# ON THE REGULAR REPRESENTATION OF ALGEBRAS

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## 1. INTRODUCTION

Let $\mathbf{A}$ be an associative algebra with unity element 1 over the field $F$. Let $\{a_1, \ldots, a_n\}$ be a base of $\mathbf{A}$, $\mathbf{RR}_l(\mathbf{A}) = \{A_a \mid a \in \mathbf{A}\}$ and $\mathbf{RR}_r(\mathbf{A}) = \{A^a \mid a \in \mathbf{A}\}$ be the left and right regular representation matrices of $\mathbf{A}$, respectively. Let $x = \sum_{i=1}^{n} a_i x_i$ and $y = \sum_{i=1}^{n} a_i y_i$ be two elements in $\mathbf{A}$ and $xy = \sum_{i=1}^{n} (\mathbf{x}^t B_i \mathbf{y}) a_i$ where $\mathbf{x} = (x_1, \ldots, x_n)^t$ and $\mathbf{y} = (y_1, \ldots, y_n)^t$. In this paper we study some properties and the connection between the sets $\{A_{a_1}, \ldots, A_{a_n}\}$, $\{A^{a_1}, \ldots, A^{a_n}\}$ and $\{B_1, \ldots, B_n\}$. These properties has various applications in the theory of multiplicative complexity of algebras.

One of our purposes will be to classify all algebras $\mathbf{A}$ that satisfy: There exist a nonsingular matrix $N$ such that $\{B_1 N, \ldots, B_n N\}$ is a base for an algebra that is isomorphic to $\mathbf{A}$ or to $\mathbf{A}^*$, the reciprocal algebra. For commutative algebra and semisimple algebras this is completely classified

## 2. PRELIMINARY RESULTS

In this section we give some preliminary results

**Definition 1 .** Let $B = \{B_1, \ldots, B_n\}$ be a set of $n \times n$ matrices. We define the *T-dual* and *D- dual sets* $B^T$ and $B^D$ of $B$ as follows:

$$B^T = \{B_1^T, \ldots, B_n^T\} \quad , \quad B^D = \{C_1, \ldots, C_n\},$$

Here $B_i^T$ is the transpose of $B_i$ and $B^D$ denotes the set of $n \times n$ matrices that satisfy

$$C_i^j = B_j^i \, , i \, , j = 1, \ldots, n \, ,$$

where $B_i^j$ is the $j$-th column of $B_i$, i.e

$$C_i = [B_1 e_i \mid \cdots \mid B_n e_i].$$

where $e_i$ is the $i$-th column unit vector of order $n$.

We also define $B^E = B^{TDT}$, i.e $B^E = \{D_1, \ldots, D_n\}$ where

$$D_i = \begin{bmatrix} e_i^T B_1 \\ \cdot \\ \cdot \\ e_i^T B_n \end{bmatrix}.$$

**Definition 2 .** Let $B = \{B_1, \ldots, B_n\}$ be a set of $n \times n$ matrices. Let $M$, $N$ and $K = (K_{i,j})$ be $n \times n$ matrices. We define

$$N\,B\,M = \{N B_1 M, \ldots, N B_n M\}, \; B\,[K] = \{\sum_{j=1}^{n} K_{1,j} B_j, \ldots, \sum_{j=1}^{n} K_{n,j} B_j\}.$$

**Definition 3 .** Let $B = \{B_1, \ldots, B_n\}$ and $C = \{C_1, \ldots, C_m\}$ be sets of $n \times n$ and $m \times m$ matrices, respectively. We define

$$B \oplus C = \{\tilde{B}_1, \ldots, \tilde{B}_n, \tilde{C}_1, \ldots, \tilde{C}_m\}$$

where

$$\tilde{B}_i = \begin{bmatrix} B_i & 0_{n \times m} \\ 0_{m \times n} & 0_{m \times m} \end{bmatrix} \quad , \quad \tilde{C}_l = \begin{bmatrix} 0_{n \times n} & 0_{n \times m} \\ 0_{m \times n} & C_l \end{bmatrix}$$

and $0_{s \times r}$ is the zero $s \times r$ matrix.

We also define

$$B \otimes C = \{B_a \otimes C_b \mid a = 1, \ldots, n \, , b = 1, \ldots, m\}.$$

where $\otimes$ is the Kronecker product of matrices.

**Lemma 1 .** Let $A_1, A_2$ and $A$ be sets of $n_1$, $n_2$ and $n$, $n_1 \times n_1$, $n_2 \times n_2$ and $n \times n$ matrices, respectively. Then

$(i)$ $A^{TT} = A$ , $A^{DD} = A$ , $A^{EE} = A$ , $A^E = A^{TDT} = A^{DTD}$.

$(ii)$ $A[K][J] = A[JK]$, $(NAM)[K] = N(A[K])M$.

$(iii)$ $(NAM)^T = M^T A^T N^T$, $(A[K])^T = A^T[K]$

$(iv)$ $(NA)^D = NA^D$, $(AM)^D = A^D[M^T]$, $(A[K])^D = (A^D K^T)$.

$(v)$ $(A_1 \oplus A_2)^T = A_1^T \oplus A_2^T$, $(A_1 \oplus A_2)^D = A_1^D \oplus A_2^D$.

$(vi)$ $(A_1 \otimes A_2)^T = A_1^T \otimes A_2^T$, $(A_1 \otimes A_2)^D = A_1^D \otimes A_2^D$.

$(vii)$ $A_1[K] \oplus A_2 = (A_1 \oplus A_2)[diag(K, I_{n_2})]$ , $NA_1M \oplus A_2 = diag(N, I_{n_2})(A_1 \oplus A_2) diag(M, I_{n_2})$.

$(viii)$ $A_1[K] \otimes A_2 = (A_1 \otimes A_2)[K \otimes I_{n_2}]$ , $NA_1M \otimes A_2 = (N \otimes I_{n_2})(A_1 \otimes A_2)(I_{n_1} \otimes M)$

where

$$diag(B_1, B_2) = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}.$$

**Proof** . In $(i)$ the first three equations are trivial. Let $A = \{(a_{k,i,j})_{i,j}\}_k$, i.e $A = \{(a_{1,i,j})_{i,j}, \ldots, (a_{n,i,j})_{i,j}\}$ where the indices $i$ and $j$ are for the rows and the columns, respectively. Then $A^D = \{(a_{k,i,j})_{i,k}\}_j$ and $A^T = \{(a_{k,i,j})_{j,i}\}_k$. Using those properties of the indices the fourth equation in $(i)$ follows. $(ii)$ and $(iii)$ are trivial. To prove $(iv)$ observe that if $M = (m_{i,j})$ and $A = \{B_1, \ldots, B_n\} = \{(a_{k,i,j})_{i,j}\}_k$ then

$$(NA)^D = \{[NB_i^{(1)} \mid \cdots \mid NB_i^{(n)}]\}_i^D = \{[NB_1^{(i)} \mid \cdots \mid NB_n^{(i)}]\}_i = N\{[B_1^{(i)} \mid \cdots \mid B_n^{(i)}]\}_i = NA^D,$$

$$(AM)^D = \{(\sum_{l=1}^n a_{k,i,l}\, m_{l,j})_{i,j}\}_k^D = \{(\sum_{l=1}^n a_{k,i,l}\, m_{l,j})_{i,k}\}_j = \{\sum_{l=1}^n m_{l,j}\,(a_{k,i,l})_{i,k}\}_j = A^D[M^T],$$

and

$$A^D K^T = (A^D K^T)^{DD} = (A[K])^D.$$

$(v)$ Follows immediately from definition 3. The first equation in $(vi)$ is well known and the second follows from

$$(B_1 \otimes B_2)^{(i+n(j-1))} = B_1^{(i)} \otimes B_2^{(j)}$$

where $B_2$ is $n \times n$ matrix. (*vii*) is trivial.

