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Cutsets and Fibres in Partially Ordered Sets

by

Roy Manley Maltby

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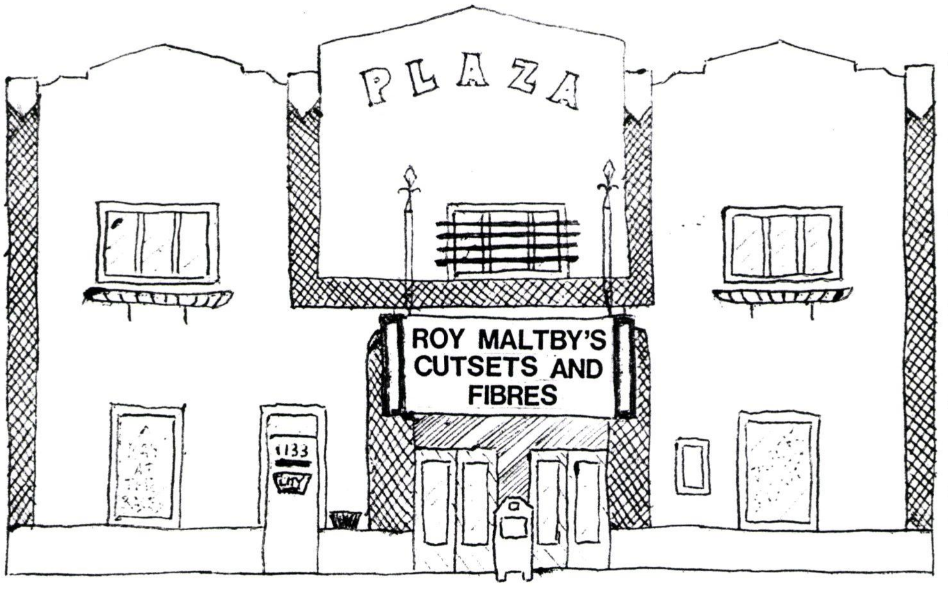

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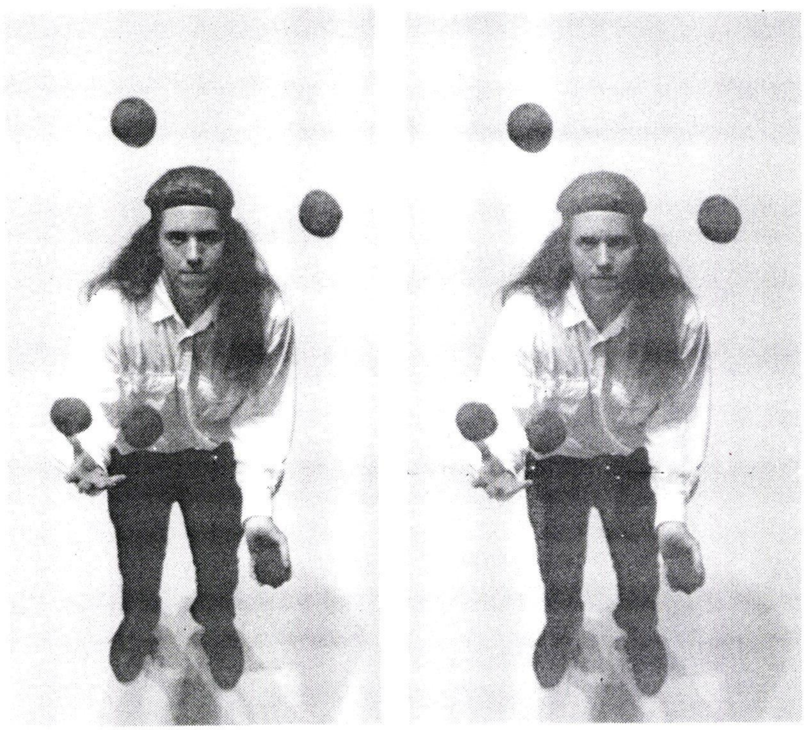

The author juggling a 5-ball cascade at the
46th Annual International Jugglers' Association Convention
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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Cutsets and Fibres in Partially Ordered Sets" submitted by Roy Manley Maltby in partial fulfillment of the requirements for the degree of Master of Science.

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Abstract

A *cutset* (*fibre*) of a poset is a subset which meets every maximal chain (antichain). We examine posets in which every cutset contains a maximal antichain, determining some operations which preserve or destroy this property. We examine the question of what properties guarantee that fence-free is equivalent to every minimal cutset being an antichain or every minimal fibre being a chain. We describe a method for partitioning any antichain-finite poset into three subsets so that the union of any two is a fibre. We also summarise what is known about cutsets for elements.

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My thanks go to Prof. Bill Sands for many hours of meticulously reading, correcting, critiquing, and discussing this work. Without the improvements resulting from his advice, it would require a hardy reader indeed to persevere much beyond this page.

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To every juggler whose place I ever stayed at, and even some whose I didn't, but who enriched my experience of juggling and life by sharing knowledge and friendship.

Contents

Figures

Symbols

$<, \leq, >, \geq, \parallel$	order relations, 1
$\oplus$	linear sum, 2
$A \setminus B$	set difference, 2
$\uparrow, \underset{\circ}{\uparrow}, \downarrow, \overset{\circ}{\downarrow}, \updownarrow$	order-related sets, 2
P^{d}	dual of P, 2
$\subset$	proper set containment, 3
$V(G), E(G), (x, y)$	graph notation, 4
$\mathbf{N}, \omega, \mathbf{n}, \mathbf{Z}$	chains, 5
$\overline{n}$	n-element antichain, 5
(a, b, c, d)	fence, 6
$\prec, \succ$	covering relations, 6
$\sum, +, \oplus$	lexicographic sums, 10
$P \times Q$	direct product, 14
$K_{r,s}$	$r \oplus s$, 19
$(K_{1,1})^r$	$K_{1,1} \times \ldots \times K_{1,1}$ (r times), 23
P^Q	exponent, 31
$A(x), K(x), r(K), s(K)$	cutset construction, 69
$\mathrm{UC}(x), \mathrm{LC}(x)$	covering sets, 70
$\mathcal{C}(X)$	set of maximal chains containing X, 70
$\mathcal{M}(P)$	set of maximal chains of P, 103
$\mathrm{disj}(P), \mathrm{cut}(P)$	Menger measurements, 104
$\mathbf{R}$	chain of real numbers, 105
$V(\mathcal{S})$	set of variables used in $\mathcal{S}$, 118

Chapter 1

Introduction

"An ancestor of mine maintained that if you eliminate the impossible, whatever remains, *however improbable*, must be the truth."
— Spock [1991] quoting Sherlock Holmes [1890]

A binary relation "$<$" on a set P is called a *partial order* if it satisfies the following conditions.

(i) For all $x, y, z \in P$, $x < y$ and $y < z$ implies $x < z$.

(ii) There are no $x, y \in P$ such that $x < y$ and $y < x$.

Notice that a consequence of (ii) is that there is no $x \in P$ such that $x < x$.

A set P with which a partial order is associated is called a *partially ordered set*, which we will usually abbreviate to *poset.*

Suppose P is a poset with partial order $<$. The following symbols will be convenient. For any $x, y \in P$,

$x \leq y$ means $x < y$ or $x = y$,

$x > y$ means $y < x$,

$x \geq y$ means $x > y$ or $x = y$, and

$x \parallel y$ means $x \neq y$, $x \not< y$, and $y \not< x$.

For subsets $X, Y \subseteq P$, $X < Y$ means that $x < y$ for all $x \in X$ and $y \in Y$. $X \leq Y$, $X > Y$, $X \geq Y$, and $X \parallel Y$ are defined similarly. When X or Y is a singleton, we may write just the element rather than the set. For instance, if $X = \{x\}$, then $x \lessdot Y$ means $\{x\} < Y$. If a poset P has nonempty disjoint subsets X and Y such that $X < Y$ and $X \cup Y = P$, then we say that P is *linearly decomposable* and write $P = X \oplus Y$.

To denote set difference, we will use "$\smallsetminus$". That is, $A \smallsetminus B$ is the set of elements of A which are not elements of B.

Define the following notation when X is a subset of P.

$$X{\uparrow} = \{y \in P : y \geq x \text{ for some } x \in X\},$$

$$X\underset{\circ}{\uparrow} = \{y \in P : y > x \text{ for some } x \in X\} \smallsetminus X,$$

$$X{\downarrow} = \{y \in P : y \leq x \text{ for some } x \in X\},$$

$$X\overset{\circ}{\downarrow} = \{y \in P : y < x \text{ for some } x \in X\} \smallsetminus X,$$

$$X{\updownarrow} = X{\uparrow} \cup X{\downarrow}.$$

Actually, the symbols $\underset{\circ}{\uparrow}$ and $\overset{\circ}{\downarrow}$ will only be used with antichains in this thesis, making the "$\smallsetminus X$" parts of the definitions unnecessary. We will abbreviate this notation slightly for $x \in P$: $x{\uparrow} = \{x\}{\uparrow}$, $x\underset{\circ}{\uparrow} = \{x\}\underset{\circ}{\uparrow}$, $x{\downarrow} = \{x\}{\downarrow}$, $x\overset{\circ}{\downarrow} = \{x\}\overset{\circ}{\downarrow}$, and $x{\updownarrow} = \{x\}{\updownarrow}$.

If P is a poset, then the dual of P, denoted P^{d}, is the poset having the same elements as P, but with the elements ordered so that $x < y$ in P^{d} if and only if $y < x$ in P.

If $C \subseteq P$ then C is called a *chain* if for all $x, y \in C$, $x \leq y$ or $y \leq x$. A subset A of P is called an *antichain* if for all distinct $x, y \in A$, $x \parallel y$. A chain (respectively, antichain) S of P is called a *maximal chain* (respectively, *maximal antichain*) of P if there is no $S' \subseteq P$ such that S' is also a chain (respectively, antichain) and $S \subset S'$.

(In this thesis, the symbol $\subset$ will always mean "is a proper subset of".) It is an easy consequence of Zorn's Lemma that if S is a chain (respectively, antichain) in a poset P, then there exists S' a maximal chain (respectively, maximal antichain) of P which contains S. An antichain A in a poset P is a maximal antichain if and only if $A\updownarrow = P$. In set theory, authors tend to keep track of the use of Zorn's Lemma so that they know what can be achieved without it. In this thesis, maximal chains and antichains are so important that there is no hope of doing anything without Zorn's Lemma, so there is no point making a fuss about using it.

If P is a poset with subset K, then K is called a *cutset* of P if K has non-empty intersection with every maximal chain of P. A cutset K is called a *minimal cutset* if no proper subset of K is a cutset. A cutset K is a minimal cutset of P if and only if for every $x \in K$, there is a maximal chain C such that $C \cap K = \{x\}$. Similarly, a subset F of P is called a *fibre* of P if F has non-empty intersection with every maximal antichain of P. A fibre F is called a *minimal fibre* if no proper subset of F is a fibre. A fibre F is a minimal fibre of P if and only if for every $x \in F$, there is a maximal antichain A such that $A \cap F = \{x\}$. In an infinite poset, a fibre (respectively, cutset) might not contain any minimal fibre (respectively, minimal cutset). It is fairly easy to see that when a fibre (respectively, cutset) is a chain (respectively, antichain), it is a minimal fibre (respectively, minimal cutset) and a maximal chain (respectively, maximal antichain).

Suppose P is a poset and $X \subseteq P$. Then $\max X$ is the set of points $x \in X$ such that there is no $y > x$ where $y \in X$ as well. $\min X$ is defined dually. We will call $\max P$ the set of *maximal elements* of P, or just the *maximals* of P, and the *minimals* dually. Sometimes when $\max X$ is a singleton set, we will use $\max X$ to refer just to the element of the singleton set rather than the set.

A poset is called *chain-finite* (respectively, *antichain-finite*) if all of its chains (respectively, antichains) are finite. Using the usual definition of sup for cardinalities, define the *height* and *width* of a poset P by

$$\text{height}(P) = \sup\{|C| : C \text{ is a chain in } P\}, \text{ and}$$

$$\text{width}(P) = \sup\{|A| : A \text{ is an antichain in } P\}.$$

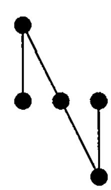
A chain-finite poset may have infinite height, and an antichain-finite poset may have infinite width. A poset P is called *well-founded* if it contains no infinite decreasing chain $x_1 > x_2 > x_3 > \ldots$. This is equivalent to saying that for every non-empty chain C in P, $|\min C| = 1$. If C is a chain, $X = \{x \in P : x \geq C\}$, and there exists $y \in X$ such that $X = y\!\uparrow$, then we say that y is the *supremum* of C, abbreviated $y = \sup C$. The *infimum* of C, denoted $\inf C$, is defined dually. We say that a poset is *chain-complete* if every chain has a supremum and an infimum.

A *graph* G is a set of vertices $V(G)$ and a set of edges $E(G)$. Each edge is a 2-element subset of $V(G)$. We will write edges delimited by parentheses rather than curly braces. Readers who insist that (x, y) is not the same as (y, x) can use the implicit assumption that $E(G)$ is a symmetric set. That is, for any $x, y \in V(G)$ such that $(x, y) \in E(G)$, (y, x) is also in $E(G)$. $C \subseteq V(G)$ is called a *clique* of the graph G if $(x, y) \in E(G)$ for all distinct $x, y \in C$. $C \subseteq V(G)$ is called a *maximal clique* of G if C is a clique and C is not properly contained in any clique. By Zorn's Lemma, any clique is contained in a maximal clique. A *transversal* of a graph G is a subset of $V(G)$ which has non-empty intersection with every maximal clique. A *minimal transversal* is a transversal which does not properly contain any transversal. A transversal T of a graph G is a minimal transversal if and only if for every $x \in T$, there is a maximal clique C such that $C \cap T = \{x\}$. In infinite graphs, transversals do not always contain minimal transversals. A subset $S \subseteq V(G)$ is called *independent* if for all $x, y \in S$, $(x, y) \notin E(G)$. When a transversal is independent, it is a minimal transversal and a maximal independent set.

The *comparability graph* G of a poset P is the graph such that $V(G) = P$ and for all distinct $x, y \in P$, $(x, y) \in E(G)$ if and only if $x < y$ or $y < x$ in P. Any subset C of P is a (maximal) chain of P if and only if $C \subseteq V(G)$ is a (maximal) clique of G. Any subset K of P is a (minimal) cutset of P if and only if $K \subseteq V(G)$ is a (minimal) transversal of G. The *complement* of a graph G is the graph G' such that $V(G') = V(G)$ and for all distinct $x, y \in V(G')$, $(x, y) \in V(G')$ if and only if $(x, y) \notin V(G)$. Let P be a poset with comparability graph G, and G' the complement of G. A subset A of P is a (maximal) antichain of P if and only if $A \subseteq V(G)$ is a (maximal) independent set of G, which is equivalent to $A \subseteq V(G')$ being a (maximal) clique of G'. F is a (minimal) fibre of P if and only if $F \subseteq V(G')$ is a (minimal) transversal of G'.

A poset which is a chain will be called a *totally ordered* set. Some totally ordered sets which warrant names are the set of natural numbers, $\mathbf{N}$, whose elements are $1 < 2 < 3 < \ldots$; the set of whole numbers, ω, whose elements are $0 < 1 < 2 < 3 < \ldots$; the set of the first n whole numbers, $\mathbf{n}$, whose elements are $0 < 1 < 2 < \ldots < n-1$; and the set of integers, $\mathbf{Z}$, whose elements are $\ldots -2 < -1 < 0 < 1 < 2 < \ldots$. Also, the symbol $\overline{n}$ will denote an n-element antichain. Ordinarily, when we talk about elements of a dual poset, we will use the same labels as in the original poset. The only exception is ω^{d}, whose elements we will label by $0 > -1 > -2 > -3 > \ldots$.

(a, b, c, d) is called a *4-fence* of a poset P if a, b, c, and d are distinct elements of P such that $a < b > c < d$ and these elements are otherwise not comparable to each other. A poset is called *fence-free* if it has no 4-fence. If a poset is not fence-free, we will save a syllable by saying that it *has a fence*, rather than saying it has a 4-fence. The poset consisting of just a 4-fence will be called F_4. $x \prec y$ (read "*x is covered by y*") means that $x < y$ and $x\uparrow \cap y\downarrow = \emptyset$. $x \succ y$ (read "*x covers y*") means $y \prec x$. (a, b, c, d) is called an N of a poset P if a, b, c, and d are distinct elements of P such that $a \prec b \succ c \prec d$ and these elements are otherwise not comparable to each other. A poset is called *N-free* if it has no N. Every fence-free poset is N-free, but the N-free poset below is not fence-free.



(a, b, c, d) is called a P_3 (a path with 3 edges) of a graph G if the edges on these vertices are just (a, b), (b, c), and (c, d). A graph is called *P_3-free* if it has no P_3. If G is the comparability graph of a poset P, then G is P_3-free if and only if P is fence-free. A graph is P_3-free if and only if its complement is P_3-free.

The earliest paper this author has found which mentions a result concerning cutsets and/or fibres is a 1969 paper by Grillet [G69]. We refer the reader to the chapter on cutset- and fibre-straight posets (Chapter 3) for the definition of "regular", since we will not prove and will quickly forget the following result. Suffice it to say, for now, that all finite posets are regular. The equivalence of conditions (i)–(iii) below is trivial. The main relevant result of [G69] is that for any regular poset P, these conditions are equivalent to (iv):

(i) Every maximal chain of P meets every maximal antichain of P.

(ii) Every maximal chain of P is a fibre.

(iii) Every maximal antichain of P is a cutset.

(iv) P is N-free.

Actually, Grillet never used the words "cutset" or "fibre", but he did coin the term *chain-antichain-complete*, abbreviated *CAC*, for posets satisfying conditions (i)–(iii). A 1973 paper by Leclerc and Monjardet [LM73] characterised finite CAC lattices and finite CAC graded posets in terms less convenient than N-free. A result by Rival and Zaguia [RZ87] is a direct extension of Grillet's theorem. They define a structure more general than N and show that in any poset, (i)–(iii) above hold if and only if the poset does not contain this more general forbidden configuration. Aside from [LM73] and [RZ87], Grillet's paper seems to have been quite ineffective at generating interest in cutsets and fibres.

The paper which seems to have initiated most current cutset research is the 1984 paper by Bell and Ginsburg [BG84]. This paper and the majority of those since which have mentioned cutsets have been concerned primarily or exclusively with cutsets for elements. We refer the reader to the chapter on that subject (Chapter 5) for more information since that subject has nothing to do with most of this thesis.

The genesis of this thesis was research relating to a paper by Duffus, Sands, and Winkler [DSW90], showing that Boolean lattices are *hockey* posets, one characterisation of which is that every cutset contains a maximal antichain. In the summer of 1990, Peter Gibson and I looked into what else could be said about hockey posets. The majority of the results in the chapter on hockey posets (Chapter 2) are products of this research. A finite poset is hockey if and only if every *minimal* cutset contains a maximal antichain. I got to wondering what could be said about posets in which every minimal cutset *is* a maximal antichain. Hence, the chapter on the subject of cutset-straight and fibre-straight posets (Chapter 3).

On the whole, cutsets have been a much more popular topic of research than fibres have. However, the chapter on "Partitions and Fibres" (Chapter 4) is exclusively about fibres, and the questions addressed really are fibre questions. Many questions about fibres, cutsets, or transversals can trivially be rephrased using either of the other two terms, but this does not apply to material in the "Partitions and Fibres" chapter.

Chapter 2

Hockey Posets

comprising research undertaken with Peter Gibson

"Ad hoc, ad loc, and quid pro quo. So little time, so much to know."
— Jeremy Hillary Boob, Ph.D. in *Yellow Submarine* [1968]

2.1 Introduction

Call a poset a *hockey* poset if it satisfies the following equivalent conditions:

(i) Every fibre intersects every cutset.

(ii) Every red-blue coloring of the elements of the poset has a red maximal chain or a blue maximal antichain.

(iii) Every fibre contains a maximal chain.

(iv) Every cutset contains a maximal antichain.

The equivalence of (ii), (iii), and (iv) is explained in [DSW90]. It is easy to see that (iii) implies (i). Conversely, if (iii) is false then so is (i) since there is a fibre whose complement is a cutset.

The property of hockeyness was discussed in [DSW90] but it was not given a name. Also, characterisation (i) was not mentioned. We introduce the term "hockey" with the following motivation. By (ii), a poset is hockey if and only if every red-blue coloring of its elements results in a red maximal chain — that is, a red line going all the way across the poset vertically — or a blue maximal antichain

— that is, a blue line going all the way across the poset horizontally. The only naturally-occurring object with red and blue lines going all the way across it is a hockey rink. There is a weakness here in that the lines on a hockey rink are parallel rather than orthogonal, but this seems to be the best we can do.

In the remainder of the chapter, we will determine when sums of posets are hockey. We will classify certain direct products of posets as being hockey or not hockey. Among other results, we find that if a direct product of two posets is hockey, where one is well-founded and the other chain-complete, then the well-founded factor must be hockey. We will show that the only finite exponents $\mathbf{2}^P$ of the two-element chain that are hockey are those in which P is a linear sum of antichains. We will also determine which zigzags and cycles are hockey.

2.2 Lexicographic Sums

Let P be a poset with a poset Q_x associated with each $x \in P$ and such that $Q_y \cap Q_z = \emptyset$ for all distinct $y, z \in P$. The *lexicographic sum* $\sum_{x \in P} Q_x$, is the set $\bigcup_{x \in P} Q_x$ with the following ordering. If $a, b \in Q_y$ then $a < b$ in $\sum_{x \in P} Q_x$ if and only if $a < b$ in Q_y. If $a \in Q_y$, $b \in Q_z$, and $y \neq z$, then $a < b$ if and only if $y < z$.

The lexicographic sum can be used to describe both disjoint (*cardinal*) and linear (*ordinal*) sums:

$$\begin{aligned} Q_0 + \ldots + Q_{n-1} &= \sum_{x \in \overline{n}} Q_x; \\ Q_0 \oplus \ldots \oplus Q_{n-1} &= \sum_{x \in \mathbf{n}} Q_x. \end{aligned}$$

(In the case of the n-element chain, we are using the conventional labelling $0 < 1 < 2 < \ldots < n-1$.)

Lemma 2.2.1 *Let $A \subseteq \sum_{x\in P} Q_x$. A is a maximal antichain of $\sum_{x\in P} Q_x$ if and only if both of the following are true:*

(i) for every $y \in P$, $A \cap Q_y$ is either empty or a maximal antichain of Q_y.

(ii) $\{x \in P : A \cap Q_x \neq \emptyset\}$ is a maximal antichain of P.

Proof. Let A be a maximal antichain of $\sum_{x\in P} Q_x$. Let $y \in P$ such that $A \cap Q_y \neq \emptyset$. Then $A \cap Q_y$ is an antichain. Suppose $A \cap Q_y$ is not a maximal antichain of Q_y. Then there is some $z \in Q_y$ such that $z \notin (A \cap Q_y)\updownarrow$. Let $a \in A$ such that $z \in a\updownarrow$. Then $a \in Q_{y'}$ for some $y' \in y\updownarrow \setminus \{y\}$. But this is impossible since it would imply $A \cap Q_y \subseteq a\updownarrow$. Thus, $A \cap Q_y$ must be a maximal antichain of Q_y and so A satisfies (i).

Clearly, $\{x \in P : A \cap Q_x \neq \emptyset\}$ is an antichain in P, but suppose it is not a maximal antichain of P. Let $y \in P$ such that $y \notin \{x \in P : A \cap Q_x \neq \emptyset\}\updownarrow$. Then $Q_y \cap A\updownarrow = \emptyset$, impossible. Thus, $\{x \in P : A \cap Q_x \neq \emptyset\}$ must be a maximal antichain of P and so A satisfies (ii).

We now prove the converse. Let $A \subseteq \sum_{x\in P} Q_x$ satisfying (i) and (ii). First we prove that A is an antichain. Suppose that $z, z' \in A$ such that $z > z'$. Let $y, y' \in P$ be such that $z \in Q_y$ and $z' \in Q_{y'}$. By (i), it is impossible to have $y = y'$. So $y > y'$. But then y and y' are comparable elements of $\{x \in P : A \cap Q_x \neq \emptyset\}$, violating (ii). So A is an antichain. Now assume that A is not a maximal antichain. So there exists some $z \in (\sum_{x\in P} Q_x) \setminus A\updownarrow$. Let $y \in P$ such that $z \in Q_y$. By (ii), there is some $y' \in P$ such that $A \cap Q_{y'} \neq \emptyset$ and $y \in y'\updownarrow$. If $y' \neq y$, then $Q_y < Q_{y'}$ or $Q_y > Q_{y'}$, either of which yields $z \in Q_y \subset (A \cap Q_{y'})\updownarrow$, contradicting $z \notin A\updownarrow$. So $y' = y$. That is, $A \cap Q_y \neq \emptyset$. Since $z \notin (A \cap Q_y)\updownarrow$, $A \cap Q_y$ is not a maximal antichain of Q_y. But this violates (i). Thus, A must be a maximal antichain. $\square$

Lemma 2.2.2 *Let $C \subseteq \sum_{x\in P} Q_x$. C is a maximal chain of $\sum_{x\in P} Q_x$ if and only if both of the following are true:*

(i) for every $y \in P$, $C \cap Q_y$ is either empty or a maximal chain of Q_y.

(ii) $\{x \in P : C \cap Q_x \neq \emptyset\}$ is a maximal chain of P.

Proof. Similar to proof of Lemma 2.2.1. □

Theorem 2.2.3 *The poset $\sum_{x\in P} Q_x$ is hockey if and only if P and each of the components Q_x are hockey.*

Proof. Suppose Q_y is not hockey for some fixed $y \in P$. Let F_y be a fibre of Q_y which contains no maximal chain of Q_y. Let

$$F = F_y \cup \bigcup\{Q_x : x \in y\updownarrow \setminus \{y\}\}.$$

F is a fibre of $\sum_{x\in P} Q_x$ by the following argument. Let A be a maximal antichain of $\sum_{x\in P} Q_x$. By Lemma 2.2.1, if $A \cap Q_y \neq \emptyset$, then $A \cap Q_y$ is a maximal antichain of Q_y and so meets F_y and F. So suppose $A \cap Q_y = \emptyset$ and $q \in Q_y$. Let $a \in A \cap q\updownarrow$. Then $a \in Q_z$ for some $z \in y\updownarrow \setminus \{y\}$ and so $a \in F$. Thus, F is a fibre.

If $C \subset F$ were a maximal chain of $\sum_{x\in P} Q_x$, then by Lemma 2.2.2, as $C \cap Q_y \neq \emptyset$, $C \cap Q_y$ would be a maximal chain of Q_y contained in F_y, a contradiction. So F contains no maximal chain. This shows that if $\sum_{x\in P} Q_x$ is hockey, then so is each Q_x.

Suppose P is not hockey. Let S be a cutset of P which contains no maximal antichain of P. Let

$$K = \bigcup_{x\in S} Q_x.$$

K is a cutset of $\sum_{x\in P} Q_x$ by the following. Let C be a maximal chain of $\sum_{x\in P} Q_x$. By Lemma 2.2.2, $\{x \in P : C \cap Q_x \neq \emptyset\}$ is a maximal chain of P, and therefore meets S. Thus, C must meet K. However, by Lemma 2.2.1, K contains no maximal antichain of $\sum_{x\in P} Q_x$. This shows that if $\sum_{x\in P} Q_x$ is hockey, then so is P.

It remains to show that $\sum_{x\in P} Q_x$ is hockey if P and each of the components Q_x are hockey. Suppose then that P is hockey and Q_x is hockey for each $x \in P$. Let K be a cutset of $\sum_{x\in P} Q_x$. Define the subset S of P by

$$S = \{x \in P : K \cap Q_x \text{ is a cutset of } Q_x\}.$$

If S were not a cutset of P, we could fix $C' \subseteq P \setminus S$ such that C' is a maximal chain of P, and let $C_x \subseteq Q_x \setminus K$ be a maximal chain of Q_x for each $x \in C'$. By Lemma 2.2.2, $\bigcup_{x\in C'} C_x$ would be a maximal chain of $\sum_{x\in P} Q_x$ and it would be disjoint from K. Since this is impossible, S must be a cutset of P.

Since P is hockey, S contains a maximal antichain T of P. For each $x \in T$, $Q_x \cap K$ is a cutset of Q_x which is hockey, and so $Q_x \cap K$ contains a maximal antichain A_x of Q_x. By Lemma 2.2.1,

$$A = \bigcup_{x\in T} A_x$$

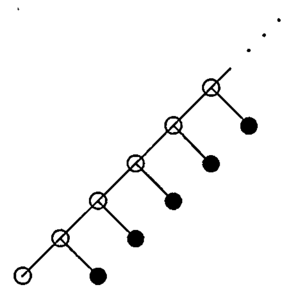
is a maximal antichain of $\sum_{x\in P} Q_x$ which is contained in K. □

Finite *series-parallel* posets are defined recursively as follows. **1** is series-parallel. If Q_0 and Q_1 are series-parallel then so are $Q_0 + Q_1$ and $Q_0 \oplus Q_1$. Only the posets which can be constructed in this manner are termed series-parallel.

Corollary 2.2.4 *Every finite series-parallel poset is hockey.*

Proof. The singleton **1** is series-parallel and hockey. If Q_0 and Q_1 are series-parallel and hockey, then $Q_0 + Q_1 = \sum_{x\in\bar{2}} Q_x$ and $Q_0 \oplus Q_1 = \sum_{x\in\mathbf{2}} Q_x$ are each hockey by Theorem 2.2.3 and series-parallel by definition. The result follows by induction. □

A finite poset is series-parallel if and only if it is fence-free [Ri86] [S73] [VTL82]. (Although everybody seems to think infinite series-parallel posets are a viable concept, there seems to be no widely accepted definition of them, or, for that matter, any definition of them anywhere in the literature.) One might wonder whether all fence-free posets are hockey. The answer is no. The poset shown below is fence-free but the hollow points form a cutset which contains no maximal antichain.



