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Valuation of Segregated Funds in India

by

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Abstract

The objective of this thesis is to develop an econometric model which is less complex than the Wilkie's model for valuing and managing financial risks associated with benefit options regarding segregated fund contracts in India.

The empirical studies conducted in this thesis revealed the following results.

- The South Asian stock markets (Sri-Lanka, India and Pakistan) did not show evidence of unit-roots, but the returns are correlated. Therefore, the most appropriate model capable of capturing the long-term equity return process for a practical dynamic hedging of segregated fund contracts in India is the VAR(1) model.
- Also, the security bonds with various maturities from the Indian money market show evidence of long-run equilibrium relationship. This characteristic makes it possible for the various yields to maturity (YTM) to be modeled jointly via a VECM representation.

Therefore, the valuation model being proposed in this thesis, combines ideas from financial engineering, life contingencies and econometrics. Assessment of the model via simulation has shown that, the net present value of outgo for a 10 year contract under the combined GMMB/GMDB for a life age 50 is mostly in the negative. This is a positive signal that, the model has the capability of meeting all the hedge cost and leave some profit.

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Chapter 1

Introduction

The purpose of life insurance is to provide financial compensation or protection to the policyholder and their dependents. Traditional actuarial practices have for a long time focused on evaluation and management of life contingent risks such as mortality and morbidity, neglecting risks associated with investment side of insurance. It is an undeniable fact that, insurance markets globally are experiencing swift transformation partly due to the growing awareness among the public as to the investment opportunities that lie outside insurance. Investors want to partake in the potential rewards of stock market as well as mortality protection. In response to these needs, insurers around the world have introduced equity-linked insurance contracts. These contracts are identified by different names in different jurisdictions. In Canada, they are known as segregated fund contracts which unequivocally are the central focus of this thesis.

1.1 What are Segregated Funds?

Segregated funds are professionally managed pooled funds, with investment potentials similar to mutual funds. These are only offered through individual variable deferred annuity contracts sold by Life Insurance companies. Governed under provincial regulations, segregated funds provide investors with unique features and guarantees not generally available in traditional market-based investments, like mutual funds. The assets of a segregated fund are held separately from the general assets of the Life Insurance company, hence the name.

Basically, a typical segregated fund used premiums from an investor to purchase mutual funds based on the investors risk preference, and the accumulated value at maturity, usually in 10 years, is guaranteed to be at least the initial principal. When the market value of the investment becomes greater than the initial principal, the investor is offered the choice to transfer the capital gain into the principal and reset the contract. Also, a guaranteed benefit amount will be paid out in the event of death of the investor prior to the maturity of the contract.

Segregated funds are similar to mutual funds in many respects but provide a number of extra features and benefits. These extra features include a maturity guarantee and mortality benefit. In addition, in many cases investors have the ability to switch from one underlying fund to another (while maintaining their guarantee levels) and also to reset their guarantee. It allows the investor to lock in market gains at any time, up to a maximum, for example, of two or four times per year.

1.2 Types of Segregated Funds

There is a wide variety of segregated funds on the market. Based on investment objectives, funds tend to fall into three (3) broad categories.

1.2.1 Growth Funds

Growth funds invest mainly in the common stock of companies with good growth prospects in order to produce capital gain.

1.2.2 Fixed Income Funds

Fixed income funds invest mainly in bonds, other debt instruments and shares of companies that pay dividends, in order to produce a stream of income while protecting investors' capital.

1.2.3 Balanced Funds

Balanced funds invest in a blend of stocks, debt instruments and dividend-bearing shares to produce a blend of capital gain and income.

1.3 Major Benefit Types under Segregated Funds

All segregated fund contracts offer benefits in a form of guarantees. These benefits fall into one of the following major categories according to Hardy (2003).

1.3.1 Guaranteed Minimum Maturity Benefit (GMMB)

The GMMB guarantees the policyholder a specified amount of money at the maturity of the contract. A simple example of GMMB might be a guaranteed return of premium if the stock index falls over the term of the insurance. The guarantee may be fixed or subject to regular or equity-dependent increases.

1.3.2 Guaranteed Minimum Death Benefit (GMDB)

The GMDB guarantees the policyholder a specified amount of money upon the occurrence of death when the contract is still in force. Here, the guarantee may be a simple return of premium or may increase at a fixed rate of interest.

1.3.3 Guaranteed Minimum Accumulated Benefit (GMAB)

Under the GMAB package, the policyholder has the option to renew the contract at the end of the original term.

1.3.4 Guaranteed Minimum Surrender Benefit (GMSB)

The GMSB is a variation of the GMMB. A guaranteed amount on the contract is payable as surrender value beyond a certain fixed date. Example of GMSB is the return of premium.

1.3.5 Guaranteed Minimum Income Benefit (GMIB)

The GMIB is a benefit package that ensures that lump sum accumulated under a separate account contract may be converted to an annuity at a guaranteed rate. A GMIB is commonly associated with variable-annuity contracts in the United States.

1.4 Provision for Segregated Funds Liabilities

Two quantitative approaches are widely used in practice to deal with guaranteed liabilities associated with segregated fund contracts. One is the dynamic hedging approach based on financial engineering and the other is the actuarial approach.

1.4.1 Dynamic Hedging Approach

This is a financial engineering technique which uses the Black-Scholes equation to find a replicating portfolio with payoff equivalent to the payoff of the guaranteed liabilities. The replicating portfolio will change continuously, so it is frequently adjusted. This approach provides a powerful way of hedging the liabilities.

1.4.2 Actuarial Approach

This approach uses simulation to establish the underlying probability distribution of the guaranteed liabilities then uses a long-term fixed rate of return to discount them to their present values. The simulation results make provision for a sufficient amount of assets which are invested in fixed income securities to accumulate. At maturity, the accumulated amount is expected to maintain the solvency of the insurer vis-à-vis the guaranteed liabilities with a high probability of say 99 percent.

1.5 Segregated Fund Contracts-Canada

Segregated fund contracts actually became popular in Canada in the late 1990's and often incorporate complex guaranteed values on death or maturity. They have been one of the most successful Canadian financial products over the past several years in terms of sales volume.

Segregated fund contracts are now very attractive to Canadians who wish to enjoy the perpetual up-side growth opportunity with a pre-fixed maximum loss, however, managing guaranteed liabilities associated with segregated fund contracts can be a very challenging task on the part of life insurers.

As mutual funds products continue to develop over the years, Canadian insurance companies offer increasingly more innovative segregated fund products. For instance, as Lui (2008) has stated, not all segregated funds are managed to trace the performance of a particular stock index. To gain more diversification, funds under one contract may consist of many sub-funds, such as a Canada equity fund, a Canadian fixed income fund or a US equity fund. These sub-funds are also segregated funds available to investors.

In the light of strict capital requirements recently imposed by Office of the Superintendent of Financial Institutions (OSFI), some companies have begun selling more restricted versions of these contracts. Other companies have discontinued the sale of these contracts all together, deeming the capital requirements to be too onerous. However, large volumes of contracts already sold remain outstanding, and companies need to manage their risk exposure, or at least allocate sufficient capital and reserves.

In this thesis, my objective is to develop an econometric model which is less complex than the Willkie's model for managing financial risk associated with the benefit options mentioned above. This model will aid emerging economies valuation of segregated fund contracts. Therefore, understanding segregated funds in Canada provides a good starting point and, in particular, actuaries in North America have not come to a general consensus as to the form of an equity return model. However, guaranteed liabilities central to the valuation of a segregated funds depend on an equity return model. Also, this thesis revolves around India which is the biggest market in South Asia. Data for other emerging economies such as Pakistan is sparse.

1.6 Segregated Funds-India

Insurance is a big opportunity in a country like India with large population and untapped potential. In India, the traditional emphasis in the past regarding life insurance particularly, endowment plan was security of capital. However, with inflationary trends witnessed all over the world, it was observed that savings through life insurance were becoming unattractive and not meeting the aspirations of policyholders.

To restore the expectations and confidence policyholders place in life insurance, insurers in India introduced the unit linked insurance plans. These plans are separate account products quite similar to segregated fund contracts. Today, the Indian life insurance market is riding high on the unit linked insurance plans.

Unit linked insurance plans are also insurance contracts that combine the benefit of investment and insurance. It provides policyholders an option to put part of their premium in various investment portfolios and derive the benefits depending upon the performance of the funds chosen by them. Unit linked insurance plans were launched at an opportune time when stock markets had just taken off in India. Ever since, unit linked plans have gained high acceptance due to the attractive features they offer. These include flexibility, transparency, liquidity and fund options.

Unit linked insurance plans have broader investment choices when compared to other forms of insurance such as endowment. They invest across the board in stocks, government securities, corporate bonds and money market instruments. The common type of charges, fees and deductions in unit linked contracts are premium allocation charges, mortality charges, surrender charges just to mention but a few.

Analysis of India's overall insurance premium from 2005 to 2008 has shown that, unit linked businesses are the main drivers of growth within the insurance industry. In spite of this impressive growth, the regulations governing unit linked products are still being developed to follow closely that of Canadian products.

1.7 Wilkie's Models

The Wilkie's model is a stochastic asset pricing model used to detect movements of multiple assets and their correlations. It is also defined as a collection of models. Even though there are many financial and econometric models for investment returns, United Kingdom (UK) actuary, David Wilkie is the first person who developed stochastic investment model for the long-term returns of multiple assets for actuarial application. His model is known as the "Wilkie's Model" Wilkie (1986 and 1995).

The Wilkie's model differs in several fundamental ways from other models. It is a multivariate model with several related economic variables. It is designed for a long-term application and is applicable to only annual data. The Wilkie's model is widely used in the developed world, particularly Canada, United States, UK and Australia.

In spite of the usefulness of the Wilkie's model for actuarial application, it has been as well subjected to a unique level of scrutiny and criticisms. Criticisms of the model are well documented in Harris (1995) and Huber (1997). These criticisms reveal the fundamental conflict that lies at the heart of any attempt to use stochastic time series to model economic data series.

In this thesis, I have employed a time series econometric model to fit the equity return (stock return) and the short rate (treasury bond) processes. The models are based on the vector autoregressive (VAR) and cointegrated VAR process developed by Johansen-Jueselius (1990 and 1991). In Chapter 2, I provide the theory underlying the modeling of the long-term equity return process and empirical results of stock market indices from India, Pakistan and Sri Lanka. In Chapter 3, I further provide the theory behind the modeling of the treasury bond and empirical results of up to 14 days, 15 to 91 days, 92 to

182 day and 183 to 364 days yields to maturity (YTM) from India's money market. Chapter 4 reviews the theory of option pricing vis-à-vis the embedded guarantees and the hedging error. It further emphasizes practical application in the context of India. Chapter 5 provides the final conclusions.

Chapter 2

Long-Term Equity Return Model

This chapter is devoted to the modeling of the long-term equity return process from the view point of a time series econometric technique, as well as the transformation method employed in performing valuation for segregated funds in India. As a first step in the modeling process, the main stock market indices of Sri Lanka (CSE), India (BSE) and Pakistan (KSE) from August 1997 to July 2007 were obtained from the Bloomberg terminals.

Theoretically, a security available at any market can be traded at any agreed price, so there is no upper bound for the price of the security. For modeling purposes however, this is an undesirable property since many probability distributions have bounded domain. This particular issue is addressed in this thesis by resorting to the logarithmic transformation. Particularly, to arrive at the log-returns, a consideration is given to the one period single return from $t-1$ to t as follows:

$$1 + r_t = \frac{S_t}{S_{t-1}} \tag{2.1}$$

$$x_t = \log(1+r_t) = \log\left(\frac{S_t}{S_{t-1}}\right) \tag{2.2}$$

Applying the logarithm function to Eq. (2.1), a new measure x_t can take any value on the positive bounded real line.

Therefore, the CSE, BSE, and the KSE are transformed to a measure similar to x_t in this thesis. As an example, consider the BSE for January 2000 and February 2000 to be 0.005 and 0.008 respectively. The log-return is given as

$$x_t = \log(1 + r_t) = \log\left(\frac{0.008}{0.005}\right) = 0.20412$$

2.1 Exploratory Analysis of South Asia Stock Markets

We now direct our attention to the empirical results of the monthly stock returns of Sri Lanka, India and Pakistan stock markets from August 1997 to July 2007. In addition to the sample mean and standard deviation, sample skewness and kurtosis are statistics used to describe and compare the empirical distributions of the index returns of the three (3) countries. Also, a further test based on Jarque-Bera (J-B) normality test is performed. Assume { $x_1, x_2, \ldots\ldots\ldots\ldots, x_T$ } be the observed log-returns for each index series over T periods.