The second property in (*viii*) follows from the following well known equation

$$N_1 M_1 \otimes N_2 M_2 = (N_1 \otimes N_2)(M_1 \otimes M_2).$$

and the first equation follows from

$$A_1[W] \otimes A_2 = (A_1^D K^T \otimes A_2^D)^D = ((A_1^D \otimes A_2^D)(K^T \otimes I_{n_2}))^D = (A_1 \otimes A_2)[K \otimes I_{n_2}]. \quad \square$$

**Definition 4 .** For two $n$-sets of $n \times n$ matrices $B$ and $C$ we write $B \equiv C$, $B$ is *equivalent* to $C$, if there exist nonsingular matrices $N$, $M$ and $K$ such that

$$B = N(C[K])M.$$

Obviously this relation is an equivalence relation.

**Lemma 2 .** Let $A_1, \ldots, A_k, B_1, \ldots, B_k$ be sets of matrices. Then

(*i*) If $A_1 \equiv B_1$ then $A_1^D \equiv B_1^D$ and $A_1^T \equiv B_1^T$.

(*ii*) $B_1 \oplus \cdots \oplus B_k \equiv B_{\phi(1)} \oplus \cdots \oplus B_{\phi(k)}$ and $B_1 \otimes \cdots \otimes B_k \equiv B_{\phi(1)} \otimes \cdots \otimes B_{\phi(k)}$ for any permutation $\phi$ on $\{1, \ldots, k\}$.

(*iii*) If $A_i \equiv B_i$, $i = 1, \ldots, k$ then $A_1 \oplus \cdots \oplus A_k \equiv B_1 \oplus \cdots \oplus B_k$ and $A_1 \otimes \cdots \otimes A_k \equiv B_1 \otimes \cdots \otimes B_k$.

**Proof .** (*i*) follows by lemma 1. Let $A$ and $B$ be sets of $n$ and $m$, $n \times n$ and $m \times m$ matrices, respectively. Then it can be easily shown that

$$\begin{bmatrix} & I_m \\ I_n & \end{bmatrix} (A \oplus B) \begin{bmatrix} & I_m \\ I_n & \end{bmatrix}^{-1} \left[ \begin{bmatrix} & I_m \\ I_n & \end{bmatrix} \right] = B \oplus A.$$

where $I_n$ is the identity matrix of order $n$.

It is known [MN] that

$$K_{m,n}(A \otimes B) K_{m,n}^{-1}[K_{m,n}] = B \otimes A$$

where $K_{m,n} = \sum_{j=1}^{n} (e_j^T \otimes I_m \otimes e_j)$. This implies (*ii*). (*iii*) follows from lemma 1. $\square$

## 3. REGULAR REPRESENTATION OF ALGEBRAS

Let A be an associative algebra with unit element 1 and $\{a_1, \ldots, a_n\}$ be a basic of the algebra A. Let

$$a_i\, a_j = \sum_{k=1}^{n} \gamma_{i,j,k}\, a_k$$

with $\gamma_{i,j,k} \in F$, $i, j, k = 1, \ldots, n$. Then for $x = \sum_{i=1}^{n} x_i\, a_i$ and $y = \sum_{j=1}^{n} y_j\, a_j$ we have

$$x\, y = \left[\sum_{i=1}^{n} x_i\, a_i\right] \left[\sum_{j=1}^{n} y_j\, a_j\right] = \sum_{k=1}^{n} \left[\sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{i,j,k}\, x_i\, y_j\right] a_k .$$

Let $a_i\, y = \sum_{k=1}^{n} \sigma_{i,k}\, a_k$ and define $A_y = (\sigma_{i,k})$ an $n \times n$ square matrix. Then it can be easily shown that $\sigma_{i,k} = \sum_{j=1}^{n} \gamma_{i,j,k}\, y_j$ and $\mathbf{RR}_l(\mathbf{A}) = \{A_a \mid a \in \mathbf{A}\}$ form an algebra over $F$ that is isomorphic to A under the corresponding $a \to A_a$ , $\{A_{a_1}, \ldots, A_{a_n}\}$ is a base for the algebra $\mathbf{RR}_l(\mathbf{A})$, $A_\lambda = \lambda I_n$, $A_a\, A_b = A_{ab}$, $A_a + A_b = A_{a+b}$, $\lambda A_a = A_{\lambda a}$ for $\lambda \in F$ and if $a\, b = 1$ then $A_a^{-1} = A_b$. The algebra $\mathbf{RR}_l(\mathbf{A})$ is called the *left regular representation* of A. The left regular representation $\mathbf{RR}_l(\mathbf{A})$ of A is depending on the chooses bases $B = \{a_1, \ldots, a_n\}$. when we want to emphasize this dependency we write $\mathbf{RR}_l(\mathbf{A}, B)$.

Let $x\, a_i = \sum_{j=1}^{n} \delta_{j,i}\, a_j$ and define $A^x = (\delta_{j,i})$ an $n \times n$ square matrix. Then $\mathbf{RR}_r(\mathbf{A}) = \{A^a \mid a \in \mathbf{A}\}$ form an algebra over $F$ that is isomorphic to A under the corresponding $a \to A^a$. The algebra $\mathbf{RR}_r(\mathbf{A})$ is called the *right regular representation* of A.

We define

$$\mathbf{B}(\mathbf{A}) = \{B_1, \ldots, B_n\}$$

where for $\mathbf{x}=(x_1, \ldots, x_n)^T$, $\mathbf{y}=(y_1, \ldots, y_n)^T$ we have

$$\mathbf{x}^T B_i\, \mathbf{y} = \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_{i,j,k}\, x_i\, y_j .$$

i.e. the $i$-th coefficient of the product $x\, y$.

Let $\mathbf{C}_l(\mathbf{A}) = \{A_{a_1}, \ldots, A_{a_n}\}$ and $\mathbf{C}_r(\mathbf{A}) = \{A^{a_1}, \ldots, A^{a_n}\}$. We now give the connection between $\mathbf{B}(\mathbf{A})$, $\mathbf{C}_l(\mathbf{A})$ and $\mathbf{C}_r(\mathbf{A})$.

<u>Lemma 3 .</u> We have

$$\mathbf{C}_l(\mathbf{A})^D = \mathbf{B}(\mathbf{A}),\ \mathbf{C}_r(\mathbf{A})^E = \mathbf{B}(\mathbf{A}),\ \mathbf{C}_r(\mathbf{A}) = \mathbf{C}_l(\mathbf{A})^{TD}.$$

**Proof .** We have

$$a_i\, y = a_i \left[ \sum_{j=1}^{n} y_j\, a_j \right] = \sum_{k=1}^{n} \left[ \sum_{j=1}^{n} \gamma_{i,j,k}\, y_j \right] a_k$$

and therefore

$$A_y = \begin{bmatrix} \sum_{j=1}^{n} \gamma_{1,j,1}\, y_j & \cdots & \sum_{j=1}^{n} \gamma_{1,j,n}\, y_j \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \sum_{j=1}^{n} \gamma_{n,j,1}\, y_j & \cdots & \sum_{j=1}^{n} \gamma_{n,j,n}\, y_j \end{bmatrix} \qquad (1)$$

The product of $x$ and $y$ is

$$x\, y = \sum_{k=1}^{n} \left[ \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{i,j,k}\, x_i\, y_j \right] a_k = \sum_{k=1}^{n} (\mathbf{x}^T B_k\, \mathbf{y})\, a_k$$

thus $\mathbf{x}^T B_k\, \mathbf{y} = \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{i,j,k}\, x_i\, y_j$ or

$$B_k = \begin{bmatrix} \gamma_{1,1,k} & \cdots & \gamma_{1,n,k} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \gamma_{n,1,k} & \cdots & \gamma_{n,n,k} \end{bmatrix}$$

and from (1) we have

$$A_{a_j} = \begin{bmatrix} \gamma_{1,j,1} & \cdots & \gamma_{1,j,n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \gamma_{n,j,1} & \cdots & \gamma_{n,j,n} \end{bmatrix}$$

and therefore $B_k = [A_{a_1} e_k \mid \cdots \mid A_{a_n} e_k\,]$.