2.3 Products

For any posets P and Q, define the *direct product* $P \times Q$ to be the set of ordered pairs (p,q) with $p \in P$ and $q \in Q$ ordered by: $(p,q) \leq (p',q')$ if and only if $p \leq p'$ in P and $q \leq q'$ in Q. Note that, up to isomorphism, direct product is a commutative and associative binary operation on posets. For $S \subseteq P \times Q$, define $\pi_1(S) = \{p \in P : (p,q) \in S \text{ for some } q \in Q\}$ and $\pi_2(S) = \{q \in Q : (p,q) \in S \text{ for some } p \in P\}$. These maps are just the projections from $P \times Q$ to P and Q, respectively.

For any chain C in any poset P, a point x is said to be the *supremum* of C if $\{x\} = \min\{y \in P : y \geq c \text{ for every } c \in C\}$. We denote this by writing $x = \sup C$.

The *infimum* of C, denoted "$\inf C$", is defined dually. A chain need not have a supremum or infimum. But if a poset P is such that every chain does have a supremum and an infimum, then P is called *chain-complete.*

Lemma 2.3.1 *If Q is chain-complete and C is a maximal chain in $P \times Q$, then $\pi_1(C)$ is a maximal chain in P.*

Proof. Let C be a maximal chain of $P \times Q$. Let $C_1 = \pi_1(C)$. Obviously, C_1 is a chain. Is C_1 a maximal chain of P? Assume not; *i.e.* assume there exists $x \in P$ such that $C_1 \subseteq x\updownarrow\setminus\{x\}$. So $C \subseteq (x\mathring{\uparrow} \times Q) \cup (x\mathring{\downarrow} \times Q)$. Either $C \cap (x\mathring{\uparrow} \times Q) \neq \emptyset$ or $C \cap (x\mathring{\downarrow} \times Q) \neq \emptyset$. Assume the former without loss of generality. Let $C^* = C \cap (x\mathring{\uparrow} \times Q) \neq \emptyset$. Let $y = \inf(\pi_2(C^*))$. Is $C \cup \{(x, y)\}$ a chain? $(x, y) < (p, q)$ for every $(p, q) \in C^*$. If $(p, q) \in C\setminus C^*$ then $p < x$ and $q \leq \inf(\pi_2(C^*)) = y$. So $(p, q) < (x, y)$. So $C \cup \{(x, y)\}$ is a chain. But $(x, y) \notin C$ a maximal chain, a contradiction. So $\pi_1(C)$ is a maximal chain of P. □

Chain-completeness is necessary in Lemma 2.3.1. $C = \{(0,0), (0,1), (1,1), (1,2), (2,2), (2,3), \ldots\}$ is a maximal chain of $(\omega \oplus \{\infty\}) \times \omega$, but $\pi_1(C) = \omega$ is not a maximal chain of $\omega \oplus \{\infty\}$.

Lemma 2.3.2 *If A is a maximal antichain of $Q_1 \times Q_2$, where Q_1 is well-founded and Q_2 has a maximal element (not necessarily unique), then $\pi_1(A)$ contains a maximal antichain of Q_1.*

Proof. Let $y_0 \in \max Q_2$. Let A be a maximal antichain of $Q_1 \times Q_2$, $A_0 = \{(x, y) \in A : y \leq y_0\}$, and $A_1 = \pi_1(A_0)$. For any $(q_1, y_0) \in A$, $q_1 \in \min(A_1)$ since otherwise there is some $(q_1', y') \in A_0$ with $q_1' < q_1$ and $y' \leq y_0$, contradicting that A is an antichain. Obviously, $\min(A_1)$ is an antichain. In fact, it is a maximal antichain of Q_1 by the following. For each $q \in Q_1$, we can pick an $(x, y) \in A \cap (q, y_0)\updownarrow$. Then either $(x, y) \geq (q, y_0)$ or $(x, y) < (q, y_0)$. If $(x, y) \geq (q, y_0)$ then $y = y_0$, $x \in \min(A_1)$, and $x \geq q$. If $(x, y) < (q, y_0)$ then $x \in A_1$, so there is some $x' \in \min(A_1)$ such that $x' \leq x \leq q$. Therefore $Q_1 = (\min(A_1))\updownarrow$. $\square$

Theorem 2.3.3 *If the direct product $P_1 \times P_2$ is hockey where P_2 is chain-complete and P_1 is well-founded, then P_1 is hockey.*

Proof. Suppose $P_1 \times P_2$ is hockey. Let K_1 be a cutset of P_1. Then, by Lemma 2.3.1, $K_1 \times P_2$ is a cutset of $P_1 \times P_2$. So $K_1 \times P_2$ contains a maximal antichain of $P_1 \times P_2$, call it A. By Lemma 2.3.2, $\pi_1(A)$ contains a maximal antichain of P_1 which is obviously contained in K_1. So P_1 is hockey. $\square$

The converse of Theorem 2.3.3 is not true. That is, it is possible for P and Q to be hockey posets while $P \times Q$ is not hockey. The next theorem gives a whole class of posets whose products with anything chain-complete (other than an antichain) are not hockey. Notice that this class includes one of the simplest types of hockey posets — chains of height 3 or more; in particular, the simple product $\mathbf{3} \times \mathbf{2}$ of two hockey posets is not hockey.

Theorem 2.3.4 *Let P be a hockey poset containing a chain $a \succ b \succ c$ such that $a\updownarrow = b\downarrow$ and $c\updownarrow = b\uparrow$. Then for any chain-complete poset Q, $P \times Q$ is hockey if and only if Q is an antichain.*

Proof. If Q is an antichain, then $P \times Q \cong \sum_{q \in Q} P$ so $P \times Q$ is hockey by Theorem 2.2.3.

Now suppose that Q is not an antichain. We color the elements of P as follows. If $p \in b\updownarrow \setminus \{b\}$, then color p red. Otherwise, color p blue. So P contains no red maximal chain. Now, color each (p,q) in $P \times Q$ with the color of p in P. Then, by Lemma 2.3.1, $P \times Q$ contains no red maximal chain.

Suppose B is a blue maximal antichain of $P \times Q$. Let $m \in \min(Q)$ and $M \in \max(Q)$ such that $M > m$. Then B must include a blue point comparable to (c, M), say (p,q).

<u>CASE (I): $(p,q) \leq (c,M)$</u>$\Rightarrow p \leq c \Rightarrow p$ red. Impossible.

<u>CASE (II): $(p,q) > (c,M)$</u>$\Rightarrow q = M, p > c \Rightarrow p \geq b \Rightarrow p = b$ (since $p > b \Rightarrow (p,q)$ red).

So $(b, M) \in B$. Similarly, $(b, m) \in B$. But this is impossible since B is an antichain. So $P \times Q$ does not have a blue maximal antichain. $\square$

Notice that the hypothesis for P in Theorem 2.3.4 applies to any chain-finite poset P which has an element b not in $\max(P) \cup \min(P)$ such that $b\updownarrow = P$. The next two theorems defy the general difficulty of finding hockey products. However, the difficulty of finding hockey products is evidenced to some extent by the fact that all the hockey poset products mentioned in this thesis have only factors of height 2.

Theorem 2.3.5 *If P is a hockey poset of height two, then $P \times \mathbf{2}$ is hockey.*

Proof. Let I be the set of isolated points of P; that is, $I = \{x \in P : x\updownarrow = \{x\}\}$. Then $P \times \mathbf{2} = ((P \smallsetminus I) + I) \times \mathbf{2} = ((P \smallsetminus I) \times \mathbf{2}) + (I \times \mathbf{2}) = ((P \smallsetminus I) \times \mathbf{2}) + \sum_{x \in I} \mathbf{2}$. Because of Theorem 2.2.3 and the fact that chains (in particular, $\mathbf{2}$) and antichains are hockey, $P \times \mathbf{2}$ is hockey if and only if $(P \smallsetminus I) \times \mathbf{2}$ is hockey. Therefore, it suffices to prove the theorem for posets with no isolated points. So assume that P has no isolated points.

Let $E = \{(x,0) \in P \times \mathbf{2} : x \in \min(P)\} \cup \{(y,1) \in P \times \mathbf{2} : y \in \max(P)\}$. Fix a coloring of $P \times \mathbf{2}$ so that it has no red maximal chain. Let R denote the set of red points, and B the set of blue points. Let $R' = \{r \in R \cap E : r\updownarrow \cap R \cap E = \{r\}\}$. No two points in R' are comparable, so R' contains no maximal chain of E. Since $E \cong P$ and P is hockey, this tells us that R' is not a fibre and so there exists, A, a maximal antichain of E, such that $A \cap R' = \emptyset$. Let $B_1 = A \cap (R'\updownarrow)$. So $B_1 \subseteq E$ and B_1 is a blue antichain with $R' \subseteq B_1\updownarrow$. Let B_2 be an antichain maximal in $B \cap E$ with $B_1 \subseteq B_2$. Let $B_3 = B_2 \cup (B \setminus B_2\updownarrow) \subseteq B_2 \cup ((P \times \mathbf{2}) \setminus E)$. Then B_3 is a blue antichain and $B \subseteq B_3\updownarrow$. Furthermore, by the following, $R \subseteq B_3\updownarrow$, and therefore B_3 is a maximal antichain.

We know that $R' \subseteq B_1\updownarrow \subseteq B_2\updownarrow \subseteq B_3\updownarrow$.

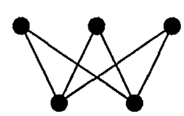
Suppose $(x,0) \in E \cap R \setminus R'$. Then $x \in \min P$ and there exists $y \in \max P$ such that $(y,1) > (x,0)$ and $(y,1) \in R \cap E$. Since $\{(x,0),(x,1),(y,1)\}$ is a maximal chain, and $P \times \mathbf{2}$ has no red maximal chain, $(x,1)$ must be blue. So $(x,1) \in B_3\updownarrow$. In fact, since the only point less than $(x,1)$ is $(x,0)$ which is red, we know that $(x,1) \in B_3\downarrow$. But then $(x,0)$ must be in $B_3\downarrow$ also. Similarly, any $(z,1) \in E \cap R \setminus R'$ must be in $B_3\uparrow$. Therefore, $E \cap R \setminus R' \subset B_3\updownarrow$. Since $R' \subset B_3\updownarrow$, it remains only to show that $R \setminus E \subset B_3\updownarrow$.

Suppose $(x,1) \in R \setminus E$, so $x \in \min P$. Now, $\{(x,0),(x,1),(y,1)\}$ is a maximal chain of $P \times \mathbf{2}$ for every $y > x$ in P. Therefore, if $(x,0)$ is red, then $(y,1)$ is blue for every $y > x$ and $(x,0) \in R' \subseteq B_1\updownarrow$. If this were the case, then the element of B_1 comparable to $(x,0)$ would have to be some $(y,1)$ such that $(y,1) > (x,0)$, but then $(y,1) > (x,1)$ also and thus $(x,1) \in B_1{\updownarrow} \subseteq B_2{\updownarrow} \subseteq B_3{\updownarrow}$. So assume $(x,0)$ is blue. Then $(x,0) \in B_2{\updownarrow}$. If $(x,0) \in B_2$ then $(x,1) \in B_2{\uparrow}$ and if $(y,1) \in B_2$ such that $(y,1) > (x,0)$ then $(x,1) \in (y,1){\downarrow} \subseteq B_2{\downarrow}$. Therefore, $R \subseteq B_3{\updownarrow}$. □

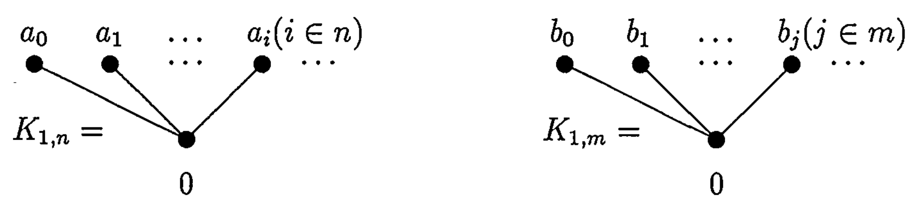
It may be possible to generalise this result as indicated in Question 2.3.6. A positive answer to Question 2.3.6 would provide more evidence in favor of a positive answer to Question 2.3.9.

Question 2.3.6 *Is $P \times \mathbf{2}^n$ hockey for every hockey poset P of height 2 and every $n \in \mathbf{N}$?*

Define $K_{r,s} = \overline{r} \oplus \overline{s}$, which is hockey by Corollary 2.2.4. The Hasse diagram of $K_{2,3}$, for instance, is:



Theorem 2.3.7 *$K_{1,n} \times K_{1,m}$ is hockey for any cardinals m, n.*



Proof. Label the elements of $K_{1,n}$ and $K_{1,m}$ as above. Let $P = K_{1,n} \times K_{1,m}$. Let $A = \{(a_i, 0) : i \in n\}$ and $B = \{(0, b_j) : j \in m\}$. Notice that $A \cup B$ is a maximal antichain. Let F be a fibre of P. So $(0,0) \in F$ since $\{(0,0)\}$ is a maximal antichain of P. If we can show that F contains a maximal chain, then P is hockey.

<u>Case 1: $F \cap A \neq \emptyset$, $F \cap B \neq \emptyset$.</u>

In this case, let $M = ((A \cup B) \setminus F) \cup \{(a_i, b_j) : (a_i, 0) \in F, (0, b_j) \in F\}$. Since M is a maximal antichain, we know that $M \cap F \neq \emptyset$. Let $x \in M \cap F$. So x is of the form $(a_{i'}, b_{j'})$. But then $(a_{i'}, 0) \in F$. So $\{(0,0), (a_{i'}, 0), (a_{i'}, b_{j'})\}$ is a maximal chain contained in F.

<u>Case 2: $B \cap F = \emptyset$.</u>

Let $M = (A \setminus F) \cup \{(a_i, b_0) : (a_i, 0) \in F\} \cup (B \setminus \{(0, b_0)\})$. Since M is a maximal antichain, we know that $M \cap F \neq \emptyset$. Let $x \in M \cap F$. So x is of the form $(a_{i'}, b_0)$. But then $(a_{i'}, 0) \in F$. So $\{(0,0), (a_{i'}, 0), (a_{i'}, b_0)\}$ is a maximal chain contained in F.

<u>Case 3: $A \cap F = \emptyset$.</u> This is analogous to Case 2.

So F contains a maximal chain. Hence, P is hockey. $\square$

The previous two theorems show some cases where products of hockey height-2 posets are hockey. One might start wondering if the product of any two hockey height-2 posets is hockey. But this turns out not to be true. For a counterexample, we need look no further than the simplest product we have not yet classified: $K_{1,2} \times K_{2,1}$. The fact that this product is not hockey is a corollary of the following theorem.

Theorem 2.3.8 *Let P be a poset with $p_1, p_2 \in \min(P) \setminus \max(P)$, $p_1 \neq p_2$, and $p_1\updownarrow \subseteq p_2\uparrow$. Let Q be a poset with $q_1, q_2 \in \max(Q) \setminus \min(Q)$, $q_1 \neq q_2$, and $q_1\updownarrow \subseteq q_2\downarrow$. If P and Q are chain-finite, then $P \times Q$ is not hockey.*

Proof. Color (p_1, q_1) blue. For every other $(p,q) \in P \times Q$, color (p,q) blue unless $p = p_1$ or $q = q_1$. In those cases color (p,q) red. That is, the set of red points is $((\{p_1\} \times Q) \cup (P \times \{q_1\})) \setminus \{(p_1, q_1)\}$.

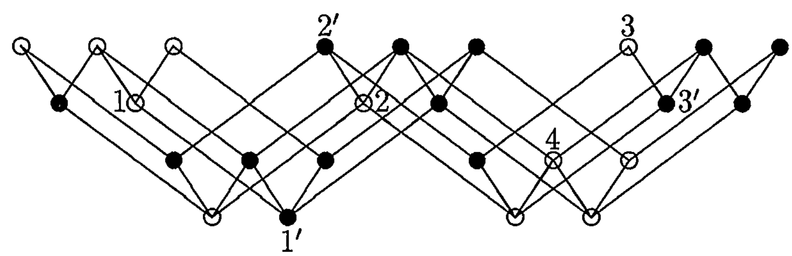
Does $P \times Q$ have a red maximal chain? Suppose it does, call it R. Each point in R has p_1 or q_1 as a coordinate. Each point of R having q_1 as a coordinate is greater than each point of R having p_1 as a coordinate. Let (p, q_1) be the least point of R having q_1 as a coordinate. Let (p_1, q) be the greatest point of R having p_1 as a coordinate. So $(p, q_1) > (p, q) > (p_1, q)$. But then $R \cup \{(p,q)\}$ is a chain. This contradicts our assumption that R is a maximal chain since $(p,q) \notin R$. So $P \times Q$ does not have a red maximal chain.

Does $P \times Q$ have a blue maximal antichain? Suppose it does. Call it B. B must include a point comparable to or equal to (p_2, q_1). But B cannot include (p_2, q_1) itself since it is red. And B cannot include a point greater than (p_2, q_1) since $(p,q) > (p_2, q_1) \Rightarrow q = q_1, p \neq p_1 \Rightarrow (p,q)$ red. So if $(p,q) \in B$ is comparable to (p_2, q_1), then $(p,q) < (p_2, q_1)$. So $p = p_2$, $q < q_1$. That is, $(p_2, q) \in B$ for some $q < q_1$.

Similarly, B must include a point (p', q_2) with $p' > p_1$. By hypothesis, $p' > p_1 \Rightarrow p' > p_2$, and $q < q_1 \Rightarrow q \not< q_2$. So $(p', q_2) > (p_2, q_2) > (p_2, q)$. This is a contradiction since (p_2, q) and (p', q_2) are in B, an antichain. □

Remark Theorems 2.3.5, 2.3.7, and 2.3.8 show that $K_{n,m} \times K_{n',m'}$ is not hockey if and only if $n, m' > 1$ or (symmetrically) $n', m > 1$.

There may be more interesting theorems like 2.3.5, 2.3.7, and 2.3.8 which apply to products of height-2 posets. For instance, the diagram below shows that the direct product of the five-element fence ($x_1 < x_2 > x_3 < x_4 > x_5$) with itself is not hockey, and this result is not covered by Theorems 2.3.5, 2.3.7, and 2.3.8. The points marked with hollow circles obviously do not yield any maximal chains, and they form a fibre by the following. Call the set of hollow points F. Suppose A is a maximal antichain disjoint from F. $1\updownarrow\setminus F = \{1'\}$ so $1' \in A$. Then $2\updownarrow\setminus(F \cup A\updownarrow) \subseteq \{2'\}$ so $2' \in A$, thus $3\updownarrow\setminus(F \cup A\updownarrow) \subseteq \{3'\}$ so $3' \in A$. This makes $4\updownarrow\setminus(F \cup A\updownarrow) = \emptyset$. Therefore, it is impossible for a maximal antichain to be disjoint from F and so F is a fibre.



Question 2.3.9 *Is it true that for any posets P_1, P_2, P_3 such that $P_i \times P_j$ is hockey for all distinct i, j in $\{1, 2, 3\}$, $P_1 \times P_2 \times P_3$ is hockey also?*

We have been unable to find a counterexample. But, by induction, if the answer is yes, then it is yes for the product of any finite number of posets, not just 3. Furthermore, if the answer to this question is yes, then so are the answers to Questions 2.3.6 and 2.4.1. In Section 2.5, we get a positive answer to Question 2.3.9 for finite posets representable as powers of **2** (*i.e.* distributive lattices).

Question 2.3.10 *If Q is not an antichain and $P \times Q$ is hockey, is $P \times \mathbf{2}$ necessarily hockey?*

2.4 Products of Claws

Question 2.4.1 *Is $K_{1,m_1} \times \ldots \times K_{1,m_k}$ hockey for every $m_1, \ldots, m_k \in \mathbf{N}$?*

Corollary 2.2.4 and Theorem 2.3.7 show that the answer to Question 2.4.1 is yes if $k \leq 2$. Duffus, Sands, and Winkler [DSW90] showed that the answer is yes when $m_1 = \ldots = m_k = 1$ (*i.e.* for $\mathbf{2} \times \ldots \times \mathbf{2}$). We will now prove positive answers in two more special cases. We abbreviate $K_{1,1} \times \ldots \times K_{1,1}$ (r times) by $(K_{1,1})^r$. In the first case, we adapt the method of [DSW90] to show that two of the factors of $(K_{1,1})^r$ may be replaced by $K_{1,m} \times K_{1,n}$.

Theorem 2.4.2 *Let r, m, and n be natural numbers. Then $(K_{1,1})^r \times K_{1,m} \times K_{1,n}$ is hockey.*

Proof. Assume for a contradiction that the theorem is false. Then there is some $(r, m, n) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N}$ for which the theorem fails. Pick (r, m, n) minimal in $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$ so that $(K_{1,1})^r \times K_{1,m} \times K_{1,n}$ is not hockey. Since direct product is commutative, we may assume without loss of generality that $n \geq m$.

Let $s = r + 1$ and $t = r + 2$. Let $E_1 = \{1\}$, $E_2 = \{2\}$, ..., $E_r = \{r\}$, $E_s = \{s_1, \ldots, s_m\}$, and $E_t = \{t_1, \ldots, t_n\}$. Define $\mathcal{P} \cong (K_{1,1})^r \times K_{1,m} \times K_{1,n}$ by

$$\mathcal{P} = \{X \subseteq \bigcup_{i=1}^{t} E_i : |X \cap E_i| \leq 1 \text{ for } i = 1, \ldots, t\}$$

ordered by set containment.

Since $\mathcal{P}$ is not hockey, it has a fibre $\mathcal{F}$ which contains no maximal chain. $\emptyset \in \mathcal{F}$ since $\{\emptyset\}$ is a maximal antichain. Furthermore, if $m > 1$ then $\{s_i\} \in \mathcal{F}$ for each $i \in \{1, \ldots, m\}$ by the following argument. Suppose $m > 1$, $i \in \{1, \ldots, m\}$, and $\{s_i\} \notin \mathcal{F}$. Let $\mathcal{P}' = \mathcal{P} \setminus \{s_i\}\uparrow$. So $\mathcal{P}' \cong (K_{1,1})^r \times K_{1,m-1} \times K_{1,n}$. $\mathcal{P}'$ is hockey by the minimality of (r, m, n). $\mathcal{F} \cap \mathcal{P}'$ is a fibre of $\mathcal{P}'$ since if $\mathcal{A}$ is a maximal antichain of $\mathcal{P}'$ disjoint from $\mathcal{F}$, then $\mathcal{A} \cup \{\{s_i\}\}$ is a maximal antichain of $\mathcal{P}$ and $(\mathcal{A} \cup \{\{s_i\}\}) \cap \mathcal{F} = \emptyset$, impossible. So there exists $\mathcal{C} \subseteq \mathcal{P}' \cap \mathcal{F}$ a maximal chain of $\mathcal{P}'$. But then $\mathcal{C} \subseteq \mathcal{F}$ is a maximal chain of $\mathcal{P}$, a contradiction. Thus, if $m > 1$ then $\{s_i\} \in \mathcal{F}$ for each $i = 1, \ldots, m$. Similarly, if $n > 1$ then $\{t_i\} \in \mathcal{F}$ for each $i = 1, \ldots, n$.

We now proceed with the method of [DSW90]. We define sets $X\updownarrow$ analogous to the "lexical chains" used in [DSW90]. Let $X \in \mathcal{P}$. Define

$$X\downarrow = \{X, X \setminus E_1, X \setminus (E_1 \cup E_2), \ldots, X \setminus (E_1 \cup E_2 \cup \ldots \cup E_t) = \emptyset\},$$

$$X\uparrow = \{X, X \cup E_1, X \cup E_1 \cup E_2, \ldots, X \cup E_1 \cup E_2 \cup \ldots \cup E_r\} \cup$$

$$\{X \cup E_1 \cup \ldots \cup E_r \cup \{s_i\} \in \mathcal{P} : 1 \leq i \leq m\} \cup$$

$$\{X \cup E_1 \cup \ldots \cup E_r \cup \{s_i\} \cup \{t_j\} \in \mathcal{P} : 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Put $X\updownarrow = X\uparrow \cup X\downarrow$, so every $X\updownarrow$ is a union of maximal chains of $\mathcal{P}$. For all $\mathcal{X} \subseteq \mathcal{P}$, define $\mathcal{X}\uparrow = \bigcup_{X \in \mathcal{X}} X\uparrow$, $\mathcal{X}\downarrow = \bigcup_{X \in \mathcal{X}} X\downarrow$, and $\mathcal{X}\updownarrow = \mathcal{X}\uparrow \cup \mathcal{X}\downarrow$.

For any $\mathcal{S} \subseteq \mathcal{F}$, call $\mathcal{S}$ *critical* if there do not exist $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}$ such that

(1) $\mathcal{A} \cup \mathcal{B}$ is an antichain disjoint from $\mathcal{F}$;

(2) $\mathcal{S} \subseteq \mathcal{A}\downarrow \cup \mathcal{B}\uparrow$;

(3) $\mathcal{A} \subseteq \mathcal{S}\uparrow$, $\mathcal{B} \subseteq \mathcal{S}\downarrow$.

Since $\mathcal{F}$ is a fibre, no $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}$ can satisfy (1) and (2) for $\mathcal{S} = \mathcal{F}$, so $\mathcal{F}$ is critical. $\emptyset$ is not critical since $\mathcal{A} = \mathcal{B} = \emptyset$ satisfy (1)–(3) for $\mathcal{S} = \emptyset$. Since $\mathcal{F}$ is finite, $\mathcal{F}$ must contain a minimal critical set $\mathcal{M}$. That is, $\mathcal{M}$ is critical but no proper subset of $\mathcal{M}$ is critical.

For each $X \in \mathcal{M}$ and each $Y \in X{\Uparrow}\smallsetminus\mathcal{F}$, define $\operatorname{rank}(X,Y)$ to be the least i such that $Y \subseteq X \cup E_1 \cup \ldots \cup E_i$. For each $X \in \mathcal{M}$ and each $Y \in X{\Downarrow}\smallsetminus\mathcal{F}$, define $\operatorname{rank}(X,Y)$ to be the least i such that $Y = X\smallsetminus(E_1 \cup E_2 \cup \ldots \cup E_i)$. For each $X \in \mathcal{M}$, define $\operatorname{rank}(X) = \min\{\operatorname{rank}(X,Y) : Y \in X{\Updownarrow}\smallsetminus\mathcal{F}\}$ — we know that $X{\Updownarrow}\smallsetminus\mathcal{F} \neq \emptyset$ since $X{\Updownarrow}$ is a union of maximal chains of $\mathcal{P}$, and $\mathcal{F}$ contains no maximal chain of $\mathcal{P}$.

Let $M \in \mathcal{M}$ be such that $\operatorname{rank}(M) \leq \operatorname{rank}(X)$ for every $X \in \mathcal{M}$. Let $M' \in M{\Updownarrow}\smallsetminus M$ be such that $\operatorname{rank}(M, M') = \operatorname{rank}(M)$. Since $\mathcal{M}\smallsetminus\{M\}$ is not critical, we can pick $\mathcal{A}, \mathcal{B}$ satisfying conditions (1)–(3) for $\mathcal{S} = \mathcal{M}\smallsetminus\{M\}$. Then $\mathcal{A}, \mathcal{B}$ satisfy (1) and (3) for $\mathcal{S} = \mathcal{M}$ also. $\mathcal{A}, \mathcal{B}$ cannot also satisfy (2) since $\mathcal{M}$ is critical, so $M \notin \mathcal{A}{\downarrow} \cup \mathcal{B}{\uparrow}$.

There are two cases to consider: $M \subset M'$ and $M' \subset M$.

First suppose $M \subset M'$. Let $\mathcal{A}' = (\mathcal{A}\smallsetminus M'{\downarrow}) \cup \{M'\}$. We now derive a contradiction by showing that $\mathcal{A}', \mathcal{B}$ satisfy (1)–(3) for $\mathcal{S} = \mathcal{M}$. Since $M' \in \mathcal{M}{\Uparrow}$ and $\mathcal{A}'\smallsetminus\{M'\} \subseteq \mathcal{A} \subseteq \mathcal{M}{\Uparrow}$, we know that $\mathcal{A}' \subseteq \mathcal{M}{\Uparrow}$. And we already knew that $\mathcal{B} \subseteq \mathcal{M}{\Downarrow}$, so (3) is satisfied.

$\mathcal{A}'{\downarrow} \cup \mathcal{B}{\uparrow} = ((\mathcal{A}\smallsetminus M'{\downarrow}) \cup \{M'\}){\downarrow} \cup \mathcal{B}{\uparrow} \supseteq \mathcal{A}{\downarrow} \cup \mathcal{B}{\uparrow} \supseteq \mathcal{M}\smallsetminus\{M\}$, and $M \in M'{\downarrow} \subseteq \mathcal{A}'{\downarrow}$. So $\mathcal{M} \subseteq \mathcal{A}'{\downarrow} \cup \mathcal{B}{\uparrow}$. That is, (2) is satisfied.