The sample mean is

$$\hat{\mu}_t = \frac{1}{T}\sum_{t=1}^{T} x_t, \tag{2.3}$$

The sample standard deviation is

$$\hat{\sigma}_t = \left(\frac{1}{T-1}\sum_{t=1}^{T}\left(x_t - \hat{\mu}_x\right)^2\right)^{1/2}, \tag{2.4}$$

The sample skewness is

$$\hat{S}(x) = \frac{1}{(T-1)\hat{\sigma}_x^3}\sum_{t=1}^{T}\left(x_t - \hat{\mu}_x\right)^3 \tag{2.5}$$

The sample kurtosis is

$$\hat{K}(x) = \frac{1}{(T-1)\hat{\sigma}_x^4} \sum_{t=1}^{T} \left(x_t - \hat{\mu}_x \right)^4 \tag{2.6}$$

The test statistic for the Jarque-Bera (J-B) test is defined as

$$J - B = \frac{T}{6} \left(S^2 + \frac{(K-3)^2}{4} \right), \tag{2.7}$$

where T is the number of observations, S and K are the sample skewness and kurtosis respectively. The statistic *J-B* has an asymptotic chi-square distribution with 2 degrees of freedom and is used to test the hypothesis that a distribution is normal.

Table 2.1 presents the summary statistics for the monthly stock returns of Sri-Lanka, India and Pakistan stock markets. The table shows that, the highest mean return is reported for KSE followed by BSE and CSE. However, these markets exhibit higher volatility (conditional standard deviation) when compared with developed countries stock markets such as New York Stock Exchange (NYSE) where volatilities are always below two decimals. This is not surprising since high volatility is a common indicator of developing markets where market prices are more sensitive to domestic as well as foreign economic and political shocks. KSE exhibits the highest volatility, followed by CSE and BSE in that order.

Table 2.1 further reveals that, all three (3) national stock markets are negatively skewed with the degree of skewness being more pronounced in KSE and BSE. The highest kurtosis is reported for KSE, CSE and BSE respectively. However, BSE is not heavy tailed as compared to KSE and CSE.

The normality test based on the Jarque-Bera (J-B) statistics is also shown in table 2.1. Apart from KSE, the rest showed a probability value greater than the five (5) percent significant level. On the basis of this information, it can be said that KSE is not normal.

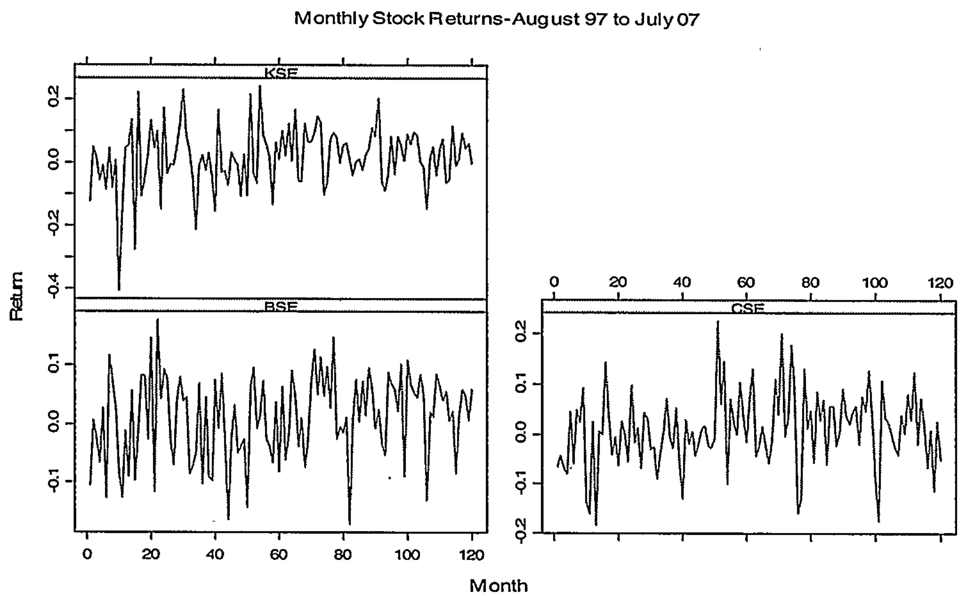

Figure 2.1: Monthly Stock Return

Table 2.1: Statistics of Monthly Stock Market Returns from Aug. 1997 to Jul. 2007

Stock Market	Mean	Volatility	Skewness	Kurtosis	J-B Statistic	P-Value
CSE	0.008604	0.074455	- 0.004597	3.535	1.4331	0.4884
BSE	0.010701	0.071986	- 0.398787	2.589	3.9440	0.1392
KSE	0.016103	0.098903	- 0.709801	5.380	38.3996	0.0000

2.2 Tests for Nonstationarity and Stationarity

To better understand the dynamic structure of the Sri-Lankan, Indian, and Pakistani stock markets in the modeling process, it is essential to confirm the nonstationarity of each of these markets. Basically, when two or more component series exhibit nonstationarity property, it is most likely that they are more dependent on each other in the long run. If this is the case, then it is appropriate to study these series jointly and in econometric literature, the cointegrated VAR model is most applicable. Should each of the series be stationary, then it is most likely that they are not dependent on each other in the long run. If this is the case, it is not appropriate to study these series jointly, and the VAR model offers the best option in the econometric literature.

In order to check the existence or otherwise of the nonstationarity property in the stock markets of Sri-Lanka, India and Pakistan, widely used tests are employed. They are the Augmented Dickey Fuller (ADF) by Dickey and Fuller (1981) and Phillip and Perron (1988) tests.

2.2.1 ADF Test

The ADF tests the null hypothesis that a time series y_t is *I(1)* (there is unit-root) against the alternative that it is *I(0)* (there is no unit-root), assuming that the data has an autoregressive moving average (ARMA) structure. The ADF test is based on estimating the test regression

$$y_t = \beta' D_t + \phi y_{t-1} + \sum_{j=1}^{p} \psi_j \Delta y_{t-j} + \varepsilon_t \tag{2.8}$$

where D_t is a vector of deterministic terms (constant, trend etc.), p is the lagged difference of terms, Δy_{t-j} are used to approximate the ARMA structure of the errors, and ε_t are serially uncorrelated errors. The ADF test statistic is based on the least squares estimates of Eq. (2.8). The ADF test statistic is the usual t-test statistic.

$$\text{ADF - test} = \frac{\hat{\phi}-1}{\text{SE}(\phi)} \tag{2.9}$$

Another important step in the implementation of the ADF test is the specification of the lag length p. Ng and Perron (1995, 2001) put forward the following procedure when confronted with the choice of appropriate lag length. The first step is to set an upper bound p_{max}. The next step is to estimate the ADF test regression with the $p = p_{max}$. If the absolute value of the t-test statistic is greater than 1.6, then set $p = p_{max}$ and the unit-root test is performed. Otherwise, p is reduced by 1 and the process repeated. Applying the ADF test (with and without drift) to the CSE, BSE, and the KSE data confirm stationarity of the markets, or all are not *I(1)*, as shown in tables 2.2 and 2.3.

Table 2.2: Augmented Dickey Fuller (ADF) Test for Unit-Roots without Drift

Variable	Hypothesis	Test Statistic	Probability Value
CSE	$CSE_{return}(\sim I(1))$	-4.903	7.586-5
BSE	$BSE_{return}(\sim I(1))$	-3.882	0.002966
KSE	$KSE_{return}(\sim I(1))$	-4.184	0.001078

Table 2.3: Augmented Dickey Fuller (ADF) Test for Unit-Roots with Drift

Variable	Hypothesis	Test Statistic	Probability Value
CSE	$CSE_{return}(\sim I(1))$	-5.268	1.501e-4
BSE	$BSE_{return}(\sim I(1))$	-4.075	0.009007
KSE	$KSE_{return}(\sim I(1))$	-4.494	0.002384

2.2.2 PP Test

The PP test as mentioned earlier provides an alternative check for nonstationarity. Phillips and Perron (1988) came up with a number of unit-root tests that have become standard in the analysis of financial time series. The test regression for the PP test is

$$\Delta x_t = \beta' D_t + \Pi x_{t-1} + e_t \tag{2.10}$$

where e_t is $I(0)$ and heteroscedastic.

The PP tests correct for any serial correlation and heteroscedasticity in the errors e_t of the test regression by directly modifying the test statistics $t_{\Pi=0}$ and $T\hat{\Pi}$. These modified statistics, denoted z_t and z_Π are respectively

$$z_t = \left(\frac{\hat{\sigma}^2}{\hat{\lambda}^2}\right) \cdot t_{\Pi=0} - \frac{1}{2}\left(\frac{\hat{\lambda}^2 - \hat{\sigma}^2}{\hat{\lambda}^2}\right) \cdot \left(T.SE\left(\frac{T.SE(\hat{\Pi})}{\hat{\sigma}^2}\right)\right) \quad \text{and}$$

$$z_\Pi = T_\Pi - \frac{1}{2}\frac{T^2.SE\left(\hat{\Pi}\right)\left(\hat{\lambda}^2 - \hat{\sigma}^2\right)}{\hat{\sigma}^2}$$

The terms $\hat{\sigma}^2$ and $\hat{\lambda}^2$ are consistent estimates of the variance parameters.

$$\sigma^2 \rightarrow T^{-1}\sum_{t=1}^{T} E\left[u_t^2\right] \quad \text{as} \quad T \rightarrow \infty$$

$$\lambda^2 \rightarrow \sum_{t=1}^{T} \mathrm{E}\left[T^{-1} S_T^2\right] \quad \text{as} \quad T \rightarrow \infty$$

$$S_T = \sum_{t=1}^{T} u_t$$

The sample variance of the least squares residuals $\hat{u}_t$ is a consistent estimate of σ^2, and the Newey-West long-run variance estimate of $\hat{u}_t$ is a consistent estimate of λ^2.

Under the null hypothesis, $\Pi = 0$, the PP z_t and z_Π statistics have the same asymptotic distribution as the ADF t-statistic.

The PP test is more robust to general forms of heteroscedasticity in the error term e_t than the ADF test. Also, one need not specify a lag length for the test regression as opposed to the situation in the ADF test.

Table 2.4 displays the PP test results. The results further confirmed the non-existence of unit-root nonstationarity process in the CSE, BSE and KSE data. All tests are performed at five (5) percent significance level. Therefore, it is not possible to perform cointegration test to determine whether these markets bear at least one long-run equilibrium relation or a common stochastic trend in the long-term. However, the VAR model readily comes to mind should different component series show evidence of stationarity but jointly dependent in a linear fashion. To comfortably endorse the VAR model as the most appropriate model for the long-term equity return process, it is necessary to establish if there exists any form of linear dependence among the 3 national stock markets.

Table 2.4: Phillip-Perron (PP) Test for Unit-Roots

Variable	Hypothesis	Test Statistic	Probability Value
CSE	$CSE_{return}(\sim I(1))$	-111.2	8.935e-14
BSE	$BSE_{return}(\sim I(1))$	-125..2	8.969e-16
KSE	$KSE_{return}(\sim I(1))$	-122.9	2.052e-15

2.3 Linear Dependence Structure using Cross-Correlation

Of equal importance is the degree of correlation to ascertain if there exists any form of linear dependence among these markets. The dependence structure can be established using cross-correlation analysis.

Mathematically, the cross-correlation matrix ρ_l is estimated by

$$\hat{\rho}_l = \hat{D}^{-1}\hat{\Gamma}_l\hat{D}^{-1}, \quad l \geq 0 \tag{2.11}$$

where

$$\hat{\Gamma}_l = \frac{1}{T}\sum_{t=l+1}^{T}(r_t - \bar{r})(r_{t-l} - \bar{r}), \quad l \geq 0$$

$$\bar{r} = \left(\sum_{t=1}^{T} r_t\right) \Big/ T$$

$\hat{D}$ is the k x k diagonal matrix of the sample standard deviations of the component series. Also $1/\sqrt{T}$ is the asymptotic α percent critical value of the sample correlation (See Tiao and Box (1981)).

The asymptotically 5 percent critical value of the sample correlation is 0.09. It is easily seen from table 2.5 that, significant cross-correlation at the approximate 5 percent level appears at lag one (1). An examination of the sample cross-correlation matrices at this lag indicates strong linear dependence between BSE and KSE, and marginal linear dependence between CSE and BSE.