The second equalition can be proved in the same manner. Now $\mathbf{C}_l\,(\mathbf{A})^D = \mathbf{B}\,(\mathbf{A}) = \mathbf{C}_r\,(\mathbf{A})^E$ follows the third equalition. □

Obviously, $\mathbf{C}_l\,(\mathbf{A})$, $\mathbf{C}_r\,(\mathbf{A})$ , $\mathbf{RR}_l(\mathbf{A})$ , $\mathbf{RR}_r(\mathbf{A})$ and $\mathbf{B}\,(\mathbf{A})$ is depending on the chooses bases $B = \{a_1, \ldots, a_n\}$. When we want to emphasis this dependency we write $\mathbf{C}_l\,(\mathbf{A}, B\,)$, $\mathbf{C}_r(\mathbf{A}, B\,)$ , $\mathbf{RR}_l(\mathbf{A}, B\,)$ , $\mathbf{RR}_r(\mathbf{A}, B\,)$ and $\mathbf{B}\,(\mathbf{A}, B\,)$.

Lemma 4 . Let **A** and **A′** be algebras. If **A** is isomorphic to **A′** then there exist bases

$A = \{a_1, \ldots, a_n\}$ and $A' = \{a_1', \ldots, a_n'\}$ for **A** and **A'**, respectively, such that

$$\mathbf{C}_l(\mathbf{A}, A) = \mathbf{C}_l(\mathbf{A}', A'),\ \mathbf{B}(\mathbf{A}, A) = \mathbf{B}(\mathbf{A}', A').$$

**Proof** . Let $\phi : \mathbf{A} \to \mathbf{A}'$ be an isomorphism. Let $A = \{a_1, \ldots, a_n\}$ be a base for **A**. Then $A' = \{\phi(a_1), \ldots, \phi(a_n)\}$ is a base for **A'** and for $x = \sum x_i a_i$, $y = \sum y_i a_i$, $x' = \sum x_i \phi(a_i)$ and $y' = \sum y_i \phi(a_i)$ we have

$$x\,y = \sum (\mathbf{x}^T B_i\, \mathbf{y}) a_i$$

and

$$x' y' = \phi(x\,y) = \sum (\mathbf{x}^T B_i\, \mathbf{y}) \phi(a_i).$$

This implies that $\mathbf{B}(\mathbf{A}, A) = \mathbf{B}(\mathbf{A}', A')$. By lemma 3 the first equalition follows. □

<u>**Lemma 5 .**</u> [FZ] . Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ be bases for the algebra **A**. Then

$$\mathbf{C}_l(\mathbf{A}, A) \equiv \mathbf{C}_l(\mathbf{A}, B),\quad \mathbf{B}(\mathbf{A}, A) \equiv \mathbf{B}(\mathbf{A}, B).$$

To find the exact connection between $\mathbf{C}_l(\mathbf{A}, A)$ and $\mathbf{C}_l(\mathbf{A}, B)$ we prove the following

<u>**Lemma 6 .**</u> Let $A$ and $B$ be as in lemma 5. If $B = A[M]$ then

$$\mathbf{B}(\mathbf{A}, B) = M\, \mathbf{B}(\mathbf{A}, A)[(M^T)^{-1}]M^T,\ \ \mathbf{C}_l(\mathbf{A}, B) = M\, \mathbf{C}_l(\mathbf{A}, A)[M]M^{-1}.$$

and

$$\mathbf{C}_r(\mathbf{A}, B) = (M^T)^{-1}\mathbf{C}_r(\mathbf{A}, A)[M]M^T.$$

**Proof** . Let $M = (m_{i,j})$. Then $b_i = \sum_{j=1}^n m_{i,j}\, a_j$ and for $x = \sum_{i=1}^n x_i\, b_i$, $y = \sum_{i=1}^n y_i b_i$ we have

$$x\,y = (\sum_{i=1}^n x_i \sum_{j=1}^n m_{i,j}\, a_j)(\sum_{i=1}^n y_i \sum_{j=1}^n m_{i,j}\, a_i) = (\sum_{j=1}^n (\sum_{i=1}^n x_i\, m_{i,j}) a_j)(\sum_{j=1}^n (\sum_{i=1}^n y_i m_{i,j}) a_j) = \sum_{j=1}^n (M^T \mathbf{x})^T B_j\, (M^T \mathbf{y}) a_j$$
$$= \sum_{j=1}^n \mathbf{x}^T (M\, B_j\, M^T)\mathbf{y}\, a_j.$$

where $\mathbf{B}(\mathbf{A}, A) = \{B_j\}_{j=1,\ldots,n}$.

Let $M^{-1} = (n_{i,j})$. Then $a_i = \sum_{j=1}^n n_{i,j}\, b_j$ and

$$x\,y = \sum_{i=1}^n \mathbf{x}^T (M\, B_i\, M^T)\mathbf{y} \sum_{j=1}^n n_{i,j}\, b_j = \sum_{j=1}^n \mathbf{x}^T (M\, (\sum_{i=1}^n n_{i,j}\, B_i) M^T)\mathbf{y}\, b_j.$$

This implies that

$$\mathbf{B}(\mathbf{A}, B) = M\, \mathbf{B}(\mathbf{A}, A)[\,(M^T)^{-1}]M^T.$$

By lemma 3 and 1 we obtain the other equations. □

**Lemma 7 .** Let $\mathbf{A}$ and $\mathbf{A}'$ be algebras. If $\mathbf{A}$ is isomorphic to $\mathbf{A}'$ then there exist a nonsingular matrix $M$ such that

$$\mathbf{RR}_l(\mathbf{A}) = M\, \mathbf{RR}_l(\mathbf{A}')M^{-1}.$$

Also for $B = A[M]$ where $A$ and $B$ are as in lemma 5

$$\mathbf{RR}_l(\mathbf{A}, B) = M\, \mathbf{RR}_l(\mathbf{A}, A)M^{-1}.$$

**Proof .** Observing that $L(M\,\mathbf{C}_l(\mathbf{A},A)[M]M^{-1}) = M\,\mathbf{RR}(\mathbf{A},A)M^{-1}$, [$L(H)$ is the linear space spanned by the elements of $H$] and by lemma 4 and 6 the result follows . □

**Lemma 8 .** Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be algebras. Then

$$\mathbf{C}_l(\mathbf{A}_1\times\mathbf{A}_2) \equiv \mathbf{C}_l(\mathbf{A}_1)\oplus\mathbf{C}_l(\mathbf{A}_2), \quad \mathbf{B}(\mathbf{A}_1\times\mathbf{A}_2)\equiv\mathbf{B}(\mathbf{A}_1)\oplus\mathbf{B}(\mathbf{A}_2),$$
$$\mathbf{C}_l(\mathbf{A}_1\otimes\mathbf{A}_2) \equiv \mathbf{C}_l(\mathbf{A}_1)\otimes\mathbf{C}_l(\mathbf{A}_2), \quad \mathbf{B}(\mathbf{A}_1\otimes\mathbf{A}_2)\equiv\mathbf{B}(\mathbf{A}_1)\otimes\mathbf{B}(\mathbf{A}_2).$$