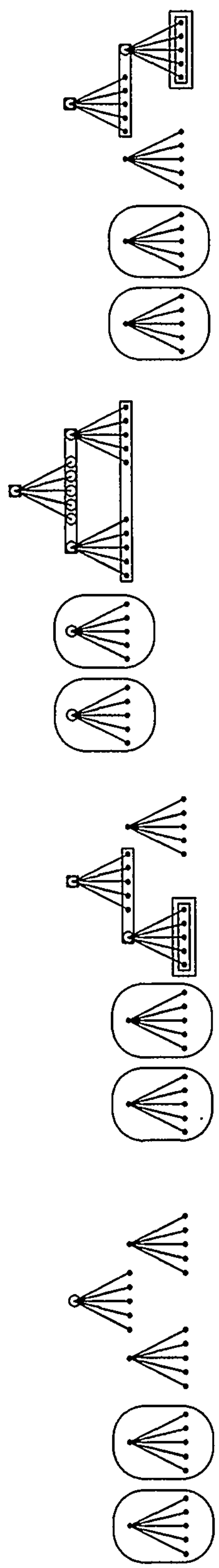
It is obvious that $\mathcal{A}' \cup \mathcal{B}$ is disjoint from $\mathcal{F}$. Since $\mathcal{A} \cup \mathcal{B}$ is an antichain, we only need to verify that $M' \notin (\mathcal{A}'\smallsetminus\{M'\}){\updownarrow}$ and $M' \notin \mathcal{B}{\updownarrow}$ to show that (1) is satisfied. Since $M \subset M'$ and $M \notin \mathcal{A}{\downarrow}$, we know that $M' \notin \mathcal{A}{\downarrow} \supset (\mathcal{A}'\smallsetminus\{M'\}){\downarrow}$. Since $\mathcal{A}'\smallsetminus\{M'\} = \mathcal{A}\smallsetminus\{M'\}{\downarrow}$, obviously $M' \notin (\mathcal{A}'\smallsetminus\{M'\}){\uparrow}$. So $\mathcal{A}'$ is an antichain. Why is $M' \in \mathcal{B}{\updownarrow}$ impossible? Let $B \in \mathcal{B}$. $B \in \mathcal{M}{\Downarrow}\smallsetminus\mathcal{M}$ and so $B \cap E_1 = \emptyset$. But $M' \in M{\Uparrow}\smallsetminus\{M\}$ and so $|M' \cap E_1| = 1$. Hence $M' \not\subseteq B$. Since there exists $B' \in \mathcal{M}$ such that $B \in B'{\Downarrow}$ and $\operatorname{rank}(B') \geq \operatorname{rank}(M)$, we know that $B \cap (E_1 \cup \ldots \cup E_{\operatorname{rank}(M)}) = \emptyset$.

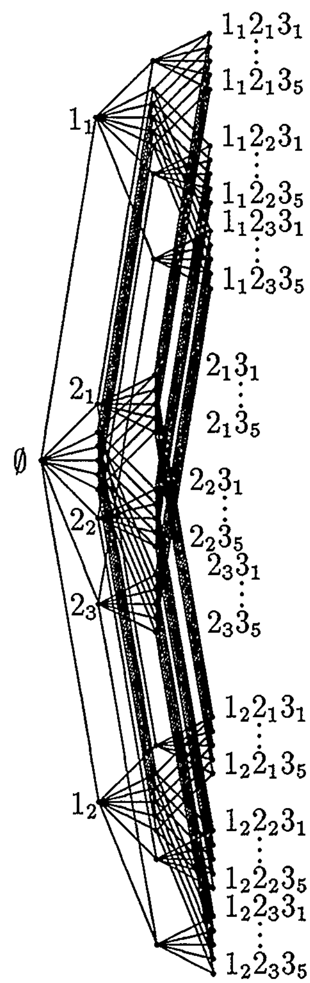
So $B \setminus M' \supseteq B \setminus (M \cup E_1 \cup E_2 \cup \ldots \cup E_{\text{rank}(M)}) = (B \setminus (E_1 \cup E_2 \cup \ldots \cup E_{\text{rank}(M)})) \setminus M = B \setminus M$. Since $M \notin \mathcal{B}\uparrow$, this says $B \setminus M' \supseteq B \setminus M \neq \emptyset$. So $B \not\subseteq M'$ and thus $M' \notin \mathcal{B}\updownarrow$. But then $\mathcal{A}', \mathcal{B}$ satisfy (1)–(3) for $\mathcal{S} = \mathcal{M}$, a contradiction. So the case $M \subset M'$ cannot occur.

Now suppose that $M' \subset M$. So $M' = M \setminus (E_1 \cup E_2 \cup \ldots \cup E_{\text{rank}(M)})$. Let $\mathcal{B}' = (\mathcal{B} \setminus M'\uparrow) \cup \{M'\}$. Let $\mathcal{S} = \mathcal{M}$. Then, dually to the case $M \subset M'$, (2) and (3) are satisfied by $\mathcal{A}, \mathcal{B}'$. That $\mathcal{A} \cup \mathcal{B}'$ is disjoint from $\mathcal{F}$ and $\mathcal{B}' \cap M'\updownarrow = \{M'\}$ are also dual to facts in the case $M \subset M'$. But to show that $\mathcal{A} \cap M'\updownarrow = \emptyset$ and therefore (1) is satisfied requires more work in this case. Let $A \in \mathcal{A}$. $A \in \mathcal{M}\Uparrow \setminus \mathcal{M}$ and so $|A \cap E_1| = 1$. But $M' \in M\downarrow \setminus \{M\}$ and so $M' \cap E_1 = \emptyset$. Hence $A \not\subseteq M'$. We know that $M \not\subset A$. Assume $M' \subset A$. Then $(M \setminus M') \setminus A = M \setminus (M' \cup A) = M \setminus A \neq \emptyset$. $M \setminus M' \subseteq E_1 \cup \ldots \cup E_{\text{rank}(M)}$. So $(E_1 \cup E_2 \cup \ldots \cup E_{\text{rank}(M)}) \setminus A \neq \emptyset$. But since there exists $A' \in \mathcal{M}$ such that $A \in A'\Uparrow$ and $\text{rank}(A') \geq \text{rank}(M)$, we know that $|A \cap E_1| = 1$, $|A \cap E_2| = 1$, …, $|A \cap E_{\text{rank}(M)}| = 1$. Therefore, there must be some i such that $1 \leq i \leq \text{rank}(M)$ for which $|E_i| > 1$. Since $|E_1| = |E_2| = \ldots = |E_r| = 1$, this tells us that $\text{rank}(M) \geq i \geq r + 1$. Since $M' = M \setminus (E_1 \cup \ldots \cup E_{\text{rank}(M)})$, this tells us that M' either is a singleton subset of $E_{r+2} = E_t$ or is $\emptyset$. Since $\emptyset \in \mathcal{F}$ and $M' \notin \mathcal{F}$, M' must be a singleton subset of E_t and $\text{rank}(M)$ must be $i = s = t - 1$. So $|E_t| = n \geq m = |E_s| > 1$. But recall the first result following from the minimality of (r, m, n): if $n = |E_t| > 1$ then every singleton subset of E_t is in $\mathcal{F}$. So $M' \in \mathcal{F}$, a contradiction. So the case $M' \subset M$ cannot occur.

With this contradiction, we have proven that $\mathcal{F}$ contains a maximal chain. $\square$

Theorem 2.4.3 *Let l, m, and n be natural numbers. Then $K_{1,l} \times K_{1,m} \times K_{1,n}$ is hockey.*





This side up $\longrightarrow$

Proof. Assume for a contradiction that the theorem is false. Then there is some $(l,m,n) \in \mathbf{N} \times \mathbf{N} \times \mathbf{N}$ for which the theorem fails. Pick (l,m,n) minimal in $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$ so that $K_{1,l} \times K_{1,m} \times K_{1,n}$ is not hockey. We know that $l,m,n > 1$ since otherwise $K_{1,l} \times K_{1,m} \times K_{1,n}$ is hockey by Theorem 2.4.2.

Let $E_1 = \{1_1, 1_2, \ldots, 1_l\}$, $E_2 = \{2_1, 2_2, \ldots, 2_m\}$, and $E_3 = \{3_1, 3_2, \ldots, 3_n\}$. Let $\mathcal{P} = \{X \subseteq \bigcup_{i=1}^{3} E_i : |X \cap E_i| \leq 1 \text{ for } i = 1,2,3\}$. Order $\mathcal{P}$ by set containment. Then $\mathcal{P} \cong K_{1,l} \times K_{1,m} \times K_{1,n}$. The two diagrams on the previous page are the Hasse diagram of $K_{1,2} \times K_{1,3} \times K_{1,5}$, and a less cluttered diagram of $K_{1,3} \times K_{1,4} \times K_{1,5}$, graphically embellished to illuminate the proof. We will abbreviate set notation by omitting commas and parentheses. For instance, $1_1 2_1$ will stand for $\{1_1, 2_1\}$.

Let $\mathcal{F}$ be a fibre of $\mathcal{P}$ which contains no maximal chain of $\mathcal{P}$. Then $\emptyset \in \mathcal{F}$ since $\{\emptyset\}$ is a maximal antichain of $\mathcal{P}$. As in the proof of Theorem 2.4.2, the minimality of (l,m,n) tells us that $\mathcal{F}$ must include every singleton in $\mathcal{P}$ by the following. $\mathcal{P} \setminus (1_1\uparrow) \cong K_{1,l-1} \times K_{1,m} \times K_{1,n}$ is hockey by the minimality of (l,m,n). So if $\mathcal{F} \setminus (1_1\uparrow)$ is a fibre of $\mathcal{P} \setminus (1_1\uparrow)$, then it contains a maximal chain of $\mathcal{P}$, contradicting our assumption that $\mathcal{F}$ contains no maximal chain of $\mathcal{P}$. Thus, there is some $\mathcal{A}$ a maximal antichain of $\mathcal{P} \setminus (1_1\uparrow)$ which is disjoint from $\mathcal{F}$. But then $\mathcal{A} \cup \{1_1\}$ is a maximal antichain of $\mathcal{P}$ and so 1_1 must be in $\mathcal{F}$. By symmetry, every singleton in $\mathcal{P}$ must be in $\mathcal{F}$.

Since the set of all doubletons in $\mathcal{P}$ is a maximal antichain, one of them must be in $\mathcal{F}$. Assume without loss of generality that $1_1 2_1$ is in $\mathcal{F}$. In the diagram, the points marked with hollow circles are some points of a particular choice of $\mathcal{F}$ for $K_{1,3} \times K_{1,4} \times K_{1,5}$.

We will now construct a maximal antichain of $\mathcal{P}$ disjoint from $\mathcal{F}$. Make the following definitions.

$$\mathcal{A}_1 = \{1_i 2_j 3_k \in \mathcal{P} : 1_i 2_j \in \mathcal{F}\},\ \mathcal{P}_1 = \mathcal{A}_1\downarrow,$$
$$\mathcal{B}_1 = \{2_j : 2_j \notin \mathcal{A}_1\downarrow\},\ \mathcal{P}_2 = \mathcal{B}_1\uparrow,$$
$$\mathcal{P}_3 = \mathcal{P} \setminus (\mathcal{P}_1 \cup \mathcal{P}_2).$$

It is easy to see that $\{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3\}$ is a partition of $\mathcal{P}$. $\mathcal{P}_2 \cong \sum_{\bar{b}}(K_{1,l} \times K_{1,n})$, where $b = |\mathcal{B}_1|$. In the diagram, the points in $\mathcal{A}_1$ are surrounded by double-walled rectangles. The points in $\mathcal{P}_1$ are surrounded by rectangles. The points in $\mathcal{P}_2$ are surrounded by ovals.

Obviously, $\mathcal{A}_1$ is an antichain and $\mathcal{P}_1 = \mathcal{A}_1\updownarrow$. $\mathcal{A}_1$ is disjoint from $\mathcal{F}$ since $\emptyset$ and all the singletons are in $\mathcal{F}$, and $\mathcal{F}$ contains no maximal chain, so each $1_i 2_j \in \mathcal{F}$ implies $1_i 2_j 3_k \notin \mathcal{F}$ for $k = 1, \ldots, n$.

Next, we find an analogous antichain $\mathcal{A}_2$ in $\mathcal{P}_2$. That is, we will find an antichain $\mathcal{A}_2$ disjoint from $\mathcal{F}$ such that $\mathcal{P}_2 \subseteq \mathcal{A}_2\updownarrow$. To do this, we shall break down $\mathcal{P}_2$ into smaller pieces. For each $2_j \in \mathcal{P}_2$, define $\mathcal{P}_{2,j} = 2_j\uparrow$. Then $\mathcal{P}_2 = \bigcup_j \mathcal{P}_{2,j}$. Choose a particular $\mathcal{P}_{2,j}$. We want to find an antichain $\mathcal{A}_{2,j} \subset \mathcal{P}_{2,j}$ such that $\mathcal{P}_{2,j} \subseteq \mathcal{A}_{2,j}\updownarrow$ and $\mathcal{A}_{2,j}$ is disjoint from $\mathcal{F}$. $\{2_j\}$ is not a satisfactory choice for $\mathcal{A}_{2,j}$ since $2_j \in \mathcal{F}$ (remember that all singletons are in $\mathcal{F}$). The next obvious choice to check is the set of all doubletons in $\mathcal{P}_{2,j}$. We know that each $1_i 2_j \notin \mathcal{F}$ since otherwise we would have $2_j \in \mathcal{P}_1$. Unfortunately, there is no guarantee that every $2_j 3_k \notin \mathcal{F}$. But we will make this choice whenever possible; that is, if $\{2_j 3_k \in \mathcal{F}\} = \emptyset$ then let

$$\mathcal{A}_{2,j} = \{1_i 2_j \in \mathcal{P}\} \cup \{2_j 3_k \in \mathcal{P}\}.$$

When $\{2_j 3_k \in \mathcal{F}\} \neq \emptyset$, we will choose $\mathcal{A}_{2,j}$ as close as possible to the choice just described. We will modify the choice by replacing $2_j 3_k$ by $1_1 2_j 3_k$ for each $2_j 3_k \in \mathcal{F}$. Since $\mathcal{F}$ contains no maximal chain of $\mathcal{P}$, we know that $1_1 2_j 3_k \notin \mathcal{F}$

whenever $2_j3_k \in \mathcal{F}$. This choice necessitates dropping 1_12_j from $\mathcal{A}_{2,j}$ to keep it an antichain. To put this in the proper notation, if $\{2_j3_k \in \mathcal{F}\} \neq \emptyset$, then let

$$\mathcal{A}_{2,j} = \{1_12_j3_k \in \mathcal{P} : 2_j3_k \in \mathcal{F}\} \cup \{2_j3_k \in \mathcal{P} : 2_j3_k \notin \mathcal{F}\} \cup \{1_i2_j \in \mathcal{P} : i > 1\}.$$

By either definition, $\mathcal{A}_{2,j}$ is an antichain disjoint from $\mathcal{F}$, and $2_j\!\uparrow \subseteq \mathcal{A}_{2,j}\!\updownarrow$. We have just described the choice of a particular $\mathcal{A}_{2,j}$. Apply the same method for every j for which $\mathcal{P}_{2,j}$ is defined. Then let $\mathcal{A}_2$ be the union of the $\mathcal{A}_{2,j}$'s. $\mathcal{A}_2$ is an antichain since every element of any $\mathcal{A}_{2,j}$ includes 2_j and no $2_{j'}$ for any $j' \neq j$. Thus $\mathcal{A}_2$ is an antichain disjoint from $\mathcal{F}$ and $\mathcal{P}_2 \subseteq \mathcal{A}_2\!\updownarrow$. In fact, $\mathcal{A}_1 \cup \mathcal{A}_2$ is an antichain since $\mathcal{A}_1 \subset \max \mathcal{P}$ and each element of $\mathcal{A}_2$ includes a 2_j such that $2_j \notin \mathcal{A}_1\!\downarrow$. So $\mathcal{A}_1 \cup \mathcal{A}_2$ is an antichain disjoint from $\mathcal{F}$ and $\mathcal{P}_1 \cup \mathcal{P}_2 \subseteq (\mathcal{A}_1 \cup \mathcal{A}_2)\!\updownarrow$.

Another fact we will need is that $\mathcal{A}_2\!\updownarrow \cap \mathcal{P}_3 \subset \mathcal{F}$. $\mathcal{A}_2\!\uparrow \subset \mathcal{P}_2\!\uparrow = \mathcal{P}_2$, leaving just $\mathcal{A}_2\!\downarrow \cap \mathcal{P}_3 \subset \mathcal{F}$ to be verified. Since $\emptyset$ and all singletons are in $\mathcal{F}$, the only way this could fail is if there is some $X \in (\mathcal{A}_2\!\updownarrow \cap \mathcal{P}_3) \setminus \mathcal{F}$ where $|X| = 2$. Assume such an X exists. Then there exists $Y \in \mathcal{A}_2$ such that $X \subset Y$ and $|Y| = 3$. $|Y| = 3$ and $Y \in \mathcal{A}_2$ imply that $Y = 1_12_j3_k$ for some j, k such that $2_j3_k \in \mathcal{F}$. So $X \in \{1_12_j, 1_13_k, 2_j3_k\}$. We can eliminate the case $X = 2_j3_k$ since $2_j3_k \in \mathcal{F}$ (also $2_j3_k \in \mathcal{P}_2$). We can eliminate the case $X = 1_13_k$ since $1_12_1 \in \mathcal{F}$, so $1_12_13_k \in \mathcal{A}_1$ and $1_13_k \in \mathcal{A}_1\!\downarrow = \mathcal{P}_1$. So $X = 1_12_j$. $1_12_j = X \in \mathcal{P}_3$ implies $1_12_j \notin \mathcal{P}_2$, so $2_j \in \mathcal{A}_1\!\downarrow$. But $Y = 1_12_j3_k \in \mathcal{A}_2 \subset \mathcal{P}_2$ implies $2_j \notin \mathcal{A}_1\!\downarrow$. With this contradiction, we conclude that $\mathcal{A}_2\!\updownarrow \cap \mathcal{P}_3 \subset \mathcal{F}$.

Finally, we find an antichain $\mathcal{A}_3$ in $\mathcal{P}_3$ such that $\mathcal{A}_3$ is disjoint from $\mathcal{F}$ and $\mathcal{P}_3 \subset \mathcal{A}_3\!\updownarrow$. For each $i \in \{1, \ldots, l\}$ such that $1_i\!\uparrow \cap \mathcal{P}_3 \neq \emptyset$, $1_i\!\uparrow \cap \mathcal{P}_3$ is hockey as the following two cases show. If $1_i\!\uparrow \cap \mathcal{P}_1 = \emptyset$, then $1_i\!\uparrow \cap \mathcal{P}_3 = 1_i\!\uparrow \setminus (\mathcal{P}_1 \cup \mathcal{P}_2) = 1_i\!\uparrow \setminus \mathcal{P}_2 \cong K_{1,m-|\mathcal{B}_1|} \times K_{1,n}$. If $1_i\!\uparrow \cap \mathcal{P}_1 \neq \emptyset$, then $1_i\!\uparrow \cap \mathcal{P}_3 = \sum_{\mathcal{T}} K_{1,n}$ where

$r = m - |\mathcal{B}_1| - |\{1_i 2_j \in \mathcal{F}\}|$. The correctness of these claims is fairly clear if one consults the diagram. If $1_i\!\uparrow \cap \mathcal{P}_3 \cap \mathcal{F}$ is a fibre of $1_i\!\uparrow \cap \mathcal{P}_3 \neq \emptyset$, then it contains a maximal chain of $1_i\!\uparrow \cap \mathcal{P}_3$ whose union with $\{\emptyset, 1_i\}$ is a maximal chain of $\mathcal{P}$ contained in $\mathcal{F}$, a contradiction. Thus, $1_i\!\uparrow \cap \mathcal{P}_3 \cap \mathcal{F}$ is not a fibre of $1_i\!\uparrow \cap \mathcal{P}_3$ and we can pick $\mathcal{A}_{3,i}$ a maximal antichain of $1_i\!\uparrow \cap \mathcal{P}_3$ disjoint from $\mathcal{F}$. Let $\mathcal{A}_{3,i} = \emptyset$ for each $i \in \{1, \ldots, l\}$ such that $1_i\!\uparrow \cap \mathcal{P}_3 = \emptyset$. Let $\mathcal{A}_3 = \bigcup_{i=1}^{l} \mathcal{A}_{3,i}$. $\mathcal{A}_3$ is an antichain since each $\mathcal{A}_{3,i}$ is an antichain, and each element of any $\mathcal{A}_{3,i}$ includes 1_i, making it impossible for elements of distinct $\mathcal{A}_{3,i}$'s to be comparable. So $\mathcal{A}_3$ is an antichain disjoint from $\mathcal{F}$ and $\mathcal{P}_3 \subseteq \mathcal{A}_3\!\updownarrow$. Recall $\mathcal{A}_2\!\updownarrow \cap \mathcal{P}_3 \subset \mathcal{F}$, and $\mathcal{A}_1\!\updownarrow = \mathcal{P}_1$, so $\mathcal{A}_3 \subset \mathcal{P}_3 \setminus (\mathcal{A}_1\!\updownarrow \cup \mathcal{A}_2\!\updownarrow)$. Thus, $\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_3$ is a maximal antichain of $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ and is disjoint from $\mathcal{F}$, which we assumed was a fibre.

This contradiction completes the proof. $\square$

2.5 Exponentiation

For posets P and Q, a function $f : Q \to P$ is called *order-preserving* if $q \leq q'$ in Q implies $f(q) \leq f(q')$ in P. Define the exponent P^Q to be the set of order-preserving functions $f : Q \to P$ ordered as follows: for $f, g \in P^Q$, $f \leq g$ if and only if $f(q) \leq g(q)$ in P for every $q \in Q$. Notice that $\mathbf{2} \times \mathbf{2} \times \ldots \times \mathbf{2}$ (n times), which was shown to be hockey in [DSW90], is isomorphic to $\mathbf{2}^{\overline{n}}$, a Boolean lattice.

For any posets P_1 and P_2 with subsets P_1' and P_2' such that $P_1 = \mathbf{1} \oplus P_1' \oplus \mathbf{1}$ and $P_2 = \mathbf{1} \oplus P_2' \oplus \mathbf{1}$, define $P_1 \overline{\oplus} P_2 = \mathbf{1} \oplus P_1' \oplus \mathbf{1} \oplus P_2' \oplus \mathbf{1}$. In other words, if P_1 and P_2 are bounded posets, then $P_1 \overline{\oplus} P_2$ is the poset obtained by placing a copy of P_2 above a copy of P_1 and identifying the top point of P_1 with the bottom point of P_2. Notice that $\overline{\oplus}$ is associative, so it makes sense to write $P_1 \overline{\oplus} P_2 \overline{\oplus} \ldots \overline{\oplus} P_n$ without parentheses. $P_1 \overline{\oplus} P_2$ is hockey if and only if both P_1 and P_2 are. This follows from Theorem 2.2.3 with the observation that $P_1 \oplus P_2 \cong \sum_{x \in P_1 \overline{\oplus} P_2} C_x$ where each $C_x = \mathbf{1}$ except for $C_m = \mathbf{2}$ where m is the middle (so to speak) element of $P_1 \overline{\oplus} P_2$. Furthermore, for any posets Q_1 and Q_2, $\mathbf{2}^{Q_1 \oplus Q_2} \cong \mathbf{2}^{Q_2} \overline{\oplus} \mathbf{2}^{Q_1}$.

Theorem 2.5.1 *For any finite poset P, $\mathbf{2}^P$ is hockey if and only if P is a linear sum of antichains.*

Proof. Suppose that $P = A_1 \oplus \ldots \oplus A_k$ where each A_i is a finite antichain. Then

$$\mathbf{2}^P = \mathbf{2}^{(A_1 \oplus \ldots \oplus A_k)} = \mathbf{2}^{(A_2 \oplus \ldots \oplus A_k)} \overline{\oplus} \mathbf{2}^{A_1} = \mathbf{2}^{A_k} \overline{\oplus} \mathbf{2}^{A_{k-1}} \overline{\oplus} \ldots \overline{\oplus} \mathbf{2}^{A_2} \overline{\oplus} \mathbf{2}^{A_1}.$$

The subposets $\mathbf{2}^{A_i}$ are finite Boolean lattices. From [DSW90], or Theorem 2.4.2, we know that every finite Boolean lattice is hockey; therefore $\mathbf{2}^P$ is hockey.

Now suppose that P is not a linear sum of antichains. Then it has three points x, y, z such that $z < x$, $y \parallel z$, and $y \parallel x$. Define

$$K = \{f \in \mathbf{2}^P : f(t) = 1 \text{ for every } t > z, f(z) = 0\}.$$

K is a cutset of $\mathbf{2}^P$ since any maximal chain in $\mathbf{2}^P$ has a greatest element mapping z to 0 and this element is in K. However, K contains no maximal antichain by the following.

Define $f, g, h \in \mathbf{2}^P$ by

$f(t) = 1$ if and only if $t > z$.

$g(t) = 1$ if and only if $t \geq z$.

$h(t) = 1$ if and only if $t \geq y$.

Suppose $A \subseteq K$ is a maximal antichain of $\mathbf{2}^P$. Since $g\updownarrow \cap K = \{f\}$, we must have $f \in A$. But $\{f\} = \min K$, so we must have $A = \{f\}$. This is impossible since $h \parallel f$ and so $\{f\}$ is not a maximal antichain. Thus, K is a cutset of $\mathbf{2}^P$ which contains no maximal antichain. □

The finite distributive lattices are precisely the posets which can be expressed as $\mathbf{2}^P$ for a finite poset P ([DP90] Corollary 8.18 and Exercise 8.18). Thus the finite distributive lattices which are hockey are characterized. They are the ones which are isomorphic to linear sums of Boolean lattices with 0's and 1's of vertically adjacent lattices identified.

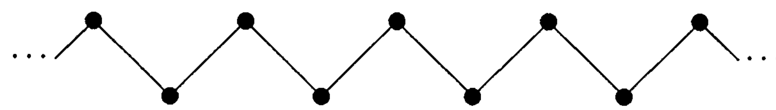
For any posets P_1 and P_2, $\mathbf{2}^{P_1} \times \mathbf{2}^{P_2} \cong \mathbf{2}^{P_1+P_2}$. So Theorem 2.5.1 shows that $\mathbf{2}^{P_1} \times \mathbf{2}^{P_2}$ is hockey if and only if P_1 and P_2 are antichains. This gives us the following positive answer to Question 2.3.9 for finite distributive lattices.

Corollary 2.5.2 *Let Q_1, Q_2, and Q_3 be finite distributive lattices. If $Q_1 \times Q_2$, $Q_2 \times Q_3$, and $Q_1 \times Q_3$ are all hockey, then $Q_1 \times Q_2 \times Q_3$ is hockey.*

Proof. Since Q_1, Q_2, and Q_3 are finite distributive lattices, there are finite posets P_1, P_2, and P_3 such that $Q_1 \cong \mathbf{2}^{P_1}$, $Q_2 \cong \mathbf{2}^{P_2}$, and $Q_3 \cong \mathbf{2}^{P_3}$. For all distinct $i, j \in \{1,2,3\}$, $Q_i \times Q_j \cong \mathbf{2}^{P_i} \times \mathbf{2}^{P_j} \cong \mathbf{2}^{P_i+P_j}$, so $P_i + P_j$ must be an antichain by Theorem 2.5.1. But then $P_1 + P_2 + P_3$ is an antichain and so $Q_1 \times Q_2 \times Q_3 \cong \mathbf{2}^{P_1} \times \mathbf{2}^{P_2} \times \mathbf{2}^{P_3} \cong \mathbf{2}^{P_1+P_2+P_3}$ is hockey by Theorem 2.5.1. □

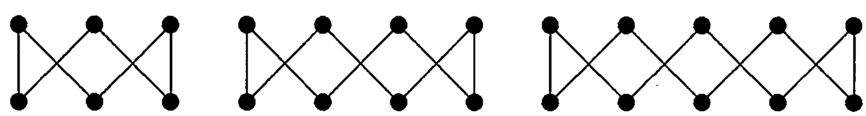
2.6 Zigzags and Cycles

A *fence* is a connected subset of:



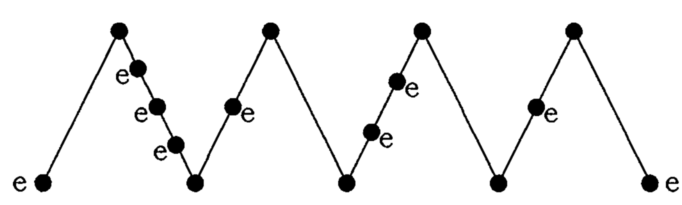
An *endpoint* of a fence is just what one would expect. A fence has zero, one, or two endpoints, according as it is two-way infinite, one-way infinite, or finite.

A *crown* is constructed by identifying the two endpoints of a finite fence which has an odd number of elements, at least 5. Any crown has an even number of elements. Some crowns are:

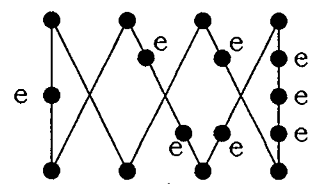


Zigzags and *cycles* are constructed by adding points on edges of the Hasse diagrams of fences and crowns, respectively. These added edge-points, together with endpoints of zigzags, are called *e-points.*

In the next two diagrams, the e's indicate the e-points. An example of a zigzag is:



An example of a cycle is:



A finite fence P' is said to be *e-embedded* in a zigzag or a crown P if P' is a subposet of P and both endpoints of P' are e-points in P. The *distance* between two distinct points in a zigzag (respectively, cycle) is the number of maximal and minimal points strictly between the two points on the path (respectively, a path) connecting them. For $d \in \{0,1,2\}$, two points are said to be at *reduced distance* d from each other if the distance between them (either distance in the case of a cycle) is congruent to d modulo 3.

Lemma 2.6.1 *Suppose $k > 1$ and P is the k-element fence $x_1 < x_2 > x_3 < \ldots x_k$. P is hockey if and only if $k \not\equiv 1$ (mod 3). Furthermore, when $k \equiv 1$ (mod 3), the only fibre of P containing no maximal chain is $\{x_1, x_4, x_7, \ldots, x_k\}$.*

Proof. The lemma is true for $k \in \{2,3\}$ by Corollary 2.2.4.

Consider the case $k = 4$. $\{x_1, x_3\}$ and $\{x_2, x_4\}$ are maximal antichains. Clearly any set other than $\{x_1, x_4\}$ which meets both these maximal antichains contains a maximal chain. And $\{x_1, x_4\}$ is a fibre since any maximal antichain disjoint from it would have to meet $x_1{\updownarrow}\setminus\{x_1\} = \{x_2\}$ and $x_4{\updownarrow}\setminus\{x_4\} = \{x_3\}$, but $x_2 > x_3$. So the lemma is true up to $k = 4$.

