THE UNIVERSITY OF CALGARY

IMPARTIAL AND PARTISAN GAMES

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE

OF MASTER OF SCIENCE

DEPARTMENT OF MATHEMATICS AND STATISTICS

CALGARY, ALBERTA

JUNE, 1976

Abstract

Conway has recently developed a theory particularly well suited to the analysis of two-person games that are completely determined. Using this theory we consolidate some results due to Conway and Guy about the partisan game Col, as well as proving some new results for take and break games. In Chapter 4, the results obtained by Guy and Smith, and Kenyon for octal games are generalized to arbitrary take and break games. Chapter 5 discusses subtraction games. We show that all subtraction games are periodic, and prove that in certain circumstances it is possible to determine the period length exactly. We also state the rules, due to Conway and Guy respectively, for writing down the period of the games $S(a,b)$, $S(a,b,2b-a)$. Using Ferguson's Pairing Property, we give the analysis, again due to Conway and Guy, of $S(a,b,a+b)$. Chapter 6 deals with arithmetico-periodicity. Conway's proof that no octal game is arithmetico-periodic is given. We prove new arithmetico-periodicity theorems for sedecimal and infinite recurring octal and tetral games. Chapter 7 contains Tables that list the G-sequence of certain types of games. With the exception of Table 7.7, the basis for these was provided by Guy. Table 7.1 was expanded by the author to include all subtraction games in which the subtrahends do not exceed 8. The games .055, .165, .356 and .644 were also solved by the author.

Acknowledgements

I would like to take this opportunity to express my sincere gratitude to my supervisor, Professor Richard K. Guy, for the constant help he has provided. His suggestions enabled me to overcome difficulties too numerous to mention. His interest and enthusiasm brought me through many periods of frustration. As well, his concentrated attention to all aspects of this thesis (not excluding the spelling of the word 'argument') was invaluable.

I am also indebted to Dr. John Conway for his willingness to discuss all aspects of this thesis, and the many helpful suggestions that came out of these discussions.

I would also like to thank Karen McDermid for the superb job she did in deciphering and typing this manuscript, as well as for her patience in correcting the copy.

Financial support was provided by the University of Calgary and the Province of Alberta.

TABLE OF CONTENTS

List of Tables

List of Figures

List of Figures

"There is plenty of time to win this game,
and to thrash the Spaniards too."

Sir Francis Drake, 20 July 1588.

Chapter 1

The Classes $\underset{\sim}{\mathrm{Ug}}$ and $\underset{\sim}{\mathrm{No}}$

1.1. Introduction

Our aim in this chapter is to develop a theory that will enable us to evaluate positions in games, so that we may determine what advantage, if any, a position confers upon a particular player. To achieve this end, we define a class $\underset{\sim}{\mathrm{Ug}}$ of games, as well as addition and a partial order on this class. It turns out that the advantage conferred upon a player by some positions can be thought of as a number of moves advantage to one of the players, Left or Right. As a result, we find that the class $\underset{\sim}{\mathrm{Ug}}$ strictly contains a real ordered field $\underset{\sim}{\mathrm{No}}$ as a subclass, which in turn strictly contains the real numbers.

Our discussion is necessarily brief. In most instances we omit proofs so that we may more quickly apply the techniques to the analysis of games. For a more complete treatment, we refer the reader to [5].

1.2. Games

By a *game* G we mean a set of positions together with rules which say for any two positions P, Q and either of the two players, Left and Right, whether it is legal for the player to move from P to Q. We require that the state of play be known to both players, and that moves be determined only by the rules, not by any external conditions such as the throwing of dice. The games under discussion bear more similarity to Chess or Checkers than to Bridge or Monopoly.

The *initial position* of a game G is the position from which play starts. If from the initial position, Left has moves only to positions $A_1, A_2, \ldots$ and Right has moves only to positions $B_1, B_2, \ldots$, we write $G = \{A_1, A_2, \ldots | B_1, B_2, \ldots\}$ and refer to $A_1, A_2, \ldots$ as the *left options* of G, $B_1, B_2, \ldots$ as the *right options* of G. The typical left or right option will be denoted by G^L or G^R respectively, so that $G = \{G^L | G^R\}$. Note that G^L, G^R here represent *sets*, empty, finite, or infinite. For simplicity we have omitted the usual braces; we will also, by a common abuse of notation, often use G^L to denote a particular option, rather than the set of all left options.

The game G will end when the player whose turn it is to move cannot do so. For example, if from $G = \{A_1, A_2 | \ \}$ it is Right to move, then the game G has ended, as the set of options available to Right is empty. In the case of an ended game, the outcome depends upon the convention under which the game is being played. In *normal play*, a player loses if it is his turn to move and he is unable to do so, i.e. these games are *last player winning*. Under *misère play*, the last player able to make a legal move loses, i.e. these games are *last player losing*.

A game G is said to be *impartial* if from each position of G, exactly the same moves are available to each player. A game that is not impartial is said to be *partisan*. For example, Col (see Chapter 2) is a partisan game. An example of an impartial game is *Nim*. It is played with a finite number of heaps of tokens, each heap containing a finite number of tokens. The players move alternately, choosing one heap and removing at least one token from that heap.

If in Nim an infinite number of heaps were allowed, the game would not terminate. A game G is said to satisfy the *terminating play condition* if there is no infinite sequence $P_0, P_1, P_2, \ldots$ of positions for which there exist legal moves from P_n to P_{n+1}, $n = 0,1,2,\ldots$. Observe that a game in which there is an infinite sequence of moves for just one of the players does not satisfy the terminating play condition. The reason we do not restrict the condition to *alternating sequences* (Left, Right, Left,...) will become clear when we define addition of games. In the following we restrict ourselves to last player winning games that satisfy the terminating play condition.

A *disjunctive compound* (*sum*) of the games $\{G_0, G_1, \ldots, G_n\}$ is played in the following manner. The player whose turn it is to move selects one of the component games, $G_0, G_1, \ldots, G_n$, and makes a legal move in that component. The disjunctive compound, denoted by $G_0 + G_1 + \ldots + G_n$, ends when each of the components had ended. Nim is a disjunctive compound of component games of one-heap Nim. If G, H are games, the positions of $G+H$ are ordered pairs (P,Q) where P is a position of G, Q is a position of H. From (P,Q), Left may move to (P^L,Q) or (P,Q^L), and Right may move to (P^R,Q) or (P,Q^R).

For each game G, there is a set of positions G^L to which Left may move, and a set of positions G^R to which Right may move. Each $P \in G^L \cup G^R$ is a shortened game, so that G is determined by the games that form its left and right options. This observation provides us with a definition of a game.

DEFINITION 1.1 (Conway [5]). If G^L and G^R are two sets of games, then there is a game $\{G^L|G^R\}$. All games are constructed in this way.

DEFINITION 1.2. (i) If $G = \{G^L|G^R\}$, then $-G = \{-G^R|-G^L\}$.

(ii) If $G = \{G^L|G^R\}$, $H = \{H^L|H^R\}$ then

$$G+H = \{G^L+H, G+H^L | G^R+H, G+H^R\}.$$

Note that the game $-G$ is obtained from G by reversing the rôles of Left and Right throughout.

Definitions 1.1 and 1.2 are inductive definitions. We show the operation of the induction by consideration of some simple games. The simplest of all games is the *Endgame* $\{\ |\ \}$. It is reasonable to denote this by 0 (take $G^L = G^R = \emptyset$ in Definition 1.2) since $-0 = 0$, and $0+H = \{0+H^L|0+H^R\} = \{H^L|H^R\} = H$. As no player may make a legal move, the one required to move first loses, i.e. this is a *second player winning game*. Consider $\{0|\ \}$. Moving first, Left may make a legal move to 0, which ends the game, so that Left wins. If Right is required to move first, Left also wins, as the set of right options is empty. Similarly in the game $\{\ |0\}$, Right wins regardless of which player starts. However $* = \{0|0\}$ (pronounced *star*) is a *first player winning game*, since the first player moves to 0, and becomes the second player in the shortened game.

To illustrate the play of games, we represent the game as a tree. The positions are represented by nodes, and a legal move from P to Q is represented by a line joining P to Q. We draw the tree so that moves for Left are represented by lines sloping upward to the left, and moves for Right are represented by lines sloping upward to the right. Figure 1.1 shows the trees of the games discussed above.

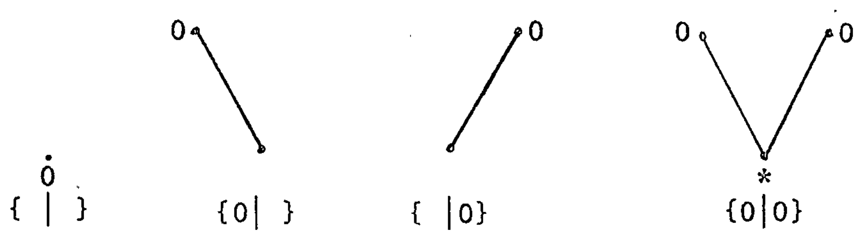


Figure 1.1. The trees of some simple games.

Any game G that satisfies the terminating play condition belongs to one of the outcome classes listed above. We define these classes more formally in the following manner.

DEFINITION 1.3. The four outcome classes are:

$G > 0$ if Left can win no matter who starts.

$G < 0$ if Right can win no matter who starts.

$G \doteq 0$ if the second player can win.

$G \parallel 0$ (G is *fuzzy*, or G is *confused* with 0) if the first player can win.

These symbols combine in a natural way.

$G \gtrsim 0$ means that if Right starts, Left wins.

$G \lesssim 0$ means that if Left starts, Right wins.

$G \,|\!\triangleright\, 0$ means that if Left starts, Left wins.

$G \,\triangleleft\!|\, 0$ means that if Right starts, Right wins.

We let Ug denote the class of all games. Equality in Ug is defined in terms of equivalence classes. We first introduce the concept of isomorphic games. For G, $H \in$ Ug, $G \equiv H$ (G is *isomorphic* to H) if there is a one to one correspondence between the legal moves of G and H.

LEMMA 1.4. (i) $0+G \equiv G$

(ii) $G+H \equiv H+G$

(iii) $(G+H)+K \equiv G+(H+K)$.

PROOF. We prove (i) to provide an example of the general inductive argument.

$$\begin{aligned} 0+G &= \{0^L+G, 0+G^L \mid 0^R+G, 0+G^R\} \\ &= \{0+G^L \mid 0+G^R\} \\ &\equiv \{G^L \mid G^R\} \\ &= G. \qquad \square \end{aligned}$$

Suppose we wish to establish a proposition $\Gamma(G)$ for all games G. It suffices to prove that $\Gamma(G^L)$, $\Gamma(G^R)$ imply $\Gamma(G)$. What is perhaps not so clear is that these inductions never require a basis, since statements about the empty set are vacuously true.

LEMMA 1.5. If $H \doteq 0$ then $G+H$ has the same outcome as G.

LEMMA 1.6 (Tweedledum and Tweedledee Principle). $\forall G \in \underset{\sim}{\mathrm{Ug}}$, $G+(-G) \doteq 0$.

PROOF. The second player mimics his opponent's move in the opposite component of the disjunctive sum.

(Lemma 1.6 explains our formulation of the Terminating Play Condition. If an infinite sequence $P_0, P_1, P_2, \ldots$ of moves for one player was permitted, then the game $G+(-G)$ might never end.)

LEMMA 1.7. If $G+(-H) \doteq 0$ then $G+K$ has the same outcome as $H+K$, $\forall K \in \underset{\sim}{\mathrm{Ug}}$.

PROOF. By Lemma 1.5, $H+K$ has the same outcome as $G+(-H)+H+K$, and $G+(-H)+H+K \equiv G+K+(H+(-H))$. Then by Lemmas 1.5 and 1.6 $G+K+(H+(-H))$ has the same outcome as $G+K$.

DEFINITION 1.8. $G = H$ if $G+(-H) \doteq 0$.

The definition of equality is based on the observation (Lemma 1.7) that if $G+(-H) \doteq 0$, it will not affect the outcome of a disjunctive sum that includes G as one of the component games if G is replaced by H. Notice in particular $G \doteq 0$ implies $G = 0$. For example, consider the game $G = \{\{\ |0\}|\{0|\ \}\}$. If Right starts, we go to $\{0|\ \}$. Left now moves to 0 and wins. Similarly if Left starts, Right wins, so that G is a second player winning game. Hence $\{\{\ |0\}|\{0|\ \}\} = 0$. In future when we speak of a game G, we mean all games H such that $G+(-H) = 0$. For example, by 0 we mean not only $\{\ |\ \}$ but also the games $\{\{\ |0\}|\{0|\ \}\}$ and $* + * = \{0|0\} + \{0|0\}$, illustrated in Figure 1.2, and $G+(-G)$ for any game G.

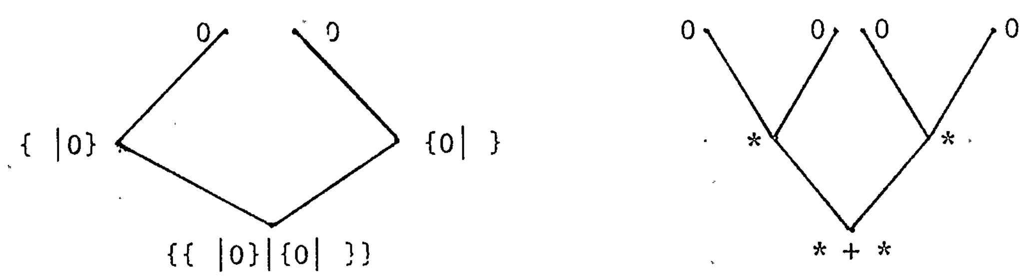


Figure 1.2. Games equivalent to 0.

Definition 1.3 enables us to define a partial order on $\underset{\sim}{\mathrm{Ug}}$. For two games, G, H, $G \geqslant H$ if $G+(-H) \geqslant 0$, i.e. the game $G+(-H)$ is Left to win if Right starts. By $G > H$ we mean $G \geqslant H$ and $G \neq H$.

LEMMA 1.9. If $G \geqslant H$, $H \geqslant K$, then $G \geqslant K$.

Lemma 1.9 assures us that there is no ambiguity in the use of the symbol '$\geqslant$' to denote the partial order.

There is an alternative formulation of the partial order that we will often use. For $G = \{G^L | G^R\}$, $H = \{H^L | H^R\}$, we have $G \geqslant H$ if there is no $H^L \geqslant G$ and there is no G^R such that $H \geqslant G^R$. This formulation, like the method of construction of games, is inductive. To decide whether $G \geqslant H$ it is first necessary to determine the order relations that hold between all the H^L and G, and the order relations that hold between H and all the G^R. If it is the case that no $H^L \geqslant G$ and $H \geqslant$ no G^R, then $G \geqslant H$.

By $G \,|\!\!\rhd H$ we shall mean $G+(-H) \,|\!\!\rhd 0$, i.e. the game $G+(-H)$ is Left to win if Left starts. As an immediate consequence of the definition we have

LEMMA 1.10. For all games G, $G^R \,|\!\!\rhd G \,|\!\!\rhd G^L$.

There are some games that behave like numbers, i.e. they provide a certain number of free moves to one of the players. We can consider n to be the game with n successive moves available to Left, and no moves to Right. Figure 1.3 illustrates the tree of moves of n.

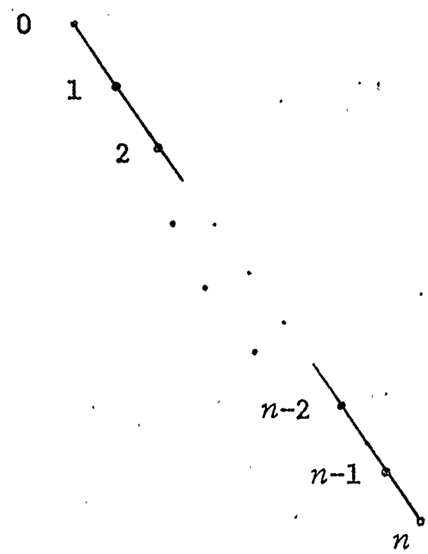


Figure 1.3. The tree of moves of n.

In the game n, Left moves to the position n-1, which suggests the following inductive definition:

$$n = \{n-1 \mid \ \}.$$

For example, $1 = \{0 \mid \ \}$ so that by Definition 1.2 (i), $-1 = \{ \ \mid 0\}$.

If we play the game $\{0|1\} + \{0|1\} + (-1)$ we discover that this is a second player winning game, so that it seems $\{0|1\}$ provides Left with 1/2 move advantage. For this reason, we call $\{0|1\} = 1/2$, so that by Definition 1.2 (i), $-1/2 = \{-1|0\}$.

It turns out that we can define a subclass No of Ug which is a real ordered field that strictly contains the real numbers. In [5], Conway details the construction of the class No. We limit ourselves to a discussion of the rôle of numbers within Ug and a statement of several of the results.

DEFINITION 1.11. $x = \{x^L | x^R\}$ is a number provided it has a form in which

(i) all the x^L, x^R are numbers

(ii) $x^L < x^R$ (for each pair x^L, x^R).

Note that $0 = \{ \mid \}$, $n = \{n-1 \mid \}$ are numbers (in each case (ii) is vacuously true) so that there are some numbers. In verifying that a number $x = \{x^L | x^R\}$ satisfies (ii), we consider x^L, x^R as games (for numbers are also games) and show that $x^R - x^L > 0$. For example $1/2 = \{0|1\}$. Since $1-0 = 1$ is Left to win, regardless of which player starts, $1-0 > 0$ and $1/2$ is a number. But $* = \{0|0\}$ is *not* a number, since $0 \not> 0$.

We have already defined addition, and a partial order on games. These are inherited by the class $\underset{\sim}{No}$ from $\underset{\sim}{Ug}$. For completeness we restate these in terms of numbers.

DEFINITION 1.12 (Conway [4]). Let x, y be numbers.

(i) $x \geq y$ if $\forall x^R$, $y \not\geq x^R$, $\forall y^L$, $y^L \not\geq x$, $y \leq x$ if $x \geq y$,

(ii) $x = y$ if $x \geq y$ and $y \geq x$,

(iii) $x+y = \{x^L+y, x+y^L | x^R+y, x+y^R\}$,

(iv) $-x = \{-x^R | -x^L\}$.

We also have

LEMMA 1.13. $\underset{\sim}{No}$ is totally ordered.

LEMMA 1.14. For any number x, $x^L < x < x^R$.

Consider $x = \{0|\frac{1}{2}\}$. Since this satisfies Definition 1.11 x is a number. If we play $\{0|\frac{1}{2}\} + \{0|\frac{1}{2}\} - \frac{1}{2}$, we see that it is second player winning. For this reason we call $\{0|\frac{1}{2}\} = \frac{1}{4}$.

From the examples considered so far, it might be thought that $\{-\frac{1}{2}|1\}$ is also equal to $\frac{1}{4}$. If Left moves first he goes to $-\frac{1}{2} = \{-1|0\}$ and Right wins, while if Right moves first, he goes to 1 and Left wins. Hence $\{-\frac{1}{2}|1\}$ is a second player win, so that $\{-\frac{1}{2}|1\} = 0$.

Therefore we cannot answer the question 'What number is x?' when x is a number by taking the arithmetic mean of x^L and x^R. By way of the Creation Story (cf. Knuth [14]) we are able to provide an answer.

We think of games as being created on consecutive days, where each day is numbered with an ordinal α. On day α we create (by Definition 1.1) all games $G = \{G^L|G^R\}$, for which each member of $G^L \cup G^R$ has been previously created. Since No is strictly contained in Ug, we know that every number has associated with it a birthday, the day on which it was created. On day 0 we create the number $0 = \{\ |\ \}$. On day 1, we use 0 to create $1 = \{0|\ \}$, $-1 = \{\ |0\}$ ($* = \{0|0\}$ is also created on day 1). On day 2, the numbers $-2 = \{\ |-1\}$, $-\frac{1}{2} = \{-1|0\}$, $\frac{1}{2} = \{0|1\}$, $2 = \{1|\ \}$ are created. Figure 1.4 illustrates the tree of numbers.

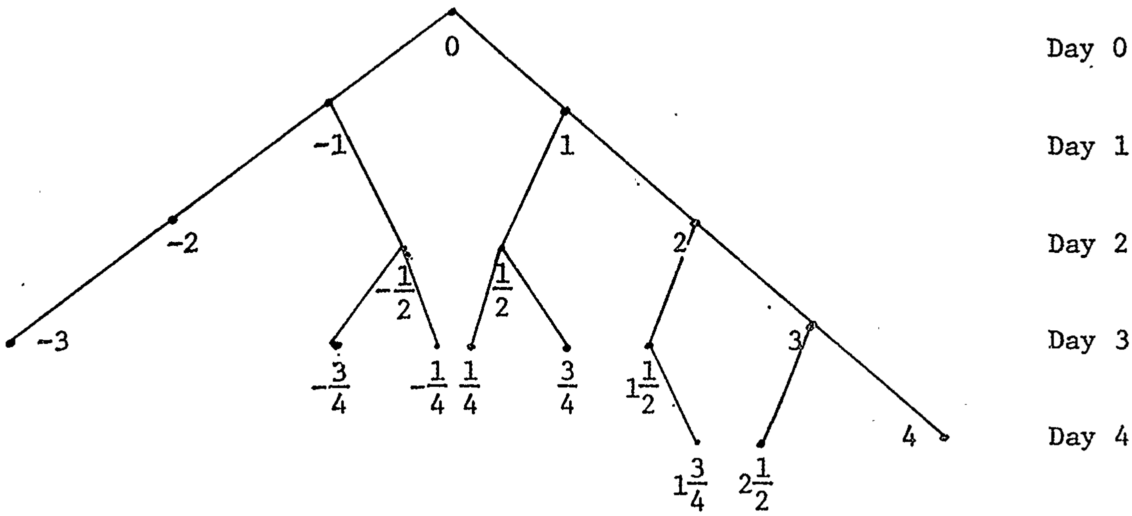


Figure 1.4. The tree of numbers.

On day n, the largest number created is n, and the least number is $-n$. Every other number created on day n is the arithmetic mean of two numbers adjacent in the chain of all numbers previously created. Hence on day 3, we create the numbers $3 = \{2|\ \}$, $1\frac{1}{2} = \{1|2\}$, $\frac{3}{4} = \{\frac{1}{2}|1\}$, $\frac{1}{4} = \{0|1\}$, and their negatives.

On day 3, we also create the number $x = \{0,1,2|\ \}$. By Lemma 1.14, we have $x > 2$, $x > 1$, $x > 0$. However $x > 2$ implies $x > 1$, $x > 0$, so that 1 and 0 seem redundant in some sense. Lemma 1.15 shows this to be so.

LEMMA 1.15. (i) If $G = \{G^L,H|G^R\}$, and $G^L \geq H$, then $G = \{G^L|G^R\}$.

(ii) If $G = \{G^L|G^R,H\}$, and $H \geq G^R$, then $G = \{G^L|G^R\}$.

Such an option H, for either Left or Right, is said to be a *dominated option*, e.g. $3 = \{2|\ \} = \{1,2|\ \} = \{0,1,2|\ \}$.

The Gift Horse Principle enables us to simplify numbers further. We state it in its most general form in terms of games. The Gift Horse Principle asserts that it is always possible to give a player a new move without affecting the value of the game, provided that it does him no good.

LEMMA 1.16 (The Gift Horse Principle). Let $G = \{G^L | G^R\}$.

(i) If $H \lhd| G$, then $G = \{G^L, H | G^R\}$.

(ii) If $H |\rhd G$, then $G = \{G^L | G^R, H\}$.

For example, $0 \| *$, so that $0 = \{\ |\ \} = \{*|\ \} = \{*|*\}$. Such a 'Gift Horse' is referred to as an *irrelevant option*. The Gift Horse Principle is usually applied in reverse to simplify games. Suppose $G = \{G^L, H | G^R\}$. If for the game $G' = \{G^L | G^R\}$, $H \lhd| G'$, then by Lemma 1.16

$$\begin{aligned} G' &= \{G^L | G^R\} \\ &= \{G^L, H | G^R\} \\ &= G. \end{aligned}$$

Figure 1.5 illustrates the effect of irrelevant and dominated options by displaying some equivalent forms of some simple games.

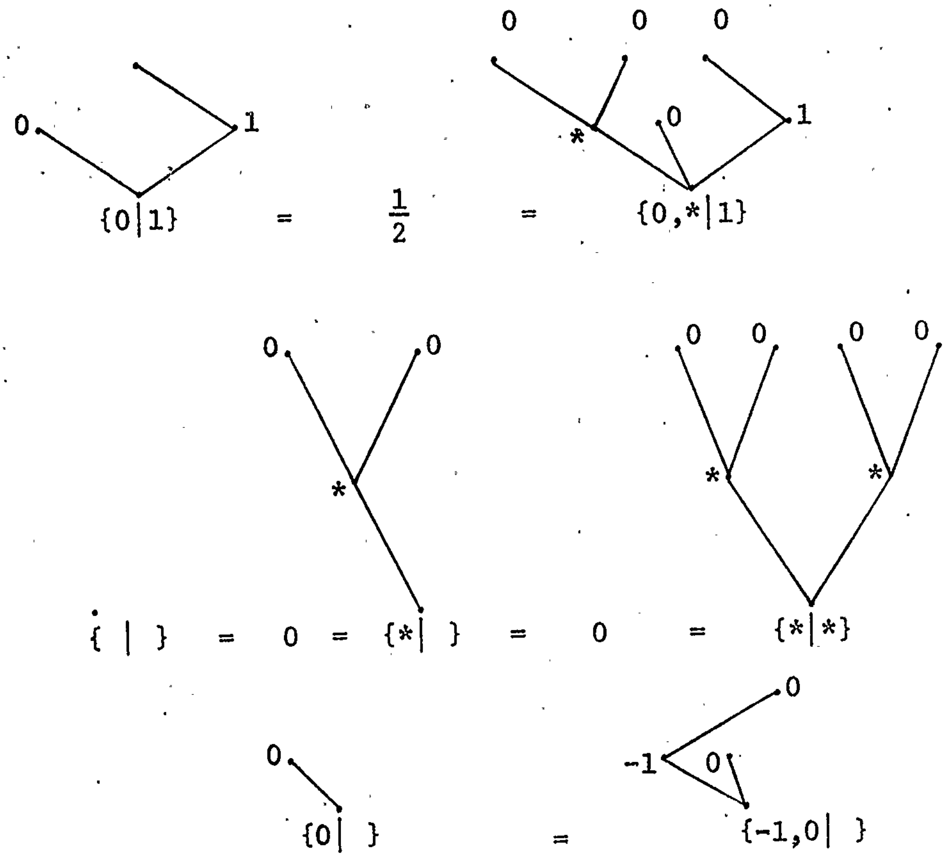


Figure 1.5. Equivalent forms of some simple games.

The form of a game may be simplified by eliminating irrelevant and dominated options. It may be the case that after such simplifications have been made, that the game is a number. The Simplicity Theorem enables us to state precisely what number $G = \{G^L | G^R\}$ is when G is a number. The word *simplest* is taken as synonymous with *earliest created.*

THEOREM 1.17 (The Simplicity Theorem). Let $G = \{G^L | G^R\}$. If there is any number x such that $\forall G^L$, $\forall G^R$, $G^L \lhd\| \; x \lhd\| \, G^R$, then G is the simplest such x, i.e. if there is an integer x, then G is the integer nearest to 0, and if there is no integer, G is the binary fraction with least denominator.

For example:

$0 = \{-47|12\}$ since 0 is the simplest (earliest created) number such that $-47 < 0 < 12$.

$1 = \{1+*|1+*\}$

$\frac{1}{2} = \{\frac{1}{8}|\frac{9}{16}\}$.

We have already seen one game that is not a number. We list several others.

$\uparrow = \{0|*\}$ (pronounced "up")

$\downarrow = \{*|0\}$ (down)

$\pm 1 = \{+1|-1\}$ (plus or minus one)

$n* = n+* = \{n|n\}$ (n star)

$+_2 = \{0|\{0|-2\}\}$ (tiny two).

Our treatment of the classes $\underset{\sim}{\mathrm{Ug}}$ and $\underset{\sim}{\mathrm{No}}$ is by no means complete. However we now have sufficient information to begin our analysis of games.

Chapter 2

The Game of Col

2.1. Introduction

Col is a partisan game suggested by Colin Vout. We may imagine the game as being played on a brown paper map. The two players Black (Left) and White (Right) equipped with pots of black and white paint in turn paint countries on the map subject to the restrictions that no country already painted may be repainted, and no two adjacent regions may be painted the same colour.

In passing we mention *Snort*, a companion game to Col, but that we now require that no two adjacent regions be painted opposite colours. The theory of Snort appears much more difficult than that of Col, and no results analogous to those presented here for Col have been discovered. However, the general *character* of Snort is well understood, namely that most positions are "hot", i.e. the first player often has a considerable advantage.

Col and Snort are typical (if not the actual prototypes) of the two very different classes of partisan games, *cool* and *snorting*, i.e. cold and hot. In the first a player usually does himself harm by moving (helps his opponent); in the second he gains some advantage by doing so (harms his opponent).

The latter are the "good" (worthwhile) games, like Chess, where the move is all-important. The *Zugzwang* positions in which it is actually a disadvantage to move seldom occur.

We follow an analysis of Col due to J.H. Conway. First, simple positions were analyzed and used to build a dictionary of values. The table of values suggested certain theorems which once proved were used to condense the table.

In the game of Col, when Black paints a region, in the play that follows he is not permitted to move in any of the contiguous regions. We speak of regions as having a white *tint* to indicate that they are reserved for White. Similarly if White paints a region, we speak of contiguous countries as having a black tint. The map may be simplified by deleting regions already painted, or regions that are tinted both colours as neither player is permitted to move in them.

We represent arbitrary maps by graphs in the following manner. To each country of the map not already painted there corresponds a node, and an edge joins two nodes that correspond to adjacent regions. The nodes are labelled to correspond to the states of their respective regions according to the following scheme:

● - tinted black

○ - tinted white

\+ - untinted

⊙ - tinted either black or white

⊗ - tinted both black and white

⦶ - tinted black or white, or untinted.

In the actual play of games, we often represent a region in which a player has moved by $\otimes$, since this prohibits either player from moving there in the play that follows, just as the move does.

The graph may be simplified by deleting edges joining oppositely tinted nodes. An edge affects the graph by preventing adjacent nodes from being similarly painted. As the tinting already accomplishes this, the edge is redundant. Such an edge is called *explosive*.

Figure 2.1 shows a map with one region painted black (represented by 'b') and one region painted white (represented by 'w') as well as the graph that corresponds to it. We analyze a slightly more general game than the one with which we started. The "brown paper" will only generate *planar* graphs, while the theory applies to arbitrary graphs, so that one can play on pieces of brown paper of any genus.

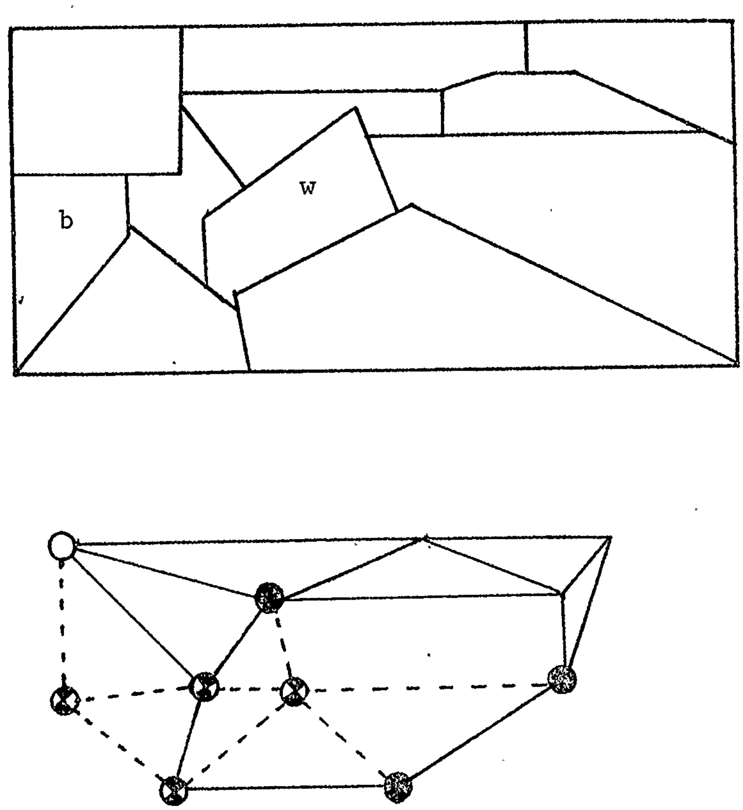
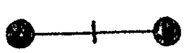
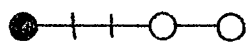
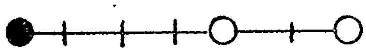


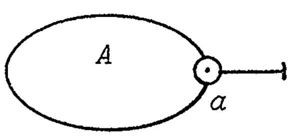
Figure 2.1. The correspondence between maps and graphs.

In the analysis presented here we primarily discuss chains, though results will be generalized to arbitrary graphs whenever possible. The following notation facilitates the description of arbitrary graphs, but is particularly appropriate when discussing chains. Let '+' correspond to a black node: $+^n$ then represents a chain of n nodes tinted black. Let '−' correspond to a white node: $-^n$ represents a chain of n nodes tinted white. Those nodes about whose tint we are uncertain are represented by 0. A string of n untinted nodes is represented by $\circ^n$. For example

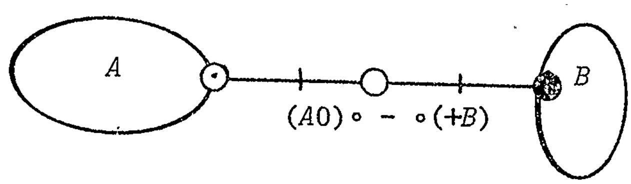
 : $+\circ+$  ; $+\circ^2-^2$

 : $+\circ^3-\circ-$

A similar technique is used when referring to a vertex, say a, joined to a set A of nodes. Note that a is not considered part of the set A, i.e. A is interpreted as the subgraph induced by the nodes other than a. Then the node a is described by the symbols outlined above. For example



: $(A0)\circ$



2.2. The Values of some Col Positions.

The analysis of Col is simpler than that of Snort as the values that arise are of a very restricted kind. Consider the values of the following simple positions:

$+ \;=\; \{0|0\} = *$ i.e. a single untinted node is a first player win.

$\bullet \;=\; \{\otimes \mid \;\} = \{0\mid\;\} = 1 \qquad \circ \;=\; \{\;\mid \otimes\} = -1$

$\bullet\!\!-\!\!\bullet \;=\; \{\otimes \mid \;\} = \{0\mid\;\} = 1 \qquad \bullet\!\!-\!\!\circ \;=\; \bullet \;\; \circ = 0$

$\bullet\!\!-\!\!\dashv \;=\; \{\circ\,,\, \otimes \mid \bullet\} = \{-1,0|1\} = \frac{1}{2}$

$\bullet\!\!-\!\!\bullet\!\!-\!\!\dashv = \{\otimes\!\!-\!\!\dashv,\; \circ\,,\; \bullet \mid \bullet\!\!-\!\!\otimes\} = \{*,-1,1|1\} =$
$= 1 + * = 1*$

In the analysis of more complicated positions, no values but x or $x+*$ where x is a number were observed. Conway and Guy have proved this is always the case. The proof depends upon the following lemmas.

LEMMA 2.1. (Hindering One's Opponent is No Harm). The value of a position is

(i) unaltered or increased by tinting a node black,

(ii) unaltered or decreased by tinting a node white,

(iii) unaltered or increased by deleting a node tinted white,

(iv) unaltered or decreased by deleting a node tinted black.

PROOF. (i), (ii) follow from the observation that tinting a node black decreases the number of right options, while tinting it white decreases the number of left options. To establish (iii) observe that if a node v is already tinted white, we may tint v black by (i) and the value of the position is unaltered or increased. However the node v then is doubly tinted and may be deleted. A similar argument establishes (iv).

LEMMA 2.2. The value of a graph is

(i) unaltered or increased by deleting any edge one end of which is tinted black,

(ii) unaltered or decreased by deleting any edge one end of which is tinted white.

PROOF. (i) The deletion of an edge, one end of which is tinted black, cannot hinder Black since it may provide Black with an extra move in an adjacent node, while if White plays in the node at the other end of such an edge, the tinted node is unaffected.

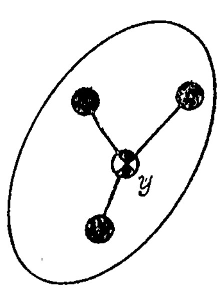
(ii) is the same statement with colours reversed.

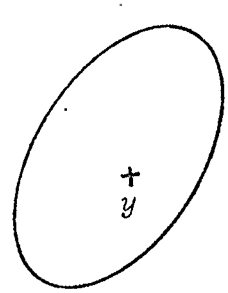
THEOREM 2.3. The value of any position G in Col is either x or $x*$ ($= x+*$) where x is a number.

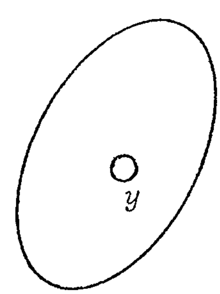
PROOF. By Lemma 1.9, we know that $G^L \lhd\!|\; G \lhd\!|\; G^R$. It suffices to prove that

$$G^L + * \leqslant G \leqslant G^R + *.$$

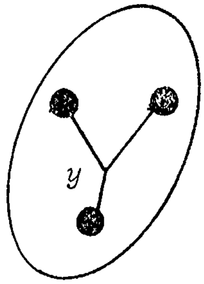
Suppose White moves by painting y in G, i.e.

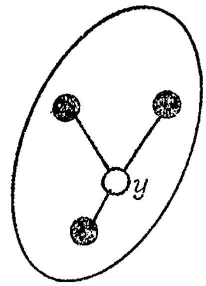
$G^R =$ 

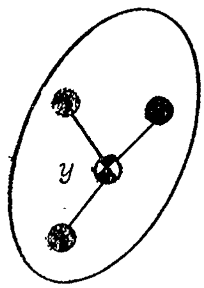
where $G =$ 

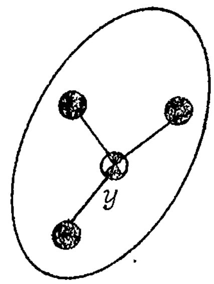
or 

Then $G \leqslant G_y$ where G_y is the position obtained by tinting black those nodes adjacent to y

i.e. G_y = 

or 

$\leqslant$ 

$+$ or 

 by Lemma 2.2(i)

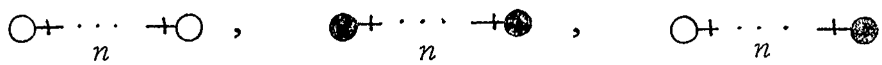
$= G^R + *$ or $G^R - 1$

and since $G^R - 1 < G^R + *$, we have

$$G \leqslant G_y \leqslant G^R + *. \qquad \square$$

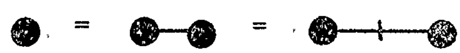
When evaluating positions it is normal to consider all Left and Right options. However it can be shown that in certain positions this is unnecessary. Some of the moves are dominated, and certain options are equivalent to other positions which are easier to evaluate. In special circumstances we can completely determine the values assumed by classes of positions.

In the case that $n = -1$

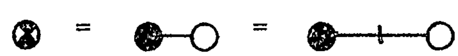


are interpreted as ○ , ● , ⊗ , with values -1, 1, 0.

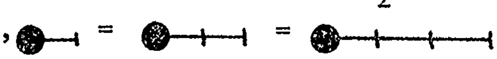
THEOREM 2.4. (i) If $n \geq -1$, then $+\circ^n+ = 1$;

i.e.,  $= \ldots = 1$.

(ii) If $n \geq -1$, then $(+\circ^n-) = 0$;

i.e.,  $= \ldots = 0$.

(iii) If $n \geq 1$, then $(+\circ^n) = \frac{1}{2}$;

i.e.,  $= \ldots = \frac{1}{2}$.

(iv) If $n \geq 2$, then $(\circ^n) = 0$;

i.e.,  $= \ldots = 0$.