Table 2.5: Cross-Correlation Matrices

Lag	Zero	One	Two	Three	Four	Five	Six
CSE/BSE	0.19	0.09	0.022	-0.1053	0.0083	-0.0639	-0.0905
CSE/KSE	0.21	0.05	0.1859	0.0407	-0.0597	0.0587	0.0106
BSE/KSE	0.37	0.11	0.0478	-0.0422	0.109	0.0763	0.0349

In the light of the above results, it is possible to model the long-term equity return process of the South Asia stock markets as a VAR model.

2.4 The Stationary Vector Autoregressive (VAR) Model

If $Y_t = (y_{1t}, y_{2t}, \ldots\ldots\ldots\ldots, y_{nt})'$ denote an (n x 1) vector of time series variables, then the basic p-lag vector (VAR(p)) model has the form

$$Y_t = c + \Pi_1 Y_{t-1} + \Pi_2 Y_{t-2} + \ldots\ldots\ldots + \Pi_p Y_{t-p} + \varepsilon_t \tag{2.12}$$

where Π_i are (n x n) coefficient matrices and ε_t is an (n x 1) unobservable zero mean white noise vector process (serially uncorrelated or independent) with time invariant covariance matrix Σ.

2.4.1 Estimation

The estimation process considers the basic VAR(p) model in Eq. (2.12). In a seemingly unrelated regression notation, each equation in the VAR(p) may be written as

$$y_t = Z\pi_i + e_i, \qquad i = 1, \ldots\ldots\ldots\ldots\ldots, n \tag{2.13}$$

Where y_t is a (T x 1) vector of observations on the i^{th} equation, Z is a (T x k) matrix with t^{th} row given by $Z'_t = (1, Y'_{t-1} \ldots\ldots\ldots, Y'_{t-p})$, $k = np+1$, π_i is a (k x 1) vector of parameters and e_i is a (T x 1) error with covariance matrix $\sigma_i^2 I_T$.

Let $\hat{\Pi} = [\hat{\pi}_1, \ldots\ldots\ldots, \hat{\pi}_n]$ denote the (k x n) matrix of LS coefficients of the n equations.

$$\text{Let } vec(\hat{\Pi}) = \begin{pmatrix} \hat{\pi}_1 \\ \cdot \\ \cdot \\ \cdot \\ \hat{\pi}_n \end{pmatrix}$$

Under standard assumptions regarding the behaviour of stationary and ergodic VAR models (see Hamilton (1994) or Lutkepohl (1991)), $vec(\hat{\Pi})$ is consistent and asymptotically normally distributed with asymptotic covariance matrix

$$a\hat{\mathrm{var}}(vec(\hat{\Pi})) = \hat{\Sigma} \otimes (Z'Z)^{-1}, \tag{2.14}$$

where

$$\hat{\Sigma} = \frac{1}{T-k} \sum_{t=1}^{T} \hat{\varepsilon}_t \hat{\varepsilon}'_t$$

and $\hat{\varepsilon}_t = Y_t - \hat{\Pi}'_t Z_t$ is the multivariate LS residuals from Eq. (2.12) at time t.

2.4.2 Inference on Coefficients

Inference on the coefficients of the VAR(p) model is carried out using the following procedure.

The ith element of $vec(\hat{\Pi})$, $\hat{\pi}_i$ is asymptotically normally distributed with asymptotic standard error given by the square root of ith diagonal element of $\hat{\Sigma} \otimes (Z'Z)^{-1}$. Hence, asymptotically, valid t-tests on individual coefficients may be constructed in the usual way.

2.4.3 Lag Length Selection

The lag length selection process for a VAR(p) model is to fit VAR(p) model with orders $p = 0, \ldots\ldots\ldots, p_{max}$ and choose the value of p which minimizes some model selection criteria. In this thesis, two of the well known selection criteria are employed. They are the Akaike Information Criterion (AIC) and Bayesian Information (BIC).

1. Akaike (AIC)

$$AIC(p) = \ln\left|\tilde{\Sigma}(p)\right| + \frac{2}{T} pn^2$$

2. Schwarz-Bayesian (BIC)

$$BIC(p) = \ln\left|\tilde{\Sigma}(p)\right| + \frac{2\ln T}{T} pn^2$$

where

$$\tilde{\Sigma}(p) = \sum_{t=1}^{T} \hat{\varepsilon}_t \hat{\varepsilon}'_t$$

For more information on the use of model selection criteria in VAR models see Lutkepohl (1991).

In fitting the VAR model, both the BIC and AIC criteria are computed with a maximum lag length of 6 and $p = 1$ minimizes both criteria as shown in table 2.6.

Table 2.6: Choice Criteria for Selecting Lag Length for the VAR Model

Model	(BIC)	(AIC)
One	-718.7848	-751.6189
Two	-687.9235	-745.3837
Three	-654.0302	-736.1161
Four	-620.7357	-727.4475
Five	-582.3687	-713.7063
Six	-558.4710	-741.4344

The VAR(1) model is automatically re-estimated and results displayed in table 2.7. The second, third and fourth columns of the table gives the respective estimated coefficients of CSE, BSE and KSE equations. Also, from the regression diagnostic table (Table 2.8), it is clear that, the fit for BSE is much better than the CSE and KSE.

The estimated (fitted) equations for the three national stock markets are as follows:

$$\begin{bmatrix} CSE_t \\ BSE_t \\ KSE_t \end{bmatrix} = \begin{bmatrix} 0.0080 \\ 0.0102 \\ 0.0182 \end{bmatrix} + \begin{bmatrix} 0.0307 & 0.0773 & 0.0111 \\ 0.0642 & -0.0533 & 0.0864 \\ -0.1137 & 0.02920 & -0.0138 \end{bmatrix} \begin{bmatrix} CSE_{t-1} \\ BSE_{t-1} \\ KSE_{t-1} \end{bmatrix} + \in$$

where

$$\in \sim N(0,\Sigma) \text{ and } \Sigma = \begin{bmatrix} 0.0056393683 & 0.0009401135 & 0.001502121 \\ 0.0009401135 & 0.0051453475 & 0.002632862 \\ 0.0015021208 & 0.0026328623 & 0.009880842 \end{bmatrix}$$

Table 2.7 Coefficients of the VAR(1) Model

Coefficients	CSE	BSE	KSE
Intercept	0.0080	0.0102	0.0182
Standard Error	0.0070	0.0067	0.0093
Test Statistic	1.1344	1.5250	1.9593
CSE. Lag 1	0.0307	0.0642	-0.1137
Standard Error	0.0955	0.0912	0.1264
Test Statistic	0.3211	0.7032	-0.8989
BSE. Lag 1	0.0773	-0.0533	0.0290
Standard Error	0.1040	0.0993	0.1377
Test Statistic	0.7431	0.5362	0.2109
KSE. Lag 1	0.0111	0.0864	-0.0138
Standard Error	0.0757	0.0724	0.1003
Test Statistic	0.1462	1.1948	-0.1374

Table 2.8 Regression Diagnostics Table

Item	CSE	BSE	KSE
R-Squared	0.0087	0.0189	0.0076
Adjusted R-Squared	-0.0172	-0.0067	-0.0183
Standard Error	0.0751	0.0717	0.0994

Figures 2.2 and 2.3 are the residuals and qq plots for the VAR(1) fit to the three (3) national stock market data. Apart from the KSE where the residuals look non-random, the rest look fairly random with some heteroscedasticity. Also, the qq-plot indicates that the residuals for the KSE are highly non-normal. However, the BSE simulated values are used for equity linking in chapter 4.

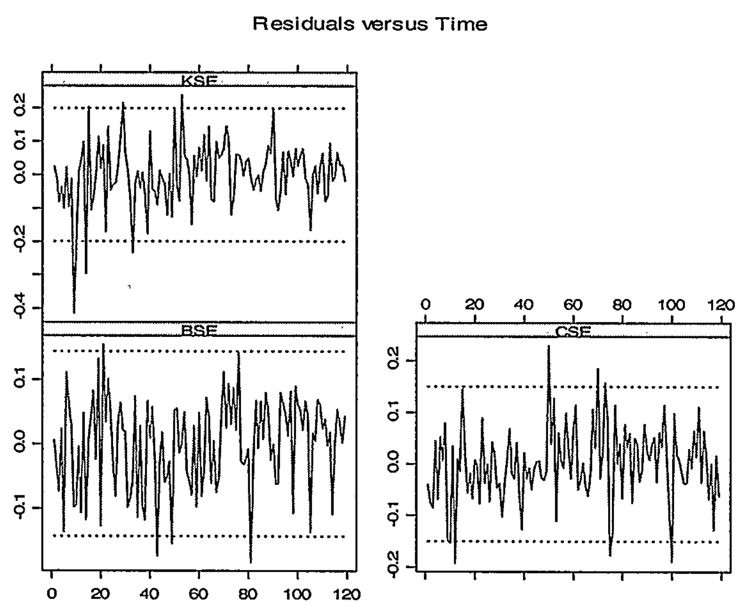

Figure 2.2: Residuals versus Time

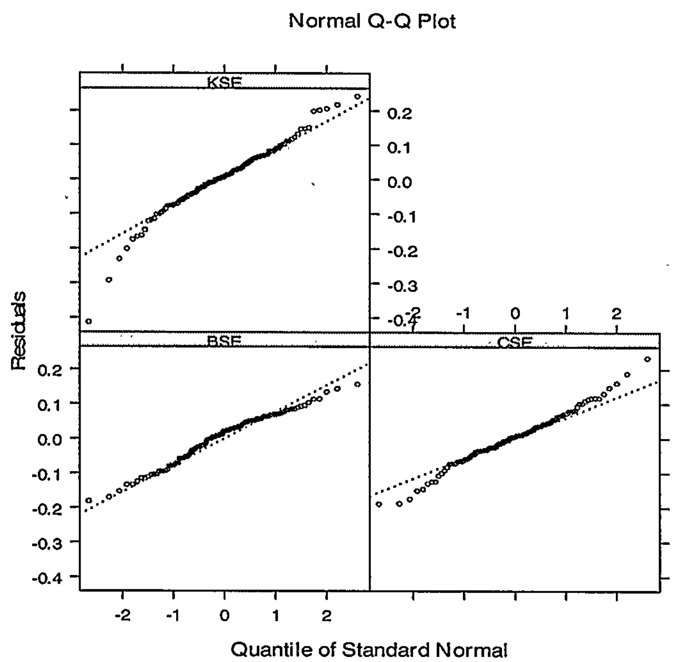

Figure 2.3: Normal Q-Q Plot for the fitted VAR(1) model

Chapter 3

Treasury Bond Model

The purpose of this chapter is to develop a treasury bond model using the same time series econometric techniques discussed in the preceding chapter. In order to construct a model that applies to India, four (4) different yields to maturity (YTM) from the India money market are studied. These comprise: up to 14 days, 15 to 91 days, 92 to 182 days and 183 to 364 days YTM spanning the period August 1997 to July 2007.

3.1 Exploratory Analysis of India's Treasury Bond Market

Taking a closer look at the India's money market, it is obvious that movements of the treasury bond rates stimulate further interest to investigate the applicability of all the 4 YTM in the valuation of segregated funds in India. However, a glance at figure 3.1, it is apparent that the movements of these yields exhibit rather a close relationship; it is possible to say that these YTM are cointegrated in the long-run.

Also, summary statistics of the 4 YTM displayed in table 3.1 indicate that the highest mean YTM is the 183 to 364 days followed by the 92 to 182 days, 15 to 91 days and up to 14 days YTM. The largest volatility is exhibited by the 92 to 182 days, followed by the 183 to 364 days, then 15 to 91 days and up to 14 days YTM. Table 3.1 further reveals that, all the 4 YTM are positively skewed. However, the only YTM which is not heavy tailed is the 183 to 364 days YTM.