Pick k such that the lemma holds for all lesser values of k. Suppose F is a fibre of P containing no maximal chain of P. Then $x_1 \in F$ since otherwise $A = \{x_{i-1} : x_i \in F\}$ would be an antichain such that $F \subseteq A{\updownarrow}\setminus A$ and so A could be extended to a maximal antichain disjoint from F. So $x_1 \in F$ and $x_2 \notin F$. If $F \setminus \{x_1, x_2, x_3\}$ is not a fibre of $P \setminus \{x_1, x_2, x_3\}$, then we can choose A a maximal antichain of $P \setminus \{x_1, x_2, x_3\}$ disjoint from $F \setminus \{x_1, x_2, x_3\}$ and then $A \cup \{x_2\}$ is a maximal antichain of P disjoint from F, a contradiction. So $F \setminus \{x_1, x_2, x_3\}$ is a fibre of $P \setminus \{x_1, x_2, x_3\}$ containing no maximal chain. Since the lemma holds for $k-3$, this tells us that $k \equiv k-3 \equiv 1 (\text{mod } 3)$ and $F \setminus \{x_1, x_2, x_3\} = \{x_4, x_7, x_{10}, \ldots, x_k\}$. Then $x_3 \notin F$ since $\{x_3, x_4\}$ is a maximal chain and $x_4 \in F$. So $F = \{x_1, x_4, x_7, \ldots, x_k\}$. It remains to show that $F = \{x_1, x_4, x_7, \ldots, x_k\}$ actually is a fibre.

Assume for a contradiction that A is a maximal antichain disjoint from F. Since A is disjoint from $F \setminus \{x_1\}$ a fibre of $P \setminus \{x_1, x_2, x_3\}$, $A \setminus \{x_1, x_2, x_3\}$ is not a maximal antichain of $P \setminus \{x_1, x_2, x_3\}$. So $(A \setminus \{x_1, x_2, x_3\})\updownarrow \neq P \setminus \{x_1, x_2, x_3\}$. Since $A\updownarrow = P$ and $\{x_1, x_2, x_3\}\updownarrow \setminus \{x_1, x_2, x_3\} = \{x_4\}$, the only way this can happen is if $x_4\updownarrow \cap A = \{x_3\}$. We must also have $x_1\updownarrow \cap A \neq \emptyset$. Since $x_1\updownarrow = \{x_1, x_2\}$ and $x_1 \in \mathcal{F}$, this tells us that $x_2 \in A$. But then x_2 and x_3 are comparable elements of A, impossible.

The lemma follows by induction. $\square$

Maltby (aka Your Humble Narrator) and Williamson stated Theorem 2.6.1 without proof in [MW92]. They went on to examine the following generalisation of fences different from zigzags. Any fence with an odd number of elements can be expressed as the union of two consecutive levels of ω^2. Maltby and Williamson addressed the question of when a union of two consecutive levels of ω^t is hockey and found that any two consecutive levels of ω^t form a hockey poset, except for the n^{th} and $(n+1)^{\text{th}}$ levels of ω^2 when $n \equiv 2 \pmod 3$ (where the lowest level is called the 0^{th} level). This leaves mostly open the question of which unions of levels of ω^t form hockey posets. This question is made a little less open by Maltby and Williamson's result that for any $r > t$, the union of levels 2 and r in ω^t is not a hockey poset [MW92].

Getting back to the main subject of this section (zigzags and cycles):

Lemma 2.6.2 *Let P and P' be posets such that $P' \subseteq P$ and every maximal chain of P contained in $P'\updownarrow$ intersects P' in a maximal chain of P'. If P is hockey, then so is P'.*

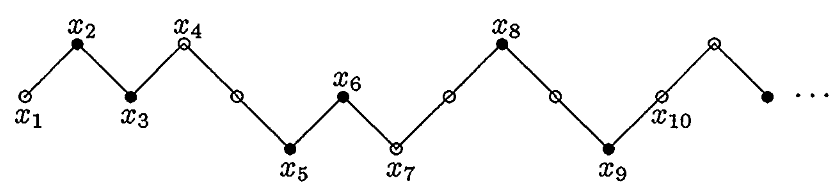
Proof. Let F' be a fibre of P'. Let $F = F' \cup (P'\updownarrow \setminus P')$ in P. We show first that F is a fibre of P. Let A be a maximal antichain of P which does not meet $P'\updownarrow \setminus P'$. Then, for each $x \in P'$, $\emptyset \neq x\updownarrow \cap A \subseteq P' \cap A$. Therefore, $P' \cap A$ is a maximal antichain of P' and so meets F'. So any maximal antichain of P meets either $P'\updownarrow \setminus P'$ or F'. This shows that $F = F' \cup (P'\updownarrow \setminus P')$ is a fibre of P. Since F is a fibre of P and P is hockey, F must contain a maximal chain of P, call it C. $C \subseteq F \subseteq P'\updownarrow$. So C contains some C' a maximal chain of P'. $C' = C \cap P' \subseteq F \cap P' = F'$. So C' is a maximal chain of P' contained in F'.

Thus, P' is hockey. □

Theorem 2.6.3 *A zigzag is hockey if and only if it has no two e-points at reduced distance 2 from each other.*

Proof. If P is a zigzag with comparable e-points, then there is a zigzag Q with no comparable e-points and a set of chains $\{C_x : x \in Q\}$ such that $\sum_{x \in Q} C_x \cong P$. The zigzag Q is obtained by identifying comparable e-points in P. This means that the maximals and minimals of Q are the same as those of P, and the distance between any two e-points in Q is the same as the distance between their antecedents in P. Since we know chains are hockey, Theorem 2.2.3 tells us that P is hockey if and only if Q is hockey. Therefore, it suffices to prove this theorem for posets having no comparable e-points.

Let P be a zigzag with no comparable e-points which has two e-points at reduced distance 2 from each other. Label these e-points and the maximals and minimals between them $x_1, \ldots, x_k$ so that $x_1 < x_2 > x_3 < \ldots x_k$ (or $x_1 > x_2 < x_3 > \ldots x_k$) with x_1 and x_k the e-points. Then $P' = \{x_i : 1 \leq i \leq k\}$ is a fence of size $k \equiv 1 \pmod 3$. By Lemma 2.6.1, P' is not hockey. It is easy to see that every maximal chain of P contained in $P'\updownarrow$ intersects P' in a maximal chain of P'. Thus, by Lemma 2.6.2, P is not hockey. An example is shown below. x_1 and x_{10} are at reduced distance 2 from each other. The x_i's are the points of P', a 10-element fence. The points marked with hollow circles form a fibre containing no maximal chain, constructed according to the proofs of Lemmas 2.6.1 and 2.6.2.

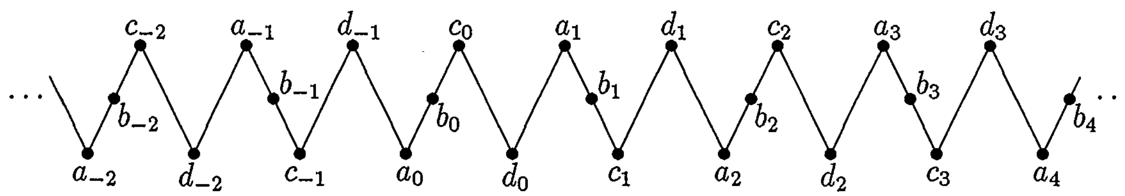


It remains to prove the "if" part of the theorem. Let P be a zigzag having no comparable e-points and no two e-points at reduced distance 2 from each other. This implies that P can be e-embedded in one of the two posets pictured below by the following.

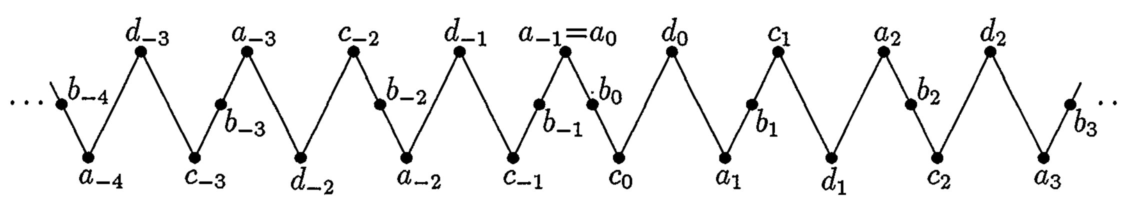
It is easy to see that if all pairs of e-points in P are at reduced distance 0, then P can be e-embedded in the first of the two posets below, P_0. But suppose P has two e-points at reduced distance 1 from each other. Let p and q be two e-points at minimum distance from each other such that their reduced distance is 1. Then P

can be e-embedded in the second of the posets below, P_1, (or its dual, P_1^{d}, which is not illustrated) with p as b_0 and q as b_n for some negative n. We know that all the other e-points in P will coincide with ones in the diagram because: an e-point on any c-d edge to the left of b_0 would be at reduced distance 2 from b_0, an e-point on any c-d edge to the right of b_0 would be at reduced distance 2 from b_n, an e-point on any a-d edge to the left of b_n would be at reduced distance 2 from b_n, an e-point on any a-d edge to the right of b_0 would be at reduced distance 2 from b_0, and an e-point on any a-d edge between b_n and b_0 would be at reduced distance 1 from b_n and violate the condition that no e-points at reduced distance 1 are closer together than b_0 and b_n.

Therefore, by Lemma 2.6.2, the theorem is true if P_0 and P_1 are hockey.



P_0



P_1

Let X be a subset of P_0 containing no maximal chain. We will prove that X cannot be a fibre by constructing a maximal antichain A disjoint from X. For each $i \in \mathbf{Z}$:

if $c_i \notin X$ then put $c_i \in A$,

if $c_i \in X$ and $b_i \notin X$ then put $b_i \in A$,

if $c_i, b_i \in X$ then put $a_i \in A$.

Then A is an antichain disjoint from X and $X \setminus A\updownarrow \subseteq \{d_i : c_i \in X\} \subseteq P_0 \setminus X$. This implies that $X \setminus A\updownarrow = \emptyset$. Therefore, A can be extended to a maximal antichain disjoint from X. So X is not a fibre, and P_0 is hockey.

We deal with P_1 in pretty much the same way as P_0. Let X be a subset of P_1 containing no maximal chain. If $d_i \in X$ for every $i \geq 0$ then put $A_d^+ = \{a_i : i \geq 1\}$. Otherwise let $i' = \min\{i : i \geq 0, d_i \notin X\}$, and put $A_d^+ = \{a_i : 1 \leq i \leq i'\}$. If $d_i \in X$ for every $i < 0$ then put $A_d^- = \{a_i : i \leq -2\}$. Otherwise let $i'' = \max\{i : i \leq -1, d_i \notin X\}$, and put $A_d^- = \{a_i : i'' \leq i \leq -2\}$. Then for each $i \notin \{0, -1\}$ such that $a_i \notin A_d^+ \cup A_d^-$:

if $c_i \notin X$ then put $c_i \in A$,

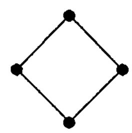
if $c_i \in X$ and $b_i \notin X$ then put $b_i \in A$,

if $c_i, b_i \in X$ then put $a_i \in A$.

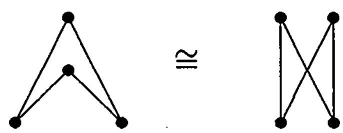
Let A_0 be a maximal antichain of $\{c_{-1}, b_{-1}, a_{-1}{=}a_0, b_0, c_0\}$ disjoint from X and let $A' = A_0 \cup A_d^+ \cup A_d^- \cup A$. Then A' is an antichain disjoint from X and $X \setminus A'\updownarrow \subseteq \{d_i : c_i \in X\} \subseteq P_1 \setminus X$. This implies that $X \setminus A'\updownarrow = \emptyset$. Therefore, A' can be extended to a maximal antichain disjoint from X. So X is not a fibre, and P_1 is hockey. $\square$

Before going on to the general result for cycles, we will examine separately each cycle with only 2 or 4 extreme points.

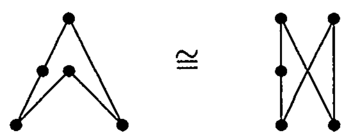
The only cycle with exactly 2 extreme points is $\mathbf{1} \oplus \overline{2} \oplus \mathbf{1}$ which is hockey. Actually, one might not want to consider this a cycle, but this is a trivial matter.



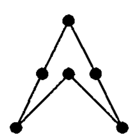
The cycle with exactly 4 extreme points and no e-points is isomorphic to $\overline{2} \oplus \overline{2}$ and so is hockey.

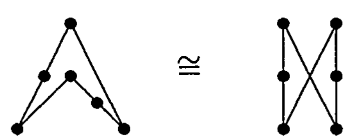


The cycle with 4 extreme points and one e-point is easily verified to be hockey.

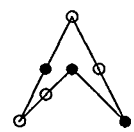


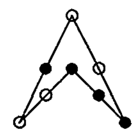
There are two non-isomorphic cycles with 4 extreme points and 2 e-points. Both are easily verified to be hockey. The second one would violate Theorem 2.6.4 if the hypothesis did not exclude cycles with only 4 extreme points.





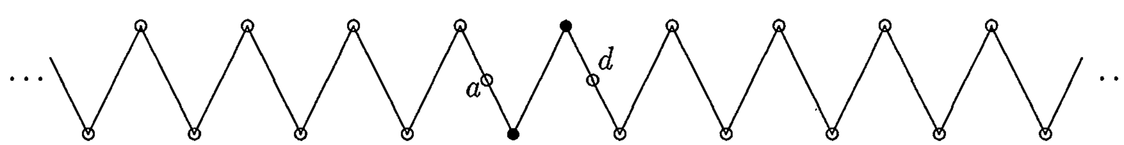
Cycles with 4 extreme points and 3 or 4 e-points are not hockey. The points drawn as hollow circles comprise fibres not containing maximal chains.





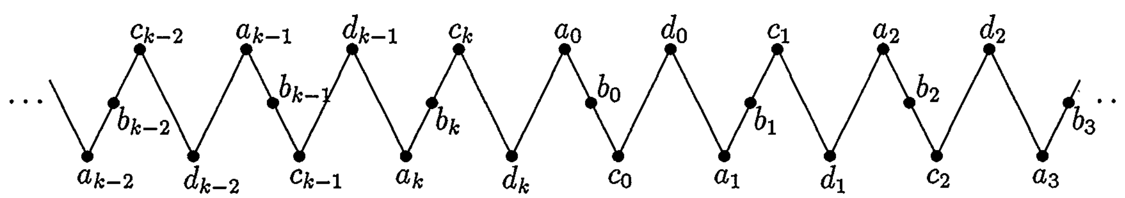
Theorem 2.6.4 *A cycle with more than* 4 *extreme points is hockey if and only if it has no two distinct e-points at reduced distance* 2 *from each other.*

Proof. We will consider only cycles with no two comparable e-points. This is sufficient for the same reason here as it was in the proof of Theorem 2.6.3. Let P be a cycle with two points at reduced distance 2 from each other. Then P has a k-element fence P' e-embedded in it for some $k > 1$, $k \equiv 1 \pmod 3$. By Lemma 2.6.1, P' is not hockey. If every maximal chain of P contained in $P'\updownarrow$ intersects P' in a maximal chain of P', then, by Lemma 2.6.2, P is not hockey. It is easy to see that the only case where this does not happen is when only two extreme points of P are not in P', as in the diagram below where the points of P' are marked with hollow circles, and the endpoints of P' are labelled a and d. It should be clear that this diagram and the ones that follow in this proof are to be interpreted as wrapping around. That is, if you go past one end of the diagram, you eventually come back on at the other end of the diagram.

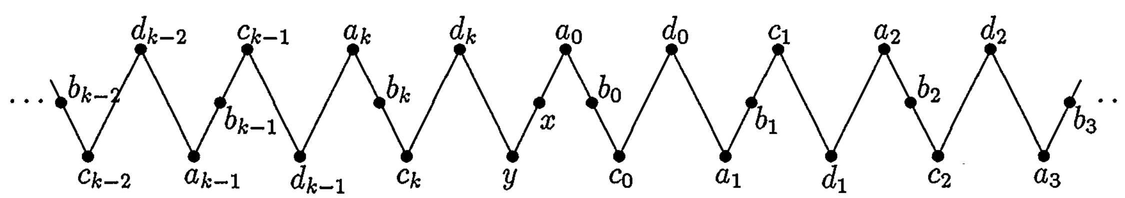


In such a case, one may let P'' be the 4-element fence consisting of the two end-points of P' and the two extreme points of P not in P'. Since we have specified that the cycle has more than 4 extreme points, we know that there are more than 2 extreme points of P which are not in P''. So every maximal chain of P contained in $P''\updownarrow$ contains a maximal chain of P'', so we can apply Theorem 2.6.2 to see that the cycle is not hockey.

It remains to prove the "if" part of the theorem. Let P' be a cycle having no two e-points at reduced distance 2 from each other. Then, for some $k \in \mathbf{N}$, P' can be e-embedded in $P_0(k)$, $P_1(k)$, or $P_1(k)^{\mathrm{d}}$, as described by the diagrams below, so that any maximal chain which is contained in $P'\updownarrow$ contains a maximal chain of P'. Therefore, by Lemma 2.6.2, the theorem is true if $P_0(k)$ and $P_1(k)$ are hockey for all $k \in \mathbf{N}$ for which they are defined. (Each diagram requires k to have a certain parity.)



$P_0(k)$



$P_1(k)$

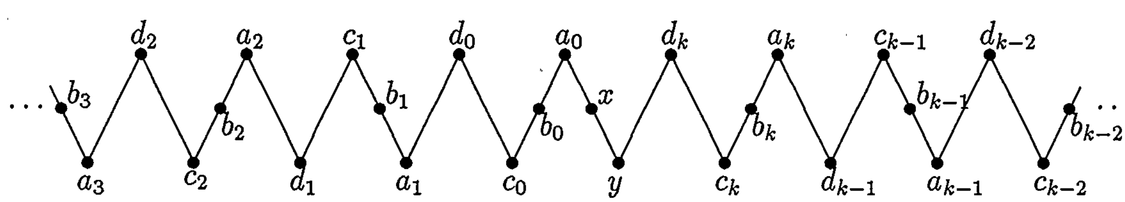
Let P be $P_0(k)$ or $P_1(k)$ for some $k \in \mathbf{N}$. Let X be a subset of P containing no maximal chain. Then for each $i \in \{0, \ldots, k\}$:

if $c_i \notin X$ then put $c_i \in A$,

if $c_i \in X$ and $b_i \notin X$ then put $b_i \in A$,

if $c_i, b_i \in X$ then put $a_i \in A$.

A is an antichain disjoint from X. Suppose P is $P_0(k)$. Then $X \setminus A\updownarrow \subseteq \{d_i : c_i \in X\} \subseteq P \setminus X$, so $X \setminus A\updownarrow = \emptyset$. Therefore, A can be extended to a maximal antichain disjoint from X. So X is not a fibre, P is hockey, and we are done. So suppose we have the case where P is $P_1(k)$. $X \setminus A\updownarrow \subseteq \{d_i : c_i \in X\} \cup \{x, y\} \subseteq (P \setminus X) \cup \{x, y\}$, so $X \setminus A\updownarrow \subseteq \{x, y\}$. If $a_0 \in A$, then $X \setminus A\updownarrow = \emptyset$, so A can be extended to a maximal antichain disjoint from X, which therefore cannot be a fibre and we are done. So assume $a_0 \notin A$. If $x \notin X$ then let $A' = A \cup \{x\}$. If $x \in X$ and $y \notin X$ then let $A' = A \cup \{y\}$. In either of these cases, we get A' an antichain disjoint from X and $X \subseteq A'\updownarrow \setminus A'$, making it impossible for X to be a fibre and thus making P hockey. So assume we have neither of these cases. That is, we have $x, y \in X$ and, therefore, $a_0 \notin X$. In this case, we can relabel the points of P, keeping the same a_0, but putting b_0 for the old x, c_0 for the old y, and so on as below. This time the construction must work since, using the new labels, we have $b_0, c_0 \in X$ and so $a_0 \in A$.



□

Chapter 3

Cutset- and Fibre-Straight Posets

"Outside of a dog, a book is a man's best friend — inside of a dog, it's too dark to read."
— Groucho Marx

3.1 Introduction

In this chapter, we consider the following three conditions on a poset P:

(i) P is fence-free.

(ii) Every minimal cutset of P is an antichain.

(iii) Every minimal fibre of P is a chain.

For finite posets, each of (ii) and (iii) is a strengthening of a characterisation of hockeyness. Rival and Zaguia [RZ85] used the condition:

$(ii)'$ Every finite minimal cutset of P is an antichain.

They showed that $(i) \Rightarrow (ii)'$. Higgs [H85] achieved the more general result $(i) \Rightarrow (ii)$. He also showed that $(ii) \Rightarrow (i)$ if P is finite and presented an infinite poset which satisfies (ii) but not (i). These implications are corollaries of a theorem of Lonc and Rival [LR87] concerning the following two conditions on a graph G:

$(i)^*$ G is P_3-free.

$(ii)^*$ Every minimal transversal of G is independent.

They showed that $(i)^* \Rightarrow (ii)^*$ and that if G is finite, then $(ii)^* \Rightarrow (i)^*$. In fact, their proof that $(ii)^* \Rightarrow (i)^*$ does not require G to be finite — only that every transversal contains a minimal transversal. Applying their result to the comparability graph of a poset P shows that $(i) \Rightarrow (ii)$, and that if every cutset of P contains a minimal cutset then $(ii) \Rightarrow (i)$. Applying their result to the complement of the comparability graph of a poset P shows that $(i) \Rightarrow (iii)$, and that if every fibre of P contains a minimal fibre, then $(iii) \Rightarrow (i)$. Before presenting Lonc and Rival's proof, we will introduce some definitions to express the problem more succinctly.

Call a poset P *cutset-straight* if it satisfies (ii), *i.e.* every minimal cutset of P is an antichain. Call a poset P *fibre-straight* if it satisfies (iii), *i.e.* every minimal fibre of P is a chain. The results mentioned above tell us that all fence-free posets are cutset-straight and fibre-straight. The problem we address in this chapter is to find as many classes as possible of posets for which cutset-straight implies fence-free or fibre-straight implies fence-free. It is this author's opinion that a poset has no business being cutset- or fibre-straight if it has a fence, yet such posets exist. Our objective is to find classes in which no such posets exist. One of the main difficulties in finding these classes is that any poset P which has no minimal cutset (respectively, fibre) is vacuously cutset-straight (respectively, fibre-straight), and so is $P + F_4$ (respectively $P \oplus F_4$) which has a fence.

3.2 Some Old Results and Easy Corollaries

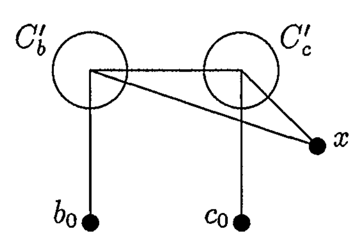
We begin by presenting Lonc and Rival's graph-theoretic results [LR87].

Theorem 3.2.1 *If G is a P_3-free graph, then every minimal transversal of G is independent.*

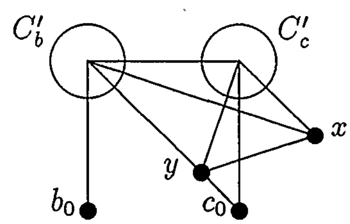
Proof. Let G be a P_3-free graph and assume for a contradiction that G has a minimal transversal T which is not independent. Let $b, c \in T$ such that (b, c) is an edge. Since T is a minimal transversal, there are maximal cliques C_b and C_c such that $C_b \cap T = \{b\}$ and $C_c \cap T = \{c\}$.

Let $C_b' = C_b \cap \{v \in V(G) : (v, c) \notin E(G)\}$. Clearly $C_b' \neq \emptyset$, or else we would have $c \in C_b$. Let $C_c' = C_c \cap \{v \in V(G) : (v, b) \notin E(G)\} \neq \emptyset$. Notice that $C_b' \cap C_c' = \emptyset$. For every $u \in C_b'$ and $v \in C_c'$, (u, v) must be an edge, otherwise (u, b, c, v) would be a P_3. So $C_b' \cup C_c'$ is a clique. But $(C_b' \cup C_c') \cap T \subseteq ((C_b \setminus \{b\}) \cup (C_c \setminus \{c\})) \cap T = \emptyset$. So $C_b' \cup C_c'$ is not a maximal clique. Let C' be a clique maximal in $C_b \cup C_c$ with $C_b' \cup C_c' \subseteq C'$. Let C be a maximal clique of G with $C' \subseteq C$. Notice $C' \neq C$ since $C' \cap T = \emptyset$ while $C \cap T \neq \emptyset$.

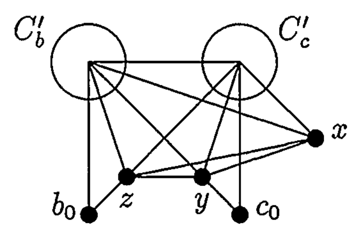
Let $x \in C \setminus C'$. So $x \notin C_b \cup C_c$, otherwise $\{x\} \cup C' \subseteq C_b \cup C_c$ would be a clique properly containing C'. Since $x \notin C_b$, we know there is some $b_0 \in C_b$ such that (b_0, x) is not an edge. Then $b_0 \notin C$, so $b_0 \notin C'$, so $b_0 \in C_b \setminus C_c$. Similarly, there is a $c_0 \in C_c$ such that (c_0, x) is not an edge. Since $c_0 \notin C'$, $c_0 \in C_c \setminus C_b$. So $b_0 \neq c_0$. We cannot be sure yet whether (b_0, c_0) is an edge.



Since $b_0 \notin C'$, we know that there exists $y \in C'$ such that (y, b_0) is not an edge. Then $y \in C_c \setminus C_b$ and $y \neq c_0$ since (y, x) is an edge (since $x, y \in C$) but (c_0, x) is not. So (y, c_0) is an edge. The diagram below shows $y \notin C'_c$ but this may not be true.



Since (x, y, c_0, b_0) cannot be a P_3, (b_0, c_0) is not an edge. Since (c_0, x) is not an edge, we know that $c_0 \notin C'$. So there exists a $z \in C'$ such that (z, c_0) is not an edge. Then $z \in C_b \setminus C_c$. And $z \neq b_0$ since (z, x) is an edge but (b_0, x) is not. So (z, b_0) is an edge. (z, y) is an edge since $\{z, y\} \subseteq C'$.



So (b_0, z, y, c_0) is a P_3. This contradiction completes the proof. □

The following two corollaries are immediate.

Corollary 3.2.2 *Every fence-free poset is cutset-straight.*

Proof. Let P be a fence-free poset and apply Theorem 3.2.1 to the comparability graph of P. □

Corollary 3.2.3 *Every fence-free poset is fibre-straight.*

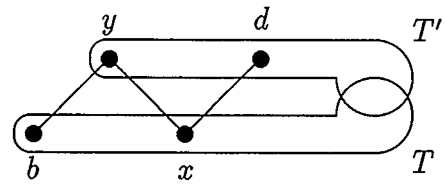
Proof. Let P be a fence-free poset and apply Theorem 3.2.1 to the complement of the comparability graph of P. □

In reading the next theorem, remember that any independent transversal must be a minimal transversal. A graph G satisfies the hypothesis of this theorem if and only if every transversal of G contains a minimal transversal, and every minimal transversal is independent.

Theorem 3.2.4 *Let G be a graph in which every transversal contains an independent transversal. Then G is P_3-free.*

Proof. Assume for a contradiction that G has a P_3 (a, b, c, d). Let C_b, C_c be maximal cliques with $\{a, b\} \subseteq C_b$ and $\{c, d\} \subseteq C_c$. Let $S = (V(G) \setminus (C_b \cup C_c)) \cup \{b, c\}$. Then $S \cap C_b = \{b\}$ and $S \cap C_c = \{c\}$. So any minimal transversal contained in S must include b and c, and therefore not be independent. So, by hypothesis, S is not a transversal. Therefore, there must be a maximal clique $C \subseteq (C_b \cup C_c) \setminus \{b, c\}$. Since $(a, d) \notin E(G)$, a and d cannot both be in C. Assume without loss of generality that $d \notin C$.

$\{b\} \cup \{v \in V(G) : (b, v) \notin E(G)\}$ is a transversal and any maximal clique which includes b meets this transversal only at b. So any minimal transversal contained in this one includes b. Let T be a minimal transversal with $b \in T$. (Similarly, it is clear that every vertex is contained in a minimal transversal.) Let $x \in T \cap C$. Then $x \neq b$ (since $b \notin C$) and (b, x) is not an edge (since T is independent). So $x \notin C_b$, $x \in C_c$. Let T' be a minimal transversal with $d \in T'$. Let $y \in C \cap T'$. Then $y \neq d$ (since $d \notin C$) and (d, y) is not an edge (since T' is independent). So $y \notin C_c$, $y \in C_b$. So (b, y, x, d) is a P_3 with $T \cap \{b, y, x, d\} = \{b, x\}$ and $T' \cap \{b, y, x, d\} = \{y, d\}$.



Let $T_0 = (T \cup T') \setminus \{b, d\}$. The only way a maximal clique K could fail to meet T_0 would be if $K \cap T' = \{d\}$ and $K \cap T = \{b\}$. This cannot happen since (b, d) is not an edge. So T_0 is a transversal. Let C_y be a maximal clique with $b, y \in C_y$. Then $C_y \cap T_0 = \{y\}$ since C_y is a clique and nothing in T_0 is adjacent to both y and b. Let C_x be a maximal clique with $x, d \in C_x$. Then $C_x \cap T_0 = \{x\}$. So any minimal transversal contained in T_0 includes x and y and therefore is not independent. But our hypothesis says every transversal contains an independent transversal, a contradiction. □

The following two corollaries are immediate.