PROOF. Straightforward calculation yields

$$+\circ^{-1}+ = +\circ^0+ = +\circ+ = +\circ^2+ = 1,$$
$$+\circ^{-1}- = +\circ^0- = +\circ- = +\circ^2- = 0,$$
$$+\circ = +\circ^2 = \frac{1}{2},$$
$$\circ^2 = 0,$$

so the above statements hold for $n \leq 2$. Let $m \geq 3$, and assume inductively that the above statements hold for $n < m$. Then

$$+\circ^m+ = \{-\circ^{m-1}+,\ -\circ^{m-2}+,\ (+\circ^i-,-\circ^j+) \text{ where } i \geq 0,\ j \geq 0,\ i+j = m-3$$
$$\big|\,(+\circ^i+,+\circ^j+) \text{ where } i \geq -1,\ j \geq -1,\ i+j = m-3\}$$
$$= \{0|2\} = 1.$$

$$\circ^{m} = \{-\circ^{m-2},\ (\circ^{i}-,-\circ^{j}) \text{ where } i \geq 0,\ j \geq 0,\ i+j = m-3 \mid -G^{L}\}$$

$$= \{-\tfrac{1}{2},\ -1\tfrac{1}{2}(i=0),\ -1(i>0) \mid \tfrac{1}{2},\ 1\tfrac{1}{2},\ 1\}$$

$$= 0.$$

$$+\circ^{m}- = \{-\circ^{m-1}-,\ (+\circ^{i}-,-\circ^{j}-) \text{ where } i \geq -1,\ j \geq 0,\ i+j = m-3 \mid -G^{L}\}$$

$$= \{-1 \mid 1\}$$

$$= 0.$$

$$+\circ^{m} = \{-\circ^{m-1},\ (+\circ^{i}-,-\circ^{j}) \text{ where } i \geq -1,\ j \geq 0,\ i+j = m-3,\ +\circ^{m-2}-,$$

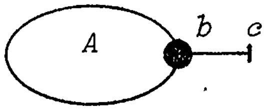
$$\mid (+\circ^{i}+,+\circ^{j}) \text{ where } i \geq -1,\ j \geq 1,\ i+j = m-3,\ (+\circ^{m-3}+,+),\ (+\circ^{m-2}+)\}$$

$$= \{-\tfrac{1}{2},-1,0 \mid 1\tfrac{1}{2},2,1\}$$

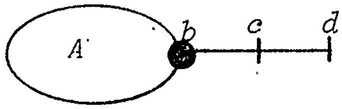
$$= \tfrac{1}{2}. \qquad \square$$

In the proof of the above theorem, the list of Left and Right options was extensive. In more complicated positions, the list of options is even longer. Fortunately, it is not always necessary to evaluate every option, as certain among them will always be dominated.

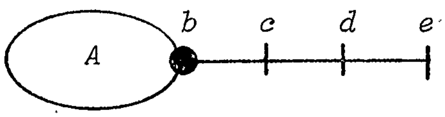
LEMMA 2.5. Let A,B be arbitrary graphs.

(i) 

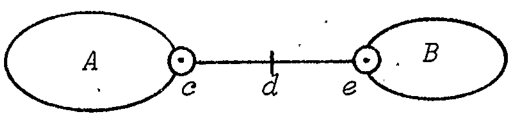
Black prefers the move c to b.

(ii) 

For both Black and White, the move d is at least as good as b or c.

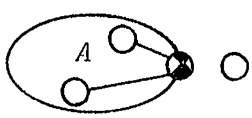
(iii) 

For both Black and White, the move e is at least as good as b, c, or d.

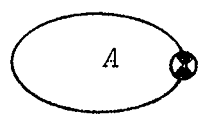
(iv) 

For both Black and White the move d is at least as good as c or e.

PROOF. (i) Black may move to

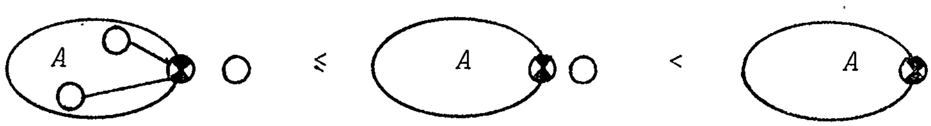


by moving in b

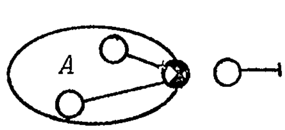


by moving in c.

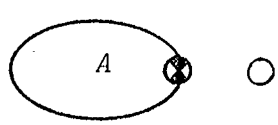
However by Lemma 2.1 (ii)



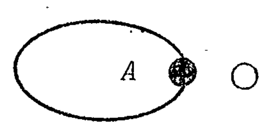
(ii) Black may move to



by moving in b

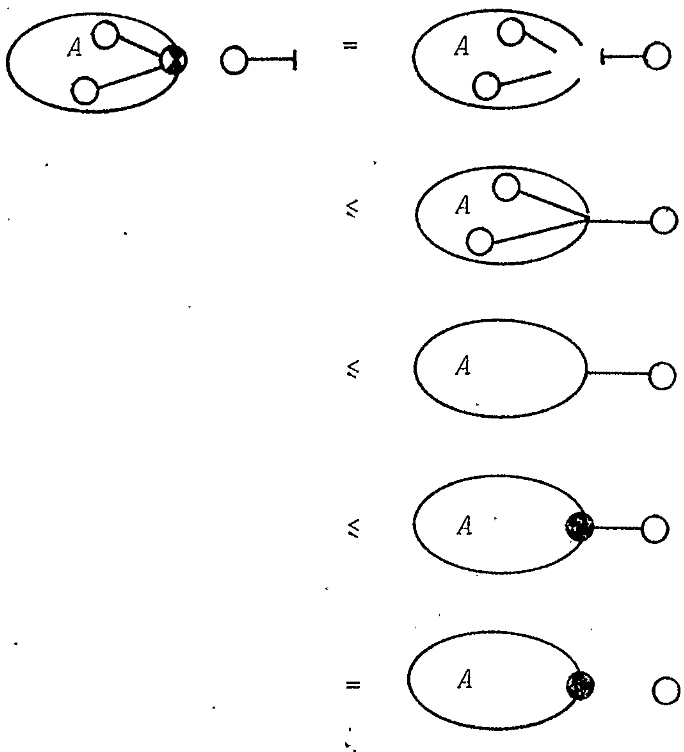


by moving in c



by moving in d.

Now

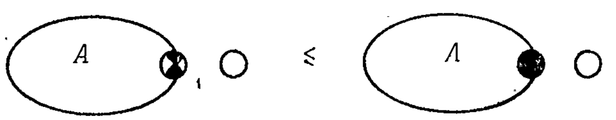


by Lemma 2.2(ii)

by Lemma 2.1(ii)

by Lemma 2.1(i)

and

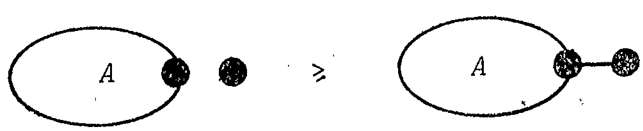


by Lemma 2.1(iv)

In the same position the white options are

by moving in c by moving in d

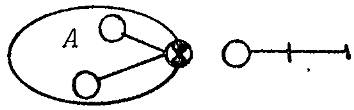
and



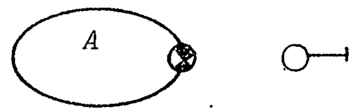
by Lemma 2.2(i).

Hence for each of the players the move in d is at least as good as the other moves.

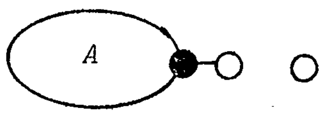
(iii) Black may move to



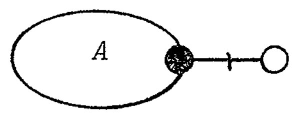
by moving in b



by moving in c

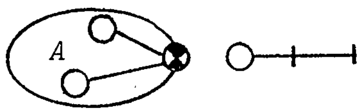


by moving in d

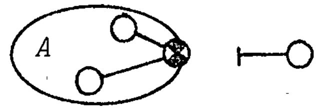


by moving in e.

Now, for the option that results from moving in b

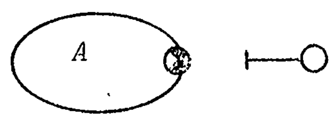


=



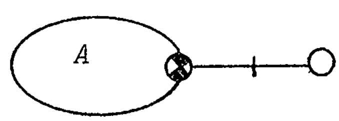
by Theorem 2.4

$\leqslant$

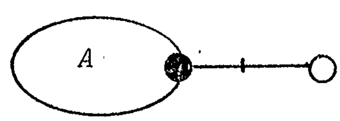


by Lemma 2.1(ii) (this is the result of painting c)

=

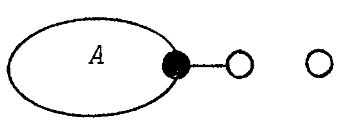


$\leqslant$

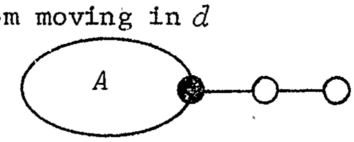


by Lemma 2.1(iv) (this is the result of painting e)

and for the option that results from moving in d

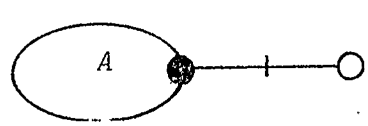


, $\leqslant$



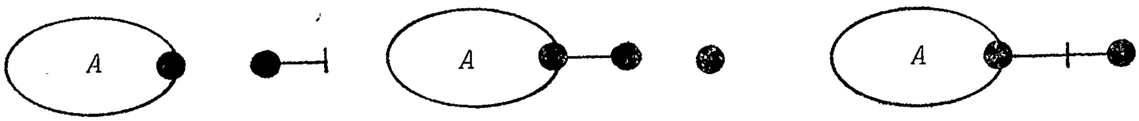
by Lemma 2.2(ii)

$\leqslant$



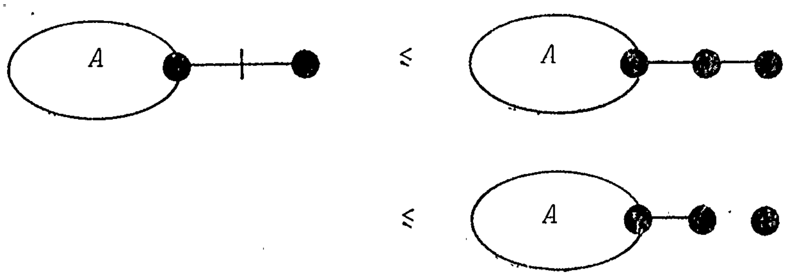
by Lemma 2.1(ii)

Hence the move in e is at least as good for Black as the other moves. To see that this is also true for White, consider his options:



by moving in c by moving in d by moving in e.

However



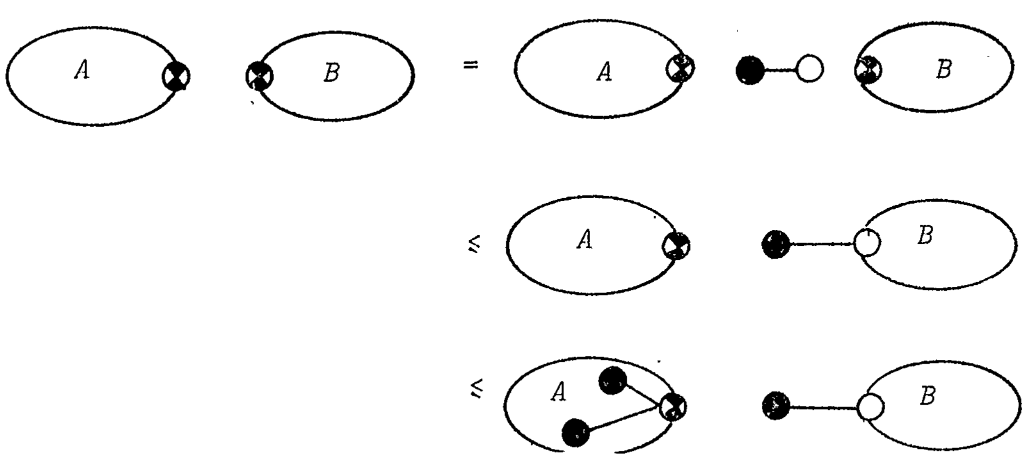
by Lemma 2.1(i)

by Lemma 2.2(i)

and

by Lemma 2.2(i).

(iv) The nodes c,e are both tinted, similarly or oppositely. If c,e are tinted similarly we may without loss of generality assume both tints are white. Then Black must, by the rules, prefer d to c or e. White also does at least as well playing d, since



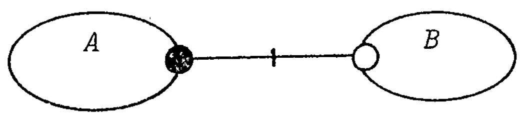
by Theorem 2.4(ii)

by Lemma 2.2(ii)

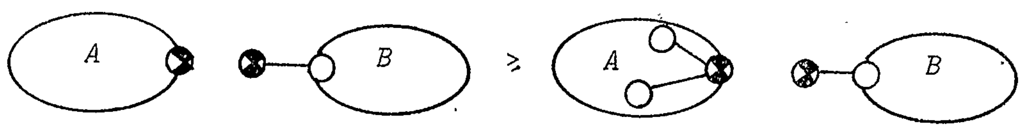
by Lemma 2.1(i)

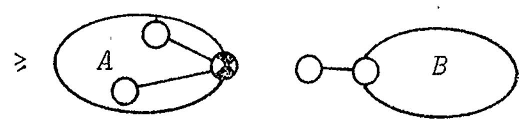
and this is the result of White's playing in c. Similarly White will not prefer e to d.

If c,e are tinted oppositely, by symmetry we need only consider Black's move in



and his move in d leads to

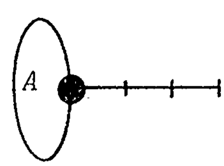
 by Lemma 2.1(ii)

 by Lemma 2.1(i)

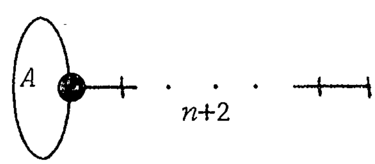
which is the result of Black's playing at c. □

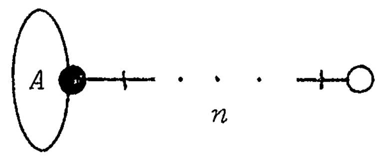
The preceding lemma enables us to prove the Half Measures and Elastic Ends Theorem which may be used to simplify the analysis of Col positions.

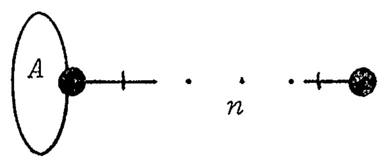
THEOREM 2.6. If A is any graph and

$(A+)\circ^3 =$  $= x$,

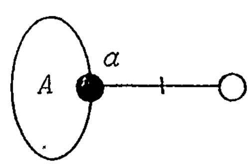
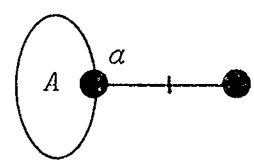
then for $n \geq 1$:

(i) $(A+)\circ^{n+2} =$  $= x$,

(ii) $(A+)\circ^{n}$ =  $= x - \frac{1}{2}$,

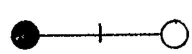
(iii) $(A+)\circ^{n}+$ =  $= x + \frac{1}{2}$.

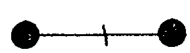
PROOF. We first show by induction on the number of nodes in A that

1 +  =  (**)

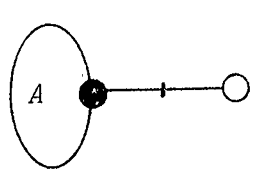
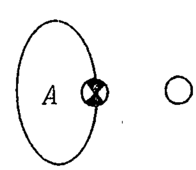
$1+((A+)\circ-) = ((A+)\circ+)$

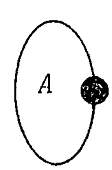
By Theorem 2.4

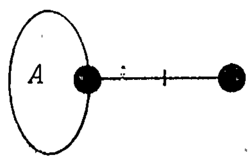
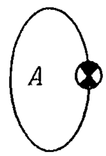
 = 0

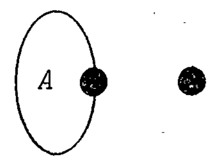
 = 1

so that (**) holds when A is empty. If (**) holds for all subgraphs A' of A, then

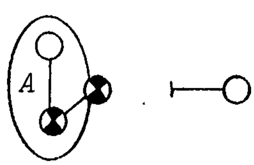
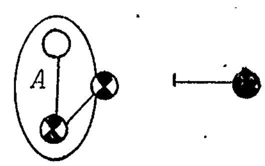
G =  = $\{G^{L1}$ =  , G^{L},

$|\, G^{R1}$ =  , $G^{R}\} = x$ by Lemma 2.5(iv) where G^L, G^R denote the options that result from moves in A and

$H =$  $= \{H^{L1} =$  $, H^L$

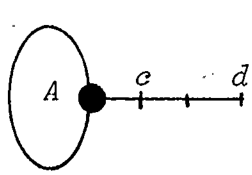
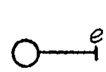
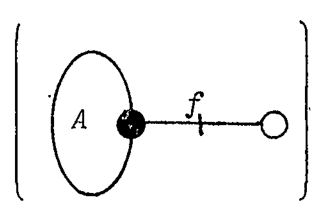
$\mid H^{R1} =$  $, H^R\} = y,$

say, where H^L, H^R denote the options that result from moves in A. By the induction hypothesis, for each H^L, H^R there exists G^L, G^R such that $H^L = G^L+1$, $H^R = G^R+1$ and vice versa. For those moves in A at a node b by Black that tint the node a white, we have options

$G^{L'} =$  , $H^{L'} =$  ,

so that $H^{L'} = G^{L'}+1$. Since $H^{L1} = G^{L1}+1$, $H^{R1} = G^{R1}+1$, the value of every option of H is 1 greater than a corresponding option in G, so that $y = x+1$.

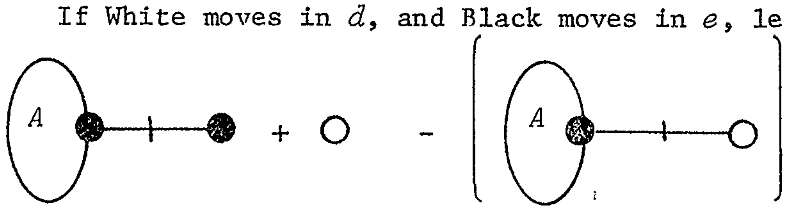
To show that $(A+)\circ^3 = ((A+)\circ-) + \frac{1}{2}$, we use the above result and play the game $((A+)\circ^3) - \frac{1}{2} - ((A+)\circ-)$, i.e.

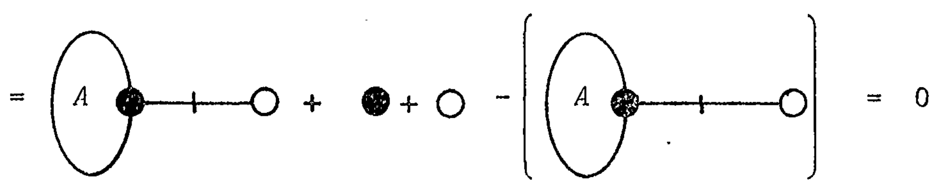
 +  - 

$H \qquad -\frac{1}{2} \qquad -G$

and show that it is a second player winning game. Moves by either player in A (in either $-G$ or H) are covered by the induction hypothesis and Lemma 1.5, the Tweedledum and Tweedledee principle. By Lemma 2.5(iii), (i) and (ii), aside from moves in A, we need only consider the moves d, e, and f.

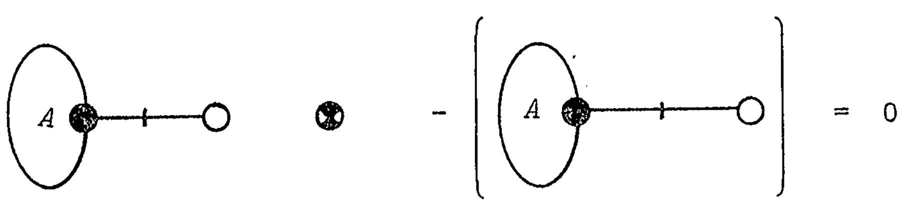
If White moves in d, and Black moves in e, leaving





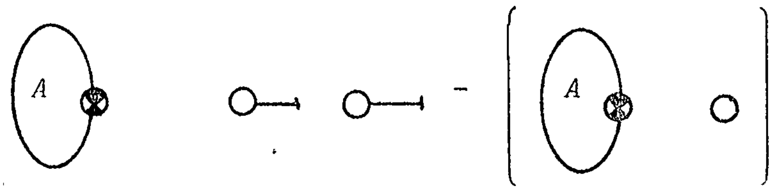
we have that the player who moved first loses because he becomes the first player in a shortened second player winning game.

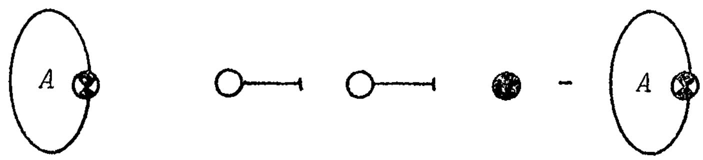
If White moves in e, and Black moves in d, leaving



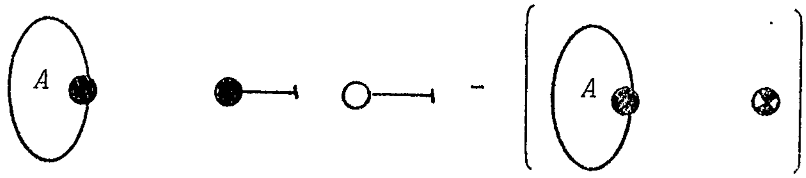
we have again that whichever player moved first, loses.

If Black moves in $-G$ at f, he may also move in H at c to leave

$H^L - \frac{1}{2} - G^L =$ 

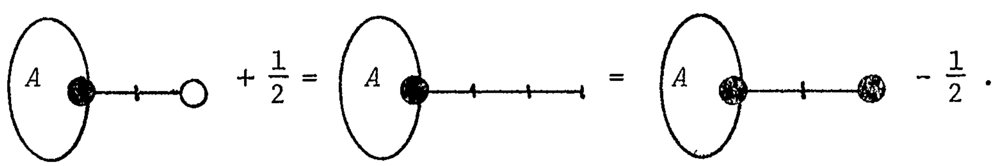
$=$  $= 0$

which is a second player winning game. Note that the move by Black in $-G$ at f is equivalent to a first move by White in the game $H - \frac{1}{2} - G$. If White moves in $-G$ at f (a first move by Black in $H - \frac{1}{2} - G$), he may also move in H at c to leave

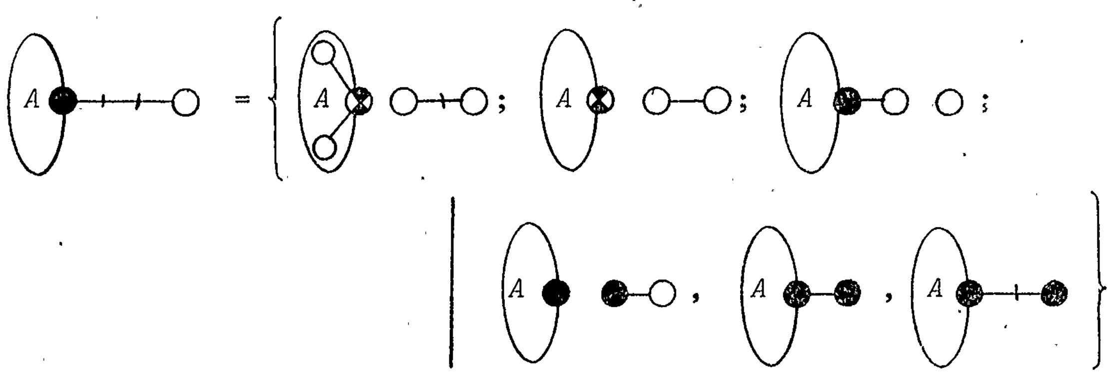
$H^R - \frac{1}{2} - G^R =$ 

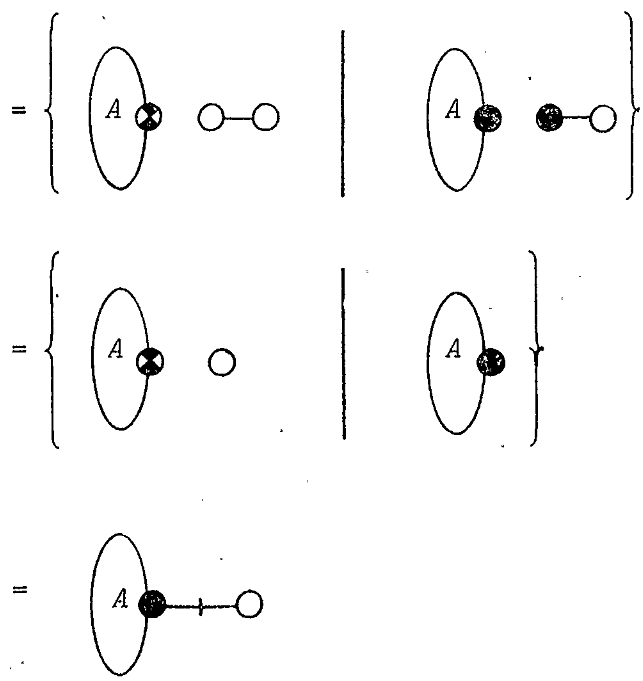
$= 0$

which is a second player winning game. Hence we have

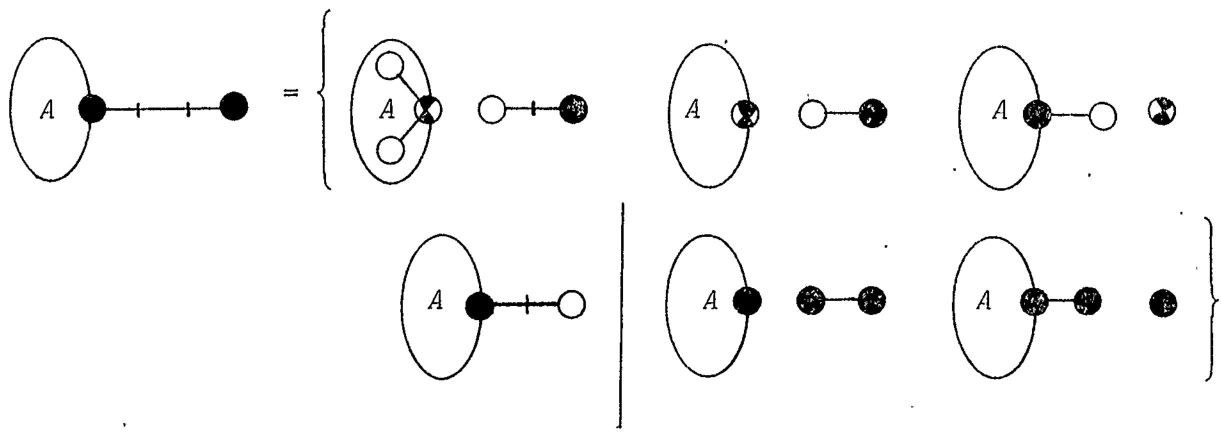


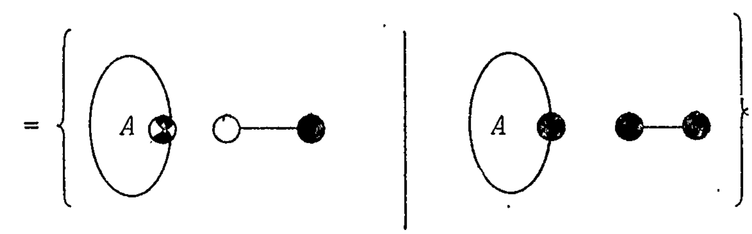
The three parts of the theorem are now proved simultaneously by induction on n. At each step we ignore moves in A, assuming that they are covered by the induction hypothesis. Note that this is not the same as the induction hypothesis on n. In reality, the proof is a double induction in which for each n, we induce on the number of nodes in A. Theorem 2.4(i), (ii), (iii) provide the basis for the induction on A at each step. When $n = 2$,



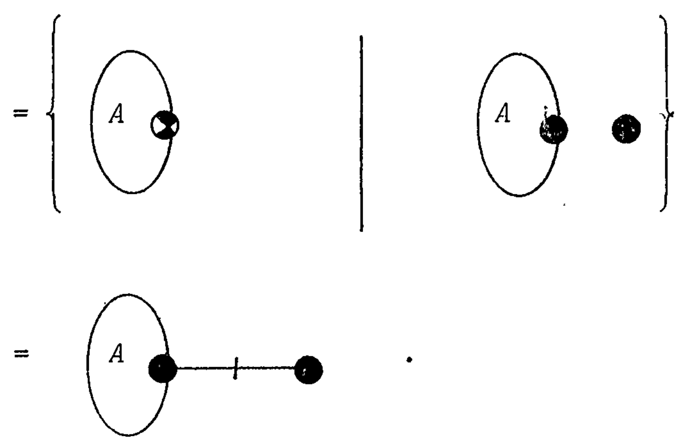


as the other options are dominated





since the other options are dominated



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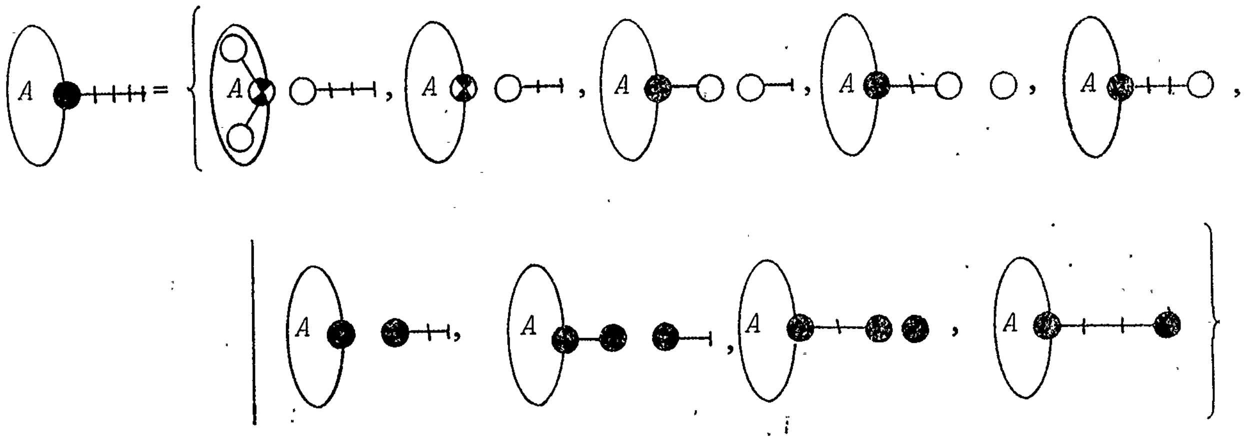
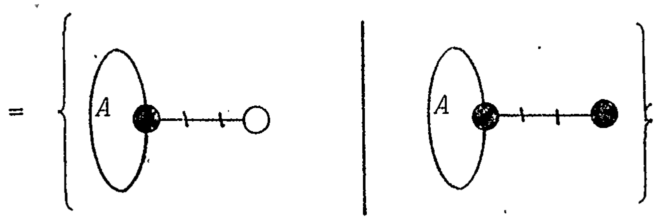
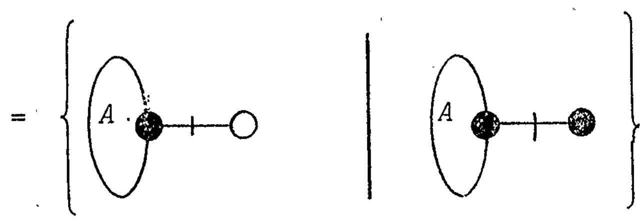
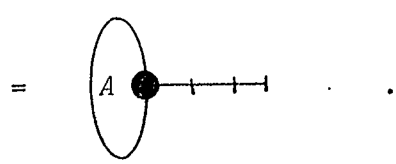


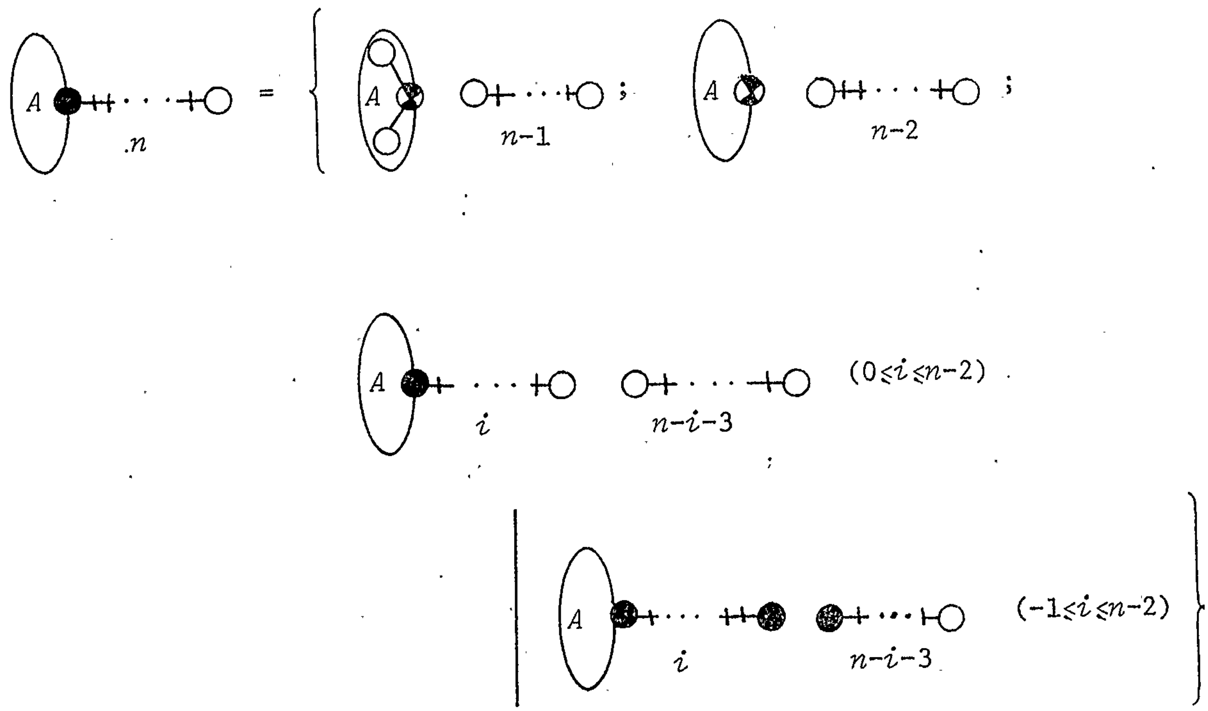
 since the other options are dominated

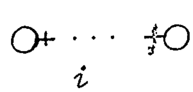
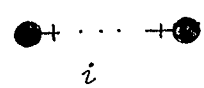
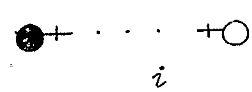
 by what has already been shown



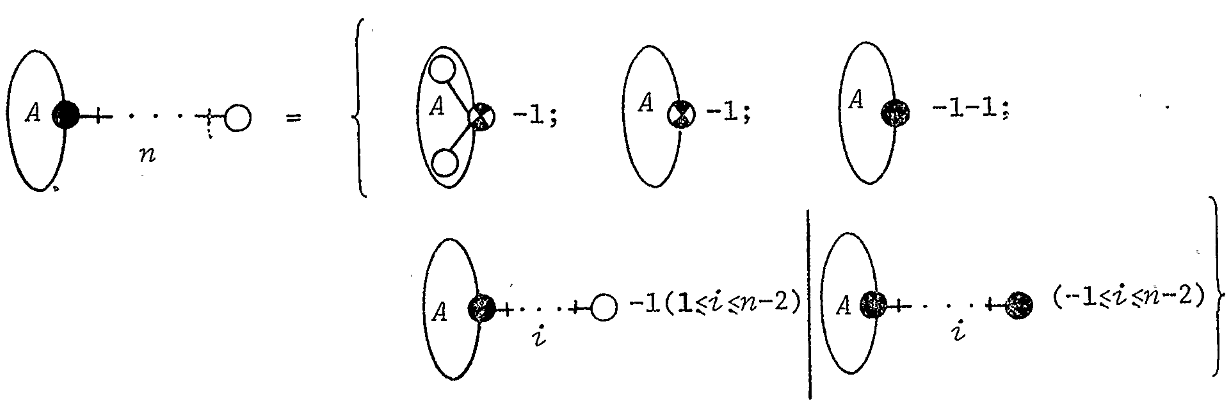
This establishes the theorem for $n = 2$.

Assume that (i), (ii), (iii) are true for $i < n$. We show the inductive step for (i). The rest follow by an equally straightforward analysis.

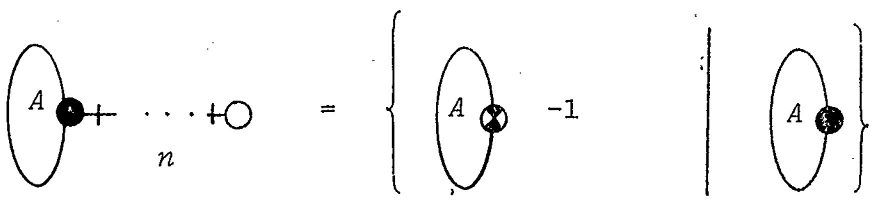


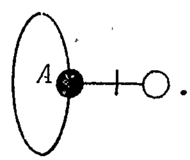
where   and  are interpreted

in the case $i = -1$, as ○ , ● and ⊗ , with values -1,1,0. Hence



For Black the second option is at least as good as the first by Lemma 2.1(ii). The third option is dominated by the second by Lemma 2.2(i), and the remaining options are no better than the second by the induction hypothesis. For White, by Lemma 2.1(ii) the option that results from taking $i = -1$ is at least as good as the others. Hence



= 

. □

2.3. Equivalent Positions

Given a graph G, a node is said to be *explosive* if the value of the graph is unaltered when we tint the node either black or white. For example,

(∘+∘): 

$$= \left\{-2,\ * \ \middle|\ \frac{1}{2}\right\}$$

$$= 0,$$

so that

Hence the middle node is explosive.

LEMMA 2.7. (i) For $n \geq 1$, $\circ^n + \circ^n = 0$.

(ii) For $n \geq 2$, $\circ + \circ^n = \frac{1}{4}$.

(iii) For $n \geq 3$, $\circ^2 + \circ^n = \frac{1}{2}$.

(iv) For $n, m \geq 3$, $\circ^n + \circ^m = 0$.

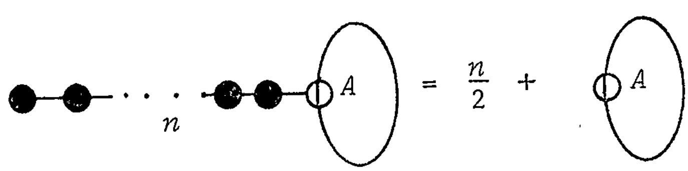
PROOF. It is easy to verify that

$$\circ + \circ = \circ^2 + \circ^2 = \circ^3 + \circ^3 = 0,$$
$$\circ + \circ^2 = \circ + \circ^3 = \frac{1}{4},$$
$$\circ^2 + \circ^3 = \frac{1}{2},$$

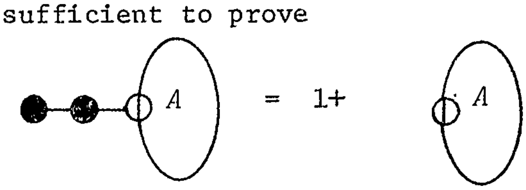
from which (i), (ii), (iii) and (iv) follow by a straightforward application of Theorem 2.6.

The following lemma enables us to simplify the evaluation of positions.

