$, and for simplicity assume that ω decomposes into two pure states $\omega=\lambda\omega_1+(1-\lambda)\omega_2$, ω_1 and ω_2 pure. Without loss of generality we may assume that the reducible G.N.S. representation $\{\mathfrak{H}_\omega,\pi_\omega\}$ corresponding to the ω can be decomposed into a direct sum of the unique irreducible representation $\{\mathfrak{H},\pi\}$ as

$$\pi_\omega(\mathfrak{A})=\begin{pmatrix}\pi(\mathfrak{A}) & 0\\ 0 & \pi(\mathfrak{A})\end{pmatrix}.$$

It easily follows that

$$\pi_\omega(\mathfrak{A})''=\begin{pmatrix}\pi(\mathfrak{A})'' & 0\\ 0 & \pi(\mathfrak{A})''\end{pmatrix},$$

so the von Neumann algebras $\pi_\omega(\mathfrak{A})''$ and $\pi(\mathfrak{A})''$ are *-isomorphic. The representations $\{\mathfrak{H}_\omega,\pi_\omega\}$ and $\{\mathfrak{H},\pi\}$ are therefore quasi-equivalent, by Theorem 1.34. This means that the set of states of $\mathfrak{A}$ that can be expressed in terms of the trace of a density operator in either of the two representations coincide. Since ω is a vector state in $\{\mathfrak{H}_\omega,\pi_\omega\}$, it can be expressed in terms of the trace of a density operator in $\{\mathfrak{H},\pi\}$. So we might as well work in the irreducible representation $\{\mathfrak{H},\pi\}$, returning to the Traditional Approach.

The C*-algebra that corresponds to an infinite system usually admits an infinite number of unitarily inequivalent irreducible representations. In Chapter 2 we shall show that this is true for the C*-algebra that corresponds to an infinite spin system and in Chapter 3 we shall show that this is true for the C*-algebra that corresponds to an infinite Bose gas. When this occurs we must work with the C*-algebra, since it provides an underlying link between all these different representations.

CHAPTER 2

QUANTUM SPIN SYSTEMS

In this chapter we will explicitly illustrate the main results of Chapter 1 by discussing the algebraic description of Quantum Spin Systems. The C*-algebra corresponding to a single spin will be constructed and analyzed. The G.N.S. representation corresponding to the canonical equilibrium state will be constructed, following the procedure given in the proof of Theorem 1.25. We will then generalize to a two spin system, a multi spin system, and finally to a system consisting of an infinite number of spins. We shall find, among other things, that the G.N.S. representations corresponding to different finite temperatures are unitarily equivalent, as long as the system remains finite, while for the infinite system these representations become unitarily inequivalent.

2.1 SINGLE SPIN SYSTEMS

Consider a system consisting of a single spin (s=1/2) with no other degrees of freedom. The C*-algebra corresponding to this system is generated by the four abstract elements σ_0, σ_1, σ_2, and σ_3 equipped with the composition laws

$$\sigma_0=(\sigma_0)^2=(\sigma_i)^2 \qquad i=1, 2, 3, \tag{2.1a}$$

$$\sigma_0=\sigma_i \qquad i=1, 2, 3, \tag{2.1b}$$

and $\sigma_1\sigma_2=i\sigma_3$, similarly for cyclic permutations of 1, 2, and 3. (2.1c)

Latin indices (i, j, k) will run from 1 to 3 while Greek indices (μ, ν, η) will run from 0 to 3. We begin by defining the algebra $\mathfrak{A}$ to be the set of all polynomials in σ_μ of finite degree with complex coefficients. Using (2.1) it is obvious that $\mathfrak{A}$ consists simply of the linear combinations of the four σ's,

$$\mathfrak{A}=\{A=\alpha^\mu\sigma_\mu\colon\ \alpha^\mu\in\mathbb{C}\}. \tag{2.2}$$

Note that σ_o is an identity for $\mathfrak{A}$ and the product of $A=\alpha^\mu\sigma_\mu$ and $B=\beta^\mu\sigma_\mu$ is

$$AB=\alpha^\mu\beta^\nu\sigma_\mu\sigma_\nu. \tag{2.3}$$

An involution of $\mathfrak{A}$ can be defined as

$$(\alpha^\mu\sigma_\mu)^*\equiv\overline{\alpha}^\mu\sigma_\mu. \tag{2.4}$$

To show that the mapping defined in (2.4) is indeed an involution of $\mathfrak{A}$ we first note that $A^*\in\mathfrak{A}$ if $A\in\mathfrak{A}$. Then

i) $(A^*)^*=A$; this is easily verified by inspection of (2.4),

ii) $(AB)^*=B^*A^*$; since the σ_μ are self adjoint we have $(AB)^*=(\alpha^\mu\beta^\nu\sigma_\mu\sigma_\nu)^*=\overline{\alpha}^\mu\overline{\beta}^\nu(\sigma_\mu\sigma_\nu)^*=\overline{\alpha}^\mu\overline{\beta}^\nu\,\sigma_\nu\sigma_\mu=B^*A^*$,

iii) $(\gamma A+\delta B)^*=\overline{\gamma}A^*+\overline{\delta}B^*$; this is easily verified by inspection of (2.4).

This shows that (2.4) does define an involution of $\mathfrak{A}$, and $\mathfrak{A}$ is therefore a *-algebra.

To define a C*-norm on $\mathfrak{A}$ we first determine the spectrum of an arbitrary $A\in\mathfrak{A}$ and then use the fact that $\|A\|=\rho(A^*A)^{1/2}$. As is evident from (2.1b) the element σ_o is an identity for $\mathfrak{A}$. It is easy to demonstrate, by direct multiplication, that the inverse of $A\in\mathfrak{A}$ is

$$A^{-1}=(\alpha^\mu\sigma_\mu)^{-1}=\frac{(\alpha^o\sigma_o-\alpha^i\sigma_i)}{(\alpha^o)^2-(\alpha^1)^2-(\alpha^2)^2-(\alpha^3)^2} \tag{2.5}$$

So A^{-1} exists if and only if $(\alpha^o)^2\neq(\alpha^1)^2+(\alpha^2)^2+(\alpha^3)^2$. Using this condition for invertibility it is easy to see that the spectrum of a general $A\in\mathfrak{A}$ is (no confusion should arise from using the symbol σ to denote the spectrum and σ_μ to denote the generators of $\mathfrak{A}$)

$$\sigma(A)=\sigma(\alpha^\mu\sigma_\mu)=\left\{\lambda=\alpha^o\pm\sqrt{(\alpha^1)^2+(\alpha^2)^2+(\alpha^3)^2}\right\}. \tag{2.6}$$

E.g., $\sigma(\sigma^o)=\{1,1\}$, $\sigma(\sigma^i)=\{1,-1\}$, $\sigma(\sigma_o+\sigma_3)=\{1\pm\sqrt{1}\}=\{0,2\}$.

Using the spectrum we then define the norm of an arbitrary $A\in\mathfrak{A}$ as

$$\|A\|=\|\alpha^\mu\sigma_\mu\|\equiv\rho(A^*A)^{1/2}=[\mathrm{Sup}\{|\lambda|:\lambda\in\sigma(A^*A)\}]^{1/2}.$$

$$=\left\{\alpha^\mu\bar{\alpha}^\mu+2\sqrt{\begin{matrix}[\mathrm{Re}(\alpha^o\bar{\alpha}^1-i\alpha^2\bar{\alpha}^3)]^2+[\mathrm{Re}(\alpha^o\bar{\alpha}^2-i\alpha^3\bar{\alpha}^1)]^2\\+[\mathrm{Re}(\alpha^o\bar{\alpha}^3-i\alpha^1\bar{\alpha}^2)]^2\end{matrix}}\right\}^{1\backslash 2} \tag{2.7}$$

E.g., $\|\sigma_\mu\|=1$, $\|\sigma_o+\sigma_3\|=(\mathrm{Sup}\{|\lambda|:\lambda=1\pm\sqrt{1}\})^{1/2}=\sqrt{2}$.

In demonstrating the connection between the norm and spectral radius of a C*-algebra ($\|A\|^2=\rho(A^*A)$) one assumes that a C*-norm exists. This means that we must demonstrate that (2.7) really does define a

C*-norm for $\mathfrak{A}$, for it may happen that our *-algebra $\mathfrak{A}$ does not admit a C*-norm (not every *-algebra will). To demonstrate that (2.7) satisfies the properties in Definition 1.4 is rather difficult (it is very messy to demonstrate that (2.7) satisfies the triangle and product inequalities) so we will instead demonstrate that the *-algebra $\mathfrak{A}$ possesses a C*-norm (with respect to which it is complete) and then use Theorem 1.6 to conclude that this C*-norm is unique and hence given by (2.7). To this end we define a norm on $\mathfrak{A}$ through the faithful representation of $\mathfrak{A}$ as the set of two by two matrices with complex entries acting on $\mathbb{C}^2$. The elements σ_i are represented by the familiar Pauli spin matrices while σ_0 is represented by the two by two unit matrix,

$$\pi(\sigma_0)=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \pi(\sigma_1)=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ \pi(\sigma_2)=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \pi(\sigma_3)=\begin{pmatrix}0&-i\\i&0\end{pmatrix}.$$

A norm on $\pi(\mathfrak{A})$ is now defined as

$$\|\pi(A)\|=\|\pi(\alpha^\mu\sigma_\mu)\|\equiv\mathrm{Sup}\{\|\pi(A)\underline{\lambda}\|:\underline{\lambda}\in\mathbb{C}^2,\ \|\underline{\lambda}\|=1\}$$

$$=\mathrm{Sup}\left\{\left\|\begin{pmatrix}\alpha^0+\alpha^3 & \alpha^1-i\alpha^2\\ \alpha^1-i\alpha^2 & \alpha^0-\alpha^3\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}\right\|:|\lambda_1|^2+|\lambda_2|^2=1\right\} \tag{2.8}$$

We now use the fact that the norm of a complex vector satisfies

(a) $\|\underline{\lambda}\|\geq 0$, $\|\underline{\lambda}\|=0$ if and only if $\underline{\lambda}=0$,

(b) $\|\alpha\underline{\lambda}\|=|\alpha|\,\|\underline{\lambda}\|$ for all $\alpha\in\mathbb{C}$,

(c) $\|\underline{\lambda}_1+\underline{\lambda}_2\|\leq\|\underline{\lambda}_1\|+\|\underline{\lambda}_2\|$,

and (d) (Cauchy inequality) $|(\underline{\lambda},\underline{\delta})|\leq\|\underline{\lambda}\|\,\|\underline{\delta}\|$, to show that (2.8) satisfies the conditions in Definition 1.4,

i) $\|\pi(A)\|\geq 0$, $\|\pi(A)\|=0$ if and only if $\pi(A)=0$; the first part is obvious. The second part follows from (a) and the fact that $\pi(A)\underline{\lambda}=0$ iff and only if $\pi(A)=0$,

ii) $\|\alpha\pi(A)\|=|\alpha|\,\|\pi(A)\|$ for all $\alpha\in\mathbb{C}$; this follows from (b),

iii) $\|\pi(A+B)\|\leq\|\pi(A)\|+\|\pi(B)\|$; using (c) and knowing that π is a representation we have

$$\begin{aligned}
\|\pi(A+B)\| &= \operatorname{Sup}\{\|\pi(A)\underline{\lambda}+\pi(B)\underline{\lambda}\| : \lambda\in\mathbb{C}^2,\ \|\underline{\lambda}\|=1\}\\
&\leq \operatorname{Sup}\{\|\pi(A)\underline{\lambda}\|+\|\pi(B)\underline{\lambda}\| : \lambda\in\mathbb{C}^2,\ \|\underline{\lambda}\|=1\}\\
&\leq \operatorname{Sup}\{\|\pi(A)\underline{\lambda}\| : \underline{\lambda}\in\mathbb{C}^2,\ \|\underline{\lambda}\|=1\}+\operatorname{Sup}\{\|\pi(B)\underline{\lambda}\| : \underline{\lambda}\in\mathbb{C}^2,\ \|\underline{\lambda}\|=1\}\\
&= \|\pi(A)\|+\|\pi(B)\|,
\end{aligned}$$

iv) $\|\pi(AB)\|\leq\|\pi(A)\|\,\|\pi(B)\|$; using (b) and the fact that π is a representation we have (we may, without loss of generality,assume that $B\neq 0$)

$$\begin{aligned}
\|\pi(AB)\| &= \operatorname{Sup}\{\|\pi(A)\pi(B)\underline{\lambda}\| : \lambda\in\mathbb{C}^2,\ \|\underline{\lambda}\|=1\}\\
&= \operatorname{Sup}\left\{\left\|\pi(A)\ \|\pi(B)\underline{\lambda}\|\frac{\pi(B)\underline{\lambda}}{\|\pi(B)\underline{\lambda}\|}\right\| : \underline{\lambda}\in\mathbb{C}^2;\ \|\underline{\lambda}\|=1\right\}\\
&= \operatorname{Sup}\{\|\pi(A)\underline{\delta}\|\ \|\pi(B)\underline{\lambda}\| : \underline{\lambda},\underline{\delta}\in\mathbb{C}^2, \|\underline{\lambda}\|=\|\underline{\delta}\|=1\}\\
&\leq \operatorname{Sup}\{\|\pi(A)\underline{\delta}\| : \underline{\delta}\in\mathbb{C}^2,\ \|\underline{\delta}\|=1\}\operatorname{Sup}\{\|\pi(B)\underline{\lambda}\| : \underline{\lambda}\in\mathbb{C}^2,\ \|\underline{\lambda}\|=1\}\\
&= \|\pi(A)\|\,\|\pi(B)\|,
\end{aligned}$$

$$\left(\text{note}: \underline{\delta}=\frac{\pi(B)\underline{\lambda}}{\|\pi(B)\underline{\lambda}\|}\right)$$

v') $\|\pi(A)\|^2=\|\pi(A^*A)\|$; denote the adjoint of $\pi(A)$ by $\pi(A)^\dagger=\pi(A^*)$. Using (d) we have

$$\begin{aligned}\|\pi(A)\|^2 &= \text{Sup}\{\|\pi(A)\underline{\lambda}\|^2 : \underline{\lambda}\in\mathbb{C}^2,\ \|\underline{\lambda}\|=1\}\\ &= \text{Sup}\{(\pi(A)\underline{\lambda}\ ,\pi(A)\underline{\lambda}) : \underline{\lambda}\in\mathbb{C}^2,\ \|\underline{\lambda}\|=1\}\\ &= \text{Sup}\{(\underline{\lambda}\ ,\pi(A)\dagger\pi(A)\underline{\lambda}) : \underline{\lambda}\in\mathbb{C}^2,\ \|\underline{\lambda}\|=1\}\\ &\le \text{Sup}\{\|\pi(A)\dagger\pi(A)\underline{\lambda}\| : \underline{\lambda}\in\mathbb{C}^2,\ \|\underline{\lambda}\|=1\}\\ &= \|\pi(A^*A)\|\\ &\le \|\pi(A^*)\|\ \|\pi(A)\|,\end{aligned}$$

so
$$\|\pi(A)\|\le\|\pi(A^*)\|.$$

By interchanging A and A* in the previous argument, it follows that $\|\pi(A^*)\|\le\|\pi(A)\|$, so $\|\pi(A^*)\|=\|\pi(A)\|$. If we now use this identity in the previous argument we have

$$\|\pi(A)\|^2 \le \|\pi(A^*A)\|\le\|\pi(A^*)\|\ \|\pi(A)\| =\|\pi(A)\|^2\ ,$$

so $\|\pi(A)\|^2 = \|\pi(A^*A)\|$.

We have shown that (2.8) defines a C*-norm for $\pi(\mathfrak{A})$. Since the representation is faithful (i.e., $\pi(A)=0$ if and only if $A=0$), $\|A\|\equiv\|\pi(A)\|$ is a C*-norm for $\mathfrak{A}$. To show that $\mathfrak{A}$ is a C*-algebra we must demonstrate that $\mathfrak{A}$ is complete with respect to this C*-norm. Specifically, we must show that the sequence $\{\alpha_m^{\mu}\,\sigma_\mu\}$ converges to an element of $\mathfrak{A}$ if it is a Cauchy sequence, i.e., if

$$\lim_{m,n\to\infty}\ \|\alpha_m^{\mu}\sigma_\mu-\alpha_n^{\mu}\sigma_\mu\|=0.$$

We have $\|\alpha_m^\mu\sigma_\mu-\alpha_n^\mu\sigma_\mu\|^2 \equiv \|\alpha_m^\mu\pi(\sigma_\mu)-\alpha_n^\mu\pi(\sigma_\mu)\|^2$

$$=\mathrm{Sup}\{\ \|(\alpha_m^\mu-\alpha_n^\mu)\pi(\sigma_\mu)\underline{\lambda}\|^2 : \underline{\lambda}\in\mathbb{C}^2,\ \|\underline{\lambda}\|=1\}$$

Letting $\underline{\lambda}=\begin{pmatrix}1\\0\end{pmatrix}$ gives

$$\|\alpha_m^\mu\sigma_\mu-\alpha_n^\mu\sigma_\mu\|^2\geq|(\alpha_m^0-\alpha_n^0)+(\alpha_m^3-\alpha_n^3)|^2+|(\alpha_m^1-\alpha_n^1)+i(\alpha_m^2-\alpha_n^2)|^2,$$

while $\underline{\lambda}=\begin{pmatrix}0\\1\end{pmatrix}$ gives

$$\|\alpha_m^\mu\sigma_\mu-\alpha_n^\mu\sigma_\mu\|^2\geq|(\alpha_m^0-\alpha_n^0)-(\alpha_m^3-\alpha_n^3)|^2+|(\alpha_m^1-\alpha_n^1)-i(\alpha_m^2-\alpha_n^2)|^2.$$

Using these inequalities we see that $\lim_{m,n\to\infty}\|\alpha_m^\mu\sigma_\mu-\alpha_n^\mu\sigma_\mu\|=0$ implies that $\lim_{m,n\to\infty}|(\alpha_m^\mu-\alpha_n^\mu)|=0$ for $\mu=0$, 1, 2, and 3, i.e., the sequences of complex numbers $\{\alpha_m^\mu\}$ are Cauchy. Since the complex numbers are complete, $\{\alpha_m^\mu\}$ converges for $\mu=0$, 1, 2, and 3. Let $\{\alpha_m^\mu\}\to\alpha^\mu$, we then see that $\{\alpha_m^\mu\,\sigma_\mu\}\to\alpha^\mu\sigma_\mu\in\mathfrak{A}$. We have shown that $\mathfrak{A}$ is complete and hence is a C*-algebra. We can now conclude that (2.7) is the unique C*-norm for $\mathfrak{A}$.[1]

The real and self adjoint elements $A\in\mathfrak{A}$ will now be classified. Since $A=\alpha^\mu\sigma_\mu$ is completely determined by the α^μ, this classification is in terms of them.

[1] What we have done here is quite general. We have shown that the usual norm on the set of bounded linear operators on a Hilbert space is a C*-norm. If we know that a *-algebra has a faithful representation then we can conclude that it possesses a C*-norm. If we can show that the *-algebra is complete with respect to this C*-norm then it is a C*-algebra and we can use the relation $\|A\|^2=\rho(A^*A)$ to derive a convenient expression for this C*-norm.

Proposition 2.9 A general element $A=\alpha^{\mu}\sigma_{\mu}\in\mathfrak{A}$ is:

i) Self adjoint if and only if the coefficients α^{μ} are real,

ii) positive if and only if the coefficients α^{μ} are real and satisfy $\alpha^{0}\geq\sqrt{(\alpha^{1})^2+(\alpha^{2})^2+(\alpha^{3})^2}$.

Proof: i) Follows trivially from $A^{*}=\bar{\alpha}^{\mu}\sigma_{\mu}$.

ii) Recall that A is positive if it is self adjoint and $\sigma(A)\in[0,\infty]$; the above condition then follows from the relation

$$\sigma(A=\alpha^{\mu}\sigma_{\mu})=\left\{\lambda=\alpha^{0}\pm\sqrt{(\alpha^{1})^2+(\alpha^{2})^2+(\alpha^{3})^2}\right\}.$$

Now that we have characterized the positive elements we can go on to construct the states over $\mathfrak{A}$. Since $A=\alpha^{\mu}\sigma_{\mu}$ is completely determined by the coefficients α^{μ}, any functional f over $\mathfrak{A}$ must be of the form $f(A)=F(\alpha^{\mu})$, where F is a complex valued function. In order for the f to be a linear functional it must satisfy $f(A+B)=f(A)+f(B)$. With $B=\beta^{\mu}\sigma_{\mu}$ this condition implies that

$$F(\alpha^{\mu}+\beta^{\mu})=F(\alpha^{\mu})+F(\beta^{\mu}),$$

so F must be of the form $F(\alpha^{\mu})=x_{\mu}\alpha^{\mu}$. Since $f(A^{*})=\overline{f(A)}$ if f is a linear functional, the coefficients x_{μ} must be real. These results are summarized in the following proposition.

Proposition 2.10 The most general linear functional f over $\mathfrak{A}$ is of the form $f(\alpha^{\mu}\sigma_{\mu})=x_{\mu}\alpha^{\mu}$, with the real coefficients x_{μ}.

A linear functional f is positive if it takes on positive values for positive elements of $\mathfrak{A}$. Using the general form for a positive element of $\mathfrak{A}$, given in Proposition 2.9, along with the previous result we see that f is positive if and only if $f(\alpha^\mu\sigma_\mu)=x_\mu\alpha^\mu\geq 0$ for all α^μ satisfying

$$\alpha^0\geq\sqrt{(\alpha^1)^2+(\alpha^2)^2+(\alpha^3)^2}\ ,$$

with the coefficients x_μ real. In particular note that $\sigma_0>0$ implies $f(\sigma_0)=x_0>0$. We can now prove the following.

Claim 2.11 A necessary and sufficient condition for a linear functional $f(\alpha^\mu\sigma_\mu)= x_\mu a^\mu$, with x^μ real, to be positive on the positive cone

$$\alpha^0\geq\sqrt{(\alpha^1)^2+(\alpha^2)^2+(\alpha^3)^2}$$

is $x_0\geq\sqrt{(x_1)^2+(x_2)^2+(x_3)^2}$.

Proof: We first show that the condition is sufficient. Given $x_0\geq\sqrt{(x_1)^2+(x_2)^2+(x_3)^2}$ we must show that $x_\mu\alpha^\mu\geq 0$ for all real α^μ satisfying $\alpha^0\geq\sqrt{(\alpha^1)^2+(\alpha^2)^2+(\alpha^3)^2}$. Combining these two conditions we have

$$(\alpha^0)^2(x_0)^2\geq[(\alpha^1)^2+(\alpha^2)^2+(\alpha^3)^2][(x_1)^2+(x_2)^2+(x_3)^2].$$

If we form the vectors $\underline{A}=(\alpha^1,\alpha^2,\alpha^3)$ and $\underline{X}=(x_1,x_2,x_3)$ we can use the Cauchy-Schwartz inequality to obtain

$$\begin{aligned}[(\alpha^1)^2+(\alpha^2)^2+(\alpha^3)^2][(x_1)^2+(x_2)^2+(x_3)^2]&=\|\underline{A}\|^2\|\underline{X}\|^2\\ &\geq\left|\underline{A}\cdot\underline{X}\right|^2\\ &=(\alpha^1x_1+\alpha^2x_2+\alpha^3x_3)^2.\end{aligned}$$

Combining the above inequalities we conclude that $x_\mu\alpha^\mu \geq 0$, and hence the condition is sufficient.

To show that the condition is necessary we assume that $x_o<\sqrt{(x_1)^2+(x_2)^2+(x_3)^2}$, i.e., the condition fails, and construct a positive A such that $f(A)<0$. Let $\alpha^o=\sqrt{(x_1)^2+(x_2)^2+(x_3)^2}$, $\alpha^1=-x_1$, $\alpha^2=-x_2$, $\alpha^3=-x_3$, and consider the element $A=\alpha^\mu\sigma_\mu$. Since $(\alpha^o)^2=(\alpha^1)^2+(\alpha^2)^2+(\alpha^3)^2$ A is positive. Now $f(A)=x_\mu\alpha^\mu=x_o\sqrt{(x_1)^2+(x_2)^2+(x_3)^2} - (x_1)^2 -(x_2)^2 -(x_3)^2 <0$, where the last inequality follows from the assumption $x_o<\sqrt{(x_1)^2+(x_2)^2+(x_3)^2}$. So we have constructed a positive A such that $f(A)<0$, and the condition is necessary.

The following "geometrical" interpretation of the condition in the previous claim will prove to be useful. Consider a linear functional $f(\alpha^\mu\sigma_\mu)=x_\mu\alpha^\mu$, with $x_o>0$, x_1, x_2, and x_3 real. We imagine that the x_1, x_2, and x_3 form a vector $\underline{X}=(x_1,x_2,x_3)$ in $\mathbb{R}^3$, and that x_o is the radius of a sphere in $\mathbb{R}^3$. We then have, from the previous claim, that f is positive if and only if $\underline{X}$ lies in or on the sphere of radius x_o.

A state ω over $\mathfrak{A}$ is a positive linear functional with unit norm. Since $\mathfrak{A}$ possesses an identity, namely σ^o, we have $\|\omega\|=\omega(\sigma^o)=x_o=1$. This immediately gives

Proposition 2.12 The most general state ω over $\mathfrak{A}$ is of the form $\omega(\alpha^\mu\sigma_\mu)=x_\mu\alpha^\mu$, with $x_o=1\geq(x_1)^2+(x_2)^2+(x_3)^2$.

In terms of our geometrical interpretation we have that a linear functional ω is a state if and only if the corresponding vector $\underline{X}$ lies in or on the unit sphere.

Finally we characterize the pure states over $\mathfrak{A}$. Recall from Definition 1.28 that a state ω is pure if it is not possible to decompose ω as $\omega=\lambda\omega_1+(1-\lambda)\omega_2$ with ω_1 and ω_2 states and $0<\lambda<1$. In terms of the vector $\underline{X}$ ω is pure if it is not possible to decompose $\underline{X}$ as $\underline{X}=\lambda\underline{X}_1+(1-\lambda)\underline{X}_2$ with $\|\underline{X}_1\|\leq 1$, $\|\underline{X}_2\|\leq 1$ and $0<\lambda<1$. So ω is pure if it is not possible to express the corresponding vector $\underline{X}$ as a non-trivial convex combination of two vectors lying in or on the unit sphere. Since $\underline{X}$ lies in or on the unit sphere this is the case only when $\underline{X}$ actually lies on the unit sphere. In terms of ω itself we then have the following proposition.

Proposition 2.13 The most general pure state ω over $\mathfrak{A}$ is of the form $\omega(\alpha^{\mu}\sigma_{\mu})=x_{\mu}\alpha^{\mu}$, where the coefficients x_{μ} are real and satisfy $x_0=1=(x_1)^2+(x_2)^2+(x_3)^2$.