Corollary 3.2.5 *If P is a cutset-straight poset in which every cutset contains a minimal cutset, then P is fence-free.*

Proof. Apply Theorem 3.2.4 to the comparability graph of P. □

Corollary 3.2.6 *If P is a fibre-straight poset in which every fibre contains a minimal fibre, then P is fence-free.*

Proof. Apply Theorem 3.2.4 to the complement of the comparability graph of P. □

Lonc and Rival present an infinite cutset-straight poset, and an infinite fibre-straight poset, each having a fence. Higgs [H85] presents an infinite cutset-straight poset with a fence from which a fibre-straight poset with a fence may be constructed. Before examining these counterexamples, we will consider a relevant and relatively easily recognised property which implies that every transversal of a particular graph contains a minimal one. The following lemma is a special case of a theorem of Li [L89].

Lemma 3.2.7 *If T is a transversal of a graph G such that every clique contained in T is finite, then T contains a minimal transversal.*

Proof. Let $\mathcal{P}$ be the partial order whose elements are the transversals of G which are subsets of T. Order $\mathcal{P}$ by set containment. Let $\mathcal{T}$ be a maximal chain (of nested transversals) in $\mathcal{P}$. Let $T_0 = \bigcap \mathcal{T}$. Let K be any maximal clique in G. Since $K \cap T$ is a clique contained in T, $K \cap T$ is finite. Then $\{K \cap T' : T' \in \mathcal{T}\}$ is a nested sequence of non-empty finite subsets of T. So $K \cap (\bigcap \mathcal{T}) = K \cap T_0 \neq \emptyset$. So T_0 is a transversal, and clearly T_0 must be a minimal transversal and $T_0 \subseteq T$. □

Applying this lemma to the comparability graph of a poset and the graph's complement, we get these corollaries.

Corollary 3.2.8 *If K is a cutset of a poset P such that every chain contained in K is finite, then K contains a minimal cutset.* □

Corollary 3.2.9 *If F is a fibre of a poset P such that every antichain contained in F is finite, then F contains a minimal fibre.* □

Applying these corollaries to Corollaries 3.2.5 and 3.2.6, we get:

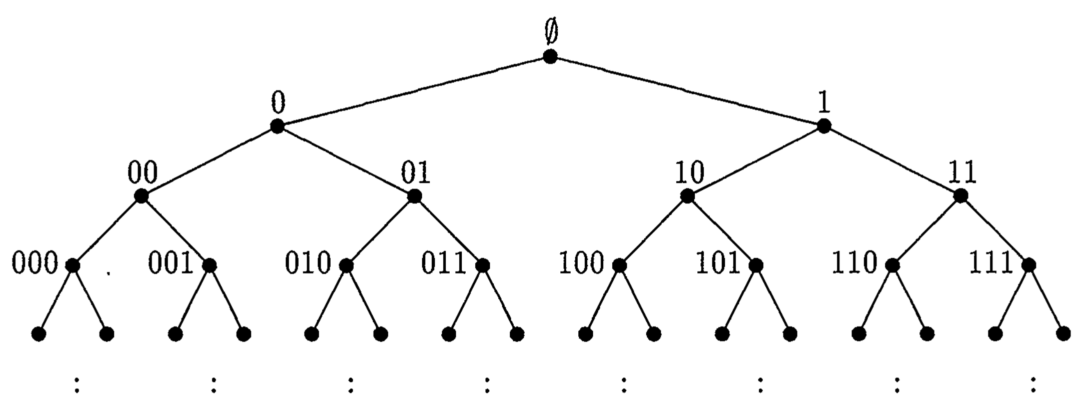
Corollary 3.2.10 *Every chain-finite, cutset-straight poset is fence-free.* □

Corollary 3.2.11 *Every antichain-finite, fibre-straight poset is fence-free.* □

From these corollaries it seems that if P is a poset which is chain-finite and antichain-finite, then P is quite well-behaved indeed. But this does not give us any new information about the issue at hand since any poset which is both chain-finite and antichain-finite is finite — a situation we have already mentioned.

Before trying to find more classes of posets in which cutset-straight implies fence-free or fibre-straight implies fence-free, we will examine some counterexamples, beginning with two fibre-straight examples with fences. Lonc and Rival [LR87] showed that if P is the rooted binary tree of height ω^{d}, then P contains no minimal fibre, and, therefore, $P \oplus F_4$ is fibre-straight, even though it has a fence.

Example 3.2.12 The binary tree of height ω^{d} shown below, call it P, is constructed as follows. The elements of P are the finite 0-1 strings, including the empty string $\emptyset$. For any $x, y \in P$ such that $x = x_1x_2\ldots x_k$ and $y = y_1y_2\ldots y_l$, $x > y$ if and only if $k < l$ and $x_i = y_i$ for $i = 1, 2, \ldots, k$. Then $P \oplus F_4$ has a fence, but is fibre-straight.



Proof. Suppose that F is a minimal fibre of P. Since P is fence-free, F is a chain by Corollary 3.2.3. And any fibre which is a chain must be a maximal chain. But no maximal chain of P is a fibre. To see this, suppose that F is a fibre of P which is a maximal chain. Then we can pick $e_1, e_2, e_3, \ldots$ an ω-sequence of 0's and 1's such that $F = \{\emptyset\} \cup \{b_i : i = 1, 2, 3, \ldots\}$ where each $b_i = e_1 e_2 \ldots e_i$. But then $A = \{b'_i : i = 1, 2, 3, \ldots\}$, where each $b'_i = e_1 e_2 \ldots e_{i-1}(1-e_i)$, is a maximal antichain disjoint from F. So F is not a fibre.

This shows that P contains no minimal fibre. So $P \oplus Q$ has no minimal fibre for any poset Q. In particular, $P \oplus F_4$ has no minimal fibre and so is fibre-straight, but it has a fence. □

Lonc and Rival's proof of the correctness of Example 3.2.12 is unnecessarily long. Rather than applying Corollary 3.2.3, which they proved earlier in the same paper, they essentially prove Corollary 3.2.3 all over again in the context of the special case of Example 3.2.12. They may have avoided using Corollary 3.2.3 since they formally stated it for finite posets only, although they observed before presenting Example 3.2.12 that their proof of Corollary 3.2.3 works for all posets, not just finite ones.

As in Example 3.2.12, any poset P having no minimal fibre gives us $P \oplus F_4$, a fibre-straight poset (because it has no minimal fibre) which is not fence-free. Similarly, if P is a poset having no minimal cutset, then $P + F_4$ is a cutset-straight poset (because it has no minimal cutset) which is not fence-free. From the facts we have already mentioned about comparability graphs, we know that if P and Q are posets with complementary comparability graphs, and P has no minimal fibre (respectively, cutset), then Q has no minimal cutset (respectively, fibre). To make use of this fact, we would like to know: For what posets do posets with complementary comparability graphs exist? Dushnik and Miller [DM41] have shown that these posets are precisely the posets of dimension ≤ 2, where dimension is defined as follows.

Let P be a poset with partial order $<_P$. Let $<_1, <_2, \ldots, <_k$ be total orderings defined on the set P. We call $\{<_1, <_2, \ldots, <_k\}$ a *realiser* of P if for all $x, y \in P$: $x <_P y$ if and only if $x <_i y$ for all $i = 1, \ldots, k$. For any poset P, if there is a finite k such that P has a realizer consisting of k total orderings of P, then the least such k is called the *dimension* of P. We can now state Dushnik and Miller's theorem, but we will not prove it.

Theorem 3.2.13 *A poset P has dimension ≤ 2 if and only if there exists a poset Q whose comparability graph is complementary to that of P.* □

One property common to all the examples in the literature of posets without minimal fibres or without minimal cutsets is that they are fence-free (until the author adds a fence to get a relevant example). We will now prove that all fence-free posets have dimension ≤ 2, thereby getting more examples from those already known. The following result is not new — a complete characterisation of the posets of dimension ≤ 2 using a list of forbidden configurations was found by Kelly [K77] and, using a different method, by Trotter and Moore [TM76]. But their proofs are much harder to read than the following one for the simple case of fence-free posets.

Lemma 3.2.14 *Every fence-free poset has dimension ≤ 2.*

Proof. We begin by proving the lemma in the case of finite posets. Any finite fence-free poset is series-parallel [VTL82] [Ri86]. Therefore, since the singleton poset has dimension 1, it suffices to prove that if P and Q are (disjoint) finite posets each having dimension ≤ 2, then $P \oplus Q$ and $P + Q$ both have dimension ≤ 2.

Let P and Q be finite posets each having dimension ≤ 2. Let $\{<_{P,1}, <_{P,2}\}$ be a realiser of P and $\{<_{Q,1}, <_{Q,2}\}$ be a realiser of Q. For $i \in \{1,2\}$, define $<_i$ a total order on $P \cup Q$ by $x <_i y$ if and only if:

(*i*) $x, y \in P$ and $x <_{P,i} y$,

(*ii*) $x, y \in Q$ and $x <_{Q,i} y$, or

(*iii*) $x \in P$ and $y \in Q$.

Define $<'_2$ a total order on $P \cup Q$ by $x <'_2 y$ if and only if:

(*i*) $x, y \in P$ and $x <_{P,2} y$,

(*ii*) $x, y \in Q$ and $x <_{Q,2} y$, or

(*iii*) $x \in Q$ and $y \in P$.

Then $\{<_1, <_2\}$ is a realiser of $P \oplus Q$ and $\{<_1, <'_2\}$ is a realiser of $P + Q$. So the lemma holds in the case of finite posets.

Extending the lemma to infinite posets is a straightforward application of the compactness theorem. Let P be an infinite fence-free poset. We define a set $\mathcal{S}$ of sentences as follows. For all $x, y \in P$ such that $x <_P y$, put the following sentence in $\mathcal{S}$:

$$L_{1,x,y} \,\&\, L_{2,x,y}$$

For all $x, y \in P$ such that $x \parallel_P y$, put the following sentence in $\mathcal{S}$:

$$(L_{1,x,y} \,\&\, L_{2,y,x}) \vee (L_{1,y,x} \,\&\, L_{2,x,y})$$

Each variable $L_{1,x,y}$ is to be interpreted as "$x <_1 y$", and similarly for $L_{2,x,y}$. So if we can give all the variables $L_{i,x,y}$ truth-values so that the above sentences are all true and $<_1$ and $<_2$ are total orders, then we will have a realiser of P. To get the total order condition, put the following sentences in $\mathcal{S}$ for all distinct $x, y, z \in P$ and $i \in \{1, 2\}$:

$$(L_{i,x,y} \;\&\; L_{i,y,z}) \Rightarrow L_{i,x,z}$$

$$L_{i,x,y} \Rightarrow \neg L_{i,y,x}$$

$$L_{i,x,y} \vee L_{i,y,x}$$

This completes the definition of $\mathcal{S}$. Let V be the set of variables appearing in $\mathcal{S}$. By the compactness theorem [Appendix A], there is an assignment of truth-functional values to the elements of V which makes all the sentences in $\mathcal{S}$ true if and only if for every finite subset of $\mathcal{S}$ there is an assignment of truth-values to its associated variables making every sentence in the subset true.

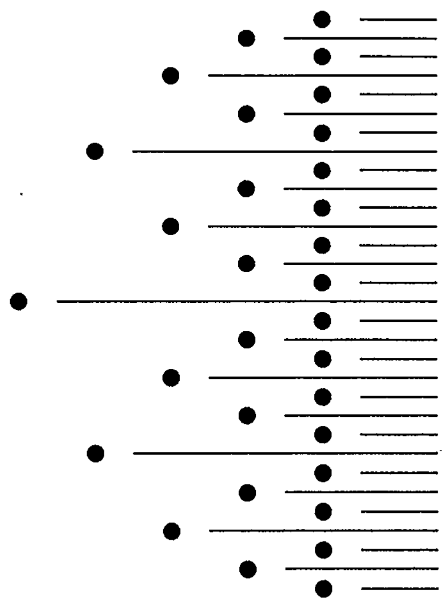
Let $\mathcal{S}'$ be a finite subset of $\mathcal{S}$ and let V' be the set of variables appearing in $\mathcal{S}'$. Let P' be the set of elements of P which appear in the names of variables in V'. V' and P' are finite. Consider P' as a subposet of P. Since P is fence-free, P' must also be fence-free. Since P' is fence-free and finite, Lemma 3.2.14 tells us that P' has dimension ≤ 2 and therefore has a realiser $\{<_1, <_2\}$. Therefore, there is an assignment of truth-values to the elements of V' that makes all the sentences in $\mathcal{S}'$ true. So, by the compactness theorem, there is an assignment of truth-values to the elements of V which makes all the sentences in $\mathcal{S}$ true. This is equivalent to saying that P has dimension ≤ 2. $\square$

By applying this lemma to Example 3.2.12, Lonc and Rival get some information about the question dealing with cutset-straight posets.

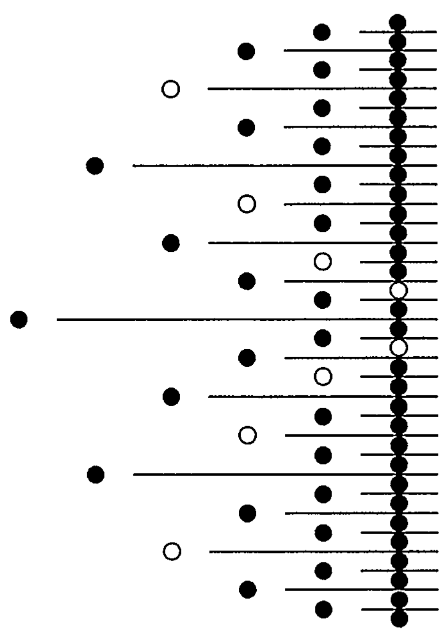
Example 3.2.15 Let P be the binary tree P of height ω^{d} as described in Example

3.2.12. By Lemma 3.2.14, P has dimension 2, so there is a poset Q whose comparability graph is complementary to that of P. Since P has no minimal fibre, Q has no minimal cutset. So $Q + F_4$ is cutset-straight but not fence-free.

Unfortunately, this author has been completely stymied by the problem of drawing an enlightening Hasse diagram of Q in Example 3.2.15. The diagram which Lonc and Rival presented was simple enough but incorrect. The incorrectness of their diagram is clear from the fact that the poset it shows is not fence-free, even though its comparability graph is supposed to be complementary to that of the binary tree of height ω which is fence-free. The diagram below depicts Q in a reasonably comprehensible, but unorthodox, format. In the diagram, the points are the points of the poset, and the horizontal lines indicate comparability. For each horizontal line, every point directly above the line is greater than every point directly below the line. Here "directly above" means that a vertical line going down from the point would hit the horizontal line in question, although it may also hit other lines and points along the way. If two points have no such line between them, then they are not comparable.



This poset has infinite antichains as well as infinite chains. Furthermore, the maximal chain indicated below by hollow circles is isomorphic to $\omega \oplus \omega^{\mathrm{d}}$, showing that the poset is not chain-complete, and also that there are chains not embeddable in the chain of integers. That is, this poset has none of the popular nice properties.

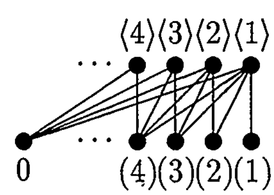


The only other example in the literature of a cutset-straight poset with a fence is due to Higgs [H85]. It is just as devoid of nice properties as Example 3.2.15, and is also constructed by adding a fence to a fence-free poset with no minimal cutset. Furthermore, the construction is harder to describe than that of Example 3.2.15, so we will not say any more about it.

3.3 A Height-3 Poset With No Minimal Fibre

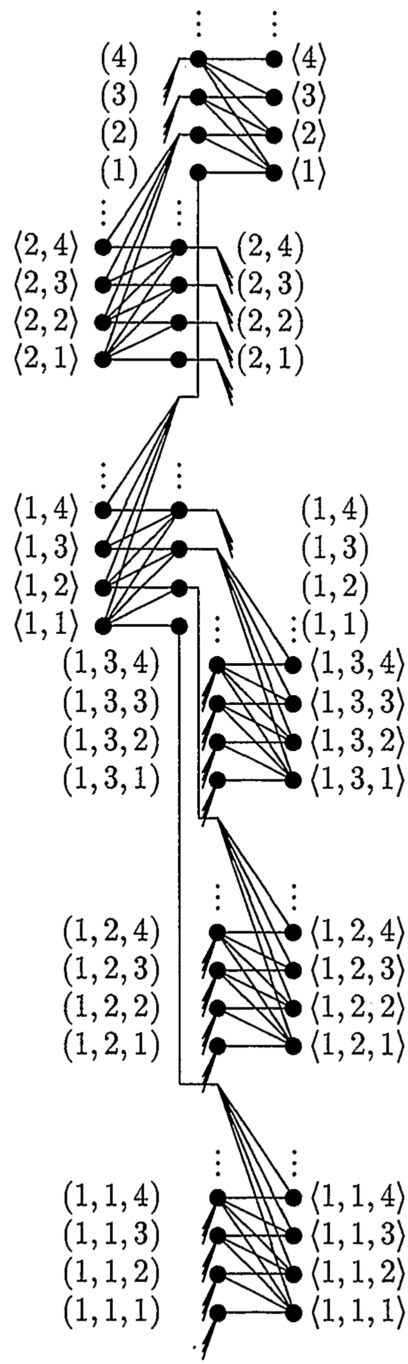
We have already seen that if a poset is antichain-finite and fibre-straight, then it is fence-free. One might hope to find a more general condition on antichains which, when combined with fibre-straight, guarantees fence-free. But since an antichain has no structure, the only conditions one can place on *individual* antichains use cardinality. Example 3.2.12 is a fibre-straight poset with a fence in which every antichain is countable. So it seems the antichain-finite condition is the best possible condition on individual antichains for assuring fibre-straight implies fence-free. What about a condition on chains? In Example 3.2.12, every maximal chain is an ω-chain. Is every chain-finite fibre-straight poset fence-free? We now present a new example of height 3 to show that the answer is no.

The example which is the basis of Example 3.3.1 is the set $\{0\} \cup \{\langle i \rangle : i \in \mathbf{N}\} \cup \{(j) : j \in \mathbf{N}\}$ ordered by $0 < \langle i \rangle$ for each $i \in \mathbf{N}$ and $\langle i \rangle > (j)$ for all $i \leq j$ in $\mathbf{N}$.



0 is not in any minimal fibre by the following. Suppose F is a minimal fibre with $0 \in F$. Since $\{\langle n\rangle : n \in \mathbf{N}\}$ is a maximal antichain, there must be some $\langle m\rangle \in F$. For every $n > m$, $\langle n\rangle\updownarrow \setminus \langle m\rangle\updownarrow = \{\langle n\rangle\}$, so $\langle n\rangle$ is in every maximal antichain $\langle m\rangle$ is in, and $\langle n\rangle$ cannot be in F, since $F\setminus\{\langle m\rangle\}$ would be a smaller fibre. Since F is a minimal fibre with $0 \in F$, there must be a maximal antichain A such that $A \cap F = \{0\}$. Since the only maximal antichain which includes 0 is $\{0\} \cup \{(n) : n \in \mathbf{N}\}$, this tells us that $\{(n) : n \in \mathbf{N}\} \cap F$ is empty. So $\{\langle i\rangle : i \geq m+1\} \cup \{(j) : j \leq m\}$ is a maximal antichain disjoint from F.

Example 3.3.1 Let P be the poset shown on the following page whose elements are the finite round-bracketed and angle-bracketed tuples all of whose coordinates are natural numbers. Order P as follows so that every maximal chain has 3 elements. The maximal elements are the angle-bracketed odd-tuples, the minimal elements are the angle-bracketed even-tuples, and the remaining elements are the round-bracketed tuples. For any $\langle x_1, x_2, x_3, \ldots, x_{2k}\rangle \in P$, the only upper covers are $(x_1, x_2, x_3, \ldots, x_{2k-1})$ and $(x_1, x_2, x_3, \ldots, x_{2k-1}, y)$ for each $y \geq x_{2k}$. For any $\langle x_1, x_2, x_3, \ldots, x_{2k+1}\rangle \in P$, the only lower covers are $(x_1, x_2, x_3, \ldots, x_{2k})$ and $(x_1, x_2, x_3, \ldots, x_{2k}, y)$ for each $y \geq x_{2k+1}$. P has no minimal fibre.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⟨1, 1, 2⟩
⟨1, 1, 1⟩

This side up →

The correctness of this example requires a relatively lengthy explanation. We will proceed with two lemmas.

Lemma 3.3.2 *Let F be a fibre of P (from Example 3.3.1) which includes two points $\langle x_1, x_2, \ldots, x_{k-1}, x_k\rangle$ and $\langle x_1, x_2, \ldots, x_{k-1}, x'_k\rangle$ with $x'_k < x_k$. Then F is not a minimal fibre.*

Proof. Let A be a maximal antichain which includes $\langle x_1, x_2, \ldots, x_{k-1}, x'_k\rangle$. $A \cap \langle x_1, x_2, \ldots, x_k\rangle\updownarrow$ cannot be empty. Therefore, since

$$\langle x_1, x_2, \ldots, x_k\rangle\updownarrow \setminus \langle x_1, x_2, \ldots, x_{k-1}, x'_k\rangle\updownarrow = \{\langle x_1, x_2, \ldots, x_k\rangle\},$$

$\langle x_1, x_2, \ldots, x_k\rangle \in A$. So there is no maximal antichain which meets F only at $\langle x_1, x_2, \ldots, x_{k-1}, x'_k\rangle$. So $F\setminus\{\langle x_1, x_2, \ldots, x_{k-1}, x'_k\rangle\}$ is a fibre. □

Lemma 3.3.3 *Let $X \subseteq P$ (from Example 3.3.1) such that, for all k, X does not include any two points $\langle x_1, x_2, \ldots, x_{k-1}, x_k\rangle$ and $\langle x_1, x_2, \ldots, x_{k-1}, x'_k\rangle$ with $x'_k < x_k$. Then X is not a fibre.*

Proof. We will construct an antichain A such that $X \subseteq A\updownarrow \setminus A$.

Let $A = \bigcup_{k=1}^{\infty} A_k$ where the A_k's are defined inductively as follows, starting with $A_0 = \emptyset$. (Define $\langle x_1, \ldots, x_k, \mathbf{N}\rangle = \{\langle x_1, \ldots, x_k, n\rangle : n \in \mathbf{N}\}$.)

$$B_k = \{(x_1, \ldots, x_k) : \langle x_1, \ldots, x_k\rangle \in X, (x_1, \ldots, x_k) \notin X\}$$
$$C_k = \{\langle x_1, \ldots, x_k, 1\rangle : (x_1, \ldots, x_k) \in X\setminus A_{k-1}\updownarrow, \langle x_1, \ldots, x_k, \mathbf{N}\rangle \cap X = \emptyset\}$$
$$D_k = \{\langle x_1, \ldots, x_k, x_{k+1}+1\rangle : (x_1, \ldots, x_k) \in X\setminus A_{k-1}\updownarrow, \langle x_1, \ldots, x_k, x_{k+1}\rangle \in X\}$$
$$A_k = B_k \cup C_k \cup D_k$$

Is it possible that two points of A are comparable? The answer is no; it is clear that each A_k is an antichain. It is clear that if two points of A are comparable, then they are of the form $\langle x_1, \ldots, x_k \rangle \in A_{k-1}$ and $(x_1, \ldots, x_{k-1}, y_k) \in A_k$ with $y_k \geq x_k$ or the second point is some $\langle x_1, \ldots, x_{k-1}, y_k, y_{k+1} \rangle \in A_k$ with $y_k \geq x_k$. The only way to get $(x_1, \ldots, x_{k-1}, y_k) \in A_k$ is if $\langle x_1, \ldots, x_{k-1}, y_k \rangle \in X$, in which case $\langle x_1, \ldots, x_k \rangle$ would not have been chosen in A_{k-1}. (Maybe $\langle x_1, \ldots, x_{k-1}, y_k + 1 \rangle$ would be in A_{k-1}, see the definition of D_k.) The only way to get $\langle x_1, \ldots, x_{k-1}, y_k, y_{k+1} \rangle \in A_k$ is if $(x_1, \ldots, x_{k-1}, y_k) \in X \setminus A_{k-1}\updownarrow$. This would not happen if $\langle x_1, \ldots, x_k \rangle \in A_{k-1}$ since $(x_1, \ldots, x_{k-1}, y_k) \in \langle x_1, \ldots, x_k \rangle\updownarrow$. So A is an antichain.

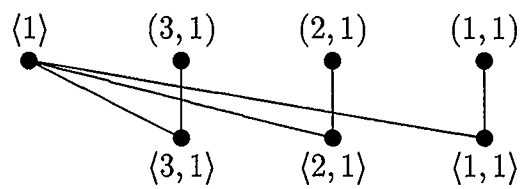
Is any point of X not in $A\updownarrow$? Obviously, each round-bracketed tuple in X is in $A\updownarrow$ because of the definitions of B_k and C_k. But suppose there is some $\langle x_1, \ldots, x_k \rangle \in X$. If $(x_1, \ldots, x_k) \notin X$, then $(x_1, \ldots, x_k) \in B_k$, so $\langle x_1, \ldots, x_k \rangle \in A\updownarrow$. So suppose $(x_1, \ldots, x_k) \in X$. If $(x_1, \ldots, x_k) \notin A_{k-1}\updownarrow$ then B_k or C_k will provide a point in A_k comparable to $\langle x_1, \ldots, x_k \rangle$ as well as $(x_1, \ldots, x_k)$. So assume that $(x_1, \ldots, x_k) \in A_{k-1}\updownarrow$. The point in A_{k-1} to which $(x_1, \ldots, x_k)$ is comparable must be of the form $\langle x_1, \ldots, x_{k-1}, y_k \rangle$ with $y_k \leq x_k$. But, because of the definition of C_k, since $\langle x_1, \ldots, x_{k-1}, x_k \rangle \in X$, the only $\langle x_1, \ldots, x_{k-1}, y_k \rangle$ which might be in A_{k-1} is the one with $y_k = x_k + 1$. With this contradiction, we may conclude that $X \subseteq A\updownarrow$.

That $A \cap X = \emptyset$ is clear from the construction. □

With these lemmas the correctness of Example 3.3.1 is clear. Since Example 3.3.1 has no minimal fibre, it is fibre-straight, but it is not fence-free. In one respect, Example 3.3.1 is nicer than Example 3.2.12. In Example 3.2.12, it is the fence-free part of the poset that makes the poset fibre-straight. In Example 3.3.1, fences are a vital part of the construction making the poset fibre-straight.

It seems to this author that there is not much left to do in investigating when fibre-straight implies fence-free, since there is a fibre-straight poset of height 3 which is not fence-free, and any antichain-finite fibre-straight poset is known to be fence-free while there is a countable fibre-straight poset which is not fence-free.

It would be nice if we could use Example 3.3.1 to construct a poset of width 3 with complementary comparability graph and having no minimal cutset, as Lonc and Rival used Example 3.2.12 to make Example 3.2.15. Because of Theorem 3.2.13, this is only possible if Example 3.3.1 has dimension ≤ 2. By [K77], Example 3.3.1 has dimension ≥ 3 since it contains the suborder



So there is no poset having a comparability graph complementary to that of Example 3.3.1.

Posets of height 1 are antichains which are fence-free. Example 3.3.1 shows that a fibre-straight poset with fences may have height as little as 3. This leaves the following question unanswered.

Question 3.3.4 *Is every fibre-straight poset of height 2 fence-free?*

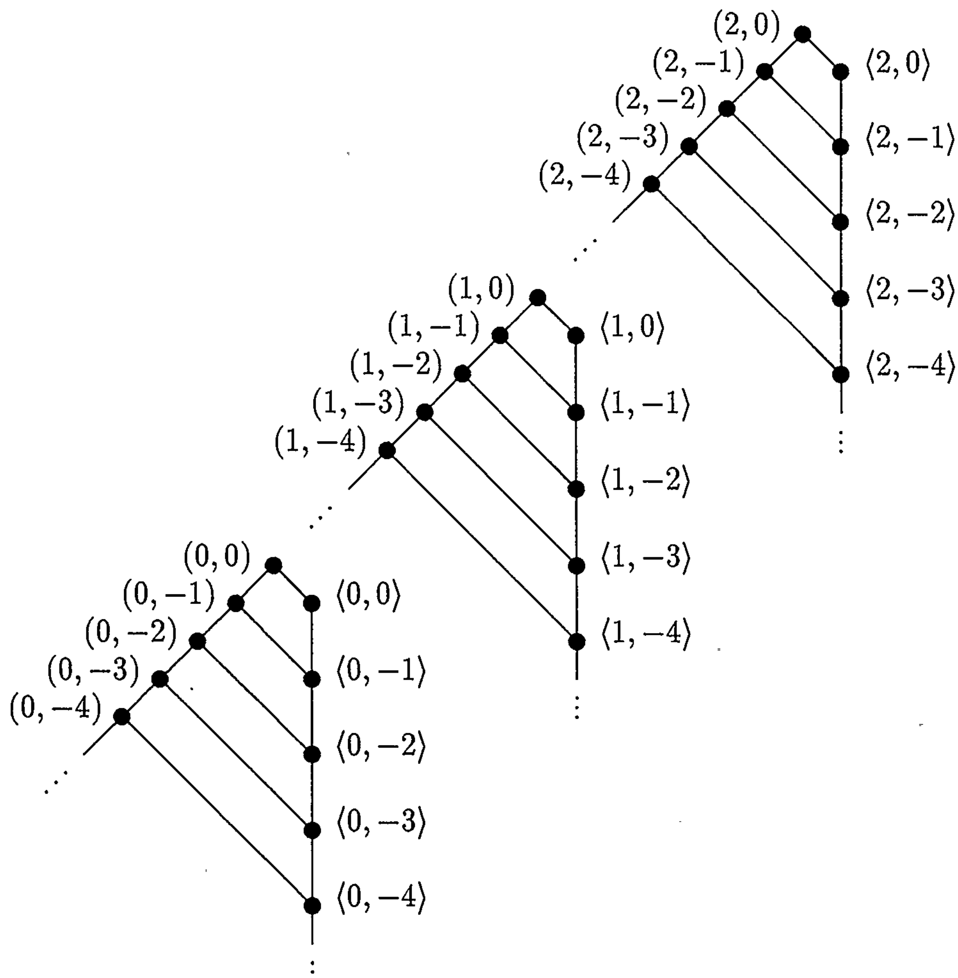
Since posets of height 2 may have dimension ≥ 3, an answer to this question need not provide an answer to the following question which we will return to at the end of the chapter.