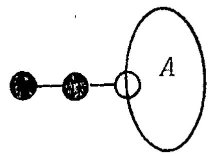
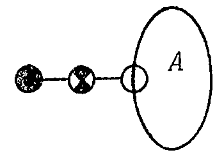
LEMMA 2.8. For n even, $n \geq 2$

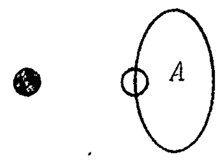


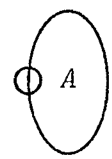
PROOF. It is sufficient to prove

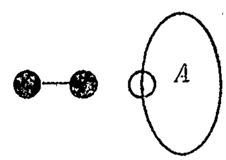


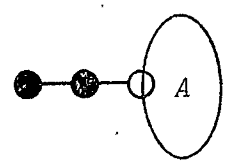
as the general case will follow from repeated applications of the above result. However

 $\geq$  by Lemma 2.1(ii)

$=$  by Lemma 2.1(ii)

$= 1 +$ 

$=$ 

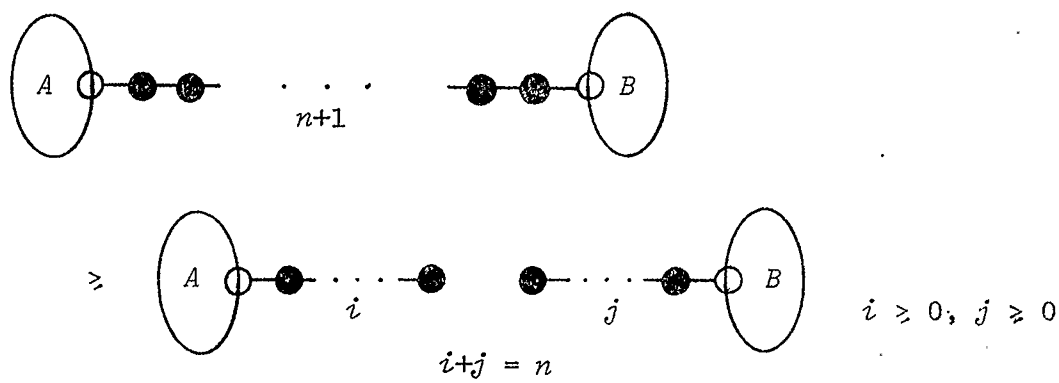
$\geq$  by Lemma 2.2(i).

□

We summarize the results established so far. In a chain of length $n > 1$, if there is no tint whatsoever, the chain has value 0. If there is a tinted node, we may assume by Theorem 2.6 that the tinted node is at most three nodes from the end. If the end node is tinted, and the penultimate node is also tinted, we may use either the remark concerning explosive nodes or the remark concerning explosive edges to simplify the chain. If the nodes are similarly tinted, the penultimate node is explosive. If the nodes are oppositely tinted, then the edge joining them is explosive and may be deleted.

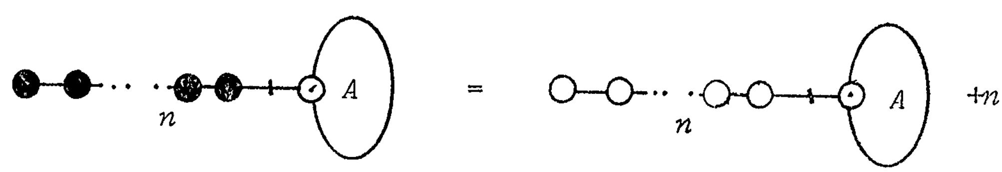
We prove another equivalence that enables us to simplify positions. However we first establish three lemmas that will be used in the proof.

LEMMA 2.9. If A is any graph, and $n \geq 0$



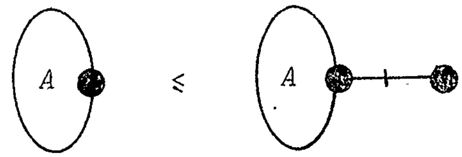
PROOF. This is an immediate consequence of Lemma 2.1(iv).

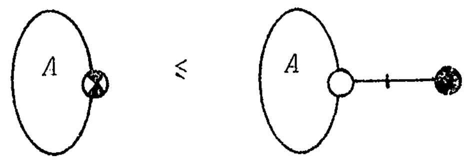
LEMMA 2.10. If A is any graph, and $n \geq 2$



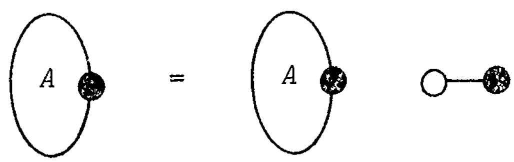
PROOF. This is an immediate consequence of Lemma 2.8, and Theorem 2.6.

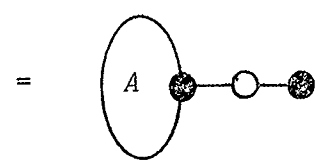
LEMMA 2.11. If A is any graph,

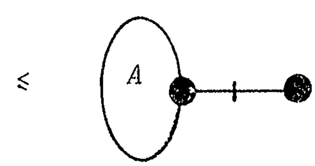
(i) 

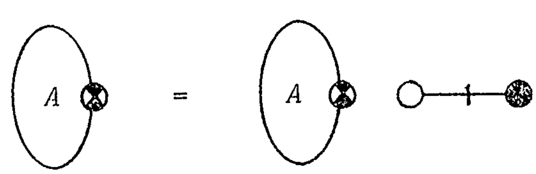
(ii) 

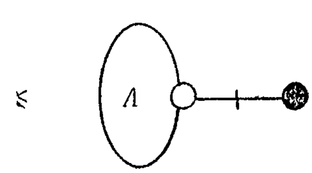
PROOF.

(i)  by Theorem 2.4(ii)

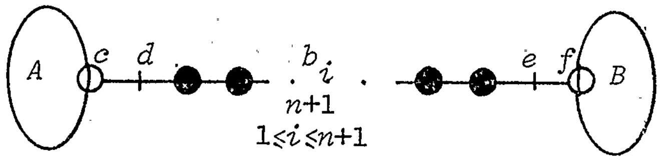


 by Lemma 2.1(ii)

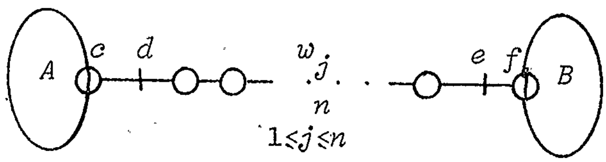
(ii)  by Theorem 2.4(ii)

 by Lemma 2.2(ii) . □

THEOREM 2.12. For any graphs A, B, and $n \geq 1$



=

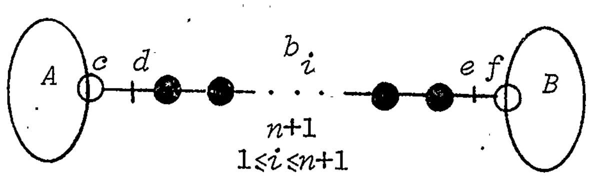


$+n$

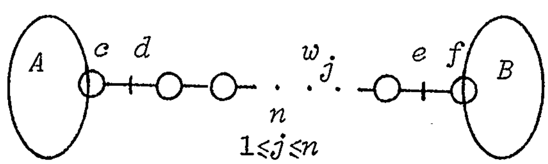
provided that if c is untinted then A is empty, and if f is untinted, then B is empty.

PROOF. Let

$G =$

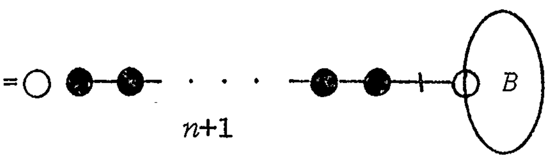


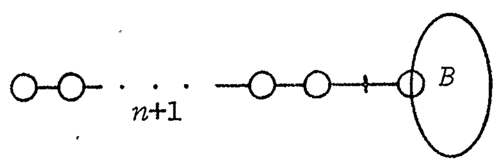
$H =$



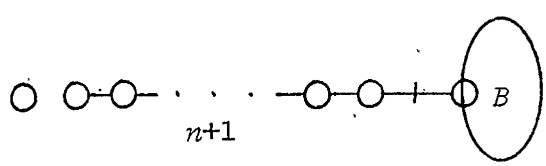
We show that $H+n-G$ is a second player winning game. For moves at c, d, e, f, we show that if the first player moves in $H(G)$, he may make a move in $G(H)$ so that either $H^L+n-G^L = 0$ or $H^R+n-G^R = 0$. This is established by showing that a move at c in G corresponds to a move at d in H, and a move at d in G corresponds to a move at d in H. The argument concerning moves at e, f will follow by a symmetrical argument. Note that by Lemma 2.5, it is only necessary to consider a move at c in the case where c is untinted, so that by hypothesis, A is empty.

Consider moves by Black at c in G and at c in H. He leaves options G^L, H^L such that

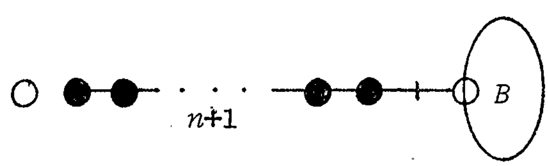
$G^L =$ 

$H^L + n =$ 

$+n$

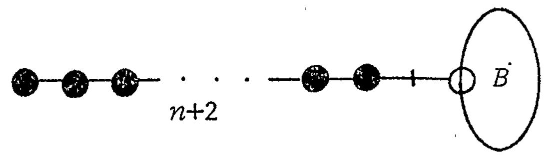
$=$ 

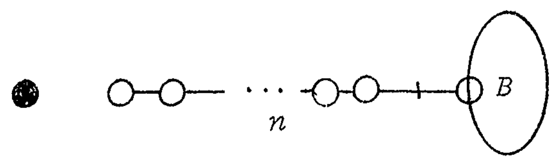
$+n+1$

$=$ 

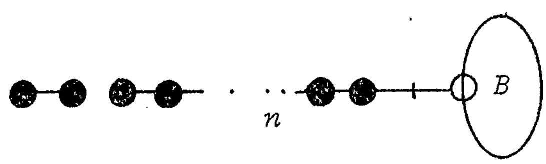
by Lemma 2.10

so $G^L = H^L+n$, and $H^L+n-G^L = 0$. If White moves at c then we have

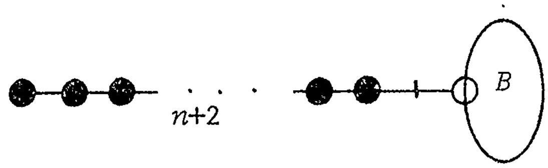
$G^R =$ 

$H^R+n =$ 

$+n$

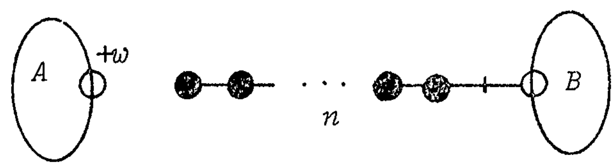
$=$ 

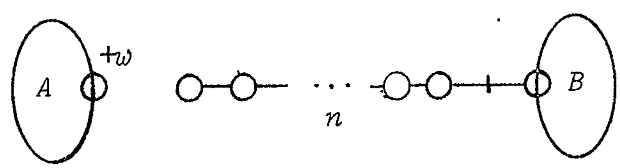
by Lemma 2.10

$=$ 

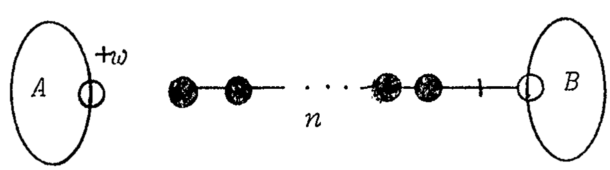
.

Moves by Black in H at d and in G at d correspond. He leaves options H^L, G^L such that

G^L = 

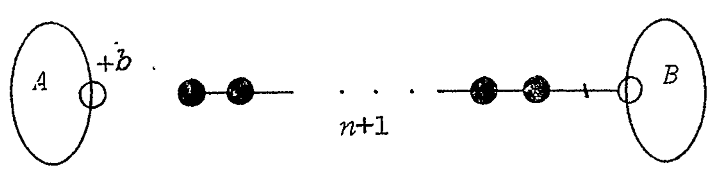
H^L+n = 

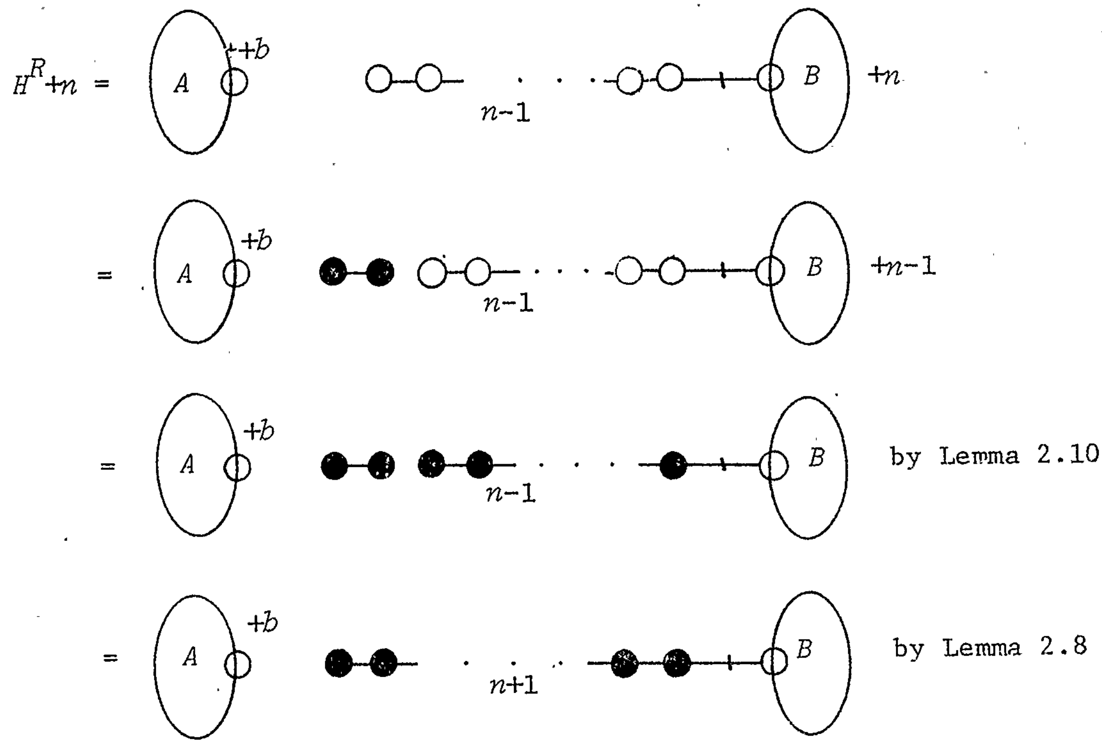
$+n$

= 

by Lemma 2.10.

where we indicate that the node c acquires a white tint in addition to whatever tint it already possesses by writing '$+w$'. Should White move at the same node we have

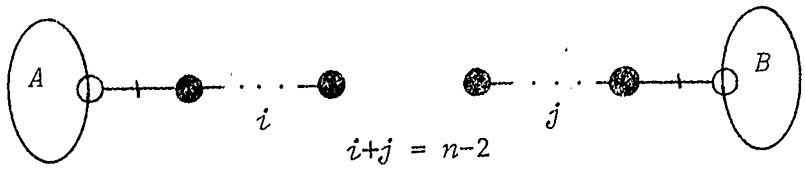
G^R = 

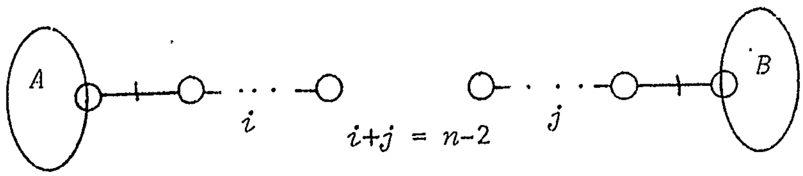


so that $G^R = H^R + n$.

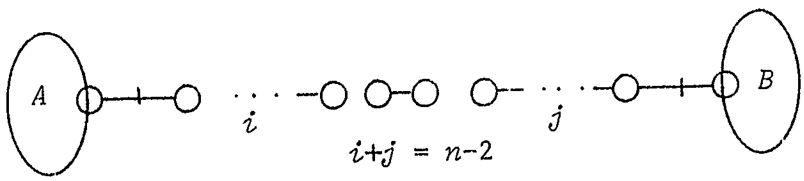
By Lemma 2.5, the legal moves at b_1, b_{n+1}, w_1, w_n are no better than moves at c or d.

We now assume that $n \geqslant 2$ and consider a move by Black in the chain of black nodes of G. This is equivalent to a first move by White in the game $H^L + n - G$. It suffices to show that $G^L < H + n$.

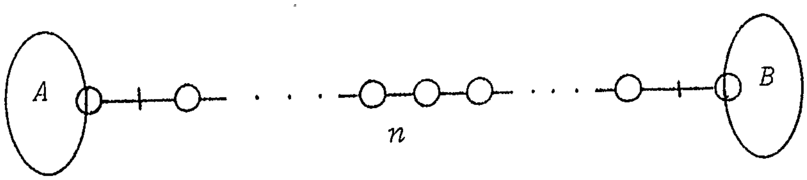
$G^L =$ 

$=$ 

$+n-2$ by Lemma 2.10

$=$ 

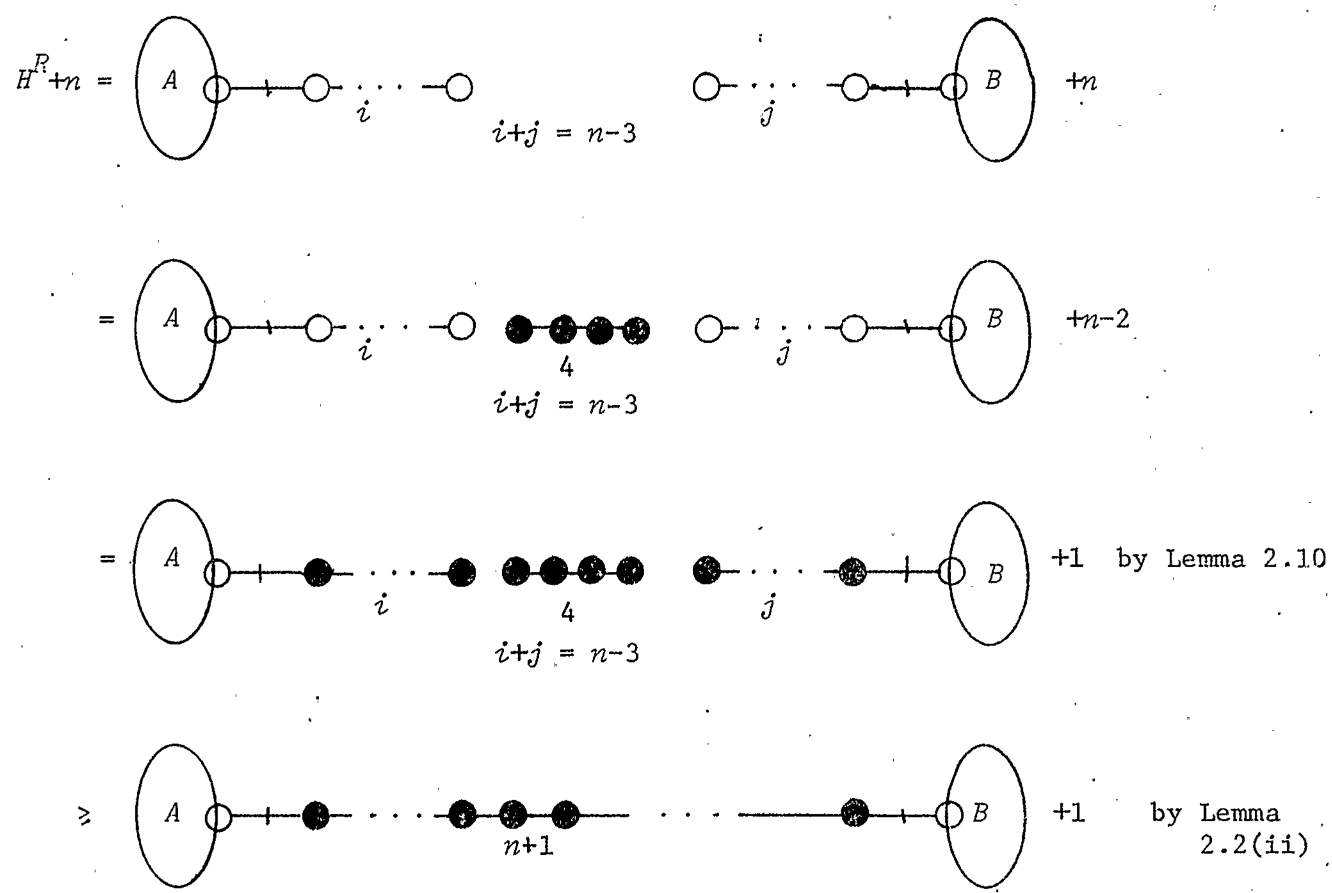
$+n-1$

$\leqslant$ 

$+n-1$ by Lemma 2.2(ii)

$< \; H^L+n$.

Similarly we show that if $n > 2$ a move in H in the chain of white nodes by White leaves an option $H^R+n > G$ so that the game H^R+n-G is Black to win.

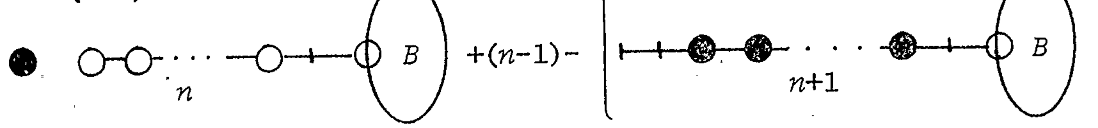


$> G.$

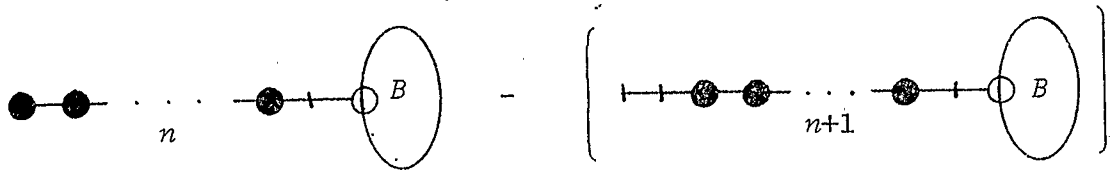
It remains to consider the situation in which Black moves in n.

Consider first the situation in which at least one of c, f is not tinted. Suppose c is not tinted, so that A is empty. To a move by Black in n White may respond by moving in c leaving

$H^R+(n-1)-G$

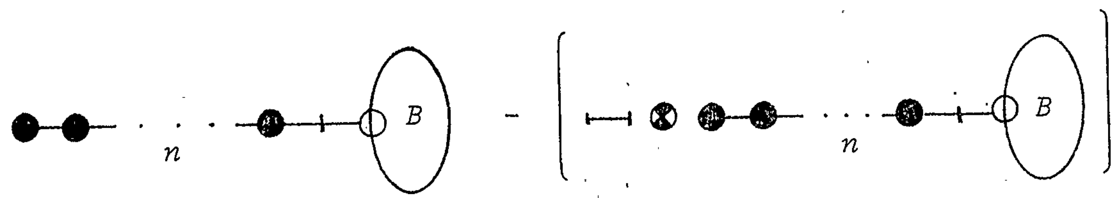


$=$



by Lemma 2.10

$\leqslant$



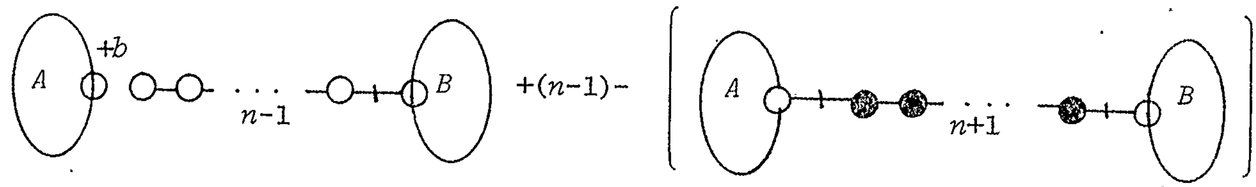
by Lemma 2.1(iv)

$= \; 0,$

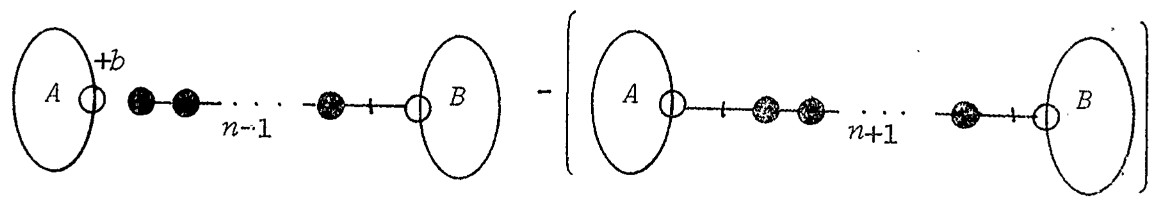
so that White as the second player can win.

If c, f are tinted, and Black moves in n to $n-1$, White answers by moving in H at d. He leaves

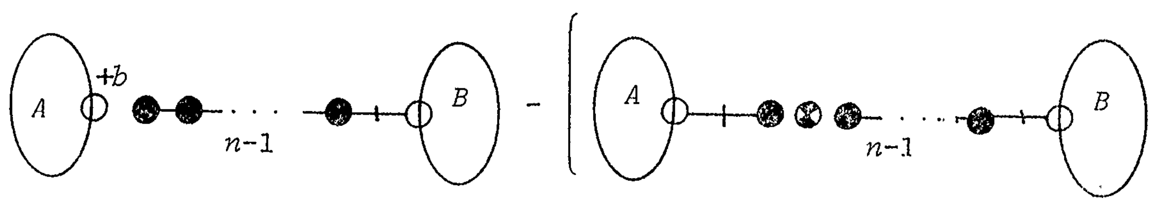
$H^R+(n-1)-G =$



=

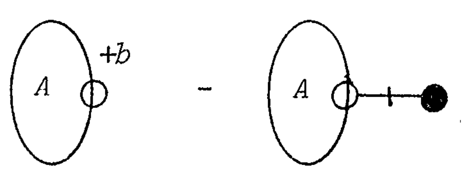


$\leq$



by Lemma 2.1(ii)

=



≤ 0 by Lemma 2.11.

□

The values of some simple positions in Col are displayed in Table 2.3. Figure 2.2 illustrates a typical element of the table. The set of nodes under consideration is described by A. Then the values of $\circ^{i}(A)\circ^{j}$, $0 \leqslant i \leqslant 3$, $0 \leqslant j \leqslant 3$ appear in the corresponding position of the array.

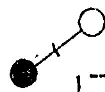
$$
\begin{array}{l|llll}
A & & & & \\
\hline
 & A & (A)\circ & (A)\circ^{2} & (A)\circ^{3} \\
 & \circ(A) & \circ(A)\circ & \circ(A)\circ^{2} & \circ(A)\circ^{3} \\
 & \circ^{2}(A) & \circ^{2}(A)\circ & \circ^{2}(A)\circ^{2} & \circ^{2}(A)\circ^{3} \\
 & \circ^{3}(A) & \circ^{3}(A)\circ & \circ^{3}(A)\circ^{2} & \circ^{3}(A)\circ^{3}
\end{array}
$$

Figure 2.2. A guide to Table 2.3.

+

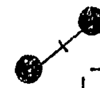
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$
$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	0

+1−



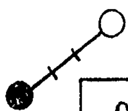
0	$\frac{1}{4}$	0	$+\frac{1}{2}$
$-\frac{1}{4}$	*	$-\frac{1}{8}$	0
0	$\frac{1}{8}$	0	0
$-\frac{1}{2}$	0	0	0

+1+



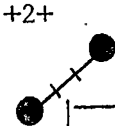
1	$\frac{3}{4}$	1	$\frac{1}{2}$
$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{1}{2}$

+2−



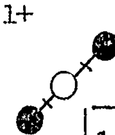
0	$\frac{1}{4}$	0	$+\frac{1}{2}$
$-\frac{1}{4}$	0	*	0
0	*	0	0
$-\frac{1}{2}$	0	0	0

+2+



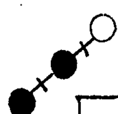
1	$\frac{3}{4}$	1	$\frac{1}{2}$
$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$*
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$*	$\frac{1}{2}$	$\frac{1}{2}$

+1−1+



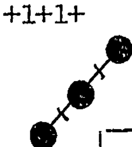
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	0

+1+1−



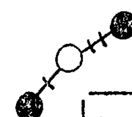
0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$-\frac{1}{8}$	0	0	$\frac{1}{4}$
0	$\frac{1}{4}$	0	$\frac{1}{2}$
0	$\frac{1}{4}$	0	0

+1+1+



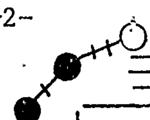
1	$\frac{7}{8}$	1	1
$\frac{7}{8}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$
1	$\frac{3}{4}$	1	$\frac{1}{2}$
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

+1−2+



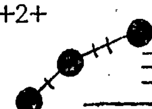
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0

+1+2−



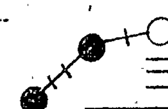
0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{4}$
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	0

+1+2+



1	1*	1	1
$\frac{7}{8}$	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{1}{2}$
1	$\frac{3}{4}$	1	$\frac{3}{4}$
1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{3}{4}$

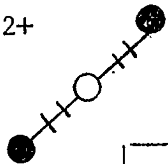
+2+1−



0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
*	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
0	$\frac{1}{4}$	0	$\frac{1}{2}$
0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

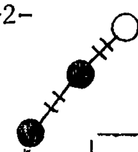
Table 2.3. The values of some Col positions

+2-2+



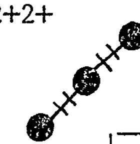
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	0	$\frac{1}{4}$	$\frac{1}{4}$
$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	0

+2+2-



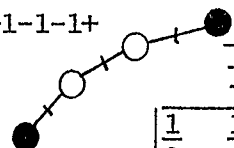
0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
*	0	$\frac{1}{4}$	$\frac{1}{4}$
0	$\frac{1}{4}$	0	$\frac{1}{2}$
0	$\frac{1}{4}$	$\frac{1}{2}$	0

+2+2+



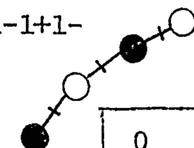
1	1*	1	1
1*	1	$\frac{3}{4}$	$\frac{3}{4}$
1	$\frac{3}{4}$	1	$\frac{1}{2}$
1	$\frac{3}{4}$	$\frac{1}{2}$	1

+1-1-1+



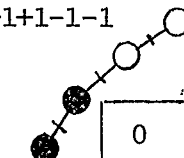
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{4}$	*	$\frac{1}{8}$	0
$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{4}$	0
$\frac{1}{2}$	0	0	0

+1-1+1-



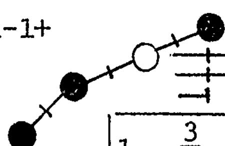
0	$\frac{1}{4}$	0	$\frac{1}{2}$
$-\frac{1}{4}$	*	$-\frac{1}{8}$	0
0	$\frac{1}{8}$	0	0
$-\frac{1}{2}$	0	0	0

+1+1-1-1



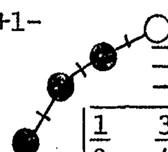
0	$\frac{1}{4}$	0	$\frac{1}{2}$
$-\frac{1}{4}$	*	$-\frac{1}{8}$	0
0	$\frac{1}{8}$	0	0
$-\frac{1}{2}$	0	0	0

+1+1-1+



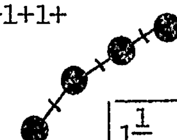
1	$\frac{3}{4}$	1	0
$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{8}$
1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{1}{2}$

+1+1+1-



$\frac{1}{2}$	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{1}{2}$
0	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{8}$
0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{2}$

+1+1+1+



$1\frac{1}{2}$	1	1	1
1	$\frac{7}{8}$	$\frac{7}{8}$	$\frac{3}{4}$
1	$\frac{7}{8}$	1	1
1	$\frac{3}{4}$	1	$\frac{3}{4}$

Table 2.3 (concluded)

Chapter 3

The Sprague-Grundy Theory

3.1. Introduction

In the remaining chapters we restrict ourselves to the class of impartial games under normal play. We still require that the state of play be known to both players, that the moves not be determined by any external means, and that the games satisfy the terminating play conditions. For reasons which will become clear, these games are known as *Nim-like games*.

The theory of the class of Nim-like games was first developed by Sprague [16] and Grundy [10] independently. We develop the theory within the more general context of Chapter 1. To facilitate the ensuing discussion we first introduce Nim-addition.

3.2. Nim-addition

For two non-negative integers a and b, the *nim-sum* of a and b, denoted by $a\overset{*}{+}b$ (pronounced "a nim b") is defined as follows: let $a = \sum_{j=0} a_j 2^j$, $b = \sum_{j=0} b_j 2^j$, $c = \sum_{j=0} c_j 2^j$, $a_j, b_j, c_j = 0$ or 1 be the binary expansions of a, b, c. Then $c = a\overset{*}{+}b$ if $c_j \equiv a_j + b_j \pmod 2$ for each j. For example consider $12\overset{*}{+}15$. In binary form $12 = 1100_2$, $15 = 1111_2$. Writing this as $\begin{matrix}1111\\1100\end{matrix}$ and adding the columns mod 2 we obtain $12\overset{*}{+}15 = 11_2 = 3$.

LEMMA 3.1. Nim addition is

(i) commutative,

(ii) associative,

(iii) distributive with respect to multiplication by powers of 2.

Further

$$\text{(iv)} \quad a \overset{*}{+} b \equiv a+b \bmod 2,$$

$$\text{(v)} \quad a \overset{*}{+} a = 0.$$

The proof follows immediately from the definition.

We also observe that $a \overset{*}{+} b \leq a+b$, and if the inequality is strict, then by Lemma 3.1(iv) the two sums differ by at least 2.

3.3. The game of Nim

Nim (see section 1.2) is an impartial game and will be used as the starting point from which we develop the theory for the class of Nim-like games. The game of Nim is actually a disjunctive compound of component games of Nim, each component consisting of a single heap of tokens.

A *position* in Nim is a set of positive integers corresponding to the number of tokens in the respective heaps. To analyze this game we let $*n$ (not to be confused with $n* = n+*$) denote the value of a nim heap of n tokens. Since any game is completely determined by its options we have

$$*0 = \{\ |\ \} = 0$$

$$*1 = \{0|0\} = *$$

$$*2 = \{0,*|0,*\}$$

so that inductively

$$*n = \{0,*,*2,\ldots,*(n-1)|0,*,*2,\ldots,*(n-1)\}.$$

This notation is consistent with that of Chapter 1 since a nim heap of size 0 is a second player win, and a nim heap of size 1 is a first player win. More generally, if $n > 0$ then $*n \| 0$ since the first player may remove all the tokens to win.

In the theory of partisan games, it is possible to speak of positions from which Left may always win, regardless of whether he moves first or second. In the game of Nim, it is only possible to speak of positions from which the first player may or may not win. If Left can win from a position G by playing first, so can Right, since the options available to either player are the same. For any impartial game, a *P-position* is a position from which the previous player (the player who moved to that position) can win, i.e. a P-position is a second player winning position, so that, if G is a P-position $G = 0$. For example in Nim, $*n + *n$ is a P-position. The second player mimics the moves of the first player in the opposite component of the disjunctive sum. An *N-position* is one from which the next player can win, i.e. it is a first player winning position. For example, in Nim, if $n \neq m$, then $*n + *m$ is an N-position. The next player equalizes the size of the two heaps, and becomes the previous player at a P-position.

3.4. The Sprague-Grundy Theory

If $\{g_1, g_2, \ldots, g_n\}$ is any set of non-negative integers, $\text{mex}\{g_1, g_2, \ldots, g_n\}$ (*m*inimal *ex*cluded value) is the least non-negative integer different from all the g_i, e.g.

$$\text{mex}\{0,2,4,1,7\} = 3$$

$$\text{mex}\ \emptyset = 0.$$

Using this definition, the theory of the game of Nim generalizes to the class of Nim-like games.

THEOREM 3.2. Let G be an impartial game whose options are all equal in value to some $*g_i$, where $g_i \geq 0$, i.e. $G = \{*g_1, *g_2, \ldots, *g_m | *g_1, *g_2, \ldots, *g_m\}$. Then $G = *g$ where $g = \text{mex}\{g_1, g_2, \ldots, g_m\}$.

PROOF. Let $g = \text{mex}\{g_1, g_2, \ldots, g_m\}$. Then from $G + *g$, the only moves are to $G + *n$ $(n<g)$, $*n + *g$ $(n<g)$, $*n + *g$ $(g<n)$, all of which are N-positions (see Figure 3.1). Hence $G + *g \doteq 0$, so that

$$\begin{aligned} G &= G + (*g+*g) \\ &= (G+*g) + *g \\ &= 0 + *g \\ &= *g. \end{aligned}$$

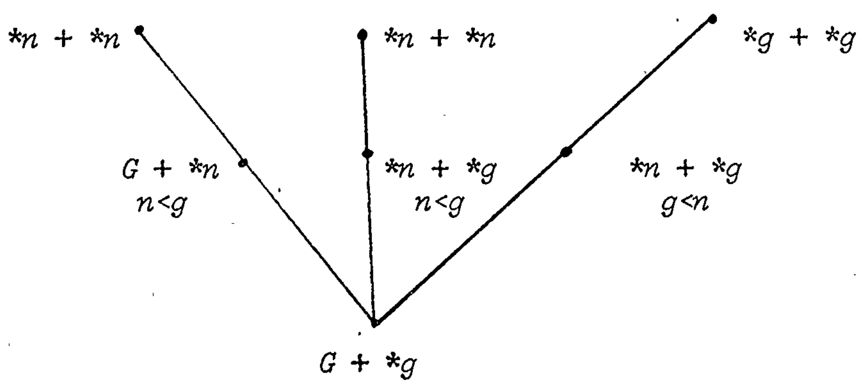


Figure 3.1. The play of $G + *g$.

As an immediate consequence of Theorem 3.2 we obtain

COROLLARY 3.3. Every Nim-like game is equal in value to $*g$ for some non-negative integer g.

In particular Theorem 3.2 and Corollary 3.3 imply that Nim itself must have a solution. Given two nim heaps of values $*n$, $*m$, their disjunctive sum $*n + *m$ is an impartial game, so that we must have $*n + *m = *g$ for some g, where g is a function of n, m. Further, for two positions $G = *n$, $H = *m$ in an impartial game, we will have evaluated the disjunctive sum $G{+}H$ the moment we have determined g. It suffices therefore to evaluate disjunctive compounds of nim heaps.

Recall that for any games G,H,

$$G{+}H = \{G^L{+}H, G{+}H^L \mid G^R{+}H, G{+}H^R\}.$$

This definition, with Corollary 3.3 and Theorem 3.2 can be used to compute $*n + *m$ inductively. Figure 3.2 lists the values of $*g = *n + *m$ for $n \leqslant 7$, $m \leqslant 7$.

	*0	*1	*2	*3	*4	*5	*6	*7
*0	*0	*1	*2	*3	*4	*5	*6	*7
*1	*1	*0	*3	*2	*5	*4	*7	*6
*2	*2	*3	*0	*1	*6	*7	*4	*5
*3	*3	*2	*1	*0	*7	*6	*5	*4
*4	*4	*5	*6	*7	*0	*1	*2	*3
*5	*5	*4	*7	*6	*1	*0	*3	*2
*6	*6	*7	*4	*5	*2	*3	*0	*1
*7	*7	*6	*5	*4	*3	*2	*1	*0

Figure 3.2. $*n + *m$, $n \leqslant 7$, $m \leqslant 7$.

Figure 3.2 suggests that given two nim heaps of values $*n$ and $*m$, $*g = *n + *m$, where $g = n \overset{*}{+} m$. The proof of this fact depends upon the following theorem.

THEOREM 3.4. (i) If $m < 2^l$, then $*2^l + *m = *(2^l+m)$

(ii) If $*g = *m + *n$ where $m,n < 2^{l+1}$, then $g < 2^{l+1}$.