**Proof .** The connection on $\mathbf{C}_l$ is well known from the theory of regular matrix representation of algebras. We have by lemma 1, 2 and 3

$$\mathbf{B}(\mathbf{A}_1\times\mathbf{A}_2) = \mathbf{C}_l(\mathbf{A}_1\times\mathbf{A}_2)^D \equiv (\mathbf{C}_l(\mathbf{A}_1)\oplus\mathbf{C}_l(\mathbf{A}_2))^D = \mathbf{C}_l(\mathbf{A}_1)^D\oplus\mathbf{C}_l(\mathbf{A}_2)^D = \mathbf{B}(\mathbf{A}_1)\oplus\mathbf{B}(\mathbf{A}_2).$$

and so for $\otimes$. □

Let $\mathbf{A}$ be an algebra. The *reciprocal algebra* $\mathbf{A}^-$ of $\mathbf{A}$ is an algebra with elements of $\mathbf{A}$ and the multiplication $*$ such that $a * b = b\,a$. We have

**Lemma 9 .** Let $\{a_1, \ldots, a_n\}$ be a base for $\mathbf{A}$. Then

$$\mathbf{B}(\mathbf{A}^-) = \mathbf{B}(\mathbf{A})^T, \quad \mathbf{C}_l(\mathbf{A}^-) = \mathbf{C}_l(\mathbf{A})^E, \; \mathbf{C}_r(\mathbf{A}^-) = \mathbf{C}_r(\mathbf{A})^D.$$

and

$$\mathbf{C}_r(\mathbf{A}^-) = \mathbf{C}_l(\mathbf{A})^T.$$

**Proof .** We have

$$x * y = y\,x = \sum_{k=1}^{n}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{i,j,k}\,x_j\,y_i\right]a_k$$
$$= \sum_{k=1}^{n}(\mathbf{y}^T B_i\,\mathbf{x})\,a_k = \sum_{k=1}^{n}(\mathbf{x}^T B_i^T\mathbf{y})\,a_k. \tag{2}$$

Therefore $\mathbf{B}(\mathbf{A}^-) = \mathbf{B}(\mathbf{A})^T$. By lemma 1, 2 and 3 we have

$$\mathbf{C}_l(\mathbf{A}^-) = \mathbf{B}(\mathbf{A}^-)^D = \mathbf{B}(\mathbf{A})^{TD} = \mathbf{C}_l(\mathbf{A})^{DTD} = \mathbf{C}_l(\mathbf{A})^E.$$

The rest follows in a similar manner. □

Observe that when **A** is commutative algebra then $\mathbf{A}^{-} = \mathbf{A}$ and therefore we have

**Lemma 10 .** We have $\mathbf{C}_l(\mathbf{A}) = \mathbf{C}_l(\mathbf{A})^E$ iff $\mathbf{C}_r(\mathbf{A}) = \mathbf{C}_r(\mathbf{A})^T$ iff **A** is commutative algebra.

**Proof .** If **A** is commutative algebra then by lemma 9 we have $\mathbf{C}_l(\mathbf{A})^E = \mathbf{C}_l(\mathbf{A}^{-}) = \mathbf{C}_l(\mathbf{A})$. If $\mathbf{C}_l(\mathbf{A}) = \mathbf{C}_l(\mathbf{A})^E$ then $\mathbf{B}(\mathbf{A}) = \mathbf{B}(\mathbf{A})^T$ and by (2) we obtain that $x\,y = y\,x$ for every $x$ and $y$ in **A**. □

**Definition 5 .** Let **A** be an algebra. For $W \in \{D, T, DT, TD, E\}$ we say that **A** is *W-algebra* (*$W^{-}$-algebra*) if

$$\mathbf{C}_l(\mathbf{A})^W \equiv \mathbf{C}_l(\mathbf{A}), \quad (\mathbf{C}_l(\mathbf{A})^W \equiv \mathbf{C}_l(\mathbf{A}^{-})).$$

We say that **A** is *W-isomorphic algebra* ( *$W^{-}$-isomorphic algebra*) if there exist matrices $N$ and $M$ such that $N\,L\,(\mathbf{C}_l(\mathbf{A})^W)M$ is an algebra that is isomorphic to **A** (to $\mathbf{A}^{-}$) [recall that $L(H)$ is the linear space spanned by the elements of $H$]. By lemma 1, we have

$$W\text{–algebra} \Rightarrow W\text{–isomorphic algebra.}$$
$$W^{-}\text{–algebra} \Rightarrow W^{-}\text{–isomorphic algebra.}$$

Obviously the following lemma follows

**Lemma 11 .** Let $W, W_1, W_2 \in \{D, T, TD, DT, E\}$. Then

(*i*) If **A** is $W_1$-algebra and $W_2$-algebra then **A** is $W_1W_2$-algebra.

(*ii*) **A** is $W^{-}$-algebra iff **A** is $WE$-algebra iff **A** is $EW$-algebra.

(*iii*) **A** is $W^{-}$-isomorphic algebra iff **A** is $WT$-isomorphic algebra.

Lemma 11 with lemma 1 follows

**Lemma 12 .** For every algebra **A** one of the following can happen

(*i*) **A** is $W$-algebra for every $W \in \{D, T, TD, E\}$.

(*ii*) **A** is $W$-algebra for only one $W \in \{D, T, TD, E\}$.

(*iii*) **A** is not $W$-algebra for every $W \in \{D, T, TD, E\}$.

**Lemma 13 .** Every algebra **A** is $TD$-isomorphic algebra.

**Proof .** Since $\mathbf{C}_l(\mathbf{A})^{TD} = \mathbf{C}_r(\mathbf{A})$ and $\mathbf{C}_r(\mathbf{A})$ is isomorphic to $\mathbf{A}$ the result follows . □

**Lemma 14 .** $\mathbf{A}$ is $E$-algebra iff $\mathbf{A}$ is isomorphic to $\mathbf{A}^-$.

**Proof .** If $\mathbf{A}$ is $E$-algebra then $\mathbf{C}_l(\mathbf{A}) \equiv \mathbf{C}_l(\mathbf{A})^E = \mathbf{C}_l(\mathbf{A}^-)$ which follows that there exist matrices $N$ and $M$ such that $\mathbf{RR}_l(\mathbf{A}) = N\,\mathbf{RR}_l(\mathbf{A}^-)M$. Since $I \in \mathbf{RR}_l(\mathbf{A}^-)$ we have $NM = A_a \in \mathbf{RR}_l(\mathbf{A})$ and therefore

$$N^{-1}\mathbf{RR}_l(\mathbf{A})N = N^{-1}\mathbf{RR}_l(\mathbf{A})A_a M^{-1} = N^{-1}\mathbf{RR}_l(\mathbf{A})M^{-1} = \mathbf{RR}_l(\mathbf{A}^-).$$

Now it can be easily show that $\phi(A_c) = N A_c N^{-1}$ is an isomorphism of $\mathbf{RR}_l(\mathbf{A})$ to $\mathbf{RR}_l(\mathbf{A}^-)$ which implies that $\mathbf{A}$ and $\mathbf{A}^-$ are isomorphic. If $\mathbf{A}$ is isomorphic to $\mathbf{A}^-$, then by lemma 4 and 6 we have $\mathbf{C}_l(\mathbf{A}) \equiv \mathbf{C}_l(\mathbf{A}^-) = \mathbf{C}_l(\mathbf{A})^E$. □

**Lemma 15 .** We have

($i$) $\mathbf{A}$ is $D$-algebra iff $\mathbf{A}^-$ is $T$-algebra.