Question 3.3.5 *Is every cutset-straight poset of width 2 fence-free?*

3.4 A New Example With No Minimal Cutset

The examples of cutset-straight posets with fences by Higgs and Lonc & Rival are both of the form $P + F_4$ where P is a fence-free poset having no minimal cutset. This is somewhat unsatisfying since it is the fence-free part of the poset that prevents the existence of a minimal cutset which is not an antichain. One might hope to get a more satisfying example by finding a poset with no minimal cutset and having this fact due to fences in the construction. The literature is no help here (as far as this author has been able to ascertain) since [H85] and [LR87] are the only papers describing posets with no minimal cutsets. (In a survey paper, El-Zahar and Zaguia [EZ86] incorrectly list [G84] as another paper describing a poset having no minimal cutset. What [G84] describes is a poset having no antichain cutset.) We now present a new example of a poset having no minimal cutset in which fences are a vital part of the construction which prevents minimal cutsets.

Example 3.4.1 Let P be the poset shown on the following page. The points of P are the elements of the cartesian product $\omega \times \omega^{\mathrm{d}}$ as well as the angle-bracketed versions of the same pairs. On the round-bracketed pairs, impose the lexicographic total order. That is, $(x, y) < (x', y')$ if $x < x'$ or if $x = x'$ and $y < y'$. For angle-bracketed pairs $\langle x, y\rangle \leq \langle x', y'\rangle$ if and only if $x = x'$ and $y \leq y'$. The only way to get $\langle x, y\rangle \leq (x', y')$ is if $(x, y) \leq (x', y')$ in which case it follows by transitivity. No round-bracketed pair is less than any angle-bracketed pair. Then P has no minimal cutset.



Proof. Suppose for a contradiction that K is a minimal cutset of P. $\{(x, y) : x \in \omega, y \in \omega^{\mathrm{d}}\}$ is a maximal chain of P. Therefore, for some $x \in \omega$ and $y \in \omega^{\mathrm{d}}$, we have $(x, y) \in K$. There must be a maximal chain C such that $C \cap K = \{(x, y)\}$. Since $(x, y)\!\uparrow$ is a chain, $C \cap (x, y)\!\uparrow \,=\, (x, y)\!\uparrow$. Let C_0 be the maximal chain $(x+1, 0)\!\uparrow \cup \{\langle x+1, y'\rangle : y' \in \omega^{\mathrm{d}}\}$. Then $C_0 \cap K \neq \emptyset$, but $(x+1, 0)\!\uparrow \subset (x, y)\!\uparrow$, so $(x+1, 0)\!\uparrow \cap K = \emptyset$. Let $\langle x+1, y'\rangle \in C_0 \cap K$. Then there must be a maximal chain C' such that $K \cap C' = \{\langle x+1, y'\rangle\}$. Since $\langle x+1, y'\rangle\!\downarrow$ is a chain, $C' \cap \langle x+1, y'\rangle\!\downarrow =$

$\langle x+1, y'\rangle\downarrow$. Let C_1 be the maximal chain $(x+1, y'-1)\uparrow \cup \langle x+1, y'-1\rangle\downarrow$. Then $C_1 \cap K = \emptyset$, a contradiction. □

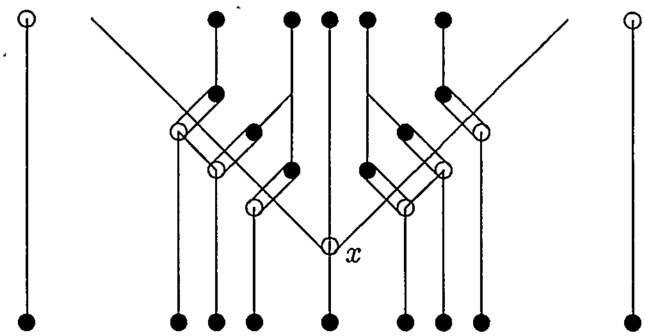
3.5 Regular Posets

A poset P is called *regular* if it is chain-complete (*i.e.* every chain has a supremum and an infimum) and for every chain $C \subseteq P$, $(\sup C)\overset{\circ}{\downarrow} \subseteq C\downarrow$ and $(\inf C)\underset{\circ}{\uparrow} \subseteq C\uparrow$. That is, in a regular poset, a supremum of a chain is not above anything that is not below an element of the chain, and dually for infima. Other than the results by Higgs and Lonc & Rival already mentioned, the only class of cutset-straight posets identified in the literature as fence-free is the class of regular, well-founded, cutset-straight posets. This result is due to Rival and Zaguia [RZ87]. In the context of their paper, this result is secondary. Their main concern is with the similar question: In a regular, well-founded poset, is every point in a minimal cutset? They show that the answer to this question is yes and then go on to apply their technique to show that every regular, well-founded, cutset-straight poset is fence-free. We will prove the same two results, but using a simpler construction.

Before stating the theorem, we will explain some notation. Let P be a regular poset and let $x \in P$. Define

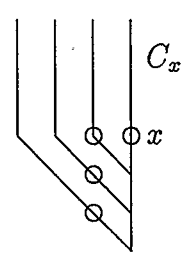
$$A(x) = \{\sup(C \setminus x\underset{\circ}{\uparrow}) : C \text{ is a maximal chain of } P\}.$$

Since no maximal chain of P is contained in $x\underset{\circ}{\uparrow}$, $A(x)$ is a cutset of P. In fact, $A(x) \setminus \{x\}$ is a cutset for x. That is, $A(x)$ is a cutset and $A(x) \setminus \{x\} \subseteq P \setminus x\underset{\circ}{\uparrow}$. Using $A(x)$ to meet maximal chains where they exit $x\underset{\circ}{\uparrow}$ is easy since the boundary of $x\underset{\circ}{\uparrow}$ is sharp in the sense that for any maximal chain C, $\sup(C \setminus x\underset{\circ}{\uparrow}) = \max(C \setminus x\underset{\circ}{\uparrow})$ and whenever $C \cap x\underset{\circ}{\uparrow} \neq \emptyset$, $\inf(C \cap x\underset{\circ}{\uparrow}) = \min(C \cap x\underset{\circ}{\uparrow})$. In the diagram below, the hollow points represent $A(x)$.



It is worthwhile contrasting this approach with that of Rival and Zaguia. Rival and Zaguia used a similar idea, but it does not take advantage of regularity as effectively as the construction of $A(x)$. What they did was to pick a maximal chain C_x through x, and then try to pick a cutset which meets other maximal chains as they diverge from C_x. This plan falls apart quite hopelessly without their additional assumption that the poset is well-founded. The cutset they use is

$$\{\min(C \setminus (C_x \cap x\updownarrow)) : C \text{ is a maximal chain of } P\}.$$



If one replaces "min" in this definition with "inf" to use in regular posets which are not well-founded, then one may get infs in $C_x \cap x\updownarrow$, so that the cutset does not meet C_x only at x. In this situation, it may be impossible to find a minimal cutset contained in this one which includes x. Even with the condition of well-foundedness, rather a lot of work is required to find a minimal cutset contained in this one. But we will see that it is quite is easy to use

$$A(x) = \{\sup(C \setminus x\updownarrow) : C \text{ is a maximal chain of } P\}$$

to get

$$K(x) = \{\min(C \cap A(x)) : C \text{ is a maximal chain of } P\}$$

a minimal cutset contained in $A(x)$.

Suppose K is a cutset of a poset P. If $\max(C \cap K)$ exists for *every* maximal chain C of P, define

$$r(K) = \{\max(C \cap K) : C \text{ is a maximal chain of } P\}.$$

If $r(K)$ is defined and $\min(C \cap r(K))$ exists for *every* maximal chain C of P, define

$$s(K) = \{\min(C \cap r(K)) : C \text{ is a maximal chain of } P\}.$$

Theorem 3.5.1 *If K is a cutset of a poset P and $s(K)$ is defined, then $s(K)$ is a minimal cutset contained in K.*

Proof. It is clear that $s(K)$ is a cutset contained in K. It remains only to verify that $s(K)$ is a minimal cutset. Let $a \in s(K)$. We have to find a maximal chain C such that $C \cap s(K) = \{a\}$. Since $a \in r(K)$, there is a maximal chain C_1 such that $a = \max(C_1 \cap K)$. Since $a \in s(K)$, there is a maximal chain C_2 such that $a = \min(C_2 \cap r(K))$. Let $C = (C_1 \cap a\uparrow) \cup (C_2 \cap a\downarrow)$. C is a maximal chain and $C \cap r(K) = \{a\} = C \cap s(K)$. Therefore, $s(K)$ is a minimal cutset. $\square$

The proof above is a (superior, I think) alternative to that already given for Corollary 3.2.8, but does not so readily yield Theorem 3.2.7 and Corollary 3.2.9. But, to get back to the issue at hand:

Corollary 3.5.2 *Let P be a regular poset and let $x \in P$. If $\min(C \cap A(x))$ exists for every maximal chain C, then each $K(x)$ is a minimal cutset of P with $x \in K(x)$.*

Proof. Let K_1 be the cutset $\{x\} \cup (P \smallsetminus x{\updownarrow})$. By regularity, $\max(C \cap K_1)$ exists for every maximal chain C. So $A(x) = r(K_1)$. So Theorem 3.5.1 tells us that $K(x) = s(K_1)$ is a minimal cutset. Furthermore, $x \in K(x)$ since if C is any maximal chain with $x \in C$, then $C \cap A(x) = \{x\}$. $\square$

In the following proof, we use these definitions:

$$\mathrm{UC}(x) = \{y : y \succ x\}$$

$$\mathrm{LC}(x) = \{y : y \prec x\}$$

For every chain $X \subseteq P$, define

$$\mathcal{C}(X) \text{ to be the set of all maximal chains containing } X.$$

We may abbreviate, for instance, $\mathcal{C}(\{x, y\})$ to just $\mathcal{C}(x, y)$.

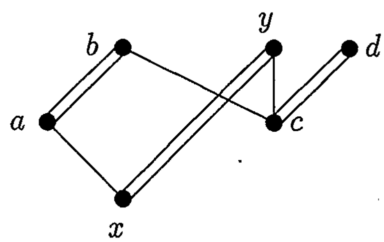
Theorem 3.5.3 *Let P be a regular poset which is not fence-free. If $\min(C \cap A(x))$ exists for every $x \in P$ and every maximal chain C, then P has a minimal cutset which is not an antichain.*

Proof. This proof will be easier if we can pick (a, b, c, d) an N. If this is not possible (*i.e.* if P is N-free), pick a fence $a \prec b > c \prec d$ as follows. Let $a' < b' > c' < d'$ be a fence. Let $b \in \min(a'\underset{\circ}{\uparrow} \cap c'\underset{\circ}{\uparrow})$, $a \in \max(a'{\uparrow} \cap b\overset{\circ}{\downarrow})$, $c \in \max(b\overset{\circ}{\downarrow} \cap d'\overset{\circ}{\downarrow})$, and $d \in \min(d'{\downarrow} \cap c\underset{\circ}{\uparrow})$. Regularity guarantees that this construction is possible. This shows P contains a fence of the form $a \prec b > c \prec d$.

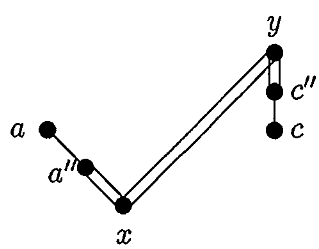
By Corollary 3.5.2, $K(c)$ is a minimal cutset containing c. Is it possible that $a \notin K(c)$? The answer is that it is possible only if P has an N by the following.

We know that $a \in A(c)$ by applying the definition of $A(c)$ to any maximal chain which includes a and b.

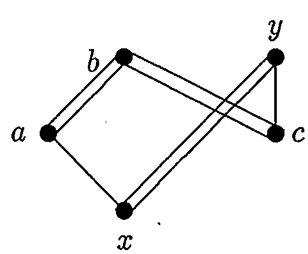
Suppose $a \notin K(c)$. Then there exists $x \in K(c) \cap a\updownarrow$. $x \parallel c$ since $K(c) \setminus \{c\} \subseteq P \setminus c\updownarrow$. Since $x \in K(c) \subseteq A(c)$, we can let C be a maximal chain of P such that $x = \sup(C \setminus c\uparrow)$. Since P is regular, we know that $y = \inf(C \cap c\uparrow) > c$, so $c < y \succ x$.



By regularity, $y \succ x$ implies that there exist $a'' \in \mathrm{UC}(x) \cap a\downarrow$ and $c'' \in \mathrm{LC}(y) \cap c\uparrow$.



Then (a'', x, y, c'') is an N, so we may also assume that (a, b, c, d) is an N. So we have:



Now we can verify that $K(a)$ is a minimal cutset which is not an antichain. $K(a)$ is a minimal cutset. Let C_c be a maximal chain having $b, c \in C_c$. Then $C_c \cap K(a) \neq \emptyset$

but $K(a) \cap a\updownarrow = \emptyset$. So $C_c \cap K(a) \subseteq c\downarrow$. Let C_y be a maximal chain having $x, y \in C_y$. Then $C_y \cap K(a) \neq \emptyset$ but $K(a) \cap a\updownarrow = \emptyset$. So $C_y \cap K(a) \subseteq y\uparrow$. Thus, $K(a)$ includes an element of $y\uparrow$ and an element of $c\downarrow$ and cannot be an antichain. That is, the conclusion of the theorem holds if $a \notin K(c)$.

Now consider what happens if $a \in K(c)$. We will find that this case also implies that P has an N or that the conclusion of the theorem holds. Since $K(c)$ is a minimal cutset, we will assume that it is an antichain, since otherwise the conclusion of the theorem holds. By regularity, $b \succ a$ implies that for every $x > a$, there is a y such that $x \geq y \succ a$. Let

$$K_2 = (K(c) \setminus \{a\}) \cup \mathrm{UC}(a),$$

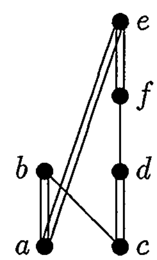
$$K_2' = K_2 \setminus \{z \in K(c) : C \cap \mathrm{UC}(a) \neq \emptyset \text{ for every } C \in \mathcal{C}(z)\}.$$

We know that $b \in K_2'$ since $b \succ a$ implies that $b \in K_2$ and $b \notin K(c)$.

Since $K(c)$ is a cutset, it is clear that K_2 and K_2' are cutsets. In fact, K_2' is a minimal cutset by the following. Let $z \in K_2' \subset K_2$. Then $z \in K(c) \setminus \{a\}$ or $z \succ a$. If $z \in K(c) \setminus \{a\}$, then, since $K(c)$ is a minimal cutset, there is a maximal chain C which meets $K(c)$ only at z. There must be some such chain C which does not meet $\mathrm{UC}(a)$ or else z would have been thrown out of K_2 to make K_2', and this C is a maximal chain which meets K_2' only at z. Now consider what happens if $z \succ a$. Let C_z be a maximal chain with $\{a, z\} \subseteq C_z$. Then $C_z \cap K(c) = \{a\}$ and $C_z \cap \mathrm{UC}(a) = \{z\}$, so $C_z \cap K_2' = \{z\}$. Thus, K_2' is a minimal cutset.

If $c \in K_2'$ then the conclusion of the theorem holds (remember $b \in K_2'$) and we are done, so assume $c \notin K_2'$. Since $c \in K(c)$, we know that $c \in K_2$, so $c \notin K_2'$ tells us that every maximal chain through c includes an upper cover of a. Extend $\{c, d\}$

to a maximal chain and let e be the point of UC(a) that it meets. Then there is a point f such that $e \succ f \geq d$. Now, b is not comparable to f.



So (f, e, a, b) is an N. But at the start of this proof we said we would pick (a, b, c, d) an N if possible and now we see that it was possible. So assume that (a, b, c, d) is an N.

Consider K_2' as defined earlier in this proof. The only way K_2' could fail to witness the conclusion of the theorem is if the diagram above holds. So assume the diagram is correct. We may also assume that $b \succ c$.

By Corollary 3.5.2, $K(f)$ is a minimal cutset. Let C_a be a maximal chain with $a, e \in C_a$. $K(f) \cap f\updownarrow = \emptyset$ so $\emptyset \neq K(f) \cap C_a \subseteq C_a \cap a\downarrow$. Let C_b be a maximal chain with $b, c \in C_b$. $K(f) \cap f\updownarrow = \emptyset$ so $\emptyset \neq K(f) \cap C_b \subseteq C_b \cap b\uparrow$. Since $b > a$, this tells us that $K(f)$ is a minimal cutset which is not an antichain. $\square$

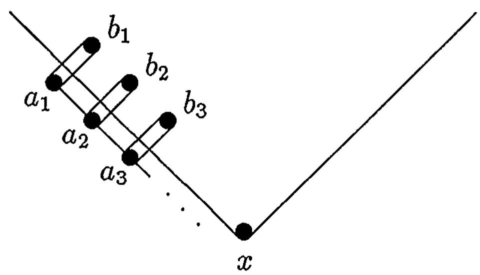
Thus, we have recovered, by an easier method, Rival and Zaguia's result [RZ87]:

Corollary 3.5.4 *Every regular, well-founded, cutset-straight poset is fence-free.* $\square$

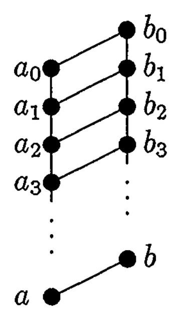
Another fairly easy corollary is:

Corollary 3.5.5 *If P is a regular, antichain-finite, cutset-straight poset, then P is fence-free.*

Proof. Let $x \in P$. Suppose C is a maximal chain of P and $C \cap A(x)$ has no least element. Let $a_0 > a_1 > a_2 > a_3 > \dots$ be an infinite descending chain in $C \cap A(x)$. For each $i \in \mathbf{N}$, let C_i be a maximal chain of P such that $a_i = \max(C_i \setminus x\mathring{\uparrow})$, and let $b_i = \min(C_i \cap x\mathring{\uparrow})$, so $b_i \succ a_i$. (We need $i \geq 1$ to make sure that $a_i \notin \max P$ so that b_i exists.)



The b_i's are all distinct so there are infinitely many of them. Since all antichains in P are finite, there is an infinite chain of b_i's. These b_i's and the corresponding a_i's must contain a copy of $\omega^{\mathrm{d}} \times \mathbf{2}$, which we will now show is impossible in a regular poset. We may assume that we chose the a_i's and b_i's so that all of them together form a copy of $\omega^{\mathrm{d}} \times \mathbf{2}$ as in the diagram below. Let $a = \inf_i a_i$ and $b = \inf_i b_i$.



Clearly, $b \geq a$. By regularity, if $b > a = \inf_i a_i$, then $b > a_j$ for some particular a_j. But then we would have $a_j < b < b_i$ for every b_i which obviously is not the case. Therefore, $b = a$. So each $a_j > a = b = \inf_i b_i$. By regularity, this says that each a_j is greater than some b_i, which obviously is not true. So P cannot contain any copy of $\omega^{\mathrm{d}} \times \mathbf{2}$.

With this contradiction, we have shown that Theorem 3.5.3 applies to P. □

This author believes that if one wishes to prove that every regular cutset-straight poset is fence-free, then the proof above is the most promising starting point. Were it not for this, it would have been more expedient to replace some of this proof with references to certain facts in the literature. This author has not found any mention in the literature of the fact that a regular poset cannot contain any copy of $\omega \times \mathbf{2}$ or its dual, but Ginsburg [G84] showed that if an antichain-finite poset P contains no copy of $\omega \times \mathbf{2}$ or its dual, then $\mathcal{M}(P)$ is compact, and this is equivalent to saying that every cutset of P contains a finite cutset [BG84]. With Corollary 3.2.5, this gives us Corollary 3.5.5.

Although we will not use the term "special" for a little while yet, we define it here to avoid disrupting the beautifully written paragraph where it appears. Let P be a poset and C a chain in P. If $x = \sup C$, then x is called a *special supremum* if there exists $c \in C$ such that $c\updownarrow \subseteq x\updownarrow$. *Special infimum* is defined dually. A poset is called *special* if all its suprema and infima are special.

We are concerned in this chapter with finding minimal cutsets of a certain form. In posets with fences, we try to find minimal cutsets which are not antichains. A similar problem is to find classes of posets in which every element is contained in a minimal cutset. In [RZ87], Rival and Zaguia tackled the following question.

Question 3.5.6 *Is every element of a regular poset contained in a minimal cutset?*

Rival and Zaguia achieved an answer only with the additional condition of well-foundedness, as we just did in Corollary 3.5.2 using a different construction. Rival and Zaguia [RZ87] conjectured that the answer to this question is yes. Li erroneously attributes more to these authors. Paraphrasing Li [L89]: Rival and Zaguia [RZ87] observed that if, in addition to being regular, a poset P has the property that every element of P belongs to some minimal cutset, then there is a positive answer to *every minimal cutset is an antichain implies P is fence-free.* (This is as close to a direct quote as is conveniently possible.) There is no such observation in [RZ87] and Li repeats this error in [L92]. It seems Li was confusing [RZ87] with [EZ86], in which El-Zahar and Zaguia say: "if we could generalize [*Every element of a regular, well-founded poset is in a minimal cutset*] to regular posets, then the same proof [as in [RZ87]] of [*Every regular, well-founded, cutset-straight poset is fence-free*] could be generalized to regular ordered sets too." But regardless of where Li picked up this idea, it is odd that he did not use it to state: *Every regular, special, cutset-straight poset is fence-free,* since in [L89] he proved: *Every point in a regular, special poset is in a minimal cutset.* One possible problem is that El-Zahar and Zaguia [EZ86] gave no reference or justification for their statement, and it seems to this author that the proof in [RZ87] to which El-Zahar and Zaguia were referring would require considerable enhancement to achieve what they say it could achieve, if it could be made to work at all in a more general context. Perhaps a regular, cutset-straight poset is fence-free if every element is in a minimal cutset of a type similar to that used in [RZ87], or some other special type. For instance, it seems reasonable to think that previously known constructions could be used to show that a regular, cutset-straight poset is fence-free if every element x is contained in a minimal cutset K such that $K \cap x\updownarrow = \{x\}$

($i.e.$ there is a minimal cutset for x), but in fact, Li [L89] shows that every element x in a regular, special poset is in a minimal cutset K such that $K \cap x\updownarrow = \{x\}$. So maybe this "reasonable" idea is not as simple as it seems, and apparently Question 3.5.7 is still open, along with the more difficult Question 3.5.8.

Question 3.5.7 *Is every regular, special, cutset-straight poset fence-free?*

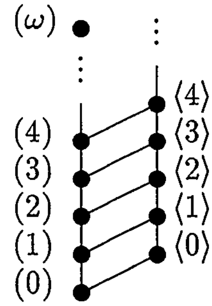
Question 3.5.8 *Is every regular, cutset-straight poset fence-free?*

Example 3.4.1 has some relevance to Question 3.5.8. If P is the poset in Example 3.4.1, then $P \oplus \{1\}$ is cutset-straight but has a fence. But it is *sup-regular* according to Grillet's definition [G69]. That is, every chain C in $P \oplus \{1\}$ has a supremum and $(\sup C)\updownarrow \subseteq C\downarrow$.

Question 3.5.8 was first raised by Higgs [H85] who observed that his proof that finite cutset-straight posets are fence-free would also work for regular posets if they had the property that every cutset of a particular form contains a minimal cutset. In [H85], Higgs says "I do not know whether [cutset-straight implies fence-free] for all regular posets". This author sees no way to interpret Higgs's statement as a conjecture. However, El-Zahar and Zaguia [EZ86], and Li [L89] have said that Higgs conjectured that all regular, cutset-straight posets are fence-free and they give only [H85] as a reference. (Li does not list [EZ86] as a reference.) This questionable interpretation does not appear in [RZ87] which extends Higgs's result to regular, well-founded posets.

3.6 Posets with Integer Chains

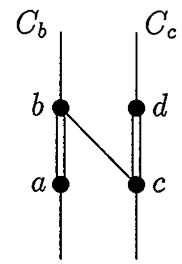
The literature does not address the question of whether all well-founded cutset-straight posets are fence-free. In the question for well-founded, regular posets, progress began with finding that every element is a member of a minimal cutset. If we only have the condition that a poset is well-founded, then we cannot get the same result. Below is a well-founded poset in which the point (ω) does not belong to any minimal cutset. The comparability graph of this poset is the complement of the one on which Example 3.3.1 is based.



This author has been unable to make any progress with respect to the question of cutset-straight implying fence-free in well-founded posets in general. However, the following theorem solves a simple case, and improves slightly on the result (Corollary 3.2.10) that all chain-finite cutset-straight posets are fence-free.

Theorem 3.6.1 *Let P be a poset all of whose chains are embeddable in ω and which has a fence. Then P has a minimal cutset which is not an antichain.*

Proof. Since every chain is embeddable in ω, P has a fence of the form $a \prec b > c \prec d$. In fact, choose (a, b, c, d) an N if possible. Let C_b, C_c be maximal chains with $a, b \in C_b$ and $c, d \in C_c$.



Let $C'_b = C_b \cap b\updownarrow$ and $C'_c = C_c \cap c\updownarrow$. So C'_b and C'_c are finite chains. Let

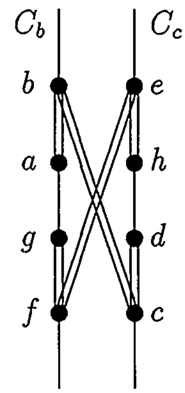
$$K_1 = \{y : y \succ x \text{ for some } x \in C'_b\},$$

$$K_2 = \{y : y \succ x \text{ for some } x \in C'_c\},$$

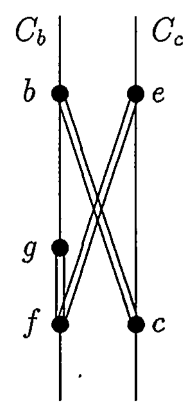
$$K = ((\min P) \cup K_1 \cup K_2) \smallsetminus (C'_b \cup C'_c).$$

Notice $b, c \in K$. We prove that K is a cutset. Let C be a maximal chain of P. If $\min C \notin \{\min C_b, \min C_c\}$, then $\min C \in K$. So assume that $\min C$ is $\min C_b$ or $\min C_c$. Let $x = \max((C'_b \cup C'_c) \cap C)$. (Since C'_b and C'_c are finite, this maximum exists. Also, $x \neq \max C$.) Let $y \succ x$ in C. Then $y \in K$. So K is a cutset. For any $z \in P$, $\mathrm{UC}(z)$ is an antichain. So K_1 and K_2 are each a union of a finite number of antichains. So K contains no infinite chain. Therefore, by Corollary 3.2.8, K contains a minimal cutset K'.

If K' is not an antichain, then we are done. So assume that K' is an antichain. Then we cannot have both b and c in K'. Since $K \cap C_b = \{b\}$, we know that $b \in K'$ and $c \notin K'$. But $c \in K$. Therefore, $C_c \cap K \neq \{c\}$. Let $e = \min(C_c \cap K \smallsetminus \{c\})$. $K \cap C'_c = \emptyset$, so $e > c$. Since $e \in K_1$, there is an $f \in C'_b$ such that $e \succ f$. The diagram below should be taken with a grain of salt, since it is possible that $f = a$ or $e = d$, but not both since $a \parallel d$. Let $g \succ f$ in C_b and $h \prec e$ in C_c. Then (g, f, e, h) is an N. We said at the start of this proof that we would choose (a, b, c, d) an N if possible. So we may assume that $b \succ c$. So we have the situation depicted below with the possibility that $f = a$ or $e = d$, but not both.



We may assume without loss of generality that $f \neq a$ and henceforth use the following simpler diagram with all points shown as distinct guaranteed to actually be distinct.



Let $C_g'' = C_b \cap f\downarrow$ and $C_c'' = C_c \cap c\updownarrow$. Let

$$K_1' = \{y \notin C_g'' : y \succ x \text{ for some } x \in C_g''\},$$

$$K_2' = \{y \notin C_c'' : y \succ x \text{ for some } x \in C_c''\},$$

$$L = ((\min P) \cup K_1' \cup K_2') \setminus (C_g'' \cup C_c'').$$

By the same argument as for K, we can say that L is a cutset which contains a minimal cutset L'.

Let $C_e = (C_c \cap e\uparrow) \cup C''_g$. Then C_e is a maximal chain and $C_e \cap L = \{e\}$. Therefore, $e \in L'$. Let $C^*_c = (C_b \cap b\uparrow) \cup (C_c \cap c\downarrow)$. Then C^*_c is a maximal chain and $C^*_c \cap L = \{c\}$. Therefore, $c \in L'$. Since $c, e \in L'$, L' is a minimal cutset which is not an antichain. $\square$

Because of Corollary 3.2.5, the next theorem is stronger than a "cutset-straight implies fence-free" theorem.

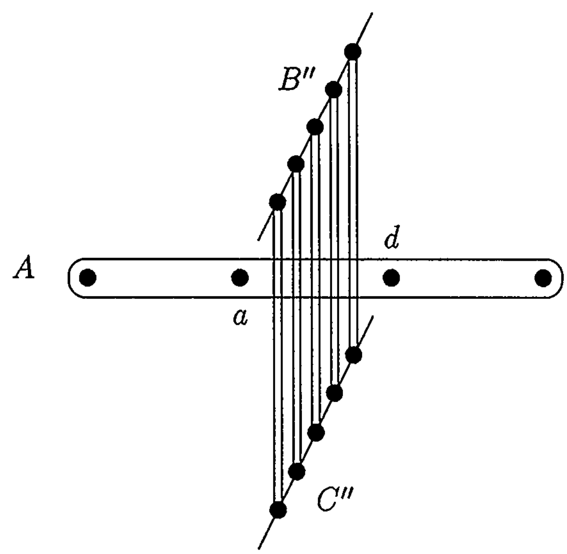
Theorem 3.6.2 *Let P be a poset in which every chain is embeddable in the chain of integers and every antichain is finite. Then every cutset of P contains a finite cutset.*

Proof. Let A be a maximal antichain of P. Then for any maximal chain C, $C \cap A\uparrow \neq \emptyset$ and $C \cap A\downarrow \neq \emptyset$. Let

$$X_0 = \{\min(C \cap A\uparrow) : C \text{ is a maximal chain of } P\}.$$

Then, obviously, X_0 is a cutset of P. A less obvious fact is that X_0 is finite, which we now prove.