Suppose that we place our system in a uniform magnetic field in the 3- direction, so that the Hamiltonian is $H=-B\sigma_3$. In the concrete Hilbert space approach, the canonical density matrix is then

$$\rho=\frac{\exp(\beta B\sigma_3)}{\operatorname{Tr}\exp(\beta B\sigma_3)},$$

where $(\beta=(kT)^{-1})$. We shall assume that this is also the case in the Algebraic Approach, with $\operatorname{Tr}(A)\equiv\{\sum\lambda:\ \lambda\in\sigma(A)\}$. This gives rise to the state

$\omega(A)=\mathrm{Tr}(A\rho)$. Using the properties of the spectrum in Theorem 1.13 we have

$$\begin{aligned}\mathrm{Tr}(\exp(\beta B\sigma_3)) &= \{\textstyle\sum\lambda : \lambda\in\sigma(\exp(\beta B\sigma_3))\}\\ &= \{\textstyle\sum\lambda : \lambda\in\exp(\sigma(\beta B\sigma_3))\}\\ &= \exp(\beta B)+\exp(-\beta B)\\ &= 2\cosh\beta B,\end{aligned}$$

and
$$\begin{aligned}\mathrm{Tr}(A\exp(\beta B\sigma_3)) &= \mathrm{Tr}(\alpha^\mu\sigma_\mu\exp(\beta B\sigma_3))\\ &= \mathrm{Tr}(\alpha^\mu\sigma_\mu(\cosh\beta B\,\sigma_0+\sinh\beta B\,\sigma_3))\\ &= (\alpha^0\cosh\beta B+\alpha^3\sinh\beta B)\,\mathrm{Tr}\,\sigma_0\\ &\quad + (X)\,\mathrm{Tr}\,\sigma_1 + (Y)\,\mathrm{Tr}\,\sigma_2 + (Z)\,\mathrm{Tr}\,\sigma_3,\end{aligned}$$

where X, Y, and Z depend on the α^μ and $\cosh\beta B$, $\sinh\beta B$. Since $\mathrm{Tr}\,\sigma_0=2$ and $\mathrm{Tr}\,\sigma_1=\mathrm{Tr}\,\sigma_2=\mathrm{Tr}\,\sigma_3=0$ we have

$$\mathrm{Tr}(\alpha^\mu\sigma_\mu\exp(\beta B\sigma_3)) = 2\,(\alpha^0\cosh\beta B+\alpha^3\sinh\beta B).$$

Therefore $\omega(\alpha^\mu\sigma_\mu)=\mathrm{Tr}(\alpha^\mu\sigma_\mu\rho)=\dfrac{\alpha^0\cosh\beta B+\alpha^3\sinh\beta B}{\cosh\beta B}$,

or
$$\omega(A)=\omega(\alpha^\mu\sigma_\mu)=\alpha^0+\alpha^3 t,\quad t=\tanh\beta B\ (0\le|t|\le 1) \tag{2.14}$$

This equilibrium state was obtained by using the canonical density matrix ρ from the concrete Hilbert space approach. If we are true disciples of the Algebraic Approach we should demand an algebraic characterization of this equilibrium state. We only mention here that this might be possible following the methods of Hagg/Trych-Pohlmeyer ([Haag1]).

We now make the following observations. Since $|t|\leq 1$, $\omega(A)$ is a state by Proposition 2.12. From Proposition 2.13 we conclude that $\omega(A)$ is pure only when $t=\pm 1$. Denote these states by $\omega_{\pm}(A)$ (i.e., $\omega_{+}(\alpha^{\mu}\sigma_{\mu})=\alpha^{0}+\alpha^{3}$ and $\omega_{-}(\alpha^{\mu}\sigma_{\mu})=\alpha^{0}-\alpha^{3}$). The state $\omega_{+}(A)$ corresponds to the zero temperature case. Finally a mixed state $\omega(A)$ can be decomposed in terms of the pure states $\omega_{\pm}$ as

$$\omega(A)=u^2\omega_{+}(A) + v^2\omega_{-}(A), \tag{2.15}$$

with $u^2=(1+t)/2$ and $v^2=(1-t)/2$.

The G.N.S. representations associated with these states will now be constructed. First consider the state $\omega(\alpha^{\mu}\sigma_{\mu})=\alpha^{0}+\alpha^{3}\,t$, $0\leq|t|<1$. We begin by constructing the left ideal $\mathfrak{I}_{\omega}=\{A\in\mathfrak{A} : \omega(A^*A)=0\}$. A simple calculation shows that

$$\mathfrak{I}_{\omega}=\{\alpha^{\mu}\sigma_{\mu}\in\mathfrak{A} : |\alpha^{0}|^2+|\alpha^{1}|^2+|\alpha^{2}|^2+|\alpha^{3}|^2+t(\alpha^{0}\overline{\alpha}^{3}+\overline{\alpha}^{0}\alpha^{3}+i\overline{\alpha}^{1}\alpha^{2}-i\alpha^{1}\overline{\alpha}^{2})=0\}$$

$$=\{\alpha^{\mu}\sigma_{\mu}\in\mathfrak{A} : [|\alpha^{0}|^2+|\alpha^{3}|^2+t\,(\alpha^{0}\overline{\alpha}^{3}+\overline{\alpha}^{0}\alpha^{3})]+[|\alpha^{1}|^2+|\alpha^{2}|^2+t\,(i\overline{\alpha}^{1}\alpha^{2}-i\alpha^{1}\overline{\alpha}^{2})]=0\}.$$

Now $|\alpha^{0}+\alpha^{3}|^2\geq 0$ implies $|\alpha^{0}|^2+|\alpha^{3}|^2\geq -\alpha^{0}\overline{\alpha}^{3}-\overline{\alpha}^{0}\alpha^{3}\geq t(\alpha^{0}\overline{\alpha}^{3}+\overline{\alpha}^{0}\alpha^{3})$, where the last inequality follows from $|t|<1$ and is an equality only when $\alpha^{0}=\alpha^{3}=0$. So we see that

$$[|\alpha^o|^2+|\alpha^3|^2+t\ (\alpha^o\overline{\alpha}^3+\overline{\alpha}^o\alpha^3)]\geq 0,$$

equality only for $\alpha^o=\alpha^3=0$. In a similar fashion we have

$$[|\alpha^1|^2+|\alpha^2|^2+t\ (i\overline{\alpha}^1\alpha^2-i\alpha^1\overline{\alpha}^2)]\geq 0,$$

equality only for $\alpha^1=\alpha^2=0$. Since equality occurs in the above expression only when all the α^μ vanish, we conclude that

$$\mathfrak{I}_\omega=\{A\in\mathfrak{A}:A=0\}. \tag{2.16}$$

The pre-Hilbert space H_ω is the span of the set $\{\Psi_A : A\in\mathfrak{A}\}$, where Ψ_A is the equivalence class $\Psi_A=\{A+I\ : I\in\mathfrak{I}_\omega\}$. A scalar product over H_ω is then defined using the state ω, $(\Psi_A,\Psi_B)=\omega(A^*B)$. We now show that H_ω is four-dimensional by demonstrating that the set $\{\Psi_{\sigma_o},\Psi_{\sigma_1},\Psi_{\sigma_2},\Psi_{\sigma_3}\}$ is linearly independent while the set $\{\Psi_{\sigma_o},\Psi_{\sigma_1},\Psi_{\sigma_2},\Psi_{\sigma_3},\Psi_B\}$ is not for arbitrary $B\in\mathfrak{A}$. Recall that $\Psi_A=0$ if and only if $A\in\mathfrak{I}_\omega$, i.e., if and only if $A=0$. Now

$$\alpha^o\Psi_{\sigma_o}+\alpha^1\Psi_{\sigma_1}+\alpha^2\Psi_{\sigma_2}+\alpha^3\Psi_{\sigma_3}=\Psi_{\alpha^\mu\sigma_\mu},$$

so that a linear combination of the Ψ_{σ_μ} vanishes if and only if the coefficients α^μ are identically zero, hence the set $\{\Psi_{\sigma_o},\Psi_{\sigma_1},\Psi_{\sigma_2},\Psi_{\sigma_3}\}$is linearly independent. Now consider the linear combination $\alpha^o\Psi_{\sigma_o}+\alpha^1\Psi_{\sigma_1}+\alpha^2\Psi_{\sigma_2}+\alpha^3\Psi_{\sigma_3}+\beta\Psi_B$ for arbitrary $B=\beta^\mu\sigma_\mu\neq 0$. We have

$$\begin{aligned}\alpha^o\Psi_{\sigma_o}+\alpha^1\Psi_{\sigma_1}+\alpha^2\Psi_{\sigma_2}+\alpha^3\Psi_{\sigma_3}+\beta\Psi_B&=\Psi_{\alpha^\mu\sigma_\mu+\beta\beta^\mu\sigma_\mu}\\&=\Psi_{(\alpha^\mu+\beta\beta^\mu)\sigma_\mu},\end{aligned}$$

so that this linear combination of the $\Psi_{\sigma_\mu}, \Psi_B$ vanishes if the coefficients are chosen to be $\beta=1$ and $\alpha^\mu = -\beta^\mu$, hence the set $\{\Psi_{\sigma_0}, \Psi_{\sigma_1}, \Psi_{\sigma_2}, \Psi_{\sigma_3}, \Psi_B\}$ is not linearly independent for arbitrary $B \neq 0$. This shows that the pre-Hilbert space H_ω is four-dimensional.

To obtain an orthonormal basis for H_ω we apply the Gram-Schmidt process to the linearly independent set $\{\Psi_{\sigma_0}, \Psi_{\sigma_1}, \Psi_{\sigma_2}, \Psi_{\sigma_3}\}$. This produces the orthonormal set $\{\Psi_1, \Psi_2, \Psi_3, \Psi_4\}$, where

$$\Psi_1 = \Psi_{\sigma_0},$$

$$\Psi_2 = (-t\Psi_{\sigma_0} + \Psi_{\sigma_3})/\sqrt{1-t^2},$$

$$\Psi_3 = \Psi_{\sigma_1},$$

and

$$\Psi_4 = (-it\Psi_{\sigma_1} + \Psi_{\sigma_2})/\sqrt{1-t^2}.$$

Since the set $\{\Psi_1, \Psi_2, \Psi_3, \Psi_4\}$ is an orthonormal basis for a four-dimensional space we let

$$\Psi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \Psi_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \Psi_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } \Psi_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

Now the representatives $\pi_\omega(A)$ are defined by $\pi_\omega(A)\Psi_B = \Psi_{AB}$, so $\pi_\omega(A)_{ij} = (\Psi_i, \pi_\omega(A)\Psi_j)$. For example

$$\pi_\omega(\sigma_3)_{34}=(\Psi_3,\pi_\omega(\sigma_3)\Psi_4)$$

$$=\frac{(\Psi_{\sigma_1},\pi_\omega(\sigma_3)[-it\Psi_{\sigma_1}+\Psi_{\sigma_2}])}{\sqrt{1-t^2}}$$

$$=\frac{(\Psi_{\sigma_1},t\Psi_{\sigma_2})}{\sqrt{1-t^2}}+\frac{(\Psi_{\sigma_1},-i\Psi_{\sigma_1})}{\sqrt{1-t^2}}$$

$$=t\frac{\omega(\sigma_1\sigma_2)}{\sqrt{1-t^2}}-i\frac{\omega(\sigma_1\sigma_1)}{\sqrt{1-t^2}}$$

$$=\frac{(it^2-i)}{\sqrt{1-t^2}}$$

$$=-i\sqrt{1-t^2}.$$

Continuing in this manner we find

$$\left.\begin{array}{l}
\pi_\omega(\sigma_0)=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\pi_\omega(\sigma_1)=\begin{pmatrix}0&0&1&0\\0&0&0&i\\1&0&0&0\\0&-i&0&0\end{pmatrix}\\[2ex]
\pi_\omega(\sigma_2)=\begin{pmatrix}0&0&-it&\sqrt{1-t^2}\\0&0&-i\sqrt{1-t^2}&-t\\it&i\sqrt{1-t^2}&0&0\\\sqrt{1-t^2}&-t&0&0\end{pmatrix}\\[2ex]
\pi_\omega(\sigma_3)=\begin{pmatrix}t&\sqrt{1-t^2}&0&0\\\sqrt{1-t^2}&-t&0&0\\0&0&-t&-i\sqrt{1-t^2}\\0&0&i\sqrt{1-t^2}&t\end{pmatrix}
\end{array}\right\}.$$

Since the pre-Hilbert space H_ω is equal to $\mathbb{C}^4$, it is already complete, so $\mathfrak{H}_\omega = \mathbb{C}^4$. The cyclic vector Ω_ω is given by $\Omega_\omega \equiv \Psi_{\sigma_0} = \Psi_1$. At this point we should have a cyclic representation $\{\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega\}$ such that $\omega(A) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega)$ for all $A \in \mathfrak{A}$. To demonstrate that this is the case we first put $\{\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega\}$ in a more convenient form by diagonalizing $\pi_\omega(\sigma_3)$. The solutions to the secular equation $0 = \left|\pi_\omega(\sigma_3) - \lambda I\right| = (\lambda^2 - 1)^2$ are $\lambda = 1, 1, -1, -1$ (I is the four by four unit matrix). The corresponding eigenvectors are found to be

$$v^1_{\lambda=1} = \begin{pmatrix} u \\ v \\ 0 \\ 0 \end{pmatrix}, \; v^2_{\lambda=1} = \begin{pmatrix} 0 \\ 0 \\ v \\ -iu \end{pmatrix}, \; v^1_{\lambda=-1} = \begin{pmatrix} v \\ -u \\ 0 \\ 0 \end{pmatrix}, \text{ and } v^2_{\lambda=-1} = \begin{pmatrix} 0 \\ 0 \\ u \\ iv \end{pmatrix},$$

where $u = \sqrt{\frac{1+t}{2}}$ and $v = \sqrt{\frac{1-t}{2}}$. The matrix U that diagonalizes $\pi_\omega(\sigma_3)$ is therefore

$$U = \begin{pmatrix} u & v & 0 & 0 \\ 0 & 0 & u & iv \\ 0 & 0 & v & -iu \\ v & -u & 0 & 0 \end{pmatrix}.$$

Applying U to our representation $\{\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega\}$ gives rise to the equivalent representation $\{\mathfrak{H}'_\omega, \pi'_\omega, \Omega'_\omega\}$, where

$$\mathfrak{H}'_\omega = U\mathfrak{H}_\omega = \mathfrak{H}_\omega = \mathbb{C}^4,$$

$$\pi'_\omega(\sigma_o) = U\pi_\omega(\sigma_o)U^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\pi'_\omega(\sigma_1) = U\pi_\omega(\sigma_1)U^\dagger = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\pi'_\omega(\sigma_2) = U\pi_\omega(\sigma_2)U^\dagger = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

$$\pi'_\omega(\sigma_3) = U\pi_\omega(\sigma_3)U^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and
$$\Omega'_\omega = U\Omega_\omega = \begin{pmatrix} u \\ 0 \\ 0 \\ v \end{pmatrix}.$$

Dropping the primes, we arrive at our final form for the G.N.S. representation corresponding to the state ω, $\omega(A)=\omega(\alpha^\mu\sigma_\mu)=\alpha^o+t\alpha^3$:

$$\left\{\mathfrak{H}_\omega = \mathbb{C}^4, \pi_\omega(\alpha^\mu\sigma_\mu) = \begin{pmatrix} \pi_o(\alpha^\mu\sigma_\mu) & 0 \\ 0 & \pi_o(\alpha^\mu\sigma_\mu) \end{pmatrix}; \pi_o(\alpha^\mu\sigma_\mu) = \begin{pmatrix} \alpha^o+\alpha^3 & \alpha^1-i\alpha^2 \\ \alpha^1+i\alpha^2 & \alpha^o-\alpha^3 \end{pmatrix}, \Omega_\omega = \begin{pmatrix} u \\ 0 \\ 0 \\ v \end{pmatrix}\right\}.$$

where $u=\sqrt{\frac{1+t}{2}}$, $v=\sqrt{\frac{1-t}{2}}$, and $t=\tanh \beta B$.

We now make the following observations about the above representation (it is easy to verify that it is indeed a representation of $\mathfrak{A}$). Let Ψ be an arbitrary vector in $\mathfrak{H}_\omega$. Then by direct multiplication it is easy to show that

$$\Psi\equiv\begin{pmatrix}\alpha\\ \beta\\ \gamma\\ \delta\end{pmatrix}=\pi_\omega\left(\frac{1}{2}\left[\frac{\alpha}{u}+\frac{\delta}{v}\right]\sigma_0+\frac{1}{2}\left[\frac{\beta}{u}+\frac{\gamma}{v}\right]\sigma_1-\frac{i}{2}\left[\frac{\beta}{u}-\frac{\gamma}{v}\right]\sigma_2+\frac{1}{2}\left[\frac{\alpha}{u}-\frac{\delta}{v}\right]\sigma_3\right)\begin{pmatrix}u\\ 0\\ 0\\ v\end{pmatrix},$$

so Ω_ω is a cyclic vector and the representation $\{\mathfrak{H}_\omega,\pi_\omega,\Omega_\omega\}$ is cyclic. Furthermore, it is easy to show that the vector state $(\Omega_\omega,\pi_\omega(A)\Omega_\omega)$ is equal to the state ω (i.e., $(\Omega_\omega,\pi_\omega(\alpha^\mu\sigma_\mu)\Omega_\omega)=\alpha^o+t\alpha^3=\omega(\alpha^\mu\sigma_\mu)$).

The representation is obviously reducible, as is to be expected since the state ω is mixed (Theorem 1.28). Since the representation is reducible, there must exist a non-zero $\Psi\in\mathfrak{H}_\omega$ that is not cyclic, by Lemma 1.22. An example of such a vector is

$$\Psi=\begin{pmatrix}1\\ 0\\ 0\\ 0\end{pmatrix},\quad \pi_\omega(\alpha^\mu\sigma_\mu)\Psi=\begin{pmatrix}\alpha^o+\alpha^3\\ \alpha^1+i\alpha^2\\ 0\\ 0\end{pmatrix}.$$

For example, it is not possible to obtain the cyclic vector Ω_ω by applying $\pi_\omega(\mathfrak{A})$ to Ψ (although there exists an $A\in\mathfrak{A}$ such that $\Psi=\pi_\omega(A)\Omega_\omega$; this A is not invertible).

The dependence of the representation on temperature is entirely contained in the cyclic vector Ω_ω, so that representations corresponding to different (nonzero) temperatures are unitarily equivalent (the representing matrices are in fact identically equal). The representation $\{\mathfrak{H}_\omega,\pi_\omega\}$ will be referred to as the finite temperature representation. We will discuss what happens to the cyclic representations $\{\mathfrak{H}_\omega,\pi_\omega,\Omega_\omega\}$ as the temperature goes to zero after we have constructed the G.N.S. representation corresponding to the zero temperature state ω_+.

The G.N.S. construction for the pure states $\omega_\pm(\alpha^\mu\sigma_\mu)=\alpha^0\pm\alpha^3$ is basically the same as that for the mixed state ω, only now the left ideals $\mathfrak{I}_{\omega\pm}$ are not trivial:

$$\begin{aligned}\mathfrak{I}_{\omega\pm}&=\{A\in\mathfrak{A}:\omega_\pm(A^*A)=0\}\\&=\{\alpha^\mu\sigma_\mu\in\mathfrak{A}:|\alpha^0\pm\alpha^3|^2+|\alpha^1\pm i\alpha^2|^2=0\}\\&=\{\alpha(\sigma_0\pm\sigma_3)+\beta(\sigma_1\pm i\sigma_2):\alpha,\beta\in\mathbb{C}\}.\end{aligned}$$

The pre-Hilbert spaces $H_{\omega\pm}=\text{span}\,\{\Psi_A: A\in\mathfrak{A}\}$, where Ψ_A is the equivalence class $\Psi_A=\{A+\alpha(\sigma_0\pm\sigma_3)+\beta(\sigma_1\pm i\sigma_2):A\in\mathfrak{A};\alpha,\beta\in\mathbb{C}\}$, are now two-dimensional. To show this we demonstrate that the set $\{\Psi\sigma_0,\Psi\sigma_1\}$ is linearly independent while the set $\{\Psi\sigma_0,\Psi\sigma_1,\Psi_B\}$ is not for arbitrary non-zero Ψ_B. Recall that the zero vector is any element of $\mathfrak{I}_{\omega\pm}$, the equivalence class corresponding to A=0, so that the linear combination $\gamma\Psi\sigma_0+\delta\Psi\sigma_1=\Psi\gamma\sigma_0+\delta\sigma_1$ vanishes if and only if $\Psi\gamma\sigma_0+\delta\sigma_1\in\mathfrak{I}_{\omega\pm}$. This is the case only when $\gamma=\delta=0$, so the set $\{\Psi\sigma_0,\Psi\sigma_1\}$ is linearly independent.

Now let $\Psi_{\beta^\mu\sigma_\mu}$ be an arbitrary non-zero vector (i.e., $\beta^o \neq \pm\beta^3$ and $\beta^1 \neq \pm i\beta^2$) and consider the linear combination

$$\gamma\Psi_{\sigma_o}+\delta\Psi_{\sigma_1}+\varepsilon\Psi_{\beta^\mu\sigma_\mu}=\Psi_{\gamma\sigma_o+\delta\sigma_1+\varepsilon\beta^\mu\sigma_\mu}.$$

This linear combination vanishes if $\varepsilon=1$, $\gamma=\beta^o\pm\beta^3(\neq 0)$, and $\delta=\beta^1\pm i\beta^2(\neq 0)$, so the set $\{\Psi_{\sigma_o},\Psi_{\sigma_1},\Psi_B\}$ is not linearly independent for arbitrary non-zero Ψ_B, and the pre-Hilbert spaces $H_{\omega\pm}$ are therefore two-dimensional. We in fact have $H_{\omega\pm}=\mathbb{C}^2$, so the $H_{\omega\pm}$ are already complete and therefore $\mathfrak{H}_{\omega\pm}=\mathbb{C}^2$.

Since the linearly independent set $\{\Psi_{\sigma_o},\Psi_{\sigma_1}\}$ is orthonormal $((\Psi_{\sigma_o},\Psi_{\sigma_1})=\omega_\pm(\sigma_o\sigma_1)=0$, $\|\Psi_{\sigma_o}\|^2=\omega_\pm(\sigma_o\sigma_o)=1$, and $\|\Psi_{\sigma_1}\|^2=\omega_\pm(\sigma_1\sigma_1)=1)$, we choose it for a basis , $\{\Psi_1=\Psi_{\sigma_o},\Psi_2=\Psi_{\sigma_1}\}$. Since the $\mathfrak{H}_{\omega\pm}$ are two-dimensional let

$$\Psi_1=\begin{pmatrix}1\\0\end{pmatrix}, \qquad \text{and } \Psi_2=\begin{pmatrix}0\\1\end{pmatrix}.$$

With this choice of basis, the representatives $\pi_{\omega\pm}$ and cyclic vectors $\Omega_{\omega\pm}$ are

$\underline{\omega_+}$: $\pi_{\omega_+}(\sigma_o)=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\pi_{\omega_+}(\sigma_1)=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\pi_{\omega_+}(\sigma_2)=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\pi_{\omega_+}(\sigma_3)=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$,

and $\Omega_{\omega_+}=\Psi_1=\begin{pmatrix}1\\0\end{pmatrix}$.

$\underline{\omega_-}$: $\pi_{\omega_-}(\sigma_o)=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $\pi_{\omega_-}(\sigma_1)=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\pi_{\omega_-}(\sigma_2)=\begin{pmatrix}0&i\\-i&0\end{pmatrix}$, $\pi_{\omega_-}(\sigma_3)=\begin{pmatrix}-1&0\\0&1\end{pmatrix}$,

and $\Omega_{\omega_-}=\Psi_1=\begin{pmatrix}1\\0\end{pmatrix}$.

Using the unitary transformation $U=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ we see that the ω_- representation is unitarily equivalent to

$$\underline{\omega_-'}:\pi_{\omega_-}(\sigma_0)'=\begin{pmatrix}1&0\\0&1\end{pmatrix},\ \pi_{\omega_-}(\sigma_1)'=\begin{pmatrix}0&1\\1&0\end{pmatrix},\ \pi_{\omega_-}(\sigma_2)'=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\ \pi_{\omega_-}(\sigma_3)'=\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

$$\text{and } \Omega_{\omega_-}=\begin{pmatrix}0\\1\end{pmatrix}.$$

Our final form for the G.N.S. representations corresponding to the pure states $\omega_\pm$ are then (dropping the prime from the ω_- representation)

$$\underline{\omega_+}:\left\{\mathfrak{H}_{\omega_+}=\mathbb{C}^2,\ \pi_{\omega_+}(\alpha^\mu\sigma_\mu)=\begin{pmatrix}\alpha^0+\alpha^3&\alpha^1-i\alpha^2\\\alpha^1+i\alpha^2&\alpha^0-\alpha^3\end{pmatrix},\ \Omega_{\omega_+}=\begin{pmatrix}1\\0\end{pmatrix}\right\}$$

$$\underline{\omega_-}:\left\{\mathfrak{H}_{\omega_-}=\mathbb{C}^2,\ \pi_{\omega_-}(\alpha^\mu\sigma_\mu)=\begin{pmatrix}\alpha^0+\alpha^3&\alpha^1-i\alpha^2\\\alpha^1+i\alpha^2&\alpha^0-\alpha^3\end{pmatrix},\ \Omega_{\omega_-}=\begin{pmatrix}0\\1\end{pmatrix}\right\}$$

We now make some observations about the above representations (it is easy to verify that they are both representations of $\mathfrak{A}$). Let Ψ be an arbitrary vector in $\mathbb{C}^2$, then

$$\Psi\equiv\begin{pmatrix}\gamma\\\delta\end{pmatrix}=\pi_{\omega_+}(\gamma\sigma_0+\delta\sigma_1)\begin{pmatrix}1\\0\end{pmatrix}$$

$$=\pi_{\omega_-}(\delta\sigma_0+\gamma\sigma_1)\begin{pmatrix}0\\1\end{pmatrix},$$

so the vectors Ω_{ω_+} and Ω_{ω_-} are cyclic for their respective representations, and hence both representations are cyclic. Both representations are also irreducible, as is to be expected since the states $\omega_{\pm}$ are pure. Since the two representations differ only in their cyclic vectors, we will concentrate on the representation corresponding to the zero temperature state ω_{+}. To demonstrate the irreducibility we use Schur's Lemma (Lemma 1.24), specifically we show that any two by two matrix that commutes with all the representatives $\pi_{\omega_+}(\mathfrak{A})$ is necessarily a multiple of the unit matrix. Let M be an arbitrary two by two matrix. Then

$$M\pi_{\omega_+}(\alpha^o\sigma_o+\alpha^1\sigma_1)=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha^o & \alpha^1 \\ \alpha^1 & \alpha^o \end{pmatrix}=\begin{pmatrix} a\alpha^o+b\alpha^1 & b\alpha^o+a\alpha^1 \\ c\alpha^o+d\alpha^1 & d\alpha^o+c\alpha^1 \end{pmatrix},$$

and
$$\pi_{\omega_+}(\alpha^o\sigma_o+\alpha^1\sigma_1)\, M=\begin{pmatrix} \alpha^o & \alpha^1 \\ \alpha^1 & \alpha^o \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a\alpha^o+c\alpha^1 & b\alpha^o+d\alpha^1 \\ c\alpha^o+a\alpha^1 & d\alpha^o+b\alpha^1 \end{pmatrix}.$$

This shows that M will commute with $\pi_{\omega_+}(\alpha^o\sigma_o+\alpha^1\sigma_1)$ only when a=d and b=c. We now require M to also commute with $\pi_{\omega_+}(\alpha^2\sigma_2+\alpha^3\sigma_3)$,

$$M\pi_{\omega_+}(\alpha^2\sigma_2+\alpha^3\sigma_3)=\begin{pmatrix} a & b \\ b & a \end{pmatrix}\begin{pmatrix} \alpha^3 & -i\alpha^2 \\ i\alpha^2 & -\alpha^3 \end{pmatrix}=\begin{pmatrix} a\alpha^3+ib\alpha^2 & -b\alpha^3-ia\alpha^2 \\ b\alpha^3+ia\alpha^2 & -a\alpha^3-ib\alpha^2 \end{pmatrix},$$

and
$$\pi_{\omega_+}(\alpha^2\sigma_2+\alpha^3\sigma_3)M=\begin{pmatrix} \alpha^3 & -i\alpha^2 \\ i\alpha^2 & -\alpha^3 \end{pmatrix}\begin{pmatrix} a & b \\ b & a \end{pmatrix}=\begin{pmatrix} a\alpha^3-ib\alpha^2 & b\alpha^3-ia\alpha^2 \\ -b\alpha^3+ia\alpha^2 & -a\alpha^3+ib\alpha^2 \end{pmatrix}.$$

We have now reached the desired result, for M will commute with $\pi_{\omega_+}(\alpha^2\sigma_2+\alpha^3\sigma_3)$ and $\pi_{\omega_+}(\alpha^o\sigma_o+\alpha^1\sigma_1)$ only when b=0, i.e., when M is a multiple of the unit matrix. Schur's Lemma then allows us to conclude that the representation corresponding to the state ω_+ is irreducible (since

$\pi_{\omega_+}(A)=\pi_{\omega_-}(A)$ for all $A\in\mathfrak{A}$, the representation corresponding to the state ω_- is also irreducible).