PROOF. By induction. Figure 3.2 establishes (i), (ii) for $l = 0,1,2$. Assume inductively that (i), (ii) hold for all $l < k$ where $k \geq 3$. To establish (i) requires a further induction on m. Note that $*2^k + *0 =$ $= *(2^k+0)$. Assume therefore that (i) holds for $m' < m$ where $l = k$. If $m_1 < 2^k$, and $*g = *m_1 + *m$, then by (ii), $g < 2^k$. Hence, $\forall m_1 < 2^k$

$$\begin{aligned}(*m_1+*m) + *m &= *m_1 + (*m+*m)\\ &= *m_1 + *0\\ &= *m_1\end{aligned}$$

so that there are moves to $*m_1$ by moving in $*2^k$. Further if we move in $*m$ to $*m'$ where $m' < m$, then by the induction hypothesis on m, $*2^k + *m' = *(2^k+m')$

$$\begin{aligned}*2^k + *m &= \operatorname{mex}\{(*m_1+*m)+*m, *2^k+*m' \mid 0 \leq m_1 < 2^k, 0 \leq m' < m\}\\ &= \operatorname{mex}\{*m_1, *(2^k+m') \mid 0 \leq m_1 < 2^k, 0 \leq m' < m\}\\ &= *(2^k+m).\end{aligned}$$

If $m < 2^k$, $n < 2^k$, then by the induction hypothesis $g < 2^k$ where $*g = *m + *n$. Let $2^k \leq m < 2^{k+1}$, $m = 2^k + m'$. If $n < 2^k$

$$\begin{aligned}
*g &= *m + *n \\
&= *(2^k+m') + *n \\
&= *2^k + (*m'+*n) \\
&= *2^k + *g' = *(2^k+g').
\end{aligned}$$

By the induction hypothesis $g' < 2^k$ so that $g < 2^{k+1}$. If $2^k \leq n < 2^{k+1}$, let $*n = *(2^k+n')$. Then

$$\begin{aligned}
*g &= *m + *n \\
&= *(2^k+m') + *(2^k+n') \\
&= *2^k + *m' + *2^k + *n' \\
&= *2^k + *2^k + (*m'+*n') \\
&= *m' + *n',
\end{aligned}$$

and $g < 2^k$ by the induction hypothesis. □

We use this result to prove that the value of a disjunctive sum of nim heaps is just the nim sum of the values of the individual heaps.

THEOREM 3.5. The value of the position $\{n,m\}$ in the game of Nim is $*g$, where $g = n \overset{*}{+} m$.

PROOF. Let $n = \sum_j n_j 2^j$, $m = \sum_j m_j 2^j$, $g = \sum g_j 2^j$ be the binary expansions of n, m, g where $g = n \overset{*}{+} m$.

$$\begin{aligned}
*n + *m &= *(\sum_j n_j 2^j) + *(\sum_j m_j 2^j) \\
&= \sum_j *n_j 2^j + \sum_j *m_j 2^j \quad \text{by Theorem 3.4} \\
&= \sum_j (*n_j 2^j + *m_j 2^j) \\
&= \sum_j *g_j 2^j \\
&= *\sum_j g_j 2^j \qquad \text{by Theorem 3.4} \\
&= *g. \qquad \square
\end{aligned}$$

By repeated applications of Theorem 3.5 arbitrary positions in Nim can be evaluated. More important, Theorem 3.5 allows us to evaluate arbitrary positions in disjunctive sums of Nim-like games. We summarize the results in the following theorem.

THEOREM 3.6. Let G be a Nim-like game. Then all the options G are equal in value to $*g$, for some $g \geqslant 0$.

If $G = \{*g_1, *g_2, \ldots, *g_j \mid *g_1, *g_2, \ldots, *g_j\}$, then $G = *n$ where $n = \text{mex}\{g_1, g_2, \ldots, g_j\}$. Moreover, if H is another Nim-like game and $H = *m$ then $G+H = *k$, where $k = n \overset{*}{+} m$.

3.5. The Sprague-Grundy Function

Consider a Nim-like game $\mathfrak{T}$ played with heaps of tokens. By Theorem 3.6 we know that each position of G is equivalent to a nim heap of size g, for some g. To avoid confusing the number of tokens in a heap and the size of the nim heap to which it is equivalent, we introduce the *Sprague-Grundy function* $G(x)$ of the positions x of $\mathfrak{T}$. It is defined by

$$G(x) = g \text{ if } x \text{ is equivalent to a nim heap of size } g.$$

The following properties are immediate consequences of the definition and Theorem 3.7:

(i) For all positions x, $G(x) = \text{mex}\{G(y)\,|\,y \text{ is an option of } x\}$

(ii) For the disjunctive sum of positions $x_1, x_2, \ldots, x_n$

$$G(x_1+x_2+\ldots+x_n) = G(x_1) \overset{*}{+} G(x_2) \overset{*}{+} \ldots \overset{*}{+} G(x_n)$$

(iii) A player wins by consistently moving to a position x for which $G(x) = 0$.

Note that (i) implies $G(x) = 0$ for all terminal positions x.

Consider the Nim-like $\mathfrak{T}_2$ played with heaps of tokens in which a legal move affects only one heap. A player may, in his turn

(i) remove one token from a heap, provided that the remaining tokens in the heap (if any) are left in at most two heaps

(ii) remove two tokens from a heap provided that some remain.

Suppose we play this game with a heap of eight tokens. Then we have

Position	Options
{0}	
{1}	{0}
{2}	{1}
{3}	{2}, {1,1}, {1}
{4}	{3}, {1,2}, {2}
{5}	{4}, {1,3}, {2,2}, {3}
{6}	{5}, {1,4}, {2,3}, {4}
{7}	{6}, {1,5}, {2,4}, {3,3}, {5}
{8}	{7}, {1,6}, {2,5}, {3,4}, {6}

so that

$$G(0) = 0$$

$$G(1) = \mathrm{mex}(G(0)) = \mathrm{mex}(0) = 1$$

$$G(2) = \mathrm{mex}(G(1)) = \mathrm{mex}(1) = 0$$

$$G(3) = \mathrm{mex}(G(2),G(1,1),G(1)) = \mathrm{mex}(0,1\overset{*}{+}1,1) = 2$$

$$G(4) = \mathrm{mex}(G(3),G(1,2),G(2)) = \mathrm{mex}(2,1\overset{*}{+}0,0) = 3$$

$$G(5) = \mathrm{mex}(G(4),G(1,3),G(2,2),G(3)) = \mathrm{mex}(3,1\overset{*}{+}2,0\overset{*}{+}0,2) = 1$$

$$G(6) = \mathrm{mex}(G(5),G(1,4),G(2,3),G(4)) = \mathrm{mex}(1,1\overset{*}{+}3,0\overset{*}{+}2,3) = 0$$

$$G(7) = \mathrm{mex}(G(6),G(1,5),G(2,4),G(3,3),G(5)) = \mathrm{mex}(0,1\overset{*}{+}1,0\overset{*}{+}3,2\overset{*}{+}2,1) = 2$$

$$G(8) = \mathrm{mex}(G(7),G(1,6),G(2,5),G(3,4),G(6)) = \mathrm{mex}(2,1\overset{*}{+}0,0\overset{*}{+}1,2\overset{*}{+}3,0) = 3.$$

Those positions x for which $G(x) = 0$ are the P-positions. For example {0}, {2}, {6}, {1,5}, {3,3}, {1,3,4} are P-positions.

If we allow heaps of tokens of arbitrary size, then this game has G-sequence 0102310231023... where the *G-sequence* is the sequence of *G-values* $G(0)$, $G(1)$, $G(2)$,..., for those games in which disjunctive compounds of heaps of tokens $\{n\}$, $n = 0,1,2,...$ are possible positions.

Chapter 4

Take and Break Games

4.1. Introduction

We now consider an infinite class of Nim-like games with a particularly concise description. The method of description was first suggested by Guy and Smith [11] and later generalized by Guy [v. 13]. These games are played with a finite number of heaps of tokens, each heap containing a finite number of tokens. A legal move affects only one of the heaps, removing some of the tokens and possibly splitting those remaining in the heap into a number of heaps.

For the class of 'octal games', the legal moves are described by the following octal notation. Consider any infinite sequence of numerals $\cdot d_1 d_2 d_3 \ldots$, where $0 \leq d_u \leq 7$. The uth numeral describes the conditions under which we may remove u tokens from a single heap as follows.

Value of d_u	Conditions for removal of u tokens from a single heap
0	Not permitted.
1	Only if the heap contains exactly u tokens.
2	Only if, after removing u, the remaining tokens in the heap are left as a single non-empty heap.
3	Only if the remaining tokens in the heap are left as a single (possibly empty) heap.
4	Only if, after removing u, the remaining tokens in the heap are left as two non-empty heaps.

Value of d_u	Conditions for removal of u tokens from a single heap
5	Only if, after removing u, the remaining tokens in the heap (if any) are left as two non-empty heaps.
6	Only if, after removing u, the remaining tokens in the heap are left as one or two non-empty heaps.
7	Only if, after removing u, the remaining tokens in the heap (if any) are left as at most two heaps.

For example, in Nim we remove any number (possibly all) of the tokens from a heap so that Nim is denoted as $.333\ldots = .\dot{3}$.

Kayles [8] is denoted by $.77$. It is the game in which we may remove one or two tokens from a heap, leaving the remaining tokens in that heap as at most two heaps.

For conciseness we express the fact that the removal of the entire heap of u is permitted by saying 'remove u tokens to leave 0 heaps'. Then unless stated otherwise, we assume that for $k > 0$, k heaps means k non-empty heaps. $.156$ is the game in which we may remove 1 token to leave 0 heaps, two tokens to leave zero or two heaps, or three tokens to leave one or two heaps.

We allow digits $d_u = 4$ before the point. If $d_u = 4$, $(u \leq 0)$ then a heap of n tokens may be replaced by two heaps of i and $n-u-i$, where we maintain the terminating play condition by requiring that both $n-u-i$ and i be less than n, so that $-u < i < n$. For example 44.3 is the game in which we may divide a heap of n tokens into two heaps of i and $n+1-i$ $(1<i<n)$, or divide a heap of n tokens into two heaps of i and $n-i$ $(0<i<n)$ or remove one token from a heap.

4.2. Take and Break Games

Let $c = c_0 2^0 + c_1 2^1 + \ldots + c_k 2^k$ be the binary expansions of c, $c_h = 0$ or 1. We say that c *contains* 2^h (or c *includes* 2^h) if $c_h = 1$.

e.g. $\underset{\sim}{5}$ contains $\underset{\sim}{1}$ and $\underset{\sim}{4}$

$\underset{\sim}{6}$ contains $\underset{\sim}{2}$ and $\underset{\sim}{4}$.

The notation for Nim-like games introduced above can be generalized to arbitrary *take and break games*. Express the code digits $\underset{\sim}{d}_u$ $(u = 1,2,\ldots)$ in binary form as

$$d_u = d_{u,0} 2^0 + d_{u,1} 2^1 + \ldots + d_{u.k} 2^k.$$

Then in a move a heap of n tokens may be replaced by exactly h heaps of $i_1, i_2, \ldots, i_h$ $(i_1+i_2+\ldots+i_h = n-u)$ if and only if $d_{u,h} = 1$. We write $\underset{\sim}{A}, \underset{\sim}{B}, \underset{\sim}{C}, \underset{\sim}{D}, \underset{\sim}{E}, \underset{\sim}{F}$ in place of $\underset{\sim}{10}, \underset{\sim}{11}, \underset{\sim}{12}, \underset{\sim}{13}, \underset{\sim}{14}, \underset{\sim}{15}$ respectively.

For example, $.\underset{\sim}{FF}$ is the game in which we can remove one or two tokens from a heap, and leave it as zero, one, two, or three heaps. $.\underset{\sim}{63A}$ is the game in which we can remove one token from a heap and leave the remaining tokens in the heap as one heap or two, remove two tokens from a heap and leave the remaining tokens in the heap, if any, as one heap, or remove three tokens from a heap and leave the remaining tokens in the heap as three heaps or one.

A digit $\underset{\sim}{d}_u$ with $\underset{\sim}{u} \leq 0$ may be allowed provided that $\underset{\sim}{d}_u$ does not contain $\underset{\sim}{2}$ or $\underset{\sim}{1}$, and provided that the terminating play condition is still satisfied. For example if $\underset{\sim}{d}_u$ $(u \leq 0)$ contains $\underset{\sim}{2}^h$ $(h \geq 2)$, a heap of n tokens may be replaced by h heaps of $i_1, i_2, \ldots, i_h$, where $i_1 + i_2 + \ldots i_h = n-u$ and for $1 \leq j \leq h$, $1 \leq i_j < n$.

For example: $\mathbf{80.4}$ is the game in which we can remove one token from a heap and split the remainder into two non-zero heaps, or add a token to a heap of n and divide it into three non-zero heaps of i_1, i_2, i_3 where $i_1 + i_2 + i_3 = n+1$.

To see that the terminating play condition is still satisfied, consider the (even larger) class of games in which any move replaces a heap of n by at most h heaps with at most $n-1$ tokens in a heap. Let m_n be the maximum number of possible moves starting from a heap of n. Then

$$m_n \leq 1 + hm_{n-1}.$$

Since $m_0 = 0$, $h \geq 2$ implies that $m_n \leq (h^n-1)/(h-1)$ and $h = 1$ implies that $m_n \leq n$.

Let $\mathbf{T}$ be a take and break game. If $\forall u, \mathbf{d}_u \leq \mathbf{3}$ (and $\mathbf{d}_u = \mathbf{0}$ for $u \leq 0$), then $\mathbf{T}$ is called a *tetral* game. Nim, $.\dot{\mathbf{3}}$, is a tetral game. If $\forall u, \mathbf{d}_u \leq \mathbf{7}$, then $\mathbf{T}$ is called an *octal* game. If $\forall u, \mathbf{d}_u \leq \mathbf{F}$ (= $\mathbf{15}$), then $\mathbf{T}$ is called a *sedecimal* game. In each case, if there are only a finite number of non-zero code digits, we call the game finite.

4.3. Periodic G-Sequences

Let $\mathbf{T}$ be a take and break game. If a heap of n tokens may be replaced by h heaps of $i_1, i_2, \ldots, i_h$ tokens in a legal move, then $G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} \ldots \overset{*}{+} G(i_h)$ is an *excluded value* for $G(n)$. To show that

$G(n) = g$ it is necessary and sufficient to show that every non-negative integer less than g is an excluded value, but that g is not an excluded value.

Consider the game .F3. We list the options of the first few positions as well as their G-values. Beneath the options of n we write the excluded values.

Position	Options	G-values
{0}		$G(0) = 0$
{1}	{0} 0	$G(1) = 1$
{2}	{0},{1} 0 1	$G(2) = 2$
{3}	{1},{2},{1,1} 1 2 0	$G(3) = 3$
{4}	{2},{3},{1,2},{1,1,1} 2 3 3 1	$G(4) = 0$
{5}	{3},{4},{1,3},{2,2},{1,1,2} 3 0 2 0 2	$G(5) = 1$
{6}	{4},{5},{1,4},{2,3},{1,1,3},{1,2,2} 0 1 1 1 3 1	$G(6) = 2$
{7}	{5},{6},{1,5},{2,4},{3,3},{1,1,4},{1,2,3},{2,2,2} 1 2 0 2 0 0 0 2	$G(7) = 3$

The G-sequence for .F3 appears to be 012301230123... . If a take and break game has the property that there exists integers $p > 0$ and $e \geq 0$ such that

$$G(n+p) = G(n) \quad \text{for all } n > e \qquad (*)$$

we say that the G-sequence is *periodic* with *period* p. In each case we choose the least integers e,p satisfying (*). Then e is called the *last irregular value*, and p is referred to as the period. We indicate the periodic values by writing a dot over the first and last members of the period. For example the G-sequence of $\cdot\underset{\sim}{F}\underset{\sim}{3}$ appears to be $\dot{0}12\dot{3}$. Guy and Smith [11] proved a periodicity theorem for octal games of the form $\cdot\underset{\sim}{d}_1\underset{\sim}{d}_2\cdots\underset{\sim}{d}_t$ which was modified by Kenyon [13] to include those octal games where $\underset{\sim}{4}$'s occur before the octal point. We generalize Kenyon's proof to arbitrary take and break games.

THEOREM 4.1. Suppose that $\underset{\sim}{T} = \underset{\sim}{d}_v\underset{\sim}{d}_{v+1}\cdots\underset{\sim}{d}_0\cdot\underset{\sim}{d}_1\underset{\sim}{d}_2\cdots\underset{\sim}{d}_w$ is a finite take and break game, in which a move replaces just one heap by at most h heaps, i.e. for $v \leqslant u \leqslant w$, $d_u < 2^{h+1}$, and that there exist integers $p > 0$ and $e \geqslant 0$ such that

$$G(i+p) = G(i) \text{ for all } i, \text{ such that } e < i \leqslant he+(h-1)p+t$$

where $t = \max\{|v|, w\}$. Then $G(i+p) = G(i)$ for all $i > e$.

PROOF. Assume inductively that $\forall i$ satisfying $e < i < n$ we have $G(i+p) = G(i)$ where $n > he+(h-1)p+t$. To show that $G(n+p) = G(n)$ we show that $G(n+p)$ and $G(n)$ have the same set of excluded values.

Suppose we can remove u tokens from a heap of $n+p$ to leave heaps of $i_1, i_2, \ldots, i_h$ where $0 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_h < n+p$, and $i_1 + i_2 + \ldots + i_h = n+p-u$. Then $G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} \ldots \overset{*}{+} G(i_h) = g$ is an excluded value for $n+p$. But $G(i_h) = G(i_h-p)$ since $i_h-p < n$, and

$$\begin{aligned} i_h &\geqslant \frac{1}{h}(n+p-u) \\ &> \frac{1}{h}(he+(h-1)p+w+p-u) \\ &\geqslant e+p, \text{ since } w-u \geqslant 0. \end{aligned}$$

Moreover, if $h \geqslant 2$, the heaps i_j, $1 \leqslant j \leqslant h-1$ are of size $< n$, since $i_{h-1} \geqslant n$ would imply that $n+p-u = \sum_{j=1}^{h} i_j \geqslant i_h + i_{h-1} \geqslant 2_{i_{h-1}} \geqslant 2n$, contradicting our assumption that $n > he+(h-1)p+t$ $(\geqslant p+|v| \geqslant p-u)$. Hence $G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} \ldots \overset{*}{+} G(i_h-p) = g$ is an excluded value for $G(n)$.

On the other hand, if $G(i_1') \overset{*}{+} G(i_2') \overset{*}{+} \ldots \overset{*}{+} G(i_h')$ is any excluded value for $G(n)$ where $i_1' \leqslant i_2' \leqslant \ldots \leqslant i_h'$, then $i_h' > e$ since $n > he+(h-1)p+t$. Then

$$G(i_1') \overset{*}{+} G(i_2') \overset{*}{+} \ldots \overset{*}{+} G(i_h') = G(i_1') \overset{*}{+} G(i_2') \overset{*}{+} \ldots \overset{*}{+} G(i_h'+p)$$

so that this is also an excluded value for $G(n+p)$. Thus $G(n)$, $G(n+p)$ have the same set of excluded values, so that they are equal. □

For example, consider the game .772. The G-sequence begins

$$012341624163416341634163416\ldots$$

and appears to periodic with period 4 and last irregular value $G(7) = 2$. Since .772 is an octal game we need only calculate $G(n)$ for $n \leqslant 2\cdot 7 + 2\cdot 4 + 3 = 25$ to establish that the game is periodic.

4.4. The Standard Form of Take and Break Games

The G-sequences for the games 4.02 and .73 are $0\dot{0}12\dot{3}$ and $\dot{0}12\dot{3}$ respectively, so that $G_{.73}(n) = G_{4.02}(n+1)$, $n = 0,1,2,\ldots$. In this section we specify the sense in which 4.02 is a disguised form of .73.

We write $\underset{\sim}{T} \equiv \underset{\sim}{U}$ if $G_{\underset{\sim}{T}}(n) = G_{\underset{\sim}{U}}(n)$ for all n, and $T \equiv_r U$ if $G_{\underset{\sim}{T}}(n) = G_{\underset{\sim}{U}}(n+r)$, $n \geq 0$ and $G_{\underset{\sim}{U}}(n) = 0$, $0 \leq n < r$. For example $.\underset{\sim}{73} \equiv_1 4.\underset{\sim}{02}$. Since $G_{.\underset{\sim}{137}}(n) = G_{.\underset{\sim}{07}}(n+1) = G_{.\underset{\sim}{4}}(n+2)$, $n = 0,1,2,\ldots$ and $G_{.\underset{\sim}{4}}(0) = G_{.\underset{\sim}{4}}(1) = 0$ we have $.\underset{\sim}{137} \equiv_1 .\underset{\sim}{07} \equiv_1 .\underset{\sim}{4}$ and $.\underset{\sim}{137} \equiv_2 .\underset{\sim}{4}$. Equivalently if $\underset{\sim}{T} \equiv_r \underset{\sim}{U}$, we may write $\underset{\sim}{U} \equiv_{-r} \underset{\sim}{T}$. Hence $.\underset{\sim}{4} \equiv_{-2} .\underset{\sim}{137}$. If $\underset{\sim}{T} \equiv_r \underset{\sim}{U}$ we refer to $\underset{\sim}{U}$ as the rth *cousin* of $\underset{\sim}{T}$. Then $.\underset{\sim}{07}$ is a first cousin of $.\underset{\sim}{137}$ and $.\underset{\sim}{4}$ is a second cousin of $.\underset{\sim}{137}$.

THEOREM 4.2. [13, p.37, Theorem 14]. If $\underset{\sim}{d}_1$ is even, and $\underset{\sim}{d}_u$ includes $\underset{\sim}{2}$ $(u>0)$, then the G-sequence is not affected by the inclusion of $\underset{\sim}{1}$ in $\underset{\sim}{d}_{u+1}$.

PROOF. If $\underset{\sim}{d}_1$ is even, then $G(1) = 0$.

If $\underset{\sim}{d}_u$ includes $\underset{\sim}{2}$ $(u>0)$, then $\{1\}$ is an option of $\{u+1\}$, so that $G(u+1) \neq G(1) = 0$, regardless of whether $\underset{\sim}{d}_{u+1}$ includes $\underset{\sim}{1}$ or not. □

E.g. $.\underset{\sim}{66} \equiv .\underset{\sim}{67} \equiv .\underset{\sim}{671}$.

Theorem 4.2 generalizes in the following manner.

THEOREM 4.3. If $\underset{\sim}{d}_1$ is even, and $\underset{\sim}{d}_u$ includes $\underset{\sim}{2}^k$, $u > 1-k$, $k \geq 1$ then the G-sequence is not affected by the inclusion of $\underset{\sim}{2}^{k-j}$, $1 \leq j \leq k$, in $\underset{\sim}{d}_{u+j}$.

PROOF. If $\underset{\sim}{d}_1$ is even, then $G(1) = 0$.

If $\underset{\sim}{d}_u$ includes $\underset{\sim}{2}^k$, $u > 1-k$, $k \geq 1$, then for $n+u \geq k+u$, $\{i_1,i_2,\ldots,i_j,i_{j+1},\ldots,i_{k-1},n-(i_1+i_2+\ldots+i_{k-1})\}$ is an option of $n+u$ where $1 = i_1 = i_2 = \ldots = i_j \leq i_{j+1} \leq \ldots \leq i_{k-1} \leq n-(i_1+\ldots+i_{k-1})$.

$$G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} \dots \overset{*}{+} G(i_{k-1}) \overset{*}{+} G(n-(i_1+i_2+\dots+i_{k-1})) =$$

$$= G(i_{j+1}) \overset{*}{+} \dots \overset{*}{+} G(i_{k-1}) \overset{*}{+} G(n-(i_1+\dots+i_{k-1}))$$

so $G(i_{j+1}) \overset{*}{+} \dots \overset{*}{+} G(i_{k-1}) \overset{*}{+} G(n-(i_1+\dots+i_{k-1}))$ is an excluded value for $G(n+u)$.

Now the additional moves made available by the possible inclusion of $\mathbf{2}^{k-j}$ in $\mathbf{d}_{u+j}$ are to replace $\{n+u\}$ by

$$\{i_{j+1}, i_{j+2}, \dots, i_{k-1}, n-j-(i_{j+1}+\dots+i_{k-1})\}$$

$$= \{i_{j+1}, i_{j+2}, \dots, i_{k-1}, n-(i_1+i_2+\dots+i_{k-1})\}$$

where $1 \leqslant i_{j+1} \leqslant i_{j+2} \leqslant \dots \leqslant i_{k-1} \leqslant n-(i_1+\dots+i_{k-1})$, which exclude the same values as before. □

For example, $\cdot\mathbf{A} \equiv \cdot\mathbf{A4} \equiv \cdot\mathbf{A42} \equiv \cdot\mathbf{A421}$.

THEOREM 4.4. [**13**, p.38, Theorem 15]. If $\mathbf{d}_u$ includes $\mathbf{4}$ $(u \geqslant 0)$, then the G-sequence is not affected by the inclusion of $\mathbf{1}$ in $\mathbf{d}_{u+2v}$ for $v > 0$.

PROOF. If $\mathbf{d}_u$ includes $\mathbf{4}$, then $\{v,v\}$ is an option of $\{u+2v\}$, so $G(u+2v) \neq 0$ regardless of whether $\mathbf{d}_{u+2v}$ includes $\mathbf{1}$ or not.

THEOREM 4.5. If $\mathbf{d}_u$ includes $\mathbf{2}^k$, $k \geqslant 2$, $u > 1-k$, then the G-sequence is not affected by the inclusion of $\mathbf{2}^{k-2j}$ $(0 \leqslant 2j \leqslant k)$ in $\mathbf{d}_{u+2v}$, where $v \geqslant j$.

PROOF. If $\underset{\sim}{d}_u$ includes $\underset{\sim}{2}^k$, $k \geq 2$, $u > 1-k$, then for $n \geq k$ $\{i_1, i_2, \ldots, i_{2j}, i_{2j+1}, \ldots, i_{k-1}, n-(i_1+i_2+\ldots+i_{k-1})\}$ is an option of $n+u$ where $1 \leq i_1 = i_2 \leq i_3 = i_4 \leq \ldots \leq i_{2j-1} = i_{2j} \leq i_{2j+1} \leq \ldots \leq i_{k-1} \leq$ $\leq n-(i_1+\ldots+i_{k-1})$, and $i_1 + i_2 + \ldots + i_{2j} = 2v$.

$$G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} \ldots + G(i_{2j-1}) \overset{*}{+} G(i_{2j}) \overset{*}{+} \ldots \overset{*}{+} G(i_{k-1}) \overset{*}{+} G(n-(i_1+i_2+\ldots+i_{k-1}))$$
$$= G(i_{2j+1}) \overset{*}{+} \ldots \overset{*}{+} G(i_{k-1}) \overset{*}{+} G(n-(i_1+\ldots+i_k))$$

so that $G(i_{2j+1}) \overset{*}{+} \ldots \overset{*}{+} G(i_{k-1}) \overset{*}{+} G(n-(i_1+\ldots i_{k-1}))$ is an excluded value for $n+u$.

Now the additional moves made available by the possible inclusion of $\underset{\sim}{2}^{k-2j}$ in $\underset{\sim}{d}_{u+2v}$ are to replace $\{n+u\}$ by

$$\{i_{2j+1}, i_{2j+2}, \ldots, i_{k-1}, n-2v-(i_{2j+1}+\ldots+i_{k-1})\} =$$
$$= \{i_{2j+1}, i_{2j+2}, \ldots, i_{k-1}, n-(i_1+\ldots+i_{k-1})\}$$

where $1 \leq i_{2j+1} \leq \ldots \leq i_{k-1} \leq n-(i_1+\ldots+i_{k-1})$, which exclude the same values as before.

For example $\cdot\underset{\sim}{F} \equiv \cdot\underset{\sim}{F03} \equiv \cdot\underset{\sim}{F0302}\cdot$

THEOREM 4.6 (cf. [13, p.39, Theorem 17]). If for the game $\underset{\sim}{T}$, $\underset{\sim}{d}_1$ is even, and we define a second take and break game $\underset{\sim}{U}$ as follows:

(i) If $\underset{\sim}{d}_v$ includes $\underset{\sim}{2}^k$ $(k \geq 0, v > 1-k)$ then $\underset{\sim}{e}_{v+k-1}$ includes $\underset{\sim}{2}^k, \underset{\sim}{2}^{k-1}, \ldots, \underset{\sim}{1}$.

(ii) If $\underset{\sim}{d}_v$ includes $\underset{\sim}{2}^k$ $(k \geq 2, v \leq 1-k)$ then $\underset{\sim}{e}_{v+k-1}$ includes $\underset{\sim}{2}^k, \underset{\sim}{2}^{k-1}, \ldots, \underset{\sim}{4}$.

Then $\underset{\sim}{U} \equiv_1 \underset{\sim}{T}$, i.e. $G_{\underset{\sim}{U}}(n) = G_{\underset{\sim}{T}}(n+1)$, $n = 0,1,2,\ldots$.

PROOF. We prove $\underset{\sim}{U} \equiv_1 \underset{\sim}{T}$ by showing that $G_{\underset{\sim}{U}}(n)$ and $G_{\underset{\sim}{T}}(n+1)$ have the same set of excluded values.

(i) We consider separately the cases $k = 0$ and $k \geq 1$. If $\underset{\sim}{d}_v$ includes $\underset{\sim}{2}^0 = \underset{\sim}{1}$ $(v\geq 2)$ and $\underset{\sim}{e}_{v-1}$ includes $\underset{\sim}{1}$, then $G_{\underset{\sim}{T}}(v) \neq 0$, $G_{\underset{\sim}{T}}(v-1) \neq 0$, i.e. $G_{\underset{\sim}{T}}(n+1) \neq 0$, $G_{\underset{\sim}{U}}(n) \neq 0$ for $n = v-1$.

If $\underset{\sim}{d}_v$ includes $\underset{\sim}{2}^k$ $(k\geq 1, v>1-k)$ and $\underset{\sim}{e}_{v+k-1}$ includes $\underset{\sim}{2}^k, \underset{\sim}{2}^{k-1}, \ldots, \underset{\sim}{1}$ then for $n \geq 0$,

$$G_{\underset{\sim}{T}}(n+1+v+k-1) \neq G_{\underset{\sim}{T}}(i_1+1) \overset{*}{+} G_{\underset{\sim}{T}}(i_2+1) \overset{*}{+} \ldots \overset{*}{+} G_{\underset{\sim}{T}}(i_{k-1}+1) \overset{*}{+} G_{\underset{\sim}{T}}(n+1-(i_1+\ldots+i_{k-1}))$$

where $0 \leq i_1 \leq i_2 \leq \ldots \leq i_{k-1} \leq n-(i_1+\ldots+i_{k-1})$, and

$$G_{\underset{\sim}{U}}(n+v+k-1) \neq G_{\underset{\sim}{U}}(i_1) \overset{*}{+} G_{\underset{\sim}{U}}(i_2) \overset{*}{+} \ldots \overset{*}{+} G_{\underset{\sim}{U}}(i_{k-1}) \overset{*}{+} G_{\underset{\sim}{U}}(n-(i_1+\ldots+i_{k-1}))$$

where $0 \leq i_1 \leq i_2 \leq \ldots \leq i_{k-1} \leq n-(i_1+\ldots+i_{k-1})$.

(ii) If $\underset{\sim}{d}_v$ includes $\underset{\sim}{2}^k$ $(k\geq 2, v\leq 1-k)$, $\underset{\sim}{e}_{v+k-1}$ includes $\underset{\sim}{2}^k, \underset{\sim}{2}^{k+1}, \ldots, \underset{\sim}{4}$, then

$$G_{\underset{\sim}{T}}((n+1)+v+k-1) \neq G_{\underset{\sim}{T}}(i_1+1) \overset{*}{+} G_{\underset{\sim}{T}}(i_2+1) \overset{*}{+} \ldots \overset{*}{+} G_{\underset{\sim}{T}}(i_{k-1}+1) \overset{*}{+}$$
$$\overset{*}{+} G_{\underset{\sim}{T}}(n+1-(i_1+i_2+\ldots+i_{k-1})),$$

where $1 \leq i_1+1 \leq i_2+1 \leq \ldots \leq i_{k-1} \leq n+1-(i_1+\ldots+i_{k-1})$, and each heap is strictly less than the original:

$$1+i_j < n+1+v+k-1, \quad 1 \leqslant j \leqslant k-1,$$

$$n+1-(i_1+i_2+\ldots+i_k) < n+1+v+k-1,$$

i.e.
$$i_j \leqslant n+v+k-2,$$

so that
$$\sum_{j=1}^{k-1} i_j \leqslant (k-1)(n+v+k-2),$$

or
$$2-v-k \leqslant \sum_{j=1}^{k-1} i_j \leqslant (k-1)(n+v+k-2).$$

On the other hand $G_{\underset{\sim}{U}}(n+v+k-1) \neq G_{\underset{\sim}{U}}(i_1) \overset{*}{+} G_{\underset{\sim}{U}}(i_2) \overset{*}{+} \ldots \overset{*}{+} G_{\underset{\sim}{U}}(i_{k-1}) \overset{*}{+}$ $\overset{*}{+} G_{\underset{\sim}{U}}(n-(i_1+\ldots+i_{k-1}))$ where $0 \leqslant i_1 \leqslant i_2 \leqslant \ldots \leqslant i_{k-1} \leqslant n-(i_1+i_2+\ldots+i_{k-1})$, and each heap is strictly less than the original:

$$i_j < n+v+k-1, \quad 1 \leqslant j \leqslant k-1,$$

$$i_j \leqslant n+v+k-2,$$

$$\sum_{j=1}^{k-1} \leqslant (k-1)(n+v+k-2),$$

$$n-(i_1+i_2+\ldots+i_{k-1}) \leqslant n+v+k-2,$$

so that
$$2-v-k \leqslant \sum_{j=1}^{k-1} \leqslant (k-1)(n+v+k-2)$$

as before. Note that here $k \geqslant 2$ and

$$i_{k-1} \geqslant \frac{1}{k-1} \sum_{j=1}^{k-1} i_j \geqslant \frac{2-v-k}{k-1} \geqslant \frac{2-(1-k)-k}{k-1} = \frac{1}{k-1} > 0$$

so that at least 2 non-empty heaps result from the heap of $n+v+k-1$. $\square$

We may repeat the above process until the first code digit is odd, i.e. $\underset{\sim}{T} \equiv_{-1} \underset{\sim}{U} \equiv_{-1} \ \ldots \equiv_{-1} \underset{\sim}{W}$ with $\underset{\sim}{w}_1$ odd. Then $\underset{\sim}{W}$ is in *standard form* and its *G*-sequence begins 01...

E.g. $\cdot\underset{\sim}{08} \equiv_{-1} \cdot\underset{\sim}{000F} \equiv_{-1} \cdot\underset{\sim}{00137F} \equiv_{-1} \cdot\underset{\sim}{0113377F} \equiv_{-1} \cdot\underset{\sim}{111333777F}$;

$\underset{\sim}{80000}\cdot\underset{\sim}{02} \equiv_{-1} \cdot\underset{\sim}{C00}\cdot\underset{\sim}{03} \equiv_{-1} \cdot\underset{\sim}{4C}\cdot\underset{\sim}{13}$.

4.5. Periodicity of Take and Break Games

It is not yet known whether all finite take and break games are either periodic, as described in Section 4.3, or arithmetico-periodic (see Section 6.1). We have so far analyzed only octal and sedecimal games, and even for these classes the question is still undecided. Information about octal games of the form $\underset{\sim}{4}\cdot\underset{\sim}{d}_1\underset{\sim}{d}_2$ or $\cdot\underset{\sim}{d}_1\underset{\sim}{d}_2\underset{\sim}{d}_3$ is contained in Tables 7.2 and 7.3. Some of the games are periodic with very few irregularities. There are many however, which so far show no sign of periodicity though the *G*-values have been calculated to or beyond $n = 9999$. No octal game has been shown not to be ultimately periodic.

We may also ask whether all take and break games that are not arithmetico-periodic are bounded.

THEOREM 4.7. Let $\underset{\sim}{T}$ be a take and break game, $\underset{\sim}{T} = \cdot\underset{\sim}{d}_1\underset{\sim}{d}_2\underset{\sim}{d}_3\ldots$ $(\forall u \leq 0, \underset{\sim}{d}_u = 0)$. Then for all n, $G(n) \leq n$.

PROOF. By induction. $G(0) = 0$, $G(1) \leq 1$ for any such game $\underset{\sim}{T}$. Assume inductively that $G(k) \leq k$, $\forall k < n$. We show that if g is an excluded value for $G(n)$, then $g \leq n-1$. If by a legal move we may take u counters from a heap of n to leave heaps of $i_1, i_2, \ldots, i_h$, then by the remark after Lemma 3.1,

$$g = G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} \dots \overset{*}{+} G(i_h) \leq G(i_1) + G(i_2) + \dots + G(i_h)$$

$$\leq i_1+i_2+\dots+i_h$$

$$= n-u$$

$$\leq n-1. \qquad \square$$

The periods of the games .356, .165, .055 $\equiv_{-1}$ .1177, .644 $\equiv_{-1}$ .3777 are displayed in Figures 4.1-4.6. These games will be used to illustrate certain patterns that have been observed in some of the octal games known to be periodic. Though the significance of these patterns is not yet known, they occur sufficiently often to be worthy of note. We discuss them briefly, and then examine the periods of the aforementioned games in more detail.

To simplify the ensuing discussion, we let T be a periodic octal game with last irregular value e and period p. If p is even, it is sometimes the case that there exists k such that for $n > e$

$$G(n+p/2) = G(n) \overset{*}{+} k.$$

The game .34 has G-sequence 010120103121203 with last irregular value $G(6) = 1$ and period 8. For $n > 6$, $G(n+4) = G(n) \overset{*}{+} 1$. This game also exhibits another feature: observe that $G(7) = G(14)\overset{*}{+}3$, $G(8) = G(13)\overset{*}{+}3$, $G(9) = G(12)\overset{*}{+}3$, $G(10) = G(11)\overset{*}{+}3$. The period is symmetrical in the following sense: if $n_1,n_2 > 6$, $n_1 \equiv a \pmod 8$, $n_2 \equiv 5-a \pmod 8$, then $G(n_1)\overset{*}{+}G(n_2) = 3$.

Those games $\underset{\sim}{T}$ in which p is large but $G(n)$ is small often exhibit *subperiodicity*, i.e. for some p', $0 < p' < p$ there is a strong tendency toward $G(n+p') = G(n)$ though this is not exact.

Figure 4.1 displays the period of $.\underset{\sim}{356}$ which has last irregular value $G(7314) = 2$ and period 142. G-values greater than 9 are denoted by the following symbols:

$$x = 11, \quad T = 12, \quad f = 15, \quad S = 16.$$

We list $G(n)$, $n > 7314$, $n \equiv 14,15,\ldots,141,0,1,\ldots,13 \pmod{142}$ in rows of 26 (excepting the first row which has only 12 entries) to illustrate the subperiodicity.

																	x		5	1	5	1			T	8	6	2	6	2	S	
x	f		1	5	1	5			8	T	2	6	2	6	x	f	x		5	1	5	1			T	8	6	2	6	2		f
x	f		1	5	1	5			8	T	2	6	2	6	x	f	x		5	1	5	1			T	8	6	2	6	2		f
x	f		1	5	1		S		8	T	2	6	2	6	x	f		1	5	1	5		S		T	8	6	2	6	2		f
x		5	1	5	1			T	8	T	2	6	2	6	x	f		1	5	1	5			8	T	8	6	2	6	2		f
x		5	1	5	1			T	8	T	2	6	2	6	x	f		1	5	1	5			8	T	8	6	2	6		S	f

Figure 4.1. The period of $.\underset{\sim}{356}$.

For $n > 7314$, if $G(n) = 16$, then $G(n+71) = 16$. For all other $n > 7314$, $G(n+71) = G(n) \overset{*}{+} 7$. A cursory look at the period of this game reveals that the distribution of the G-values is abnormal. The only values that occur in the period are 1,2,5,6,8,11,12,15,16.

For any take and break game $\underset{\sim}{U}$ (not necessarily periodic) we define a G-value g to be *rare* if $\lim_{n\to\infty} g_n/n = 0$ where $g_n = |\{m \mid G(m)=g, m \leq n\}|$. For periodic games this is equivalent to requiring that g appear only a finite number of times in the G-sequence. It sometimes happens that while a G-value appears in the period, the frequency of its occurrence is small. Such a G-value is called *sparse*. A G-value that is not sparse or rare is said to be *common*. For $\underset{\sim}{.356}$ the rare G-values, written in binary, are 0, 11, 100, 111, 1001, 1010, 1101, and 1110, i.e. those with an even number of 1's, ignoring the 2^2 bit. Note that if g is rare, then $g\overset{*}{+}1$, $g\overset{*}{+}2$, $g\overset{*}{+}8$ are common, and $g\overset{*}{+}4$ is rare.