Suppose for a contradiction that X_0 is infinite. Since $X_0 \subseteq A\uparrow$ and A is finite, this tells us that there is some $a \in A$ such that $|a\uparrow \cap X_0|$ is infinite. Let $B = X_0 \cap a\updownarrow$, obviously B is infinite. For each $b \in B$, there is a maximal chain C such that $b = \min(C \cap A\uparrow)$. So for each $b \in B$, we can choose some $f(b) \prec b$ such that $f(b) \in A\updownarrow$. B is an infinite set having no infinite antichains. Therefore, B contains an infinite chain B'. Let $C' = f(B')$. Since $b'_1 \neq b'_2 \Rightarrow f(b'_1) \neq f(b'_2)$ for $b'_1, b'_2 \in B'$, C' is also infinite. Since C' cannot contain any infinite antichains, this tells us that C' contains an infinite chain C''. There must be some $d \in A$ such that $C'' \subseteq d{\downarrow}$. Let $B'' = \{b \in B' : f(b) \in C''\}$. Then f determines a 1-1 correspondence between B'' and C''. In fact, f determines an order-isomorphism between B'' and C''. Since B'' is an infinite chain, and every chain is embeddable in the integers, B'' must be isomorphic to ω, ω^{d}, or the chain of integers. Since a is a lower bound of B'', and every chain is embeddable in the integers, this tells us that $B'' \cong \omega$. Similarly, we get $C'' \cong \omega^{\mathrm{d}}$. But this contradicts $B'' \cong C''$. Therefore, X_0 is finite.



The rest of this proof is like a proof of König's Lemma.

Let $L_0 = X_0$. Define subsequent L_i's by $L_i = \min(L_0{\uparrow} \setminus \bigcup_{j<i} L_j)$ and $L'_{-i} = \max(L_0{\downarrow} \setminus \bigcup_{j<i} L_{-j})$ for each $i \in \omega$. So every L_i is an antichain except for L_0, and every L_i is finite. Since every chain is embeddable in $\mathbf{Z}$, $P = \bigcup_{i \in \mathbf{Z}} L_i$.

Let K be a cutset of P. For each chain $X \subseteq P$, define $g(X)$ in $\omega \oplus \{\infty\}$ by

$$g(X) = \sup\{k : \exists C \in \mathcal{C}(X) \text{ such that } C \cap K \cap (\bigcup_{-k<i<k} L_i) = \emptyset\}.$$

So $g(X)$ tells us how far we can extend X before hitting K. If $g(\{x\})$ is finite for every $x \in X_0$ then, since X_0 is a finite cutset, there exists $k = \max\{g(\{x\}) :$

$x \in X_0\}$, and $K \cap (\bigcup_{-k \leq i \leq k} L_i)$ is a finite cutset of P. So assume there is some $x_0 \in X_0$, such that $g(\{x_0\}) = \infty$, so $x_0 \notin K$. Clearly, $g(\{x_0\}) = \sup\{g(\{x_0, x_1\}) : x_1 \succ x_0\}$, unless $x_0 \in \max P$. Since $\mathrm{UC}(x_0)$ is an antichain and therefore finite, this tells us that if $x_0 \notin \max P$, then there exists $x_1 \succ x_0$ such that $g(\{x_0, x_1\}) = \infty$, so $x_1 \notin K$. Similarly, if $x_0 \notin \min P$, then there exists $x_{-1} \prec x_0$ such that $g(\{x_{-1}, x_0, x_1\}) = \infty$ with $x_{-1} \notin K$, and so on. So we have a chain $\ldots \prec x_{-2} \prec x_{-1} \prec x_0 \prec x_1 \prec x_2 \prec \ldots$ disjoint from K with the possible restriction that this chain terminates at a minimal or maximal element of P or both. Since every chain in P is embeddable in the integers, this chain must be maximal, and therefore must meet K, a contradiction. So K contains a finite cutset. □

Applying Corollary 3.2.5 to Theorem 3.6.2, we get:

Corollary 3.6.3 *If P is an antichain-finite, cutset-straight poset in which every chain is embeddable in the chain $\mathbf{Z}$, then P is fence-free.* □

It is unknown whether the antichain-finite condition in Corollary 3.6.3 can be dropped:

Question 3.6.4 *Is every cutset-straight poset in which every chain is embeddable in the integers fence-free?*

3.7 Summarising the Known and the Unknown

As mentioned earlier, the following question is not addressed in the literature, and this author has not been able to find an answer.

Question 3.7.1 *Is every well-founded, cutset-straight poset fence-free?*

Another surprisingly slippery question is:

Question 3.7.2 *Is every antichain-finite, cutset-straight poset fence-free?*

In fact, as mentioned earlier, it is even unknown whether every cutset-straight poset of width 2 is fence-free. One might hope to get an easier question by combining the last two. The following is also unknown.

Question 3.7.3 *Is every well-founded, antichain-finite, cutset-straight poset fence-free?*

To summarise the posets we do know about, a cutset-straight poset is fence-free if it satisfies any of the following conditions, none of which implies either of the others.

(*i*) every cutset contains a minimal cutset (Corollary 3.2.5)

(*ii*) regular and well-founded (Corollary 3.5.4)

(*iii*) every chain embeddable in ω (Theorem 3.6.1)

We could also put the following three conditions in this list, but they would be redundant since each of them implies (*i*).

(*iv*) chain-finite (Corollary 3.2.8)

(*v*) regular and antichain-finite (comments following Corollary 3.5.4)

(*vi*) antichain-finite and every chain embeddable in $\mathbf{Z}$ (Theorem 3.6.2)

The only nicely-defined condition for which it is known that cutset-straight does not imply fence-free is sup-regular.

As mentioned before, if a poset P has no minimal cutset, then $P + F_4$ has no minimal cutset and so is cutset-straight, but it is not fence-free. Thus, a negative answer to Question 3.7.4 or 3.7.5 implies a negative answer to the corresponding one of Question 3.6.4 or 3.7.2.

Question 3.7.4 *Does every poset in which every chain is embeddable in the integers have a minimal cutset?*

Question 3.7.5 *Does every antichain-finite poset have a minimal cutset?*

Since the set of minimal elements of a well-founded poset is a minimal cutset, the issue addressed in Questions 3.7.4 and 3.7.5 is not an issue for the posets in Questions 3.7.1 and 3.7.3.

Chapter 4

Partitions and Fibres

"The footprints of delivery vans corrugated the slush."
— Salman Rushdie, *The Satanic Verses* [1988]

In this chapter we look at some of the research which has followed from a paper by Aigner and Andreae [AA86]. It is perhaps surprising how many interesting results have followed from the paper considering it was never published in a journal. The main point of the paper was a proof of a conjecture of Gallai which was communicated to Aigner and Andreae by Erdős: *Let G be a triangulated graph on n vertices without isolated points. Then there is a set of at most $\frac{n}{2}$ vertices that meets all maximal cliques of G.*

In [AA86], Aigner and Andreae prove the following theorem. The reader does not need to know the meanings of all the terms mentioned in the theorem to discern the pattern that gives rise to a question.

Theorem 4.1 *A finite graph G with no isolated points contains a set of at most $\frac{|V(G)|}{2}$ vertices that meets all maximal cliques of G if G satisfies any of the following conditions:*

(*i*) *G is a triangulated graph.*

(*ii*) *G is the complement of a triangulated graph.*

(*iii*) *G is a bipartite graph.*

(*iv*) *G is the complement of a bipartite graph.*

(*v*) *G is the line graph of a bipartite graph.*

(*vi*) *G is the complement of the line graph of a bipartite graph.*

(*vii*) *G is the comparability graph of a poset.* □

The question which immediately leaps to mind is: *What about the complement of the comparability graph of a poset?* Aigner and Andreae left this as an open question. Call a point x in a poset P a *splitting element* if $x\updownarrow = P$. Then in terms of fibres, the question is:

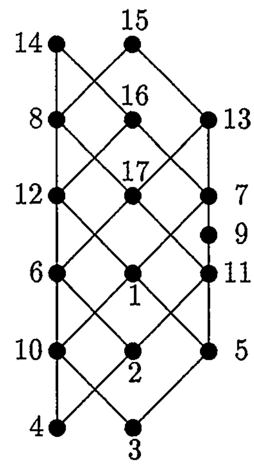
Question 4.2 *Does every finite poset P with no splitting element have a fibre of size $\leq \frac{|P|}{2}$?*

Lonc and Rival [LR87] erroneously reported that Aigner and Andreae conjectured in [AA86] that the answer to this question is yes. In fact, Aigner and Andreae expressed no opinion regarding the answer. But Lonc and Rival [LR87] made a conjecture even stronger than a positive answer to Question 4.2 [AA86]. Their conjecture was: *For every ordered set P without any splitting element, there is a subset F such that both F and $P \setminus F$ are fibres.* Although this conjecture turned out to be false in general [DSSW91], Lonc and Rival described four cases in which it is true. One of these brings the following question to mind:

Question 4.3 *What is the least possible height of a finite poset P with no splitting element and no fibre F such that $P \setminus F$ is also a fibre?*

The relevant result of Lonc and Rival [LR87] shows that the answer is at least 5, but we will not repeat the proof here since it is quite long. In [DSSW91], Duffus, Sands, Sauer, and Woodrow presented Example 4.4, a 17-element poset in which every fibre has at least 9 elements. So Example 4.4 disproves the conjecture of Lonc and Rival and gives a negative answer to Question 4.2. Example 4.4 has height 7 and so the cases of heights 5 and 6 are left unanswered for Question 4.3.

Example 4.4 [DSSW91] Let P be the poset whose Hasse diagram appears below. P has 17 elements but has no fibre with fewer than 9 elements.



Proof. Each of $\{1,2\}$, $\{2,3\}$, $\{3,4\}$, $\ldots$, $\{16,17\}$ is a maximal antichain. The only set with fewer than 9 elements that meets all of these maximal antichains is $\{2,4,6,8,10,12,14,16\}$. So this 8-element set is the only candidate to be a fibre having fewer than 9 elements. But this set is not a fibre since it does not meet the maximal antichain $\{1,9,17\}$. Thus, every fibre of this poset has at least 9 elements. Notice that $\{1,3,5,7,\ldots,17\}$ is a fibre having exactly 9 elements. $\square$

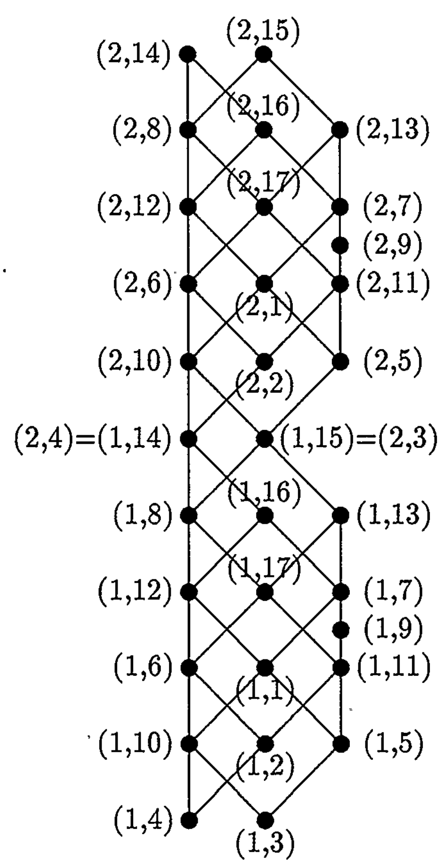
This result led to the following question in [DSSW91]:

Question 4.5 *Let λ be the smallest real number such that every finite poset P with no splitting element has a fibre of size at most $\lambda|P|$. What is the exact value of λ?*

Example 4.4 shows that $\lambda \geq \frac{9}{17} \approx .5294$. Later on we will see a result of Duffus, Kierstead, and Trotter showing that $\lambda \leq \frac{2}{3}$. But first we will examine an example by this author [M92], showing how to stack copies of Example 4.4 to demonstrate that $\lambda \geq \frac{8}{15} \approx .5333$.

Example 4.6 Let n be a positive integer. Let $P_n = \{(i,j) : 1 \leq i \leq n, 1 \leq j \leq 17\}$ with each $(i,3)$ identified with $(i-1,15)$ and each $(i,4)$ identified with $(i-1,14)$. Let the ordering on P be the transitive closure of that induced by putting $(i,j) \leq (i,j')$ in P whenever $j \leq j'$ in Example 4.4. Then P_n has $15n+2$ elements, but has no fibre with fewer than $8n+1$ elements.

Proof. P_2 is shown below as an example of the construction.



For $1 \leq i \leq n$, let $Q_i = \{(i,j) : j = 1,2,5,6,7,\ldots,17\}$. Let $(0,15)$ be another label for $(1,3)$ and $(0,14)$ another label for $(1,4)$. Let $Q_0 = \{(0,14),(0,15)\}$. Then $\{Q_i : 0 \leq i \leq n\}$ is a partition of P_n and each $|Q_i| = 15$ except for $|Q_0| = 2$. When referring to points in a particular Q_i, it will be convenient to refer to the points using only their second coordinates. Suppose F is a fibre of P_n. Since Q_0 is a maximal antichain of P_n, we know that $|F \cap Q_0| \geq 1$. Now suppose $i \in \{1,\ldots,n\}$ and consider $Q_i \cap F$. In Q_i, $\{1,2\}$, $\{5,6\}$, $\{6,7\}$, $\{7,8\}$, ..., $\{16,17\}$ are maximal antichains of Q_i, and in fact are maximal antichains of P_n since $P_n \smallsetminus Q_i \subseteq (\max Q_i)\uparrow \cup (\min Q_i)\downarrow$. Thus, F must include one of $\{1,2\}$ and six of $\{5,6,7,\ldots,17\}$. So $|F \cap Q_i| \geq 7$. If we could show that $|F \cap Q_i| \geq 8$ then we would be done. In fact, it is not quite that simple. What we shall do instead is to show that if $|F \cap Q_i| < 8$ then $|F \cap Q_i| = 7$ and $|F \cap Q_{i-1}| \geq 9$, unless $i = 1$ in which case we will get $|F \cap Q_{i-1}| = 2$. Suppose $|F \cap Q_i| < 8$. Since F includes one of $\{1,2\}$ and at least six of $\{5,6,\ldots,17\}$, the only way this can happen is if F contains exactly six of $\{5,6,\ldots,17\}$. And since $\{5,6\}$, $\{6,7\}$, ..., $\{16,17\}$ are maximal antichains, these six points in F must be $6,8,10,12,14,16$. And since $\{1,9,17\}$ is a maximal antichain, this means that 1 must be the point of $\{1,2\}$ which is in F. $\{2,3\}$ and $\{4,5\}$ are maximal antichains and 2 and 5 are not in F. So $3,4 \in F$. That is, $(i-1,15),(i-1,14) \in F$. If $i = 1$ then this shows that $|F \cap Q_{i-1}| = 2$ as we wanted. So assume that $i > 1$. $\{(i-1,16),(i,5)\}$ is a maximal antichain and $(i,5) \notin F$ so $(i-1,16) \in F$. $\{(i-1,13),(i,2)\}$ is a maximal antichain and $(i,2) \notin F$ so $(i-1,13) \in F$. So far we have $13,14,15,16 \in Q_{i-1} \cap F$. Because of the maximal antichains $\{1,2\}$, $\{5,6\}$, $\{6,7\}$, ..., $\{11,12\}$, $Q_{i-1} \cap F$ must include one of $1,2$ and four of $5,6,\ldots,12$. Thus, $|Q_{i-1} \cap F| \geq 9$, and we are done. That is, it is clear that $|P_n| = 15n + 2$ and we have shown that every fibre of P_n has at least $8n + 1$ points.

Notice that $\{(i, j) : 1 \leq i \leq n, 1 \leq j \leq 17, j \text{ odd}\}$ is a fibre of P_n of size exactly $8n + 1$. $\square$

Example 4.6 gives the best known lower bound for the value λ of Question 4.5, that bound being $\frac{8}{15}$. The best known upper bound of λ is $\frac{2}{3}$. This result was achieved by Duffus, Kierstead, and Trotter [DKT91] using a coloring technique. They described a way to partition any finite poset with no splitting element into three color classes so that the union of any two of the classes is a fibre. Taking the two smallest color classes yields a fibre of size at most $\frac{2}{3}$ the size of the whole poset. In infinite cases, such fractions are meaningless, but we can find interesting results about colorings. For any poset P, a *good* [G92] coloring is one such that P has no monochromatic maximal antichain with more than one element. In a poset with no splitting element, a good k-coloring is one such that the union of any $k - 1$ of the color classes is a fibre. (A splitting element comprises a one-element maximal antichain.) Goddard [G92] determined a class of infinite posets to which the construction in [DKT91] may be applied:

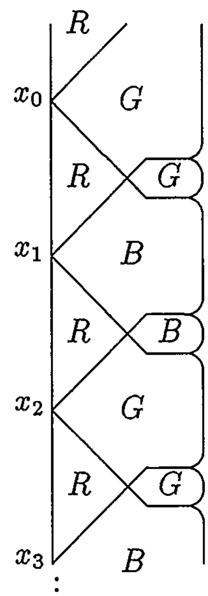
Theorem 4.7 *Let P be a well-founded (finite or infinite) poset which contains an element x such that no maximal antichain of P is contained in $x\uparrow$. If P cannot be expressed as a linear sum of non-empty subsets, then P has a good 3-coloring.*

Proof. Notice that the hypothesis implies that P has no splitting element, and that $|P| > 1$.

Let $x_0 \in P$ such that there is no maximal antichain of P contained in $x_0\uparrow$. Then we choose $x_1 > x_2 > \ldots$ as follows until we choose some $x_i \in \min P$.

Suppose x_i has been chosen, x_{i+1} has not, and $x_i \notin \min P$. Then we choose x_{i+1} as follows. First, notice that by the following there must be an element $y < x_i$ such that $P \setminus (x_i \updownarrow \cup y \updownarrow) \neq \emptyset$. Suppose for a contradiction that no such y exists. Then $P \setminus x_i \updownarrow \subseteq y \updownarrow$ for every $y < x_i$. In fact, this means that $P \setminus x_i \updownarrow \subseteq y \uparrow$ for every $y < x_i$. Thus $P = x_i \updownarrow \oplus (P \setminus x_i \updownarrow)$. This contradicts our hypothesis that P cannot be expressed as a linear sum of non-empty subsets. Therefore, there is some $y < x_i$ such that $P \setminus (x_i \updownarrow \cup y \updownarrow) \neq \emptyset$. Choose x_{i+1} to be a minimal such y. In this way we get a decreasing sequence $x_0 > x_1 > x_2 > \ldots > x_n$ such that no maximal antichain is contained in $x_0 \uparrow$ and $x_n \in \min P$. The sequence must reach $x_n \in \min P$ in a finite number of steps since P is well-founded.

Next, we partition P into 3 color classes. Color $x \in P$ red if x is comparable to all the x_i's. Denote the set of red points by R. If $x \in P \setminus R$, then color x blue or green according as the least i such that $x \parallel x_i$ is odd or even.



Suppose A is a red maximal antichain. Then $A \not\subseteq x_0 \uparrow$. Since A is an antichain and $\{x_0, x_1, \ldots, x_n\}$ is a chain with $x_n \in \min P$, there exists $i \in \{0, \ldots, n-1\}$ such that $x_{i+1} < A < x_i$. But now A cannot be a maximal antichain since $A\updownarrow = A\uparrow \cup A\downarrow \subseteq x_{i+1}\uparrow \cup x_i\downarrow$. We chose x_i and x_{i+1} so that there are elements comparable to neither of them, so these elements will not be in $A\updownarrow$. Therefore, there is no red maximal antichain.

Let D be a set of blue points such that $D\updownarrow = P$. For each $b \in D$, there is a least i such that $b \parallel x_i$. These i's cannot all be the same. Let $b, c \in D$, and let i least such that $b \parallel x_i$ and j least such that $c \parallel x_j$. We may assume that we chose b and c so that $i < j$. Since b and c are blue, i and j are both odd, so in fact we have $i < j - 1$. Our choice of j ensures that $c < x_{j-1} \leq x_{i+1} < x_i$. Since x_{i+1} was chosen minimal having incomparable points in common with x_i, this tells us that there are no points incomparable to both c and x_i. Since $b \parallel x_i$, b and c must be comparable. So D is not an antichain. Thus, there is no blue maximal antichain.

The argument that there is no green maximal antichain is analogous to that for blue. $\square$

Lemma 4.8 *Let $P = \sum_{c \in C} P_c$ be a poset where C is a chain and each P_c has a good 3-coloring. Then so does P.*

Proof. For each P_c, choose a good 3-coloring using red, blue, and green. Then the 3-coloring induced on P is good. $\square$

The following corollary is Duffus, Kierstead, and Trotter's result [DKT91].

Corollary 4.9 *If P is a finite poset, then it has a good 3-coloring.*

Proof. Let $\sum_{c \in C} P_c$ be the linear decomposition of P. By Lemma 4.8, it suffices to prove this corollary for each P_c. Let $c \in C$. If $P_c = \mathbf{1}$, then any coloring is a good 3-coloring. If $P_c \neq \mathbf{1}$, then P_c has more than one maximal element, so any maximal element of P_c can stand for x so that Theorem 4.7 may be applied. □

The next corollary is due to Goddard [G92].

Corollary 4.10 *Every antichain-finite poset P has a good 3-coloring.*

Proof. Apply the compactness theorem and Corollary 4.9 to the following set of sentences $\mathcal{S}$. For each $x \in P$, let R_x, G_x, and B_x be truth-functional variables and put the following sentence in $\mathcal{S}$:

$$(R_x \vee G_x \vee B_x) \mathbin{\&} \neg(R_x \mathbin{\&} G_x) \mathbin{\&} \neg(R_x \mathbin{\&} B_x) \mathbin{\&} \neg(B_x \mathbin{\&} G_x)$$

R_x is to be interpreted as "x is red", and B_x and G_x similarly for blue and green. Thus, any assignment of truth-functional values to all the R_x's, G_x's, and B_x's which makes all the above sentences true corresponds to a 3-coloring of P.

Also put the following sentence in $\mathcal{S}$ for every $\{x_1, x_2, \ldots, x_n\}$ a maximal antichain of P:

$$\neg(R_{x_1} \mathbin{\&} R_{x_2} \mathbin{\&} \ldots \mathbin{\&} R_{x_n}) \mathbin{\&} \neg(G_{x_1} \mathbin{\&} G_{x_2} \mathbin{\&} \ldots \mathbin{\&} G_{x_n}) \mathbin{\&} \neg(B_{x_1} \mathbin{\&} B_{x_2} \mathbin{\&} \ldots \mathbin{\&} B_{x_n})$$

This completes the description of $\mathcal{S}$.

Let $V = \{R_x : x \in P\} \cup \{G_x : x \in P\} \cup \{B_x : x \in P\}$. There is an assignment of truth values to all the elements of V which makes all the sentences in $\mathcal{S}$ true if and only if there is a good 3-coloring of P. By the compactness theorem [Appendix A], there is an assignment of truth values to all the elements of V which makes all the sentences in $\mathcal{S}$ true if and only if the same is true for every finite subset of $\mathcal{S}$ and its associated variables. Let $\mathcal{S}'$ be a finite subset of $\mathcal{S}$, and let V' be the set of variables appearing in sentences in $\mathcal{S}'$. Let X be the set of points $x \in P$ such that R_x, B_x, or G_x appears in $\mathcal{S}$. (In fact, if one appears, they all do, but this is not important.) Consider X as a poset with the ordering induced by P. By Corollary 4.9, X has a good 3-coloring. With this coloring is associated an assignment of truth-values to the elements of V' which makes all the sentences in $\mathcal{S}'$ true. Therefore, by the compactness theorem, there is an assignment of truth-values to all the elements of V which makes every sentence in $\mathcal{S}$ true. This is equivalent to saying that P has a good 3-coloring. $\square$

The proof above shows that a good 3-coloring exists, but it does not give one any sort of image of how the coloring looks. We now present a more constructive proof of Corollary 4.10. This proof involves picking maximal chains. According to Woodrow, the necessity of using Zorn's Lemma to be sure that maximal chains exist disqualifies this step from being considered constructive, but he claims that it is "not too terribly sinful, either". If one agrees with this assessment, it seems the following proof is about as close to being constructive as one could hope for.

Proof. By Lemma 4.8, we may assume that P is not linearly decomposable.

Let C_0 be a singleton subset of P. Define sequences of C_α's, X_α's, and Y_α's indexed by ordinals ≥ 1 as follows. Let

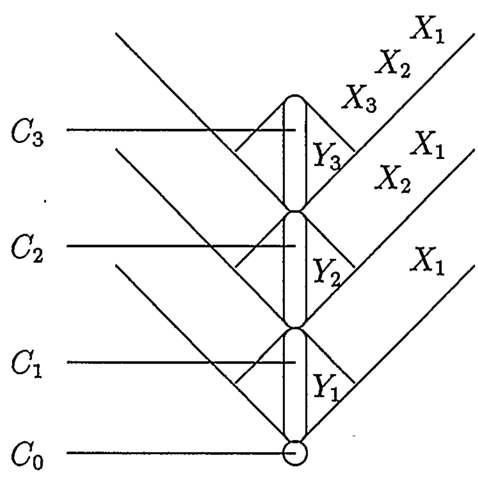
$$X_\alpha = \{x \in P : x > C_\beta \text{ for every } \beta < \alpha\},$$

$$Y_\alpha = \{y \in X_\alpha : y{\downarrow} \cap X_\alpha \text{ contains no maximal antichain of } P\}, \text{ and}$$

$$C_\alpha \text{ a chain maximal in } Y_\alpha.$$

Apply these definitions by transfinite induction until $X_\alpha = \emptyset$. Notice that $X_\alpha{\uparrow} = X_\alpha$ for every X_α. In other words (for those who like such words) X_α is an up-set or a dual order ideal. Notice that whenever $Y_\alpha \neq \emptyset$ and $\beta > \alpha$, we get $X_\beta \subset X_\alpha$ and $C_\beta > C_\alpha$. We should verify that $Y_\alpha \neq \emptyset$ whenever $X_\alpha \neq \emptyset$. Suppose α is an ordinal such that X_α is not empty but Y_α is. Let $a \in X_\alpha$ and $b \in P \setminus X_\alpha$. Since $a \notin Y_\alpha$, there is a maximal antichain A of P contained in $a{\downarrow} \cap X_\alpha$. Then $b \in A{\updownarrow}$. Since $b \notin X_\alpha$, we cannot have $b \in A{\uparrow}$. So $b \in A{\updownarrow}$. Since $A \subset a{\downarrow}$, this tells us that $a > b$. So $P = (P \setminus X_\alpha) \oplus X_\alpha$. This contradicts our assumption that P is not linearly decomposable, and so $Y_\alpha \neq \emptyset$ whenever $X_\alpha \neq \emptyset$. So for each $X_\alpha \neq \emptyset$, we also have $C_\alpha \neq \emptyset$. And the C_α's are disjoint since each $C_\alpha \subseteq X_\alpha$ and each X_α is defined to be disjoint from all previous C_α's. Therefore, we will eventually get $X_\alpha = \emptyset$ for some ordinal α such that $|\alpha| \leq |P|$.

Before trusting the following diagram, we should verify that the Y_α's are disjoint. Suppose $y \in Y_\alpha$ and $\beta < \alpha$. Then $y \in X_\alpha$, so $y > C_\beta$. Thus, $C_\beta \cup \{y\}$ is a chain strictly containing C_β which is a chain maximal in Y_β, so $y \notin Y_\beta$. Therefore, $Y_\alpha \cap Y_\beta = \emptyset$.



Next, define sequences dual to these according to the following. Let $C'_0 = C_0$,

$$X'_\alpha = \{x \in P : x < C'_\beta \text{ for every } \beta < \alpha\},$$

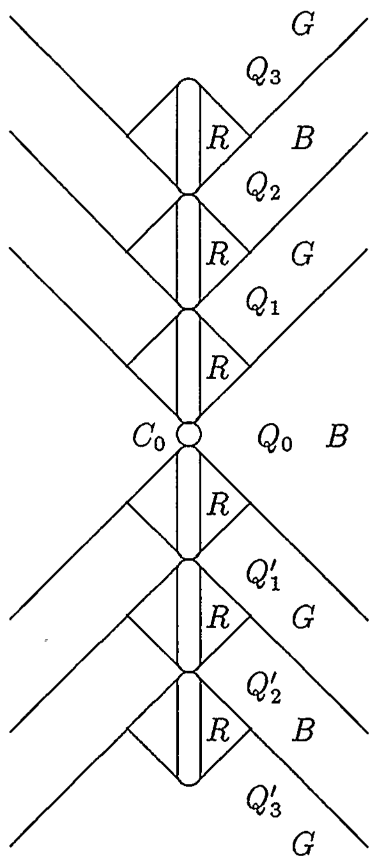
$$Y'_\alpha = \{y \in X'_\alpha : y\!\uparrow \cap X'_\alpha \text{ contains no maximal antichain of } P\}, \text{ and}$$

$$C'_\alpha \text{ a chain maximal in } Y'_\alpha.$$

Apply these definitions by transfinite induction until $X'_\alpha = \emptyset$.