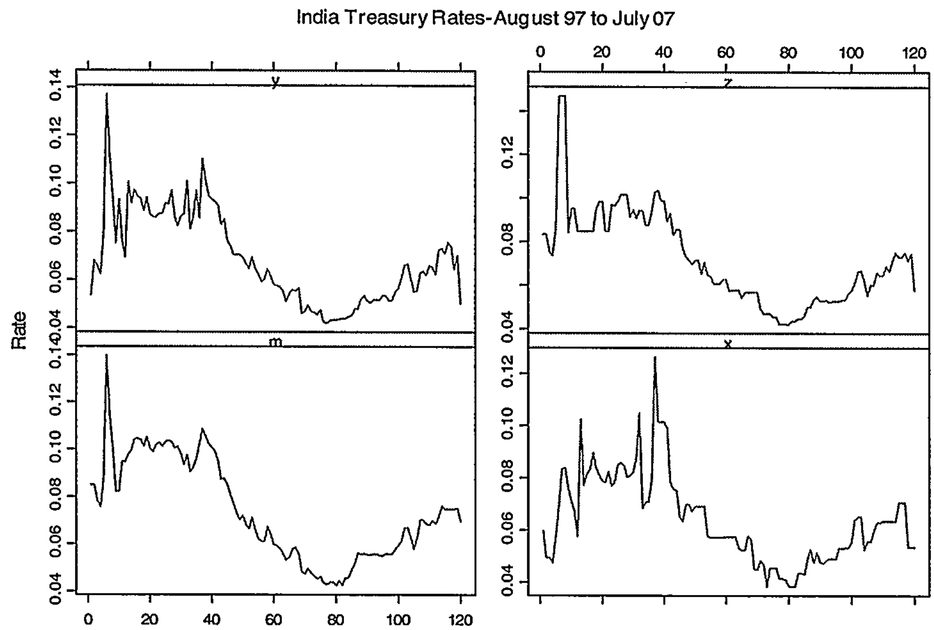

Figure 3.1: India Treasury Bonds: August 97 to July 07

Table 3.1: Statistics of India's Treasury Bond from Aug. 97 to Jul. 07

YTM (Days)	Mean	Volatility	Skewness	Kurtosis
Up to 14	0.06407	0.01678	0.8447	3.709
15 to 91	0.06869	0.01895	0.6664	3.040
92 to 182	0.07199	0.02146	1.0388	4.665
183 to 364	0.07390	0.02122	0.4045	2.246

Normality checks based on the J-B statistic performed on the YTM, show that, the 4 YTM do not follow the normal distribution when the test is done at the 5 percent significant level. However, at the 1 percent level of significance, only the 15 to 91 days and the 92 to 182 days YTM are normally distributed. Results of this test are shown in table 3.2.

Table 3.2: Normality Test Table

Jarque-Bera (JB) Test	Up to 14 Days	15 to 91Days	92 to 182 Days	183 to 364 Days
Test Statistic	16.7839	8.8895	35.4428	6.1146
Prob. Value	0.0002	0.0117	0.0000	0.0470

As mentioned earlier in this chapter, it is possible that these 4 YTM could bear a common long run equilibrium relationship or are cointegrated.

3.2 Tests for Nonstationarity and Stationarity

To ensure the correct specification of the treasury bond model, it is necessary to first check the existence of unit-root nonstationary processes. Here, both the ADF and PP tests are performed as before.

3.2.1 ADF Test

The ADF test with and without drift using $p = 6$ are shown in tables 3.3 and 3.4 respectively. Both tests confirm that, all the 4 YTM under the null hypothesis of existence of a unit-root are not rejected at the 5 percent significant level. This means that, all the 4 YTM follow the unit-root nonstationarity process or are *I(1)*.

Table 3.3: Augmented Dickey Fuller (ADF) Test for Unit-Roots without Drift

YTM (Days)	Hypothesis	Test Statistic	Probability Value
Up to 14	*Up to* $14_{return}(\sim I(1))$	-1.421	0.5698
15 to 91	15 *to* $91_{return}(\sim I(1))$	-1.773	0.3922
92 to 182	92 *to* $182_{return}(\sim I(1))$	-2.620	0.0919
183 to 364	183 *to* $364_{return}(\sim I(1))$	-2.362	0.1549

Table 3.4: Augmented Dickey Fuller (ADF) Test for Unit-Roots with Drift

YTM (Days)	Hypothesis	Test Statistic	Probability Value
Up to 14	*Up to* $14_{return}(\sim I(1))$	-2.239	0.4631
15 to 91	15 *to* $91_{return}(\sim I(1))$	-1.239	0.8972
92 to 182	92 *to* $182_{return}(\sim I(1))$	-2.086	0.5476
183 to 364	183 *to* $364_{return}(\sim I(1))$	-0.2237	0.9918

3.2.2 PP Test

To verify and confirm the preceding results, the PP test is performed at the 5 percent level of significance and results displayed in table 3.5. It is evident from the table that, all the 4 YTM under the null hypothesis cannot be rejected at the 5 percent significant level confirming the outcome of the ADF test.

Table 3.5: Phillip-Perron (PP) Test for Unit-Roots

YTM (Days)	Hypothesis	Test Statistic	Probability Value
Up to 14	*Up to* $14_{return}(\sim I(1))$	-13.130	0.0568
15 to 91	*15 to* $91_{return}(\sim I(1))$	-10.930	0.1001
92 to 182	*92 to* $182_{return}(\sim I(1))$	-8.814	0.1710
183 to 364	*183 to* $364_{return}(\sim I(1))$	-3.892	0.5453

It is clear at this stage that, all the 4 YTM follow the unit-root processes or all are $I(1)$. Therefore, modeling them jointly through cointegrated VAR process will be the most appropriate thing to do.

3.3 Cointegration

This section first outlines Johansen's approach to cointegration modeling, then a summary of the results from the modeling process and finally the specification of the treasury bond model for the valuation of segregated fund contracts in India.

3.3.1 Cointegrated VAR Model

Given a k-dimensional VAR(p) model x_t, then the model with possible time trend is given as

$$x_t = \mu_t + \Phi_1 x_{t-1} + \dots\dots\dots\dots\dots + \Phi_p x_{t-p} + a_t \tag{3.1}$$

where a_t the innovation is assumed to be Gaussian and $\mu_t = \mu_0 + \mu_1 t$, where μ_0 and μ_1 are k-dimensional constant vectors. Now, we write $\Phi(B) = I - \Phi_1 B - \dots\dots\dots\dots\dots - \Phi_p B^p$ if all the zeros of the determinant $|\Phi(B)|$ are outside the unit circle, then x_t is unit-root stationary. In the literature, a unit-root stationary series is said to be *I(0)* process or it is not integrated. If $|\Phi(B)| = 0$, then x_t is unit-root nonstationary or at most an integrated process of order 1 that is an *I(1)* process.

3.3.2 Error Correction Model (ECM) for VAR(p) Process

An ECM process for VAR(p) model x_t is given as

$$\Delta x_t = \mu_t + \Pi x_{t-1} + \Phi_1^* \Delta x_{t-1} + \ldots\ldots\ldots\ldots\ldots + \Phi_{p-1}^* \Delta x_{t-p+1} + a_t \qquad (3.2)$$

where $\Phi_j^* = \sum_{i=j+1}^{p} \Phi_i$ and $j = 1, \ldots\ldots\ldots\ldots\ldots\ldots, p\text{-}1$ and $\Pi = \alpha\beta'$ is obtained from a cointegrated VARMA (p,q) model with m cointegrating factors ($m<k$) with an ECM representation as

$$\Delta x_t = \alpha\beta' x_{t-1} + \sum_{i=1}^{p-1} \Phi_i^* \Delta x_{t-i} + a_t - \sum_{j=1}^{q} \Theta_j a_{t-j} \qquad (3.3)$$

In Eq. (3.2), when the Rank ($\prod$) = 0 it implies $\prod = 0$ and x_t is not cointegrated. When Rank ($\prod$) = k, it implies that $|\Phi(1)| \neq 0$ and x_t contains no unit-roots (*I(0)*). Also, when $0 <$ Rank ($\prod$) $< k$, in this situation, one can write $\prod$ as $\alpha\beta'$ where α and β are k x m matrices with rank (α) = rank (β) = m. The ECM of Eq. (3.2) becomes

$$\Delta x_t = \mu_t + \alpha\beta' x_{t-1} + \Phi_1^* \Delta x_{t-1} + \ldots.. + \Phi_{p-1}^* \Delta x_{t-p+1} + a_t \qquad (3.4)$$

This means that x_t is cointegrated with m linearly independent cointegrating vectors, $w_t = \beta' x_t$, and has $k - m$ unit-roots that give $k - m$ common stochastic trends of x_t. This is the approach followed by Johansen (1988, 1995). The same approach is adopted in this thesis with a slight adjustment of the deterministic term by restricting it to a constant to eliminate any quadratic term in the time series x_t. Moreover, the restricted constant form of the deterministic term, is appropriate for non-trending *I(1)* data such as interest rates and exchange rates (See Zivot and Wang (2003)). The restricted constant VECM adopted in this thesis is of the form

$$\Delta x_t = \alpha(\beta' x_{t-1} + c_0) + \Phi_1^* \Delta x_{t-1} + \dots\dots\dots + \Phi_{p-1}^* \Delta x_{t-p+1} + a_t \quad (3.5)$$

where c_0 is an m-dimensional nonzero constant vector. The series in x_t are still *I(1)* without drift and the cointegrating relations $\beta' x_t$ have non-zero means c_0.

3.3.3 Cointegration Test

Now, the goal is to test the rank of Π to know the number of cointegrating vectors. Mathematically, the rank of Π is the dimension of Π, and is number of nonzero eigenvalues of Π.

Consider the hypotheses

$$H_0 : \text{Rank}(\Pi) = m \text{ versus } H_a : \text{Rank}(\Pi) > m .$$

Johansen (1988) proposes the likelihood ratio (LR) statistic

$$LK_{tr}(m) = -(T-p) \sum_{i=m+1}^{k} \ln(1 - \hat{\lambda}_i) \quad (3.6)$$

to perform the test. If the Rank $(\Pi) = m$, then $\hat{\lambda}_i$ should be small for $i > m$ and hence $LK_{tr}(m)$ should be small. This test is referred to as the *trace cointegration test.* Due to the presence of unit-roots, the asymptotic distribution of $LK_{tr}(m)$ is not chi-squared but a function of standard Brownian motions. Thus, critical values of $LK_{tr}(m)$ must be obtained via simulation.

Johansen (1988) also, considers a sequential procedure to determine the number of cointegrating vectors. Specifically, the hypotheses of interest are

$$H_0: \text{Rank}(\Pi) = m \text{ Versus } H_a: \text{Rank}(\Pi) = m+1$$

The *LK* ratio test statistic, called the *maximum eigenvalue statistic*, is

$$LK_{\max}(m) = -(T-p)\ln(1-\hat{\lambda}_{m+1}). \tag{3.7}$$

Again, critical values of the test statistics are nonstandard and must be evaluated via simulations. Critical values for these tests are provided in Osterwald-Lenum (1992). In performing cointegration test for the up to 14 days, 15 to 91 days, 92 to 182 days and 183 to 364 days YTM, the lag length that minimizes both the Bayesian information criterion (BIC) and the Akaike information criterion (AIC) with a maximum lag of 6 are p =1 and p =6 respectively. However, priority is given to the BIC, hence p =1 is selected. The lag length for the VECM is then $p - 1 = 0$.

Table 3.6: Choice Criteria for Selecting Lag Length for the VECM

Model	(BIC)	(AIC)
One	-3537.360	-2686.647
Two	-3497.833	-2673.962
Three	-3508.769	-2679.051
Four	-3478.888	-2688.445
Five	-3431.984	-2704.912
Six	-3407.565	-2708.243

Table 3.7 focuses on the tests for cointegration ranks. To eliminate any quadratic term in the vector series, the Johansen likelihood ratio (LR) tests are computed by assuming the restricted constant. From the table, the 4 estimated eigenvalues are less than 1, indicating

that the test is stable. Both trace and maximum tests reject H (0), H (1), and H (2) but fail to reject H (3) at the 1percent significance level. Therefore, there exist 3 linearly independent cointegrating vectors (rank of $\prod$) and 1 common stochastic trend (unit-root).