The irreducibility of the representation $\{\mathfrak{H}_{\omega_+},\pi_{\omega_+}\}$ implies that every non-zero vector $\Phi\in\mathfrak{H}_{\omega_+}$ is cyclic, and every vector state $\phi(A)=(\Phi,\pi_{\omega_+}(A)\Phi)$ is pure. To demonstrate this, consider an arbitrary non-zero vector $\Phi\in\mathfrak{H}_{\omega_+}$, $\Phi=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$. We may assume, without loss of generality, that Φ is normalized to unity. The following result shows that Φ is cyclic. Let Ψ be an arbitrary vector in $\mathfrak{H}_{\omega_+}$, then

$$\Psi\equiv\begin{pmatrix}a\\ b\end{pmatrix}=\pi_{\omega_+}(\alpha^2\sigma_2+\alpha^3\sigma_3)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}.$$

Next we consider the vector state $\phi(A)=(\Phi,\pi_{\omega_+}(A)\Phi)$,

$$\phi(\alpha^\mu\sigma_\mu)=(\overline{\alpha},\ \overline{\beta})\begin{pmatrix}\alpha^o+\alpha^3 & \alpha^1-i\alpha^2\\ \alpha^1+i\alpha^2 & \alpha^o-\alpha^3\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$=(|\alpha|^2+|\beta|^2)\alpha^o+(\overline{\alpha}\beta+\alpha\overline{\beta})\alpha^1+(\alpha\overline{\beta}-i\overline{\alpha}\beta)\alpha^2+(|\alpha|^2-|\beta|^2)\alpha^3.$$

Now the state $\omega(\alpha^\mu\sigma_\mu)=x^\mu\alpha^\mu$ is pure if and only if $x^o=1=\sqrt{(x^1)^2+(x^2)^2+(x^3)^2}$ (Proposition 2.13). We have $x^o=|\alpha|^2+|\beta|^2=1$, $x^1=\overline{\alpha}\beta+\alpha\overline{\beta}=2\mathrm{Re}[\alpha\overline{\beta}]$, $x^2=\alpha\overline{\beta}-i\overline{\alpha}\beta=2\mathrm{Re}[i\alpha\overline{\beta}]$, and $x^3=|\alpha|^2-|\beta|^2$, so

$$\begin{aligned}(x^1)^2+(x^2)^2+(x^3)^2&=(|\alpha|^2-|\beta|^2)^2+4\mathrm{Re}[\alpha\overline{\beta}]^2+4\mathrm{Re}[i\alpha\overline{\beta}]^2\\ &=(|\alpha|^2-|\beta|^2)^2+4|\alpha|^2|\beta|^2\\ &=(|\alpha|^2+|\beta|^2)^2\\ &=1.\end{aligned}$$

The vector state $\phi(A)$ is therefore a pure state. This was to be expected from the general theory of C*-algebras. The C*-algebra $\mathfrak{A}$ has the additional property that all pure states are vector states in the representation $\{\mathfrak{H}_{\omega_+}, \pi_{\omega_+}\}$.

Theorem 2.17 Let ϕ be an arbitrary pure state over $\mathfrak{A}$. Then there exists a cyclic vector $\Phi \in \mathfrak{H}_{\omega_+}$ such that $\phi(A)=(\Phi,\pi_{\omega_+}(A)\Phi)$ for all $A \in \mathfrak{A}$.

Proof: An arbitrary pure state ϕ over $\mathfrak{A}$ is of the form $\phi(\alpha^\mu\sigma_\mu)=x^\mu\alpha^\mu$ with $x^o=1=\sqrt{(x^1)^2+(x^2)^2+(x^3)^2}$, and the x^μ real. We first consider the general case in which $x^o=\sqrt{(x^1)^2+(x^2)^2+(x^3)^2}$. If we assume that $x_3 \neq -x_o$, then the vector

$$\Phi=\sqrt{\frac{x^o+x^3}{2}}\begin{pmatrix} 1 \\ \frac{x^1+ix^2}{x^o+x^3} \end{pmatrix}$$

is well defined and

$$2(\Phi,\pi_\omega(\alpha^\mu\sigma_\mu)\Phi)=(x^o+x^3)\left(1,\ \frac{x^1-ix^2}{x^o+x^3}\right)\begin{pmatrix} \alpha^o+\alpha^3 & \alpha^1-i\alpha^2 \\ \alpha^1+i\alpha^2 & \alpha^o-\alpha^3 \end{pmatrix}\begin{pmatrix} 1 \\ \frac{x^1+ix^2}{x^o+x^3} \end{pmatrix}$$

$$=(\alpha^o+\alpha^3)(x^o+x^3)+(\alpha^1-i\alpha^2)(x^1+ix^2)+(\alpha^1+i\alpha^2)(x^1-ix^2)+(\alpha^o-\alpha^3)\left(\frac{(x^1)^2+(x^2)^2}{x^o+x^3}\right)$$

$$=\alpha^o\frac{(x^o)^2+(x^1)^2+(x^2)^2+(x^3)^2+2x^ox^3}{x^o+x^3}+2\alpha^1x^1+2\alpha^2x^2$$

$$+\alpha^3\frac{(x^o)^2-(x^1)^2-(x^2)^2+(x^3)^2+2x^ox^3}{x^o+x^3}$$

$$=2(x^\mu\alpha^\mu) \qquad \text{(since } (x^o)^2=(x^1)^2+(x^2)^2+(x^3)^2\text{)},$$

$$=2\phi(\alpha^\mu\sigma_\mu)$$

So $(\Phi,\pi_\omega(A)\Phi)=\phi(A)$ for all $A\in\mathfrak{A}$. A pure state has the further property that $x^0=1$, so the above vector will work for all pure states that have $x^3\neq-1$. The case $x^3=-1$ corresponds to the state ω_-, which we have seen is produced by the vector $\Omega_{\omega_-}=\begin{pmatrix}0\\1\end{pmatrix}$ in the representation $\{\mathfrak{H}_{\omega_-},\pi_{\omega_-}\}$. Finally Φ is cyclic because the representation $\{\mathfrak{H}_{\omega_+},\pi_{\omega_+}\}$ is irreducible. This completes the proof.

Corollary 2.18 Every G.N.S. representation $\{\mathfrak{H}_\phi,\pi_\phi,\Omega_\phi\}$ arising from a pure state ϕ is unitarily equivalent to the representation $\{\mathfrak{H}_{\omega_+},\pi_{\omega_+},\phi\}$, for some cyclic vector $\phi\in\mathfrak{H}_{\omega_+}$. In particular the representation $\{\mathfrak{H}_\phi,\pi_\phi\}$is unitarily equivalent to $\{\mathfrak{H}_{\omega_+},\pi_{\omega_+}\}$.

Proof: From Theorem 2.17 we know that there is a cyclic $\Phi\in\mathfrak{H}_{\omega_+}$ such that $\phi(A)=(\Phi,\pi_{\omega_+}(A)\Phi)$ for all $A\in\mathfrak{A}$. The cyclic representations $\{\mathfrak{H}_\phi,\pi_\phi,\Omega_\phi\}$ and $\{\mathfrak{H}_{\omega_+},\pi_{\omega_+},\Phi\}$ are therefore unitarily equivalent by Theorem 1.25. In particular the representations $\{\mathfrak{H}_\phi,\pi_\phi\}$ and $\{\mathfrak{H}_{\omega_+},\pi_{\omega_+}\}$ are unitarily equivalent.

Corollary 2.19 The representation $\{\mathfrak{H}_{\omega_+},\pi_{\omega_+}\}$ of the C*-algebra $\mathfrak{A}$ is the only irreducible representation of $\mathfrak{A}$ (up to unitary equivalence).

Proof: Let $\{\mathfrak{H},\pi\}$ be an irreducible representation and $\Phi\in\mathfrak{H}$ be an arbitrary non-zero vector. Since the representation $\{\mathfrak{H},\pi\}$ is irreducible Φ is a cyclic vector, and the representation $\{\mathfrak{H},\pi,\Phi\}$ is cyclic. Also the vector state $\phi(A)=(\Phi,\pi(A)\Phi)$ is pure. The cyclic representation $\{\mathfrak{H},\pi,\Phi\}$ is therefore unitarily equivalent to the cyclic G.N.S. representation

$\{\mathfrak{H}_\phi,\pi_\phi,\Omega_\phi\}$ by Theorem 1.25. It then follows that the representation $\{\mathfrak{H},\pi\}$ is unitarily equivalent to the G.N.S. representation $\{\mathfrak{H}_\phi,\pi_\phi\}$, which in turn is unitarily equivalent to $\{\mathfrak{H}_{\omega_+},\pi_{\omega_+}\}$ by Corollary 2.18. The representation $\{\mathfrak{H},\pi\}$ is therefore unitarily equivalent to $\{\mathfrak{H}_{\omega_+},\pi_{\omega_+}\}$.

The following discussion is intended to illustrate the unitary equivalence of representations mentioned in Theorem 1.27. The mixed state ω for finite temperatures can be decomposed into the pure states $\omega_\pm$, $\omega=u^2\omega_+ + v^2\omega_-$, so the representation $\{\mathfrak{H}_\omega,\pi_\omega,\Omega_\omega\}$ should be a direct sum of the representations $\{\mathfrak{H}_{\omega\pm},\pi_{\omega\pm},\Omega_{\omega\pm}\}$. Inspection of the respective representations shows that this is indeed the case, with $\Omega_\omega = u\Omega_{\omega_+} \oplus v\Omega_{\omega_-}$. We first examine the physical meaning of this. Introduce the parameter $\eta=-\frac{1}{T}$. Then η goes from $-\infty$ to ∞ as T goes from 0 to ∞, jumps to $-\infty$ and increases to -0. The values $\eta=-\infty$ and $\eta=\infty$ correspond to $t=1$ and $t=-1$, respectively. So we see that $t=-1$ corresponds to the negative zero temperature state. This is the pure state corresponding to the spin being anti-aligned with the magnetic field. Every finite temperature state is then a statistical mixture of these two pure states (the spin aligned with the magnetic field and the spin anti-aligned with the magnetic field). Note that the probability of obtaining the value 1 when measuring the z-component of the spin in the state $\omega=u^2\omega_+ + v^2\omega_-$ is u^2 while the probability of measuring the value -1 for the z-component of the spin is v^2.

Now consider what happens to the representation $\{\mathfrak{H}_\omega,\pi_\omega,\Omega_\omega\}$ as the temperature goes to zero, i.e., as $t\to 1$ and $\omega\to\omega_+$. The dependence of the representation $\{\mathfrak{H}_\omega,\pi_\omega,\Omega_\omega\}$ on temperature is entirely contained in Ω_ω. As $t\to 1$, $u\to 1$ and $v\to 0$, so that

$$\Omega_\omega \to \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

In this limit $\{\mathfrak{H}_\omega,\pi_\omega,\Omega_\omega\}$ is still a representation of $\mathfrak{A}$, and $\{\mathfrak{H}_\omega,\pi_\omega\}$ is still a direct sum of the representations $\{\mathfrak{H}_{\omega\pm},\pi_{\omega\pm}\}$, with $\Omega_\omega=\Omega_{\omega_+}\oplus 0\Omega_{\omega_-}$. Furthermore, ω is still the vector state $\omega(A)=(\Omega_\omega,\pi_\omega(A)\Omega_\omega)$. Despite this, the representation $\{\mathfrak{H}_\omega,\pi_\omega,\Omega_\omega\}$ is not, in the limit $t\to 1$, unitarily equivalent to the G.N.S. zero temperature representation $\{\mathfrak{H}_{\omega_+},\pi_{\omega_+},\Omega_{\omega_+}\}$ (one is four-dimensional and the other is two-dimensional). The reason for this is that, even though the representation $\{\mathfrak{H}_\omega,\pi_\omega,\Omega_\omega\}$ produces the correct vector state in the limit $t\to 1$, the vector $\Omega_\omega=\Omega_{\omega_+}\oplus 0\Omega_{\omega_-}$ is not cyclic in this limit, and so the conditions of Theorem 1.25 do not hold.

The representations $\{\mathfrak{H}_\omega,\pi_\omega\}$ and $\{\mathfrak{H}_{\omega_+},\pi_{\omega_+}\}$ are both faithful and so they are physically equivalent (Ker π_ω =Ker π_{ω_+} =0). Every state ϕ over $\mathfrak{A}$ is a state over $\pi_\omega(\mathfrak{A})$ and $\pi_{\omega_+}(\mathfrak{A})$ by the definitions $\phi_\omega(\pi_\omega(A))\equiv\phi(A)$ and $\phi_{\omega_+}(\pi_{\omega_+}(A))\equiv\phi(A)$. The set $\pi_{\omega_+}(\mathfrak{A})$ consists of all two by two matrices with complex entries. This set is irreducible to that $\pi_{\omega_+}(\mathfrak{A})'$ consists of only multiples of the two by two unit matrix. The bi-commutant $\pi_{\omega_+}(\mathfrak{A})''$ is then equal to the set of all two by two matrices with complex entries so $\pi_{\omega_+}(\mathfrak{A})=\pi_{\omega_+}(\mathfrak{A})''$, and the set $\pi_{\omega_+}(\mathfrak{A})$ is therefore a von Neumann algebra. The set $\pi_\omega(\mathfrak{A})$ is also equal to its bi-commutant $\pi_\omega(\mathfrak{A})''$ and therefore a von Neumann algebra. To see this we note that every element of $\pi_\omega(\mathfrak{A})$ is of the form $\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$ with M an arbitrary complex two by two matrix. Writing an arbitrary complex four by four matrix as $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where A, B, C, and D are complex two by two matrices, gives

$$\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}=\begin{pmatrix} MA & MB \\ MC & MD \end{pmatrix},$$

and
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}=\begin{pmatrix} AM & BM \\ CM & DM \end{pmatrix}.$$

So an arbitrary matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ will commute with every $\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$ if and only if each of the two by two matrices A, B, C, and D commute with every two by two matrix M. As we have seen above the set of complex two by two matrices are irreducible so that the matrices A, B, C, and D must each be a multiple of the two by two unit matrix I. So the commutant is $\pi_\omega(\mathfrak{A})'=\left\{\begin{pmatrix} \alpha I & \beta I \\ \gamma I & \delta I \end{pmatrix}:\alpha,\beta,\gamma,\delta\in\mathbb{C}\right\}$. We now require an arbitrary complex four by four matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ to commute with all members of $\pi_\omega(\mathfrak{A})'$. We have

$$\begin{pmatrix} \alpha I & \beta I \\ \gamma I & \delta I \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix}=\begin{pmatrix} \alpha A+\beta C & \alpha B+\beta D \\ \gamma A+\delta C & \gamma B+\delta D \end{pmatrix},$$

and
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \alpha I & \beta I \\ \gamma I & \delta I \end{pmatrix}=\begin{pmatrix} \alpha A+\gamma B & \beta A+\delta B \\ \alpha C+\gamma D & \beta C+\gamma D \end{pmatrix}.$$

So the two matrices will commute if and only if B=C=0 and A=D. The bi-commutant is then $\pi_\omega(\mathfrak{A})''=\left\{\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}\right\}=\pi_\omega(\mathfrak{A})$ and the set $\pi_\omega(\mathfrak{A})$ is therefore a von Neumann algebra. Now every element of

$\pi_{\omega_+}(\mathfrak{A})=\pi_{\omega_+}(\mathfrak{A})''$ or $\pi_\omega(\mathfrak{A})=\pi_\omega(\mathfrak{A})''$ has finite trace and is therefore a density matrix. Assume that the state ϕ over $\mathfrak{A}$ is a π_{ω_+}-normal state, i.e., there exists a density matrix $\rho\in\pi_{\omega_+}(\mathfrak{A})$ such that $\phi(A)=\frac{\mathrm{Tr}\rho\pi_{\omega_+}(A)}{\mathrm{Tr}\rho}$ for all $A\in\mathfrak{A}$. The element $\rho\oplus\rho\in\pi_\omega(\mathfrak{A})$ is a density matrix and

$$\frac{\mathrm{Tr}\,(\rho\oplus\rho)\pi_\omega(A)}{\mathrm{Tr}(\rho\oplus\rho)}=\frac{\mathrm{Tr}\,(\rho\oplus\rho)(\pi_{\omega_+}(A)\oplus\pi_{\omega_+}(A))}{\mathrm{Tr}(\rho\oplus\rho)}$$

$$=\frac{2\mathrm{Tr}\rho\pi_{\omega_+}(A)}{2\mathrm{Tr}\rho}$$

$$=\phi(A)$$

for all $A\in\mathfrak{A}$. The state ϕ is therefore π_ω-normal. In a similar fashion we can show that every π_{ω_+}-normal state is a π_ω-normal state so the two representations are quasi-equivalent. We could have reached this conclusion by observing that the sets $\pi_{\omega_+}(\mathfrak{A})''=\pi_{\omega_+}(\mathfrak{A})$ and $\pi_\omega(\mathfrak{A})''=\pi_\omega(\mathfrak{A})$ are *-isomorphic.

For completeness we briefly review how this system is treated in the Traditional Approach. The Hilbert space corresponding to a single spin ($s=1\backslash 2$) system is a two-dimensional complex space, $\mathbb{C}^2$. The observables of the system correspond to the set of real two by two matrices, which are linear transformations on $\mathbb{C}^2$. This set is generated by the Pauli spin matrices s_1, s_2, s_3 and the identity I ($\equiv s_0$). If we work in a basis that diagonalizes s_3 and $s^2=s_1^2+s_2^2+s_3^2$ then the Pauli spin matrices are the same as the matrices representing the elements $\sigma_\mu\in\mathfrak{A}$ in the G.N.S. representation for the states $\omega_\pm$ ($s_\mu=\pi_{\omega\pm}(\sigma_\mu)$, etc.). A general element of this set is a real linear combination of the Pauli spin matrices, $a=x^{\mu}s_{\mu}$.

If we now place the system in a uniform magnetic field in the 3-direction the Hamiltonian is $H=-Bs_3$ and the canonical density matrix is $\rho=\frac{e^{\beta Bs_3}}{Tr(e^{\beta Bs_3})}$. We will denote the state that ρ produces ($\langle a\rangle=Tr(a\rho)$) by the same symbol ρ. The density matrix ρ is now an operator on the concrete Hilbert space $\mathbb{C}^2$. Working with the orthonormal basis $e_1=\begin{pmatrix}1\\0\end{pmatrix}$ and $e_2=\begin{pmatrix}0\\1\end{pmatrix}$ (which are the normalized eigenvectors of s^2 and s_3) we have

$$e^{\beta Bs_3}=e^{\beta B\begin{pmatrix}1&0\\0&-1\end{pmatrix}}$$

$$=\cosh\beta B\begin{pmatrix}1&0\\0&1\end{pmatrix}+\sinh\beta B\begin{pmatrix}1&0\\0&-1\end{pmatrix},$$

and

$$Tr\, e^{\beta Bs_3}=2\cosh\beta B\,,$$

therefore

$$\rho=\frac{1}{2}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{t}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

$$=\frac{1+t}{2}\begin{pmatrix}1&0\\0&0\end{pmatrix}+\frac{1-t}{2}\begin{pmatrix}0&0\\0&1\end{pmatrix}.$$

The matrix $\begin{pmatrix}1&0\\0&0\end{pmatrix}$ is a projection operator onto the vector e_1 while the matrix $\begin{pmatrix}0&0\\0&1\end{pmatrix}$ is a projection operator onto the vector e_2. The decomposition of ρ into projection operators allows us to express the expectation value of an operator $a=x^{\mu}s_{\mu}$ as a linear combination of vector states,

$$\mathrm{Tr}\,(a\rho)=(e_1,a\rho e_1)+(e_2,a\rho e_2)$$

$$=\frac{1+t}{2}(e_1,ae_1)+\frac{1-t}{2}(e_2,ae_2).$$

It is now easy to see that the state ρ is pure only in the zero temperature case (t=1), in which case it is the vector state (e_1,ae_1).

In the Traditional Approach one is working in the unique irreducible representation of the C*-algebra $\mathfrak{A}$. This is the same representation that arises, via the G.N.S. construction, from the pure states $\omega_{\pm}$. The mathematical structure of the Traditional and Algebraic Approaches is therefore the same in the zero temperature case. For finite temperatures the mathematical structure is different for the two approaches. In the Traditional Approach one is still working in the same irreducible representation. The state ρ is a linear combination of vector states in this representation. In the Algebraic Approach the state ρ is a vector state in a reducible representation. This reducible representation is a direct sum of the above irreducible representation with itself.

Despite the mathematical differences between the two approaches they give the same physical predictions (i.e., the expectation values of observables and the probability to observe a given eigenvalue of an observable are the same). We illustrate this by calculating the probability that the value 1 will be obtained when the 3-component of the spin is measured when the system in a finite temperature state. In the Traditional Approach the eigenvector of s_3 corresponding to the eigenvalue 1 is e_1. The probability of finding the value 1 when measuring the 3-component is then the expectation value of the projection operator onto e_1, $P_{e_1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,

$$\mathrm{Prob}(s_3=1)=\mathrm{Tr}(P_{e_1}\rho)=\frac{1+t}{2}(e_1,P_{e_1}e_1)+\frac{1-t}{2}(e_2,P_{e_1}e_2)$$

$$=\frac{1+t}{2}$$

In the Algebraic Approach the eigenvalue 1 of $\pi_\omega(\sigma_3)$ is doubly degenerate, with corresponding eigenvectors

$$\Psi_1=\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \text{and } \Psi_2=\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

The probability to find the value 1 when measuring the 3-component of the spin is then the expectation value of the projection operator onto the subspace spanned by Ψ_1 and Ψ_2,

$$P_{\{\Psi_1,\Psi_2\}}=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(this corresponds to the direct sum $P_{e_1}\oplus P_{e_1}$). We have

$$\mathrm{Prob}(\pi_\omega(\sigma_3))=(\Omega_\omega, P_{\{\Psi_1,\Psi_2\}}\, \Omega_\omega)$$

$$=(u,0,0,v)\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}u\\0\\0\\v\end{pmatrix}$$

$$=u^2=\frac{1+t}{2}.$$

2.2 MANY SPIN SYSTEMS

The algebraic description of systems which consist of arrays of spins will now be considered. We will begin with a system that contains two spins and then generalize first to a finite array of spins, and then to an infinite array of spins.

The first step is to construct the appropriate C*-algebra. To each spin corresponds a copy of the C*-algebra $\mathfrak{A}$. The C*-algebra corresponding to the entire system is the direct product $\mathfrak{A}^2 \equiv \mathfrak{A} \otimes \mathfrak{A}$. A general element of $\mathfrak{A}^2$ is of the form $A=\alpha^{\mu\nu}\sigma_\mu \otimes \sigma_\nu$.

Let $A=\alpha^{\mu\nu}\sigma_\mu \otimes \sigma_\nu$ and $B=\beta^{\eta\kappa}\sigma_\eta \otimes \sigma_\kappa$, then the following operations are defined in $\mathfrak{A}^2$:

i) $A+B=(\alpha^{\mu\nu}+\beta^{\mu\nu})\sigma_\mu\otimes\sigma_\nu$,

ii) $AB=\alpha^{\mu\nu}\beta^{\eta\kappa}\,\sigma_\mu\sigma_\eta\otimes\sigma_\nu\sigma_\kappa$,

iii) $A^*=\bar{\alpha}^{\mu\nu}\sigma_\mu\otimes\sigma_\nu$.

So $\mathfrak{A}^2$ is a $*$-algebra. To show that $\mathfrak{A}^2$ is a C*-algebra we must demonstrate the existence of a C*-norm with respect to which $\mathfrak{A}^2$ is complete. To do this we turn to the faithful two-dimensional representation $\{\mathfrak{H}_{\omega_+},\pi_{\omega_+}\}$ of $\mathfrak{A}$ and define the norm of a general element of $\mathfrak{A}$ as

$$\|\alpha^{\mu\nu}\sigma_\mu\otimes\sigma_\nu\|\equiv\sup\{\|\alpha^{\mu\nu}\pi_{\omega_+}(\sigma_\mu)\otimes\pi_{\omega_+}(\sigma_\nu)\underline{\lambda}\|:\underline{\lambda}\in\mathfrak{H}_{\omega_+}\otimes\mathfrak{H}_{\omega_+};\|\underline{\lambda}\|=1\} \quad (2.20)$$

The representation $\{\mathfrak{H},\pi\}$,with $\mathfrak{H}\equiv\mathfrak{H}_{\omega_+}\otimes\mathfrak{H}_{\omega_+}$ and

$$\pi(\alpha^{\mu\nu}\sigma_\mu\otimes\sigma_\nu)\equiv\alpha^{\mu\nu}\pi_{\omega_+}(\sigma_\mu)\otimes\pi_{\omega_+}(\sigma_\nu))$$

is faithful so we know that (2.20) defines a C*-norm for $\mathfrak{A}^2$. It is also possible to show that the sequence $\{\alpha_n^{\mu\nu}\sigma_\mu\otimes\sigma_\nu\}$ is Cauchy (with respect to the above norm) if and only if the sequences $\{\alpha_n^{\mu\nu}\}$ of complex numbers are Cauchy for all $\mu,\nu=0,1,2,3$. Thus the sequence $\{\alpha_n^{\mu\nu}\sigma_\mu\otimes\sigma_\nu\}$ converges to $\alpha^{\mu\nu}\sigma_\mu\otimes\sigma_\nu$, where $\alpha^{\mu\nu}=\lim_{n\to\infty}\alpha_n^{\mu\nu}=\alpha^{\mu\nu}$. $\mathfrak{A}^2$ is therefore complete with respect to the above norm, and hence $\mathfrak{A}^2$ is a C*-algebra.