For each α for which X_α is defined, let $R_\alpha = X_\alpha \cap C_\alpha\!\downarrow$, and for each α for which X'_α is defined, let $R'_\alpha = X'_\alpha \cap C'_\alpha\!\uparrow$. Let $R_0 = R'_0 = C_0 = C'_0$. Notice that whenever $\beta > \alpha$, $R_\beta > R_\alpha$ and $R'_\beta < R'_\alpha$. Let $R = (\bigcup_\alpha R_\alpha) \cup (\bigcup_\alpha R'_\alpha)$. For each ordinal $\alpha \neq 0$, let $Q_\alpha = X_\alpha \setminus (R \cup X_{\alpha+1})$ and let $Q'_\alpha = X'_\alpha \setminus (R \cup X'_{\alpha+1})$. Let $Q_0 = Q'_0 = P \setminus C_0\!\updownarrow$. Notice that $\{Q_\alpha : Q_\alpha \neq \emptyset\} \cup \{Q'_\alpha : Q'_\alpha \neq \emptyset\}$ is a partition of $P \setminus R$.

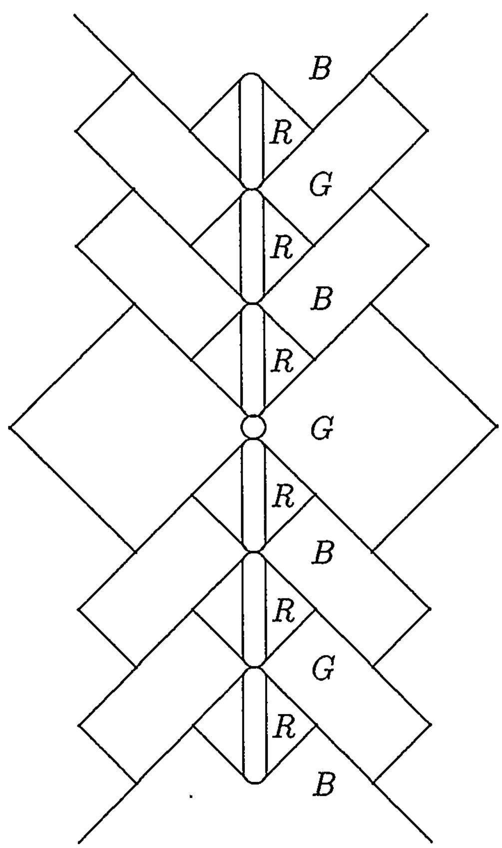
Any ordinal has a unique representation $\alpha + k$ where α is a limit ordinal or 0 and k is a finite ordinal. Call $\alpha + k$ *even* or *odd* according as k is even or odd in the usual sense. Let $B = (\bigcup_{\alpha \text{ even}} Q_\alpha) \cup (\bigcup_{\alpha \text{ even}} Q'_\alpha)$, $G = (\bigcup_{\alpha \text{ odd}} Q_\alpha) \cup (\bigcup_{\alpha \text{ odd}} Q'_\alpha)$. Color the elements of R red, B blue, and G green.



Is there a red maximal antichain? Suppose A is a red antichain. Then, clearly $A \subset R_\alpha$ or $A \subset R'_\alpha$ for some α. Assume without loss of generality that $A \subset R_\alpha$ for some α. That is, $A \subset X_\alpha \cap C_\alpha\downarrow$. Say $A = \{a_1, a_2, \ldots, a_k\}$. For each $a_i \in A$, let $c_i \in C_\alpha$ such that $a_i \leq c_i$. Then let $c = \max\{c_i : i = 1, \ldots, k\}$. So $c \in C_\alpha \subseteq Y_\alpha$. So $A \subset X_\alpha \cap c\downarrow$ cannot be a maximal antichain. Therefore, there is no red maximal antichain.

Is there a blue maximal antichain? First, we show that any antichain contained in the Q_α's and Q'_α's must be contained in two consecutive such sets. Since all the Q_α's other than $Q_0 = Q'_0$ are contained in $C_0\uparrow$ and all the Q'_α's other than $Q'_0 = Q_0$

are contained in $C_0\downarrow$, we know that any antichain contained in $(\bigcup_\alpha Q_\alpha) \cup (\bigcup_\alpha Q'_\alpha)$ must, in fact, be contained in $\bigcup_\alpha Q_\alpha$ or $\bigcup_\alpha Q'_\alpha$. We will consider the $\bigcup_\alpha Q_\alpha$ case, the other is dual. Let $a \in Q_\alpha$ and $b \in Q_\beta$ where $\beta > \alpha + 1$. $b \in Q_\beta \subset X_\beta \subset X_{\alpha+1}$. $b \in X_\beta$ implies $b > C_{\alpha+1}$. Since $C_{\alpha+1}$ is a maximal chain of $Y_{\alpha+1}$, this tells us that $b \notin Y_{\alpha+1}$. So $b \in X_{\alpha+1} \setminus Y_{\alpha+1}$. Therefore, by the definition of $Y_{\alpha+1}$, there exists $A \subset b\updownarrow \cap X_{\alpha+1}$ such that A is a maximal antichain of P. $a \in P = A\updownarrow$. $A\uparrow \subset X_{\alpha+1}\uparrow = X_{\alpha+1}$, so $a \notin A\uparrow$. So $a \in A\updownarrow \subset b\updownarrow$. That is, $a < b$. So any antichain contained in $(\bigcup_\alpha Q_\alpha) \cup (\bigcup_\alpha Q'_\alpha)$ must, in fact, be contained in two consecutive Q_α's or Q'_α's. This information allows us to improve the accuracy of the diagram.



Since any two consecutive Q_α's or Q'_α's have different colors, we can prove that there is no monochromatic maximal antichain by showing that there is no maximal antichain contained in any single Q_α or Q'_α. Suppose A is a maximal antichain contained in some Q_α. The case where a maximal antichain is contained in some Q'_β is dual, so it suffices to prove only the case just described. Obviously $\alpha \neq 0$ since $C_0 \not\subset Q_0\updownarrow$. Since $A \subset Q_\alpha \subset X_\alpha$, we know that $\bigcup_{\beta<\alpha} C_\beta < A$. Also, $C_\alpha \subset A\updownarrow$ but $C_\alpha \cap A\uparrow$ must be empty since $A \subset X_\alpha$ and $C_\alpha\downarrow \cap X_\alpha \subset R$. So $C_\alpha \subset A\downarrow$. Since A is finite and $\bigcup_{\beta\leq\alpha} C_\beta$ is a chain contained in $A\downarrow$, there exists $a \in A$ such that $\bigcup_{\beta\leq\alpha} C_\beta < a$. But then $a \in X_{\alpha+1}$, a contradiction. So no maximal antichain is contained in a single Q_α. So there is no blue or green maximal antichain. □

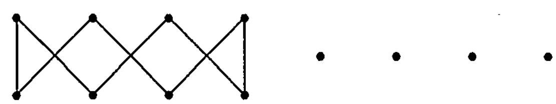
In [G92], Goddard presented several results about good colorings, but we will not mention any more of them here. A question he was unable to answer is the following:

Question 4.11 *Does every poset have a good 3-coloring?*

In fact, it is even unknown whether there is a poset which has no good finite-coloring, and this seems to be quite a difficult point to resolve. Superficially, Lemma 2.3.2 seems to indicate that products are unlikely to make useful examples of posets requiring many colors for a good coloring. This is because if P has no splitting element and P and Q satisfy the hypothesis of Lemma 2.3.2 (Q has a maximal element and P is well-founded), then it is easy to see by Lemma 2.3.2 that any good coloring of P provides a good coloring of $P \times Q$ by coloring each (p, q) in $P \times Q$ with the color of p in P. Thus, $P \times Q$ requires no more colors for a good coloring than P does. However, any posets satisfying the hypothesis of Lemma 2.3.2 are probably not posets one would try to use to resolve this issue.

So I suspect that this observation will be irrelevant to any reasonable attempt to resolve this issue.

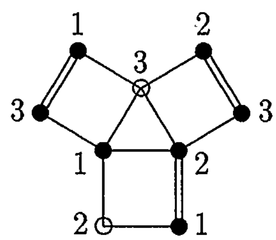
Another antichain-coloring result is: *Every poset has a 2-coloring so that every 2-element maximal antichain is 2-colored.* This result is due to Duffus, Sands, Sauer, and Woodrow [DSSW91]. One might wonder whether the 2's in this statement could be replaced by any natural number k. An unpublished example by Sands shows that this is not possible for $k \geq 3$. The example for $k = 3$ is the 8-element crown. For larger values of k, add $k-3$ isolated points to the 8-element crown.



The other open question in [AA86] pushes the boundary of relevance to this thesis. However, I think it is worth mentioning, since the only journal article to make relevant comments made apparently contradictory comments which might leave a reader wondering whether there is any point looking at the question. We need a couple of definitions for this question. The *chromatic number* of a graph G is the least number k so that the vertices of the graph can be k-colored so that no edge connects two vertices of the same color. A *perfect graph* is one in which every induced subgraph has its chromatic number equal to the size of its largest clique.

Question 4.12 *Suppose G is a perfect graph with no isolated points and T is a minimum-sized transversal of G. What is the greatest possible value of $\frac{|T|}{|V(G)|}$?*

In [LR87], Lonc and Rival say: "As every connected comparability graph [with n vertices] is perfect it contains a subset of at most $\frac{n}{2}$ vertices which meets every maximal clique". This statement seems implicitly to contradict something they say three sentences earlier: "... not every perfect graph [with n vertices] has a subset of at most $\frac{n}{2}$ vertices which meets every maximal clique." Can a minimum-sized transversal of a finite perfect graph include more than half the vertices? The following example in [AA86] shows that the answer is yes.



The graph above has only one 3-element clique and a 3-coloring is indicated which does not give the same color to any adjacent vertices. Thus any induced subgraph which contains the 3-element clique has chromatic number 3. And any induced subgraph which does not contain the 3-element clique is bipartite and so has chromatic number equal to the size of the largest clique. So the graph is perfect. However, we will now show that the graph has no transversal of fewer than 5 elements. Let T be a transversal of the graph. T must include one of the vertices of the triangle in the center of the diagram — assume without loss of generality that it includes the one marked with a hollow circle. T must also include one of the vertices in the edge most distant in the diagram from the vertex just mentioned — assume without loss of generality that T includes the other point marked with a hollow circle. Now the edges marked with double lines indicate three maximal cliques having no elements in common with each other or with the two points marked with hollow circles. Thus every transversal of this graph has at least 5 vertices.

Chapter 5

Cutsets for Elements

"... it is very easy to be blinded to the essential uselessness of them by the sense of achievement you get from getting them to work at all."
— Douglas Adams, *So Long, And Thanks For All The Fish* [1984]

In this chapter we look at some results concerning cutsets for elements. This author has not found cutsets for elements a particularly exciting concept, but because of their popularity with other authors, they warrant some discussion here. If P is a poset and $x \in P$, then we say that K is a *cutset for* x if $K \subseteq P \setminus x\updownarrow$ and $K \cup \{x\}$ is a cutset of P. We say that a poset P has the *n-cutset property* if for every $x \in P$, there is some $K \subset P$ which is a cutset for x and $|K| \leq n$. We say that a poset P has the *finite-cutset property* if for every $x \in P$, there is a cutset for x which is finite. (In [LR87], Lonc and Rival accidentally omitted the word "finite" from their definition of the finite-cutset property.)

These concepts were first discussed by Bell and Ginsburg [BG84] who related them to a topology. For any poset P, let $\mathcal{M}(P)$ be the set of maximal chains of P, and recall that for any $x \in P$, $\mathcal{C}(x)$ is the set of maximal chains which include x. Impose on $\mathcal{M}(P)$ the topology with subbase $\{\mathcal{C}(x) : x \in P\}$.

Theorem 5.1 [BG84] *For any poset P, the following are equivalent:*

(*i*) *Every cutset of P contains a finite cutset of P.*

(*ii*) *P has the finite-cutset property.*

(*iii*) $\mathcal{M}(P)$ *is compact.* □

Actually, Bell and Ginsburg proved the analogous theorem for graphs, getting Theorem 5.1 as an immediate corollary by way of comparability graphs. They also showed that for chain-complete posets, the following characterisation can be added to the theorem, where the interval $[x, y]$ is the subposet $x{\uparrow} \cap y{\downarrow}$.

(*iv*) *P is special and for all* $x \leq y$ *in P, there is a finite cutset of* $[x, y]$.

Part of this result applies to posets in general, regardless of whether they are chain-complete:

Theorem 5.2 *Every poset with the finite-cutset property is special.* □

El-Zahar and Zaguia mentioned Theorem 5.2 in the introduction to their survey paper [EZ86], but when they stated it in the main body of the paper [EZ86, Proposition 2.4], they added the unnecessary condition that the poset be chain-complete.

Aharoni, Brochet, and Pouzet [ABP88] related the finite-cutset property to an adaptation of the Menger property for graphs. For P any poset, define

$$\mathrm{cut}(P) = \min\{|K| : K \text{ is a cutset of } P\}, \text{ and}$$

$$\mathrm{disj}(P) = \sup\{|\mathcal{C}| : \mathcal{C} \text{ is a set of pairwise disjoint maximal chains of } P\}.$$

Call P *Menger* if $\mathrm{disj}(P) = \mathrm{cut}(P)$, and *finitely Menger* if these numbers are equal and finite. The most popular way of stating Menger's Theorem is: *If* V_1 *and* V_2 *are* *disjoint sets of vertices in a finite graph G, and k is the size of the smallest subset of $V(G) \setminus (V_1 \cup V_2)$ which meets every path beginning in V_1 and ending in V_2, then there are k paths from V_1 to V_2 which do not meet pairwise except in $V_1 \cup V_2$.* Most authors cite [M27] as the source of this theorem, but the formulation in [M27] is in the context of one-dimensional spaces. This author prefers the graph-theoretic description given by König [K36], but those who do not read German will probably not have a preference. Menger's Theorem tells us that any finite poset P is Menger by applying the theorem to the graph whose vertices are the points of $\mathbf{1} \oplus P \oplus \mathbf{1}$ and in which (x, y) is an edge if and only if $x \prec y$ or $y \prec x$. Apply the theorem with V_1 the singleton set containing just the maximal element and V_2 the singleton set containing just the minimal element.

Aharoni, Brochet, and Pouzet [ABP88] showed that:

Theorem 5.3 *If P is a poset with the finite-cutset property, then P is finitely Menger.* □

Another Menger result was proved by Li [L89]:

Theorem 5.4 *If P is chain-complete and special, then P is Menger.* □

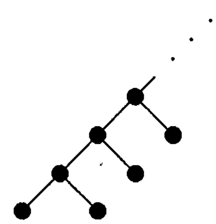
The next two theorems concern situations where $\mathrm{disj}(P) = \mathrm{cut}(P) = 1$. The results bear some intuitive resemblance to the one-dimensional version of Helly's Theorem: *If $\{K_i\}$ is a finite set of intervals in $\mathbf{R}$ such that no two are disjoint, then $\bigcap_i K_i \neq \emptyset$* [CFG91]. The following results were proved by Sands (unpublished) and Brochet and Pouzet [BP88], respectively:

Theorem 5.5 *If a poset P is chain-complete and no two maximal chains of P are* *disjoint, then* $\bigcap \mathcal{M}(P) \neq \emptyset$. □

Theorem 5.6 *If a poset* P *has finite width and no two maximal chains of* P *are disjoint, then* $\bigcap \mathcal{M}(P) \neq \emptyset$. □

In [K88], Kierstead proved a related result for intersections of maximal chains with arbitrary subsets.

Nobody seems to have addressed the issue of whether the finite-width condition in Theorem 5.6 can be weakened to antichain-finite, but we do know that the finite-width condition cannot be dropped altogether, as this example shows:



In [N86], Nowakowski found the smallest cutsets for elements of Boolean lattices. In almost all cases, the smallest cutset for an element x in a Boolean lattice is $\{y \in P \setminus x\uparrow : y \prec z \text{ for some } z \in x\updownarrow\}$ or, dually, $\{y \in P \setminus x\downarrow : y \succ z \text{ for some } z \in x\updownarrow\}$. Griggs and Kleitman [GK89] gave a different proof of the same result. At the other extreme, Füredi, Griggs, and Kleitman [FGK89] found minimal cutsets of Boolean lattices whose size as a fraction of the lattice's size approaches 1 as the lattice's size goes to infinity. The issue of finding bounds on the size of minimal fibres of Boolean lattices was mentioned by Lonc & Rival in [LR87] and Duffus, Sands, & Winkler in [DSW91], but neither of these papers has a complete solution.

Ginsburg, Rival, and Sands [GRS86] showed that if a chain-finite poset has the finite-cutset property, then it must be finite. Lonc and Rival [LR87] defined analogous properties for transversals and fibres and proved related results. Say that a graph G has the *finite-transversal property* if for every $x \in V(G)$, there is some finite $T \subset V(G)$ such that $\{x\} \cup T$ is a transversal and there is no $(x, y) \in E(G)$ where $y \in T$. A poset P has the *finite-fibre property* if for every $x \in P$, there is some finite $F \subset P \setminus x\updownarrow$ such that $\{x\} \cup F$ is a fibre. Lonc and Rival showed:

Theorem 5.7 *If G is a graph in which every clique is finite and G has the finite-transversal property, then G is finite.* □

Lonc and Rival observed that applying this result to the comparability graph of a poset yields the aforementioned result of Ginsburg, Rival, and Sands, and applying it to the complement of the comparability graph of a poset shows that if a poset has the finite-fibre property and every antichain is finite, then the poset is finite. This is the only mention of the finite-fibre property in the literature. The n-fibre property appears nowhere in the literature in spite of the popularity of the n-cutset property which has been mentioned in at least nine articles.

It is clear that a poset has the 0-cutset property if and only if it is a chain. A poset with the 1-cutset property has width 1 or 2. Ginsburg [G] calls this fact "well-known and often rediscovered". Interesting results regarding the n-cutset property begin at $n = 2$.

In [G86b], Ginsburg describes a simple configuration which he calls a ladder, and shows that a poset P with the 2-cutset property must contain a ladder of height $\lfloor \frac{1}{2}(\text{width}(P) - 3) \rfloor + 1$.

The first result to say something about the n-cutset property for arbitrary finite n was proven by Ginsburg [G87]. He showed that if P has the n-cutset property, where n is any natural number, then $|\max P| \leq 2^n$. Later, Arpin and Ginsburg [AG91] improved this result by weakening the hypothesis:

Theorem 5.8 *Let P be a poset and $n \in \mathbf{N}$ such that for every $x \in \max P$, there is a cutset K for x satisfying $|K| \leq n$. Then $|\max P| \leq 2^n$.* □

Two similar results in the same paper are:

Theorem 5.9 *Let $n \in \mathbf{N}$ and P a poset with the n-cutset property satisfying $|\max P| = 2^n$. Then P contains a complete binary tree T of height n such that $\max T = \max P$.* □

Theorem 5.10 *Let P be a poset with the $\aleph_0$-cutset property satisfying $|\max P| = 2^{\aleph_0}$. Then P contains a complete binary tree of height ω.* □

Other questions have been posed concerning the n-cutset property for arbitrary n but answered in only a few cases. For one of these questions, the few cases that have been answered are the only interesting ones. Sauer and Woodrow [SW84] addressed the question: *For n a natural number, what is the least m such that for every poset P with the n-cutset property, every $x \in P$ is contained in a maximal antichain of $\leq m$ points?* They showed that for $n = 0$, $m = 1$; for $n = 1$, $m = 2$; for $n = 2$, $m = 4$; and for $n = 3$, $m \geq \aleph_0$. Call a poset P *conditionally chain-complete* if every bounded chain has a supremum and an infimum. Sauer and Woodrow [SW84] showed that if a conditionally chain-complete poset P has the finite-cutset property, then every finite antichain of P is contained in a finite maximal antichain.

A similar question is: *What is the maximum possible width of a poset P with the*

n-cutset property? Sauer and Woodrow [SW84] provide the following best possible answers: for $n = 0$, width$(P) = 1$; for $n = 1$, width$(P) \leq 2$; and for $n = 2$, no bound exists. Kezdy, Markert, and West [KMW90] examined the question of approximating the maximum possible width of a poset for finite fixed n-cutset property and height. Hajnal and Sauer [HS93] addressed the same question for infinite fixed κ-cutset property and height. An early result was proved by El-Zahar & Zaguia [EZ86] and Ginsburg, Sands, & West [GSW89] using different methods:

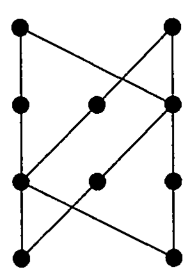
Theorem 5.11 width$(P) \leq$ height$(P) + 2$ *for any poset P with the 2-cutset property.* □

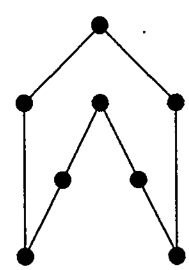
The dimension of a poset cannot exceed its width if its width is finite. We will use this fact in the next paragraph, but first we will say something about the difficulty of attributing it. This fact is stated without proof in [D50] (in a footnote!), which causes this author to question the judgement of Trotter who listed only [D50] as a reference when he stated and proved this fact in [T92]. Also, this seems to indicate some change of opinion since Trotter and Kelly listed only Hiraguchi's [H55] as a reference when they stated and proved this fact in [KT82]. The rest of the literature is also divided between attribution to [H55] and [D50]. I cannot say which attribution is more appropriate since I have not seen [H55] and its description in *Mathematical Reviews* does not specifically mention this result. Furthermore, many authors have not bothered to explicitly mention the finite width condition, without which the statement is false; Nation, Pickering, and Schmerl have shown that a poset with no infinite antichains may have arbitrarily large infinite dimension [NPS88].

Ginsburg [G] addressed a simple question without a simple answer: *If $n \in \mathbf{N}$ and P is a poset with the n-cutset property, then what is the maximal possible dimension of P?* For $n = 0$, the answer is trivially 1. For $n = 1$, the answer is 2 as is easily verified using the knowledge that posets with the 1-cutset property all have width less than 3, and posets of dimension 3 all have width at least 3. The first interesting case is $n = 2$, for which Ginsburg shows that the answer is no more than 41 and says that this result "can undoubtedly be improved". In fact, the greatest dimension for which a poset with the 2-cutset property is known to exist is 4.

In [G89], Ginsburg characterises by means of forbidden configurations the finite posets which can be embedded in posets with the 1-cutset property. A shorter version of the proof is provided by Rutkowski [Ru92].

Say that a poset P has the *chain-cutset property* if for every $x \in P$, there is a cutset for x which is a chain. Ginsburg, Rival, and Sands [GRS86] ask: *What is the maximum possible width (if any exists) of a poset with the chain-cutset property?* It is known only that the answer is at least 4, as demonstrated by examples in [GRS86] (on the left) and El-Zahar and Zaguia's [EZ86] (on the right).





In his Ph.D. dissertation [Z85], Zaguia showed that the answer to this question restricted to lattices is 2.

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Appendix A

The Compactness Theorem

The compactness theorem is used twice in the thesis. Even though anyone likely to read this thesis probably already knows the compactness theorem, I think the following proof is worthy of being included in this thesis since it is both correct and comprehensible, and this is more than I can say of any of the proofs I read in the literature.

We will not subject ourselves to the tedium of a rigorous description of sentential logic. We will just point out, although it probably is already clear, that the sentences referred to in the theorem have no quantifiers, only truth-functional variables and the usual unary and binary truth-functional operators.

For any set $\mathcal{S}$ of sentences, let $V(\mathcal{S})$ be the set of truth-functional variables appearing in the sentences in $\mathcal{S}$. Call $\mathcal{S}$ *satisfiable* if there is an assignment of truth-functional values to the elements of $V(\mathcal{S})$ making all elements of $\mathcal{S}$ evaluate to true. Call $\mathcal{S}$ *finitely satisfiable* if every finite subset of $\mathcal{S}$ is satisfiable.

Theorem (The Compactness Theorem)

Let $\mathcal{S}$ be any set of sentences. $\mathcal{S}$ is satisfiable if and only if $\mathcal{S}$ is finitely satisfiable.

Proof. The theorem is obvious when $\mathcal{S}$ is finite, so assume $\mathcal{S}$ is infinite. The "only if" part is obvious in any case, so we will prove only the "if" part.

Let $\kappa = |V(\mathcal{S})|$. Let $f : \kappa \to V(\mathcal{S})$ be one-to-one and onto. For each $\alpha \in \kappa$, write v_α for $f(\alpha)$. So $V(\mathcal{S}) = \{v_\alpha : \alpha < \kappa\}$. Define two increasing (with respect to $\subseteq$) sequences of finitely satisfiable sets of sentences by transfinite induction as follows. Let $\mathcal{S}_0 = \mathcal{S}$. For each $\alpha \leq \kappa$ such that $\alpha \neq 0$, let $\mathcal{S}_\alpha = \bigcup_{\beta<\alpha} \mathcal{T}_\beta$. For each $\alpha \in \kappa$, put $\mathcal{T}_\alpha = \mathcal{S}_\alpha \cup$ "v_α" if $\mathcal{S} \cup$ "v_α" is finitely satisfiable, otherwise put $\mathcal{T}_\alpha = \mathcal{S}_\alpha \cup$ "$\neg v_\alpha$". I claim that, in fact, all the $\mathcal{T}_\alpha$'s and $\mathcal{S}_\alpha$'s are finitely satisfiable. If there is any $\alpha \in \kappa$ such that $\mathcal{T}_\alpha$ is not finitely satisfiable, then $\mathcal{S}_{\alpha+1} = \mathcal{T}_\alpha$ is not finitely satisfiable. Thus, to show that all the $\mathcal{S}_\alpha$'s and $\mathcal{T}_\alpha$'s are finitely satisfiable, it suffices just to show that all the $\mathcal{S}_\alpha$'s are finitely satisfiable. Suppose for a contradiction that there is some $\alpha \leq \kappa$ such that $\mathcal{S}_\alpha$ is not finitely satisfiable. We may assume that α is least so that $\mathcal{S}_\alpha$ is not finitely satisfiable. Let $\mathcal{A}$ be a finite subset of $\mathcal{S}_\alpha$ which is not satisfiable. Then there exists $\beta < \alpha$ such that $\mathcal{A} \subseteq \mathcal{T}_\beta$. So $\mathcal{T}_\beta$ is not finitely satisfiable, but $\mathcal{S}_\beta$ is, so "v_β", "$\neg v_\beta$" $\notin \mathcal{S}_\beta$. Since $\mathcal{T}_\beta$ is not finitely satisfiable, neither $\mathcal{S}_\beta \cup \{$"$v_\beta$"$\}$ nor $\mathcal{S}_\beta \cup \{$"$\neg v_\beta$"$\}$ is. Let $\mathcal{B}$ and $\mathcal{C}$ be finite unsatisfiable subsets of $\mathcal{S}_\beta \cup \{$"$v_\beta$"$\}$ and $\mathcal{S}_\beta \cup \{$"$\neg v_\beta$"$\}$ respectively. Since $\mathcal{S}_\beta$ is finitely satisfiable, we know that "v_β" $\in \mathcal{B}$ and "$\neg v_\beta$" $\in \mathcal{C}$. Furthermore, we know that $v_\beta \in V(\mathcal{B} \setminus \{$"$v_\beta$"$\})$ and every assignment of truth-functional values to the elements of $V(\mathcal{B} \setminus \{$"$v_\beta$"$\})$ that makes every sentence in $\mathcal{B} \setminus \{$"v_β"$\}$ true assigns v_β false. Similarly, $v_\beta \in V(\mathcal{C} \setminus \{$"$\neg v_\beta$"$\})$ and every assignment of truth-functional values to the elements of $V(\mathcal{C} \setminus \{$"$\neg v_\beta$"$\})$ that makes every sentence in $\mathcal{C} \setminus \{$"$\neg v_\beta$"$\}$ true assigns v_β true. But then $(\mathcal{B} \setminus$ "v_β"$) \cup (\mathcal{C} \setminus$ "$\neg v_\beta$"$)$ is a finite subset of $\mathcal{S}_\beta$ such that there is no assignment of truth-functional values to the elements of $V((\mathcal{B} \setminus$ "v_β"$) \cup (\mathcal{C} \setminus$ "$\neg v_\beta$"$))$ making all the sentences in $\mathcal{S}_\beta$ true. So $\mathcal{S}_\beta$ is not finitely satisfiable, a contradiction. So all the $\mathcal{S}_\alpha$'s are finitely satisfiable. In particular, $\mathcal{S}_\kappa$ is finitely satisfiable.

For each $\alpha < \kappa$, let $S(v_\alpha)$ be either "v_α" or "$\neg v_\alpha$", whichever of these is in

$\mathcal{S}_\kappa$. For each $\alpha < \kappa$, assign v_α the value true or false according as $S(v_\alpha)$ is "v_α" or "$\neg v_\alpha$". Does this assignment of truth-values make every sentence in $\mathcal{S}_\kappa$ evaluate to true? Let $A \in \mathcal{S}_\kappa$. Let $\mathcal{A} = \{A\} \cup \{S(v) : v \in V(\{A\})\}$. $\mathcal{A}$ is a finite subset of $\mathcal{S}_\kappa$, so there is a way to assign truth-values to the elements of $V(\mathcal{A})$ so that all the sentences in $\mathcal{A}$ evaluate to true. Obviously, this assignment is the one which coincides with the one we have chosen for all of $V(\mathcal{S}_\kappa)$. So this assignment does make A evaluate to true. That is, the assignment makes every sentence in $\mathcal{S}_\kappa$ true. So it makes every sentence in $\mathcal{S}$ true.

Q(uite) E(asily) D(one). (Not really.)$\square$

Appendix B

permission from Peter Gibson to use joint research

To whom it may concern:

Most of the results in Chapter 2 of Roy Maltby's Master's thesis (results which are concerned with posets in which every cutset meets every fibre) are from joint research undertaken by me (Peter Gibson) and Roy Maltby during the summer of 1990, except for the theorem concerning posets of the form $(K_{1,1})^r \times K_{1,m} \times K_{1,n}$ and the theorem concerning posets of the form $K_{1,l} \times K_{1,m} \times K_{1,n}$ which are more recent results obtained by Roy Maltby alone. Roy Maltby has my permission to use results from our joint research in his Master's thesis.

Peter Gibson

June 29, 1993

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No animals were harmed in the typing of this thesis.

Have a nice day.