($ii$) If $\mathbf{A}$ is $D$-algebra then $\mathbf{A}$ is isomorphic to $\mathbf{A}^-$.

**Proof .** We have

$$\mathbf{C}_l(\mathbf{A}^-) = \mathbf{C}_l(\mathbf{A})^{DTD} \equiv \mathbf{C}_l(\mathbf{A})^{TD} = \mathbf{C}_l(\mathbf{A}^-)^{TDTTD} = \mathbf{C}_l(\mathbf{A}^-)^T.$$

Since $\mathbf{C}_l(\mathbf{A}^-) \equiv \mathbf{C}_l(\mathbf{A}^-)^T$ then there exist nonsingular matrices $K$, $N$ and $M$ such that

$$\mathbf{C}_l(\mathbf{A}^-)[K] = N\,\mathbf{C}_l(\mathbf{A}^-)^T M.$$

Since $L(\mathbf{C}_l(\mathbf{A}^-)[K]) = \mathbf{RR}_l(\mathbf{A}^-)$ and $L(N\,\mathbf{C}_l(\mathbf{A}^-)^T M) = N\,\mathbf{RR}_l(\mathbf{A}^-)^T M$ we have

$$\mathbf{RR}_l(\mathbf{A}^-) = N\,\mathbf{RR}_l(\mathbf{A}^-)^T M.$$

Since $I \in \mathbf{RR}_l(\mathbf{A}^-)$ then $NM \in \mathbf{RR}_l(\mathbf{A}^-)$. Let $NM = A_a$ for some $a \in \mathbf{A}^-$. Since $N$ is nonsingular we have $M = N^{-1}A_a$ and then

$$\mathbf{RR}_l(\mathbf{A}^-) = N\,\mathbf{RR}_l(\mathbf{A}^-)N^{-1}A_a$$

Since $A_a = NM$ is nonsingular and $\mathbf{RR}(\mathbf{A}^-)A_a^{-1} = \mathbf{RR}_l(\mathbf{A}^-)$ we have

$$\mathbf{RR}_l(\mathbf{A}^-) = N\,\mathbf{RR}_l(\mathbf{A}^-)^T N^{-1}.$$

Since $\mathbf{RR}_l(\mathbf{A}^-)^T$ is isomorphic to $\mathbf{A}$ and $\mathbf{RR}(\mathbf{A}^-)$ is isomorphic to $\mathbf{A}^-$ we have $\mathbf{A}$ is isomorphic to $\mathbf{A}^-$. □

The proofs of the following two lemmas are similar to the proof of lemma 14 and 15

Lemma 16 . We have

(*i*) A is *DT*-algebra iff A is *TD*-algebra iff $\mathbf{A}^-$ is *DT*-algebra.

(*ii*) A is *DT*-algebra iff there exist a nonsingular matrix $N$ such that $\mathbf{RR}_r(\mathbf{A}) = N\,\mathbf{RR}_l(\mathbf{A})N^{-1}$.

Lemma 17 . A is *E*-algebra iff A is *E*- isomorphic algebra iff A is *T*- isomorphic algebra iff A is isomorphic to $\mathbf{A}^-$.

Lemma 18 . If $\mathbf{A}_1$ and $\mathbf{A}_2$ are *W*-algebra (*W*-isomorphic algebra) then are the algebras $\mathbf{A}_1 \times \mathbf{A}_2$ and $\mathbf{A}_1 \otimes \mathbf{A}_2$.

**Proof .** By lemma 8 we have

$$\mathbf{C}_l(\mathbf{A}_1 \times \mathbf{A}_2)^W \equiv \mathbf{C}_l(\mathbf{A}_1)^W \oplus \mathbf{C}_l(\mathbf{A}_2)^W \equiv \mathbf{C}_l(\mathbf{A}_1) \oplus \mathbf{C}_l(\mathbf{A}_2) \equiv \mathbf{C}_l(\mathbf{A}_1 \times \mathbf{A}_2).$$

and so for $\otimes$. □

Let A be an algebra. Let A′ be subalgebra of A. We define

$$U(\mathbf{A}') = \{v \in F^n \mid \ a \in \mathbf{A}' - \{0\} : A_a v = 0\}.$$

Define

$$P_{\mathbf{A}} = \{a \in \mathbf{A} \mid (rad\,\mathbf{A})a = 0\}.$$

Obviously, $P_{\mathbf{A}}$ is subalgebra of A.

Lemma 19 . We have

$$U(P_{\mathbf{A}}) = U(\mathbf{A}).$$

**Proof .** Obviously, $U(P_{\mathbf{A}}) \subseteq U(\mathbf{A})$. If $0 \neq v \in U(\mathbf{A})$ then there exist $a \in \mathbf{A}$ such that $A_a v = 0$. If $(rad\,\mathbf{A})a \neq 0$ then we can find $b_1 \in rad\,\mathbf{A}$ such that $b_1 a \neq 0$. If $(rad\,\mathbf{A})b_1 a \neq 0$ then we can find $b_2 \in rad\,\mathbf{A}$ such that $b_2 b_1 a \neq 0$ and so on. Since $b_1, b_2, \cdots \in rad\,\mathbf{A}$ we have $b_t b_{t-1} \cdots b_1 a \in (rad\,\mathbf{A})^t$ and since for $t = index\,(rad\,\mathbf{A})$ we have $(rad\,\mathbf{A})^t = 0$ there exist $r$ such that $b_r \cdots b_1 a \neq 0$ and $(rad\,\mathbf{A})b_r \cdots b_1 a = 0$. Therefore $b_r \cdots b_1 a \in P_{\mathbf{A}}$. Now since

$$A_{b_r \cdots b_1 a} v = (A_{b_r \cdots b_1})(A_a v) = 0$$

we have $v \in U(P_{\mathbf{A}})$. □

We say that A is *weakly* *W*-algebra if there exist nonsingular matrices $N$ and $M$ such that $N\,L(\mathbf{C}_l(\mathbf{A})^W)M$ is an associative algebra with unity $I$.

$$W\text{-algebra} \Rightarrow W\text{-isomorphic algebra} \Rightarrow \text{weakly } W\text{-algebra} .$$

By lemma 9 every algebra is weakly $W$-algebra for $W \in \{T, TD, E\}$ and an algebra $\mathbf{A}$ is weakly $D$-algebra iff $\mathbf{A}$ is weakly $DT$-algebra.

We call $\mathbf{A}$ *D-regular* algebra if $U(\mathbf{A}) = F^n$.

<u>**Lemma 20 .**</u> If $\mathbf{A}$ is weakly $D$-algebra then $\mathbf{A}$ is $D$-regular.

**Proof .** Let $\mathbf{C}_l(\mathbf{A})^D = \{B_1, \ldots, B_n\}$. Then

$$L(\mathbf{C}_l(\mathbf{A})^D) = \{B_v = [A_{a_1} v \mid \cdots \mid A_{a_n} v] \mid v \in F^n\}.$$

If $U(\mathbf{A}) = F^n$ then for every $v$ there exist $a \in \mathbf{A}$ such that $A_a v = 0$. If $a = \sum_{i=1}^n \lambda_i a_i$ then $A_a v = \sum_{i=1}^n \lambda_i A_{a_i} v = 0$ which implies that $B_v$ is singular matrix for every $v$. Therefore for every non-singular matrices $N$ and $M$ the set $N L(\mathbf{B}(\mathbf{A}))M$ cannot contain the identity matrix. Hence $\mathbf{A}$ is not weakly $D$-algebra. □

It follows from the proof of lemma 20 that

<u>**Lemma 21 .**</u> The algebra $\mathbf{A}$ is not $D$-regular if and only if $L(\mathbf{C}_l(\mathbf{A})^D)$ contains no nonsingular element.