Place the system in a uniform magnetic field in the 3-direction. The Hamiltonian H^2 and canonical density matrix ρ^2 are then

$$H^2=-B(\sigma_o\otimes\sigma_3+\sigma_3\otimes\sigma_o),$$

and

$$\rho^2=\frac{e^{\beta B(\sigma_o\otimes\sigma_3+\sigma_3\otimes\sigma_o)}}{Tre^{\beta B(\sigma_o\otimes\sigma_3+\sigma_3\otimes\sigma_o)}}$$

$$=\frac{\cosh^2\beta B\sigma_o\otimes\sigma_o+\sinh^2\beta B\sigma_3\otimes\sigma_3+\sinh\beta B\cosh\beta B(\sigma_o\otimes\sigma_3+\sigma_3\otimes\sigma_o)}{Tr[\cosh^2\beta B\sigma_o\otimes\sigma_o+\sinh^2\beta B\sigma_3\otimes\sigma_3+\sinh\beta B\cosh\beta B(\sigma_o\otimes\sigma_3+\sigma_3\otimes\sigma_o)]}$$

$$=\frac{(\cosh\beta B\sigma_o+\sinh\beta B\sigma_3)\otimes(\cosh\beta B\sigma_o+\sinh\beta B\sigma_3)}{Tr[\cosh\beta B\sigma_o+\sinh\beta B\sigma_3]\;Tr[\cosh\beta B\sigma_o+\sinh\beta B\sigma_3]}$$

$$=\frac{e^{\beta B\sigma_3}\otimes e^{\beta B\sigma_3}}{Tre^{\beta B\sigma_3}Tre^{\beta B\sigma_3}}$$

$$=\rho\otimes\rho, \tag{2.21}$$

where ρ is the canonical density matrix for the single spin system. ρ^2 gives rise to the state ω^2 over $\mathfrak{A}^2$,

$$\omega^2(\alpha^{\mu\nu}\sigma_\mu\otimes\sigma_\nu)=Tr[\alpha^{\mu\nu}\sigma_\mu\otimes\sigma_\nu\,\rho^2]=\alpha^{\mu\nu}Tr[\sigma_\mu\rho]Tr[\sigma_\nu\rho]$$

$$=\alpha^{\mu\nu}\omega(\sigma_\mu)\omega(\sigma_\nu)$$

$$=\alpha^{oo}+t(\alpha^{o3}+\alpha^{3o})+t^2\alpha^{33},\quad t=\omega(\sigma_3). \tag{2.22}$$

We can then write the state ω^2 as $\omega^2=\omega\otimes\omega$ where $\omega\otimes\omega(\alpha^{\mu\nu}\sigma_\mu\otimes\sigma_\nu)\equiv\alpha^{\mu\nu}\omega(\sigma_\mu)\omega(\sigma_\nu)$. We have referred to ω^2 as a state but this really must be shown. We need to demonstrate that ω^2 is a positive linear functional over $\mathfrak{A}^2$ with unit norm. It is obviously a linear functional over $\mathfrak{A}^2$. The following calculation shows that it is positive. Consider an arbitrary element $A=\alpha^{\mu\nu}\sigma_\mu\otimes\sigma_\nu\equiv B^\nu\otimes\sigma_\nu$, where $B^\nu\equiv\alpha^{\mu\nu}\sigma_\mu$, then

$$\omega^2(A^*A)=\omega^2(B^{\mu*}B^\nu\otimes\sigma_\mu\sigma_\nu)$$

$$=\omega(B^{\mu*}B^\nu\omega(\sigma_\mu\sigma_\nu)$$

$$=\omega([B^{o*}B^o+tB^{o*}B^3+tB^{3*}B^o+B^{3*}B^3]+[B^{1*}B^1+itB^{1*}B^2\ itB^{2*}B^1+B^{2*}B^2])$$

$$=\omega([B^o+tB^3][B^o+tB^3]^*+(1-t^2)B^{3*}B^3+[B^1+itB^2][B^1+itB^2]^*+(1-t^2)B^{2*}B^2)$$

$$=\omega([B^o+tB^3][B^o+tB^3]^*)+(1-t^2)\omega(B^{3*}B^3)$$

$$+\omega([B^1+itB^2][B^1+itB^2]^*)+(1-t^2)\omega(B^{2*}B^2). \tag{2.23}$$

Now ω is a state over $\mathfrak{A}$ and $(1-t^2)\geq 0$ so each of the above terms is non-negative; ω^2 is therefore positive. Since ω^2 is positive, its norm is given by its value on the identity, $\|\omega^2\|=\omega^2(\sigma_o\otimes\sigma_o)=1$. So ω^2 is normalized and therefore a state over $\mathfrak{A}^2$.

We now consider the G.N.S. representation of $\mathfrak{A}^2$ associated with the state ω^2, for different temperatures. Instead of constructing these representations from the G.N.S. prescription, as was done in the single spin case, we will postulate a cyclic representation that produces the correct vector state over $\mathfrak{A}^2$, and use Theorem 1.27 to conclude that this representation is unitarily equivalent to the G.N.S. representation.

First consider the case $t\neq 1$ (i.e., the finite temperature case). The representation $\{\mathfrak{H}_\omega\otimes\mathfrak{H}_\omega,\pi_\omega\otimes\pi_\omega,\Omega_\omega\otimes\Omega_\omega\}$ produces the correct vector state over $\mathfrak{A}^2$,

$$(\Omega_\omega\otimes\Omega_\omega,\alpha^{\mu\nu}\pi_\omega(\sigma_\mu)\otimes\pi_\omega(\sigma_\nu)\Omega_\omega\otimes\Omega_\omega)=\alpha^{\mu\nu}(\Omega_\omega,\pi_\omega(\sigma_\mu)\Omega_\omega)(\Omega_\omega,\pi_\omega(\sigma_\nu)\Omega_\omega)$$

$$=\alpha^{\mu\nu}\omega(\sigma_\mu)\omega(\sigma_\nu)$$

$$=\omega^2(\alpha^{\mu\nu}\sigma_\mu\otimes\sigma_\nu). \qquad (2.24)$$

Since Ω_ω is cyclic for $\mathfrak{H}_\omega$, $\Omega_\omega\otimes\Omega_\omega$ is cyclic for $\mathfrak{H}_\omega\otimes\mathfrak{H}_\omega$. So the representation $\{\mathfrak{H}_\omega\otimes\mathfrak{H}_\omega,\pi_\omega\otimes\pi_\omega,\Omega_\omega\otimes\Omega_\omega\}$ is unitarily equivalent to the G.N.S. representation $\{\mathfrak{H}_{\omega 2},\pi_{\omega 2},\Omega_{\omega 2}\}$. This representation is obviously reducible, as it is the direct product of two reducible representations.

For the zero temperature case, $t=1$, the representation $\{\mathfrak{H}_{\omega_+}\otimes\mathfrak{H}_{\omega_+},\pi_{\omega_+}\otimes\pi_{\omega_+},\Omega_{\omega_+}\otimes\Omega_{\omega_+}\}$ is unitarily equivalent to the G.N.S. representation $\{\mathfrak{H}_{\omega_+^2},\pi_{\omega_+^2},\Omega_{\omega_+^2})$ corresponding to the state $\omega_+^2(\alpha^{\mu\nu}\sigma_\mu\otimes\sigma_\nu)=\alpha^{oo}+\alpha^{o3}+\alpha^{3o}+\alpha^{33}$. We can use Schur's Lemma to show

that this representation is irreducible if we can show that an arbitrary element $A=\alpha^{\mu\nu}\pi_{\omega_+}(\sigma_\mu)\otimes\pi_{\omega_+}(\sigma_\nu)$ that commutes with all elements of $\pi_{\omega_+}(\mathfrak{A})\otimes\pi_{\omega_+}(\mathfrak{A})$ is a multiple of the identity $\pi_{\omega_+}(\sigma_o)\otimes\pi_{\omega_+}(\sigma_o)$. We can write such an element as $A=\pi_{\omega_+}(B^\nu)\otimes\pi_{\omega_+}(\sigma_\nu)$ (with $B^\nu=\alpha^{\mu\nu}\sigma_\mu$) . We now require A to commute with elements of the form $\pi_{\omega_+}(A^1)\otimes\pi_{\omega_+}(\sigma_o)$. Since the set $\pi_{\omega_+}(\mathfrak{A})$ is irreducible this is only possible if B_ν is a multiple of the identity σ_o, $B^\nu=\alpha^{\mu\nu}\sigma_\mu=\beta^\nu\sigma_o$. This means the coefficients $\alpha^{\mu\nu}$ must be of the form $\alpha^{\mu\nu}=\delta^{\mu o}\beta^\nu$, and $A=\beta^\nu\pi_{\omega_+}(\sigma_o)\otimes\pi_{\omega_+}(\sigma_\nu)$. If we also require A to commute with elements of the form $\pi_{\omega_+}(\sigma_o)\otimes\pi_{\omega_+}(A^2)$ we can conclude that $\beta^\nu=\beta\delta^{\nu o}$, and therefore $A=\beta\pi_{\omega_+}(\sigma_o)\otimes\pi_{\omega_+}(\sigma_o)$. This shows that the representation is irreducible.

We now generalize to a system consisting of a finite number of spins. Since the spins do not interact with one another, the geometry of the system is not important. Denote the spins by the parameter i, whose range is Z (=set of integers). The C*-algebra corresponding to the spin at the site i is a copy of $\mathfrak{A}$, which we will denote by $\mathfrak{A}_i$. The C*-algebra corresponding to a finite collection of spins $\zeta=[k,k+1,\ldots,l-1,l\}$ $(l>k)$ is then $\mathfrak{A}_\zeta\equiv\bigotimes_{i\in\zeta}\mathfrak{A}_i$. A general element of $\mathfrak{A}_\zeta$ is of the form

$$A=\alpha^{\mu_k\mu_{k+1}\cdots\mu_l}\ \sigma^k_{\mu_k}\otimes\sigma^{k+1}_{\mu_{k+1}}\otimes\ldots\sigma^l_{\mu_l}\ ,\quad l-k+1=\text{number of sites in }\zeta. \tag{2.25}$$

As in the two-spin case, we define a norm on $\mathfrak{A}_\zeta$ using the faithful two- dimensional representation $\{\mathfrak{H}_{\omega_+},\pi_{\omega_+}\}$ of $\mathfrak{A}$,

$$\|\alpha^{\mu_k\mu_{k+1}\cdots\mu_l}\ \sigma^k_{\mu_k}\otimes\sigma^{k+1}_{\mu_{k+1}}\otimes\ldots\sigma^l_{\mu_l}\|$$

$$\equiv\sup\left\{\|\alpha^{\mu_l\mu_{l+1}\cdots\mu_k}\pi_{\omega_+}(\sigma^k_{\mu_k})\otimes\pi_{\omega_+}(\sigma_{\mu_{k+1}})\otimes\ldots\pi_{\omega_+}(\sigma^l_{\mu_l})\,\lambda\|:\lambda\in\bigotimes_{i\in\zeta}\mathfrak{H}_{\omega_+};\|\lambda\|=1\}\right\} \tag{2.26}$$

The representation $\{\bigotimes_{i\in\zeta}\mathfrak{H}_{\omega_+},\bigotimes_{i\in\zeta}\pi_{\omega_+}\}$ is faithful so that (2.26) is a C*-norm for $\mathfrak{A}_\zeta$. In a similar fashion to the two-spin case we may conclude that $\mathfrak{A}_\zeta$ is complete with respect to this norm and therefore a C*-algebra.

If we place this system in a uniform magnetic field in the 3-direction then a generalization of the arguments used in the two-spin case shows that the canonical density matrix for this system is $\rho_\zeta \equiv \underset{i\in\zeta}{\otimes} \rho_i$, where ρ_i is the canonical density matrix for the single spin at the site i. This canonical density matrix gives rise to the state (at natural temperature β) $\omega_\zeta \equiv \underset{i\in\zeta}{\otimes} \omega_i$, where ω_i is the single spin state generated by ρ_i (at natural temperature β). Again we must show that ω_ζ defines a state over $\mathfrak{A}_\zeta$. Since ω_ζ is linear and $\omega_\zeta(\underset{i\in\zeta}{\otimes}\sigma_o)=1$ we only need to show that ω_ζ is positive. We will prove that the linear functional $\omega_n \equiv \overset{n}{\underset{i=1}{\otimes}} \omega_i$ over $\mathfrak{A}_n \equiv \overset{n}{\underset{i=1}{\otimes}} \mathfrak{A}_i$ is positive for arbitrary n by induction ($\mathfrak{A}_i$ and ω_i are copies of the single spin algebra and state). For n=1 we are in the single spin case and the linear functional is positive. We now prove that it is positive for n+1 if it is positive for n by following the method used to show that the two spin state ω^2 is positive. An arbitrary element of $A \in \mathfrak{A}_{n+1}$ can be written as $A=B^\nu \otimes \sigma_\nu$ with each $B^\nu \in \mathfrak{A}_n$. As in the two spin case, we can show that

$$\omega_{n+1}(A^*A)=\omega_{n+1}(B^{\mu *}B^\nu \otimes \sigma_\mu \sigma_\nu)$$

$$=\omega_n(B^{\mu *}B^\nu)\omega(\sigma_\mu\sigma_\nu)$$

$$=\omega_n([B^o+tB^3][B^o+tB^3]^*)+(1-t^2)\omega_n(B^{3*}B^3)$$

$$+\omega_n([B^1+tB^2][B^1+itB^2]^*)+(1-t^2)\omega_n(B^{2*}B^2)$$

$$\geq 0, \tag{2.27}$$

where the last inequality follows from the assumption that ω_n is positive. So ω_n is positive for arbitrary n by induction, and ω_ζ is therefore a state over $\mathfrak{A}_\zeta$.

The G.N.S. representation corresponding to ω_ζ is unitarily equivalent to the representation $\{\mathfrak{H},\pi,\Omega\}$ with

$$\mathfrak{H} \equiv \bigotimes_{i\in\zeta} \mathfrak{H}_i\,,\ \pi \equiv \bigotimes_{i\in\zeta} \pi_i\,,\ \text{and}\ \Omega \equiv \bigotimes_{i\in\zeta} \Omega_i\,, \tag{2.28}$$

where $\{\mathfrak{H}_i, \pi_i, \Omega_i\}$ is the G.N.S. representation corresponding to the single spin state ω_i. At this point we should mention that the dependence of the representation on temperature is still entirely contained in the cyclic vector Ω. In particular, representations $\{\mathfrak{H},\pi\}$ corresponding to different finite temperatures are still unitarily equivalent.

We now construct the quasi-local C*-algebra corresponding to the infinite system. Let Σ denote the set of all finite collections ζ of sites i in Z, and equip Σ with the ordering of set theoretic inclusion. This ordering is a partial ordering. Also, for any pair of elements ζ_1 and ζ_2 in Σ there exists a ζ_3 in Σ such that $\zeta_1 \leq \zeta_3$ and $\zeta_2 \leq \zeta_3$, so that the set Σ is a directed set. Now consider the family $\{\mathfrak{A}_\zeta : \zeta \in \Sigma\}$ of C*-algebras. The elements of any pair ζ_1 ,ζ_2 satisfying $\zeta_1 \leq \zeta_2$ (with ζ_1 containing n sites, ζ_2 containing m sites and $n \leq m$) are of the form

$$A_1 = \alpha^{\mu_k \mu_{k+1} \cdots \mu_l}\ \sigma^k_{\mu_k} \otimes \sigma^{k+1}_{\mu_{k+1}} \otimes \ldots \sigma^l_{\mu_l} \quad \text{with } n = l - k + 1,$$

and $$A_2 = \alpha^{\mu_p \mu_{p+1} \cdots \mu_q}\ \sigma^p_{\mu_p} \otimes \sigma^{p+1}_{\mu_{p+1}} \otimes \ldots \sigma^q_{\mu_q} \quad \text{with } m = q - p + 1,\ q \geq l, \text{ and } p \leq k. \tag{2.29}$$

Let $i_{2,1}$ denote the mapping from $\mathfrak{A}_{\zeta_1}$ into $\mathfrak{A}_{\zeta_2}$ defined as

$$i_{2,1}(\alpha^{\mu_k\mu_{k+1}\cdots\mu_l}\ \sigma^k_{\mu_k}\otimes\sigma^{k+1}_{\mu_{k+1}}\otimes\ldots\sigma^l_{\mu_l})$$

$$\equiv\bigotimes_{i=p}^{k}(\sigma^i_o)\otimes[\alpha^{\mu_k\mu_{k+1}\cdots\mu_l}\ \sigma^k_{\mu_k}\otimes\sigma^{k+1}_{\mu_{k+1}}\otimes\ldots\sigma^l_{\mu_l}\ )]\bigotimes_{i=l}^{q}(\sigma^i_o), \tag{2.30}$$

The mapping $i_{2,1}$ is a *-homomorphism from $\mathfrak{A}_{\zeta_1}$ into $\mathfrak{A}_{\zeta_2}$ with the properties

i) $i_{2,1}(\mathbb{1}_1)=\mathbb{1}_2$, where $\mathbb{1}_1$ and $\mathbb{1}_2$ are the identities of $\mathfrak{A}_{\zeta_1}$ and $\mathfrak{A}_{\zeta_2}$, respectively,

and ii) $i_{3,1}=i_{3,2}i_{2,1}$ whenever $\zeta_1\leq\zeta_2\leq\zeta_3$.

The above mapping shows that the family of C*-algebras $\{\mathfrak{A}_\zeta:\zeta\in\Sigma\}$ satisfies the postulate of isotony and therefore admit a C*-inductive limit (see section 1.5). Recall that this is a C*-algebra $\mathfrak{A}^\infty$ with identity $\mathbb{1}^\infty$ that has the property that for every $\zeta\in\Sigma$ there exists a *-homomorphism i_ζ from $\mathfrak{A}_\zeta$ into $\mathfrak{A}^\infty$ that satisfies

i) $i_\zeta(\mathbb{1}_\zeta)=\mathbb{1}^\infty$, where $\mathbb{1}_\zeta$ is the identity for $\mathfrak{A}_\zeta$,

ii) $i_{\zeta_2}(\mathfrak{A}_{\zeta_2})\supset i_{\zeta_1}(\mathfrak{A}_{\zeta_1})$, whenever $\zeta_2\geq\zeta_1$,

and iii) $\overline{\bigcup_{\zeta\in\Sigma} i_\zeta(\mathfrak{A}_\zeta)} = \mathfrak{A}^\infty$, where the bar denotes the uniform closure.

This mapping is given by $i_\zeta(A)\equiv(\bigotimes_{i<k}\sigma^i_o)\otimes A\otimes(\bigotimes_{i>l}\sigma^i_o)$ for all $A\in\mathfrak{A}_\zeta$, where $\zeta=[k,l]$. We will denote this C*-inductive limit by $\bigotimes_{i\in Z}\mathfrak{A}_i$ and refer to it as the infinite direct product of the C*-algebras $\{\mathfrak{A}_\zeta:\zeta\in\Sigma\}$. It is the quasi-local algebra corresponding to our infinite lattice of spins. An arbitrary element of $\bigotimes_{i\in Z} \mathfrak{A}_i$ can be approximated, in the norm, to any degree by linear combinations of elements of the form $\bigotimes_{i\in Z} A_i$, where $A_i\in \mathfrak{A}_i$ and all but a finite number of the A_i are equal to σ_0^i (this is the content of condition (iii) above). Given an element A of $\bigotimes_{i\in Z} \mathfrak{A}_i$, we define its projection onto $i_\zeta(\mathfrak{A}_\zeta)$, $\zeta=[k,l]$, as the element one obtains by replacing all of its components before the k^{th} spot and beyond the l^{th} spot with the identity σ_0, and denote it by $P_\zeta(A)$. We will refer to the element of $\mathfrak{A}_\zeta$ that is mapped, via i_ζ, into $P_\zeta(A)$ as the projection of A onto $\mathfrak{A}_\zeta$. This element will be denoted by A_ζ.

The canonical equilibrium state for the infinite system of spins will now be constructed following the procedure outlined in chapter 1, section 1.5. Define the subset Σ_0 of Σ as $\Sigma_0\equiv\{\zeta_n\in\Sigma:\zeta_n=[-n,-n+1,\dots,0,\dots,n-1,n]\}$. An arbitrary element $\zeta=[k,k+1,\dots,l]$ in Σ is contained in each ζ_n for all $n\geq\max\{|k|, |l|\}$. For example, $[-9,-8,\dots,4,5]$ is contained in each ζ_n for all $n\geq 9$. For arbitrary $A\in \mathfrak{A}_{\zeta_n}$, the canonical equilibrium state ω_{ζ_n} over $\mathfrak{A}_{\zeta_n}$ is given by

$$\omega_{\zeta_n}(A)=\left[\bigotimes_{i=-n}^{n} \omega_i\right](A). \tag{2.31}$$

Let A be an arbitrary element of $\bigotimes_{i\in Z} \mathfrak{A}_i$ and let A_{ζ_n} denote its projection onto $\mathfrak{A}_{\zeta_n}$. Note that $\lim_{n\to\infty} i_\zeta(A_{\zeta_n})=A$. We then define the equilibrium state ω^∞ for the infinite spin system as

$$\omega^\infty(A)\equiv \lim_{n\to\infty} \omega_{\zeta_n}(A_{\zeta_n}) \text{ , for all } A\in \bigotimes_{i\in Z} \mathfrak{A}_i . \tag{2.32}$$

In order to demonstrate that this limit is well defined we must show that ω^∞ is bounded with unit norm. This follows from the fact that $\omega^\infty(\mathbb{1}^\infty)=1$

where $\mathbb{1}^\infty$ is the identity for $\underset{i\in Z}{\otimes}\mathfrak{A}_i$. To show this we note that the projection of $\mathbb{1}^\infty$ onto $\mathfrak{A}_{\zeta_n}$ is the identity $\mathbb{1}_{\zeta_n}$ for $\mathfrak{A}_{\zeta_n}$, so that $\|\omega^\infty\|=\omega^\infty(\mathbb{1}^\infty)=\lim_{n\to\infty}\omega_{\zeta_n}(\mathbb{1}_{\zeta_n})=1$. So ω^∞ is normalized to unity. This of course implies that ω^∞ is bounded. Explicitly we have

$$1=\|\omega^\infty\|=\mathrm{Sup}\left\{\frac{|\omega^\infty(A)|}{\|A\|}:A\in\underset{i\in Z}{\otimes}\mathfrak{A}_i,\ A\neq 0\right\}\Rightarrow|\omega^\infty(A)|\leq\|A\|\text{ for all }A\in\underset{i\in Z}{\otimes}\mathfrak{A}_i.$$

We have proven by induction that each ω_{ζ_n} is a positive linear functional over $\mathfrak{A}_{\zeta_n}$ for arbitrary n. ω^∞ is therefore a positive linear functional with unit norm over $\underset{i\in Z}{\otimes}\mathfrak{A}_i$, i.e., ω^∞ is a state over $\underset{i\in Z}{\otimes}\mathfrak{A}_i$.

For example, consider the element $\rho(E)$ in $\underset{i\in Z}{\otimes}\mathfrak{A}_i$ corresponding to the average energy per site. The projection of $\rho(E)$ onto $\mathfrak{A}_{\zeta_n}$ is given by

$$\rho_{\zeta_n}(E)=-\frac{B}{2n+1}\sum_{i=-n}^{n}[\sigma_o^{-n}\otimes\sigma_o^{-n+1}\otimes\ldots\sigma_3^{i}\otimes\ldots\sigma_o^{n}], \tag{2.33}$$

where σ_o is replaced by σ_3 in the i^{th} spot. This gives

$$\begin{aligned}\omega^\infty(\rho(E))&=\lim_{n\to\infty}\omega_{\zeta_n}(\rho_{\zeta_n}(E))\\ &=\lim_{n\to\infty}\left[\underset{i=-n}{\overset{n}{\otimes}}\omega_i\right]\left(-\frac{B}{2n+1}\sum_{i=-n}^{n}[\sigma_o^{-n}\otimes\sigma_o^{-n+1}\otimes\ldots\sigma_3^{i}\otimes\ldots\sigma_o^{n}]\right)\\ &=\lim_{n\to\infty}\left(-\frac{B}{2n+1}\sum_{i=-n}^{n}[\omega_{-n}(\sigma_o^{-n})\omega_{-n+1}(\sigma_o^{-n+1})\ldots\omega_i(\sigma_3^{i})\ldots\omega_n(\sigma_o^{n})]\right)\\ &=\lim_{n\to\infty}\left(-\frac{B}{2n+1}\sum_{i=-n}^{n}t\right)=-Bt=-B\tanh\beta B.\end{aligned} \tag{2.34}$$

Note that the expectation value of $\rho(E)$ in the state ω^∞ is the same as the expectation value of the single spin Hamiltonian in the single spin state ω. This is to be expected since the spins are non-interacting.

We will now construct a cyclic representation $\{\mathfrak{H},\pi,\Omega\}$ of $\bigotimes_{i\in Z}\mathfrak{A}_i$ with the property $(\Omega,\pi(A)\Omega)=\omega(A)$ for all $A\in \bigotimes_{i\in Z}\mathfrak{A}_i$. This representation will therefore be unitarily equivalent to the G.N.S representation $\{\mathfrak{H}_\omega,\pi_\omega,\Omega_\omega\}$. This representation will be obtained as the infinite direct product of the single spin G.N.S. representations. The infinite direct product of a family of Hilbert spaces was originally defined by von Neumann [Neum1]. For our purposes it is more convenient to follow the definition used in [Emch], which can be shown to be equivalent to von Neumann's definition.

Consider a family of complex vector spaces $\{V_i: i\in Z\}$ (what follows is valid for an arbitrary directed index set Γ). Define the infinite direct product $\bigotimes_{i\in Z}V_i$ in analogy with the finite case. To every family $\{x_i: i\in Z; x_i\in V_i\}$ corresponds the element $\bigotimes_{i\in Z}x_i$ in $\bigotimes_{i\in Z}V_i$, and every element of $\bigotimes_{i\in Z}V_i$ is a linear combination of such elements. Let $a=\{a_i: i\in Z; a_i\in V_i; a_i\neq 0\}$ be an arbitrary family. Consider all vectors $\bigotimes_{i\in Z}x_i$ of $\bigotimes_{i\in Z}V_i$ that have $x_i=a_i$ for all but a finite number of $i\in Z$. Then define a subspace $\bigotimes_{i\in Z}^a V_i$ of $\bigotimes_{i\in Z}V_i$ that consists of all finite linear combinations of these vectors just defined.

We now consider the case where the vector spaces V_i are separable Hilbert spaces $\mathfrak{H}_i$. It is not possible to obtain a Hilbert space from the entire infinite direct product $\bigotimes_{i\in Z}\mathfrak{H}_i$. This follows since, by definition, the norm of a Hilbert space must be related to the scalar product in the usual way. The scalar product between two elements of $\otimes_{i\in Z}\mathfrak{H}_i$ is

$$\left(\otimes_{i\in Z} x_i \, , \, \otimes_{i\in Z} y_i \right) \equiv \prod_{i\in Z} (x_i, y_i), \tag{2.35}$$

and this infinite product may diverge. We can, however, define subsets of $\otimes_{i\in Z}\mathfrak{H}_i$ that are Hilbert spaces. Let $a = \{a_i : i\in Z;\ a_i\in \mathfrak{H}_i; \|a_i\|=1\}$ be an arbitrary family and define the subspace $\otimes_{i\in Z}{}^a H_i$ as above. We can define a scalar product on $\otimes_{i\in Z}{}^a H_i$ as

$$\left(\otimes_{i\in Z} x_i \, , \, \otimes_{i\in Z} y_i \right) \equiv \prod_{i\in Z} (x_i, y_i). \tag{2.36}$$

Since $x_i=a_i$ and $y_i=a_i$ for all but a finite number of $i\in Z$ the above is well defined (all but a finite number of terms in the product have the value one). At this point $\otimes_{i\in Z}{}^a H_i$ is a pre-Hilbert space which we can complete with respect to the norm obtained from this scalar product. We will denote this Hilbert space by the $\otimes_{i\in Z}{}^a\mathfrak{H}_i$. In his original paper von Neumann has shown that $\otimes_{i\in Z}{}^a\mathfrak{H}_i$ is a separable Hilbert space, i.e., it has a countable orthonormal basis. If $\{e_k^i\}$ is an orthonormal basis for $\mathfrak{H}_i$, then one such basis may be obtained by enumerating the set

$$\{\otimes_{i\in Z} x_i : x_i=a_i \text{ for all but a finite number of } i\in Z \text{ and } x_i=e_k^i, \text{ for some k, if } x_i\neq a^i\}. \tag{2.37}$$

Denote the resulting basis by $\{e_j\}$. Each of these basis vectors is of the product form $e_j=\otimes_{i\in Z} x_i^j$, as opposed to being a linear combination of such vectors. We will refer to such a basis as a product basis. Note that not every basis is a product basis.