Table 3.7: Cointegration Rank Test

Null Hypothesis	Eigen value	Trace Statistic	95% CV	99% CV	Maximum Statistic	95% CV	99% CV
H(0)++**	0.5965	189.621	53.12	60.16	108.014	28.14	33.24
H(1)++**	0.3364	81.6071	34.91	41.07	48.8048	22	26.81
H(2)++**	0.2167	32.8024	19.96	24.6	29.0579	15.67	20.2
H(3)	0.031	3.7445	9.24	12.97	3.7445	9.24	12.97

+ +and + means trace statistic is significant at one (1) and five (5) percent respectively
** and * means maximum statistic is significant at one (1) and five (5) percent respectively

Now that the number of cointegrating vectors is known, the maximum likelihood estimates of the full VECM can be obtained. A comprehensive result of the computed VECM is shown in appendix B. Since the 4 YTM are cointegrated with a common stochastic trend, then the specified stationary series is given as

$w_t \approx x_t - 2.9400y_t + 0.9616z_t + 0.8846m_t$ and the mean of w_t is about 0.004.

where

$x =$ Up to 14 Days YTM

$y =$ 15 to 91Days YTM

$z =$ 92 to 182 Days YTM

$m =$ 183 to 364 Days YTM

The fitted VECM is

$$\Delta x_t = \begin{bmatrix} 0.0038 \\ 0.2464 \\ -0.1690 \\ -0.0004 \end{bmatrix} [w_{t-1} + 0.004] + a_t \tag{3.8}$$

where

$$a_t \sim N(0, \Sigma)$$

$$\Sigma = \begin{bmatrix} 0.00008407590 & 0.00004352695 & 0.00001672633 & 0.00001570864 \\ 0.00004352695 & 0.00007684850 & 0.00006086263 & 0.00004765095 \\ 0.00001672633 & 0.00006086263 & 0.00007990423 & 0.00004476357 \\ 0.00001570864 & 0.00004765095 & 0.00004476357 & 0.00004513098 \end{bmatrix}$$

The adequacy of the model is examined via the residuals verses time and cointegrating residuals plots shown in figures 3.2 and 3.3 respectively. Some large residuals are shown in the plots which occurred prior to early 2001when interest rates were high and volatile. Again, from figure 3.2, it is apparent that, the up to 14 days YTM is the most volatile, followed by the 15 to 91 days, 92 to 182 days then 183 to 364 days YTM.

Finally, the fitted VECM is used to simulate 120 months YTM using July 2007 YTM as the initial values.

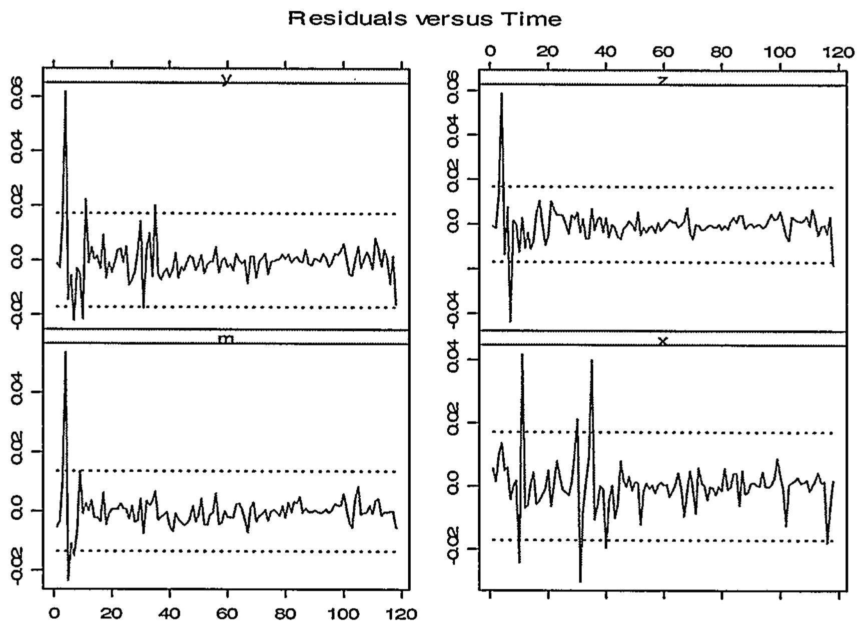

Figure 3.2: Residuals verses Time plots for the 4 YTM Series for India Market.

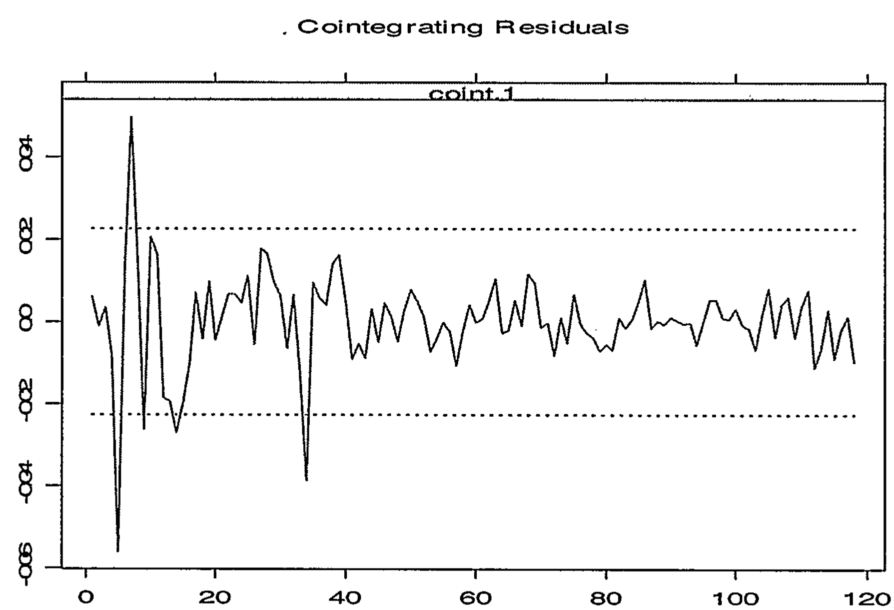

Figure 3.3: Cointegrating Residuals for a VECM fit to the monthly India TYM

However, an easy way to obtain simulated values from the VECM representation is to convert it to a VAR model. The VAR representation of the fitted VECM is shown as follows:

$$\begin{bmatrix} x_t \\ y_t \\ z_t \\ m_t \end{bmatrix} = \begin{bmatrix} 0.00001501346 \\ 0.0009770362 \\ -0.0006698636 \\ -1.707409e-006 \end{bmatrix} +$$

$$\begin{bmatrix} 0.00008407590 & 0.00004352695 & 0.00001672633 & 0.00001570864 \\ 0.00004352695 & 0.00007684850 & 0.00006086263 & 0.00004765095 \\ 0.00001672633 & 0.00006086263 & 0.00007990423 & 0.00004476357 \\ 0.00001570864 & 0.00004765095 & 0.00004476357 & 0.00004513098 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \\ m_{t-1} \end{bmatrix} + a_t$$

where

$$a_t \sim N(0, \Sigma).$$

The simulated values for the 15 to 91 day YTM are used as the risk-free rate to discount all corresponding future income (margin offset) to their present values in the next chapter.

Chapter 4

Segregated Fund Guarantees: Dynamic Hedging Approach

This chapter applies the models of the previous chapters and the theory of option pricing in the valuation of segregated fund contracts in India. It therefore provides a review of option pricing theory and the key role it plays in the derivation of the valuation formulae for the GMMB and GMDB. We include mention of the dynamic hedging approach in making provision for the hedged and unhedged liabilities.

4.1 Review of Option Pricing Theory

Since the seminal work of Black and Scholes (1973) and Merton (1973), the theory and practice of option valuation and risk management has expanded phenomenally. Boyle et al. (1998) and Hull (1989) are two excellent reference materials on option pricing theory.

4.1.1 What is an Option?

An option is a derivative security that gives the buyer (i.e., the holder) the right, but not the obligation, to buy or sell the underlier at an agreed price on or before a specified date in the future. The price at which the underlier can be purchased or sold in the future is called strike price, and the up-front cost of purchasing the flexibility provided by an option is called the premium. The item being traded is called the underlier because its value is the foundation for the value of the derivative contract.

4.1.2 Types of Options

Basically, there are two types of options. They are a call and a put option. A call option gives the option buyer the right, but not the obligation, to purchase the underlier at the agreed-on (strike) price in the future. If the buyer decides to go through with the deal, then he or she exercises the call option. There are different types of call options. The well known ones are the European and American call options. The difference between the two is that, whereas the European call option cannot be exercised until the specified future date, the American call option can be exercised any time before or on the specified future date.

A put option gives the option buyer the right, but not the obligation, to sell the underlier at the agreed-on price (strike price) in the future. If the buyer decides to go through with the deal, then he or she exercises the put option. As with the call, there are 2 common types of a put option: the European and American put options.

In this study, the European put option formula is adopted to reflect the fact that, the underlier is the segregated fund value instead of the usual stock price.

4.2 The Black-Sholes-Merton (B-S-M) Theory

In essence, the B-S-M framework for option valuation is a continuous time stochastic process when viewed from the real world application and is based on more sophisticated assumptions.

4.2.1 The B-S-M Assumptions

This sub-section outlines the major assumptions underpinning the theory. The major assumptions are as follows:

1. The asset price S_t follows a geometric Brownian motion (GBM) with constant variance σ^2. This implies that asset returns over any period have a lognormal distribution, and that asset returns over two disjoint periods of equal length are independent and identically distributed.
2. Markets are assumed to be "frictionless" – that is, no transaction costs or taxes and all securities are infinitely divisible.
3. Short selling is allowed without restriction, and borrowing and lending rates of interest are the same.
4. There are no riskless arbitrage opportunities.
5. Trading is continuous.
6. Interest rates are constant or predictable.

To some extent, all of these assumptions are impracticable because, markets are not open continuously and trading costs money. Nevertheless, the B-S-M model has proven its worth in terms of its being remarkably robust to departures from the assumptions.

4.2.2 B-S-M Results

The concept built from the assumptions enumerated in sub-section 4.2.1 can be used to value any option. The most famous equations are the Black-Scholes (B-S) equations for a European option. The B-S-M results are as follows:

1. Price

The most general result from the B-S-M framework is that any derivative security can be valued using the discounted expected pay off under the artificial, risk-neutral probability distribution, where the force of interest for discounting is the risk-free rate, denoted r. That is, for a security with a payoff W at time T, where the payoff is contingent on a risky underlier with price process S_t, the cost of the replicating portfolio at $t < T$ is

$$P_t = e^{-r(T-t)} \mathrm{E}_Q[W] \tag{4.1}$$

where Q = risk neutral measure

2. The Hedge

The general B-S result goes beyond the price of the replicating portfolio P_t at time t by further providing information on how to construct a hedging portfolio. Assume

$$\Psi_t = \frac{\partial P_t}{\partial S_t} \tag{4.2}$$

The portfolio that comprises of $\Psi_t S_t$ in the risky asset and $P_t - \Psi_t S_t$ in the risk-free asset at time t will exactly replicate the option, and will be self-financing, under the B-S assumptions. Self-financing means that, the change in value of the stock part of the hedge in each infinitesimal time step must be precisely adequate to service the change in bond price in the hedge.

3. The Risk-Neutral Probability Distribution (Q-Measure)

In mathematical finance, risk-neutral probability distribution is a probability measure that results when one assumes that the current value of all financial assets is equal to the expected value of the future payoff of the asset discounted at the risk free rate. The concept is used in pricing of derivatives. It is called the artificial probability or Q-measure in financial economics.

4.3 Derivation of the European Put Option

In this section, the value of a put option at time t using the principle of discounted expected payoff under the Q-measure. Let t denote the current time; T the time to maturity of the contract; σ^2 the constant variance per unit time of the GBM; S_t the price process of the underlier on which the option is written; and $\Phi(\ )$ the standard normal distribution function. The payoff is $(K-S_T)^+$. Let $f_Q(\ )$ denote the risk-neutral density for the accumulation factor A_{T-t}. Then the price of the replicating portfolio at time $t<T$ is denoted BSP_t:

$$\mathrm{BSP}_t = \mathrm{E}_Q\left[(K-S_T)^+\right]e^{-r(T-t)} \tag{4.3}$$

$$= S_t \mathrm{E}_Q\left[(K/S_t - A_{T-t})^+\right]e^{-r(T-t)}. \tag{4.4}$$

Since A_{T-t} has a lognormal distribution with mean parameter $(T-t)(r-\sigma^2/2)$ and variance parameter $\sigma\sqrt{T-t}$, (See Klugman, Panjer and Willmot 1998).

$$\mathrm{BSP}_t = \{K\Phi\left(\frac{\log(K/S_t)-(r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)$$

$$- S_t e^{r(T-t)}\Phi\left(\frac{\log(K/S_t)-(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\right)\}e^{-r(T-t)}$$

$$= K\,e^{-r(T-t)}\Phi(-d_2) - S_t\Phi(-d_1) \tag{4.5}$$

where d_1 and d_2 are functions:

$$d_1 = \frac{\log(K/S_t)+(T-t)(r+\sigma^2/2)}{\sigma\sqrt{T-t}} \tag{4.6}$$

$$d_2 = \frac{\log(K/S_t) + (T-t)(r-\sigma^2/2)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t} \tag{4.7}$$

To establish the hedge portfolio, the stock part of the hedge portfolio is $S_t\Psi_t$ where

$$\Psi_t = \frac{\partial}{\partial S_t}\mathrm{BSP}_t$$

$$= -\Phi(-d_1) - S_t\frac{\partial(-d_1)}{\partial S_t}\phi(-d_1) + K\,e^{-r(T-t)}\frac{\partial(-d_2)}{\partial S_t}\phi(-d_2) \tag{4.8}$$

where $\phi(\;)$ is the standard normal density function.