The following follows from the above results

$$D\text{-algebra} \Rightarrow D\text{-isomorphic algebra} \Rightarrow \text{weakly } D\text{-algebra} \Rightarrow D\text{-regular}. \qquad (3)$$

<u>**Lemma 22 .**</u> If $\mathbf{A}_1$ is not $D$-regular then for every algebra $\mathbf{A}_2$ the algebras $\mathbf{A}_1 \times \mathbf{A}_2$ and $\mathbf{A}_1 \otimes \mathbf{A}_2$ are not $D$-regular.

**Proof .** Since $\mathbf{C}_l(\mathbf{A}_1 \times \mathbf{A}_2)^D \equiv \mathbf{C}_l(\mathbf{A}_1)^D \oplus \mathbf{C}_l(\mathbf{A}_2)^D$ and $\mathbf{C}_l(\mathbf{A}_1 \otimes \mathbf{A}_2)^D = \mathbf{C}_l(\mathbf{A}_1)^D \otimes \mathbf{C}_l(\mathbf{A}_2)^D$ then $L(\mathbf{C}_l(\mathbf{A}_1 \times \mathbf{A}_2)^D)$ and $L(\mathbf{C}_l(\mathbf{A}_1 \otimes \mathbf{A}_2)^D)$ contains no nonsingular matrix. □

## 4. COMMUTATIVE ALGEBRAS

In this section we study the properties of commutative algebras. By lemma 10 we have

<u>**Theorem 1 .**</u> Every commutative algebra is $E$-algebra.

Therefore by lemma 12 we have

<u>**Theorem 2 .**</u> A commutative algebra $\mathbf{A}$ is $DT$-algebra iff $\mathbf{A}$ is $TD$-algebra iff $\mathbf{A}$ is $T$-algebra iff $\mathbf{A}$ is

$D$-algebra.

This lemma shows that it is enough to investigate the conditions where a commutative algebra is $D$-algebra. In this section we prove the following

**Theorem 3 .** A commutative local algebra is $D$-algebra iff

$$P_{\mathbf{A}} = d\,\mathbf{A}$$

where $d \in \mathbf{A}$.

Since by Artin theorem every commutative algebra is a direct sum of local commutative algebras the commutative $D$-algebras is completely classified.

**Lemma 23 .** A commutative algebra $\mathbf{A}$ is $D$-algebra if and only if $\mathbf{A}$ is $D$-regular.

**Proof .** If $\mathbf{A}$ is $D$-algebra then by (3) $\mathbf{A}$ is $D$-regular. Assume that $\mathbf{A}$ is not weak. Then there exist $v \in F^n$ such that

$$B_v = [A_{a_1} v \mid \cdots \mid A_{a_n} v]$$

is nonsingular. Consider the set

$$H = \{A_{a_1} B_v, A_{a_2} B_v, \ldots, A_{a_n} B_v\} = \mathbf{C}_l(\mathbf{A}) B_v .$$

Since $A_{a_i} B_v = [A_{a_i a_1} v \mid \cdots \mid A_{a_i a_n} v]$ and $A_{a_i a_j} = A_{a_j a_i}$ we have $H^D = H$. Hence

$$(\mathbf{C}_l(\mathbf{A}) B_v)^D = \mathbf{C}_l(\mathbf{A}) B_v .$$

By lemma 1 and since $B_v$ is nonsingular we obtain

$$\mathbf{C}_l(\mathbf{A}) = (\mathbf{C}_l(\mathbf{A})^D [B_v^T]) B_v^{-1} \equiv \mathbf{C}_l(\mathbf{A})^D . \quad \square$$

**Lemma 24 .** Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be commutative algebras. Then $\mathbf{A}_1 \otimes \mathbf{A}_2$ is $D$-algebra iff $\mathbf{A}_1 \oplus \mathbf{A}_2$ is $D$-algebra iff $\mathbf{A}_1$ and $\mathbf{A}_2$ are $D$-algebra.

**Proof .** If $\mathbf{A}_1 \otimes \mathbf{A}_2$ is $D$-algebra then $\mathbf{A}_1 \otimes \mathbf{A}_2$ is D- regular. Then by lemma 22 $\mathbf{A}_1$ and $\mathbf{A}_2$ are $D$-regular and therefore by lemma 23 they are $D$-algebra. $\square$

**Lemma 25 .** A local commutative algebra $\mathbf{A}$ is $D$-algebra if and only if $P_{\mathbf{A}} = a\,\mathbf{A}$ for some $a \in \mathbf{A}$.

**Proof .** Let $P_{\mathbf{A}} = a\,\mathbf{A}$. If $P_{\mathbf{A}} = (0)$ then $U(P_{\mathbf{A}}) = \varnothing$ and by lemma 23, $\mathbf{A}$ is $D$-algebra.

Let $0 \ne b \in P_{\mathbf{A}} = a\,\mathbf{A}$ then $b = ca$. Since $a, b \ne 0$ are in $P_{\mathbf{A}}$ we must have $c$ not in $rad\,\mathbf{A}$ and therefore $c$ is nonsingular. Hence $A_c$ is nonsingular matrix and $A_b v = A_c A_a v = 0$ is equivalent to

$A_a\, v = 0$. Therefore

$$U\,(P_{\mathbf{A}}) = \{v \mid A_a\, v = 0\}.$$

Since $a \neq 0$ the matrix $A_a \neq 0$ and therefore $U\,(P_{\mathbf{A}}) \neq F^n$. This follows that $\mathbf{A}$ is $D$-algebra.

Let $\mathbf{A}$ be a commutative local $D$-algebra. Let $a_1 \in P_{\mathbf{A}}$. Then $a_1\mathbf{A} \subseteq P_{\mathbf{A}}$. Now let $a_2 \in P_{\mathbf{A}} - a_1\mathbf{A}$. Then $a_2\mathbf{A} \subseteq P_{\mathbf{A}}$. If $a_1\mathbf{A} \cap a_2\mathbf{A} \neq (0)$. Then there exist $u_1$ and $u_2$ such that $a_1u_1 = a_2u_2$. If $u_2$ is singular then $u_2 \in rad\ \mathbf{A}$ and $a_1u_1 = 0 = a_2u_2$. If $u_2$ is nonsingular then $a_2 = a_1u_2^{-1}\,u_1$ and $a_2 \in a_1\mathbf{A}$. A contradiction. Therefore there exist $a_1, a_2, \ldots, a_w \in P_{\mathbf{A}}$ such that (Induction hypothesis)

$$P_{\mathbf{A}} = a_1\mathbf{A} \oplus a_2\mathbf{A} \oplus \cdots \oplus a_w\mathbf{A}.$$