We now return to the quasi-local algebra and consider the state $\omega^{\infty}(A) \equiv \lim_{n\to\infty} \omega_{\zeta_n}(A_{\zeta_n}) = \lim_{n\to\infty}\left[\bigotimes_{i=-n}^{n} \omega_i\right](A_{\zeta_n})$, for all $A \in \bigotimes_{i\in Z} \mathfrak{A}_i$. Denote the G.N.S. representation corresponding to ω_i by $\{\mathfrak{H}_{\omega_i}, \pi_{\omega_i}, \Omega_{\omega_i}\}$ and form the Hilbert space $\mathfrak{H}_{\omega^\infty} \equiv \bigotimes_{i\in Z}^{\Omega} \mathfrak{H}_i$ associated with the family $\Omega = \{\Omega_{\omega_i} : i \in Z\}$. Let $\{e_j = \bigotimes_{i\in Z} x_i^j\}$ be an orthonormal product basis for $\mathfrak{H}_{\omega^\infty}$. For all $A \in \bigotimes_{i\in Z} \mathfrak{A}_i$ of the form $\bigotimes_{i\in Z} A_i$, where $A_i \in \mathfrak{A}_i$ and all but a finite number of the A_i are equal to σ_0^i, we define

$$[\pi_{\omega^\infty}(\bigotimes_{i\in Z} A_i)]e_j \equiv \bigotimes_{i\in Z} \pi_{\omega_i}(A_i)x_i^j. \tag{2.38}$$

Because all but a finite number of the A_i are equal to σ_0^i and all but a finite number of the x_i^j are equal to Ω_{ω_i}, all but a finite number of the the $\pi_{\omega_i}(A_i)x_i^j$ are equal to Ω_{ω_i}. The vector $\bigotimes_{i\in Z} \pi_{\omega_i}(A_i)x_i^j$ is therefore an element $\otimes_{i\in Z}^{\Omega} \mathfrak{H}_i$. We can therefore extend this mapping by linearity to a bounded mapping from $\otimes_{i\in Z}^{\Omega} \mathfrak{H}_i$ into itself by defining

$$[\pi_{\omega^\infty}(\bigotimes_{i\in Z} A_i)]\left(\sum_j \alpha_j \otimes_{i\in Z} x_i^j\right) \equiv \sum_j a_j [\pi_{\omega^\infty}(\bigotimes_{i\in Z} A_i)](\otimes_{i\in Z}) x^j]. \tag{2.39}$$

Finally we can extend the definition of this operator by continuity to all of $\bigotimes_{i\in Z} \mathfrak{A}_i$ (and thus obtain a representation π_{ω^∞} of $\bigotimes_{i\in Z} \mathfrak{A}_i$ on $\mathfrak{H}_{\omega^\infty}$) as follows. Again let A_{ζ_n} denote the projection of $A \in \bigotimes_{i\in Z} \mathfrak{A}_i$ onto $\mathfrak{A}_{\zeta_n}$, then $i_{\zeta_n}(A_{\zeta_n})$ is a finite linear combination of elements of the form $\bigotimes_{i\in Z} A_i$ where

$A_i \in \mathfrak{A}_i$ and A_i is equal to σ_0^i for all i such that $|i|>n$. Denote this linear combination by $\sum \bigotimes_{i\in Z} A_i$. We then define

$$\pi_{\omega^\infty}(A) \equiv \lim_{n\to\infty}[\pi_{\omega^\infty}(i_{\zeta_n}(A_{\zeta_n}))] \equiv \lim_{n\to\infty}[\sum \pi_{\omega^\infty}(\bigotimes_{i\in Z} A_i)], \tag{2.40}$$

for all $A \in \bigotimes_{i\in Z} \mathfrak{A}_i$.

The vector $\Omega_{\omega^\infty} \equiv \bigotimes_{i\in Z} \Omega_{\omega_i}$ in $\mathfrak{H}_{\omega^\infty}$ is obviously cyclic for the representation π'_{ω^∞}, so that $\{\mathfrak{H}_{\omega^\infty}, \pi_{\omega^\infty}, \Omega_{\omega^\infty}\}$ is a cyclic representation of $\bigotimes_{i\in Z} \mathfrak{A}_i$. Again let A_{ζ_n} denote the projection of $A \in \bigotimes_{i\in Z} \mathfrak{A}_i$ onto $\mathfrak{A}_{\zeta_n}$. We then have

$$\begin{aligned}
(\Omega_{\omega^\infty}, \pi_{\omega^\infty}(i_{\zeta_n}(A_{\zeta_n}))\Omega_{\omega^\infty}) &= \left(\Omega_{\omega^\infty}, \sum \left[\bigotimes_{i\in Z} \pi_{\omega_i}(A_i)\right]\Omega_{\omega^\infty}\right) \\
&= \sum \prod_{i\in Z} (\Omega_{\omega_i}, \pi_{\omega_i}(A_i)\Omega_{\omega_i}) \\
&= \sum \prod_{i=-n}^{n} \omega_i(A_i) \\
&= \omega_n(A_{\zeta_n}).
\end{aligned} \tag{2.41}$$

This then gives

$$(\Omega_{\omega^\infty},\pi_{\omega^\infty}(A)\Omega_{\omega^\infty})=\lim_{n\to\infty}(\Omega_{\omega^\infty},\pi_{\omega^\infty}(i_{\zeta_n}(A_{\zeta_n}))\Omega_{\omega^\infty})$$

$$=\lim_{n\to\infty}\omega_n(A^{\zeta_n})$$

$$\equiv\omega^\infty(A), \tag{2.42}$$

for all $A\in \bigotimes_{i\in Z}\mathfrak{A}_i$. So the representation $\{\mathfrak{H}_{\omega^\infty},\pi_{\omega^\infty},\Omega_{\omega^\infty}\}$ produces the correct vector state over $\bigotimes_{i\in Z}\mathfrak{A}_i$ and is therefore unitarily equivalent to the G.N.S. representation arising from the state ω^∞ over $\bigotimes_{i\in Z}\mathfrak{A}_i$.

Finally we show that the representations $\{\mathfrak{H}_{\omega^\infty},\pi_{\omega^\infty}\}$ corresponding to different temperatures are unitarily inequivalent. Let $\{\mathfrak{H}_{\omega_1},\pi_{\omega_1},\Omega_{\omega_1}\}$ and $\{\mathfrak{H}_{\omega_2},\pi_{\omega_2},\Omega_{\omega_2}\}$ be the representations corresponding to natural temperature β_1 and β_2, respectively, with $\beta_1\neq\beta_2$ (we now drop the superscript ∞ from the state ω^∞). Our problem is this. Fix a basis $\{e_j\}$ in $\mathfrak{H}_{\omega_1}$. For each $A\in\bigotimes_{i\in Z}\mathfrak{A}_i$ we can determine the matrix elements of $\pi_{\omega_1}(A)$ with respect to the basis $\{e_j\}$. Given an orthonormal basis $\{f_j\}$ in $\mathfrak{H}_{\omega_2}$ we can determine the matrix elements of $\pi_{\omega_2}(A)$ with respect to the basis $\{f_j\}$. In order to demonstrate the unitary inequivalence of the two representations we must show that there does not exist a basis $\{f_j\}$ in $\mathfrak{H}_{\omega_2}$ such that these two matrices are identical.

Let $\{e_j=\bigotimes_{i\in Z}x_i^j\}$ be an orthonormal product basis for $\mathfrak{H}_{\omega_1}$ such that $e_1=\bigotimes_{i\in Z}\Omega_{\omega_1 i}$. Consider the set of elements $\{A_i\}\in\bigotimes_{i\in Z}\mathfrak{A}_i$ of the form

$$A_i=\sigma_0^1\otimes\sigma_0^2\cdots\sigma_0^{i-1}\otimes\sigma_3^i\otimes\sigma_0^{i+1}\cdots. \tag{2.43}$$

The diagonal matrix elements of $\pi_{\omega_1}(A_i)$ between the basis element e_1 are

$$(e_1,\pi_{\omega_1}(A_i)e_1)=(\Omega_{\omega_{1i}}\ ,\ \pi_{\omega_{1i}}(\sigma_3^i)\Omega_{\omega_{1i}})$$

$$=(u_1)^2-(v_1)^2=t_1, \tag{2.44a}$$

for all i. The diagonal matrix elements of $\mathbb{1}\equiv\sigma_0^1\otimes\sigma_0^2\cdots\otimes\sigma_0^i\otimes\cdots$ between the basis element e_1 is

$$(e_1,\pi_{\omega_1}(\mathbb{1})e_1)=(e_1,e_1)=1. \tag{2.44b}$$

We now must show that there does not exist a basis $\{f_j\}$ of $\mathfrak{H}_{\omega_2}$ such that $(f_j,\pi_{\omega_2}(A_i)f_j)=(e_1,\pi_{\omega_1}(A_i)e_1)$ and $(f_j,\pi_{\omega_2}(\mathbb{1})f_j)=(f_j,f_j)=1$ for some j and all i. To show this it is sufficient to show that there does not exist a single normalized vector g in $\mathfrak{H}_{\omega_2}$ such that

$$(g,\pi_{\omega_2}(A_i)g)=(e_1,\pi_{\omega_1}(A_i)e_1)=t_1, \tag{2.45}$$

for **all** i. One way to convince yourself that this is so will be outlined.

First, let us choose an orthonormal product basis $\{h_j=\otimes_{i\in Z}z_i^j\}$ for $\mathfrak{H}_{\omega_2}$. Note that for each j, $z_i^j=\Omega_{\omega_{2i}}$ for all but a finite number of i. We will first argue that a vector g that satisfies (2.45) cannot be one of the basis elements h_j and that it cannot be a finite linear combination of the basis elements $\{h_j\}$. Finally, we will use these results along with the fact that the set of finite linear combinations of the basis vectors is dense in $\mathfrak{H}_{\omega 2}$ to show that there does not exist a vector g in $\mathfrak{H}_{\omega 2}$ that satisfies (2.45).

First assume that $g=h_j$ for some j. Then g is of the form $g=\underset{i\in Z}{\otimes} z_i^j$, where $z_i^j=\Omega_{\omega 2i}$ for all but a finite number of i. Suppose that $z_k^j=\Omega_{\omega 2k}$, then

$$(g,\pi_{\omega 2}(A_k)g)=(z_1^j,z_1^j)(z_2^j,z_2^j)\cdots(z_{k-1}^j,z_{k-1}^j)(z_k^j,\pi_{\omega 2k}(\sigma_3^k)z_k^j)(z_{k+1}^j,z_{k+1}^j)\cdots$$

$$=(z_k^j,\ \pi_{\omega 2k}(\sigma_1^k)z_k^j)=(\Omega_{\omega 2k},\pi_{\omega 2k}(\sigma_3^k)\Omega_{\omega 2k})$$

$$=t_2, \tag{2.46}$$

which does not agree with (2.45) for i=k. Therefore g cannot be one of the basis elements h_j.

Now assume that g is a finite linear combination of the basis elements $\{h_j=\underset{i\in Z}{\otimes} z_i^j\}$. Since, for each j, $z_i^j=\Omega_{\omega 2i}$ for all but a finite number of i, such a finite linear combination is of the form

$$g=\Sigma\otimes\Omega_{\omega 2m}\otimes\Omega_{\omega 2m+1}\otimes\Omega_{\omega 2m+2}\cdots, \tag{2.47}$$

where Σ is a finite linear combination of vectors of the form $x_1\otimes x_2\otimes\cdots\otimes x_{m-1}$, where $x_i\in\mathfrak{H}_{\omega 2i}$. Now for all k>m we have

$$(g,\pi_{\omega 2}(A_k)g)=(\Omega_{\omega 2k},\pi_{\omega 2k}(\sigma_3^k)\Omega_{\omega 2k})$$

$$=t_2, \tag{2.48}$$

which does not agree with (2,27) for i=k (we have used the fact that g is normalized). Therefore g cannot be a finite linear combination of the basis elements $\{h_j\}$.

Finally, let us consider the possibility that g is an arbitrary element of $\mathfrak{H}_{\omega 2}$. Since the set of finite linear combinations of the basis vectors is dense in $\mathfrak{H}_{\omega 2}$ there must exist a sequence $\{g_n\}$ of vectors in $\mathfrak{H}_{\omega 2}$ that converges to g in the norm, where each vector g_n is a finite linear combination of the basis vectors $\{h_j\}$. The convergence of the sequence $\{g_n\}$ to g implies that we can make

$$|(g_n,\pi_{\omega 2}(A_i)g_n)-(g,\pi_{\omega 2}(A_i)g)| \tag{2.49}$$

as small as we want (for all i) by choosing n sufficiently large. The argument for finite linear combinations implies that there exists an m such that $(g_n,\pi_{\omega 2}(A_k)g_n)=t_2$ for all k>m. Furthermore we have $(g,\pi_{\omega 2}(A_k)g)=t_1$ for all k by assumption. So for all k>m we have

$$|(g_n,\pi_{\omega 2}(A_k)g_n)-(g,\pi_{\omega 2}(A_k)g)|=|t_2-t_1|, \tag{2.50}$$

which contradicts the fact that we can make $|(g_n,\pi_{\omega 2}(A_k)g_n)-(g,\pi_{\omega 2}(A_k)g)|$ as small as we want by choosing n sufficiently large. Therefore g cannot be an arbitrary element of $\mathfrak{H}_{\omega 2}$.

We have now shown that there does not exist a vector $g \in \mathfrak{H}_{\omega 2}$ that satisfies (2.45). The representations $\{\mathfrak{H}_{\omega_1}, \pi_{\omega_1}\}$ and $\{\mathfrak{H}_{\omega_2}, \pi_{\omega_2}\}$ are therefore unitarily inequivalent.

The previous result shows that the C*-algebra for the infinite spin system admits an infinite number of inequivalent representations, which are labeled by temperature, a macroscopic parameter. In the finite case the cyclic vector in the G.N.S. representation is a finite direct product of the cyclic vector for the single-spin system. This cyclic vector generates the entire representation space, which is a finite direct product of the single-spin representation space. Thus, there is only one G.N.S. representation. When we pass to the infinite system, the corresponding cyclic vector does not generate the entire infinite direct product space, but generates only a subspace, called by von Neumann an incomplete direct product space. Representations over different subspaces are unitarily inequivalent.

CHAPTER 3

INFINITE BOSE GAS

3.1 THE ALGEBRA OF THE CANONICAL COMMUTATION RELATIONS

The goal of this section is to construct the C*-algebra which is appropriate for the description of a system of non-interacting bosons. This C*-algebra is the C.C.R. (canonical commutation relations) algebra. The elements of this algebra will be labeled by the elements of an arbitrary vector space. Different vector spaces will correspond to different physical systems. We will proceed in three steps. In section 3.1a, a set of elements which are labeled by the elements of a vector space and satisfy the C.C.R.'s will be introduced. These elements will be used to construct the C*-algebra corresponding to a single point particle in section 3.1b. At this point the Weyl form of the C.C.R.'s will be introduced. Finally, the C*-algebra corresponding to an arbitrary system of bosons (finite or infinite) will be constructed in section 3.1c.

3.1a THE CANONICAL COMMUTATION RELATIONS

We begin by considering the abstract elements $\Psi(\mathbf{x})$ and $\Psi^*(\mathbf{x})$, labeled by a parameter $\mathbf{x} \in \mathbb{R}^3$, that satisfy

$$[\Psi(\mathbf{x}),\Psi^*(\mathbf{y})]=\delta(\mathbf{x}-\mathbf{y}), \tag{3.1a}$$

and

$$[\Psi(\mathbf{x}),\Psi(\mathbf{y})]=[\Psi^*(\mathbf{x}),\Psi^*(\mathbf{y})]=0, \tag{3.1b}$$

where the commutator $[A,B]\equiv AB-BA$. These elements will be used to construct a general C*-algebra, that is appropriate for the description of a (finite or infinite) Bose system of point particles. The presence of the delta function in (1) implies that these relations are to be interpreted in terms of distributions. This motivates the definitions

$$\Psi(f)\equiv\int\overline{f(\mathbf{x})}\,\Psi(\mathbf{x})d\mathbf{x}, \tag{3.2a}$$

and

$$\Psi^*(f)\equiv\int f(\mathbf{x})\Psi^*(\mathbf{x})d\mathbf{x}, \tag{3.2b}$$

for suitably vanishing $f(\mathbf{x})$. The set of functions $f(\mathbf{x})$ will be referred to as the space of test functions. It is easy to see that the "smeared" fields $\Psi(f)$ and $\Psi^*(f)$ satisfy the following commutation relations:

$$[\Psi(f),\Psi(g)^*]=(f,g);\quad (f,g)\equiv\int\overline{f(\mathbf{x})}\,g(\mathbf{x})d\mathbf{x}, \tag{3.2c}$$

and

$$[\Psi(f),\Psi(g)]=[\Psi^*(f),\Psi^*(g)]=0. \tag{3.2d}$$

These commutation relations depend solely on the inner product (f,g). We now replace the test functions with vectors in an arbitrary Hilbert space $\mathfrak{H}$, and work with the commutation relations

$$[\Psi(f),\Psi(g)^*]=\langle f,g\rangle, \tag{3.3a}$$

and

$$[\Psi(f),\Psi(g)]=[\Psi^*(f),\Psi^*(g)]=0, \tag{3.3b}$$

for all $f,g \in \mathfrak{H}$, where $\langle f,g \rangle$ is the scalar product in $\mathfrak{H}$. This will allow us to describe two different "types" of systems, one which contains a fixed finite number of particles and one which contains an arbitrary (finite or infinite) number of particles. The first case will result when $\mathfrak{H}$ is $\mathbb{C}^3$. The second case will occur when $\mathfrak{H}$ is the Hilbert space appropriate for the description (in the Traditional Approach) of a single particle of the system (see [Emch] section 3.1.c). We will justify the above interpretations by examining two familiar representations of the C.C.R. algebra, the Schrödinger and Fock representations. We will find that the one-dimensional Schrödinger representation is a representation of the C.C.R. algebra corresponding to $\mathfrak{H}=\mathbb{C}$, and the Fock representation is a representation of the C.C.R. algebra corresponding to the case when $\mathfrak{H}$ is the Hilbert space appropriate for the description of a single particle of the system.

It is often convenient to work with the elements

$$\phi(f) \equiv \frac{\Psi(f)+\Psi^*(f)}{\sqrt{2}} \tag{3.4a}$$

and

$$\pi(f) \equiv \phi(if) = \frac{\Psi(f)-\Psi^*(f)}{\sqrt{2}i}, \tag{3.4b}$$

that satisfy

$$[\phi(f),\pi(g)] = i\mathrm{Re}[\langle f,g \rangle], \tag{3.5a}$$

and

$$[\phi(f),\phi(g)] = [\pi(f),\pi(g)] = i\mathrm{Im}[\langle f,g \rangle], \tag{3.5b}$$

One is more accustomed to the commutation relations $[\phi(f),\pi(g)]=i\langle f,g\rangle$, and $[\phi(f),\phi(g)]=[\pi(f),\pi(g)]=0$, which are the same as (3.5) when $\mathfrak{H}$ is a real Hilbert space. Since this restriction seems unnecessary, we will assume that $\mathfrak{H}$ is complex and work with the commutation relations in (3.5). Note that if the set $\{h_j\}$ is an orthonormal basis for $\mathfrak{H}$, and we define $p_j\equiv\pi(h_j)$, $q_i\equiv\phi(h_j)$, we recover the familiar canonical commutation relations

$$[q_j,p_k]=i\delta_{j,k}, \tag{3.6a}$$

$$j,k=1,2,3,...$$

and

$$[q_j,q_k]=[p_j,p_k]=0 \tag{3.6b}$$

3.1b SINGLE PARTICLE IN ONE DIMENSION

We will begin by discussing the algebraic description of a single point particle. Although this case is interesting in itself, it will turn out to be needed also in constructing the very important Fock representation of a **general** C*-algebra. This representation will be constructed from a representation of the C*-algebra corresponding to a **single** particle in one dimension. The C*-algebra corresponding to a point particle in one dimension is constructed from the commutation relations (3.6) with $j=k=1$ (i.e., when $\mathfrak{H}=\mathbb{C}$). The number 1 forms a basis for $\mathbb{C}$, so we have two elements, $q=\phi(1)$ and $p=\pi(1)\equiv\phi(i)$, that satisfy

$$[q,p]=i. \tag{3.7}$$

We initially consider the algebra $\mathfrak{A}$ consisting of all polynomials in p and q with complex coefficients. Using (3.6) it is clear that $\mathfrak{A}$ is simply given by

$$\mathfrak{A}=\{A=\sum_{n,m=0}^{N,M}\alpha_{nm}p^nq^m:\alpha_{mn}\in\mathbb{C}\}. \tag{3.8}$$

We note that the number 1 is an identity for $\mathfrak{A}$. Furthermore,

$$\left(\sum_{n,m=0}^{N,M}\alpha_{nm}p^nq^m\right)^* \equiv \sum_{n,m=0}^{N,M}\bar{\alpha}_{nm}q^mp^n \tag{3.9}$$

defines an involution on $\mathfrak{A}$, so that $\mathfrak{A}$ is a *-algebra. This algebra, however, does not admit a *-norm.

Theorem 3.10 If two elements p and q of an algebra satisfy the relation (3.6) then the spectrum of at least one of the two elements p and q is necessarily unbounded.

Proof: The proof is the same as in the case when p and q are operators on a Hilbert space (see, for example, [Put] page 2). The proof is by contradiction. Assume that the spectra of both p and q are bounded, and hence the spectrum of pq is bounded. We may assume, without loss of generality, that q is invertible . If q is not invertible we can define $q'\equiv q-\lambda$ ($[q',q]=[p,q]$), for some $\lambda\notin\sigma(q)$. Since q has a bounded spectrum, such a λ must exist. Since q is invertible, $qp-\lambda=q[pq-\lambda]q^{-1}$, which implies $\sigma(qp)=\sigma(pq)$. Consider the following.

$$pq-\mu=(qp-i)-\mu$$

$$=qp-(\mu+i).$$

This shows that $\mu\in\sigma(\pi(f)\phi(f))$ implies $(\mu+i)\in\sigma(\phi(f)\pi(f))=\sigma(\pi(f)\phi(f))$. This then implies that $(\mu+in)\in\sigma(qp)=\sigma(pq)$ for arbitrary $n\geq0$, and hence $\sigma(pq)$ is unbounded, which is a contradiction. This completes the proof.

Theorem 3.10 allows us to conclude that it is not possible to define a C*-norm for $\mathfrak{A}$, for we know that such a norm is related to the spectrum through the spectral radius, which does not exist for an element whose spectrum is unbounded. To avoid these problems we define

$$U(t)\equiv e^{itq}, \tag{3.11a}$$

and

$$V(s)\equiv e^{isp}, \tag{3.11b}$$

where t and s are arbitrary real numbers. By explicitly writing out the power series in (3.11) and using the commutation relations (3.7) it is possible to show that

$$U(t_1)U(t_2)=U(t_1+t_2), \tag{3.12a}$$

$$V(s_1)V(s_2)=V(s_1+s_2), \tag{3.12b}$$

and

$$U(t)V(s)=e^{its}V(s)U(t). \tag{3.12c}$$

This form of the canonical commutation relations is known as the Weyl form. We now consider the algebra $\mathfrak{A}$ consisting of all finite polynomials in U(t) and V(s) with complex coefficients. Using (3.12) we see that $\mathfrak{A}$ is simply given by

$$\mathfrak{A}=\{A=\sum_{n,m=0}^{N,M}\alpha_{nm}U(t_n)V(s_m):\alpha_{nm}\in\mathbb{C};t_n,s_m\in\mathbb{R}\}. \tag{3.13}$$

An involution of $\mathfrak{A}$ is defined as

$$\left(\sum_{n,m=0}^{N,M}\alpha_{nm}U(t_n)V(s_m)\right)^* \equiv \sum_{n,m=0}^{N,M}\bar{\alpha}_{nm}V(-s_m)U(-t_n). \tag{3.14}$$

It is easy to verify that (3.14) defines an involution of $\mathfrak{A}$, so that $\mathfrak{A}$ is a *-algebra. We note that $U(t)U(t)^*=U(t)^*U(t)=1$ and $V(s)V(s)^*=V(s)^*V(s)=1$, which imply that U(t) and V(s) are unitary for all t,s.

A faithful representation $\{\mathfrak{H}_h,\pi_h\}$ for A, the harmonic oscillator or Schrödinger representation, is described in Appendix A. Apart from the language in which this representation is described here, the representation is well known from all quantum mechanics texts. For all $A \in \mathfrak{A}$ we define

$$\|A\| \equiv \mathrm{Sup}\left\{\frac{\|\pi_h(A)\Psi\|}{\|\Psi\|} : \Psi \in \mathfrak{H}_h; \Psi \neq 0\right\}. \tag{3.15}$$

Since $\{\mathfrak{H}_h, \pi_h\}$ is a faithful representation of $\mathfrak{A}$, (3.15) defines a C*-norm for $\mathfrak{A}$. We can then complete $\mathfrak{A}$ with respect to the uniform topology defined by this norm. We denote the resulting C*-algebra by $\mathfrak{A}(\mathbb{C})$, and refer to it as the C*-algebra of the C.C.R.'s for a single degree of freedom.

The vector $|0\rangle$, see Appendix A, is cyclic for the set $\{\pi_h(a^*), \pi_h(a)\}$, and therefore for the set $\{\pi_h(q), \pi_h(p)\}$. Since any element of this set may be approximated to any degree by elements of $\pi_h(\mathfrak{A}(\mathbb{C}))$, the vector $|0\rangle$ is cyclic for $\pi_h(\mathfrak{A}(\mathbb{C}))$, i.e., the representation $\{\mathfrak{H}_h, \pi_h, |0\rangle\}$ is cyclic. This representation is also irreducible. To show this, assume that there exists a bounded linear operator M on $\mathfrak{H}_h$ that commutes with every element of $\pi_h(\mathfrak{A}(\mathbb{C}))$. Such an element must commute with $\pi_h(a^*)$ and $\pi_h(a)$ (on $D(N^{1/2})$. The matrix elements of M, with respect to the basis $|n\rangle$, are

$$(\langle m|)(M|n\rangle) = \langle 0|\frac{\pi(a)^n}{\sqrt{n!}} M \frac{\pi(a^*)^m}{\sqrt{m!}}|0\rangle$$

$$= \langle 0|M\frac{\pi(a)^n}{\sqrt{n!}} \frac{\pi(a^*)^m}{\sqrt{m!}}|0\rangle$$

$$= \langle 0|\frac{\pi(a)^n}{\sqrt{n!}} \frac{\pi(a^*)^m}{\sqrt{m!}}M|0\rangle = \delta_{nm}\langle 0|M|0\rangle, \tag{3.16}$$

where the last equality follows from (A.9). This implies that M is a multiple of the identity, and hence the representation $\{H,\pi\}$ is irreducible, by Schur's lemma.

In Appendix A we have shown that the harmonic oscillator or Schrödinger representation is a faithful representation of the C.C.R. algebra $\mathfrak{A}(\mathbb{C})$. It is a well known result that it is the unique, up to unitary equivalence, faithful representation of $\mathfrak{A}(\mathbb{C})$ (this result is originally due to von Neumann [Neum2]). Since the Schrödinger representation is the representation that one uses for the description of a single particle in the Traditional Approach, our assumption that the C.C.R. algebra $\mathfrak{A}(\mathbb{C})$ is the C*-algebra corresponding to a single point particle in one dimension seems reasonable.