4.4 B-S Equation for GMMB

The standard put option on the segregated fund is the GMMB. Assume the following information.

F_0 = a fund value at the valuation date $t = 0$,

G = the guarantee and assume first that the guarantee is fixed,

$(G-F_T)^+$ = the insurer's liability under the GMMB at maturity in say, T years,

m = monthly management charge deducted,

S_T = the stock index for equity linking. Then,

$$F_T = F_0\frac{S_T}{S_0}(1-m)^T, \tag{4.9}$$

Let $F_0 = S_0$, then the option price is

$$P_0 = e^{-rT}\mathrm{E}_Q\left[(G-F_T)^+\right]$$

$$= e^{-rT}\mathrm{E}_Q\left[(G-S_T(1-m)^T)^+\right] \tag{4.10}$$

Utilizing Eq. (4.10) coupled with changing S_0 to $S_0(1-m)^T$ in the standard B-S formula, the put option at time $t = 0$ is:

$$P_0 = Ge^{-rT}\Phi(-d_2) - S_0(1-m)^T\Phi(-d_1) \tag{4.11}$$

where

$$d_1 = \frac{\log(S_0(1-m)^T/G) + (r+\sigma^2/2)T}{\sigma\sqrt{T}} \tag{4.12}$$

$$d_2 = \frac{\log(S_0/G) + (r+\log(1+m)+\sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T} \tag{4.13}$$

This price approach makes no allowance for mortality or lapses. More over, not all policyholders will survive to maturity. These are situations to which the fund has an exposure. The effect of these exposures can be mitigated through diversification. The use of diversification to mitigate the risk effect attributable to lapses or exit is only possible so long as lapses are independent of the guarantee liabilities.

Given the assumption that BSP_0 is the option price with no allowance for lapses, and ${}_T p_x^{\tau}$ is the probability that the contract is in force at maturity, then the option price allowing for lapses is given as

$${}_T p_x^{\tau} \mathrm{BSP}_0 \tag{4.14}$$

Obviously, the GMMB replicating portfolio allowing for exits maybe determined by taking the product of the option price and the survival probability. For instance, if the probability that the policyholder lapses or dies before maturity date is ${}_T q_x^{\tau} = 0.27$, and we are aware that BSP_0 is the amount required for a guarantee maturity benefit with no allowance for exits, then the amount required allowing for exits is

$(1-0.27)\mathrm{BSP}_0 = 0.73\mathrm{BSP}_0$.

4.5 B-S Equation for GMDB

The GMDB liability is identical to that of the GMMB, except that the maturity date is contingent on the policyholder's death rather than his or her survival. The term of the option is therefore a random variable.

Let $\mathrm{BSP}_0(T)$ represent the cost at time 0 of a put option that matures in T years. Under the GMDB, T is a random variable denoting the future lifetime of the policyholder, corresponding to T_x. let $\mathrm{E}_T[\;]$ denote expectation over the distribution of T, then the cost of the hedge portfolio is simply the expected value of $\mathrm{BSP}_0(T)$ over the distribution of T.

Let ${}_T p_x^{\tau}$ denote the double decrement survival probability, as before, and let $\mu_{x,t}^{(d)}$ represent the force of mortality at time t for a life age x at time $t = 0$. Then the cost of the hedge portfolio at time $t = 0$ for a contract with a maximum n time units is

$$\mathrm{E}_T[\mathrm{BSP}_0(T)] = \int_0^n \mathrm{BSP}_0(t)\,{}_t p_x^{\tau} \mu_{x,t}^{(d)}\,dt \tag{4.15}$$

This can be evaluated numerically by using the approximation

$$H(0) = \sum_{t=1}^{n} \mathrm{BSP}_0(t)\,{}_{t-1} p_x^{\tau}\,{}_1 q_{x,t-1}^{(d)} \tag{4.16}$$

where t is measured in a time step (usually monthly), ${}_{t-1} p_x^{\tau}$ is the survival probability for t-1time units, and ${}_1 q_{x,t-1}^{(d)}$ is the probability that the policyholder dies in the time interval t-1 to t, given that he or she has survived t-1 time units.

The hedge portfolio can be found by splitting $\mathrm{BSP}_0(t)$ in Eq. (4.15) into the risky asset part and the risk-free asset part. Therefore, the total hedge cost allowing for mortality at time 0 is

$$H(0)=\int_0^n \mathrm{BSP}_0(t)\cdot {}_{t}p_x^{\tau}\mu_{x,t}^{(d)}\,dt \tag{4.17}$$

$$=\int_0^n (G\,e^{-rT}\Phi(-d_2)-S_0(1-m)^t\Phi(-d_1))\cdot {}_{t}p_x^{\tau}\mu_{x,t}^{(d)}\,dt \tag{4.18}$$

$$\begin{aligned}&=\int_0^n (G\,e^{-rT}\Phi(-d_2))\cdot {}_{t}p_x^{\tau}\mu_{x,t}^{(d)}\,dt\\&+\int_0^n (-S_0(1-m)^t\Phi(-d_1))\cdot {}_{t}p_x^{\tau}\mu_{x,t}^{(d)}\,dt\end{aligned} \tag{4.19}$$

The first part gives the risk-free asset portion of the hedge portfolio, whereas the second part gives the risky asset portion.

Eq. (4.17) can be adapted for more complex death benefits simply by adapting the definition of $\mathrm{BSP}_0(T)$. We have assumed in Eq. (4.19) that $\mathrm{BSP}_0(T)$ is the price of a standard European put option with fixed strike price G. It is possible to have contracts where the death benefit guarantee increases at a compound rate. For instance, suppose a contract with GMDB that increases at 2 percent per year. In this instance, the put option, contingent on time Tth month, has a strike price $G_T = G_0(1.02)^{T/12}$.

4.6 Unhedged Liability and Discrete Hedging Error

In spite of the fact that, the hedge portfolio indicated by the B-S analysis will be adequate to meet the liability at maturity, there are however, costs associated with transactions which are not considered in the B-S price. The unhedged liability, therefore are additional costs on top of the hedge portfolio for a practically sustainable hedging strategy. To better quantify unhedged liability, there is the need to first study discrete hedging error.

The gap between the change in the stock part and bond part of a hedge over a discrete time interval is called the discrete hedging error. This error is introduced when the assumption of continuous trading under the B-S-M approach is relaxed.

Generally, discrete hedging error can be classified as either "with certain maturity date" or "with life-contingent maturity date". In this thesis, the latter is adopted since life contingencies are indispensible in insurance.

Boyle and Emmanuel (1990), Boyle and Vorst (1992) and Leland (1995) did much analysis of discrete hedging error and transactions costs from a financial engineering view point.

4.6.1 Discrete Hedging Error: Life-Contingent Maturity

Here, consideration is given to hedging error under the combined GMMB/GMDB contract. Under such a contract, the death benefit $(G-F_t)^+$ is paid at end of the month of death, should death occur in the month $t-1$ to t, maturity benefit $(G-F_n)^+$ on survival to the end of the contract term.

Let:

$P(t,w)$ be the B –S price at t for a put option maturity at $w \geq t$.

${}_{w-t\backslash}q^{d}_{x,t}$ denote the probability that a life age x years t months survives as a policyholder for a further $w-t$ months, and dies in the following month.

$P(t,n)$ be the B –S price at t for a put option maturity at $n \geq t$.

${}_{n-t\backslash}p^{\tau}_{x,t}$ denote the probability that a policyholder age x years t months survives, and does not lapse, for a further $n-t$ months.

The total hedge price at time t for a GMMB/GMDB contract conditional on the contract being in force at t, is

$$H^{c}(t) = \sum_{w=t}^{n-1} {}_{w-t\backslash}q_{x,t}^{d} P(t,w) + {}_{n-t\backslash}p_{x,t}^{\tau} P(t,n) \tag{4.20}$$

The total hedge price at t unconditionally is determined by multiplying ${}_{t}p_{x}^{\tau}$ to obtain

$$H(t) = \sum_{w=t}^{n-1} {}_{w\backslash}q_{x}^{d} P(t,w) + {}_{n\backslash}p_{x}^{\tau} P(t,n) \tag{4.21}$$

The hedge error is obtained by calculating the difference between the hedge required at t, including any payout at that time, and the hedge brought forward from $t-1$ to t. Under the conditional payments, the hedge $H^{c}(t)$ is split into the stock (S_t, Ψ_t^c) and bond (Υ_t^c) part. Therefore, the required hedge at t conditional on the policy being in force at that time is

$$H^{c}(t) = \Upsilon^{c} + \Psi_t^c S_t \tag{4.22}$$

where

$$\Psi_t^c = \frac{\partial}{\partial S_t} H^{c}(t) \quad \text{and} \quad \Upsilon_t^c = H^{c}(t) - S_t \Psi_t^c$$

Similarly, the unconditional payments are given as

$$H(t) = \Upsilon_t + \Psi_t S_t \tag{4.23}$$

where

$$\Psi_t = {}_{t}p_{x}^{\tau} \Psi_t^c \quad \text{and} \quad \Upsilon_t = {}_{n-t\backslash}p_{x,t}^{\tau} \Upsilon_t^c$$

The hedge portfolio brought forward whether the contract remain in force or otherwise at that time is given as

$$H(t^{-}) = \Upsilon_{t-1} e^{r/12} + \Psi_{t-1} S_t \tag{4.24}$$

Now, consider the hedging error at t given the contract is in force at $t-1$.

- If the life survives, the hedging error is

$$p^{\tau}_{x,t-1}(H^{c}(t)-H^{c}(t^{-}))$$

- If the life dies or lapses, the hedging error is

$$q^{d}_{x,t-1}((G-F_t)^{+}-H^{c}(t^{-}))-q^{l}_{x,t-1}(H^{c}(t^{-}))$$

The total hedging error at t conditional on surviving to $t-1$ is given as

$$HE^{c}_{t}=p^{\tau}_{x,t-1}H^{c}(t)+q^{d}_{x,t-1}((G-F_t)^{+})-H^{c}(t^{-})) \tag{4.25}$$

The unconditional hedging error at t is also as follows:

$$\begin{aligned} &HE_t=p^{\tau}_{x}\{p^{\tau}_{x,t-1}H^{c}(t)+q^{d}_{x,t-1}((G-F_t)^{+})-H^{c}(t^{-}))\} \\ &HE_t=H(t)+{}_{t-1\backslash}q^{d}_{x}((G-F_t)^{+})-H(t^{-}) \end{aligned} \tag{4.26}$$

This equation is used in arriving at the hedging error in this thesis.

4.6.2 Transaction Cost

Transaction costs are usually proportional to the absolute change in the stock part of the hedge. Mathematically, it is represented as

$$\tau S_t\left|\Psi_t-\Psi_{t-1}\right|$$

Also, the transaction costs at t conditional on survival to t^{-} are

$$\tau S_t\left|p^{\tau}_{x,t-1}\Psi^{c}_{t}-\Psi^{c}_{t-1}\right|$$

Further, the unconditional transactions at t:

$${}_{t-1}p^{\tau}_{x}\{\tau S_t\left|p^{\tau}_{x+t-1}\Psi^{c}_{t}-\Psi^{c}_{t-1}\right|\}$$

$$TC_t=\tau S_t\left|\Psi_t-\Psi_{t-1}\right| \tag{4.27}$$

where

τ is a percentage or proportion.

4.7 Numerical Example for Joint GMMB and GMDB Contract

The contract details are as follows:

Mortality:	See Appendix C
Premium:	\$100
Guarantee:	100 percent of premium on death or maturity
MER:	0.25 percent per month
Margin offset:	0.06 percent per month
Term:	10 years

The simulation details are as follows:

Number of simulations:	5,000
Volatility	20 percent per year
Stock price process:	VAR(1), with parameters from table 2.7
Security bond process:	VECM, with parameters from Appendix B
Transactions costs:	0.2 percent of the change in the market value of stocks
Rebalancing	Monthly

At the end of each month, the outgo is calculated as follows:

i. Sum of all mortality payout (M),

ii. plus transactions costs from rebalancing the hedge (TC),

iii. plus the hedge required in respect of future guarantees (HE), and

iv. minus the hedge brought forward from the previous month (HE^-).

$$\text{Outgo} = M + TC + HE - HE^-$$

The income at the end of each month is calculated as follows:

i. Margin offset multiplied by fund value at the end of each month, except the last,

ii. the present value is calculated using the simulated 15 to 91 day YTM.