In general the rare G-values are those for which the number of bits that are 1 in some fixed set of digits in the binary expansion, is even, and the common G-values are those for which this number is odd. Defined in this way, if $\underset{\sim}{T}$ is a game in which every G-value is either rare or common, then

(i) if g_1, g_2 are rare G-values, then $g_1 \overset{*}{+} g_2$ is rare,

(ii) if g_1, g_2 are common G-values, then $g_1 \overset{*}{+} g_2$ is rare,

(iii) if g_1 is a rare G-value, g_2 is common, then $g_1 \overset{*}{+} g_2$ is common.

For $n > 5180$, the game $\underset{\sim}{.165}$ is periodic with period 1550, but it also exhibits strong subpatterns and subperiodicities. These are illustrated in Figure 4.2, in which the 310 exhibited G-values are to be read consecutively from left to right down the page, disregarding spaces. They are the values of $G(n)$ for $n \equiv -47,-46,\ldots,-1,0,1,\ldots,262$, mod 310, and must be repeated four more times to produce the complete period of length 1550. They are displayed in 14 rows of 24 values, except that rows 4 and 11 each contain 15 values instead of 12 in the first "half" and rows 7 and 14 contain only 8 values. The array is divided horizontally to illustrate the subperiodicity of 155 with "saltus nim 7": i.e. for most n

$$G(n+155) = G(n) \overset{*}{+} 7;$$

```
1 8 2 1 -       τ 6   5   T     S - 1 | 2 8 1 2 8 1   - 6   -     - 6 2
1 8 2 1 8 2   -       5 6 - 5       1 | 2 8 1 2 8 1  S 6 5 T 6    -
1 8 2 1 8 2 1         5 6 4 5 6       | 2 - 1 2 8 1 2  6 5 T 6 5
1 - 2 1 8 2 1 8 6 - 5 6 -       S   1 | 2 8 1 2 -          5 T 6 5 - 6 -
1 - 2 1 -       6 f 5 6 T 5     S     | 2 8 1 2 -          5 - 6 5 T 6 -
1 - 2 1 8       6 - 5 6 - 5 6         | 2 - 1 2 8          5 - 6 5 T 6 5
1 S 2 1 8       6 - 5                 |

6 T 5 6 T     S 1   2   8       - - 6 | 5 T 6 5 f 6   - 1   -     8 1 5
6 T 5 6 T 5   S     2 1 8 2         6 | 5 T 6 5 - 6   - 1 2 8 1   -
6 - 5 6 T 5 6       2 1 - 2 1         | 5 - 6 5 T 6 5  1 2 8 1 2
6 S 5 6 T 5 6 - 1 - 2 1 -       τ   6 | 5 T 6 5 f          2 8 1 2 8 1 S
6 T 5 6 -       1 - 2 1 - 2     -     | 5 T 6 5 f          2 8 1 2 8 1 S
6 - 5 6 -       1 - 2 1 3 2 1         | 5 - 6 5 -          2 8 1 2 8 1 2
6 - 5 6 -       1 - 2                 |
```

Figure 4.2. The period of .165.

Residue class of n, mod 1550	-43	-37	-29	-27	-26	-17	-14	0	14	26	34	37	44	49	51	53	56	68	70	75	77	82	85	89	94	106
usual G-value	8	8	S	f	T	S	T	φ	8	S	f	f	8	T	S	8	8	8	T	S	8	f	4	8	T	f
unusual G-values $G(n)$	-	-	-	-	S	f	f	f	-	-	-	T	-	f	-	-	-	-	f	-	x	-	-	-	f	-
$G(n+310)$	-	S	T	-	-	-	f	-	x	8	T	-	S	-	8	x	x	a	-	φ	-	τ	T	x	-	V
$G(n+620)$	-	-	-	T	S	-	-	-	-	-	T	T	-	-	8	x	x	-	f	-	x	S	T	S	f	φ
$G(n+930)$	x	-	-	-	S	-	f	f	-	-	-	T	-	f	-	-	-	-	f	-	x	-	-	-	f	-
$G(n+1240)$	-	S	-	T	-	-	-	-	x	-	T	T	-	-	8	x	x	-	f	-	x	S	T	S	f	φ

Residue class of n, mod 1550	117	118	126	128	148	150	155	157	165	169	187	189	192	211	213	216	218	232	235	237	244	247	256	259	261
usual G-value	8	f	S	8	f	S	φ	f	3	f	8	8	8	f	8	8	S	f	f	8	f	f	S	f	8
unusual G-values $G(n)$	-	-	-	-	T	-	-	-	8	τ	T	-	-	T	x	-	φ	-	T	x	τ	T	-	T	τ
$G(n+310)$	-	φ	8	S	-	8	-	-	-	τ	T	-	-	T	x	-	-	-	T	x	τ	-	-	T	τ
$G(n+620)$	-	-	8	S	-	8	S	-	-	τ	-	x	x	-	-	3	φ	T	-	-	τ	-	T	-	-
$G(n+930)$	τ	-	-	-	T	-	-	T	-	-	T	-	x	T	x	-	φ	T	T	x	-	T	-	T	x
$G(n+1240)$	-	τ	8	S	-	8	S	-	-	τ	-	x	x	-	-	-	-	T	-	-	τ	-	-	-	-

Table 4.3. G-values missing from Figure 4.2.

indeed this is always true for $G(n) = 1, 2, 5,$ or 6. The diagram is also divided vertically to illustrate the relation between the "NW quarter" and the "SE" one, and between the "NE quarter" and the "SW" one, i.e. it is often the case that

$$G(n+143) \text{ \&/or } G(n+167) \doteq G(n) \overset{*}{+} 4.$$

The following symbols are used to denote G-values greater than 9:

x = 11, T = 12, f = 15, S = 16, a = 19, V = 20, τ = 23, ϕ = 25.

If one of these symbols, or a single digit appears in Figure 4.2, then these are G-values with a true subperiodicity of 310. Where values do not always exhibit this subperiodicity, a hyphen appears. The G-values so represented can be found in Table 4.3, whose rows are the residue classes of n, mod 1550; the usual value of $G(n)$ insofar as it can be determined; $G(n)$; $G(n+310)$; $G(n+620)$; $G(n+930)$; and $G(n+1240)$. These last five rows contain a hyphen if the G-value is usual, and the actual G-value otherwise. To facilitate the reading of Table 4.3, vertical bars separate values from different rows of Figure 4.2, the double bar occurring after the seventh row. E.g., there are 5 hyphens in the first row of Figure 4.2, corresponding to the first five columns (before the vertical bar) in Table 4.3.

For example $7707 \equiv -43 \pmod{1550}$. Since the usual G-value for $n \equiv -43$ is 8, we have

$$G(7707+1550k) = 8 \qquad \text{for } k \geq -1$$

$$G(7707+310+1550k) = 8 \qquad \text{for } k \geq -1$$

$$G(7707+620+1550k) = 8 \qquad \text{for } k \geq -2$$

$$G(7707+930+1550k) = 11 \qquad \text{for } k \geq -2$$

$$G(7707+1240+1550k) = 8 \qquad \text{for } k \geq -2.$$

For .165 we have that 3 occurs 10 times in a period of 1550, 4 occurs 7 times, and 19 and 20 occur once. Those G-values that contain an even number of 1 bits in their binary expansions, omitting the coefficient of 2^2 are either sparse or rare.

				7	4			7	8
1	1				4	4	4	7	2
1	1	1			4	4	4	$\boxed{}$	8
1	1		2		4	4	4		
1	1	1	2	7	4	4	4		
1	1	1		7	4			7	2
1	1	$\boxed{1}$			4	4	4	7	$\boxed{}$
1	1		2	8	4	4			
1	1	1	2	7	4	4	4		
1	1	1		8	4	4			2
1	1	1			4	4	4	$\boxed{7}$	

				2	1			2	8
4	4				1	1	1	2	7
4	4	4			1	1	1	$\boxed{2}$	8
4	4		7		1	1	1		
4	4	4	7	2	1	1	1		
4	4	4		2	1			2	7
4	4	$\boxed{}$			1	1	1	2	$\boxed{2}$
4	4		7	8	1	1			
4	4	4	7	2	1	1	1		
4	4	4		8	1	1			7
4	4	4			1	1	1	$\boxed{}$	

Figure 4.4. The period of .055.

					53	54			55	56
A	57	58				59	60	61	62	63
	64	65	66			67	68	69	$\boxed{}$	70
	71	72		73		74	*73*	*72*		
B	*71*	*70*	*69*	*68*	*67*	*66*	*65*	*64*		
	63	*62*	*61*		*60*	*59*			*58*	*57*
A	*56*	*55*	$\boxed{\mathit{54}}$			*53*	*52*	*51*	*50*	$\boxed{}$
	49	*48*		*47*	*46*	*45*	*44*			
B	*43*	*42*	*41*	*40*	*39*	*38*	*37*	*36*		
	35	*34*	*33*		*32*	*31*	*30*			*29*
	28	*27*	*26*			*25*	*24*	*23*	$\boxed{\mathit{22}}$	

					21	*20*			*19*	*18*
A	*17*	*16*				*15*	*14*	*13*	*12*	*11*
	10	*9*	*8*			*7*	*6*	*5*	$\boxed{\mathit{4}}$	*3*
C	*2*	*1*				1	2	3		
	4	5	6	7	8	9	10	11		
	12	13	14		15	16			17	18
A	19	20	$\boxed{}$			21	22	23	24	$\boxed{25}$
	26	27		28	29	30	31			
C	32	33	34	35	36	37	38	39		
	40	41	42		43	44	45			46
	47	48	49			50	51	52	$\boxed{}$	

Figure 4.5. Residue classes of n (mod 148) (italic numbers are negative).

Figure 4.4 shows the period of $\cdot 055 \equiv_{-1} \cdot 1177$, which has last irregular value $G(257) = 2$ and period 148. The G-values illustrated in Figure 4.4 are to be read consecutively from left to right down the page. They are the values of $G(n)$ for $n > 257$, $n \equiv 53,54,\ldots,147,0,1,\ldots,52$ (mod 148). Figure 4.5 shows the residue class modulo 148 to which n belongs for $G(n)$ in the corresponding position in Figure 4.4.

There is a strong tendency to subperiodicity with "saltus nim 5", and in fact

(i) if $G(n) = 8$, then $G(n+74) = 8$,

(ii) if $G(n) \neq 8$ then $G(n+74) = G(n) \overset{*}{+} 5$,

for those values $G(n)$ for which n appears in a region of Figure 4.5 beside which an A appears, i.e. for $n \equiv \alpha$ (mod 148) where $53 \leqslant \alpha \leqslant 69$, $-53 \leqslant \alpha \leqslant -50$, $-21 \leqslant \alpha \leqslant -5$ or $21 \leqslant \alpha \leqslant 24$. Pure periodicity of this kind is prevented by the appearance of the boxed values or by the absence of values in the empty boxes. For values in region B, i.e. for $-78 \leqslant \alpha \leqslant -55$ or $-49 \leqslant \alpha \leqslant -23$

(i) if $G(n) = 8$ then $G(n-73) = G(n) = G(n+75)$,

(ii) if $G(n) \neq 8$ then $G(n-73) = G(n) \overset{*}{+} 5 = G(n+75)$,

and for values in region C, i.e. for $-3 \leqslant \alpha \leqslant 20$ or $26 \leqslant \alpha \leqslant 52$

(i) if $G(n) = 8$ then $G(n-75) = G(n) = G(n+73)$,

(ii) if $G(n) \neq 8$ then $G(n-75) = G(n) \overset{*}{+} 5 = G(n+73)$.

The rare G-values are those that contain an even number of bits that are 1 in the binary expansion, i.e. $3 = 11_2$, $5 = 101_2$, and $6 = 110_2$. Those G-values that are not rare are common.

The game $.\underset{\sim}{644} \equiv_{-1} .\underset{\sim}{3777}$ has last irregular value $G(3254) = 32$ and period 442. For all $n > 3254$

$$G(n+221) = G(n) \overset{*}{+} 7.$$

S			5	F	τ			9	2	S			5	F				9	2
S			5	F	τ		2	9		S		F	5		τ				2
S		F	5		τ		2	9		S			5	F			2	9	
S	σ		5		τ		2	9		S			5	F	τ		2	9	
S			5		τ	α			2	S		F	5		τ		2	9	
S			5	F	τ		2	9					5	F	τ	α			2
S			5	F	τ		2	9		S			5	F	τ				2
S	σ		5		τ			9	2	S			5	F	τ		2	9	
S			5		τ		2	9		S		F	5				2	9	
S	σ		5		τ	α			2	S			5	F	τ		2	9	
S			5	F	τ				2	S	σ		5	F			2	9	
S			5	F	τ		2	9		S			5		τ	α			2
S		F	5		τ		2	9		S			5	F	τ				2
S	σ		5		τ	α	2	9					5	F	τ		2	9	
S			5		τ	α			2	S		F	5		τ		2	9	
S			5		τ		2	9		S		F	5		τ			9	2
S			5	F	τ				2	S	σ		5		τ			9	2
S			5	F	τ			9	2	S			5	F				9	2
S	σ		5		τ		2	9		S		F	5		τ		2	9	
		F	5		τ	α			2	S			5	F					

Figure 4.6. The period of $.\underset{\sim}{644}$.

Figure 4.6 lists $G(n)$ for $n > 3245$ and $n \equiv -7,-6,\ldots,-1,0,1,\ldots,213 \pmod{442}$. The following symbols are used to represent G-values greater than 9:

$$F = 14,\ S = 16,\ \tau = 23,\ \sigma = 27,\ \alpha = 28.$$

The subperiodicity (of 11) is illustrated by writing the G-values in rows of 11, except that rows 4, 8, 10, 14 and 19 contain 12, row 9 has 10, and the last row has 8. For $n > 3254$, $n \equiv 214,215,\ldots,434 \pmod{442}$ nim-add 7 to $G(n-221)$, obtained from Figure 4.6. This game also shows that it is not necessary that the highest power of 2 occurring in the G-sequence occur in the period. We have $G(62) = G(3254) = 32$, $G(333) = 64$ but for $n > 3254$, $G(n) < 32$.

The notes to Table 7.2 contain further observations about periodic octal games.

4.6. Relations between the G-sequence and the rules of the game.

Related to the question of whether all take and break games exhibit some form of periodicity is the question of the relationship between the G-sequence and the rules of the game. This question appears very difficult and may not be possible to answer in general. Guy has made some advances in this area with theorems concerning the G-sequences of octal games (cf. Kenyon [13]). The restatements of these theorems for more general take and break games are straightforward.

The following theorems due to Guy describe the G-sequence of certain octal games. For conciseness we represent a sequence of r identical G-values, say $g = G(n) = G(n+1) = \dots G(n+r-1)$ by g^r. For example $0\dot{1}^3 0^5 \dot{1}^2$ represents the G-sequence $0\dot{1}11000001\dot{1}$. We use a similar notation for r identical code digits.

THEOREM 4.8. For $s \geq 3$, the octal game $.1^s4$ has period $4s+5$, irregularities $G(0) = G(s+1) = G(s+2) = G(5s+6) = 0$, and $G(n)$ takes the values

$$\dot{4}1^s 4 4 1^s 2^{s+1} 1 \dot{2}^s$$

for $n \equiv 0,1,2,\dots,4s+4 \pmod{4s+5}$ otherwise.

PROOF. The legal moves are of two kinds:

(a) remove complete heaps of size at most s, and

(b) split heaps of size $n \geqslant s+3$ into two heaps of i, $n-s-1-i$, where $1 \leqslant i \leqslant n-s-2$.

The excluded values, x, for $G(n)$ are thus:

(a) $x = 0$ for $1 \leqslant n \leqslant s$, and

(b) $x = G(i) \overset{*}{+} G(n-s-1-i)$, $1 \leqslant i \leqslant n-s-2$, for $n \geqslant s+3$.

(1) $G(0) = G(s+1) = G(s+2) = 0$, and $G(n) = 1$ for $1 \leqslant n \leqslant s$.

(2) For $s+3 \leqslant n \leqslant 2s+2$, $x = 1 \overset{*}{+} 1 = 0$, so that $G(n) = 1$ in this interval.

(3) For $2s+3 \leqslant n \leqslant 3s+3$, $x = 1 \overset{*}{+} 1$ or $1 \overset{*}{+} 0$ (or $0 \overset{*}{+} 0$ in the case that $i = s+1$, and $n = 3s+3$). Moreover ($i = n-2s-2$, $n-2s-1$) both these values occur, so $G(n) = 2$ in this interval.

(4) If $n = 3s+4$, $x = G(i) \overset{*}{+} G(2s+3-i) = 1 \overset{*}{+} 1$ (or $0 \overset{*}{+} 0$ in the case that $i = s+1$, $s+2$) for all i, so $G(3s+4) = 1$.

(5) From the G-values found so far, 2 can only be excluded by $0 \overset{*}{+} 2$, and $G(n) = 0$ only for $n = s+1$, $s+2$. For $3s+5 \leqslant n \leqslant 4s+4$, $x = 0$ and 1 for two of $i = s$, $s+1$, $s+2$, $s+3$, and $x \neq 2$, so $G(n) = 2$ in this interval.

(6) For $n = 4s+5$, $x \leqslant 3$, and $x = 0,1,2,3$, for $i = s+3$, $s+2$, $s+1$, s, so $G(4s+5) = 4$.

(7) From the G-values so far found, 1 can only be excluded by $0 \overset{*}{+} 1$. For $4s+6 \leqslant n \leqslant 5s+5$, $G(n-s-1-i) = 2$ when $i = s+1$, $s+2$, so $x \neq 1$. But $x = 0$ for $i = 2s+2$ so $G(n) = 1$ in this interval.

(8) For $n = 5s+6$, $G(i) \neq G(n-s-1-i)$ so $G(5s+6) = 0$.

(9) For $s = 5s+7$, $x = 0,1,2,3$, for $i = 2s+3$, $s+2$, $s+1$, s, and $x \neq 4$ so $G(5s+7) = 4$.

(10) For $5s+8 \leqslant n \leqslant 6s+7$, $G(n) = 1$ (cf. (7) above).

(11) For $6s+8 \leqslant n \leqslant 7s+8$, $x \neq 2$ (as in (5) since $G(i) = 0$ only for $i = s+1$, $s+2$, $5s+6$). $x = 2 \overset{*}{+} 2 = 0$ for $i = 2s+3$ or $2s+4$, and $x = 0 \overset{*}{+} 1 = 1$ for $i = 5s+5$ or $5s+6$, so $G(n) = 2$ in this range.

(12) For $n = 7s+9$, $i = 1$ gives $x = 1 \overset{*}{+} 1 = 0$ and $G(7s+9) = 1$ as in (7) above.

(13) For $7s+10 \leqslant n \leqslant 8s+9$, $G(n) = 2$ (cf. (11) above).

(14) For $n = 8s+10$, $x \neq 4$ since this can only be formed by $0 \overset{*}{+} 4$. But $x = 0,1,2,3$ for $i = s+3$, $s+2$, $s+1$, s, so $G(8s+10) = 4$.

(15) For $8s+11 \leqslant n \leqslant 9s+10$, $G(n) = 1$ as in (7) above.

(16) For $n = 9s+11$, $x = 0,1,2,3$ for $i = 4s+5$, $s+1$, $s+2$, $s+3$, and $G(9s+11) = 4$ as in (14).

(17) For $n = 9s+12$, $x = 0,1,2,3$ for $i = 2s+3$, $s+1$, $s+2$, $s+3$, and $G(9s+11) = 4$ as in (14).

(18) For $9s+13 \leqslant n \leqslant 10s+12$, $G(n) = 1$ as in (7).

(19) For $10s+13 \leqslant n \leqslant 11s+13$, $G(n) = 2$ as in (11).

(20) For $n > 11s+13$ ($= 2(5s+6)+s+1$) Table 4.7 displays values of $x = G(i) \overset{*}{+} G(n-s-1-i)$, the rows corresponding to $i = s+1$ (or $5s+6$), $i = s+2$ (exceptions); $i \equiv s+3, s+4, \ldots, 4s+4, 0, \ldots, s \pmod{4s+5}$ and ($i > 5s+6$) $i \equiv s+1$, $i \equiv s+2 \pmod{4s+5}$, and the columns to $n > 11s+13$, $n \equiv 3s+4, 3s+5, \ldots, 4s+4, 0, \ldots, 3s+3 \pmod{4s+5}$. The G-values are given in the final row, being the mex of the entries in the corresponding columns.

$n>11s+13, n\equiv$	$3s+4$, $3s+5$ . . . $4s+4$, 0, 1 . . . s, $s+1$, $s+2$, $s+3$ . . . $2s+2$, $2s+3$ . . . $3s+3$
$i=s+1, 5s+6$	4 1 . . . 1 2 . . . . . 2 1 2 . . . 2 4 1 . . . 1 4
$i=s+2$	4 4 1 . . . 1 2 . . . . 2 1 2 . . . 2 4 1 . . . 1
$i\equiv s+3$	5 5 . . . 0 . . . 5
	0 0 3 3 0
$i\equiv 2s+2$	5 5 . . . 0 . . . 5
$i\equiv 2s+3$	6 . . . 6 6 . . . 3
	0 3 3 0 0
$i\equiv 3s+3$	6 . . . 6 6 . . . 3
$i\equiv 3s+4$	0 3 . . . 3 5 0 . . . 0 5 5 0 . . . 0 3 3
$i\equiv 3s+5$	3 . . . 6 . . . 6 6
	0 0 3 3 0
$i\equiv 4s+4$	3 . . . 6 . . . 6 6
$i\equiv 0$	6 . . . . 6 5 6 . . . 6 0 5 . . . 5 0 0 5 . . . 5
$i\equiv 1$	0 . . . 5 . . . 5 5
	0 3 3 0 0
$i\equiv s$	0 . . . 5 . . . 5 5
$i\equiv s+1 (>5s+6)$	0 5 . . . 5 6 . . . . 6 5 6 . . . 6 0 5 . . . 5 0
$i\equiv s+2 (>s+2)$	0 0 5 . . . 5 6 . . . 6 5 6 . . . 6 0 5 . . . 5
$G(n)=$	1 2 . . . 2 4 1 . . . 1 4 4 1 . . . 1 2 2

Table 4.7. Excluded values, $G(i) \overset{*}{+} G(n-s-1-i)$, for $\cdot 1^s 4$.

The following theorems may be established in a similar manner.

THEOREM 4.9. The G of $.1^{s}5$ is $0\dot{1}^{s+1}0\dot{1}^{s+1}2^{s+1}1\dot{2}^{s+1}$.

E.g. $.15$ has G-sequence $0\dot{1}101122 12\dot{2}$,

$.115$ has G-sequence $0\dot{1}1101112221222\dot{2}$.

THEOREM 4.10. The G-sequence of $.1^{s}44$ is $01^{s}00\dot{1}^{s}2^{2s+2}44\dot{1}^{s+2}$.

E.g. $.144$ has G-sequence $0100\dot{1}22224411\dot{1}$.

THEOREM 4.11. The G-sequence of $.1^{s}45$ is $01^{s}0\dot{1}^{s+1}2^{2s+2}4\dot{1}^{s+1}$.

E.g. $.145$ has G-sequence $010\dot{1}12222 41\dot{1}$.

THEOREM 4.12. The G-sequence of $.1^{s}53$ is

$$01^{s+2}2^{s+2}1^{s+1}02^{s+2}4^{s+1}0\dot{1}^{s+1}2^{s+2}1^{s+2}2^{s+2}4^{s+1}\dot{1}.$$

E.g. $.153$ has G-sequence 0111222110222440,
$\dot{1}1222111222 44\dot{1}$.

THEOREM 4.13. For $s \geq 0$, the G-sequence of $.1^{s}54$ is $0\dot{1}^{s+1}0\dot{1}^{s+1}2^{2s+3}4\dot{1}^{s+2}$.

E.g. $.54$ has G-sequence $010\dot{1}222 41\dot{1}$,

$.154$ has G-sequence $0110\dot{1}1222222 411\dot{1}$.

THEOREM 4.14. For $s \geq 1$, the G-sequence of $.1^{s}47$ is $01^{s}0\dot{1}2^{s+2}4^{s+1}\dot{1}^{s+1}$.

E.g. $.147$ has G-sequence $010\dot{1}222 441\dot{1}$.

THEOREM 4.15. For $s \geq 0$, the G-sequence of $.1^{s}57$ is $0\dot{1}^{s+2}\dot{2}^{s+2}$.

E.g. $.57$ has G-sequence $0\dot{1}12\dot{2}$,

$.157$ has G-sequence $0\dot{1}1122\dot{2}$.

Chapter 5

Subtraction Games

5.1. Introduction

For any set $\{s_1, s_2, \ldots, s_k\}$ of positive integers with $s_1 < s_2 < \ldots < s_k$ we define the *subtraction game* $S(s_1, s_2, \ldots, s_k)$ in which the legal moves are those that reduce a sufficiently large heap of n tokens by s_i, $1 \leqslant i \leqslant k$. The set $\{1,2,4\}$, for example, determines the subtraction game $S(1,2,4)$ in which we may remove 1, 2, or 4 tokens from a heap to leave 0 or 1 heaps, so that $S(1,2,4) \equiv .\underset{\sim}{3303}$. Because the legal moves are of a simple nature, much more is known about the class of subtraction games than about arbitrary take and break games.

LEMMA 5.1. For the game $S(s_1, s_2, \ldots, s_k)$, $G(n) \leqslant k$ for all $n \geqslant 0$.

PROOF. This is an immediate consequence of the fact that for any n, there are at most k options.

THEOREM 5.2. Every finite subtraction game is periodic.

PROOF. For $S(s_1, s_2, \ldots, s_k)$, pick n_0 sufficiently large. Then $G(n_0) = \text{mex}\{G(n_0-s_1), G(n_0-s_2), \ldots, G(n_0-s_k)\}$. Moreover there are precisely $(k+1)^{s_k}$ sequences $g_1 g_2 \cdots g_{s_k}$ where $0 \leqslant g_i \leqslant k$ for $i = 1,2,\ldots,s_k$. Hence there exists $p \leqslant (k+1)^{s_k} + s_k$ such that $G(n_0+p-s_k) = G(n_0-s_k)$, $G(n_0+p-s_k-1) = G(n_0-s_k-1), \ldots, G(n_0+p-1) = G(n_0-1)$. But then, for all $n \geqslant n_0$, $G(n+p) = G(n)$. $\square$

Although Lemma 5.2 shows that all subtraction games are periodic, the bound on the period given in the proof seems Brobdingnagian when compared with data provided by the actual analysis of games. However in certain cases it is possible to provide a more reasonable bound.

A subtraction game is said to be *exactly periodic* with period p if for all $n \geq 0$, $G(n+p) = G(n)$.

THEOREM 5.3. Let $U = \{s_1, s_2, \ldots, s_k\}$ be a non-empty set of positive integers. If there exists $p > 0$ such that $u \in U$ whenever $p-u \in U$, then $S(s_1, s_2, \ldots, s_k)$ is exactly periodic.

PROOF. By induction on the G-value g. Let $n \geq 0$. If $G(n) = 0$, then $G(n+s_1) \neq 0$, $G(n+s_2) \neq 0, \ldots, G(n+s_k) \neq 0$ since n is an option of each of $n+s_1$, $n+s_2, \ldots, n+s_k$. Moreover, for all $s_i \in U$, there exists $s_j = p-s_i \in U$ so $n+s_i = n+p-s_j$. Hence the options of $n+p$ are precisely $n+s_1, n+s_2, \ldots, n+s_k$ and

$$G(n+p) = \text{mex}\{G(n+s_1), G(n+s_2), \ldots, G(n+s_k)\} = 0.$$

Assume inductively that for $n \geq 0$, $G(n) = l$ implies $G(n+p) = l$ for $0 \leq l < g$. If $G(n) = g$, then $G(n+s_1) \neq g$, $G(n+s_2) \neq g, \ldots, G(n+s_k) \neq g$, since n is an option of $n+s_1, n+s_2, \ldots, n+s_k$.

Furthermore, since $s_i \in U$ implies $p-s_i \in U$, each $n+s_i$ is an option of $n+p$, so that g is not an excluded value for $G(n+p)$, i.e. $G(n+p) \leq g$. As $G(n) = g$, if $0 \leq l < g$, there exists s_{i_l} such that $G(n-s_{i_l}) = l$. By the induction hypothesis $G(n-s_{i_l}) = G(n+p-s_{i_l})$. For $s_{j_l} = p-s_{i_l}$,

$G(n+s_{j_l}) = G(n+p-s_{i_l}) = l$. Thus every value strictly less than g is an excluded value for $G(n+p)$, so that $G(n) = G(n+p)$.

Since $G(n) = G(n+p)$ for all $n \geq 0$, $S(s_1,s_2,\ldots,s_k)$ is exactly periodic with a period p.

Example. The game $S(2,5)$ has G-sequence $\dot{0}01102\dot{1}$, with period $7 = 2+5$. In section 5.2 we describe completely the G-sequence of the games $S(s_1)(p=2s_1)$, $S(s_1,s_2)(p=s_1+s_2)$, and $S(s_1,s_2,2s_2-s_1)(p=2s_2)$.

By Theorem 5.2, every subtraction game is ultimately periodic, though it is not the case that all subtraction games are exactly periodic. As a counter-example $S(2,3,5,8)$ has G-sequence

$$0011223041304\dot{1}2230011233021403\dot{3}\ .$$

Table 7.1 lists the G-sequences of all subtraction games in which the subtrahends do not exceed 8.

The games $S(1)$, $S(1,3)$, $S(1,3,5)$, $S(1,5)$, ... all have G-sequence $\dot{0}\dot{1}$. For the game $S(1)$, $G(n) \neq G(n+2k+1)$ for all $n,k \geq 0$. Hence we may adjoin $2k+1$ to the subtraction set of $S(1)$ without affecting the outcome of the game. More generally, if for $S(s_1,s_2,\ldots,s_k)$, $G(n) \neq G(n+s)$ for all $n \geq 0$ then we may adjoin s to the subtraction set without affecting the G-sequence of the game.

LEMMA 5.4. If $S(s_1,s_2,\ldots,s_k)$ is exactly periodic with period p and s may be adjoined to the subtraction set without affecting the G-sequence, then $p-s$ may also be adjoined to the subtraction set.

PROOF. Since s may be adjoined to the subtraction set, it must be the case that for all $n \geqslant 0$, $G(n+p) \neq G(n+p-s)$. However, since $S(s_1,s_2,\ldots,s_k)$ is exactly periodic, $G(n) = G(n+p) \neq G(n+p-s)$ so that $G(n) \neq G(n+p-s)$. Hence by the remark just before the statement of the lemma, we may adjoin $p-s$ to the subtraction set. □

As an immediate consequence of this lemma we have

LEMMA 5.5. If $S(s_1,s_2,\ldots,s_k)$ is exactly periodic with period p, then $p-s_1,p-s_2,\ldots,p-s_k$ may be adjoined to the subtraction set without affecting the G-sequence.

The condition of exact periodicity in Lemma 5.5 is necessary. Consider the game $S(2,3,5,8)$ whose G-sequence appears above. The period is 17, with last irregular value $G(12) = 4$. While we may 'adjoin' 8 to the subtraction set, $9 = 17-8$ may not be adjoined since $G(5) = G(14) = 2$. Nor is it true that if p is even, we can necessarily adjoin $p/2$ to the subtraction set. $S(3,7)$ has G-sequence $\dot{0}00111022\dot{1}$ with period 10. However $S(3,5,7)$ has G-sequence $\dot{0}00111222\dot{3}$.

5.2. <u>G-sequences of Subtraction Games</u>.

No general expression for the period length of arbitrary subtraction games is known. However in certain cases we can give rules that enable us to write down the G-sequence immediately. In doing so, it suffices to consider only those subtraction games where the greatest common divisor of the members of the subtraction set is 1.

LEMMA 5.6. Let $\underset{\sim}{T} \equiv S(s_1,s_2,\ldots,s_k)$, $\underset{\sim}{U} \equiv S(ds_1,ds_2,\ldots,ds_k)$ where $d > 1$. Then $\forall n \geq 0$

$$G_{\underset{\sim}{T}}(n) = G_{\underset{\sim}{U}}(dn) = G_{\underset{\sim}{U}}(dn+1) = \ldots = G_{\underset{\sim}{U}}(dn+d-1). \qquad (*)$$

PROOF. By induction. Since $G_{\underset{\sim}{T}}(0) = G_{\underset{\sim}{T}}(1) = \ldots G_{\underset{\sim}{T}}(s_1-1) = 0$, and $G_{\underset{\sim}{U}}(0) = G_{\underset{\sim}{U}}(1) = \ldots G_{\underset{\sim}{U}}(d(s_1-1)+d-1) = 0$, (*) holds for $n = 0,1,\ldots,s_1-1$. Assume inductively that (*) holds for $n < n_0$. It suffices to show that g is an excluded value for $G_{\underset{\sim}{T}}(n_0)$ if and only if g is an excluded value for $G_{\underset{\sim}{U}}(dn_0+r)$, where $0 \leq r < d$.

If we can remove s_i tokens from a heap of n_0, and $G_{\underset{\sim}{T}}(n_0-s_i) = g$, then we can remove ds_i tokens from a heap of dn_0+r, and by the induction hypothesis

$$\begin{aligned} G_{\underset{\sim}{U}}(dn_0+r-ds_i) &= G_{\underset{\sim}{U}}(d(n_0-s_i)+r) \\ &= G_{\underset{\sim}{T}}(n_0-s_i) \\ &= g. \end{aligned}$$

Similarly if g is an excluded value for $G_{\underset{\sim}{U}}(dn_0+r)$, then g is an excluded value for $G_{\underset{\sim}{T}}(n_0)$. □

For subtraction games $\underset{\sim}{T}$, $\underset{\sim}{U}$, defined as in Lemma 5.6, we say that $\underset{\sim}{U}$ is a *d-plicate* of $\underset{\sim}{T}$, e.g. $S(s_1)$ is an s_1-plicate of $S(1)$. The G-sequence of $S(s_1)$ is just $\dot{0}\ldots001\ldots1\dot{1}$, where each string of 0's and 1's is of length s_1.

The G-sequence of $S(1,2k+1)$ is the same as that of $S(1)$, since we may adjoin $2k+1$ to the subtraction set of $S(1)$ without affecting the outcome of the game. For $S(1,2k)$, a period is $2k+1$ by Theorem 5.3, and since the G-sequence is $\dot{0}101\ldots01\dot{2}$, the period is just $2k+1$.

For $S(a,b)$, where $1 < a < b$, we may assume that a, b are relatively prime, for if g.c.d. $(a,b) = d \gneq 1$, then $S(a,b)$ is just the d-plicate of $S(a/d,b/d)$. By Theorem 5.3, $S(a,b)$ is exactly periodic with a period $a+b$. Let $b = 2ha \pm r$, where $0 < r < a$.

We write down the G-sequence as follows. Put a 0's, then a 1's. Repeat this pattern until we have $a+b$ digits. Then change the last $a-r$ 0's into 2's. For example consider $S(4,13)$. Since $a = 4$, $b = 2.2.4-3$, so that $b = 2$ and $r = 3$. We write

$$00001111000011110$$

then change $a-r = 4-3 = 1$ 0's to 2's so that the G-sequence is

$$\dot{0}000111100001111\dot{2}.$$

For $S(4,9)$, $a = 4$, $b = 2.4+1$. We write

$$0000111100001$$

then change the last $a-r = 4-1 = 3$ 0's to 2's. Hence the G-sequence is

$$\dot{0}00011110222\dot{1}.$$

It is also possible to describe completely the period of $S(a,b,2b-a)$. If $a = 1$, and b is odd, then $2b-a$ is odd, so the G-sequence is just $\dot{0}\dot{1}$. If $a = 1$, and $b = 2$, then $2b-a = 3$ and the period is $\dot{0}12\dot{3}$. Otherwise, let $b = 2ha \pm r$ where $0 < r < a$. Write down a 0's followed by a 1's, and repeat this pattern until there are $b+a$ digits. Continue with a 0's followed by a 1's, and repeat this pattern until there are $b-a$ further digits. Then change the last $a-r$ 0's in each of the sets of $b+a$, $b-a$ digits to 2's. If $h = 1$, and $b = 2a+r$, and $a-2r > 0$, further change the first $a-2r$ 2's in the second set of $a-r$ 2's to 3's. If $h = 1$, $b = 2a-r$, and $a-2r > 0$ then replace the last $a-2r$ 2's in the second set of $a-r$ 2's to 3's. E.g. for $S(4,13,22)$, $a = 4$, $b = 2.2.4-3$. We write 4 0's, followed by 4 1's, until we have $b+a = 17$ digits, then repeat, stopping this time after $b-a = 9$ digits

$$0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0 \qquad 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0.$$

Then $a-r = 4-3 = 1$, so that we replace the last 0 in each set by a 2. The G-sequence is then

$$\dot{0}000111100001111200001111\dot{2}.$$

For $S(4,9,14)$, $a = 4$, $b = 2.4+1$. We write

$$0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 1 \qquad 0\ 0\ 0\ 0\ 1.$$

Since $a-r = 4-1 = 3$, the last 3 0's in each set are replaced by 2's, yielding

$$0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 2\ 2\ 2\ 1 \qquad 0\ 2\ 2\ 2\ 1.$$

As $h = 1$, $a-2r = 4-2 = 2 > 0$, $b = 2a+r$, we change the first 2 2's in the second set to 3's. The G-sequence is then

$$\dot{0}0001111022210332\dot{1}.$$

For $S(4,7,10)$, $a = 4$, $b = 2.4 - 7$. We write

$$0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 0\ 0\ 0 \qquad 0\ 0\ 0.$$

Since $a-r = 4-1 = 3$, the last three 0's in each set are replaced by 2's, yielding

$$0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 2\ 2\ 2 \qquad 2\ 2\ 2.$$

As $h = 1$, $a-2r = 4-2 = 2 > 0$, we change the first two 2's in the second set to 3's. The G-sequence is thus

$$\dot{0}0001111222233\dot{3}.$$

5.3. $S(a,b,a+b)$ and the Berlekamp Method.

We can determine the period of $S(a,b,a+b)$, and in some cases specify the G-values themselves. However in the general case, a concise description of the period, such as we have for $S(a,b)$ seems out of reach. The analysis which will appear in [1] rests upon the following theorem of Ferguson [9].

LEMMA 5.7. (Ferguson's Pairing Property). Let $S(s_1,s_2,\ldots,s_k)$ be a subtraction game $(s_1<s_2<\ldots<s_k)$. Then $G(n) = 1$ if and only if $G(n-s_1) = 0$.

PROOF. We give a proof by contradiction. Observe that $G(s_1) = 1$ since 0 is the only option of s_1. If the statement fails, then there is a smallest number n for which it does so, and either

(i) $G(n) = 1$ and $G(n-s_1) \neq 0$

or (ii) $G(n-s_1) = 0$ and $G(n) \neq 1$.

(i) If $G(n-s_1) \neq 0$, then for some s_j, $1 < j \leq k$, $G(n-s_1-s_j) = 0$. Since n is the least number for which the statement fails, $G(n-s_j) = 1$. But $n-s_j$ is an option of n, so that $G(n) \neq 1$.

(ii) Certainly $G(n) \neq 0$ since $n-s_1$ is an option of n, and $G(n-s_1) = 0$. Hence $G(n) > 1$, and there exists s_j, $1 < j \leq k$ such that $G(n-s_j) = 1$. Since n is the least number for which the above statement fails $G(n-s_1-s_j) = 0$. But $n-s_1-s_j$ is an option of $n-s_1$, so that $G(n-s_1) \neq 0$.

Berlekamp has suggested the following method for calculating the P- and N-positions. For $S(s_1,s_2,\ldots,s_k)$ set up $k+1$ columns. The first entries in each of the columns are the numbers $0,s_1,s_2,\ldots,s_k$. The first entry in each of the succeeding rows is the mex, say n, of those numbers already written. The remaining entries in the row are the numbers $n+s_1,n+s_2,\ldots,n+s_k$. E.g. for $S(3,10,13)$ we have

0	3	10	13
1	4	11	14
2	5	12	15
6	9	16	19
7	10	17	20
8	11	18	21
22	25	32	35
23	26	33	36
24	27	34	37
28	31	38	41
29	32	39	42
30	33	40	43
44	47	54	57
45			

Figure 5.1. Analysis of $S(3,10,13)$.

The sample table has been divided into three sections. Every number in the second section may be obtained by adding 22 to a number in the corresponding position in the first section. In this sense, the table for $S(s_1,s_2,\ldots,s_k)$ will eventually become periodic: each entry may be obtained by adding p to an earlier occurring number in a corresponding position.