3.1c ARBITRARY BOSE SYSTEM

The C.C.R. algebra $\mathfrak{A}(\mathbb{C})$ developed in the last section corresponds to a system consisting of a single particle in one dimension. We now repeat the construction carried out in 3.1.b for an arbitrary Hilbert space $\mathfrak{H}$. This C*-algebra will correspond to a system of particles whose single particle Hilbert space (in the Traditional Approach) is $\mathfrak{H}$. Fix an orthonormal basis $\{h_i\}$ in $\mathfrak{H}$ and consider the algebra $\mathfrak{A}(\mathfrak{H})$ generated by the $\phi(f)$ and $\pi(g)$. The elements in (3.3) satisfy, for $\alpha,\beta\in\mathbb{C}$ and $f=\sum_i \alpha^i h_i$,

$$\Psi(f)=\Psi(\sum_i \alpha^i h_i)=\sum_i \Psi(\alpha^i h_i)=\sum_i \bar{\alpha}^i \Psi(h_i), \tag{3.17a}$$

and
$$\psi^*(f)=\psi^*(\sum_i \alpha^i h_i)=\sum_i \alpha^i \psi^*(h_i), \tag{3.17b}$$

i.e., ψ and ψ^* are antilinear and linear in the smeared functions, respectively. However, the linear combinations (3.4), ϕ and π, are neither linear nor antilinear. We can only go as far as

$$\phi(f)=\phi(\sum_i \alpha^i h_i)=\sum_i \phi(\alpha^i h_i), \tag{3.17c}$$

$$\pi(f)=\pi(\sum_i \alpha^i h_i)=\sum_i \pi(\alpha^i h_i). \tag{3.17d}$$

The following relations are also useful.

$$\begin{aligned}\phi(\alpha f)&=\mathrm{Re}(\alpha)\phi(f)+\mathrm{Im}(\alpha)\phi(if)\\ &=\mathrm{Re}(\alpha)\phi(f)+\mathrm{Im}(\alpha)\pi(f),\end{aligned} \tag{3.18a}$$

and
$$\pi(\alpha f)=\mathrm{Re}(\alpha)\pi(f)+\mathrm{Im}(\alpha)\phi(-f). \tag{3.18b}$$

As in the case of a single degree of freedom, the spectrum of at least one of $\phi(f)$ and $\pi(g)$ must be unbounded, so we define

$$U(f) \equiv e^{i\phi(f)}, \tag{3.19a}$$

and
$$V(g) \equiv e^{i\pi(g)} = U(ig). \tag{3.19b}$$

As before we construct a C*-algebra from the elements U(f) and V(g). Using the commutation relations (3.5) along with (3.18) and the relation $e^{AB} = e^{A+B}\ e^{[A,B]/2}$ ([A,B]=complex number), we can easily deduce the following.

$$U(f_1)U(f_2) = U(f_1+f_2)e^{\frac{-i}{2}\mathrm{Im}[\langle f_1,f_2\rangle]}, \tag{3.20a}$$

$$V(g_1)V(g_2) = V(g_1+g_2)e^{\frac{-i}{2}\mathrm{Im}[\langle g_1,g_2\rangle]}, \tag{3.20b}$$

and
$$U(f)V(g) = V(g)U(f)e^{-i\mathrm{Re}(f,g)}. \tag{3.20c}$$

We can summarize these relations by introducing the element W(f), $W(f) \equiv U(f)$ and $W(if) \equiv U(if) = V(f)$, that satisfies

$$W(f)W(g) = W(f+g)e^{\frac{-i}{2}\mathrm{Im}[\langle f,g\rangle]},$$

$$= W(g)W(f)e^{-i\mathrm{Im}[\langle f,g\rangle]}. \tag{3.21}$$

The algebra $\mathfrak{A}(\mathfrak{H})$ is then given by

$$\mathfrak{A}(\mathfrak{H})=\{A=\sum_{i=1}^{N}\alpha^i W(f_i):\alpha_i\in\mathbb{C},\ f_i\in\mathfrak{H}\}\tag{3.22}$$

From (3.22) we see that W(0)=1 is an identity for $\mathfrak{A}(\mathfrak{H})$. An involution of $\mathfrak{A}(\mathfrak{H})$ is defined as

$$A^*=[\sum_{i=1}^{N}\alpha^i W(f_i)]^*\equiv\sum_{i=1}^{N}\overline{\alpha}^i W(-f_i).\tag{3.23}$$

It is easy to verify that (3.23) defines an involution of $\mathfrak{A}(\mathfrak{H})$. We have to show that (3.23) satisfies the following (Definition 1.3) with $A=\sum_{i=1}^{N}\alpha^i W(f_i)$ and $B=\sum_{j=1}^{M}\beta^j W(g_j)$:

i) $(A^*)^*=A$; this follows immediately from (3.23),

ii) $(AB)^*=B^*A^*$; $(AB)^*=\left[\sum_{i,j=1}^{N,M}\alpha^i\beta^j W(f_i)W(g_j)\right]^*$

$$=\left[\sum_{i,j=1}^{N,M}\alpha^i\beta^j W(f_i+g_j)e^{\frac{-i}{2}\mathrm{Im}[<f_i,g_j>]}\right]^*$$

$$=\sum_{i,j=1}^{N,M}\overline{\alpha^i\beta^j}\ W(-f_i-g_j)e^{\frac{-i}{2}\mathrm{Im}[<g_j,f_i>]}$$

$$=\sum_{j=1}^{M}\overline{\beta}^i W(-g_j)\sum_{i=1}^{N}\overline{\alpha}^i W(-f_i)=B^*A^*,$$

iii) $(\alpha A+\beta B)^*=\bar{\alpha}A^*+\bar{\beta}B^*$; this follows immediately from (3.23).

So (3.22) does define an involution of $\mathfrak{A}(\mathfrak{H})$, and $\mathfrak{A}(\mathfrak{H})$ is therefore a *-algebra.

To define a norm for $\mathfrak{A}(\mathfrak{H})$ we first construct a faithful representation of $\mathfrak{A}(\mathfrak{H})$, the so-called Fock representation. The first step is the definition of the representation space $\mathfrak{H}_F$. The definition uses the representation $\{\mathfrak{H}_h,\pi_h,\Psi_o\equiv|0\rangle\}$ developed for the case of a single degree of freedom. Let $\{h_i\}$ be a basis for $\mathfrak{H}$. With each h_i, associate a copy of the representation $\{\mathfrak{H}_h,\pi_h,\Psi_o\}$ denoted $\{\mathfrak{H}_{hi},\pi_{hi},\Psi_{oi}\}$. The representation space $\mathfrak{H}_F$ is then defined to be the Hilbert space $\mathfrak{H}_F\equiv\otimes_i{}^{\Psi}\mathfrak{H}_{hi}$ associated with the family $\Psi=\{\Psi_{oi}\}$ (see page 71). The set

$$\{|n_1,n_2,\ldots,n_i,\ldots\rangle\equiv|n_1\rangle\otimes|n_2\rangle\otimes\ldots\otimes|n_i\rangle\otimes\ldots:\sum_{i=1}^{\infty}n_i<\infty\} \tag{3.24}$$

forms an orthonormal basis for $\mathfrak{H}_F$. Note that the condition $\sum_{i=1}^{\infty}n_i<\infty$ implies that $n_i=0$ for all but a finite number of the i's (i.e., each basis vector is an element of the family Ψ). Using this basis, $\mathfrak{H}_F$ may be expressed as

$$\mathfrak{H}_F=\left\{\Psi=\sum_{n_1,n_2,\ldots=0}^{\infty}\alpha(n_1,n_2,\ldots)|n_1,n_2,\ldots\rangle:\alpha(n_1,n_2,\ldots)\in\mathbb{C};\ \sum_{n_1,n_2,\ldots=0}^{\infty}|\alpha(n_1,n_2,\ldots)|^2<\infty\right\}. \tag{3.25}$$

For $W(h_i)$ and $\alpha^i=a^i+ib^i$, a^i and b^i real, define

$$\pi(W(\alpha^i h_i))_F\equiv 1\otimes 1\otimes\ldots\otimes\pi_i(U(a^i)V(b^i))\otimes 1\otimes\ldots\,. \tag{3.26}$$

For arbitrary $f\in\mathfrak{H}$, with $f=\sum_i\alpha^i h_i=\sum_i(a^i+ib^i)h_i$, we then have

$$\pi_F(W(f))=\pi_F(W(\sum_i\alpha^i h_i))=\pi_F(\prod_i W(\alpha^i h_i)$$

$$\equiv\prod_i\pi_F(W(\alpha^i h_i))$$

$$=\pi_1(U(a^1)V(b^1))\otimes\pi_2(U(a^2)V(b^2))\otimes\ldots\otimes\pi_i(U(a^i)V(b^i))\otimes\ldots\,. \tag{3.27}$$

It is easy to verify that the pair $\{\mathfrak{H}_F,\pi_F\}$ form a faithful representation of $\mathfrak{A}(\mathfrak{H})$, the Fock representation, and that the representatives $\pi_F(W(f))$ are unitary operators on $\mathfrak{H}_F$. Furthermore, since the $\pi_{hi}(U(a^i)V(b^i))$ are irreducible in $\mathfrak{H}_{hi}$, the set $\pi_F(\mathfrak{A})$ is irreducible on $\mathfrak{H}_F$. This implies that the vector $\Psi_{F0}\equiv\Psi_{01}\otimes\Psi_{02}\otimes\ldots$ is cyclic for $\pi_F(\mathfrak{A})$, which is to be expected since the Ψ_{0i} is cyclic for $\pi_{hi}(U(a^i)V(b^i))$. The definition

$$\|A\|\equiv\mathrm{Sup}\left\{\frac{\|\pi_F(A)\Psi\|}{\|\Psi\|}:\Psi\in\mathfrak{H}_F;\Psi\neq 0\right\} \tag{3.28}$$

is then a C*-norm for $\mathfrak{A}(\mathfrak{H})$. Completing $\mathfrak{A}(\mathfrak{H})$ with respect to the uniform topology defined by this norm makes $\mathfrak{A}(\mathfrak{H})$ a C*-algebra, which we will denote by the same symbol. We will refer to this C*-algebra as the C.C.R. algebra, relative to the Hilbert space $\mathfrak{H}$.

3.2 STATES AND REPRESENTATIONS OF THE C.C.R. ALGEBRA

We initially attempted to construct a C*-algebra from the elements $\phi(f)$ and $\pi(g)=\phi(ig)$. Since it was not possible to define a C*-norm for these elements, we introduced the elements $W(f)=e^{i\phi(f)}$ and used them to construct a C*-algebra, the C.C.R. algebra. The elements $\phi(f)$ and $\pi(g)$ are formally given by

$$\phi(f)=\frac{1}{i}\frac{d}{dt}(W(tf))|_{t=0}=\frac{1}{i}\lim_{k\to 0}\frac{1}{k}(W(kf)-W(0)) \tag{3.29}$$

and

$$\pi(g)=\frac{1}{i}\frac{d}{dt}(W(itg))|_{t=0}=\frac{1}{i}\lim_{k\to 0}\frac{1}{k}(W(ikg)-W(0)), \tag{3.30}$$

but are not elements of the C.C.R. algebra. It is possible for one of them to have a finite norm and be an element of the C.C.R. algebra, but it is not possible for both to have a finite norm.

Although it is not possible for the C.C.R. algebra to contain both the elements $\phi(f)$ and $\pi(g)$ (Theorem 3.10), there might exist a set of unbounded linear operators on the Hilbert space of a representation of the C.C.R. algebra that forms a representation of the $\phi(f)$ and $\pi(g)$. Representations of these elements in a representation of the C.C.R. algebra are of physical; interest because with them one can form a number operator (or a density operator in the infinite case). For example, consider the Fock representation $\{\mathfrak{H}_F, \pi_F\}$ of $\mathfrak{A}(\mathfrak{H})$, and the operator $\pi_F(W(h_i))$, where $\{h_i\}$ is an orthonormal basis for $\mathfrak{H}$. From (3.26) we have, for real k,

$$\pi_F(W(kh_i))=1\otimes 1\otimes\ldots\otimes\pi_i(U(k))\otimes 1\otimes\ldots$$

$$=1\otimes 1\otimes\ldots\otimes\pi_i(e^{ikq})\otimes 1\otimes\ldots, \tag{3.31}$$

so that $$\pi_F(\phi(h_i))\equiv\frac{1}{i}\lim_{k\to 0}\frac{1}{k}(\pi_F(W(kh_i))-\pi_F(W(0)))$$

$$=1\otimes 1\otimes\ldots\otimes\pi_i(q)\otimes 1\otimes\ldots \tag{3.32a}$$

In a similar fashion, we have

$$\pi_F(\pi(h_i))\equiv 1\otimes 1\otimes\ldots\otimes\pi_i(p)\otimes 1\otimes\ldots. \tag{3.32b}$$

The operators $\pi_i(p)$ and $\pi_i(q)$ are copies of the representatives of the elements p and q in the harmonic oscillator representation developed in Appendix A. We use these to define

$$\pi_F(\Psi(h_i))\equiv\frac{\pi_F(\phi(h_i))+i\pi_F(\pi(h_i))}{\sqrt{2}}, \tag{3.33a}$$

$$\pi_F(\Psi(h_i)^*)\equiv\frac{\pi_F(\phi(h_i))-i\pi_F(\pi(h_i))}{i\sqrt{2}}, \tag{3.33b}$$

and, for arbitrary $f\in\mathfrak{H}$, with $f=\sum_i \alpha^i h_i$,

$$\pi_F(\Psi(f))\equiv\sum_i \bar{\alpha}^i\pi_F(\Psi(h_i)), \tag{3.33c}$$

and

$$\pi_F(\Psi(f)^*)\equiv\sum_i \alpha^i\pi_F(\Psi(h_i)^*), \tag{3.33d}$$

with domains to be described. The action of the $\pi_F(\Psi(h_i))$ and $\pi_F(\Psi(h_i)^*)$ on the basis vectors $\{|n_1,n_2,\ldots,n_i,\ldots\rangle\equiv|n_1\rangle\otimes|n_2\rangle\otimes\ldots\otimes|n_i\rangle\otimes\ldots : \sum_{i=1}^{\infty} n_i<\infty\}$ is

$$\pi_F(\Psi(h_i))|n_1,n_2,\ldots,n_i,\ldots\rangle\equiv\sqrt{n_i}\,|n_1,n_2,\ldots,n_i-1,\ldots\rangle, \tag{3.34a}$$

and

$$\pi_F(\Psi^*(h_i))|n_1,n_2,\ldots,n_i,\ldots\rangle\equiv\sqrt{n_i+1}\,|n_1,n_2,\ldots,n_i+1,\ldots\rangle. \tag{3.34b}$$

If we interpret the basis vectors in the usual manner, where $|n_1,n_2,\ldots,n_i,\ldots\rangle$ corresponds to the state in which there are n_i particles in the state h_i, then $\pi_F(\Psi^*(h_i))$ and $\pi_F(\Psi(h_i))$ create and destroy a particle in the state h_i, respectively. The domains of $\pi_F(\Psi(f)^*)$ and $\pi_F(\Psi(f))$ are

$D(\pi_F(\Psi(f)^*))=\{\Psi\in H_F:\|\pi_F(\Psi(f)^*)\Psi\|<\infty\}$

$$=\left\{\Psi=\sum_{n_1,n_2,\ldots=0}^{\infty}\alpha(n_1,n_2,\ldots)|n_1,n_2,\ldots\rangle : \sum_{n_1,n_2,\ldots=0}^{\infty}|\alpha(n_1,n_2,\ldots)\sum_i a_i\sqrt{n_i+1}|^2<\infty\right\}, \quad (3.35a)$$

and $D(\pi_F(\Psi(f)))=\{\Psi\in H_F:\|\pi_F(\Psi(f))\Psi\|<\infty\}$

$$=\left\{\Psi=\sum_{n_1,n_2,\ldots=0}^{\infty}\alpha(n_1,n_2,\ldots)|n_1,n_2,\ldots\rangle : \sum_{n_1,n_2,\ldots=0}^{\infty}|\alpha(n_1,n_2,\ldots)\sum_i a_i\sqrt{n_i}|^2<\infty\right\}. \quad (3.35b)$$

These domains may be different for different functions f. For example, $D(\pi_F(\Psi(h_i)))\neq D(\pi_F(\Psi(h_j)))$ for $i\neq j$. To see this, note that the vector

$$\sum_{n_i=1}^{\infty}\frac{1}{n_i}|0,0,\ldots,0,n_i,0,\ldots\rangle$$

is in $D(\pi_F(\Psi(h_j)))$, but not in $D(\pi_F(\Psi(h_i)))$. This motivates us to determine the common domain of definition of the annihilation and creation operators. Consider the number operator

$$N\equiv\sum_i\pi_F(\Psi(h_i)^*)\pi_F(\Psi(h_i)). \quad (3.36)$$

For arbitrary $\Psi\in H_F$ we have

$$N\Psi=N\sum_{n_1,n_2,\ldots=0}^{\infty}\alpha(n_1,n_2,\ldots)|n_1,n_2,\ldots\rangle, \quad (3.37)$$

so that the domain D(N) of N is

$$D(N)=\left\{\Psi=\sum_{n_1,n_2,\ldots=0}^{\infty}\alpha(n_1,n_2,\ldots)|n_1,n_2,\ldots\rangle:\ \sum_{n_1,n_2,\ldots=0}^{\infty}|\alpha(n_1,n_2,\ldots)\sum_i n_i|^2<\infty\right\},\quad(3.38a)$$

and the domain $D(N^{1/2})$ of $N^{1/2}$ is

$$D(N^{1/2})=\left\{\Psi=\sum_{n_1,n_2,\ldots=0}^{\infty}\alpha(n_1,n_2,\ldots)|n_1,n_2,\ldots\rangle:\ \sum_{n_1,n_2,\ldots=0}^{\infty}|\alpha(n_1,n_2,\ldots)\sum_i\sqrt{n_i}|^2<\infty\right\}.\quad(3.38b)$$

The common domain of definition of the set $\{\pi_F(\Psi(f)^*),\pi_F(\Psi(g)):f,g\in\mathfrak{H}\}$, and therefore of the set $\{\pi_F(\phi(f)),\pi_F(\pi(g)):f,g\in\mathfrak{H}\}$, contains $D(N^{1/2})$. Since $D(N^{1/2})$ contains all finite linear combinations of the basis vectors, it is dense in H_F. On this common, dense domain of definition, the $\pi_F(\phi(f))$ and $\pi_F(\pi(g))$ satisfy

$$[\pi_F(\phi(f)),\pi_F(\pi(g))]\Psi=i\mathrm{Re}[\langle f,g\rangle]\Psi,\qquad(3.39a)$$

$$[\pi_F(\phi(f)),\pi_F(\phi(g))]\Psi=i\mathrm{Im}[\langle f,g\rangle]\Psi,\qquad(3.39b)$$

and

$$[\pi_F(\pi(f)),\pi_F(\pi(g))]\Psi=i\mathrm{Im}[\langle f,g\rangle]\Psi,\qquad(3.39c)$$

i.e., they form a representation of the $\phi(f)$ and $\pi(g)$. Furthermore, $\pi_F(\Psi(f)^*)$ is the Hermitian adjoint of $\pi_F(\Psi(f))$, and the $\pi_F(\phi(f))$ and $\pi_F(\pi(g))$ are therefore self-adjoint.

The previous discussion shows that in the Fock representation there exist densely defined, self-adjoint operators that form a representation of the $\phi(f)$ and $\pi(g)$. The following version of Stone's Theorem ([Riesz] page 385) allows us to formulate conditions that an arbitrary representation must satisfy in order to have this property.

Theorem 3.40 (Stone's Theorem) Every weakly continuous 1-parameter group of unitary transformations $\{U_t\}$ on a Hilbert space $\mathfrak{H}$ is generated by an infinitesimal transformation A. A is a densely defined, self-adjoint transformation, which in general is not bounded, and which satisfies

$$U_t = e^{itA}, \quad A = \frac{1}{i} \lim_{k \to 0} \frac{1}{k}(U_k - \mathbb{1}).$$

note: the unitary transformations U_t are weakly or strongly continuous in the real parameter t if the sequence $\{U_{t_n}\}$ converges weakly or strongly, respectively, to U_t whenever t_n converges to t. See Definition 1.31 for a definition of weak and strong continuity.

Now, in an arbitrary representation $\{H,\pi\}$ of the C.C.R. algebra $\mathfrak{A}(\mathfrak{H})$, $\pi(W(f))$ is a unitary operator on H. The set $\{\pi(W(tf));\ t \in \mathbb{R}\}$ is therefore a 1-parameter group of unitary transformations, for each fixed f. If we require the operators $\pi(W(tf))$ to be weakly continuous in the real parameter t, for all f, then Theorem 3.40 implies that there exist densely defined, self-adjoint transformations $\pi(\phi(f))$ that satisfy

$$\pi(W(tf))=e^{it\pi(\phi(f))}, \tag{3.41a}$$

and

$$\pi(\phi(f))=\frac{1}{i}\lim_{k\to 0}\frac{1}{k}(\pi(W(kf))-\pi(W(0))). \tag{3.41b}$$

With $\pi(\pi(g))\equiv\pi(\phi(ig))$, it is easy to verify (using the commutation relations (3.21) of the W(f)) that the $\pi(\phi(f))$ and $\pi(\pi(g))$ satisfy the commutation relations (3.5), i.e., they form a representation of the $\phi(f)$ and $\pi(g)$. The condition we seek is that the unitary operators $\pi(W(tf))$ be weakly continuous in the real parameter t. Since weak and strong continuity coincide for unitary operators we make the following definition.

Definition 3.42 A representation $\{H,\pi\}$ of the C.C.R. algebra $\mathfrak{A}(\mathfrak{H})$ with the property that the unitary operators $\pi(W(tf))$ are strongly continuous in the real parameter t, for fixed $f\in\mathfrak{H}$, is said to be a regular representation.

Thus, a regular representation of A(h) allows for a representation of the operators f(f) and p(f) and for the existence of a number or density operator.

One of the reasons a relatively simple discussion of spin systems was possible in Chapter 2 was the simple characterization of a state in Proposition 2.12. An arbitrary state ω over the C.C.R. algebra $\mathfrak{A}(\mathfrak{H})$ is determined by its values on the W(f). We will denote these values by

$E_\omega(f) \equiv \omega(W(f))$. $E_\omega(f)$ is a functional on the set of vectors $\{f : f \in \mathfrak{H}\}$. The following conditions characterize functionals E(f) that define a state over $\mathfrak{A}(\mathfrak{H})$.

Proposition 3.43 A functional E(f) defined for all $f \in \mathfrak{H}$ that satisfies

i) $E_\omega(0)=1$,

ii) $\overline{E_\omega(f)} = E_\omega(-f)$,

iii) $\sum_{i,j} \bar{\alpha}^i \alpha^j \exp[\frac{-i}{2} \mathrm{Im}(<f_j, f_i>)] E_\omega(f_j - f_i) \geq 0$

for all $\{f_i\} \in \mathfrak{H}$ and $\{\alpha^i\} \in \mathbb{C}$, $i=1,2,\ldots,n$,

and is continuous defines a state over the C.C.R. algebra $\mathfrak{A}(\mathfrak{H})$. This state is obtained by defining

$$\omega(W(f)) \equiv E(f), \qquad (3.43a)$$

and then extending this definition to all of $\mathfrak{A}(\mathfrak{H})$ by linearity.

Proof: First note that it is possible to extend (3.43a) to all of $\mathfrak{A}(\mathfrak{H})$ since $E_\omega(f)$ is continuous. By definition, ω is a linear functional over $\mathfrak{A}(\mathfrak{H})$. Condition i) guarantees that ω is normalized, condition ii) guarantees that the ω is real and condition iii) guarantees that ω is positive.

Araki and Woods constructed representations of the C.C.R. algebra which they claimed were appropriate for the description of the infinite Bose gas ([Arak]). They worked solely with functionals over the algebra, and never introduced the concept of a state. They give the conditions of Proposition 3.43 as being necessary and sufficient for a functional to define a cyclic representation of the C.C.R. algebra. By dealing with states we obtain a simple proof of this claim (just use Proposition 3.43 and the G.N.S. construction) along with a physical interpretation of these conditions. Note that Araki and Woods actually dealt with the case when $\mathfrak{H}$ is a pre-Hilbert space and so did not require the functional to be continuous. They showed that it is possible to extend (in the strong operator topology) a representation of $\mathfrak{A}(\mathfrak{H})$ to a representation of $\mathfrak{A}(\mathfrak{H}_1)$, where $\mathfrak{H}_1$ is a subset of the completion $\overline{\mathfrak{H}}$ of $\mathfrak{H}$. This subset contains all vectors $f \in \overline{\mathfrak{H}}$ for which there exists a sequence $\{f_n\} \in \overline{\mathfrak{H}}$ such that $\lim_{n,m \to \infty} E_\omega(f_n - f_m) = 0$. If $E_\omega(f)$ is continuous then $\mathfrak{H}_1 = \overline{\mathfrak{H}}$. Since the functionals that we shall postulate are continuous our methods and results are the same as Araki and Woods. We extend the algebra to the completed Hilbert space $\overline{\mathfrak{H}}$ and work with a C*-algebra while they extend the representations of the algebra to the completed Hilbert space $\overline{\mathfrak{H}}$ and never use the concept of a C*-algebra. We should also note that the inequality of two functionals implies that the corresponding cyclic G.N.S. representations are unitarily inequivalent. It does not imply that the corresponding G.N.S. representations are unitarily inequivalent. For example, consider a representation $\{\mathfrak{H}, \pi\}$ and two cyclic vectors $\Psi_1, \Psi_2 \in \mathfrak{H}$. The cyclic representations $\{\mathfrak{H}, \pi, \Psi_1\}$ and $\{\mathfrak{H}, \pi, \Psi_2\}$ are the cyclic G.N.S. representations corresponding to the functionals $E_1(f) = (\Psi_1, \pi(W(f))\Psi_1)$ and $E_2(f) = (\Psi_2, \pi(W(f))\Psi_2)$. In general

$E_1(f) \neq E_2(f)$, but $\{\mathfrak{H},\pi\}$ is the G.N.S. representation corresponding to both functionals. At the end of this chapter a proof will be suggested to show that the G.N.S. representations corresponding to states of an infinite Bose gas with different densities are unitarily inequivalent. this claim goes beyond the statement that the corresponding cyclic representations are unitarily inequivalent.

A state ω over $\mathfrak{A}(\mathfrak{H})$ is said to be regular if the corresponding G.N.S. representation $\{\mathfrak{H}_\omega,\pi_\omega,\Omega_\omega\}$ is regular (see [Brat2] section 5.2.3. for a discussion of regular states). It is possible to formulate a condition on the functional $E_\omega(f)$ in order for the state ω to be regular. In the representation $\{\mathfrak{H}_\omega,\pi_\omega,\Omega_\omega\}$ one has

$$\|(\pi_\omega(W(tf))-\mathbb{1})\pi_\omega(W(g))\Omega_\omega\|^2=2-e^{-it\,\mathrm{Im}\langle f,g\rangle}\omega(W(tf))-e^{it\,\mathrm{Im}\langle f,g\rangle}\omega(W(-tf))$$

$$=2-e^{-it\,\mathrm{Im}\langle f,g\rangle}E_\omega(tf)-e^{it\,\mathrm{Im}\langle f,g\rangle}E_\omega(-tf).\quad(3.44)$$

This leads to the following Proposition.

Proposition 3.45 A state ω over the C.C.R. algebra $\mathfrak{A}(\mathfrak{H})$ is regular if and only if $E_\omega(tf)=\omega(W(tf))$ is continuous in the real parameter t, for all $f\in\mathfrak{H}$.

Proof: The proof follows immediately from (3.44).