(direct sum of subspaces). Consider the following base of $\mathbf{A}$

$$A = \{u_1, \ldots, u_s, t_1, \ldots, t_r, a_1u_1, \ldots, a_1u_s, \ldots, a_wu_1, \ldots, a_wu_s\}$$

where $u_1, \ldots, u_s \in \mathbf{A}$ are nonsingular elements $s = dim\ (\mathbf{A}/rad\ \mathbf{A})$, $\{t_1, \ldots, t_r\}$ is a base for $(rad\ \mathbf{A}) - P_{\mathbf{A}}$ and $\{a_1u_1, \ldots, a_wu_s\}$ is a base for $P_{\mathbf{A}}$. Let $\phi : \mathbf{A} \to \mathbf{A}/rad\ \mathbf{A}$ be a canonical homomorphism. Since $\mathbf{A}$ is local commutative algebra we have $\mathbf{A}/rad\ \mathbf{A}$ is a field. Let $\phi(u_i) = d_i \in \mathbf{A}/rad\ \mathbf{A}$. If

$$d_i\, d_j = \textstyle\sum_{k=1}^{s} \gamma_{i,j,k}\, d_k \tag{4}$$

then because

$$\phi(u_i u_j) = \phi(u_i)\,\phi(u_j) = d_i\, d_j = \textstyle\sum_{k=1}^{s}\gamma_{i,j,k} d_k = \phi(\sum_{k=1}^{s}\gamma_{i,j,k}\, u_k) = \sum_{k=1}^{s}\gamma_{i,j,k}\, d_k$$

we have

$$u_i\, u_j = \sum_{k=1}^{s} \gamma_{i,j,k}\, u_k + h$$

where $h \in rad\ \mathbf{A}$. Therefore

$$u_i\,(a_l\, u_j) = a_l \sum_{k=1}^{s} \gamma_{i,j,k}\, u_k + a_l\, h = \sum_{k=1}^{s} \gamma_{i,j,k} a_l\, u_k\,.$$

This with (4) implies

$$A_{a_l u_j} = \begin{bmatrix} & \tilde{A}_{d_j} & 0_{s\times s} & \underset{w-l}{\cdots} & 0_{s\times s} \\ 0 & & & & \\ & & & & \end{bmatrix}$$

where $\mathbf{C}_l\,(\mathbf{A}/rad\ \mathbf{A}\,, \{d_1, \ldots, d_s\}) = \{\tilde{A}_{d_1}, \ldots, \tilde{A}_{d_s}\}$.

Let $v = (v_0, v_1, v_2, v_3)$ be any vector of length $n$ where $v_0$ is of length $n - dim\ rad\ \mathbf{A}$ and $v_1, v_2$ are of length $s$. Then

$$A_{\sum\lambda_i a_1 u_i + \sum\delta_i a_2 u_i} v = \tilde{A}_{\sum\lambda_i d_i} v_1 + \tilde{A}_{\sum\delta_i d_i} v_2.$$

We shall show that for any $v_1,\ v_2 \in F^s$ there exist $\tilde{A}_{f_1}, \tilde{A}_{f_2} \in \mathbf{C}_l\,(\mathbf{A}/rad\ \mathbf{A})$ not both zero such that $\tilde{A}_{f_1} v_1 + \tilde{A}_{f_2} v_2 = 0$. This follows that any $v \in F^n$ is in $U\,(\mathbf{A})$ which complete the proof.

If $v_1 = 0$ or $v_2 = 0$ then this result is trivial. Assume $v_1, v_2 \neq 0$. Consider the set

$$H = \{\tilde{A}_a v_1 \mid a \in \mathbf{A}/rad\ \mathbf{A}\}$$

If $\tilde{A}_{d_1} v, \ldots, \tilde{A}_{d_s} v \in H$ are linearly dependent then there exist $(\psi_1, \ldots, \psi_s) \in F^s$ such that $\sum_{i=1}^{s} \psi_i \tilde{A}_{d_i} v = 0$ which implies that $\tilde{A}_d\, v = 0$ for $d = \sum_{i=1}^{n} \psi_i\, d_i$ and $\tilde{A}_d$ is singular. Since $\mathbf{A}/rad\ \mathbf{A}$ is a field we have a contradiction. Therefore $H = F^s$ and therefore there exist $d \in \mathbf{A}/rad\ \mathbf{A}$ such that $\tilde{A}_d\, v_1 = -v_2$. Now this implies $\tilde{A}_d v_1 + \tilde{A}_1 v_2 = 0$. □

We now give some examples of commutative $D$-algebras.

**Example 1 . Polynomial algebras**

The polynomial algebra is the algebra $F\,(p) = F\,[\alpha]/(p\,(\alpha))$ where $p\,(\alpha) \in F\,[\alpha]$. It can be easily shown that

$$\mathbf{C}_l\,(F\,[\alpha]/(p\,(\alpha))) \equiv \{C_p^0, C_p^1, \ldots, C_p^{deg\,p-1}\}.$$

where $C_p$ is the companion matrix of $p$. If $p\,(\alpha) = p_1(\alpha)^{d_1} \cdots p_r\,(\alpha)^{d_r}$ where $p_1(\alpha), \ldots, p_r\,(\alpha)$ are distinct irreducible polynomials. Then

$$F\,(p) = F(p_1^{d_1}) \times \cdots \times F(p_r^{d_r}). \tag{5}$$

and it is well known that

$$F\,(p_1^{d_1}) = F\,(p_1) \otimes F\,(\alpha^{d_1}). \tag{6}$$

This algebra is also satisfy

**Corollary 1 .** The algebra $F\,(p)$ is $D$-algebra.

**Proof .** By (5) and (6) and lemma 24 it is enough to prove that $\mathbf{A} = F\,(\alpha^d)$ is $D$-algebra. Since $P_{\mathbf{A}} = \alpha^{d-1}\mathbf{A}$ by lemma 25 the result follows. □

**Example 2 .**

Let $\{1, p_1, p_2, d\}$ be a base for the algebra **A** that satisfies: 1 is the unit element,

$$p_1 p_2 = p_2 p_1 = d\ ,\ p_1^2 = p_2^2 = d^2 = 0.$$

Then **A** is local commutative algebra and for $y = y_1 + y_2 p_1 + y_3 p_2 + y_4 d$ we have

$$A_y = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ 0 & y_1 & 0 & y_3 \\ 0 & 0 & y_1 & y_2 \\ 0 & 0 & 0 & y_1 \end{bmatrix}$$

Since $P_{\mathbf{A}} = d\,\mathbf{A}$ the algebra **A** is $D$-algebra. □

**Lemma 26 .** If **A** is commutative local $D$-algebra and $k = index\,(rad\,\mathbf{A})$ is the least integer such that $(rad\,\mathbf{A})^k = 0$ then

$$P_{\mathbf{A}} = (rad\,\mathbf{A})^{k-1}.$$

**Proof .** Since for every $a \in (rad\,\mathbf{A})^{k-1}$ we have $a\ rad\,\mathbf{A} = 0$ then $(rad\,\mathbf{A})^{k-1} \subseteq P_{\mathbf{A}}$. If **A** is $D$-algebra then $P_{\mathbf{A}} = d\,\mathbf{A}$ for some $d \in P_{\mathbf{A}}$.

If $e \in (rad\,\mathbf{A})^{k-1}$ then $e = d\,u$ for some nonsingular $u$ and then $P_{\mathbf{A}} = d\,\mathbf{A} \subseteq (rad\,\mathbf{A})^{k-1}$. □

Notice that in the end of the proof of lemma 23 we have the exact connection between $\mathbf{C}_l(\mathbf{A})$ and $\mathbf{C}_l(\mathbf{A})^D$ for commutative $D$-algebras. The exact connection between $\mathbf{C}_l(\mathbf{A})$ and $\mathbf{C}_l(\mathbf{A})^{DT}$, $\mathbf{C}_l(\mathbf{A})^T \cdots$ can be obtain from this by using lemma 1.

## 5. SEMISIMPLE ALGEBRAS

We shall begin to investigate the case when **A** is simple algebra. If **A** is a simple algebra then it is well known that

$$\mathbf{A} = \mathbf{M}_n \otimes \mathbf{P}$$

where $\mathbf{M}_n$ is the total matrix algebra of order $n$ and **P** is a division algebra. By lemma 14 we have

**Lemma 27 .** The simple algebra $\mathbf{M}_n \otimes \mathbf{P}$ is $E$-algebra if and only if **P** is isomorphic to $\mathbf{P}^-$.