While Berlekamp's method does not describe the period completely, the first column contains all numbers n such that $G(n) = 0$. By Ferguson's pairing property the second column contains those n for which $G(n) = 1$. The remaining columns contain those n such that $G(n) \geq 2$, unless the entry is a duplicate of an entry occurring in an earlier column. In Figure 5.1, the numbers 10, 11, 32, 33 appear in both the second and the third column, so that $G(10) = G(11) = G(32) = G(33) = 1$.

For $S(a,b,a+b)$, if duplicates occur, it must be the case that a number occurring in the second column is a duplicate of a number in the third. By definition, no number in the first column is a duplicate of a number in the others. If a number n occurred in both the second and fourth columns, then $n-a$ would appear in the first and third columns. If n occurred in the third and fourth columns, then $n-b$ would appear in the first and second.

The analyses of $S(1,2k,2k+1)$, and $S(1,2k+1,2k+2)$ are straightforward. Figure 5.2 illustrates the Berlekamp analysis of $S(1,2k,2k+1)$. Since there are no repetitions, the game is exactly periodic, with period $2b = 4k$ and the G-sequence is $\dot{0}101\ldots012323\ldots2\dot{3}$, the period consisting of k 0's, k 1's , k 2's and k 3's.

0	1	$2k$	$2k+1$
2	3	$2k+2$	$2k+3$
4	5	$2k+4$	$2k+6$
.	.	.	.
.	.	.	.
.	.	.	.
$2k-4$	$2k-3$	$4k-4$	$4k-3$
$2k-2$	$2k-1$	$4k-2$	$4k-1$

Figure 5.2. Analysis of $S(1,2k,2k+1)$.

0	1	$\boxed{2k+1}$	$2k+2$
2	3	$2k+3$	$2k+4$
4	5	$2k+5$	$2k+6$
.	.	.	.
.	.	.	.
.	.	.	.
$2k-2$	$2k-1$	$4k-1$	$4k$
$2k$	$\boxed{2k+1}$	$4k+1$	$4k+2$

Figure 5.3. Analysis of $S(1,2k+1,2k+2)$.

Figure 5.3 illustrates the Berlekamp analysis of $S(1,2k+1,2k+2)$. For each set of $4k+4$ entries there is just one repetition, so that the period is $4k+3 = 2b+1$, and the G-sequence is

$$\dot{0}101\ldots0101232\dot{3}\ldots23\dot{2},$$

where there are $k+1$ 0's, 1's, and 2's, and k 3's. The Berlekamp analysis of $S(1,13,14)$ is illustrated in Figure 5.4.

0	1	$\boxed{13}$	14
2	3	15	16
4	5	17	18
6	7	19	20
8	9	21	22
10	11	23	24
12	$\boxed{13}$	25	26

Figure 5.4. Analysis of $S(1,13,14)$.

The 13 is repeated in the second and third columns, so that the period is 27, and the G-sequence is

$$\dot{0}1010101010101232323232323\dot{2}.$$

For $a > 1$, we assume that a, b are relatively prime, and consider separately the cases $b = 2ha-r$, $b = 2ha+r$ where $0 < r < a$. The case where $b = 2ha-r$ is reasonably straightforward. The diagram of the Berlekamp analysis is illustrated in Figure 5.5.

There are h sections to the diagram, where a section consists of a rows, so that there are $4ha$ entries in total. However the r boxed numbers in the second column are duplicates of the boxed numbers in the third column. Allowing for these r repetitions, the period is $4ha-r = 2b+r$.

The analysis of the case $b = 2ha+r$ is more complicated as the period is a times as long. It is best described with reference to a specific example. Figure 5.6 illustrates the diagram of the Berlekamp analysis of $S(5,22,27)$.

For $S(a,2ha+r,(2h+1)a+r)$, $0 < r < a$, the diagram consists of a sets of four columns. Within each set there are h or $h+1$ sections of a rows. In Figure 5.6 there are 5 sets of 4 columns and each set contains either 2 or 3 sections of 5 rows. Further, 2 of the sets contain 3 sections. In general r of the sets of columns contain $h+1$ sections of a rows, and $a-r$ of the sets contain h sections. In each set of 4 columns, the last section of a rows may be divided into 2 subsections. For the kth set of columns, the subsections contain $(kr)_a$ and $a-(kr)_a$ rows respectively, where $(kr)_a$ denotes the least non-negative residue of kr, mod a.

0	a	$2ha-r$	$(2h+1)a-r$
1	$a+1$	$2ha-r+1$	$(2h+1)a-r+1$
2	$a+2$	$2ha-r+2$	$(2h+1)a-r+2$
…	…	…	…
$r-1$	$a+r-1$	$2ha-1$	$(2h+1)a-1$
r	$a+r$	$2ha$	$(2h+1)a$
…	…	…	…
$a-1$	$2a-1$	$(2h+1)a-r-1$	$(2h+2)a-r-1$
$2a$	$3a$	$(2h+2)a-r$	$(2h+3)a-r$
$2a+1$	$3a+1$	$(2h+2)a-r+1$	$(2h+3)a-r+1$
…	…	…	…
$3a-1$	$4a-1$	$(2h+3)a-r-1$	$(2h+4)a-r-1$
$4a$	$5a$	$(2h+4)a-r$	$(2h+5)a-r$
$4a+1$	$5a+1$	$(2h+4)a-r+1$	$(2h+5)a-r+1$
…	…	…	…
$5a-1$	$6a-1$	$(2h+5)a-r-1$	$(2h+6)a-r-1$
…	…	…	…
$(2h-2)a$	$(2h-1)a$	$(4h-2)a-r$	$(4h-1)a-r$
$(2h-2)a+1$	$(2h-1)a+1$	$(4h-2)a-r+1$	$(4h-1)a-r+1$
…	…	…	…
$(2h-1)a-r-1$	$2ha-r-1$	$(4h-2)a-2r-1$	$(4h-1)a-2r-1$
$(2h-1)a-r$	$2ha-r$	$(4h-2)a-2r$	$(4h-1)a-2r$
…	…	…	…
$(2h-1)a-1$	$2ha-1$	$(4h-1)a-r-1$	$4ha-r-1$

Figure 5.5. Analysis of $S(a,2ha-r,(2h+1)a-r)$.

0	5	22	27					94	99	116	121								
1	6	23	28					95	100	117	122								
2	7	24	29					96	101	118	123								
3	8	$\boxed{25}$	30					97	102	$\boxed{119}$	124								
4	9	$\boxed{26}$	31					98	103	120	125								
10	15	32	37	52	57	74	79	104	109	126	131	146	151	168	173	188	193	210	215
11	16	33	38	53	58	75	80	105	$\boxed{110}$	127	132	147	152	169	174	189	194	211	216
12	17	34	39	54	59	76	81	106	111	128	133	148	153	170	175	190	195	212	217
13	18	35	40	55	60	77	82	107	112	129	134	149	154	171	176	191	196	213	218
14	19	36	41	56	61	78	83	108	113	130	135	150	155	172	177	192	197	214	219
20	$\boxed{25}$	42	47	62	$\boxed{67}$	84	89	114	$\boxed{119}$	136	141	156	$\boxed{161}$	178	183				
21	$\boxed{26}$	43	48	63	$\boxed{68}$	85	90					157	$\boxed{162}$	179	184	198	$\boxed{203}$	220	225
				64	69	86	91	137	142	154	164	158	163	180	185	199	$\boxed{204}$	221	226
44	49	66	71	65	70	87	92	138	143	160	165					200	205	222	227
45	50	$\boxed{67}$	72					139	144	$\boxed{161}$	166	181	186	$\boxed{203}$	208	201	206	223	228
46	51	$\boxed{68}$	73	88	93	$\boxed{110}$	115	140	145	$\boxed{162}$	167	182	187	$\boxed{204}$	209	202	207	224	229

Figure 5.6. Analysis of $S(5,22,27)$.

	0	a	$2ha+r$	$(2h+1)a+r$
	1	$a+1$	$2ha+r-1$	$(2h+1)a+r-1$
	$\cdots$	$\cdots$	$\cdots$	$\cdots$
a	$a-r-1$	$2a-r-1$	$(2h+1)a-1$	$(2h+2)a-1$
	$a-r$	$2a-r$	$\boxed{(2h+1)a}$	$(2h+2)a$
	$a-r+1$	$2a-r+1$	$(2h+1)a+1$	$(2h+2)a+1$
	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	$a-1$	$2a-1$	$(2h+1)a+r-1$	$(2h+2)a+r-1$
	$2a$	$3a$	$(2h+2)a+r$	$(2h+3)a+r$
	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	$3a-1$	$4a-1$	$(2h+3)a+r-1$	$(2h+3)a+r-1$
	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	$(2h-2)a$	$(2h-1)a$	$(4h-2)a+r$	$(4h-1)a+r$
	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	$(2h-1)a-1$	$2ha-1$	$(4h-1)a+r-1$	$4ha+r-1$
r	$2ha$	$(2h+1)a$	$4ha+r$	$(4h+1)a+r$
	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	$2ha+r-1$	$(2h+1)a+r-1$	$4ha+2r-1$	$(4h+1)a+2r-1$
$a-r$	$4ha+2r$	$(4h+1)a+2r$	$6ha+3r$	$(6h+1)a+3r$
	$\cdots$	$\cdots$	$\cdots$	$\cdots$
	$(4h+1)a+r-1$	$(4h+2)a+r-1$	$(6h+1)a+2r-1$	$(6h+2)a+2r-1$

Figure 5.7. Part of analysis of $S(a,2ha+r,(2h+1)a+r)$.

In each set of 4 columns, there are exactly r duplicates. These occur in the second column, duplicating numbers that have already occurred in the third. (For $S(5,22,27)$, $r = 2$ and in each of the sets of columns displayed in Figure 5.6 there are 2 duplicates.) Hence the total number of entries is $4(ha)(a-r) + 4(ha+a)r = 4(ha+r)a$. Allowing for the r duplicates occurring in each set of columns, the period is $4(ha+r)a - ar = {}$ $= (4ha+3r)a = (2b+r)a$.

Figure 5.7 shows the diagram of the first set of a columns. There are $h+1$ sections of a rows, and the last section is divided into two subsections of r and $a-r$ rows respectively. The duplicates that occur are boxed.

Consider now the $(k+1)^{\text{st}}$ set of 4 columns. If $a-(kr)_a \geq r$, then $a \geq (kr)_a + r = ((k+1)r)_a$, and there will be no split in the duplicates. The $(k+1)^{\text{st}}$ set contains h sections of a rows, and the last section is divided into subsections of $((k+1)r)_a$, and $a-((k+1)r)_a$ rows where $r < ((k+1)r)_a \leq a$. The first r entries in the second column of the $((k+1)r)_a$ rows are duplicates as the last r entries occurring in the third column of the kth set of 4 columns. Figure 5.8 illustrates the situation when $a-(kr)_a \geq r$. The upper portion of the diagram shows the last $a-(kr)_a$ rows of the kth set of 4 columns. In the $(k+1)^{\text{st}}$ set of columns, the entries have been grouped in sections of a rows, but only the first $((k+1)r)_a$ rows of the last section are shown. The boxed numbers in the second column are duplicates of earlier occurring numbers in the third column.

Last $a-(kr)_a$ rows of kth set of columns

$$
a-(kr)_a \left\{ \begin{array}{cccc}
n & n+a & n+2ha+r & n+(2h+1)a+r \\
n+1 & n+a+1 & n+2ha+r+1 & n+(2h+1)a+r+1 \\
\cdots & \cdots & \cdots & \cdots \\
n+a-r-(kr)_a-1 & n+2a-r-(kr)_a-1 & n+(2h+1)a-(kr)_a-1 & n+(2h+2)a-(kr)_a-1 \\
n+a-r-(kr)_a & n+2a-r-(kr)_a & n+(2h+1)a-(kr)_a & n+(2h+2)a-(kr)_a \\
\cdots & \cdots & \cdots & \cdots \\
n+a-(kr)_a-1 & n+2a-(kr)_a-1 & n+(2h+1)a+r-(kr)_a-1 & n+(2h+2)a+r-(kr)_a-1
\end{array} \right.
$$

$(k+1)$st set of 4 columns

$$
a \left\{ \begin{array}{cccc}
n+2a-(kr)_a & n+3a-(kr)_a & n+(2h+2)a+r-(kr)_a & n+(2h+3)a+r-(kr)_a \\
\cdots & \cdots & \cdots & \cdots \\
n+3a-(kr)_a-1 & n+3a-(kr)_a-1 & n+(2h+3)a+r-(kr)_a-1 & n+(2h+4)a+r-(kr)_a-1
\end{array} \right.
$$

$$
a \left\{ \begin{array}{cccc}
n+4a-(kr)_a & n+5a-(kr)_a & n+(2h+4)a+r-(kr)_a & n+(2h+5)a+r-(kr)_a \\
\cdots & \cdots & \cdots & \cdots \\
n+5a-(kr)_a-1 & n+6a-(kr)_a-1 & n+(2h+5)a+r-(kr)_a & n+(2h+6)a+r-(kr)_a
\end{array} \right.
$$

$$
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
$$

$$
a \left\{ \begin{array}{cccc}
n+(2h-2)a-(kr)_a & n+(2h-1)a-(kr)_a & n+(4h-2)a+r-(kr)_a & n+(4h-1)a+r-(kr)_a \\
\cdots & \cdots & \cdots & \cdots \\
n+(2h-1)a-(kr)_a & n+2ha-(kr)_a-1 & n+(4h-1)a+r-(kr)_a-1 & n+4ha+r-(kr)_a-1
\end{array} \right.
$$

$$
(k+1)r)_a \left\{ \begin{array}{cccc}
n+2ha-(kr)_a & n+(2h+1)a-(kr)_a & n+4ha+r-(kr)_a & n+(4h+1)a+r-(kr)_a \\
\cdots & \cdots & \cdots & \cdots \\
n+2ha+r-2-(kr)_a & n+(2h+1)a-r-2-(kr)_a & n+4ha+2r-2-(kr)_a & n+(4h+1)a+2r-2-(kr)_a \\
n+2ha+r-1-(kr)_a & n+(2h+1)a-r-1-(kr)_a & n+4ha+2r-1-(kr)_a & n+(4h+1)a+2r-1-(kr)_a \\
n+2ha+r-(kr)_a & n+(2h+1)a-r-(kr)_a & n+4ha+2r-(kr)_a & n+(4h+1)a+2r-(kr)_a \\
\cdots & \cdots & \cdots & \cdots \\
n+2ha+r-1 & n+(2h+1)a+r-1 & n+4ha+2r-1 & n+(4h+1)a+2r-1
\end{array} \right.
$$

Figure 5.8. Part of the analysis of $S(a,2ha+r,(2h+1)a+r)$ when $a-(kr)_a \geq r$.

Last $a-(kr)_a$ rows of kth set of columns.

$a-(kr)_a$	n	$n+a$	$\boxed{n+2ha+r}$	$n+(2h+1)a+r$
	....	....	....	....
	$n+a-(kr)_a-1$	$n+a+(a-(kr)_a)-1$	$\boxed{n+2ha+r+(a-(kr)_a)-1}$	$n+(2h+1)a+r+(a-(kr)_a)-1$

$(k+1)$st set of columns.

a	$n+a+(a-(kr)_a)$	$n+2a+(a-(kr)_a)$	$n+(2h+1)a+r+(a-(kr)_a)$	$n+(2h+2)a+r+(a-(kr)_a)$
	....	....	....	....
	$n+2a+(a-(kr)_a-r)$	$n+3a+(a-(kr)_a-r)$	$\boxed{n+(2h+2)a+(a-(kr)_a)}$	$n+(2h+3)a+(a-(kr)_a)$
	....	....	....	....
	$n+2a-1$	$n+3a-1$	$\boxed{n+(2h+2)a+r-1}$	$n+(2h+3)a+r-1$
	....	....	....	....
	$n+2a+(a-(kr)_a)-1$	$n+3a-(a-(kr)_a)-1$	$n+(2h+2)a+r+(a-(kr)_a)-1$	$n+(2h+3)a+r+(a-(kr)_a)-1$
a	$n+3a+(a-(kr)_a)$	$n+4a+(a-(kr)_a)$	$n+(2h+3)a+r+(a-(kr)_a)$	$n+(2h+4)a+r+(a-(kr)_a)$
	....	....	....	....
	$n+4a+(a-(kr)_a)-1$	$n+4a+(a-(kr)_a-1$	$n+(2h+4)a+r+(a+(kr)_a)-1$	$n+(2h+5)a+r+(a-(kr)_a)-1$
	....	....	....	....
a	$n+(2h-3)a+(a-(kr)_a)$	$n+(2h-2)a+(a-(kr)_a)$	$n+(4h-3)a+r+(a-(kr)_a)$	$n+(4h-2)a+r+(a-(kr)_a)$
	....	....	....	....
	$n+(2h-2)a+(a-(kr)_a)-1$	$n+(2h-1)a+(a-(kr)_a)-1$	$n+(4h-2)a+r+(a-(kr)_a)-1$	$n+(4h-1)a+r+(a-(kr)_a)-1$
a	$n+(2h-1)a+(a-(kr)_a)$	$n+2ha+(a-(kr)_a)$	$n+(4h-1)a+r+(a-(kr)_a)$	$n+4ha+r+(a-(kr)_a)$
	....	....	....	....
	$n+(2h-1)a+r-1$	$n+2ha+r-1$	$n+(4h-1)a+2r-1$	$n+4ha+2r-1$
	$n+(2h-1)a+r$	$\boxed{n+2ha+r}$	$n+(4h-1)a+2r$	$n+4ha+2r$
	....	....	....	....
	$n+(2h-1)a+r+(a-(kr)_a)-1$	$\boxed{n+2ha+r+(a-(kr)_a)-1}$	$n+(4h-1)a+2r+(a-(kr)_a)-1$	$n+4ha+2r+(a-(kr)_a)-1$
	....	....	....	....
	$n+2ha+(a-(kr)_a)-1$	$n+(2h+1)a+(a-(kr)_a)-1$	$n+4ha+r+(a-(kr)_a)-1$	$n+(4h+1)a+r+(a-(kr)_a)-1$
$r-(kr)_a$	$n+(2h+1)a+(a-(kr)_a)$	$\boxed{n+(2h+2)a+(a-(kr)_a)}$	$n+(4h+1)a+r+(a-(kr)_a)$	$n+(4h+2)a+r+(a-(kr)_a)$
	....	....	....	....
	$n+(2h+1)a+r-1$	$\boxed{n+(2h+2)+r-1}$	$n+(4h+1)a+2r-1$	$n+(4h+2)a+2r-1$

Figure 5.9. Part of the analysis of $S(a,2ha+r,(2h+1)a+r)$ when $a-(kr)_a < r$.

If $a-(kr)_a < r$, then $a < (kr)_a+r = a+((k+1)r)_a$. In this case the $(k+1)$st set contains $h+1$ sections of a rows, and the r duplicates are split into two groups, one of size $a-(kr)_a$ and the other of size $r-(a-(kr)_a)$. Their relative positions are illustrated in Figure 5.9. Once again the upper portion of the diagram shows the last $a-(kr)_a$ rows of the kth set of 4 columns. In the $(k+1)$st set of columns the entries have been grouped in sections of a rows, but only the first $r-(kr)_a$ rows of the last section are shown. The boxed numbers represent those entries that are duplicates.

5.4. Tetral Games

The subtraction game $S(s_1,s_2,\ldots,s_k)$ is equivalent to the tetral game $\cdot\underset{\sim}{d}_1\underset{\sim}{d}_2\underset{\sim}{d}_3\ldots$ where $\underset{\sim}{d}_u = \underset{\sim}{3}$ whenever $u \in (s_1,s_2,\ldots,s_k)$, $\underset{\sim}{d}_u = \underset{\sim}{0}$ for all other u. Some of the results proved here hold for finite tetral games in which we also allow digits $\underset{\sim}{d}_u = 1$ or $\underset{\sim}{d}_u = 2$.

The proof that every finite subtraction game is ultimately periodic rested upon the fact that for all n, the number of options of n was bounded by an integer k. Since this is also true of finite tetral games, a similar argument shows that every finite tetral game is ultimately periodic.

THEOREM 5.8. Let $\underset{\sim}{T} = \cdot\underset{\sim}{d}_1\underset{\sim}{d}_2\ldots\underset{\sim}{d}_v$ $(\forall u, \underset{\sim}{d}_u \leq \underset{\sim}{3}$, and for $u \leq 0$, $u > v$, $\underset{\sim}{d}_u = \underset{\sim}{0})$ and $k = |\{u|\underset{\sim}{d}_u \text{ contains } \underset{\sim}{2}\}|$. If there exists p such that $\underset{\sim}{d}_u$ contains $\underset{\sim}{2}$ whenever $\underset{\sim}{d}_{p-u}$ contains $\underset{\sim}{2}$, then for all $n > v+kp$, $G(n+p) = G(n)$.

PROOF. Observe that for $n > v$, $G(n) \leqslant k$, since n has at most k options. For $n > v$, an argument identical to that of Theorem 5.3 shows that if $G(n) = 0$, then $G(n+p) = 0$. We assume inductively that if $n > v+jp$, $0 \leqslant j < g$, then $G(n) = j$ implies $G(n+p) = j$, and show that if $n > v+gp$, $G(n) = g$ implies $G(n+p) = g$ by an argument similar to that of Theorem 5.3.

The remaining results proved for subtraction games do not necessarily hold for arbitrary tetral games. Ferguson's pairing property does not hold, as .1223 shows. This game has G-sequence $0\dot{1}00221\dot{1}$ and $G(2) = 0$, $G(4) \neq 1$. Consequently no results about the G-sequence of tetral games analogous to those of sections 5.2 and 5.3 have been established.

Chapter 6

Arithmetico-periodicity

6.1. Introduction

There are numerous games for which $G(n)$ is unbounded. The game of Nim, $\cdot\mathbf{3}$, has G-sequence 012345... . It is periodic in the following generalized sense. A game $\mathbf{D}$ is said to be *arithmetico-periodic* if there exist e, p, s $(s>0)$ such that for all $n > e$, $G(n+p) = G(n)+s$. The least e,p,s for which this is true are called the *last irregular value* e, the *period* p, and the *saltus* s. For $n > e$, we may write $G(n) = \dfrac{s(n-c_n)}{p}$ where c_n depends only on the residue class to which n belongs modulo p.

6.2. Finite Octals and Arithmetico-Periodicity

In his analyses of octal and sedecimal games, Kenyon [13] observed that no finite octal appeared to be arithmetico-periodic. To establish this, we follow an analysis due to J. Conway.

The *Fibonacci numbers* are defined by the following recurrence relation: $F_0 = 0$, $F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$, e.g. $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$.

For $n \geq 0$, let $f(n)$ be the number of distinct values assumed by $a \overset{*}{+} b$, where $a \geq 0$, $b \geq 0$, $a+b = n-1$, e.g. $f(0) = 0$, $f(1) = f(2) = 1$, $f(3) = 2$, $f(4) = 1$, $f(5) = 3$, $f(6) = 2$.

LEMMA 6.1. (i) $f(2n) = f(n)$

(ii) $f(2n+1) = f(n+1) + f(n)$

(iii) if $n \leq 2^k$, then $f(n) \leq F_{k+1}$

PROOF. (i) $f(2n) = |\{a \overset{*}{+} b \,|\, a+b = 2n-1\}|$. If $a+b = 2n-1$, without loss of generality we may write $a = 2a'+1$, $b = 2b'$, where $a'+b' = n-1$. Then $a \overset{*}{+} b = (2a'+1) \overset{*}{+} 2b' = 2(a' \overset{*}{+} b') + 1$ by Lemma 3.1 (iii). Hence there is a bijection between $\{a \overset{*}{+} b \,|\, a+b = 2n-1\}$ and $\{a' \overset{*}{+} b' \,|\, a'+b' = n-1\}$, so that $f(2n) = f(n)$.

(ii) $f(2n+1) = |\{a \overset{*}{+} b \,|\, a+b = 2n\}|$. If a,b are both even, $a = 2a'$, $b = 2b'$ where $a'+b' = n$, then $a \overset{*}{+} b = 2a' \overset{*}{+} 2b' = 2(a' \overset{*}{+} b')$. If a,b are both odd $a = 2a''+1$, $b = 2b''+1$ where $a''+b'' = n-1$, then $a \overset{*}{+} b = (2a''+1) \overset{*}{+} (2b''+1) = 2(a'' \overset{*}{+} b'')$. If n is even, $a'+b' = n$, then $a'+b' \equiv 0 \pmod 2$ so that $2(a' \overset{*}{+} b') \equiv 0 \pmod 4$, and $2(a'' \overset{*}{+} b'') \equiv 2 \pmod 4$. Similarly, if n is odd the sets $\{2(a' \overset{*}{+} b') \,|\, a'+b' = n\}$, $\{2(a'' \overset{*}{+} b'') \,|\, a''+b'' = n-1\}$ are distinct. Hence $f(2n+1) = f(n) + f(n+1)$.

(iii) The result is true for $n = 0,1,2$. Assume inductively that (iii) holds for $n \leqslant 2^k$, $k \geqslant 1$. If $n \leqslant 2^{k+1}$, and $n = 2n'$, $n' \leqslant 2^k$, then by (i) and the inductive hypothesis

$$f(n) = f(2n') = f(n') \leqslant F_k. \tag{1}$$

If $n = 2n'+1$, then $n'+1 \leqslant 2^k$, $f(n) = f(n') + f(n'+1)$. Just one of n', $n'+1$ is even, so by (1) and the inductive hypothesis, $f(n) \leqslant F_k + F_{k+1} = F_{k+2}$.

THEOREM 6.2. No finite octal game is arithmetico-periodic.

PROOF. Suppose on the contrary that a finite octal game $\underset{\sim}{D}$ has period p, saltus $s \geqslant 1$, and is (ultimately) arithmetico-periodic. Choose c such that

(a) $\underset{\sim}{d}_u = \underset{\sim}{0}$ for $u < -2^c$ and $u \geq 2^c$

(b) $p \leq 2^c$

(c) $G(n+p) = G(n)+s$ for all $n \geq (2^c-1)p$.

By (a), there are at most 2^{c+1} *splitting* moves ($\underset{\sim}{d}_u$ contains $\underset{\sim}{4}$) and at most 2^c *taking* moves ($\underset{\sim}{d}_u$ contains $\underset{\sim}{2}$). The total number of different moves from a heap of n tokens is thus at most $2^{c+1}(n/2)+2^c = 2^c(n+1)$. Therefore

$$G(n) \leq 2^c(n+1) \qquad (*)$$

and

$$s/p = \lim_{n\to\infty} G(n)/n \leq 2^c$$

so that $s \leq 2^c p$.

Let $2^h \leq n < 2^{h+1}$, where $h \geq 2c+2$. The number of distinct G-values arising from taking moves is at most 2^c. The number of distinct G-values arising from splitting moves in which one of the resulting heaps has size less than $2^c p$ is at most $2^c p 2^{c+1} = 2^{2c+1}p$. Other moves consist in choosing a splitting move (in one of at most 2^{c+1} ways) and choosing a residue class, μ_1, mod p, $0 \leq \mu_1 \leq p-1$ (in one of at most p ways) and replacing a heap of n tokens by two heaps of a and b, where $a+b = n-u$, $-2^c < u \leq 2^c$, $a,b \geq 2^c p$, and $a = \lambda_1 p + \mu_1$, $\lambda_1 \geq 2^c$. Write $b = \lambda_2 p + \mu_2$, where $0 \leq \mu_2 \leq p-1$, $\lambda_2 \geq 2^c$. Note that $n-u = a+b = \lambda_1 p + \mu_1 + \lambda_2 p + \mu_2$ so that when n, u, μ_1 are chosen, μ_2 is fixed, and $\lambda_1 + \lambda_2 = (n-u-\mu_1-\mu_2)/p$.

By (c), $\exists g_1, g_2$ such that $G(a) = G(\lambda_1 p+\mu_1) = \lambda_1 s + g_1 = \alpha$, say, and $G(b) = G(\lambda_1 p+\mu_2) = \lambda_2 s + g_2 = \beta$, say. We observe

$$
\begin{aligned}
2^c s + g_1 &= G(2^c p+\mu_1) && \text{by (c)}\\
&\leq 2^c(2^c p+\mu_1+1) && \text{by (*)}\\
&\leq 2^c(2^c+1)p && \text{by (b)}\\
&\leq 2^{3c+1} && \text{by (b).}
\end{aligned}
$$

Hence $g_1 \leq 2^{3c+1}$, and by a similar argument, $g_2 \leq 2^{3c+1}$. Further $\alpha+\beta = \lambda_1 s + g_1 + \lambda_2 s + g_2 = (\lambda_1+\lambda_2)s + g_1 + g_2 = \frac{s}{p}(n-u-\mu_1-\mu_2) + g_1 + g_2$ is a fixed integer, m say, where

$$m \leq 2^c(n+2^c-0-0) + 2^{3c+1} + 2^{3c+1} < 2^{h+c+2} + 2^{3c+2} < 2^{h+c+3}.$$

The G-values resulting from such moves are a subset of $\{\alpha \overset{*}{+} \beta \mid \alpha+\beta = m\}$ whose cardinality is $f(m+1)$, which by Lemma 6.1 (iii) is less than or equal to F_{h+c+4}. Therefore

$$G(n) \leq 2^{c+1} p F_{h+c+4} + 2^{2c+1} p + 2^c \leq 2^{2c+1} F_{h+c+4} + 2^{3c+1} + 2^c \quad \text{by (b).}$$

Now it is easy to see by induction that $F_h < \tau^h$ where $\tau^2 = \tau+1$, $\tau = \frac{1}{2}(1+\sqrt{5}) = 1.618\ldots < 2$, so

$$\frac{s}{p} = \lim_{n\to\infty} G(n)/n \leq \lim_{h\to\infty} \frac{2^{2c+1}\tau^{h+c+4} + 2^{3c+1} + 2^c}{2^h} = 0$$

contrary to our assumption that $s \geq 1$.

6.3. An Arithmetico-Periodicity Theorem for Sedecimal Games

We now prove a theorem for arithmetico-periodicity analogous to Theorem 4.2 for normal periodicity. We first establish several lemmas that will be used in the proof.

LEMMA 6.3. Suppose that $\underset{\sim}{T} = \underset{\sim}{d}_v\underset{\sim}{d}_{v+1} \;\cdots\; \underset{\sim}{d}_{-1}\underset{\sim}{d}_0 \cdot \underset{\sim}{d}_1\underset{\sim}{d}_2\underset{\sim}{d}_3 \cdots$ is a take and break game, and that for some integer n there exist integers e, p, and s such that

(1) $G(i+p) = G(i)+s$, for all i, $e < i \leqslant n$

(2) $G(i) < s$ for all $i \leqslant e$

(3) $G(i) < 2s$ for all $i \leqslant e+p$.

Then

(i) if $i > e+qp$ and $q \geqslant 0$ then $G(i) \geqslant qs$

(ii) if $G(i) \geqslant qs$ and $q \geqslant 1$ then $i > e+(q-1)p$.

PROOF. (i) Let $i = e+ap+r$ where $a \geqslant q \geqslant 0$, $0 < r \leqslant p$. Then

$$G(i) = G(e+ap+r) = G(e+r)+as \geqslant 0+qs = qs. \qquad \text{by (1)}$$

(ii) $q = 1$. If $G(i) \geqslant s$, then (2) implies $i > e = e+(q-1)p$.
$q = 2$. If $G(i) \geqslant 2s$, then (3) implies $i > e+p = e+(q-1)p$ so we may assume $q > 2$ and

$$G(i) \geqslant qs > 2s. \qquad (4)$$

Then by (3), $i = e+ap+r$ where $a \geqslant 1$, $p \geqslant r > 0$, so that

$$
\begin{aligned}
G(i) &= G(e+ap+r) & \\
&= G(e+r)+as & \text{by (1)} \\
&< 2s+as & \text{by (3)} \\
&= (a+2)s. & (5)
\end{aligned}
$$

Inequalities (4) and (5) yield

$$(a+2)s > qs \Rightarrow a+2 > q \Rightarrow a > q-2$$

and since a is an integer $a \geqslant q-1$.

Thus $i = e+ap+r \geqslant e+(q-1)p+r > e+(q-1)p$.

COROLLARY 6.4. Suppose that $\underset{\sim}{T} = \underset{\sim}{d}_v\underset{\sim}{d}_{v+1} \cdots\cdots \underset{\sim}{d}_0 \cdot \underset{\sim}{d}_1\underset{\sim}{d}_2 \cdots$ is a take and break game, and that for some integer n, there exist integers e, p, and s (assumed to be a power of 2, $s = 2^k$) such that

(1) $G(i+p) = G(i)+s$ for all $e < i \leqslant n$

(2) $G(i) < s$ for all $i \leqslant e$

(3) $G(i) < 2s$ for all $i \leqslant e+p$.

If $G(i)$ contains 2^m, $m \geqslant k$, then $i > e+(2^{m-k}-1)p$. If $G(i)$ contains 2^m, 2^l, where $m > l \geqslant k$, then $i > e+(2^{m-k}+2^{l-k}-1)p$.

PROOF. This follows as an immediate consequence of Lemma 6.3 (ii) by taking $s = 2^k$.

For such a game $\underset{\sim}{T}$, if $G(i) > g > G(i)-3s$, then g is an excluded value for $G(i)$. If $\forall u, d_u \leqslant 15$, there is a move taking u tokens from a heap of i and leaving three non-negative heaps of i_1, i_2, i_3 tokens so that $i-u = i_1+i_2+i_3$ and $g = G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3)$. The next lemma provides information about the binary expansions of $G(i_1)$, $G(i_2)$, $G(i_3)$.

LEMMA 6.5. Let $\underset{\sim}{T} = \underset{\sim}{d}_0 \cdot \underset{\sim}{d}_1 \underset{\sim}{d}_2 \underset{\sim}{d}_3 \ldots$ be a take and break game in which a move replaces one heap by at most three heaps (i.e. $d_u \leqslant 15$) and suppose $\underset{\sim}{T}$ satisfies the assumptions of Corollary 6.4. Let

$$G(i) > g > G(i)-3s,$$
$$i-u = i_1+i_2+i_3,$$
$$g = G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3).$$

If 2^l is the largest power of 2 contained in g, and $l \geqslant k+1$ then 2^{l+1} is not contained in $G(i_1)$, $G(i_2)$, $G(i_3)$.

PROOF. Since $g > G(i)-3s$, and 2^l is the largest power of 2 contained in g,

$$2^{l+1} > G(i)-3s. \tag{6}$$

As g contains 2^l, $g = G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3)$, 2^l is contained in an odd number of $G(i_1)$, $G(i_2)$, $G(i_3)$. Without loss of generality we may assume that 2^l is contained in $G(i_1)$. Either 2^{l+1} is not contained in any of $G(i_1)$, $G(i_2)$, $G(i_3)$ and there is nothing to prove, or 2^{l+1} is contained in just two of them. We give an argument by contradiction to show the latter is not possible. It suffices to consider the two cases where

(i) $G(i_1)$, $G(i_2)$ contain 2^{l+1},
(ii) $G(i_2)$, $G(i_3)$ contain 2^{l+1}.

(i) If $G(i_1)$ contains 2^l, 2^{l+1}, $G(i_2)$ contains 2^{l+1}, then

$$\begin{aligned} i &\geq i-u \\ &= i_1+i_2+i_3 \\ &> e+(2^{l+1-k}+2^{l-k}-1)p+e+(2^{l+1-k}-1)p \quad \text{by Corollary 6.4} \\ &\geq e+(2^{l+2-k}+2^{l-k}-2)p \end{aligned}$$

so that by Lemma 6.3 (i),

$$G(i) \geq (2^{l+2-k}+2^{l-k}-2)s = 2^{l+2} + 2^l - 2s.$$

Therefore

$$\begin{aligned} G(i)-3s &\geq 2^{l+2} + 2^l - 5s \\ &= 2^{l+1} + 6.2^{l-1} - 5s \\ &\geq 2^{l+1} + s \qquad \text{since } l \geq k+1 \end{aligned}$$

which contradicts (6).

(ii) If $G(i_1)$ contains 2^l, $G(i_2)$, $G(i_3)$ contain 2^{l+1} then

$$\begin{aligned} i &\geq i-u \\ &= i_1+i_2+i_3 \\ &> e+(2^{l-k}-1)p+e+(2^{l+1-k}-1)p+e+(2^{l+1-k}-1)p \quad \text{by Corollary 6.4} \\ &\geq e+(2^{l+2-k}+2^{l-k}-3)p \end{aligned}$$

so that by Lemma 6.3 (i)

$$G(i) \geq (2^{l+2-k}+2^{l-k}-3)s = 2^{l+2} + 2^l - 3s.$$

Therefore

$$G(i)-3s \geq 2^{l+2} + 2^{l} - 6s = 2^{l+1} + 6.2^{l-1} - 6s \geq 2^{l+1} \quad \text{since } l \geq k+1$$

which contradicts (6).

LEMMA 6.6. Under the assumptions of Lemma 6.3 suppose that for each g, $0 \leq g < 2s$ there exists i such that $G(i) = g$. If $i_1 > e+2p$ and $G(i_1-2p) \geq g_1$, then there exists $i_2 < i_1$ such that $G(i_2) = g_1$.

PROOF. Since $i_1 > e+2p$,

$$i_1-2p > e \tag{7}$$

so that by (1), $G(i_1-2p) = G(i_1)-2s$. Let $g_1 \leq G(i_1-2p)$. If $0 \leq g_1 < 2s$ then by hypothesis there exists i such that $G(i) = g_1$, and

$$\begin{aligned} i &\leq e+2p \qquad \text{by Lemma 6.3 (i)} \\ &< i_1. \end{aligned}$$

Take $i_2 = i$.

If $2s \leq g_1 \leq G(i_1-2p) = G(i_1)-2s$, then let $g_1 = qs+r$, where $q \geq 2$, $0 \leq r < s$. Thus $G(i_1)-2s \geq qs+r$ so that $G(i_1) \geq (q+2)s$. By Lemma 6.3 (i) $i_1 > e+(q+1)p$. By hypothesis there exists i such that $G(i) = s+r$ where $e < i \leq e+2p$, so that

$$\begin{aligned} G(i+(q-1)p) &= G(i) + (q-1)s \qquad \text{by (1)} \\ &= s + r + (q-1)s \\ &= g_1 \end{aligned}$$

where

$$i + (q-1)p \leq e + 2p + (q-1)p = e + (q+1)p < i_1 .$$

Take $i_2 = i + (q-1)p$.

LEMMA 6.7. Under the assumptions of Corollary 6.4, suppose that for each g, $0 \leq g < 2s$, there exists $2v+1 > 0$, $2w > 0$, such that $G(2v+1) = G(2w) = g$. If $i_1 > e+2p$ and $G(i_1-2p) \geq g_1$ then there exist $2v_1+1$, $2w_1$, $0 < 2w_1, 2v_1+1 < i_1$ such that $G(2w_1) = G(2v_1+1) = g_1$.

PROOF. The proof is similar to that of Lemma 6.6, but it is necessary to consider separately the cases where p is even, p is odd.

THEOREM 6.8. Suppose that $\underset{\sim}{T} = d_0.d_1d_2\ldots d_t$ ($d_u = 0$ for $u > t \geq 1$, $u < 0$) is a take and break game in which a move replaces just one heap by at most three heaps, i.e. $d_u \leq 15$ and that there exist integers e (the last irregular value), $p \geq t+2$ (a period) and $s \geq 1$ (a saltus, assumed to be a power of 2, $s = 2^k$) such that

(1) $G(i+p) = G(i)+s$ for all i, $e < i < e+7p+t$

(2) $G(i) < s$ for all $i \leq e$

(3) $G(i) < 2s$ for all $i \leq e+p$

(4) either there exist $\underset{\sim}{d}_{2v+1}, \underset{\sim}{d}_{2w}$ both of which contain $\underset{\sim}{8}$, and for each g, $0 \leq g < 2s$, there exists $i > 0$, such that $G(i) = g$ or there exists $\underset{\sim}{d}_u$ which contains $\underset{\sim}{8}$, and for each g, $0 \leq g < 2s$, there exist $2v+1$, $2w \geq 0$ such that $G(2v+1) = G(2w) = g$.

Then for all $i > e$

$$G(i+p) = G(i)+s. \qquad (*)$$

Note that in order to satisfy (2), (3), and the condition $p \geq t+2$, it may be necessary to choose appropriate multiples of *the* period, and *the* saltus which are defined as the least p and s satisfying (1), e.g. the game .F8 has G-sequence $0101023234\dot{5}4567\dot{8}(+4)$, where *the* period is 6, and *the* saltus 4 is indicated in parentheses. In order to apply Theorem 6.8 it was necessary to choose a period of 12, and a saltus of 8.