For example, consider the Fock representation $\{H_F,\pi_F,\Psi_{Fo}\}$. We can use the cyclic vector Ψ_{Fo} to define the vector state $\phi_{Fo}(A)=(\Psi_{Fo},\pi_F(A)\Psi_{Fo})$, which we refer to as the Fock ground state. The

Fock representation is unitarily equivalent to the G.N.S. representation corresponding to the Fock ground state ϕ_{Fo}. The linear functional $E_{Fo}(f)=\phi_{Fo}(W(f))$ is calculated as follows. It is possible to show that (see equation (4.6c) in the fourth section of this chapter)

$$W(f)_F=e^{-\|f\|^2/4}\exp[\frac{i}{\sqrt{2}}\Psi(f)_F^*]\exp[\frac{i}{\sqrt{2}}\Psi(f)_F],$$

so that
$$E_{Fo}(f)=e^{-\|f\|^2/4}(\Psi_{Fo},\exp[\frac{i}{\sqrt{2}}\Psi(f)_F^*]\exp[\frac{i}{\sqrt{2}}\Psi(f)_F]\Psi_{Fo})$$

$$=e^{-\|f\|^2/4}(\exp[\frac{i}{\sqrt{2}}\Psi(f)_F]\Psi_{Fo},\exp[\frac{i}{\sqrt{2}}\Psi(f)_F]\Psi_{Fo})$$

$$=e^{-\|f\|^2/4}. \tag{3.46}$$

It is obvious that $E_{Fo}(tf)$ is continuous in the real parameter t, for each fixed $f\in\mathfrak{H}$. This implies that the Fock ground state ϕ_{Fo} is a regular state, and hence the Fock representation is a regular representation. This, of course, is the reason why we were able to recover the infinitesimal generators $\pi_F(\phi(f))$ of the unitary operators $\pi_F(W(f))$ in the Fock representation (Stones Theorem 3.40 guarantees that these infinitesimal generators exist).

When $\mathfrak{H}$ is the Hilbert space appropriate for the description of a single point particle, the C.C.R. algebra $\mathfrak{A}(\mathfrak{H})$ corresponds to a system containing an arbitrary, finite or infinite, number of these point particles. The number of particles in the system depends upon the state the system is in. Different representations will then correspond to different particle numbers. For example, every state that is a vector state in the Fock representation is a finite particle state. To see this, let Ψ be an arbitrary vector in the densely defined domain of the number operator N on $\mathfrak{H}_F$ (see (3.36)). The expectation value $(\Psi,N\Psi)=(N^{1/2}\Psi,N^{1/2}\Psi)$ of the number operator gives the average number of particles in the state Ψ. Equation (3.38) guarantees that this number is finite.

3.3 QUASI-LOCAL C.C.R. ALGEBRAS

We now analyze the quasi-local structure of the C.C.R. algebra $\mathfrak{A}(\mathfrak{H})$ over the Hilbert space $\mathfrak{H}=\mathcal{L}^2(\mathbb{R}^3)$. This is the C.C.R. algebra corresponding to an infinitely extended system of bosons. We will denote it by $\mathfrak{A}_{\mathbb{R}^3}$. For an arbitrary bounded region Z in $\mathbb{R}^3$, denote the subspace of $\mathfrak{H}$ formed by the functions whose support is in Z by H_Z, and the corresponding C.C.R. algebra $\mathfrak{A}(H_Z)$ by $\mathfrak{A}_Z$. The collection Σ of all finite regions Z, ordered by inclusion, is a directed set. If $Z_1 \leq Z_2$, then an arbitrary element $f_1 \in H_{Z_1}$ is an element of H_{Z_2}, and we can construct the following mapping $i_{2,1}$ from $\mathfrak{A}_{Z_1}$ into $\mathfrak{A}_{Z_2}$. For arbitrary $W_1(f_1) \in \mathfrak{A}_{Z_1}$ define

$$i_{2,1}(W_1(f_1)) \equiv W_2(f_1). \tag{3.47}$$

This mapping may be extended to all of $\mathfrak{A}_{Z_1}$. The resulting mapping is a *-homomorphism from $\mathfrak{A}_{Z_1}$ into $\mathfrak{A}_{Z_2}$ that satisfies

i) $i_{2,1}(\mathbb{1}_1)=\mathbb{1}_2$, where $\mathbb{1}_1=W_1(0)$ and $\mathbb{1}_2=W_2(0)$,

ii) $i_{3,2}i_{2,1}=i_{3,1}$ whenever $Z_1 \leq Z_2 \leq Z_3$.

The family of C*-algebras $\{\mathfrak{A}_Z : Z \in \mathbb{R}^3\}$ satisfies the postulate of isotony (see pg. 25), and therefore admits a C*-inductive limit $\mathfrak{A}$. Recall that this is a C*-algebra with identity $\mathbb{1}$ that has the property that for every $Z \in \Sigma$ there exists a *-homomorphism i_Z from $\mathfrak{A}_Z$ into $\mathfrak{A}$ that satisfies

i) $i_Z(1_Z)=1$, where 1_Z is the identity for $\mathfrak{A}_Z$,

ii) $i_{Z_2}(\mathfrak{A}_{Z_2}) \supset i_{Z_1}(\mathfrak{A}_{Z_1})$, whenever $Z_2 \supset Z_1$,

and iii) $\overline{\bigcup_{Z \in \Sigma} i_Z(\mathfrak{A}_Z)} = \mathfrak{A}$, where the bar denotes the uniform closure.

Any element of $\mathfrak{H}$ may be approximated to any degree by functions with bounded support. This result suggests that $\mathfrak{H} = \overline{\bigcup_{Z \in \Sigma} H_Z}$. Hence this C*-inductive limit is equal to $\mathfrak{A}_{\mathbb{R}^3}$, so that the set $\{\mathfrak{A}_{\mathbb{R}^3}; \mathfrak{A}_Z, Z \in \Sigma\}$ is a quasi-local algebra (see section 1.5). Since every element of $\mathfrak{A}_{Z_1}$ is also an element of $\mathfrak{A}_{\mathbb{R}^3}$, the *-homomorphism i_Z from $\mathfrak{A}_Z$ into $\mathfrak{A}_{\mathbb{R}^3}$ is simply given by $i_Z(A)=A$ for all $A \in \mathfrak{A}_Z$.

Let ω be a regular state over the quasi-local algebra $\{\mathfrak{A}_{\mathbb{R}^3}; \mathfrak{A}_Z, Z \in \Sigma\}$ and $\{\mathfrak{H}_\omega, \pi_\omega, \Omega_\omega\}$ the corresponding G.N.S. representation. Let $\{h_{Zi}\}$ be an orthonormal basis for H_Z. We formally define an operator N_Z for the region Z as

$$N_Z \equiv \sum_i \pi_\omega(\Psi^*(h_i)\Psi(h_i)). \tag{3.48}$$

If this operator exists we interpret it as the number operator for the region Z. Although we are not guaranteed that N_Z exists, we expect that it will for states which correspond to an infinite system with finite density (see [Brat2] section 5.2.3). If a density operator exists in $\mathfrak{A}_{\mathbb{R}^3}$ it will be given by

$$\rho_{op} = \lim_{Z \to \mathbb{R}^3} \frac{N_Z}{V_Z}, \tag{3.49}$$

where V_Z is the volume of the region Z.

3.4 THE INFINITE FREE BOSE GAS

In this section we examine the algebraic description of an infinite Bose gas of finite density, when all particles are in the zero momentum state. The C*-algebra corresponding to this system is the quasi-local algebra $\{\mathfrak{A}_{\mathbb{R}^3}; \mathfrak{A}_Z, Z \in \Sigma\}$. The state ω_ρ corresponding to a given density ρ of particles in the zero momentum state will be constructed in terms of its generating functional $E_\rho(f)$, following the methods of Araki and Woods ([Arak]). The generating functional $E_\rho(f)$ will be dependent on the density ρ. This causes Araki and Woods to conclude that different densities give rise to unitarily inequivalent representations. From the discussion following Proposition 3.43 we know that this conclusion is not necessarily true, so we will explicitly show that different densities give rise to unitarily inequivalent representations. We will also find (following Araki and Woods) that the G.N.S representation corresponding to a fixed density is reducible, and express the representation as a direct integral of unitarily inequivalent irreducible representations. This shows, among other things, that the C.C.R. algebra $\mathfrak{A}(\mathbb{C})$ admits an infinite number of unitarily inequivalent irreducible representations.

Let Λ^L denote the cube in $\mathbb{R}^3$ centered on the origin with sides of length L (and volume L^3). The restriction $E_{restL}(f)$ of $E_\rho(f)$ to the local algebra $\mathfrak{A}_{\Lambda^L}$ is obtained by restricting the functions f to the set of functions whose support lies in Λ^L, i.e., we define

$$E_{restL}(f)\equiv E_\rho(f) \text{ for all } f(\mathbf{x}) \text{ with Supp}(f) \text{ in } \Lambda^L. \tag{3.50}$$

The generating functional E_ρ is then given by $E_\rho(f)=\lim_{L\to\infty} E_{restL}(f)(f_{\Lambda^L})$, where f_{Λ^L} is the restriction of f to the region Λ^L,

$$f_{\Lambda^L}(\mathbf{x})=\begin{cases} f(x) & \text{if } \mathbf{x}\in \Lambda^L \\ 0 & \text{if } \mathbf{x}\notin \Lambda^L \end{cases} . \tag{3.51}$$

To determine E_ρ we could postulate a form for E_L and then use $E_\rho(f)=\lim_{L\to\infty} E_{restL}(f)(f_{\Lambda^L})$. The problem with this, of course, is how does one determine $E_{restL}(f)$ without knowing E_ρ. For this reason we will follow Araki and Woods and postulate that the functional E_ρ is the limit of the functionals E_L that describe a system of N bosons in the box Λ^L such that $\frac{V_L}{N}=\rho$, where V_L is the volume of Λ^L. Note that the description of a gas in a finite box is not the same as the description of a finite portion of an infinite gas; after E_ρ has been calculated we will find that $E_{restL} \neq E_L$.

Consider N particles in the box Λ^L. Let all particles be in the zero momentum state and let the system have a mean particle density $\rho = \frac{V_L}{N}$, where V_L is the volume of Λ^L. In section 3.2, equation (3.46), we calculated the Fock space functional $E_{Fo}(f)$. It corresponds to the no particle case (zero density). Now we are interested in calculating the functional $E_L(f)$ which corresponds to the N-particle case, although we are taking the special case when all particles have zero momentum. To calculate $E_L(f)$ we use the Fock representation $\{\mathfrak{H}_F, \pi_F, \Omega_{Fo}\}$ of the local algebra $\mathfrak{A}_{\Lambda^L}$. To simplify notation we will denote the element $\pi_F(A)$ by A_F. For example $\pi_F(W(f)) = W(f)_F$ and $\pi_F(\Psi(f)^*) = \Psi(f)^*_F$. The functional $E_L(f)$ is the given by

$$E_L(f) \equiv \omega(W(f)) = (\Omega_N, W(f)_F \Omega_N), \tag{3.52}$$

where Ω_N is the vector in the Fock representation corresponding to the state which contains $N = \rho L^3 = \rho V$ particles in the zero momentum state in Λ^L. This vector is given by

$$\Omega_N = (N!)^{-1/2} (\Psi(f_V)^*_F)^N \Omega_{Fo}, \tag{3.53a}$$

where
$$f_V(x) = V^{-1/2} = \left(\frac{\rho}{N}\right)^{1\backslash 2},\ x \in \Lambda^L, \tag{3.53b}$$

i.e., we create N particles in the state f_V, which is the zero momentum state for a single particle in a box of volume V. This gives

$$E_L(f)=(N!)^{-1}(\Psi(f_V)_F^*)^N\Omega_{Fo}\,,\,W(f)_F(\Psi(f_V)_F^*)^N\Omega_{Fo}). \tag{3.54}$$

To evaluate (3.54) we need the following results.

$$\|(\Psi(f)_F^*)^N\Omega_{Fo}\|^2=\|f\|^{2N}N!, \tag{3.55a}$$

$$\exp[\Psi(f)_F]\,[\Psi(g)_F^*]^N\exp[\Psi(-f)_F]=[\Psi(g)_F^*+(f,g)]^N, \tag{3.55b}$$

and

$$W(f)_F=E_{Fo}(f)\,\exp[\tfrac{i}{\sqrt{2}}\Psi(f)_F^*]\,\exp[\tfrac{i}{\sqrt{2}}\Psi(f)_F], \tag{3.55c}$$

where $E_{Fo}(f)=\exp[-\frac{1}{4}\|f\|^2]$ is the generating functional for the Fock ground state, (3.46). To show (3.55a), we note that the commutation relations $[\Psi(f)_F,\Psi(f)_F^*]=\|f\|^2$ imply

$$[(\Psi(f)_F,(\Psi(f)_F^*)^N]=N\|f\|^2(\Psi(f)_F^*)^{N-1}. \tag{3.56}$$

Using this along with the result that $\Psi(f)_F\Omega_{Fo}=0$ we obtain the following

$$(\Psi(f)_F)^N(\Psi(f)_F^*)^N\Omega_{Fo}=(\Psi(f)_F)^{N-1}\{(\Psi(f)_F^*)^N\Psi(f)_F+N\|f\|^2(\Psi(f)_F^*)^{N-1}\}\Omega_{Fo}$$

$$=N\|f\|^2(\Psi(f)_F)^{N-1}(\Psi(f)_F^*)^{N-1}\Omega_{Fo}. \tag{3.57}$$

Repeated application of (3.57) gives

$$(\Psi(f)_F)^N(\Psi(f)_F^*)^N\Omega_{Fo}=N!\|f\|^{2N}\Omega_{Fo}. \qquad (3.58)$$

Using (3.58) we have

$$\|(\Psi(f)_F^*)^N\Omega_{Fo}\|^2=((\Psi(f)_F^*)^N\Omega_{Fo}\,,\,(\Psi(f)_F^*)^N\Omega_{Fo})$$

$$=(\Omega_{Fo}\,,\,(\Psi(f)_F)^N(\Psi(f)_F^*)^N\Omega_{Fo})$$

$$=N!\|f\|^{2N} \qquad (3.55a)$$

(3.55b) follows from the commutation relations of the $\Psi(f)_F^*$ and $\Psi(f)_F$ and the relation $e^AB^Ne^{-A}=[e^ABe^{-A}]^N=\left[B+[A,B]\right]^N$, valid when [A,B] is a constant. (3.55c) follows from the relation $e^{A+B}=e^{-1/2[A,B]}e^A\,e^B$ and the commutation relations of the $\Psi(f)_F^*$ and $\Psi(f)_F$,

$$W(f)_F=\exp[i\phi(f)_F]=\exp[\tfrac{i}{\sqrt{2}}\Psi(f)_F^*+\tfrac{i}{\sqrt{2}}\Psi(f)_F]$$

$$=\exp\left[-\tfrac{1}{2}[\Psi(f)_F^*,\Psi(f)_F]\right]\exp[\tfrac{i}{\sqrt{2}}\Psi(f)_F^*]\,\exp[\tfrac{i}{\sqrt{2}}\Psi(f)_F]$$

$$=\exp[-\tfrac{1}{4}\|f\|^2]\exp[\tfrac{i}{\sqrt{2}}\Psi(f)_F^*]\,\exp[\tfrac{i}{\sqrt{2}}\Psi(f)_F]. \qquad (3.55c)$$

We can now evaluate (3.54).

$$E_L(f)=(N!)^{-1}(\Psi(fV)_F^*)^N\Omega_{Fo}\,,\,W(f)_F(\Psi(fV)_F^*)^N\Omega_{Fo})$$

(using (3.55c))

$$=E_{Fo}(f)(N!)^{-1}\Big((\Psi(fV)_F^*)^N\Omega_{Fo}\,,\,\exp[\tfrac{i}{\sqrt{2}}\Psi(f)_F^*]\,\exp[\tfrac{i}{\sqrt{2}}\Psi(f)_F](\Psi(fV)_F^*)^N\Omega_{Fo}\Big)$$

$$=E_{Fo}(f)(N!)^{-1}\Big(\exp[\tfrac{i}{\sqrt{2}}\Psi(-f)_F](\Psi(fV)_F^*))^N\Omega_{Fo}\,,\,\exp[\tfrac{i}{\sqrt{2}}\Psi(f)_F](\Psi(fV)_F^*)^N\Omega_{Fo}\Big),$$

$$(3.59)$$

where the last equality follows from the relation

$$\exp[\tfrac{i}{\sqrt{2}}\Psi(f)_F^*]^\dagger=\exp[-\tfrac{i}{\sqrt{2}}\Psi(f)_F]=\exp[\tfrac{i}{\sqrt{2}}\Psi(-f)_F].$$

We now put the right hand side of this scalar product into a more convenient form. Since $\exp[\frac{i}{\sqrt{2}}\Psi(-f)_F]\Omega_{Fo}=\Omega_{Fo}$, we have

$$\exp[\tfrac{i}{\sqrt{2}}\Psi(f)_F](\Psi(fV)_F^*)^N\Omega_{Fo}=\exp[\tfrac{i}{\sqrt{2}}\Psi(f)_F](\Psi(fV)_F^*)^N\exp[\tfrac{i}{\sqrt{2}}\Psi(-f)_F]\Omega_{Fo},$$

(using (3.55b) $\quad =[\Psi(fV)_F^*-\frac{i}{\sqrt{2}}(f,fV)]^N\Omega_{Fo}$.

$$=\sum_{r=0}^{N}(-1)^r\Big(\frac{N!}{r!(N-r)!}\Big)(\Psi(fV)_F^*)^{N-r}[-\tfrac{i}{\sqrt{2}}(f,fV)]^r\Omega_{Fo},\quad(3.60a)$$

where the last equality follows from the binomial theorem. In a similar fashion we find

$$\exp[\tfrac{i}{\sqrt{2}}\Psi(-f)_F](\Psi(fv)_F^*)^N\Omega_{Fo}=\sum_{r=0}^{N}(-1)^r\Big(\frac{N!}{r!(N-r)!}\Big)(\Psi(fv)_F^*)^{N-r}[\tfrac{i}{\sqrt{2}}(f,fv)]^r\Omega_{Fo}. \quad (3.60b)$$

Substituting (3.60) into (3.59) gives

$$E_L(f)=E_{Fo}(f)(N!)^{-1}$$
$$\times\sum_{r,s=0}^{N}\left\{\begin{array}{c}(-1)^r(-1)^s\Big(\frac{N!}{r!(N-r)!}\Big)\Big(\frac{N!}{s!(N-s)!}\Big)[-\frac{i}{\sqrt{2}}(f,fv)]^r[-\frac{i}{\sqrt{2}}\overline{(f,fv)}]^s\\ \times\Big(\Psi(fv)_F^*)^{N-s}\Omega_{Fo},(\Psi(fv)_F^*)^{N-r}\Omega_{Fo}\Big)\end{array}\right\}. \quad (3.61)$$

Using (3.55a) and the result $\|f_V\|=1$we have

$$(\Psi(fv)_F^*)^{N-s}\Omega_{Fo},(\Psi(fv)_F^*)^{N-r}\Omega_{Fo})=\delta_{r,s}\|(\Psi(f)_F^*)^{N-r}\Omega_{Fo}\|^2$$

$$=\delta_{r,s}(N-r)!. \quad (3.62)$$

Substituting (3.62) into (3.61) gives

$$E_L(f)=E_{Fo}(f)(N!)^{-1}\sum_{r=0}^{N}(-1)^r\Big(\frac{N!}{r!(N-r)!}\Big)^2(N-r)!\left[\frac{|(f,f_V)|^2}{2}\right]^r$$

$$=E_{Fo}(f)\sum_{r=0}^{N}(-1)^r\frac{N!}{(r!)^2(N-r)!}\left[\frac{|(f,f_V)|^2}{2}\right]^r. \quad (3.63)$$

We can put the above expression into a more convenient form by noting that the N^{th} Laguerre polynomial is given by the power series

$$L_N(x)=\sum_{r=0}^{N}(-1)^r\frac{N!}{(r!)^2(N-r)!}x^r. \tag{3.64}$$

This allows us to write $E_L(f)$ as

$$E_L(f)=E_{Fo}(f)L_N\Big(\tfrac{1}{2}\,|(f,f_V)|^2\Big). \tag{3.65}$$

Since the support of f is contained in Λ^L we can write the scalar product (f,f_V) as

$$(f,f_V)=\frac{1}{(V)^{1/2}}\int_{\mathbb{R}^3}\overline{f(\mathbf{x})}\,d\mathbf{x}=\frac{\rho}{N}\tilde{f}(0),$$

where $\tilde{f}(0)$ is the Fourier transform $\tilde{f}(\mathbf{k})=\int_{\mathbb{R}^3}\overline{f(\mathbf{x})}\,e^{i\mathbf{k}\cdot\mathbf{x}}d\mathbf{x}$ of $f(\mathbf{x})$ evaluated at $\mathbf{k}$=0. The quantity $|(f,f_V)|^2$ may therefore be written as

$$|(f,f_V)|^2=\frac{\rho}{N}|\tilde{f}(0)|^2,\quad \tilde{f}(0)=\int_{\mathbb{R}^3}f(x)d\mathbf{x}. \tag{3.66}$$

Substituting (3.66) into (3.65) gives

$$E_L(f)=E_{Fo}(f)L_N\Big(\tfrac{1}{2}\tfrac{\rho}{N}|\tilde{f}(0)|^2\Big). \tag{3.67}$$

The limit V and $N\rightarrow\infty$ is now taken, holding the density $\rho=N/V$ constant. The Laguerre polynomials have the property (see [Szeg] Theorem 8.1.3)

$$\lim_{N\to\infty} L_N(z/N)=J_o(2z^{1/2}), \tag{3.68a}$$

where $J_o(x)$ is the zero order Bessel function,

$$J_o(x)=\sum_{n=0}^{\infty}\frac{\left[\left[\frac{x}{2}\right]^2\right]^n}{(n!)^2}. \tag{3.68b}$$

Using this property we have

$$E_\rho(f)=E_{Fo}(f)\lim_{N\to\infty} L_N\left(\frac{1}{2}\frac{\rho}{N}|\tilde{f}(0)|^2\right)$$

$$=\exp[-\tfrac{1}{4}\|f\|^2]J_o\left((2\rho)^{1/2}|\tilde{f}(0)|\right). \tag{3.69}$$

Note that $E_{restL}(f)$ is simply obtained by restricting the functions f to the region L^L, and this does not give the functional $E_L(f)$.

Equation (4,20) is the expression that we postulate for the functional corresponding to the ground state of the infinite bose gas with density ρ. At this point we should demonstrate that this functional satisfies the conditions of Proposition 3.43 and hence defines a state of $\mathfrak{A}_{\mathbb{R}^3}$. Since it is quite difficult to show that this functional satisfies condition iii) of Proposition 3.43 we we first proceed to construct a cyclic representation. In terms of this representation this proof will be easy.

The G.N.S. representation corresponding to the functional $E_\rho(f)$ will now be constructed. The functional $E_\rho(f)$ is a product of the Fock functional with another functional. This suggests that the representation that we are looking for is the direct product of two representations, one of which is the Fock representation. We first define the representation space $\mathfrak{H}_\rho$ to be

$$\mathfrak{H}_\rho=\mathfrak{H}_F\otimes\mathcal{L}^2(S^1), \tag{3.70a}$$

where $\mathfrak{H}_F$ is the Fock space and $\mathcal{L}^2(S^1)$ is the space of square integrable functions on the unit circle with respect to the Lebesgue measure $(d\theta/2\pi)$. The representatives $\pi_\rho(W(f))$ are then defined to be

$$\pi_\rho(W(f))=W(f)_F\otimes\exp\{i(2\rho)^{1/2}|\tilde{f}(0)|A\}, \tag{3.70b}$$

where A is the operator

$$(Af)(\theta)=\cos\theta f(\theta) \tag{3.70c}$$

defined on $\mathcal{L}^2(S^1)$. It is easy to verify that the pair $\{\mathfrak{H}_\rho,\pi_\rho\}$ forms a representation of the C.C.R. algebra. Next consider the vector

$$\Omega_\rho=\Omega_{F_0}\otimes 1. \tag{3.70d}$$

The vector 1 is cyclic for $\mathcal{L}^2(S^1)$ with respect to the algebra generated by A and Ω_{F_0} is cyclic for the Fock representation. This implies that the vector Ω_ρ is cyclic for the representation $\{\mathfrak{H}_\rho,\pi_\rho\}$, hence $\{\mathfrak{H}_\rho,\pi_\rho,\Omega_\rho\}$ is a cyclic representation. Now

$$(\Omega_\rho,\pi_\rho(W(f))\Omega_\rho)=(W_{Fo},W(f)_F W_{Fo})\,(1,\exp\{(2\rho)^{1/2}|\tilde{f}(0)|A\}1)$$

$$=E_{Fo}(f)\int_0^{2\pi}\exp\{i(2\rho)^{1/2}|\tilde{f}(0)|\cos\theta\}(d\theta/2\pi)$$

$$=\exp[-\tfrac{1}{4}\|f\|^2]J_o\big((2\rho)^{1/2}|\tilde{f}(0)|\big)$$

$$=E_\rho(f), \tag{3.71}$$

where we have used a standard integral formula for J_o that may be found in most texts on mathematical physics (see, for example, [Arfk} page 580). So the representation $\{\mathfrak{H}_\rho,\pi_\rho,\Omega_\rho\}$ produces the correct state over the C.C.R. algebra. Since $J_o(0)=1$, $E_{\rho=0}=E_{Fo}$.

At this point we should mention we have yet to show that this functional satisfies the conditions of Proposition 3.43 and hence defines a state of $\mathfrak{A}_{\mathbb{R}^3}$. It is quite difficult to show directly that the functional $E_\rho(f)$ satisfies condition iii) of Proposition 3.43. Now that we have written the functional in the form $E_\rho(f)=(\Omega_\rho,\pi_\rho(W(f))\Omega_\rho)$, it is trivial to verify that it satisfies the conditions of Proposition 3.43. Furthermore, since the representation $\{\mathfrak{H}_\rho,\pi_\rho,\Omega_\rho\}$ is cyclic we may use Theorem 1.25 to conclude that $\{\mathfrak{H}_\rho,\pi_\rho,\Omega_\rho\}$ is unitarily equivalent to the G.N.S. representation arising from the state that corresponds to the functional $E_\rho(f)$.