Now we shall use Noether-Skolem theorem, [H], to prove

**Lemma 28 .** Any simple algebra is $TD$-algebra.

**Proof .** Since $\mathbf{RR}_l(\mathbf{A})$ is isomorphic to $\mathbf{RR}_r(\mathbf{A})$ and the unit element in both of them coincide we have by Noether-Skolem theorem $\mathbf{RR}_r(\mathbf{A}) = N^{-1}\mathbf{RR}_l(\mathbf{A})N$ which follows that $\mathbf{C}_l(\mathbf{A})^{TD} = N^{-1}\mathbf{C}_l(\mathbf{A})N\,[M]$ for some nonsingular matrix $M$. □

Since every semisimple algebra $\mathbf{A}$ is

$$\mathbf{A} = \mathop{\times}_{i=1}^{t} \mathbf{M}_{n_i} \otimes \mathbf{P}_i$$

where $\mathbf{P}_i$ , $i = 1, \ldots, t$ are division algebras we have

<u>**Theorem 4 .**</u> Any semisimple algebra is $TD$-algebra.

Since

$$\mathbf{A}^{-} \text{ isomorphic to } \mathop{\times}_{i=1}^{t} \mathbf{M}_{n_i}^{-} \otimes \mathbf{P}_i^{-} \text{ isomorphic to } \mathop{\times}_{i=1}^{t} \mathbf{M}_{n_i} \otimes \mathbf{P}_i^{-}$$

we have

<u>**Theorem 5 .**</u> A semisimple algebra $\mathop{\times}_{i=1}^{t} \mathbf{M}_{n_i} \otimes \mathbf{P}_i$ is $E$-algebra iff there exist a permutation $\Phi$ on $\{1, \ldots, t\}$ such that: For $i = 1, \ldots, t$ we have

$(i)$ $n_i = n_{\Phi(i)}$.

$(ii)$ $P_i$ is isomorphic to $P_{\Phi(i)}^{-}$.

By lemma 12 we have

<u>**Theorem 6 .**</u> A semisimple algebra is $D$- algebra iff $\mathbf{A}$ is $E$-algebra.

By lemma 11 we have

<u>**Theorem 7 .**</u> Every semisimple algebra is $T^{-}$-algebra and $D^{-}$-algebra.

## 6. APPLICATIONS

In this section we shall give some applications of the results in sections 4 and 5.

Let $B = \{B_1, \ldots, B_k\}$ be a set of $n \times m$ matrices. In a similar manner as in definitions 1 and 2 we can define $B^T, B^D, NBM[K]$. For $C$ a $k'$ set of $n' \times m'$ matrices we also can define, as in definition 3, $B \oplus C$ and $B \otimes C$. Then all the equations in lemma 1 and 2 are true for these extended definitions.

The *multiplicative complexity* of $B$ is the minimal integer $t$ such that there exist matrices $L_1, L_2$ and $L_3$ of order $t \times k$, $t \times n$ and $t \times m$, respectively, where

$$\begin{bmatrix} \mathbf{x}^T B_1 \mathbf{y} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{x}^T B_k \mathbf{y} \end{bmatrix} = L_1^T (L_2 \mathbf{x} * L_3 \mathbf{y}).$$

where for $u = (u_1, \ldots, u_t)^T$, $v = (v_1, \ldots, v_t)$ the componentwise product $u * v$ is $(u_1 v_1, \ldots, u_t v_t)^T$. The triple $(L_1, L_2, L_3)$ is called a *minimal bilinear algorithm* for $B$ and the multiplicative complexity is denoted by $\delta(B)$.

It is known that if $(L_1, L_2, L_3)$ is a bilinear algorithm for $B$ then $(L_1, L_3, L_2)$ and $(L_2, L_1, L_3)$ is minimal bilinear algorithm for $B^T$ and $B^D$ respectively. If $N$, $M$ and $K$ are nonsingular matrices of order $n \times n$, $m \times m$ and $k \times k$, respectively, then $(L_1 K^T, L_2 N, L_3 M)$ is a minimal bilinear algorithm for $N\, B\, M\, [K]$. Therefore

$$\delta(B) = \delta(B^T) = \delta(B^D) = \delta(N\, B\, M\, [K]). \tag{7}$$

For algebra $\mathbf{A}$ the multiplicative complexity of $\mathbf{A}$ is

$$\delta(\mathbf{A}) = \delta(\mathbf{B}(\mathbf{A})).$$

The applied meaning of the multiplicative complexity of $B$ is the number of multiplications and divisions needed to compute $\mathbf{x}^T B_1 \mathbf{y}, \ldots, \mathbf{x}^T B_k \mathbf{y}$ by a program. Therefore $\delta(\mathbf{A})$ is the number of multiplications and divisions needed to compute the multiplication of two elements in the algebra $\mathbf{A}$.

The multiplicative complexity of computing $\mathbf{x}^T H_1 \mathbf{y}, \ldots, \mathbf{x}^T H_l \mathbf{y}$ where $\mathbf{x} = (x_1, \ldots, x_r)^T$, $\mathbf{y} = (y_1, \ldots, y_s)^T$ are vector of elements in $\mathbf{A}$ is

$$\delta(H \otimes \mathbf{B}(\mathbf{A}))$$

where $H = \{H_1, \ldots, H_l\}$.

For these problems we have

**Theorem 8 .**

$(i)$ For $D$-algebras

$$\delta(H \otimes \mathbf{B}(\mathbf{A})) = \delta(H^D \otimes \mathbf{B}(\mathbf{A})).$$

(*ii*) For *TD*-algebras

$$\delta(H \otimes \mathbf{B}(\mathbf{A})) = \delta(H^{DT} \otimes \mathbf{B}(\mathbf{A})).$$

(*iii*) For *T*-algebras

$$\delta(H \otimes \mathbf{B}(\mathbf{A})) = \delta(H^{E} \otimes \mathbf{B}(\mathbf{A})).$$

**Proof .** If **A** is *D*-algebra then

$$(H \otimes \mathbf{B}(\mathbf{A}))^D = H^D \otimes \mathbf{B}(\mathbf{A})^D = H^D \otimes \mathbf{B}(\mathbf{A}).$$

Therefore

$$\delta(H \otimes \mathbf{B}(\mathbf{A})) = \delta(H^D \otimes \mathbf{B}(\mathbf{A})).$$

If **A** is *TD*-algebra then $\delta(H \otimes \mathbf{B}(\mathbf{A})) = \delta(H^D \otimes \mathbf{B}(\mathbf{A}^-))\delta(H^{DT} \otimes \mathbf{B}(\mathbf{A})) = \delta(H^D \otimes \mathbf{B}(\mathbf{A}))$. (*ii*) follows in a similar manner. □

One application of this theorem is

**Corollary 2 .** For *D*-algebras and *TD*-algebras the multiplicative complexity of computing $x_1 y_1 + \cdots + x_n y_n$ and $x_1 y, x_2 y, \ldots, x_n y$ where $x_i, y_i \in \mathbf{A}$ are equal.

**Proof .** The complexity of $x_1 y_1 + \cdots + x_n y_n$ is $\delta(I_n \otimes \mathbf{B}(\mathbf{A}))$ and the complexity of $x_1 y, x_2 y, \ldots, x_n y$ where $x_i, y_i \in \mathbf{A}$ is $\delta(I_n^D \otimes \mathbf{B}(\mathbf{A}))$ and by Theorem 1 they are equals. □

The reader can find more application in [AW], [B3], [B4], [Gr2] and [HM].

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