Kenyon [13] has solved the game .3F and shown that the G-sequence is $\dot{0}1201\dot{2}(+3)$. Similarly we have shown the game .169 has G-sequence $0\dot{1}0210213\dot{2}(+3)$. However, .3F $\equiv$ $.3\dot{3}\dot{0}$ and .169 $\equiv$ $.161\dot{0}\dot{2}$ so that both of these games are equivalent to infinite recurring octal games (see section 6.4). No theorem for sedecimal games exhibiting arithmetico-periodicity with a saltus other than a power of 2 has been proved.

PROOF. By hypothesis, (*) holds for $e < i < e+7p+t$. Assume inductively that (*) holds for $e < i < n$ where $n \geq e+7p+t$. To show $G(n+p) = G(n)+s$ we prove that:

(i) $G(n)+s$ is not an excluded value for $G(n+p)$

(ii) For each g, $0 \leq g < G(n)+s$, g is an excluded value.

(i) We suppose that $G(n)+s$ is an excluded value for $G(n+p)$ and show that this leads to a contradiction. We consider five cases, where each case leads to a result that contradicts our induction hypothesis.

If $G(n)+s$ is an excluded value for $G(n+p)$, then it must be excluded by removing u tokens from a heap of $n+p$ to leave three non-negative heaps of i_1, i_2, i_3 tokens where $n+p-u = i_1+i_2+i_3$ and $G(n)+s = G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3)$.

Since $n \geq e+7p+t$, $n-7p \geq e+t > e$, and we have by (1),

$$\begin{aligned} G(n) &= G(n-7p+7p) \\ &= G(n-7p)+7s \\ &\geq 7s \\ \Rightarrow \quad G(n)+s &\geq 8s \end{aligned}$$

so that if 2^m is the largest power of 2 contained in $G(n)+s$, $m \geq k+3$. As $G(n)+s = G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3)$, 2^m is contained in an odd number of $G(i_1)$, $G(i_2)$, $G(i_3)$, and we may assume without loss of generality that 2^m is contained in $G(i_1)$.

<u>CASE I</u>: If $G(i_1)$ also contains 2^l, where $l \geq k$, $l \neq m$ (see Figure 6.1), then

$$n - (2^{m-k}-1)p - u = (i_1-2^{m-k}p) + i_2 + i_3$$

and

$$G(n-(2^{m-k}-1)p) = G(i_1-2^{m-k}p) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3);$$

but by definition $G(n-(2^{m-k}-1)p) \neq G(i_1-2^{m-k}p) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3)$. As $G(i_1)$ contains 2^m, 2^l, $l \geq k$, $l \neq m$,

$$G(i_1) \geq 2^m + 2^l = (2^{m-k}+2^{l-k})s$$

		2^m		2^l		2^k
$G(i_1)$	. . . .	1	. .	1	. . .	.
$G(i_2)$	. . . .	X	. .	.	. . .	.
$G(i_3)$	. . . .	X	. .	.	. . .	.
$G(n)+s$	. . . 0 0 0	1	. .	.	. . .	.

Figure 6.1. Case I.

$X = 0$ in both places or $X = 1$ in both places.

so that by Corollary 6.4

$$i_1 > e + (2^{m-k}+2^{l-k}-1)p$$

$$\geqslant e + 2^{m-k}p,$$

$$\Rightarrow \qquad i_1 - 2^{m-k}p > e,$$

and by (1)

$$G(i_1-2^{m-k}p) = G(i_1) - 2^{m-k}p$$

$$= G(i_1) - 2^m \ . \tag{8}$$

Since

$$n - (2^{m-k}-1)p - u = n + p - u - 2^{m-k}p$$

$$= i_1 + i_2 + i_3 - 2^{m-k}p$$

$$= (i_1-2^{m-k}p) + i_2 + i_3, \tag{9}$$

we see that we can remove u tokens from a heap of $n-(2^{m-k}-1)p$, leaving three non-negative heaps, the first of which contains more than e tokens. So we can apply (1) to give

$$G(n-(2^{m-k}-1)p) = G(n) - (2^{m-k}-1)s$$

$$= G(n) + s - 2^m$$

$$= G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3) - 2^m$$

$$= (G(i_1)-2^m) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3),$$

since $G(i_1)$ contains 2^m, and an even number of $G(i_2)$, $G(i_3)$ do. By (8),

$$(G(i_1)-2^m) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3) = G(i_1-2^{m-k}p) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3).$$

But by (9), $G(i_1-2^{m-k}p) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3)$ is an excluded value for $G(n-(2^{m-k}-1)p)$. Therefore $G(i_1)$ does not contain 2^l, $l \geq k$, $l \neq m$.

<u>CASE II</u>: If $G(i_1)$ contains 2^m, $G(i_1)$ does not contain 2^l, $l \geq k$, $l \neq m$, and either $G(i_2)$, $G(i_3)$ both contain 2^{m-1}, or both do not contain 2^{m-1} (see Figure 6.2), then

$$n - (2^{m-1-k}-1)p - u = (i_1-2^{m-1-k}p) + i_2 + i_3,$$

and

$$G(n-(2^{m-1-k}-1)p) = G(i_1-2^{m-1-k}p) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3);$$

but by definition

$$G(n-(2^{m-1-k}-1)p) \neq G(i_1-2^{m-1-k}p) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3).$$

		2^m	2^{m-1}		2^k
$G(i_1)$	. . . 0	1	0	0 . . . 0	0
$G(i_2)$	. . . .	X	Y	. . . . . .	.
$G(i_3)$	. . . .	X	Y	. . . . . .	.
$G(n)+s$	. . . 0	1	0	. . . . . .	.

Figure 6.2. Case II.

$X = 0$ in both positions or $X = 1$ in both positions.

$Y = 0$ in both positions or $Y = 1$ in both positions.

As $G(i_1)$ contains 2^m, $G(i_1) \geq 2^m = 2^{m-k}s$, so that by Corollary 6.4,

$$i_1 > e + (2^{m-k}-1)p > e + 2^{m-1-k}p$$

$$\Rightarrow \quad i_1 - 2^{m-1-k}p > e \tag{10}$$

and by (1)

$$G(i_1 - 2^{m-1-k}p) = G(i_1) - 2^{m-1-k}s$$

$$= G(i_1) - 2^{m-1}. \tag{11}$$

Since

$$n - (2^{m-1-k}-1)p - u = n + p - u - 2^{m-1-k}p$$

$$= i_1 + i_2 + i_3 - 2^{m-1-k}p$$

$$= (i_1 - 2^{m-1-k}p) + i_2 + i_3, \tag{12}$$

we see that we can remove u tokens from a heap of $n-(2^{m-1-k}-1)p$, leaving three non-negative heaps, the first of which contains more than e tokens by (10). So we can apply (1) to give

$$\begin{aligned} G(n-(2^{m-1-k}-1)p) &= G(n) - (2^{m-k-1}-1)s \\ &= G(n) + s - 2^{m-1} \\ &= G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3) - 2^{m-1} \\ &= (G(i_1)-2^{m-1}) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3), \end{aligned}$$

since $G(i_1)$ does not contain 2^{m-1}, and an even number of $G(i_2)$, $G(i_3)$ do. By (11)

$$(G(i_1)-2^{m-1}) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3) = G(i_1-2^{m-1-k}p) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3).$$

But by (12) $G(i_1-2^{m-1-k}p) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3)$ is an excluded value for $G(n-(2^{m-1-k}-1)p)$.

If just one of $G(i_2)$, $G(i_3)$ contains 2^{m-1} without loss of generality we may assume that $G(i_2)$ contains 2^{m-1}.

<u>CASE III</u>: $G(i_1)$ contains 2^m, $G(i_1)$ does not contain 2^l, $l \geq k$, $l \neq m$, and $G(i_2)$ contains 2^{m-1}, $G(i_3)$ does not contain 2^{m-1}. If $G(i_2)$ also contains 2^l, where $l \geq k$, $l \neq m-1$ (see Figure 6.3), then

$$n - (2^{m-1-k}-1)p - u = i_1 + (i_2-2^{m-1-k}p) + i_3$$

and

$$G(n-(2^{m- -k}-1)p) = G(i\ ) \overset{*}{+} G(i\ -2^{m- -k}p) \overset{*}{+} G(i\ );$$

but by definition

$$G(n-(2^{m-1-k}-1)p) \neq G(i_1) \overset{*}{+} G(i_2-2^{m-1-k}p) \overset{*}{+} G(i_3).$$

					2^m	2^{m-1}				2^l					2^k	
$G(i_1)$	.	.	.	0	1	0	0	.	.	0	0	.	.	0	0	.
$G(i_2)$	.	.	.	.	X	1	.	.	.	1	.	.	.	.	.	.
$G(i_3)$	.	.	.	.	X	0	.	.	.	.	.	.	.	.	.	.
$G(n)+s$	.	.	.	0	1	1	.	.	.	.	.	.	.	.	.	.

Figure 6.3. Case III.

$X = 0$ in both places or $X = 1$ in both places.

As $G(i_2)$ contains 2^{m-1}, 2^l, $l \geq k$, $l \neq m-1$

$$G(i_2) \geq 2^{m-1} + 2^l$$

$$= (2^{m-1-k}+2^{l-k})s,$$

so that by Corollary 6.4

$$i_2 > e + (2^{m-1-k}+2^{l-k}-1)p$$

$$\geq e + 2^{m-1-k}p,$$

$$\Rightarrow \qquad i_2 - 2^{m-1-k}p > e, \tag{13}$$

and by (1)

$$G(i_2-2^{m-1-k}p) = G(i_2) - 2^{m-1-k}s$$

$$= G(i_2) - 2^{m-1}. \tag{14}$$

Since

$$n - (2^{m-1-k}-1)p - u = n + p - u - 2^{m-1-k}p$$

$$= i_1 + i_2 + i_3 - 2^{m-1-k}p$$

$$= i_1 + (i_2-2^{m-1-k}p) + i_3, \qquad (15)$$

we see that we can remove u tokens from a heap of $n-(2^{m-1-k}-1)p$, leaving three non-negative heaps, the second of which contains more than e tokens by (13). So we can apply (1) to give

$$G(n-(2^{m-1-k}-1)p) = G(n) - (2^{m-1-k}-1)s$$

$$= G(n) + s - 2^{m-1}$$

$$= G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3) - 2^{m-1}$$

$$= G(i_1) \overset{*}{+} (G(i_2)-2^{m-1}) \overset{*}{+} G(i_3),$$

since $G(i_2)$ contains 2^{m-1}, $G(i_1)$ and $G(i_3)$ do not. By (14)

$$G(i_1) \overset{*}{+} (G(i_2)-2^{m-1}) \overset{*}{+} G(i_3) = G(i_1) \overset{*}{+} G(i_2-2^{m-1-k}p) \overset{*}{+} G(i_3).$$

But by (15), $G(n-(2^{m-1-k}-1)p) \neq G(i_1) \overset{*}{+} G(i_2-2^{m-1-k}p) \overset{*}{+} G(i_3)$. Therefore $G(i_2)$ does not contain 2^l, $l \geq k$, $l \neq m-1$.

<u>CASE IV</u>: $G(i_1)$ contains 2^m, $G(i_1)$ does not contain 2^l, $l \geq k$, $l \neq m$, $G(i_2)$ contains 2^{m-1}, $G(i_2)$ does not contain 2^l, $l \geq k$, $l \neq m-1$. If $G(i_3)$ does not contain 2^{m-2} (see Figure 6.4), then

$$n - (2^{m-2-k}-1)p - u = i_1 + (i_2 - 2^{m-2-k}p) + i_3,$$

and

$$G(n-(2^{m-2-k}-1)p) = G(i_1) \overset{*}{+} G(i_2 - 2^{m-2-k}p) \overset{*}{+} G(i_3);$$

but by definition

$$G(n-(2^{m-2-k}-1)p) \neq G(i_1) \overset{*}{+} G(i_2 - 2^{m-2-k}p) \overset{*}{+} G(i_3).$$

				2^m	2^{m-1}	2^{m-2}					2^l				2^k	
$G(i_1)$	.	.	0	1	0	0	0	.	.	0	0	0	.	.	0	.
$G(i_2)$	.	.	0	0	1	0	0	.	.	0	0	0	.	.	0	.
$G(i_3)$	.	.	0	0	0	0	.	.	.	.	.	.	.	.	.	.
$G(n)+s$	.	.	0	1	1	0	.	.	.	.	.	.	.	.	0	.

Figure 6.4. Case IV.

Since $G(i_2)$ contains 2^{m-1}, $G(i_2) \geq 2^{m-1} = 2^{m-1-k}s$, so that by Corollary 6.4,

$$\begin{aligned} i_2 &> e + (2^{m-1-k}-1)p \\ &> e + (2^{m-2-k}p) \end{aligned}$$

$$\Rightarrow \qquad i_2 - 2^{m-2-k} > e, \tag{16}$$

and by (1)

$$\begin{aligned} G(i_2 - 2^{m-2-k}p) &= G(i_2) - 2^{m-2-k}s \\ &= G(i_2) - 2^{m-2}. \end{aligned} \tag{17}$$

Since

$$n - (2^{m-2-k}-1)p - u = n + p - u - 2^{m-2-k}p$$

$$= i_1 + i_2 + i_3 - 2^{m-2-k}p$$

$$= i_1 + (i_2 - 2^{m-2-k}p) + i_3, \qquad (18)$$

we see that we can remove u tokens from a heap of $n-(2^{m-2-k}-1)p$, leaving three non-negative heaps, the second of which contains more than e tokens by (16). So we may apply (1) to get

$$G(n-(2^{m-2-k}-1)p) = G(n) - (2^{m-2-k}-1)s$$

$$= G(n) + s - 2^{m-2}$$

$$= G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3) - 2^{m-2}$$

$$= G(i_1) \overset{*}{+} (G(i_2)-2^{m-2}) \overset{*}{+} G(i_3),$$

since $G(i_2)$ contains 2^{m-1}, $G(i_1)$ and $G(i_3)$ do not contain $2^{m-2}, 2^{m-1}$. By (17)

$$G(i_1) \overset{*}{+} (G(i_2)-2^{m-2}) \overset{*}{+} G(i_3) = G(i_1) \overset{*}{+} G(i_2-2^{m-2-k}p) \overset{*}{+} G(i_3).$$

But by (18) $G(n-(2^{m-2-k}-1)p) \neq G(i_1) \overset{*}{+} G(i_2-2^{m-2-k}p) \overset{*}{+} G(i_3)$.

<u>CASE V</u>: $G(i_1)$ contains 2^m, $G(i_1)$ does not contain 2^l, $l \geq k$, $l \neq m$, $G(i_2)$ contains 2^{m-1}, $G(i_2)$ does not contain 2^l, $l \geq k$, $l \neq m-1$, and $G(i_3)$ contains 2^{m-2}. Then

$$n - (2^{m-3-k}-1)p - u = i_1 + i_2 + (i_3-2^{m-3-k}p),$$

and

$$G(n-(2^{m-3-k}-1)p = G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3-2^{m-3-k}p),$$

since $m \geq k+3$. But by definition

$$G(n-(2^{m-3-k}-1)p) \neq G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3-2^{m-3-k}p).$$

		2^m	2^{m-1}	2^{m-2}	2^{m-3}		2^k	
$G(i_1)$	. . . 0	1	0	0	0	. . .	0	. .
$G(i_2)$	. . . 0	0	1	0	0	. . .	0	. .
$G(i_3)$	. . . 0	0	0	1	?	. . .	.	. .
$G(n)+s$	. . . 0	1	1	1	?	. . .	.	. . .

Figure 6.5. Case V.

Since $G(i_3)$ contains 2^{m-2}, by Corollary 6.4,

$$i_3 > e + (2^{m-2-k}-1)p$$

$$\geq e + 2^{m-3-k}p$$

$$\Rightarrow \qquad i_3 - 2^{m-3-k}p > e, \tag{19}$$

and by (1)

$$G(i_3-2^{m-3-k}p) = G(i_3) - 2^{m-3-k}s$$

$$= G(i_3) - 2^{m-3}. \tag{20}$$

Since

$$n - (2^{m-3-k}-1)p - u = n + p - u - 2^{m-3-k}p$$

$$= i_1 + i_2 + i_3 - 2^{m-3-k}p$$

$$= i_1 + i_2 + (i_3-2^{m-3-k}p), \qquad (21)$$

we see that we can remove u tokens from a heap of $n-(2^{m-3-k}-1)p$, leaving three non-negative heaps, the third of which contains more than e tokens by (19). So we can apply (1) to give

$$G(n-(2^{m-3-k}-1)p) = G(n) - (2^{m-3-k}-1)s$$

$$= G(n) + s - 2^{m-3}$$

$$= G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3) - 2^{m-3}$$

$$= G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} (G(i_3)-2^{m-3}),$$

since $G(i_3)$ contains 2^{m-2}, and $G(i_1)$, $G(i_2)$ do not contain 2^{m-2}, 2^{m-3}. By (20),

$$G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} (G(i_3)-2^{m-3}) = G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} (G(i_3)-2^{m-3-k}p).$$

But by (21) $G(n-(2^{m-3-k}-1)p) \neq G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3-2^{m-3-k}p)$.

If we assume there exists a move from $n+p$ of taking u tokens to leave three non-negative heaps of i_1, i_2, i_3 tokens where $G(n)+s =$ $= G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3)$ then $G(i_1)$, $G(i_2)$, $G(i_3)$ will satisfy the conditions of one of Case I - Case V. Hence $G(n)+s$ is not an excluded value for $G(n+p)$.

(ii) We first show that (4) allows us to exclude g for $0 \leqslant g \leqslant G(n-2p)$. Then using Lemma 6.5 we find moves from $n+p$ to positions of G-value g for $G(n-2p) < g < G(n)+s$. Since $n \geqslant e+7p+t$, $n-2p \geqslant e+5p+t$, so that by (1), $G(n-2p) = G(n)-2s$.

(A) If there exist $\underset{\sim}{d}_{2v+1}$, $\underset{\sim}{d}_{2w}$, both of which contain $\underset{\sim}{8}$, and for each g $0 \leqslant g < 2s$ there exists i such that $G(i) = g$, let $0 \leqslant g \leqslant G(n)-2s$. Then by Lemma 6.6 there exists $i < n$ such that

$$G(i) = g. \tag{21}$$

As $p \geqslant t+2$, where t is the maximum number of tokens we may remove

$$n + p - (2v+1) \geqslant n + 2$$

$$n + p - 2w \geqslant n + 2.$$

For $1 \leqslant i_1 = i_2 \leqslant \frac{1}{2}(n+p-(2v+1)-1)$

$$G(n+p) \neq G(i_1) \overset{*}{+} G(i_1) \overset{*}{+} G(n+p-(2v+1)-2i_1)$$

$$= G(n+p-(2v+1)-2i_1).$$

For $1 \leqslant i_1 = i_2 \leqslant \frac{1}{2}(n+p-2w-1)$

$$G(n+p) \neq G(i_1) \overset{*}{+} G(i_1) \overset{*}{+} G(n+p-2w-2i_1)$$

$$= G(n+p-2w-2i_1).$$

Thus $G(1)$, $G(2)$, ..., $G(n)$ are excluded values. But by (21) this excludes g, $0 \leqslant g \leqslant G(n)-2s$.

(B) If there exists $\underset{\sim}{d}_u$ which contains $\underset{\sim}{8}$, and for each g $0 \leqslant g < 2s$, there exist $2v+1$, $2w > 0$ such that $G(2v+1) = G(2w) = g$, let $0 \leqslant g \leqslant G(n)-2s$. By Lemma 6.7 there exist $2v+1$, $2w$ such that $0 < 2v+1$, $2w < n$, and

$$G(2v+1) = G(2w) = g. \tag{22}$$

Since $p \geqslant t+2$,

$$n + p - u \geqslant n + 2.$$

For $1 \leqslant i_1 = i_2 \leqslant \frac{1}{2}(n+p-u-1)$,

$$\begin{aligned} G(n+p) &\neq G(i_1) + G(i_1) \overset{*}{+} G(n+p-u-2i_1) \\ &= G(n+p-u-2i_1) \end{aligned}$$

so that either $G(1)$, $G(3)$, $G(5)$,,.. or $G(2)$, $G(4)$, $G(6)$,... are excluded values. But by (22), this excludes g, $0 \leqslant g \leqslant G(n)-2s$.

Let $G(n)+s > g > G(n)-2s$. Since $G(n)+s \geqslant 8s$, $g > 5s$, so that if 2^m is the largest power of 2 contained in g, $m \geqslant k+2$.

(a) If g also contains 2^{m-1},

$$\begin{aligned} G(n) + s - 2^m &> g - 2^m \\ &\geqslant 2^{m-1} \end{aligned} \tag{23}$$

$$\Rightarrow \qquad G(n) > (2^{m-k}+2^{m-1-k}-1)s,$$

so that by Corollary 6.4,

$$n > e + (2^{m-k}+2^{m-1-k}-2)p$$

$$\geq e + 2^{m-k}p.$$

We can apply (1) to get

$$\begin{aligned} G(n-(2^{m-k}-1)p) &= G(n) - (2^{m-k}-1)s \\ &= G(n) + s - 2^m \\ &> g - 2^m \\ &> G(n) - 2s - 2^m \\ &= (G(n)+s-2^m) - 3s \\ &= G(n-(2^{m-k}-1)p) - 3s, \end{aligned}$$

or

$$G(n-(2^{m-k}-1)p) > g - 2^m > G(n-(2^{m-k}-1)p) - 3s, \tag{24}$$

so that $g-2^m$ is an excluded value for $G(n-(2^{m-k}-1)p)$. Therefore it must be the case that we can remove u tokens from a heap of $n-(2^{m-k}-1)p$ to leave three non-negative heaps of i_1, i_2, i_3 where

$$n - (2^{m-k}-1)p - u = i_1 + i_2 + i_3$$

$$g - 2^m = G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3). \tag{25}$$

As $g-2^m$ contains 2^{m-1}, an odd number of $G(i_1)$, $G(i_2)$, $G(i_3)$ contain 2^{m-1}. Without loss of generality we may assume that $G(i_1)$ contains 2^{m-1}, where $m-1 \geq k+1$, so that

$$G(i_1) \geq 2^{m-1}$$

$$\geq 2s,$$

and by (2)

$$i_1 > e + p.$$

Therefore by (1)

$$G(i_1) + 2^m = G(i_1+2^{m-k}p). \tag{26}$$

We apply Lemma 6.5 to (25), (26) with $l = m-1$, and $i = n-(2^{m-k}-1)p$ to show that 2^m is not contained in any of $G(i_1)$, $G(i_2)$, $G(i_3)$ so that

$$\begin{aligned} g &= g - 2^m + 2^m \\ &= G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3) + 2^m \\ &= (G(i_1)+2^m) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3) \\ &= G(i_1+2^{m-k}p) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3) \qquad \text{by (26).} \end{aligned}$$

Since

$$\begin{aligned} i_1 + 2^{m-k}p + i_2 + i_3 &= i_1 + i_2 + i_3 + 2^{m-k}p \\ &= n - (2^{m-k}-1)p - u + 2^{m-k}p \\ &= n + p - u, \end{aligned}$$

g is an excluded value for $G(n+p)$.

(b) Suppose g contains 2^m, but g does not contain 2^{m-1}.

$$G(n) + s - 2^{m-1} > g - 2^{m-1}$$

$$\geqslant 2^{m-1},$$

$$\Rightarrow \qquad G(n) > (2^{m-k}-1)s$$

so that by Corollary 6.4,

$$n > e + (2^{m-k}-2)p$$

$$\geqslant e + 2^{m-1-k}p.$$

We can apply (1) to obtain

$$\begin{aligned} G(n-(2^{m-1-k}-1)p) &= G(n) - (2^{m-1-k}-1)s \\ &= G(n) + s - 2^{m-1} \qquad (27) \\ &> g - 2^{m-1} \\ &> G(n) - 2s - 2^{m-1} \\ &= (G(n)+s-2^{m-1}) - 3s \\ &= G(n-(2^{m-1-k}-1)p) - 3s, \qquad (28) \end{aligned}$$

or

$$G(n-(2^{m-1-k}-1)p) > g - 2^{m-1} > G(n-(2^{m-1-k}-1)p) - 3s. \qquad (29)$$

Therefore $g-2^{m-1}$ is an excluded value for $G(n-(2^{m-1-k}-1)p)$. It must be the case that we can remove u tokens from a heap of $n-(2^{m-1-k}-1)p$ to leave three non-negative heaps of i_1, i_2, i_3 where

$$n - (2^{m-1-k}-1)p - u = i_1 + i_2 + i_3$$

$$g - 2^{m-1} = G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3). \qquad (30)$$

As $g-2^{m-1}$ contains 2^{m-1}, an odd number of $G(i_1)$, $G(i_2)$, $G(i_3)$ contain 2^{m-1}. Without loss of generality we may assume that $G(i_1)$ contains 2^{m-1}, so that $G(i_1) \geqslant 2^{m-1} \geqslant 2s$, and by (2), $i_1 > e + p$.

We can therefore apply (1) to obtain

$$G(i_1) + 2^{m-1} = G(i_1+2^{m-1-k}p). \qquad (31)$$

We apply Lemma 6.5 to (30), (31) with $l = m-1$, $i = n-(2^{m-1-k}-1)p$ to show that 2^m is not contained in any of $G(i_1)$, $G(i_2)$, $G(i_3)$, so that by (26),

$$\begin{aligned} g &= g - 2^{m-1} + 2^{m-1} \\ &= G(i_1) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3) + 2^{m-1} \\ &= (G(i_1)+2^{m-1}) \overset{*}{+} G(i_2) \overset{*}{+} G(i_3), \end{aligned}$$

where

$$(i_1+2^{m-1-k}p) + i_2 + i_3 = i_1 + i_2 + i_3 + 2^{m-1-k}p$$

$$= n - (2^{m-1-k}-1)p - u + 2^{m-1-k}p$$

$$= n + p - u.$$

Hence g is an excluded value for $G(n+p)$. □

For example, the game $\cdot\underset{\sim}{B8}$ has G-sequence $01010232\dot{3}4545676\dot{}(+4)$ with last irregular value $G(8) = 3$, period 7 and saltus 4. To apply Theorem 6.8 it was necessary to calculate $8+8\cdot7+2-1 = 65$ G values.

Section (i) of the proof of Theorem 6.8 generalizes to take and break games $\underset{\sim}{T} = \underset{\sim}{d}_0\cdot\underset{\sim}{d}_1\underset{\sim}{d}_2\cdots\underset{\sim}{d}_t$ where the saltus is a power of 2, $s = 2^k$. If we permit one heap of tokens to be replaced by h heaps, then we require

$$G(i+p) = G(i) + s, \quad e < i < e + (2^h-1)p + t$$

$$G(i) < s \quad \text{for all} \quad i \leqslant e$$

$$G(i) < 2s \quad \text{for all } i \leqslant e.$$

In fact, section (i) applies even to finite octal games. The difficulty lies in ensuring that every lesser value will be excluded. E.g. the game $\cdot\underset{\sim}{16C}$ has initial G-values 0100122344566788893... . No sedecimal game has been found which satisfies condition (1) of Theorem 6.8, but not condition (4). If one heap may be replaced by $2h+1$ heaps $(h\geq1)$, $p \geqslant t+2h$, and (4) holds, an analysis similar to (ii) may enable us to show that every lesser value is excluded.

Table 7.7 displays those sedecimal games that were discovered to be arithmetico-periodic.

6.4. Infinite Recurring Games and Arithmetico-Periodicity.

In section 5.1 we proved that no finite tetral game is arithmetico-periodic, and in section 6.2 we established the same result for finite octal games. There are numerous infinite octal games that can be shown to be arithmetico-periodic.

A take and break game $\underset{\sim}{T} = \underset{\sim}{d}_0 . \underset{\sim}{d}_1 \underset{\sim}{d}_2 \ldots$ is said to be an *infinite recurring* game if

(i) there exist v, t such that for all $u > v$, $\underset{\sim}{d}_u = \underset{\sim}{d}_{u+t}$ and

(ii) there exists $w > v$, $\underset{\sim}{d}_w \neq \underset{\sim}{0}$.

We now prove theorems concerning this class of games. As with sedecimal games, it may be necessary to choose appropriate multiples of the period and the saltus.

THEOREM 6.9. Suppose that $\underset{\sim}{T} = \underset{\sim}{d}_0 . \underset{\sim}{d}_1 \underset{\sim}{d}_2 \ldots \dot{\underset{\sim}{d}}_v \underset{\sim}{d}_{v+1} \ldots \dot{\underset{\sim}{d}}_{v+t}$ is an infinite recurring octal game satisfying:

(a) $\underset{\sim}{d}_u = \underset{\sim}{d}_{u+t}$ for all $u > v$,

(b) if $\underset{\sim}{d}_u$ contains $\underset{\sim}{4}$ $(u \geq 0)$, then $\underset{\sim}{d}_{u+t}$ contains $\underset{\sim}{4}$,

and that there exist integers e (the last irregular value), $p \geq v+t$ (a period), and $s \geq 1$ (a saltus, assumed to be a power of 2, $s = 2^k$) such that

(1) $G(i+p) = G(i) + s$ for all i, $e < i \leq e+6p$

(2) $G(i) < s$ for all $i \leq e$

(3) $G(i) < 2s$ for all $i \leq e+p$.

Then for all $i > e$,

$$G(i+p) = G(i) + s. \qquad (*)$$

PROOF. By hypothesis (*) holds for all i, $e < i \leqslant e+6p$. Assume inductively that (*) holds for all i, $e < i < n$ where $n > e+6p$. To show that $G(n+p) = G(n)+s$ we prove that:

(i) $G(n)+s$ is not an excluded value for $G(n+p)$.

(ii) For each g, $0 \leqslant g < G(n)+s$, g is an excluded value.

(i) We suppose that $G(n)+s$ is an excluded value for $G(n+p)$ and show this leads to a contradiction. If $G(n)+s$ is an excluded value for $G(n+p)$, then it must be excluded by removing u tokens from a heap of $n+p$ to leave two non-negative heaps of i_1, i_2 tokens where

$$n + p - u = i_1 + i_2$$

$$G(n) + s = G(i_1) \overset{*}{+} G(i_2).$$

Since $n > e+6p$, $n-6p > e$, and we have by (1)

$$\begin{aligned} G(n) &= G(n-6p+6p) \\ &= G(n-6p) + 6s \\ &\geqslant 6s \end{aligned}$$

$$\Rightarrow \qquad G(n) + s \geqslant 7s,$$

so that if 2^m is the largest power of 2 contained in $G(n)+s$, $m \geqslant k+2$. As $G(n)+s = G(i_1) \overset{*}{+} G(i_2)$, 2^m is contained in just one of $G(i_1)$, $G(i_2)$. Without loss of generality we may assume that 2^m is contained in $G(i_1)$ and is not contained in $G(i_2)$. There are three cases to consider, where each case leads to a result that contradicts the induction hypothesis.

<u>CASE I</u>: $G(i_1)$ contains 2^m, 2^l, $l \geqslant k$, $l \neq m$. The argument that leads to a contradiction is similar to that of Case I, Theorem 6.8 (i), since we may assume that the i_3 of Theorem 6.8 equals 0.

<u>CASE II</u>: $G(i_1)$ contains 2^m, $G(i_1)$ does not contain 2^l, $l \geqslant k$, $l \neq m$, and $G(i_2)$ does not contain 2^{m-1}. The argument that leads to a contradiction is similar to that of Case II, Theorem 6.8 (i), since we may assume that the i_3 of Theorem 6.8 equals 0.

<u>CASE III</u>: $G(i_1)$ contains 2^m, $G(i_1)$ does not contain 2^l, $l \geqslant k$, and $G(i_2)$ contains 2^{m-1}. The argument that leads to a contradiction is similar to that of Case III Theorem 6.8 (i) since we may assume that the i_3 of Theorem 6.8 equals 0.

If we assume there exists a move from $n+p$ of taking u tokens to leave two non-negative heaps of i_1, i_2, where $G(n)+s = G(i_1) \overset{*}{+} G(i_2)$ then $G(i_1)$, $G(i_2)$ will satisfy the conditions of one of Cases I to III. Hence $G(n)+s$ is not an excluded value for $G(n+p)$.

(ii) We first show that g is an excluded value $0 \leqslant g \leqslant G(n)-2s$. Then, using Lemma 6.5 we find moves from $n+p$ to positions of G-value g for $G(n)-2s < g < G(n)+s$.

Let $0 \leqslant w \leqslant v$. Since $n > e+6p$, and $p \geqslant v+t$, we have $n-2p > e+4p$, $n+p-w-t > e+6p$, so that

$$G(n-2p) = G(n) - 2s, \tag{4}$$

$$G(n+p-w-t) = G(n-w-t) + s. \tag{5}$$

There exists $q \geqslant 6$ such that $e+qp < n \leqslant e+(q+1)p$. By Lemma 6.3

$$qs \leqslant G(n) < (q+2)s$$

$$\Rightarrow \quad (q+1)s \leqslant G(n) + s < (q+3)s.$$

By (4),

$$(q-2)s \leqslant G(n-2p) < qs. \tag{6}$$

Since $0 \leqslant w \leqslant v$, $p \geqslant w+t$ so that

$$n - w - t > e + qp - w - t$$

$$\geqslant e + (q-1)p.$$

By Lemma 6.3 and (5),

$$G(n-w-t) \geqslant (q-1)s$$

$$G(n-w-t) + s \geqslant qs$$

$$G(n+p-w-t) \geqslant qs. \tag{7}$$

Then (6) and (7) yield

$$G(n+p-w-t) > G(n) - 2s \tag{8}$$

for all w, $0 \leqslant w \leqslant v$.

Let $g \leqslant G(n)-2s$. By (8), g is an excluded value for $G(n+p-t)$. Hence there exists a move taking u tokens from a heap of $n+p-t$ to leave two heaps of i_2, i_2, $i_1 \geqslant i_2 \geqslant 0$.

$$n + p - u - t = i_1 + i_2,$$

$$g = G(i_1) \overset{*}{+} G(i_2).$$

If $i_2 > 0$, then $\underset{\sim}{d}_u$ contains $\underset{\sim}{4}$. By (b), $\underset{\sim}{d}_{u+t}$ contains $\underset{\sim}{4}$, so that

$$\begin{aligned} n + p - (u+t) &= n + p - u - t \\ &= i_1 + i_2, \\ g &= G(i_1) \overset{*}{+} G(i_2). \end{aligned}$$

Hence g is an excluded value for $G(n+p)$. If $i_2 = 0$, then $\underset{\sim}{d}_u$ contains $\underset{\sim}{2}$ and $u > v$ by (8). Therefore $\underset{\sim}{d}_{u+t}$ contains $\underset{\sim}{2}$, and

$$\begin{aligned} G(n+p-(u+t)) &= G(n+p-u-t) \\ &= G(i_1) \\ &= g, \end{aligned}$$

so that g is an excluded value.

Let $G(n)+s > g > G(n)-2s$. Since $G(n)+s > 7s$, $g > 4s$. If 2^m is the largest power of 2 contained in g, $m \geq k+2$.

The remainder of the argument is identical to that of Theorem 6.8 (ii), (b), since we may take $i_3 = 0$ in Theorem 6.8.

Thus for each g, $0 \leq g < G(n)+s$, g is an excluded value. Hence $G(n+p) = G(n)+s$. □

For example, the game $\cdot\dot{\mathbf{5}}\dot{\mathbf{3}}$ has G-sequence $011\dot{2}\dot{2}(+2)$, with last irregular value $G(2) = 1$, period 2, and saltus 2. To apply Theorem 6.9 it was necessary to calculate 2+7.2 = 16 values.

That $G(i+p) = G(i)+s$ for $e < i \leq e+6p$ is used only in section (ii) of the proof. To establish (1) it suffices that $G(i+p) = G(i)+s$ for $e < i \leq e+3p$. However, to exclude g, for $G(n)-2s < g < G(n)+s$, we need that g contains 2^m, where $m \geq k+2$. This in turn requires that $G(n)-2s \geq 4s$ or $G(n) \geq 6s$. Only if $n > e+6p$ can we ensure that $G(n) \geq 6s$.

The game $\cdot\mathbf{7}\dot{\mathbf{7}}\dot{\mathbf{0}}$ appears to be arithmetico-periodic with G-sequence $\dot{0}1231432456\dot{7}(+8)$. We cannot apply Theorem 6.9 to $\cdot\mathbf{7}\dot{\mathbf{7}}\dot{\mathbf{0}}$ since it does not satisfy the assumption (b). While section (i) of the proof applies to any octal satisfying (1), (2), (3), and hence to $\cdot\mathbf{7}\dot{\mathbf{7}}\dot{\mathbf{0}}$, the argument used in section (ii) breaks down. The reason for which it fails is similar to the reason for which it was necessary to assume (4) in Theorem 6.8. If $g < G(n)-2s$ then g is an excluded value for $G(n)$. Let g be excluded by the removal of u tokens $(u \leq v)$ from a heap of n to leave two positive heaps of i_1, i_2, where $g = G(i_1) \overset{*}{+} G(i_2)$. Only if the binary expansions of $G(i_1)$, $G(i_2)$ satisfy certain conditions can we say that g will be an excluded value for $G(n+p)$. In general this is not the case.

We now prove an arithmetico-periodicity theorem for infinite recurring tetral games. We no longer require the saltus be a power of 2.

THEOREM 6.10. Let $\mathbf{T} = \cdot\mathbf{d}_1\mathbf{d}_2\cdots\mathbf{d}_v\dot{\mathbf{d}}_{v+1}\mathbf{d}_{v+2}\cdots\dot{\mathbf{d}}_{v+t}$ where for $u \geq 1$, $d_u \leq 3$, and not all $d_{v+u} \leq 1$. If there exist integers p, a period, $s \geq 1$, and e, the last irregular value such that

(1) $G(i+p) = G(i) + s$, for all i, $e < i \leq e + p + v + t$

(2) $G(i) < s$, for all $i \leq e$

(3) $G(i) < 2s$ for all $i \leq e + p$.

Then for all $i > e$,

$$G(i+p) = G(i)+s. \tag{*}$$

PROOF. By hypothesis (*) holds for all i, $e < i \leq e+p+v+t$. Assume inductively that (*) holds for $e < i < n$ where $n > e+p+v+t$. To show $G(n+p) = G(n)+s$ we prove that:

(i) $G(n)+s$ is not an excluded value for $G(n+p)$,

(ii) For each g, $0 \leq g < G(n)+s$, g is an excluded value.

(i) We suppose that $G(n)+s$ is an excluded value and show that this leads to a contradiction. If $G(n)+s$ is an excluded value then there exists a move from $n+p$ of taking u tokens, $0 < u \leq n+p$, such that $G(n+p-u) = G(n)+s$. Since $n > e+p+v+t$ let $n = e+p+c$ where $c > 0$. Then

$$\begin{aligned} G(n) &= G(e+p+c) \\ &= G(e+c) + s \qquad \text{by (1)} \\ &\geq s, \end{aligned}$$

so that

$$\begin{aligned} & G(n) + s \geq 2s \\ \Rightarrow\ & G(n+p-u) \geq 2s \\ \Rightarrow\ & n + p - u > e + p \qquad \text{by (3)} \\ \Rightarrow\ & n - u > e. \end{aligned}$$

Hence we may remove u tokens from a heap of n to leave a heap $n-u$ where

$$\begin{aligned} G(n) &= G(n) + s - s \\ &= G(n+p-u) - s \\ &= G(n+p-u-p) \\ &= G(n-u), \end{aligned}$$

which is a contradiction. Hence $G(n)+s$ is not an excluded value.

(ii) If $0 \leq g-s < G(n)$, then $g-s$ is an excluded value for $G(n)$. Therefore it must be the case that we can remove u tokens from a heap of n tokens where $G(n-u) = g-s$. If $g-s \geq s$, then by (2), $n-u > e$, so that

$$\begin{aligned} g &= g - s + s \\ &= G(n-u) + s \\ &= G(n+p-u). & \text{by (1)} \end{aligned}$$

Hence if $2s \leq g < G(n)+s$, then g is an excluded value for $G(n+p)$. Let $n' = n+p-t$. Then

$$\begin{aligned} n' &= n + p - t \\ &> e + p + v + t + p - t \\ &= e + 2p + v, \end{aligned} \tag{4}$$

so that by Lemma 6.3 $G(n') \geqslant 2s$. If $0 \leqslant g < 2s$, then g is an excluded value for $G(n')$. Therefore it must be the case that we can remove u tokens from a heap of n' where $G(n'-u) = g$. Moreover, $u > v$. If not, by (4), $n'-u > e+2p$, so that by Lemma 6.3, $G(n'-u) \geqslant 2s$, which contradicts the choice of u. Therefore $u > v$, and by hypothesis, $\underset{\sim}{d}_{u+t} = \underset{\sim}{d}_u$. There is then a move, taking $u+t$ tokens from $n+p$, where

$$n + p - (t+u) = n + p - t - u$$

$$= n' - u$$

so that $G(n+p-(t+u)) = G(n'-u) = g$. Hence g is an excluded value for $G(n+p)$.