The functional $E_\rho(tf)$ is continuous in the real parameter t and therefore defines a regular state. The infinitesimal generator $\pi_\rho(\phi(f))$ is given by

$$\pi_\rho(\phi(f))=\frac{d}{dt}\Big[\pi_\rho(W(tf))\Big]_{t=0}$$

$$=\frac{d}{dt}\Big[W(tf)_F\otimes\exp\{i(2\rho)^{1/2}t|\tilde{f}(0)|A\}\Big]_{t=0}$$

$$=\phi(f)_F\otimes 1+1\otimes i(2\rho)^{1/2}|\tilde{f}(0)|A. \qquad (3.72)$$

The annihilation and creation operators are then given by

$$\pi_\rho(\Psi(f))=\frac{1}{\sqrt{2}}[\pi_\rho(\phi(f))+i\pi_\rho(\phi(if))]$$

$$=\Psi(f)_F\otimes 1+1\otimes(i-1)(\rho)^{1/2}|\tilde{f}(0)|A, \qquad (3.73a)$$

and

$$\pi_\rho(\Psi(f)^*)=\frac{1}{\sqrt{2}}[\pi_\rho(\phi(f))-i\pi_\rho(\phi(if))]$$

$$=\Psi(f)^*{}_F\otimes 1+1\otimes(i+1)(\rho)^{1/2}|\tilde{f}(0)|A. \quad (3.73b)$$

Let $\{h_{Zi}\}$ be an orthonormal basis for H_Z, where Z is an arbitrary bounded region in $\mathbb{R}^3$ with volume V_Z (recall that the Hilbert space H_Z is the subspace of $\mathcal{L}^2(\mathbb{R}^3)$ formed by functions Whose support lies in Z. The number operator N_Z for the region Z is

$$N_z \equiv \sum_i \pi_\rho(\Psi^*(h_{zi})\Psi(h_{zi}))$$

$$= \sum_i \begin{bmatrix} \Psi(h_{zi})^*{}_F\Psi(h_{zi})_F \otimes 1 + \Psi(h_{zi})^*{}_F \otimes (i-1)(\rho)^{1/2}|\tilde{h}_{zi}(0)|A \\ +\Psi(h_{zi})_F \otimes (i+1)(\rho)^{1/2}|\tilde{h}_{zi}(0)|A + 1 \otimes 2\rho|\tilde{h}_{zi}(0)|^2 A^2 \end{bmatrix} \tag{3.74}$$

We now evaluate $(\Omega_\rho, N_z\Omega_\rho)$.

$$N_z\Omega_\rho = N_z W_{Fo} \otimes 1$$

$$= \sum_i \begin{bmatrix} 0 \otimes 1 + \Psi(h_{zi})^*{}_F \Omega_{Fo} \otimes (i-1)(\rho)^{1/2}|\tilde{h}_{zi}(0)|\cos\theta \\ 0 \otimes (i+1)(\rho)^{1/2}|\tilde{h}_{zi}(0)|\cos\theta + \Omega_{Fo} \otimes 2\rho|\tilde{h}_{zi}(0)|^2\cos^2\theta \end{bmatrix},$$

so $$(\Omega_\rho, N_z\Omega_\rho) = \sum_i (\Omega_{fo}, \Omega_{fo})(1, 2\rho|\tilde{h}_{zi}(0)|^2\cos^2\theta)$$

$$= \sum_i 2\rho|\tilde{h}_{zi}(0)|^2 \int_0^{2\pi} \cos^2\theta (d\theta/2\pi)$$

$$= \rho\sum_i |\tilde{h}_{zi}(0)|^2. \tag{3.75}$$

Consider the function $f(\mathbf{x})$ that takes on the constant value 1 for $\mathbf{x} \in Z$ and vanishes for $\mathbf{x} \notin Z$. The expansion of $f(x)$ in terms of the basis $\{h_{zi}\}$ is

$$f(\mathbf{x})=\sum_i \widetilde{h_{zi}}(0)h_{zi}. \tag{3.76}$$

We can then use Parseval's formula to conclude

$$\sum_i |\widetilde{h_{zi}}(0)|^2=\|f(\mathbf{x})\|^2=V_Z, \tag{3.77}$$

where V_Z is the volume of the region Z. Now the number density operator in the representation $\{\mathfrak{H}_\rho,\pi_\rho,\Omega_\rho\}$ is given by

$$\rho_{op}=\lim_{Z\to\mathbb{R}^3} \frac{N_Z}{V_Z}. \tag{3.78}$$

In the representations used here ρ_{op} is a constant. Thus it is clear that this limit exists. The expectation value of ρ_{op} is then (using (3.75) and (3.78))

$$\lim_{Z\to\mathbb{R}^3} (\Omega_\rho,\frac{N_Z}{V_Z}\Omega_\rho)=\rho\ , \tag{3.79}$$

so the representation $\{\mathfrak{H}_\rho,\pi_\rho,\Omega_\rho\}$ has the correct particle density.

We now discuss the inequivalence of the representations $\{\mathfrak{H}_\rho,\pi_\rho\}$ corresponding to different densities ρ. The dependence of the representation on density is entirely contained in the second factor of the direct product

$$\pi_\rho(W(f))=W(f)_F\otimes\exp\{i(2\rho)^{1/2}|\tilde{f}(0)|A\}.$$

Since the representations are all faithful they are physically equivalent. If a bounded linear operator commutes with $\pi_{\rho 1}(W(f))$ then it will obviously commute with $\pi_{\rho 2}(W(f))$. This implies that the von Neumann algebras $\{\pi_{\rho 1}(W(f))\}''$ and $\{\pi_{\rho 2}(W(f))\}''$ are equal and the representations are quasi-equivalent. They are not, however, unitarily equivalent. This is suggested by the fact that the operators $\exp\{i(2\rho)^{1/2}|\tilde{f}(0)|A\}$ on $\mathcal{L}^2(S^1)$ are unitarily inequivalent for different values of ρ. The eigenvalue equation for this operator is

$$\exp\{i(2\rho)^{1/2}|\tilde{f}(0)|A\}\delta(\theta-\lambda)=\exp\{i(2\rho)^{1/2}|\tilde{f}(0)|\cos\lambda\}\delta(\theta-\lambda). \quad (3.80)$$

This allows us to conclude that the eigenvalue $\exp\{i(2\rho_1)^{1/2}|\tilde{f}(0)|\}$ of $\exp\{i(2\rho_1)^{1/2}|\tilde{f}(0)|A\}$ is not an eigenvalue of $\exp\{i(2\rho_2)^{1/2}|\tilde{f}(0)|A\}$ if $\rho_1\neq\rho_2$. Since the operators $\exp\{i(2\rho_1)^{1/2}|\tilde{f}(0)|A\}$ and $\exp\{i(2\rho_2)^{1/2}|\tilde{f}(0)|A\}$ have different eigenvalues they cannot be unitarily equivalent. These eigenvalues do not seem to admit any physical interpretation. As we shall see below, however, they may be used to label the irreducible constituents of the representation $\{\mathfrak{H}_\rho,\pi_\rho,\Omega_\rho\}$. Note that when the system is finite we are working in the Fock representation for all densities..

For any operator T on $\mathcal{L}^2(S)$ that commutes with A, $1\otimes T$ is an element of the commutant $\{\pi_\rho(\mathfrak{A}(\mathbb{R}^3))\}'$, so the representation $\{\mathfrak{H}_\rho,\pi_\rho,\Omega_\rho\}$ is reducible. Araki and Woods have shown that it is a direct integral of irreducible representations. They write

$$\mathfrak{L}^2(S)=\int_0^{2\pi\oplus} M(\theta)d\theta/2\pi, \tag{3.81}$$

where dim $M(\theta)=1$. They then use the result [Dixm2])

$$\mathfrak{H}_F\oplus\int_0^{2\pi\oplus} M(\theta)d\theta/2\pi=\int_0^{2\pi\oplus}\mathfrak{H}_F\oplus M(\theta)d\theta/2\pi \tag{3.82a}$$

to decompose the representation space $\mathfrak{H}_\rho$. With respect to this decomposition, the operators $\pi_\rho(W(f))$ are decomposed as

$$\pi_\rho(W(f))=\int_0^{2\pi\oplus}\pi_\theta(W(f))d\theta/2\pi, \tag{3.82b}$$

where $$\pi_\theta(W(f))=W(f)_F\otimes\exp\{i(2\rho)^{1/2}|\tilde{f}(0)|\cos\theta\}. \tag{3.82c}$$

The Fock representation is irreducible. Since dim $M(\theta)=1$, it follows that the set $\{\pi_\theta(W(f))\}$ is irreducible in $\mathfrak{H}_F\otimes M(\theta)$. The representation $\{\mathfrak{H}_F\otimes M(\theta),\pi_\theta(W(f))\}$ is therefore irreducible. We may use the eigenvalues of the operator $\exp\{i(2\rho)^{1/2}|\tilde{f}(0)|A\}$ to label these irreducible representations. Since the operators $\exp\{i(2\rho)^{1/2}|\tilde{f}(0)|\cos\theta\}$ are simply complex numbers, they are unitarily inequivalent for different values of θ. This implies that the irreducible representations $\{\mathfrak{H}_F\otimes M(\theta),\pi_\theta(W(f))\}$ are unitarily inequivalent. This shows, among other things, that the C.C.R. algebra $\mathfrak{A}_{\mathbb{R}^3}$ admits an infinite number of unitarily inequivalent irreducible representations.

Araki and Woods also consider then case when the density is a function of momentum, $\rho=\rho(\mathbf{k})$. Once again the representation they construct is a direct integral of irreducible representations, which they demonstrate are unitarily inequivalent. More recently Lewis and Pulé [Lewi] and Cannon [Cann] have calculated the canonical and grand canonical equilibrium states over $\mathfrak{A}_{\mathbb{R}^3}$. The form that they obtain for the generating functional reduces to our $E_L(f)$ when the temperature goes to zero. We expect that the representations corresponding to different temperatures (and chemical potentials in the grand canonical case) are unitarily inequivalent, although neither of these papers shows this.

REFERENCES

[Arak] Araki, H., Woods, E.J.,
Representations of the Canonical Commutation Relations Describing a Nonrelativistic Infinite Free Bose Gas.
J. Math. Phys. **4** (1963), 637-662.

[Arfk] Arfken, G,
Mathematical Methods for Physicists, 3rd ed.,
Academic Press, Orlando (1985).

[Brat1] Bratteli, O., Robinson, D.W.,
Operator Algebras and Quantum Statistical Mechanics, Vol.1.
Springer-Verlag, New York- Heidelberg-Berlin (1979).

[Brat2] Bratteli, O., Robinson, D.W.,
Operator Algebras and Quantum Statistical Mechanics, Vol.2.
Springer-Verlag, New York- Heidelberg-Berlin (1979).

[Cann] Cannon, J.T.,
Infinite Volume Limits of the Canonical Free Bose Gas States on the Weyl Algebra.
Comm. Math. Phys. **29** (1972), 89-104.

REFERENCES

[Dixm1] Dixmier, J.,
Les C*-algebras et leurs Representations.
Gauthier-Villars, Paris (1964).

[Dixm2] Dixmier, J.,
Les Algebres d'operateurs dans L'espace Hilbertian.
Gauthier-Villars, Paris (1957).

[Emch] Emch, G.,
Algebraic Methods in Statistical Mechanics and Quatum Field Theory.
Wiley -Interscience, New York (1972).

[Gelf] Gelfand, I.M., Naimark, M.A.,
On the Imbedding of Normed Rings into the Ring of Operators in Hilbert Space.
Mat. Sborn.,N.S. **12** **[54]** (1943), 197-217.

[Haag] Haag, R.,
Infinite Quantum Systems.
Lectures in Theoretical Physics, Vol. XIVB.
Colorado Associated University Press,
Boulder, Colorado (1973)

[Haag1] Haag, R.,
The Algebraic Approach to Quantum Mechanics, Equilibrium States and Hierarchy of Stability.
Lectures in Physics, **4** (1976), 156-188.

[Haag2] Haag, R., Kastler, D.,
An Algebraic Approach to Quantum Field Theory.
J. Math. Phys. **5** (1964), 848-861.

[Kadi] Kadison, R.V.,
Operator Algebras.
Algebras in Analysis,
J.H. Williamson, Ed., Academic Press, London-New York-San Francisco (1975).

[Lewi] Lewis, J.T., Pule, J.V.,
The Equilibrium States of the Free Boson Gas.
Comm. Math. Phys. **36** (1974), 1-18.

[Neum1] von Neumann, J.,
On Infinite Direct Products.
Compos. Math. **6** (1938), 1-77

[Neum2] von Neumann, J.,
Die Eindeutigkeit der Schrödingerschen Operatoren.
Math. Ann. **104** (1931), 570-578.

REFERENCES

[Put] Putnam, C.R.,
Commutation Properties of Hilbert Space Operators and Related Topics.
Springer-Verlag, New York (1967).

[Riesz] Riesz, F., Sz-Nagy, B.
Functional Analysis.
Frederick Ungar Pub. Co., New York (1955).

[Sega] Segal, I.E.,
Postulates for General Quantum Mechanics.
Ann. Math. 48 (1947), 930-948.

[Szeg] Szegö, G.,
Orthogonal Polynomials.
Am. Math. Soc., New York (1959).

APPENDIX A

THE HARMONIC OSCILLATOR REPRESENTATION

In this Appendix we construct a faithful representation of the C.C.R. algebra $\mathfrak{A}(\mathbb{R})$ corresponding to a system with one degree of freedom. This representation is is constructed with the aid of the annihilation and creation operators, familiar from most modern Quantum Mechanics texts,

$$a=\frac{q+ip}{\sqrt{2}}, \tag{A.1a}$$

and

$$a^*=\frac{q-ip}{\sqrt{2}}. \tag{A.1b}$$

They satisfy

$$[a,a]=[a^*,a^*]=0, \tag{A.1c}$$

and

$$[a,a^*]=1. \tag{A.1d}$$

Consider the abstract vector $|n\rangle$, where n is an arbitrary nonnegative integer. These vectors will form a basis for the representation space $\mathfrak{H}_h$. We initially define E to be the set of all finite linear combinations of the $|n\rangle$, so that E is a complex vector space. We define linear transformations $\pi_h(a)$ and $\pi_h(a^*)$ on E by defining their action on the basis vectors,

$$\pi_h(a)|n\rangle\equiv\sqrt{n}|n-1\rangle, \tag{A.2a}$$

and

$$\pi_h(a^*)|n\rangle\equiv\sqrt{n+1}|n+1\rangle. \tag{A.2b}$$

Note that $\pi_h(a)\pi_h(a^*)|n\rangle=(n+1)|n\rangle$ and $\pi_h(a^*)\pi_h(a)|n\rangle=n|n\rangle$, so that $[\pi_h(a),\pi_h(a^*)]|n\rangle=|n\rangle$, i.e., $\pi_h(a)$ and $\pi_h(a^*)$ satisfy the commutation relations (A.1).

It is apparent that any basis vector $|n\rangle$ can be obtained by repeated application of $\pi_h(a^*)$ to $|0\rangle$,

$$|n\rangle=\frac{\pi_h(a^*)^n}{\sqrt{n!}}|0\rangle. \tag{A.3}$$

To define a scalar product for E we introduce the conjugate vectors. For the basis vectors we define

$$\begin{aligned} |n\rangle^\dagger &= \left(\frac{\pi_h(a^*)^n}{\sqrt{n!}}|0\rangle\right)^\dagger \\ &\equiv \langle 0|\frac{\pi_h(a)^n}{\sqrt{n!}} \\ &\equiv \langle n|, \end{aligned} \tag{A.4a}$$

and for arbitrary $\Psi=\sum_{i=0}^{N}\alpha_n|n\rangle$ in E

$$\Psi^\dagger\equiv\sum_{i=0}^{N}\langle n|\bar{\alpha}_n. \tag{A.4b}$$

The action of $\pi_h(a^*)$ and $\pi_h(a)$ on the conjugate basis vectors $\langle n|$ is found to be

$$\langle n|\pi_h(a)=\langle n+1|\sqrt{n+1}. \tag{A.5a}$$

and

$$\langle n|\pi_h(a^*)=\langle n-1|\sqrt{n}. \tag{A.5b}$$

The conjugate vectors $\langle n|$ can be obtained by repeated application of $\pi_h(a)$ to $\langle 0|$,

$$\langle n|=\langle 0|\frac{\pi_h(a)^n}{\sqrt{n!}}. \tag{A.6}$$

To define a scalar product for the basis vectors we define

$$\langle 0|0\rangle\equiv 1, \tag{A.7a}$$

and

$$(|n\rangle,|m\rangle)\equiv\langle n|m\rangle. \tag{A.7b}$$

Using (A.4) and (A.6) we have

$$(|n\rangle,|m\rangle)\equiv\langle n|m\rangle$$

$$=\langle 0|\frac{\pi_h(a)^n}{\sqrt{n!}}\,\frac{\pi_h(a^*)^m}{\sqrt{m!}}|0\rangle. \tag{A.8}$$

Now the commutation relations (A.1) imply that

$$\frac{\pi_h(a)^n}{\sqrt{n!}}\,\frac{\pi_h(a^*)^m}{\sqrt{m!}}|0\rangle=\begin{Bmatrix}|0\rangle \text{ if } n=m\\ 0 \text{ if } n>m\end{Bmatrix}, \tag{A.9a}$$

and
$$<0|\frac{\pi_h(a)^n}{\sqrt{n!}}\frac{\pi_h(a^*)^m}{\sqrt{m!}}=\begin{Bmatrix}<0| \text{ if } n=m\\ 0 \text{ if } m>n\end{Bmatrix}. \tag{A.9b}$$

Substituting (A.9) into (A.8) then gives

$$(|n>,|m>)\equiv<n|m>=\delta_{nm}, \tag{A.10}$$

so the basis vectors are orthonormal with respect to the scalar product (A.7).

Consider the set

$$\mathfrak{H}_h=\{\Psi=\sum_{n=0}^{\infty}\alpha_n|n>:\sum_{n=0}^{\infty}|\alpha_n|^2<\infty\}, \tag{A.11}$$

and extend the scalar product to all of $\mathfrak{H}_h$ by defining

$$(\Psi,\Phi)=(\sum_{n=0}^{\infty}\alpha_n|n>,\sum_{m=0}^{\infty}\beta_m|m>)$$

$$\equiv\sum_{n,m=0}^{\infty}\bar{\alpha}_n\beta_m<n|m>$$

$$=\sum_{n=0}^{\infty}\bar{\alpha}_n\beta_n. \tag{A.12}$$

This definition satisfies

$$i)(\Pi,\alpha\Psi+\beta\Phi)=\alpha(\Pi,\Psi)+\beta(\Pi,\Phi),$$

$$ii)\ (\Phi,\Psi)=\overline{(\Psi,\Phi)}\ ,$$

and $$iii)(\Psi,\Psi)\geq 0, \text{ and } (\Psi,\Psi)=0 \text{ only when } \Psi=0,$$

for all $\Psi,\Phi,\Pi\in\mathfrak{H}_h$ and $\alpha,\beta\in\mathbb{C}$, so it does define a scalar product for $\mathfrak{H}_h$. We can use this scalar product to define a norm on $\mathfrak{H}_h$ as

$$\|\Psi,\Psi\|^2\equiv(\Psi,\Psi). \tag{A.13}$$

Equipped with this norm, $\mathfrak{H}_h$ is a pre-Hilbert space. We now show that $\mathfrak{H}_h$ is in fact a Hilbert space, i.e., we will show that all Cauchy sequences converge to an element of $\mathfrak{H}_h$. Consider a sequence $\{\Psi_i=\sum_{n=0}^{\infty}\alpha_n^i|n\rangle\}$ in $\mathfrak{H}_h$. Using (A.13) we have

$$\|\Psi_i-\Psi_j\|^2=\sum_{n=0}^{\infty}|\alpha_n^i-\alpha_n^j|^2. \tag{A.14}$$

Assume that the sequence is Cauchy, so that for given any positive real number ε, there exists a positive integer $N(\varepsilon)$ such that $\|\Psi_i-\Psi_j\|\leq\varepsilon$ for all $i,j\geq N(\varepsilon)$. Using (A.14) we have

$$|\alpha_n^i-\alpha_n^j|^2 \le \sum_{n=0}^{\infty} |\alpha_n^i-\alpha_n^j|^2$$

$$\le \varepsilon^2, \qquad \text{(A.15)}$$

for all $i,j \ge N(\varepsilon)$. This implies that $|\alpha_n^i-\alpha_n^j| \le \varepsilon$ for all $i,j \le N(\varepsilon)$, i.e. the sequences of complex numbers $\{\alpha_n^i\}$ are Cauchy for each fixed n, and hence each α_n^i converges to a complex number α_n. For an arbitrary finite integer N>0, it follows from (A.15) that

$$\sum_{n=0}^{N} |\alpha_n^i-\alpha_n^j|^2 \le \varepsilon^2, \qquad \text{(A.16)}$$

for all $i,j \ge N(\varepsilon)$. If we then let $j \to \infty$, we obtain

$$\sum_{n=0}^{N} |\alpha_n^i-\alpha_n|^2 \le \varepsilon^2, \qquad \text{(A.17)}$$

for all $i \ge N(\varepsilon)$. Since N is arbitrary, (A.17) implies that

$$\sum_{n=0}^{\infty} |\alpha_n^i-\alpha_n|^2 \le \varepsilon^2, \qquad \text{(A.18)}$$

for all $i \ge N(\varepsilon)$. Now

$$|\alpha_n|^2=|\alpha_n-\alpha_n^i+\alpha_n^i|^2$$

$$\leq 2(|\alpha_n-\alpha_n^i|^2+|\alpha_n^i|^2), \tag{A.19}$$

so that for all $i\geq N(e)$

$$\sum_{n=0}^{\infty}|\alpha_n|^2\leq 2\sum_{n0}^{\infty}|\alpha_n-\alpha_n^i|^2+2\sum_{n=0}^{\infty}|\alpha_n^i|^2$$

$$<\infty, \tag{A.20}$$

where the last inequality follows from (A.18) and the fact that $\Psi_i\in\mathfrak{H}_h$. This then implies that $\Psi=\sum_{n=0}^{\infty}\alpha_n|n\rangle$ is an element of $\mathfrak{H}_h$. Now, for all $i\geq N(\varepsilon)$, (A.18) implies that

$$\|\Psi-\Psi_i\|^2=\sum_{n=0}^{\infty}|\alpha_n-\alpha_n^i|^2$$

$$<\varepsilon^2 \tag{A.21}$$

so that the Cauchy sequence $\{\Psi_i\}$ converges to Ψ, which is an element of $\mathfrak{H}_h$. $\mathfrak{H}_h$ is therefore complete, and hence a Hilbert space.

The action of the operators $\pi_h(a^*)$ and $\pi_h(a)$ on $\mathfrak{H}_h$ will now be discussed. They are both linear transformations on $\mathfrak{H}_h$, but they are both unbounded. For example, $\|\pi_h(a^*)|n\rangle\|=\sqrt{n+1}$ and $\|\pi_h(a)|n\rangle\|=\sqrt{n}$, which have no upper bound for all n, (note that $\||n\rangle\|=1$). Their domains are

$$D(\pi_h(a^*))=\{\Psi\in\mathfrak{H}_h:\|\pi_h(a^*)\Psi\|<\infty\}$$

$$=\{\Psi=\sum_{n=0}^{\infty}\alpha_n|n\rangle:\ \sum_{n=0}^{\infty}|\alpha_n|^2(n+1)<\infty\}, \qquad \text{(A.22a)}$$

and
$$D(\pi_h(a))=\{\Psi\in\mathfrak{H}_h:\|\pi_h(a)\Psi\|<\infty\}$$

$$=\{\Psi=\sum_{n=0}^{\infty}\alpha_n|n\rangle:\ \sum_{n=0}^{\infty}|\alpha_n|^2 n<\infty\}. \qquad \text{(A.22b)}$$

As expected, $\pi_h(a^*)$ and $\pi_h(a)$ are not defined on all of $\mathfrak{H}_h$. The domains $D(\pi_h(a^*))$ and $D(\pi_h(a))$ are , however, equal. It is convenient to characterize these domains in a different manner. Consider the "number" operator $N\equiv\pi_h(a^*)\pi_h(a)$. We have $N|n\rangle=n|n\rangle$, so the domain of N is

$$D(N)=\{\Psi=\sum_{n=0}^{\infty}\alpha_n|n\rangle:\ \sum_{n=0}^{\infty}|\alpha_n|^2 n^2<\infty\}, \qquad \text{(A.23a)}$$

and the domain of $N^{1/2}$ is

$$D(N^{1/2})=\{\Psi=\sum_{n=0}^{\infty}\alpha_n|n\rangle:\ \sum_{n=0}^{\infty}|\alpha_n|^2 n<\infty\}. \qquad \text{(A.23b)}$$

Comparing (A.23b) with (A.22), we see that $D(\pi_h(a^*))=D(\pi_h(a))=D(N^{1/2})$. Since every finite linear combination of the basis vectors is contained in $D(N^{1/2})$, $D(N^{1/2})$ is dense in $\mathfrak{H}_h$. Furthermore, for all $\Psi,\Phi\in D(N^{1/2})$ we have

$$(\pi_h(a)\Psi,\Phi)=(\Psi,\pi_h(a^*)\Phi), \tag{A.24}$$

so that $\pi_h(a^*)=\pi_h(a)^\dagger$, the Hermitian adjoint of $\pi_h(a)$.

We now construct a representation of $\mathfrak{A}(\mathbb{R})$ on $\mathfrak{H}_h$, using $\pi_h(a^*)$ and $\pi_h(a)$. For an arbitrary polynomial $\sum_{n,m=0}^{\infty}\alpha_{nm}(a^*)^n(a)^m$ we define

$$\pi_h\left(\sum_{n,m=0}^{\infty}\alpha_{nm}(a^*)^n(a)^m\right)\equiv\sum_{n,m=0}^{\infty}\alpha_{nm}(\pi_h(a^*))^n(\pi_h(a))^m. \tag{A.25}$$

In particular we have, using (A.1),

$$\pi_h(q)=\frac{\pi_h(a)+\pi_h(a^*)}{\sqrt{2}}, \tag{A.26a}$$

and

$$\pi_h(p)=\frac{\pi_h(a)-\pi_h(a^*)}{i\sqrt{2}}. \tag{A.26b}$$

Both $\pi_h(q)$ and $\pi_h(p)$ are self adjoint, with domain $D(N^{1/2})$. We now define

$$\pi_h(U(t)) \equiv e^{it\pi_h(q)}, \tag{A.27a}$$

$$\pi(V(s)) \equiv e^{is\pi_h(p)}, \tag{A.27b}$$

and

$$\pi_h\left(\sum_{n,m=0}^{N,M} \alpha_{nm} U(t_n) V(s_m)\right) \equiv \sum_{n,m=0}^{N,M} \alpha_{nm} \pi_h(U(t_n)) \pi_h(V(s_m)) \tag{A.27c}$$

For arbitrary $\Psi,\Phi \in D(N^{1/2})$ we have

$$\begin{aligned}(\pi_h(U(t))\Psi,\pi_h(U(t))\Phi) &= (e^{it\pi_h(q)}\Psi, e^{it\pi_h(q)}\Phi) \\ &= (\Psi,(e^{it\pi_h(q)})^{\dagger} e^{it\pi_h(q)}\Phi) \\ &= (\Psi, e^{-it\pi_h(q)} e^{it\pi_h(q)}\Phi) \\ &= (\Psi,\pi_h(U(-t))\pi_h(U(t))\Phi) \\ &= (\Psi,\Phi).\end{aligned} \tag{A.28a}$$

In a similar fashion we find

$$\begin{aligned}(\pi_h(V(s))\Psi,\pi_h(V(s))\Phi) &= (\Psi,\pi_h(V(-s))\pi_h(V(s))\Phi) \\ &= (\Psi,\Phi).\end{aligned} \tag{A.28b}$$

This shows that $\pi_h(U(t))$ and $\pi_h(V(s))$ are well defined on $D(N^{1/2})$, and are in fact bounded on $D(N^{1/2})$. This, along with the fact that $D(N^{1/2})$ is dense in $\mathfrak{H}_h$, shows that $\pi_h(U(t))$ and $\pi_h(V(s))$ are well defined on all of

$\mathfrak{H}_h$. In addition,. (A.28) now holds for all $\Psi,\Phi \in \mathfrak{H}_h$, which implies that $\pi_h(U(t))$ and $\pi_h(V(s))$ are unitary operators on $\mathfrak{H}_h$. Since the infinitesimal generators $\pi_h(p)$ and $\pi_h(q)$ of $\pi_h(U(t))$ and $\pi_h(V(s))$ satisfy the commutations relations (1.7) (section 3.1), $\pi_h(U(t))$ and $\pi_h(V(s))$ satisfy the commutation relations (1.12) (section 3.1). The pair $(\mathfrak{H}_h,\pi_h)$ therefore forms a representation of $\mathfrak{A}(\mathbb{R})$. Since the action of $\pi_h(U(t))$ and $\pi_h(V(s))$ on the vector $|0\rangle$ always produces a non-zero vector, this representation is faithful.