At each month end, outgo and income are calculated. Since we are simulating a loss random variable (Outgo – Income), negative values indicate that the simulated 15 to 91 YTM income exceeded outgo. We can see from figure 4.1 that the bulk of the distribution falls in the negative part of the graph. This gives us a clue that in most cases, the margin offset is adequate to meet all the hedge costs and leave some profit. However, there is a very small part of the distribution in the positive quadrant reflecting an insignificant probability of a loss.

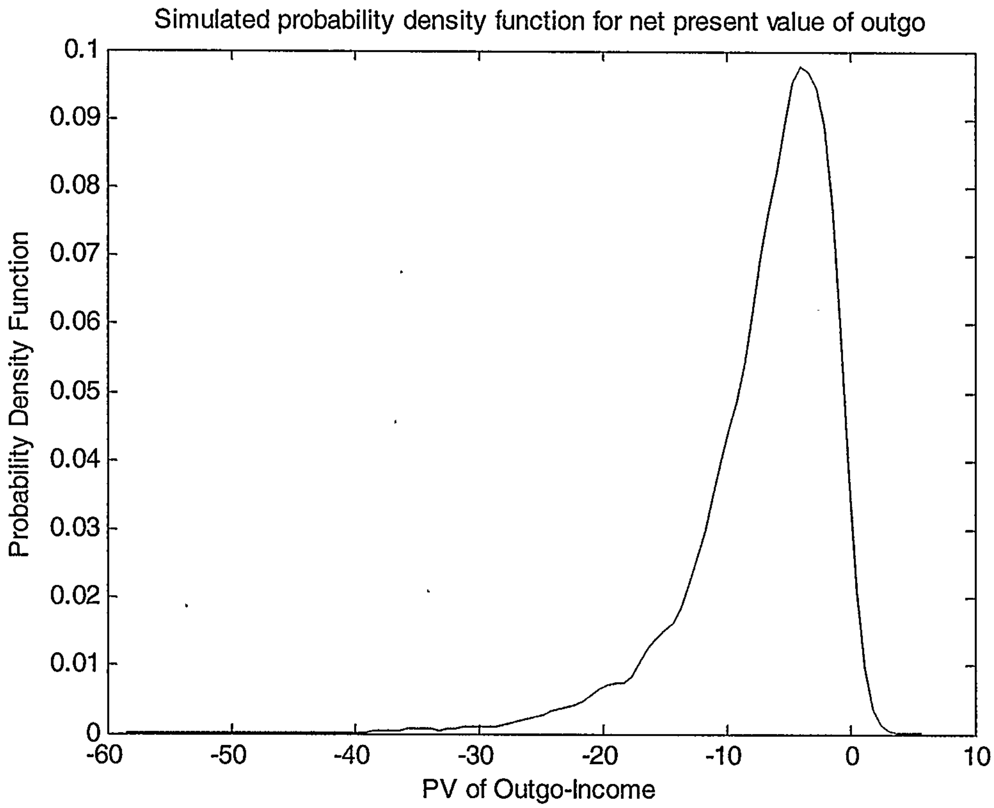

Figure 4.1: Simulated Probability Density Function for Net Present Value of Outgo of the Joint GMMB/GMDB Contract.

Chapter 5

Conclusions

From the results presented through out this thesis, the following conclusions can be drawn:

1. The South Asian stock markets (CSE, BSE and KSE) did not show evidence of unit-roots, but the returns are correlated. Therefore, the most appropriate model capable of capturing the long-term equity return process for a practical dynamic hedging of segregated fund contracts in India is the VAR(1) model.
2. The security bond market of India did provide evidence of unit-root and a long- run stochastic trend. On the basis of these, the VECM model is chosen to describe the security bond process in the valuation of segregated fund contracts in India. However, to discount all future income to their present values, the 15 to 91 YTM simulated values are used.
3. Also, the simulated valuation results using a life age 50, at a premium of $100 for a contract with combined GMMB/GMDB maturing in 10 years indicate an extremely high probability of a profit than a loss.

In the light of the foregoing developments, it can finally be concluded that, abiding by the valuation processes in this thesis and all things being equal, profitability of segregated fund contracts in India is inevitable.

Appendix A

Program Code

The S-Plus code for "Modeling the long-term equity process"

```
%%Data importation and descriptive statistics
library(finmetrics)
CSE= CIK5[,2]
BSE= CIK5[,3]
KSE= CIK5[,4]
u=cbind(CSE,BSE,KSE)
v=data.frame(u)
summaryStats(u)
par(mfrow=c(1,3))
seriesPlot(u, one.plot=F, strip.text=colIds(v),xlab='Month',ylab="Return",
main="Monthly Stock Teturns-August 97 to July 07")
normalTest(u,method='jb')

%%Unit-root tests
adft.cse = unitroot(CSE, trend = 'c', statistic="t", method = 'adf', lag = 6)
summary(adft.cse)
adft.bse = unitroot(BSE, trend = 'c', statistic="t", method = 'adf', lag = 6)
summary(adft.bse)
adft.kse = unitroot(KSE, trend = 'c', statistic="t", method = 'adf', lag = 6)
summary(adft.kse)
adft.cse = unitroot(CSE, trend = 'ct', statistic="t", method = 'adf', lag = 6)
summary(adft.cse)
adft.bse = unitroot(BSE, trend = 'ct', statistic="t", method = 'adf', lag = 6)
summary(adft.bse)
adft.kse = unitroot(KSE, trend = 'ct', statistic="t", method = 'adf', lag = 6)
summary(adft.kse)
args(unitroot)
function(CSE, trend="c", method= "adf", statistic= "t", lags = 1, bandwidth = NULL,
window = "bartlett", asymptotic = F, na.rm = F)
unitroot(CSE, trend = "c", method = "pp")
unitroot(CSE, trend = "c", method = "pp", statistic = "n")
function(BSE, trend="c", method= "adf", statistic= "t", lags = 1, bandwidth = NULL,
window = "bartlett", asymptotic = F, na.rm = F)
unitroot(BSE, trend = "c", method = "pp")
unitroot(BSE, trend = "c", method = "pp", statistic = "n")
```

```
function(KSE, trend="c", method= "adf", statistic= "t", lags = 1, bandwidth = NULL,
window = "bartlett", asymptotic = F, na.rm = F)
unitroot(KSE, trend = "c", method = "pp")
unitroot(KSE, trend = "c", method = "pp", statistic = "n")
acf(CSE, lag.n=10, method="lb", na.rm=F)
acf(BSE, lag.n=10, method="lb", na.rm=F)
acf(KSE, lag.n=10, method="lb", na.rm=F)
autocorTest(CSE, lag.n=6, method="lb", na.rm=F)
autocorTest(BSE, lag.n=6, method="lb", na.rm=F)
autocorTest(KSE, lag.n=6, method="lb", na.rm=F)
acf(u, lag.n=10, method="lb", na.rm=F, bycol=T)
autocorTest(u, lag.n=6, method="lb", na.rm=F, bycol=T)
autocorTest(u, lag.n=10, method="lb", na.rm=F, bycol=T)
autocorTest(u, lag.n=30, method="lb", na.rm=F, bycol=T)

%%Lag length selection
ord.choice=VAR(v,max.ar=6)
ord.choice=VAR(v,max.ar=6,criterion='AIC')
ord.choice$info

%%Fitting a VAR(1) model
var1.fit=VAR(u~ar(1))
summary(var1.fit)
plot(var1.fit) %Residual plots
(var1.fit$Sigma)/115 %Covariance matrix
```

The S-Plus code for "Modeling the security bond model"

```
%%Data importation and descriptive statistics
x= INDIA.TREASURY.TESTDATA1[,2]
y= INDIA.TREASURY.TESTDATA1[,3]
z= INDIA.TREASURY.TESTDATA1[,4]
m= INDIA.TREASURY.TESTDATA1[,5]
a=cbind(x,y,z,m)
b=data.frame(a)
summaryStats(b)
par(mfrow=c(2,2))
seriesPlot(b, one.plot=F, strip.text=colIds(b),xlab='Month',ylab="Rate", main="India
Treasury Rates-August 97 to July 07")
normalTest(b,method='jb')

%%Unit-root tests
adft.x = unitroot(x, trend = 'c', statistic="t", method = 'adf', lag = 6)
summary(adft.x)
adft.y = unitroot(y, trend = 'c', statistic="t", method = 'adf', lag = 6)
```

```
summary(adft.y)
adft.z = unitroot(z, trend = 'c', statistic="t", method = 'adf', lag = 6)
summary(adft.z)
adft.m = unitroot(m, trend = 'c', statistic="t", method = 'adf', lag = 6)
summary(adft.m)
adft.x = unitroot(x, trend = 'ct', statistic="t", method = 'adf', lag = 6)
summary(adft.x)
adft.y = unitroot(y, trend = 'ct', statistic="t", method = 'adf', lag = 6)
summary(adft.y)
adft.z = unitroot(z, trend = 'ct', statistic="t", method = 'adf', lag = 6)
summary(adft.z)
adft.m = unitroot(m, trend = 'ct', statistic="t", method = 'adf', lag = 6)
summary(adft.m)
args(unitroot)
function(x, trend="c", method= "adf", statistic= "t", lags = 1, bandwidth = NULL,
window = "bartlett", asymptotic = F, na.rm = F)
unitroot(x, trend = "c", method = "pp")
unitroot(x, trend = "c", method = "pp", statistic = "n")
function(y, trend="c", method= "adf", statistic= "t", lags = 1, bandwidth = NULL,
window = "bartlett", asymptotic = F, na.rm = F)
unitroot(y, trend = "c", method = "pp")
unitroot(y, trend = "c", method = "pp", statistic = "n")
function(z, trend="c", method= "adf", statistic= "t", lags = 1, bandwidth = NULL,
window = "bartlett", asymptotic = F, na.rm = F)
unitroot(z, trend = "c", method = "pp")
unitroot(z, trend = "c", method = "pp", statistic = "n")
function(m, trend="c", method= "adf", statistic= "t", lags = 1, bandwidth = NULL,
window = "bartlett", asymptotic = F, na.rm = F)
unitroot(m, trend = "c", method = "pp")
unitroot(m, trend = "c", method = "pp", statistic = "n")

%%Lag length selection
ord.choice=VAR(b, max.ar=6)
ord.choice=VAR(b, max.ar=6,criterion="AIC")
ord.choice$info

%%Cointegration rank test
cointst.rc=coint(a, trend="rc", lag = 0)
(cointst.rc)

%%Fitting the VECM representation
vecm.fit= VECM(cointst.rc)
summary(vecm.fit)
plot(vecm.fit) %Residual plots
(vecm.fit$Sigma)/118 %Covariance matrix
VECM2VAR(vecm.fit) %Converting VECM to a VAR model
```

Matlab code for "Valuation of segregated fund contracts"

```
%%Simulation of the stock return
function [LSIG,mut,Phi1,xt1] = stock_ind_init()
mut = [0.008; 0.0102;0.0182];
Phi1 =[ 0.0307 0.0773 0.0111
       0.0642 -0.0533 0.0864
       -0.1137 0.0290 -0.0138];
SIGMA =[0.6485274 0.1081131 0.1727439
      0.1081131 0.5917150 0.3027792
      0.1727439 0.3027792 1.1362969];
SIGMA = SIGMA/115;
LSIG = chol(SIGMA,'lower');
Burns = 1000;
xt1= [-0.051899144; 0.059649155; -0.002393867];
Norms = randn(3,Burns);
for i=1:Burns
  xt2 = mut + Phi1*xt1 + LSIG*Norms(:,i);
  xt1 =xt2;
end
end

%%Simulation of the treasury bond
function [LSIG,mut,Phi1,xt1] = interest_rate_int()
%wt = xt - 2.94yt + 0.961zt + 0.8846mt
AA = [0.0038;0.2464;-0.1690;-0.004];
%xt1 is the rates on 5/1/2008
xt1 = [5.7524; 7.4170; 7.4500; 7.5234];
mut = 0.004*AA;
BB = [1 -2.94 0.961 0.8846];
BigPi = AA*BB;
Phi1 = eye(4) +BigPi;
% SIGMA estimated from S-Plus
SIGMA =[0.009920957 0.005136180 0.001973707 0.001853620
  0.005136180 0.009068123 0.007181791 0.005622812
 0.001973707 0.007181791 0.009428699 0.005282101
 0.001853620 0.005622812 0.005282101 0.005325456];
SIGMA = SIGMA/118;
LSIG = chol(SIGMA,'lower');
Burns = 1000;
Norms = randn(4,Burns);
for i=1:Burns
```

```
  xt2 = mut + Phi1*xt1 + LSIG*Norms(:,i);
  xt1 =xt2;

end
end

%%Black Sholes price
function  [Stock_part, Bond_part] = Black_Schole_put_m(S,K,r,q,sigma,T)
d1 = (log(S/K) + (r-q+sigma^2/2)*T)./(sigma*T.^0.5);
d2 = d1 - sigma*T.^0.5;
Bond_part = K*exp(-r*T).*normcdf(-d2,0,1);
Stock_part=-S.*exp(-q*T).*normcdf(-d1,0,1);
end

function  [Stock_part, Bond_part] = BS_time_0(S,K)
if S>= K
   Stock_part =0;
   Bond_part =0;
else
  Stock_part= -S;
  Bond_part = K;
end
end

%%Joint GMMB and GMDB
%Lines 2-12 initialize the values of the example
MER = 0.25/100;
Marg_offset = 0.06/100; % try a few different values for this

tau = 0.2/100;
q = -12*log(1-MER); % use MER as ctsly compounded dividends rate
S0 = 100;
r=.06;
K=100;
sigma = 0.2;
Term = 10; % may use a different value for Term, but maximum value is 22;
nmonths = Term*12;

%Mortality.txt is a text file containing mortality rates
load Mortality.txt
tpx = Mortality(:,1);
tqx = Mortality(:,2);
```

```
% initilize stock index model variables
% stock_ind_init() contains estimated model

SLSIG =zeros(3,3);
Smut = zeros(3,1);
Sphi1 = zeros(3,3);
Sxt1 = zeros(3,1);
[SLSIG,Smut,SPhi1,Sxt1] = stock_ind_init();
Sx = zeros(3,nmonths);
Sx(:,1) = Sxt1;

% initialize interest rate model based on estimates
ILSIG =zeros(4,4);
Imut = zeros(4,1);
IPhi1 = zeros(4,4);
Ixt1 = zeros(4,1);
[ILSIG,Imut,IPhi1,Itx1] = interest_rate_int();
Itx = zeros(4,nmonths);
Itx(:,1) = Itx1;

Nsimulas = 5000;
PV = zeros(Nsimulas,1);
% next 9 lines compute (8.17) of the text for t=0; it separates stock and
% bond parts
T =(1:nmonths)';
t = T/12;
F0 = 100;
[Sp0(2:nmonths+1,1),Bp0(2:nmonths+1,1)] = Black_Schole_put_m(S0,K,r,q,sigma,t);
[Sp0(1,1),Bp0(1,1)] = BS_time_0(S0,K);
HSp = zeros(nmonths+1,1);
HBp = zeros(nmonths+1,1);
HSp(1,1) = Sp0(1:nmonths,1)'*Mortality(1:nmonths,2) +
Sp0(nmonths+1,1)*Mortality(nmonths+1,1);
HBp(1,1)= Bp0(1:nmonths,1)'*Mortality(1:nmonths,2) +
Bp0(nmonths+1,1)*Mortality(nmonths+1,1);

% Initialize values for simulations
% In the simulation equation 8.17 is implemented in vectorized form.

Values = zeros(nmonths,1);
SV = zeros(nmonths+1,1);
Htm = zeros(nmonths+1,1);
Ht = zeros(nmonths+1,1);
TACost = zeros(nmonths+1,1);
SV(1,1) = S0;
```

```
Time = zeros(nmonths,nmonths);
for i=1:nmonths-1
  for j=i+1:nmonths
     Time(i,j) = j-i;
  end
end;

diagonal_ind = (1:nmonths:nmonths*nmonths)+(0:nmonths-1);
non_zero_Time = find(Time);
Timeu = Time(non_zero_Time)/12;
pqs = [Mortality(2:nmonths,2);Mortality(nmonths+1,1)];
Htm = zeros(nmonths+1,1);
HE_t = zeros(nmonths+1,1);
Psi(1,1) = HSp(1,1)/S0;
TACost(1,1) = abs(Psi(1,1))*tau*S0;
Income = zeros(nmonths+1,1);
Sp = zeros(nmonths,nmonths);
Bp = zeros(nmonths,nmonths);
Divid = exp(-q*(0:nmonths)'/12);

for NSims=1:Nsimulas

% for each simulation generate one stock index return path
 Norms = randn(3,nmonths);
 for i=1:nmonths-1
  Sx(:,i+1) = Smut+ SPhi1*Sx(:,i) + SLSIG*Norms(:,i);
 end
 Values = exp(cumsum(Sx(2,:)'));

 % generate one path of interest rates

 Norms = randn(4,nmonths);
 for i=1:nmonths-1
  Itx(:,i+1) = Imut+ IPhi1*Itx(:,i) + ILSIG*Norms(:,i);
 end

 % the following would be used if the rates are cts compounded
 %Dfactors = exp(-[0 Itx(2,:)].*(0:nmonths)/1200);
 % the following is used assuming rates are annual effective
 Dfactors = (1+[0 Itx(2,:)]/100).^(-(0:nmonths)/12);

 SV(2:nmonths+1,1) = S0*Values;
 F =F0.*[1; Values].*Divid;
 Smat = repmat(SV(2:nmonths+1,1),1,nmonths);
 Smatu = Smat(non_zero_Time);
```

```
 [Sp(non_zero_Time),Bp(non_zero_Time)] =
Black_Schole_put_m(Smatu,K,r,q,sigma,Timeu);
 [Sp(diagonal_ind),Bp(diagonal_ind)] = BS_time_0(SV(2:nmonths+1,1),K);
 [Sp(nmonths,nmonths),Bp(nmonths,nmonths)] = BS_time_0(SV(nmonths+1,1),K);
 HSp(2:nmonths+1,1) = Sp*pqs;
 HBp(2:nmonths+1,1) = Bp*pqs;
 Psi = HSp./SV;
 Ht = HSp+ HBp;
 Htm(2:nmonths+1,1) = HBp(1:nmonths,1)*exp(r/12) +
Psi(1:nmonths,1).*SV(2:nmonths+1,1);
 TACost(2:nmonths+1,1) = tau*SV(2:nmonths+1,1).*abs(Psi(2:nmonths+1,1)-
Psi(1:nmonths,1));
 HE_t = Ht-Htm + Mortality(1:nmonths+1,2).*max(0,(K-F))+ TACost;
 Income = F*Marg_offset;
 Income(nmonths+1,1) =0;
 PV(NSims,1) = Dfactors*(HE_t - Income);
End

[PVx,xrange] =ksdensity(PV);
plot(xrange,PVx);
xlabel('PV of Outgo-Income')
ylabel('Probability Density Function')
title('Simulated probability density function for net present value of outgo')
```

Appendix B

S-Plus output for the VECM representation

```
Cointegrating Vectors:
          coint.1
           1.0000

        y -2.9400
(std.err)  0.3354
 (t.stat) -8.7664

        z  0.9618
(std.err)  0.1683
 (t.stat)  5.7143

        m  0.8846
(std.err)  0.2395
 (t.stat)  3.6941

Intercept*  0.0040
  (std.err) 0.0042
   (t.stat) 0.9450

          VECM Coefficients:
                x       y       z       m
  coint.1  0.0038  0.2464 -0.1690 -0.0004
(std.err)  0.0639  0.0611  0.0623  0.0468
 (t.stat)  0.0593  4.0340 -2.7124 -0.0092

Regression Diagnostics:
                     x      y      z      m
     R-squared  0.0000 0.1214 0.0584 0.0000
Adj. R-squared  0.0000 0.1214 0.0584 0.0000
  Resid. Scale  0.0092 0.0088 0.0089 0.0067

Information Criteria:
   logL       AIC       BIC        HQ
 1766.214 -3530.429 -3527.650 -3529.300

                     total  residual
Degree of freedom:    119     118
```

Appendix C

Mortality and Survival Probabilities

In this appendix, we give the mortality and survival rates used in the valuation of the segregated funds under the combined GMMB/GMDB contract. At $t = 0$, the life is assumed to be age 50, time t is in months. Independent mortality rates are from the Canadian Institute of Actuaries male annuitants' mortality rates.

t	${}_{t}p^{T}_{x}$	${}_{t\mid1}q^{d}_{x}$	t	${}_{t}p^{T}_{x}$	${}_{t\mid1}q^{d}_{x}$	t	${}_{t}p^{T}_{x}$	${}_{t\mid1}q^{d}_{x}$
0	1	0.00029	21	0.86361	0.0003	42	0.74479	0.00031
1	0.99307	0.00029	22	0.85757	0.0003	43	0.73953	0.00031
2	0.98618	0.00029	23	0.85157	0.0003	44	0.7343	0.00031
3	0.97934	0.00029	24	0.84561	0.0003	45	0.72911	0.00031
4	0.97255	0.00029	25	0.8397	0.0003	46	0.72396	0.00031
5	0.9658	0.00029	26	0.83382	0.0003	47	0.71883	0.00031
6	0.95909	0.00029	27	0.82797	0.0003	48	0.71374	0.00031
7	0.95243	0.00029	28	0.82217	0.0003	49	0.70869	0.00032
8	0.94581	0.00029	29	0.8164	0.0003	50	0.70366	0.00032
9	0.93923	0.00029	30	0.81067	0.00031	51	0.69867	0.00032
10	0.9327	0.00029	31	0.80498	0.00031	52	0.69372	0.00032
11	0.92621	0.00029	32	0.79933	0.00031	53	0.68879	0.00032
12	0.91976	0.00029	33	0.79371	0.00031	54	0.6839	0.00032
13	0.91336	0.00029	34	0.78813	0.00031	55	0.67903	0.00032
14	0.907	0.0003	35	0.78259	0.00031	56	0.6742	0.00032
15	0.90067	0.0003	36	0.77708	0.00031	57	0.66941	0.00032
16	0.89439	0.0003	37	0.77161	0.00031	58	0.66464	0.00032
17	0.88816	0.0003	38	0.76618	0.00031	59	0.6599	0.00032
18	0.88196	0.0003	39	0.76078	0.00031	60	0.6552	0.00032
19	0.8758	0.0003	40	0.75541	0.00031	61	0.65052	0.00032
20	0.86968	0.0003	41	0.75008	0.00031	62	0.64588	0.00032

t	${}_tp^T_x$	${}_{t\|1}q^d_x$	t	${}_tp^T_x$	${}_{t\|1}q^d_x$
63	0.64127	0.00032	101	0.48655	0.00034
64	0.63668	0.00032	102	0.48297	0.00034
65	0.63213	0.00032	103	0.47942	0.00034
66	0.62761	0.00033	104	0.47589	0.00034
67	0.62311	0.00033	105	0.47239	0.00035
68	0.61865	0.00033	106	0.46891	0.00035
69	0.61421	0.00033	107	0.46545	0.00035
70	0.6098	0.00033	108	0.46201	0.00035
71	0.60542	0.00033	109	0.45859	0.00035
72	0.60107	0.00033	110	0.4552	0.00035
73	0.59675	0.00033	111	0.45183	0.00035
74	0.59246	0.00033	112	0.44848	0.00035
75	0.5882	0.00033	113	0.44515	0.00035
76	0.58396	0.00033	114	0.44185	0.00035
77	0.57975	0.00033	115	0.43857	0.00035
78	0.57557	0.00033	116	0.4353	0.00035
79	0.57141	0.00033	117	0.43206	0.00035
80	0.56728	0.00033	118	0.42884	0.00035
81	0.56318	0.00033	119	0.42564	0.00035
82	0.55911	0.00033	120	0.42247	0.00035
83	0.55506	0.00033	121	0.41931	0.00035
84	0.55104	0.00033	122	0.41617	0.00035
85	0.54704	0.00034	123	0.41306	0.00035
86	0.54307	0.00034			
87	0.53913	0.00034			
88	0.53521	0.00034			
89	0.53132	0.00034			
90	0.52745	0.00034			
91	0.52361	0.00034			
92	0.5198	0.00034			
93	0.516	0.00034			
94	0.51224	0.00034			
95	0.5085	0.00034			
96	0.50478	0.00034			
97	0.50108	0.00034			
98	0.49742	0.00034			
99	0.49377	0.00034			
100	0.49015	0.00034			

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