Since $G(n)+s$ is not an excluded value for $G(n+p)$ and every value strictly less than $G(n)+s$ is an excluded value, $G(n+p) = G(n)+s$. E.g. the game $\underset{\sim}{\cdot 3\dot{3}\dot{0}}$ has G-sequence $\dot{0}1201\dot{2}(+3)$. To apply Theorem 6.10 it was necessary to calculate 0+12+1+2 = 15 G-values.

Table 7.4 displays those infinite recurring octal and tetral games that exhibit arithmetico-periodicity.

Chapter 7

The G-sequences of Take and Break Games

7.1. Introduction

The tables of this chapter contain information about the G-values of take and break games. Table 7.1 displays the G-sequence of all subtraction games whose subtrahends do not exceed 8. The initial G-values of some octal games are listed in Table 7.2. Where the G-sequence is known to be periodic, the length of the period is listed. Table 7.3 indexes Table 7.2, enabling us to find the initial G-values of any octal game of the form $\underset{\sim}{4}.\underset{\sim}{d}_1\underset{\sim}{d}_2$ or $.\underset{\sim}{d}_1\underset{\sim}{d}_2\underset{\sim}{d}_3$. Table 7.4 contains information about infinite recurring octal games that exhibit arithmetico periodicity. Tables 7.5 and 7.6 complement Table 7.4 as Table 7.3 complements Table 7.2. The G-sequences of those sedecimal games that were discovered to be arithmetico-periodic are displayed in Table 7.7.

Tables 7.2 to 7.6 were compiled by Guy [1]. Additions and corrections to Table 7.2 were made by the author.

7.2. Subtraction Games

Table 7.1 lists the G-sequences of some subtraction games. The first column contains the members of the subtraction set. The second column displays the numbers that we may adjoin to the subtraction set without affecting the outcome of the game. The table therefore includes all subtraction games that may be described by a subtraction set, the members of which do not exceed 8. The third column contains the G-sequence, where a dot is placed over the first and last members of the period. The period is listed in the last column.

Table 7.1. G-sequences of subtraction games.

Subtraction Set	Optional Members	G-sequence	Period
1	(3,5,7,9,11,...)	$\dot{0}\dot{1}$	2
2	(6,10,14,18,...)	$\dot{0}01\dot{1}$	4
1,2	(4,5,7,8,10,11,...)	$\dot{0}1\dot{2}$	3
3	(9,15,21,27,...)	$\dot{0}0011\dot{1}$	6
2,3	(7,8,12,13,17,18,22,23,...)	$\dot{0}011\dot{2}$	5
1,2,3	(5,6,7,9,10,11,13,14,15,...)	$\dot{0}12\dot{3}$	4
4	(12,20,28,36,44,52,...)	$\dot{0}000111\dot{1}$	8
1,4	(6,9,11,14,16,19,21,24,...)	$\dot{0}101\dot{2}$	5
2,4	(3,8,9,10,14,15,16,20,21,22,...)	$\dot{0}0112\dot{2}$	6
3,4	(10,11,17,18,24,25,31,32,...)	$\dot{0}00111\dot{2}$	7
1,3,4	(6,8,10,11,13,15,17,18,20,...)	$\dot{0}10123\dot{2}$	7
1,2,3,4	(6,7,8,9,11,12,13,14,...)	$\dot{0}123\dot{4}$	5
5	(15,25,35,45,55,...)	$\dot{0}00001111\dot{1}$	10
2,5	(9,12,16,19,23,26,30,33,...)	$\dot{0}01102\dot{1}$	7
3,5	(4,11,12,13,19,20,21,...)	$\dot{0}001112\dot{2}$	8
2,3,5	(4,9,10,11,12,16,17,18,19,...)	$\dot{0}01122\dot{3}$	7
4,5	(13,14,22,23,31,32,40,41,...)	$\dot{0}0001111\dot{2}$	9
1,4,5	(3,7,9,11,12,13,15,17,19,20,...)	$\dot{0}101232\dot{3}$	8
2,4,5	(3,9,10,11,12,16,17,18,19,...)	$\dot{0}01122\dot{3}$	7
1,2,3,4,5	(7,8,9,10,11,13,14,15,16,17,...)	$\dot{0}1234\dot{5}$	6
6	(18,30,42,...)	$\dot{0}0000011111\dot{1}$	12
1,6	(8,13,15,20,22,27,...)	$\dot{0}10101\dot{2}$	7
1,2,6	(5,8,9,12,13,15,16,19,20,...)	$\dot{0}12012\dot{3}$	7
3,6	(4,5,12,13,14,15,21,22,23,24,...)	$\dot{0}0011122\dot{2}$	9
1,3,6	(8,10,12,15,17,19,21,24,26,...)	$\dot{0}1010123\dot{2}$	9
2,3,6	(7,11,12,15,16,20,21,24,25,...)	$\dot{0}0112031\dot{2}$	9
4,6	(5,14,15,16,24,25,26,...)	$\dot{0}00011112\dot{2}$	10
2,4,6	(3,5,10,11,12,13,14,18,19,...)	$\dot{0}011223\dot{3}$	8
1,2,4,6	(7,9,10,12,14,15,17,18,20,22,...)	$\dot{0}120123\dot{4}$	8
5,6	(16,17,27,28,38,39,49,50,60,...)	$\dot{0}000011111\dot{2}$	11
1,5,6	(3,8,10,12,14,16,17,19,21,...)	$\dot{0}101012323\dot{2}$	11
2,5,6	(9,13,16,17,20,24,27,28,...)	$\dot{0}011021302\dot{1}$	11
2,3,5,6	(4,10,11,12,13,14,18,19,20,...)	$\dot{0}011223\dot{3}$	8
1,4,5,6	(3,8,10,12,13,14,15,17,19,21,...)	$\dot{0}1012323\dot{4}$	9
1,2,4,5,6	(8,9,11,12,14,15,16,18,19,21,...)	$\dot{0}12012345\dot{3}$	10
1,2,3,4,5,6	(8,9,10,11,12,13,15,16,17,18,...)	$\dot{0}12345\dot{6}$	7

(continued)

Subtraction Set	Optional Members	G-sequence	Period
7	(21,35,49,63,77,91,...)	00000001111111	14
2,7	(11,16,20,25,29,34,38,...)	001100112	9
3,7	(13,17,23,27,33,...)	0001110221	10
4,7	(5,6,15,16,17,18,26,...)	00001111222	11
1,4,7	(9,12,15,17,20,23,...)	01012012	8
2,4,7	(10,13,16,19,22,25,28,...)	00112203102	3
3,4,7	(5,6,13,14,15,16,17,...)	0001112223	10
1,3,4,7	(5,9,11,12,13,15,...)	01012323	8
2,3,4,7	(8,9,13,14,15,18,19,...)	00112203142	11
5,7	(6,17,18,19,29,...)	000001111122	12
2,5,7	(11,15,17,20,24,27,...)	0011021322031001122332	22
3,5,7	(4,6,13,14,15,16,17,...)	0001112223	10
2,3,5,7	(4,6,11,12,13,14,15,16,20,21,...)	001122334	9
2,4,5,7	(3,6,11,12,13,14,15,16,20,...)	001122334	9
6,7	(19,20,32,33,45,46,58,...)	0000001111112	13
1,6,7	(3,5,9,11,13,15,17,...)	010101232323	12
2,6,7	(11,15,19,20,24,28,32,33,...)	0011001120312	13
1,2,6,7	(4,9,10,12,14,15,17,18,20,...)	01201234	8
3,6,7	(4,5,13,14,15,16,17,23,24,...)	0001112223	10
1,4,6,7	(9,12,14,17,19,20,...)	0101201232012	13
2,4,6,7	(3,5,11,12,13,14,15,16,20,...)	001122334	9
1,3,4,6,7	(5,9,11,13,14,15,16,17,19,21,...)	0101232345	10
2,5,6,7	(10,14,17,18,19,22,26,29,...)	001102132233	12
1,2,5,6,7	(4,9,10,12,13,15,16,17,18,...)	01201234534	11
1,4,5,6,7	(3,9,11,13,14,15,16,17,19,21,...)	0101232345	10
1,2,3,4,5,6,7	(9,10,11,12,13,14,15,17,18,19,..)	01234567	8
8	(24,40,56,72,...)	0000000011111111	16
1,8	(10,17,19,26,28,...)	010101012	9
2,8	(12,18,22,28,32,38,...)	0011001122	10
3,8	(14,19,25,30,36,...)	00011100211	11
1,3,8	(10,12,14,19,21,23,25,...)	01010101232	11
1,2,3,8	(6,7,10,11,12,15,16,17,19,...)	012301234	9
4,8	(5,6,7,16,17,18,19,20,28,...)	000011112222	12
1,4,8	(6,11,13,16,18,20,23,...)	010120101232	12
3,4,8	(9,15,16,20,21,27,...)	000111202313	12
5,8	(6,7,18,19,20,21,31,...)	0000011111222	13
1,5,8	(3,10,12,14,16,18,21,...)	0101010123232	13
2,5,8	(12,15,18,22,25,28,...)	0011021021	10
3,5,8	(4,6,7,14,15,16,17,18,19,25,...)	00011122233	11
2,3,5,8	(14,22,25,31,39,...)	0011223041304 12230011233021403	17
1,2,3,5,8	(7,9,11,12,13,15,17,18,19,...)	0123012345	10
1,4,5,8	(3,6,10,12,13,14,15,17,...)	010123234	9
2,4,5,8	(11,14,17,20,23,...)	00112230410 2	3
2,3,4,5,8	(9,10,11,15,16,17,18,21,22,...)	0011223041523	13
6,8	(7,20,21,22,34,35,36,...)	00000011111122	14

Table 7.1 (continued)

Subtraction Set	Optional Members	G-sequence	Period
2,6,8	(7,12,16,20,21,22,26,30,...)	00110011223322	14
3,6,8	(4,5,7,14,15,16,17,18,19,25,...)	00011122233	11
2,3,6,8	(7,11,12,16,17,20,21,22,...)	00112031220312	14
2,4,6,8	(3,5,7,12,13,14,15,16,17,18,...)	0011223344	10
1,2,4,6,8	(5,9,11,12,14,15,16,18,19,21,...)	0120123453	10
2,5,6,8	(9,12,16,19,20,22,23,26,30,...)	00110213223021	14
2,3,5,6,8	(4,7,12,13,14,15,16,17,18,22,...)	0011223344	10
1,2,3,5,6,8	(9,10,12,13,14,16,17,19,20,21,..)	01230123456	11
7,8	(22,23,37,38,52,53,...)	000000011111112	15
1,7,8	(3,5,10,12,14,16,18,20,22,23,...)	010101012323232	15
2,7,8	(12,17,22,27,32,...)	00110011220312001	5
3,7,8	(13,18,23,28,33,...)	000111022130021 1	5
1,4,7,8	(10,15,17,18,21,24,26,29,32,...)	010120123230130101 2324323	25
2,4,7,8	(3,9,13,14,15,18,19,20,...)	00112203142	11
3,4,7,8	(5,6,14,15,16,17,18,19,...)	00011122233	11
1,3,4,7,8	(5,6,10,12,14,15,16,17,18,19,...)	01012323454	11
2,5,7,8	(15,16,18,21,25,28,30,31,...)	00110213220310011322332	23
2,3,5,7,8	(4,6,12,13,14,15,16,17,18,22,...)	0011223344	10
1,4,5,7,8	(3,6,10,12,14,15,16,17,18,19,...)	01012323454	11
2,4,5,7,8	(3,6,12,13,14,15,16,17,18,22,...)	0011223344	10
1,6,7,8	(3,5,10,12,14,16,18,19,20,21,...)	0101012323234	13
1,2,6,7,8	(4,5,10,11,13,14,16,17,18,19,...)	012012345345	12
1,4,6,7,8	(10,13,15,18,20,21,22,24,27,...)	01012012323453	14
2,5,6,7,8	(11,15,18,19,20,21,24,28,...)	0011021322334	13
1,2,3,5,6,7,8	(10,11,12,14,15,16,18,19,20,...)	0123012345674	13
1,2,3,4,5,6,7,8	(10,11,12,13,14,15,16,17,19,20,.)	012345678	9

Table 7.1. (concluded).

7.3. Octal Games

Table 7.2 contains information about octal games. When used in conjunction with Table 7.3, it lists the initial G-values for all octal games of the form $\mathbf{4}\cdot\mathbf{d}_1\mathbf{d}_2$ or $\cdot\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3$. Each row of the table contains information about one game. The row is indexed by a number ($d_1d_2d_3$ or $4.d_1d_2$) appearing in the first column, and the row refers to the standard form of the game $\cdot\mathbf{d}_1\mathbf{d}_2\mathbf{d}_3$ or $\mathbf{4}\cdot\mathbf{d}_1\mathbf{d}_2$. The second column contains the name of a first cousin if any, and the third column lists the standard form, e.g. $\cdot\mathbf{002} \equiv_{-1} \cdot\mathbf{013} \equiv_{-1} \cdot\mathbf{113}$ so that the row indexed by 002 contains the G-sequence of the game $\cdot\mathbf{113}$, and has $\cdot\mathbf{013}$ listed as a first cousin.

The main entry consists of the initial G-values. As $G(0) = 0$ always, it has been omitted except in the first two rows. We list 30 G-values, unless the G-sequence is periodic, and it may be described in less. For those games that are periodic, the beginning and end of the period are indicated by dots over the first and last members of the period. G-values greater than 9 are represented by the following symbols:

X	x	T	t	F	f	S	s	A	a	V
10	11	12	13	14	15	16	17	18	19	20 .

In the case of those games which have essentially the same G-sequence, but different code digits, only one reference appears, e.g. the games $\cdot\mathbf{151}$, $\mathbf{4}\cdot\mathbf{1}$, $\cdot\mathbf{51}$ all have G-sequence $0.\dot{1}$, but only $\cdot\mathbf{51}$ is displayed.

The last column contains the period, p, and a reference to the notes that follow. If there is no entry in the column for the period, this indicates that the period, if any, is not yet known.

Table 7.2. *G*-sequences of octal games.

	1st cousins	Standard Form	*G*-sequence	Period	Notes
001	.01	.1	0.10	1	
002	.013	.113	0.111000	6	
004	.011337	.1113337	.1111222033 3111104433 3322224440		(3)
005	.0107	.10137	.1011222033 4110154333 2221601045		(4)
006	.01337	.113337	.1112220331 1122433355 2144333222		(5)
014	.014	.1007	.1001012212 3401051212 5303451211		
015	.015	.1107	.1101021223 0142145122 3234014512		
016	.016	.1037	.1012220101 4422161604 2127661512		(8)
017	.017	.1137	.1112023114 0451320211 1402616404	60	(9)
02	.03	.13	.1100	4	
022	.033	.133	.11200	5	
024	.0307	.13137	.1122304112 5324115560 3125148142		
026	.0337	.13337	.1122304112 5334112530 4421133442		
034	.034	.1307	.1102231401 4312210514 5632481402		
04	.0137	.11337	.1112203311 1043332224 4055222330		(15)
044	.01377	.113377	.1112223331 1144433322 2111444222	36	(16)
045	.0177	.11377	.1112223311 1444332221 1144222664	32	(17)
05	.05	.107	.10	2	(18)
051	.053	.117	.1110221340 1113222340 1543222310	48	(19)
054	.056	.1077	.1012223441 1163222411 6667344511		
055	.057	.1177	.1112223111 4443222111 4222644411	148	(21)
06	.036	.1337	.1122031122 3344053342 2113022114		(22)
064	.0377	.13377	.1122334115 5332211544 2266841122		
07	.071	.137	.1120311033 2240522330 1130211045	34	(24)
101		.101	.1010	1	
102		.102	.100011	6	
104		.104	.1000102212 2410401566 1228104015		(27)
106		.106	.1000122214 4010621242 1045166512		(28)
11	.011	.11	.110	1	
111		.111	.1110	1	
112		.112	.110001	6	
114		.114	.1100112021 2041104115 2415241120		
115		.115	.1110111222 1222	14	(33)
116		.116	.1100212021 1044152411 2041204115	96	(34)
12		.12	.1001	4	
121		.121	.1021001	4	
122		.122	.10021	5	
123		.123	.102210021	5	
124		.124	.1001102130 2130113023 3223425042	62	(39)
125		.125	.1021102130 1130234223 4253225320		
126		.126	.1002133210 4250315041 5041304130		
127		.127	.1022104412 2014461770 1226144812		(42)
131		.131	.1120011	4	
132		.132	.11002	5	

(continued)

	1st cousins	Standard Form	G-sequence	Period	Notes
134		.134	.1100112031 2031103122 3322435143	62	(45)
135		.135	.1120112031 1031224322 4352235221		
136		.136	.1100213021 1022334251 4223342011		
14		.14	.1001021221 0414412212 0104126164		(48)
141		.141	.1011012212 4101121221 2412	11	
142		.142	.1002221103 3241063231 0162240115		
143		.143	.1012220104 2215047228 0412228104		(51)
144		.144	.1001222244 111	10	(52)
145		.145	.1011222241 1	9	(53)
146		.146	.1002224111 3324446662 3111766842		
147		.147	.1012224411	8	(55)
15		.15	.1101122122	10	(56)
152		.152	.1102220104 3231013224 0104223101	48	(57)
153		.153	.1112221102 2244011222 111222441	14	(58)
154		.154	.1101122222 4111	11	(59)
156		.156	.1102224411 1322444666 2111576688	349	(60)
157		.157	.111222	6	(61)
16		.16	.1001221401 4214014214 2102142145		(62)
161		.161	.1021021321 3243043241 2312012415		(63)
162		.162	.1002231104 2261034266 0542330142		
163		.163	.1022310422 6104226104 3221043265		(65)
164		.164	.1001223445 1163223415 66738211X7		
165		.165	.1021321344 3623128126 5445182182	1550	(67)
166		.166	.1002234116 6224411338 5446633118		
167		.167	.1022341162 2441133544 663315866X		
17	.43	.17	.1102130113 2234153223 1103120114	34	(70)
171		.171	.1122110214 0112211221 42	11	
172		.172	.1102230113 2244063224 0163220116		
173	.432	.173	.1122310432 0112235143 2211023741	40	(73)
174		.174	.1102132214 4564223115 4128865741		
176		.176	.1102234411 6223441166 33241166334	8	(75)
204	.204	.3007	.1012010123 1212314303 1432324323		
205	.205	.3107	.1201012312 3134034532 3253210202		
206	.206	.3037	.1012320101 2323451232 3454010342		
207	.207	.3137	.1212030124 5312124303 0214358213		
22	.22	.33	.120	3	
224	.224	.3307	.1201231231 4304314213 2102142641		
226	.226	.3337	.1234012345 1234512305 1234	5	
244	.244	.3077	.1012323451 5673232158 9767654548		
245	.245	.3177	.1212345156 7321289765 64T9212X74		
26	.26	.333	.1230	4	(85)
264	.264	.3377	.1234516325 1867524816 X45267X518		

Table 7.2 (continued)

	1st cousins	Standard Form	*G*-sequence	Period	Notes
31	.201	.31	.1201	2	
312		.312	.120201	2	
316		.316	.1202123010 30123	12	(89)
32		.32	.102	3	
324		.324	.1021301340 2342132034 1346201253		(91)
331		.331	.123012	3	
332		.332	.1203	4	
334		.334	.1201203123 1243503426 1241302172		
336		.336	.1203124031 2034123612 3051306413		(95)
34		.34	.1012010312 1203	8	(96)
342		.342	.1012320103 2345023254 0102321456		
344		.344	.1012324514 6232145876 7X14123264		
346		.346	.1012324516 7232158676 X548923Xx4		
35		.35	.120102	6	
351		.351	.12120102	8	
353		.353	.12120	2	
354		.354	.1201243123 5243513524 7247864762		
356		.356	.1202124516 7512826281 5x79581212	142	(104)
36		.36	.1021021321 3243043241 2312012415		(105)
362		.362	.1023410234 1523714237 0123750132		
364		.364	.1021321345 3423125125 7457482962		
366		.366	.1023451623 4576891276 85432915x3		
37	.6	.37	.1201231234 0342132102 1451451201		(109)
371		.371	.1231032402 3401241632 0123413421		
373	.603	.373	.1234012341 5231472104 321402640		
374		.374	.1201243123 5243513524 7247864762		
375		.375	.1231243213 4274814812 4814381482	18	(113)
376		.376	.1203124352 4351432645 867X827362		
404	.0707	.13737	.1122334115 6332211087 7255401122		(115)
414	.414	.1707	.1102234401 1322344566 3223118763		
416	.416	.1737	.1122341166 3221066844 5X17833241		(117)
44	.077	.1377	.1122331144 3322114422 6644112277	24	(118)
444	.0777	.13777	.1122334115 6332211887 7655441122		
45	.45	.177	.1122311443 2211422644 1122711443	20	(120)
454	.454	.1777	.1122341166 3221166844 5X11833447		
51		.51	.1	1	
512		.512	.11122210	6	
52		.52	.1022103	4	
524		.524	.1022104416 7012261446 1870187614	52	(125)
53		.53	.1122102240 122112241	9	
532		.532	.1122401224 1	5	
536		.536	.11224	5	
54		.54	.101222411	7	
544		.544	.1012224411	8	

Table 7.2 (continued)

	1st cousins	Standard Form	G-sequence	Period	Notes
56		.56	.1022411324 4662117684 11654811T4		(131)
564		.564	.1022441132 5476823X76 8932T65432		
57		.57	.1122	4	
604	.604	.3707	.1201231234 5345321321 0254754768		
606	.606	.3737	.1234012345 1234562345 6734167891		
64	.64	.377	.1234153215 4268123745 8295476814		
644	.644	.3777	.1234516325 896X5496FX 42367S49FX	442	(137)
71	.203	.71	.1210	2	
72		.72	.1023	4	
74		.74	.1012324146 2321517685 1Xx26845X6		
744		.744	.1012324516 723218967X 45981XxX45		
75		.75	.12	2	
76		.76	.1023416234 1673216752 89652871X4		
764		.764	.1023451623 4576891X76 8543261543		
77	4.4	.77	.1231432142 6412714321 4674128547	12	(145)
772		.772	.1234162416 3	4	
774		.774	.1231456713 289546T219 645Tt298X5		
776	4.44	.776	.1234163216 74581X5476 1236143218		
4.12		4.12	.1122042112 21	7	
4.3		4.3	.120	2	
4.72		4.72	.124	3	

Table 7.2. (concluded).

Notes to Table 7.2.

Unless otherwise indicated, all games have been analyzed to $n = 9999$, where n pertains to the form of the game listed in the first column.

(3) $n = 14{,}999$. Notes in this form indicate that $G(n)$ has been calculated to or beyond the indicated value, and periodicity has not been observed.

(4) $n = 19{,}999$.

(5) $n = 14{,}999$.

(8) $n = 19{,}999$.

(9) $G(0) = 0$, $G(7) = 3$, $G(13) = 5$. Otherwise for $n \equiv 0,1,2,\dots,59$ (mod 60), $G(n) =$

4111202611 4046132021 1140261640
4111202615 4046132021 1180261640.

There is a strong tendency towards a period of 30.

(15) $n = 3216$.

(16) Triplicate Kayles, see Guy and Smith [11].

(17) The last irregular value is $G(186) = 6$. For $n > 186$, $n \equiv 28,29,\dots,31,0,1,\dots,27$ (mod 32), $G(n) =$

7744411122288111
4447722211188222.

There is a strong resemblence to '8/3 -plicate Kayles'. For **.77** the last irregular value is 70, and $\left\lfloor \frac{8\times 70}{3} \right\rfloor = 186$. Exactly the same values appear in the period of the two games, so that in each case the rare G-values are those that contain an even number of 1's in their binary expansions. Furthermore, in each case there is a strong tendency, for $n > e$, to $G(n+\frac{p}{2}) = G(n) \overset{*}{+} 3$.

(18) "She loves me, She loves me not".

(19) $G(7) = G(12) = 1$; $G(6) = G(16) = G(26) = G(36) = 2$; $G(22) = G(45) = 5$; otherwise, for $n \equiv 0,1,2,\dots,47$, (mod 48), $G(n) =$

01010232 34010132 32340104
32323101 04323201 01043234.

For $n > e = 45$, $n \not\equiv 9,19,23$ (mod 24), $G(n+24) = G(n) \overset{*}{+} 3$.

(21) The last irregular value is $G(257) = 2$. There are 128 irregular values. For $n > 257$, there is a strong tendency towards $G(n+74) = G(n)\overset{*}{+}5$. For more information about the period see section 4.5.

(22) $n = 17,999$.

(24) Dawson's Kayles. See Guy and Smith [11], Dawson [6],[7].

(27) $n = 47,549$.

(28) $n = 42,724$.

(33) See Theorem 4.9.

(34) $G(3) = 0$; $G(88) = 1$; $G(n) = 2$ for $n = 5,9,25,35,37,47$; $G(31) = G(41) = 4$; $G(42) = G(94) = G(138) = 8$; otherwise, for $n \equiv 0,1,2,\ldots,95$, (mod 96) $G(n) =$

```
01120X12 06110441 52411204
15041152 425X0X15 42T58285
524X1X0X 52425114 05120211
4X514201 120X120X 818981T2
```

where X = 10, T = 12.

(39) $G(n) = 0$ for $n = 0,2,3,28,64$; $G(1) = 1$; $G(n) = 2$ for $n = 26,30,34,59,95$; $G(n) = 3$ for $n = 24,32,121$; otherwise for $n \equiv 0,1,2,\ldots,61$, (mod 62), $G(n) =$

```
7584110213 0213011302 33227465445
5796332031 1031203120 11405547564.
```

For $n > 121$, there is a strong tendency towards $G(n+31) = G(n)\overset{*}{+}2$.

(42) $n = 17,999$.

(45) $G(0) = G(3) = 0$; $G(1) = G(28) = 1$; $G(24) = G(32) = G(59) = 2$; $G(26) = G(30) = G(34) = 3$; otherwise, for $n \equiv 0,1,2,\ldots,61$, (mod 62) $G(n) =$

```
6514011203 1203110312 23326475447
5627322130 1130213021 10415446374.
```

For $n > 59$, there is a strong tendency towards $G(n+31) = G(n) \overset{*}{+} 3$. Note the similarity between .134 and .124. It is often the case that for n odd, $G_{.134}(n) = G_{.124}(n)$, and for n even $G_{.134}(n) = G_{.124}(n) \overset{*}{+} 1$.

(48) $n = 35{,}949$.

(51) $n = 34{,}874$.

(52) See Theorem 4.10.

(53) See Theorem 4.11.

(55) See Theorem 4.14.

(56) Guiles; see Guy and Smith [11].

(57) The only irregular values are $G(0) = 0$, $G(1) = 1$. Otherwise for $n \equiv 0,1,\ldots,47$, (mod 48) $G(n) =$

401022201043 231013224010
422310132340 102220104323.

For $n > 1$, there is a tendency towards $G(n+24) = G(n) \overset{*}{+} 3$.

(58) See Theorem 4.12.

(59) See Theorem 4.13.

(60) See J.C. Kenyon [13]. The last irregular value is $G(3478) = 8$. The G-values illustrate a remarkable tendency to a period of 10, and for $n > 3478$, to $G(n+174) = G(n) \overset{*}{+} 4$.

(61) See Theorem 4.15.

(62) $n = 50{,}174$.

(63) The G-sequence of .36 and .161 agree as far as $n = 518$. $G_{.161}(518) = G_{.36}(518) = 2$, but $G_{.161}(519) = 2$, $G_{.36}(519) = 4$.

(65) $n = 54{,}424$.

(67) The last two irregular values are $G(5180) = G(3495) = 4$. There are 251 irregular values. See section 4.5.

(70) See Guy and Smith [11]. The irregular values are $G(0) = 0$, $G(15) = 1$, $G(17) = 3$, $G(32) = 2$; otherwise, for $n \equiv 0,1,2,\ldots,33 \pmod{34}$, $G(n) =$

41102130113223445
72231103120114436.

For $n > 32$, there is a strong tendency toward $G(n+17) = G(n) \overset{*}{+} 3$.

(73) $G(0) = 0$; $G(1) = 1$; $G(9) = G(16) = G(20) = 3$; otherwise, for $n \equiv 0,1,2,\ldots,39$, (mod 40) $G(n) =$

4012231046 2011227514
7221102374 1322104627.

For $n > 20$, $n \not\equiv 1,9,15 \pmod{20}$, $G(n+20) = G(n) \overset{*}{+} 3$.

(75) For $n > 23$, $G(n+4) = G(n) \overset{*}{+} 2$.

(85) See Ferguson [9]. When played under misère rules, .73 and .333 are not equivalent.

(89) For $n > 3$, $G(n+6) = G(n) \overset{*}{+} 2$.

(91) $n = 29{,}999$.

(95) $n = 29{,}999$.

(96) Except for $n = 0,2$, and 6, $G(n+4) = G(n) \overset{*}{+} 1$.

(104) The last irregular value is $G(7314) = 2$. There are 6419 irregular values. These are part of prior attempts to establish a period. For $n > 7314$, if $G(n) = 16$, then $G(n+71) = 16$. If $G(n) \neq 16$, then $G(n+71) = G(n) \overset{*}{+} 7$.

(105) $n = 17,999$.

(109) $n = 10,342$, $G(10,342) = 256$.

(113) $G(0) = 0$; $G(4) = 1$; $G(5) = G(8) = 2$; $G(n) = 3$ for $n = 3,7,10,25$; $G(n) = 4$ for $n = 11,17$, and 35; $G(13) = 7$ and $G(18) = G(36) = 8$; otherwise, for $n \equiv 0,1,2,\ldots,17$, (mod 18) $G(n) =$

$$\dot{4}1248147814821481\dot{7}.$$

(115) The G-sequences of .404 and .444 agree as far as $G(19)$.

(117) The G-sequences of .416 and .454 agree as fas as $G(15)$.

(118) Duplicate Kayles, see Guy and Smith [11].

(120) See Guy and Smith [11]. The ultimate period is 20; the last irregular value is $G(497) = 8$. In some sense this is "5/3-plicate Kayles" [cf. note (17)].

(125) The only irregular value is $G(0) = 0$. Otherwise, for $n \equiv 0,1,2,\ldots,51$, (mod 52), $G(n) =$

8102210441670 1226144618701
8761476107816 7410721078167.

For $n > 0$, there is a tendency towards $G(n+26) = G(n) \overset{*}{+} 6$.

(131) $n = 49,999$.

(137) The last irregular value is $G(3254) = 32$. There are 2179 irregular values. See section 4.5.

(145) Kayles, see Guy and Smith [11], Dudeney [8], and Loyd [15].

The G-sequence of the standard form of any octal game $\cdot\underset{\sim}{d}_1\underset{\sim}{d}_2\underset{\sim}{d}_3$ or $\underset{\sim}{4}\cdot\underset{\sim}{d}_1\underset{\sim}{d}_2$ appears in Table 7.2. To find the G-sequence, look in Table 7.2 for the row $d_1d_2d_3$ or $4.d_1d_2$. If this does not appear, find that entry in Table 7.3 and consult the row of Table 7.2 to which the entry refers.

A '-' in the entry of Table 7.3 indicates that the row appears in Table 7.2. The * indicates that the G-sequence of the game $.\underset{\sim}{\dot{0}}$ is just $\dot{0}$.

Table 7.3. Guide to Table 7.2.

d_3 / d_1d_2	0	1	2	3	4	5	6	7
00	*	—	—	002	—	—	—	04
01	001	11	05	002	—	—	—	—
02	—	02	—	022	—	024	—	026
03	02	02	022	022	—	034	06	06
04	—	017	04	017	—	—	044	045
05	—	—	05	051	—	—	054	055
06	—	06	06	06	—	064	064	064
07	—	07	07	07	44	44	44	44
10	001	—	—	05	—	05	—	05
11	—	—	—	002	—	—	—	051
12	—	—	—	—	—	—	—	—
13	02	—	—	022	—	—	—	07
14	—	—	—	—	—	—	—	—
15	—	51	—	—	—	51	—	—
16	—	—	—	—	—	—	—	—
17	—	—	—	—	—	57	—	45
20	05	31	05	71	—	—	—	—
21	05	31	05	71	204	205	206	207
22	—	22	26	26	—	224	—	226
23	22	22	26	26	224	224	226	226
24	05	71	05	71	—	—	244	245
25	05	71	05	71	244	245	244	245
26	—	26	26	26	—	264	264	264
27	26	26	26	26	264	264	264	264
30	05	05	05	05	05	05	05	05
31	—	71	—	71	31	71	—	71
32	—	32	72	72	—	324	72	72
33	22	—	—	26	—	26	—	26
34	—	34	—	342	—	344	—	346
35	—	—	4.3	—	—	75	—	75
36	—	36	—	362	—	364	—	366
37	—	—	332	—	—	—	—	64
40	07	07	07	07	—	404	404	404
41	17	17	173	173	—	414	—	416
42	07	07	07	07	404	404	404	404
43	17	17	173	173	414	414	416	416
44	—	44	44	44	—	444	444	444
45	—	45	45	45	—	454	454	454
46	44	44	44	44	444	444	444	444
47	45	45	45	45	454	454	454	454

(continued)

d_3 / $d_1 d_2$	0	1	2	3	4	5	6	7
50	05	05	05	05	05	05	05	05
51	—	51	—	512	51	51	157	157
52	—	52	52	52	—	524	524	524
53	—	53	—	532	57	57	—	536
54	—	54	54	54	—	544	544	544
55	51	51	157	157	51	51	157	157
56	—	56	56	56	—	564	564	564
57	—	57	536	536	57	57	536	536
60	37	37	373	373	—	604	—	606
61	37	37	373	373	604	604	606	606
62	37	37	373	373	604	604	606	606
63	37	37	373	373	604	604	606	606
64	—	64	64	64	—	644	644	644
65	64	64	64	64	644	644	644	644
66	64	64	64	64	644	644	644	644
67	64	64	64	64	644	644	644	644
70	05	05	05	05	05	05	05	07
71	—	71	71	71	71	71	71	71
72	—	72	72	72	72	72	72	72
73	26	73	26	26	26	26	26	26
74	—	74	74	74	—	744	744	744
75	—	75	75	75	75	75	75	75
76	—	76	76	76	—	764	764	764
77	—	77	—	772	—	774	—	776
4.0	05	05	26	26	05	05	26	26
4.1	51	51	—	4.12	51	51	57	57
4.2	05	05	26	26	05	05	26	26
4.3	—	4.3	332	332	4.3	4.3	332	332
4.4	77	77	77	77	776	776	776	776
4.5	51	51	57	57	51	51	57	57
4.6	77	77	77	77	776	776	776	776
4.7	75	75	—	4.72	75	75	4.72	4.72

Table 7.3. (concluded).

7.4. <u>Infinite recurring octal games</u>.

Table 7.4 contains information about infinite recurring octal games of the form $.\mathbf{\dot{d}_1\dot{d}_2}$, $.\mathbf{d_1\dot{d}_2}$, and $\mathbf{4.\dot{d}_1}$. The organization of the table is similar to the organization of Table 7.2. The number in parentheses following the G-sequence indicates the saltus. An extra column has been added adjacent to the column for the period to permit inclusion of the saltus. G-values greater than 9 are represented by the following symbols

10	11	12	13	14	15	16	17	18	19	20
X	x	T	t	F	f	S	s	A	a	V

For completeness we have included $.\mathbf{5\dot{2}}$ and $.\mathbf{5\dot{3}}$ in the table. These games appear to be arithmetico-periodic, but this has not been established. We have also included $.\mathbf{3\dot{3}\dot{0}}$ and $.\mathbf{\dot{3}00300\dot{3}}$. The former is equivalent to the sedecimal game $.\mathbf{3F}$ first analyzed by J.C. Kenyon [13]. The latter provides another example of a game whose saltus is not a power of 2.

To find the G-sequence for any game $.\mathbf{d_1\dot{d}_2}$ or $\mathbf{4.\dot{d}_1}$ look in Table 7.5. The entry refers to the row of the table in which the G-sequence of the game in standard form may be found. An asterisk indicates that the G-values are bounded. For example, $.\mathbf{1}$, $.\mathbf{5}$, $.\mathbf{1\dot{5}}$, $.\mathbf{\dot{1}\dot{5}}$, $.\mathbf{5\dot{1}}$, $.\mathbf{\dot{5}\dot{1}}$, $\mathbf{4.1}$ and $\mathbf{4.5}$ all have G-sequence $0\dot{1}$, $.\mathbf{3\dot{1}}$, $.\mathbf{\dot{3}\dot{1}}$, $.\mathbf{3\dot{5}}$, $.\mathbf{7\dot{1}}$, $.\mathbf{\dot{7}\dot{1}}$, $.\mathbf{7\dot{5}}$, $.\mathbf{\dot{7}\dot{5}}$ all have G-sequence $0\dot{1}\dot{2}$, and $.\mathbf{\dot{0}\dot{5}}$, $.\mathbf{\dot{2}\dot{0}}$, $.\mathbf{\dot{2}\dot{1}}$, $.\mathbf{\dot{2}\dot{4}}$ and $.\mathbf{\dot{2}\dot{5}}$ are all first cousins of $.\mathbf{\dot{1}0\dot{7}}$, $.\mathbf{\dot{3}\dot{0}}$, $.\mathbf{\dot{3}0\dot{7}}$, $.\mathbf{\dot{5}\dot{0}}$ and $.\mathbf{\dot{7}\dot{0}}$ which have G-sequence $\dot{0}\dot{1}$; $.\mathbf{4\dot{1}}$ has G-sequence $001\dot{1}2\dot{2}$. The ? corresponding to $.\mathbf{61}$ indicates that this game is as yet unsolved, though it has been analyzed to n = 14,999.

To find the G-sequence for any game $.\mathbf{\dot{d}_1\dot{d}_2}$, apply an analogous procedure to 7.6.

	1st cousins	standard form	G-Sequence	Period	Saltus	Notes
$\dot{0}1\dot{2}$	$.\dot{0}1\dot{2}$	$.\dot{1}0\dot{3}$	$\dot{0}.1\dot{0}(+1)$	3	1	
$0\dot{2}$	$.0\dot{3}$	$.1\dot{3}$	$0.\dot{1}\dot{1}(+1)$	2	1	(a)
$0\dot{4}$	$.013\dot{7}$	$.1133\dot{7}$	$0.\dot{1}1\dot{1}(+1)$	3	1	(b)
$0\dot{5}$	$.0\dot{5}$	$.11\dot{7}$	$0.11\dot{1}22\dot{2}(+2)$	4	2	
$1\dot{2}$		$.1\dot{2}$	$0.100\dot{2}\dot{2}(+1)$	2	1	
$1\dot{4}$		$.1\dot{4}$	$0.10012\dot{2}224444(+4)$	7	4	
$1\dot{6}$		$.1\dot{6}$	$0.100\dot{2}2\dot{3}(+2)$	3	2	
$1\dot{7}$	$.4\dot{7}$	$.1\dot{7}$	$0.11\dot{2}2\dot{3}(+2)$	3	2	
$\dot{2}$	$.\dot{2}$	$.\dot{3}$	$0.\dot{1}(+1)$	1	1	(c)
$2\dot{4}$	$.2\dot{4}$	$.\dot{3}0\dot{7}$	$0.\dot{1}01\dot{2}(+2)$	4	2	(d)
$2\dot{5}$	$.2\dot{5}$	$.31\dot{7}$	$0.12\dot{1}2345\dot{4}(+4)$	6	4	
$\dot{3}00300\dot{3}$		$.\dot{3}00300\dot{3}$	$\dot{0}.1012012323453454\dot{5}(+6)$	18	6	
$3\dot{2}$		$.3\dot{2}$	$0.10\dot{2}(+1)$	1	1	
$3\dot{3}\dot{0}$		$.3\dot{3}\dot{0}$	$\dot{0}.1201\dot{2}(+3)$	6	3	(e)
$3\dot{4}$		$.3\dot{4}$	$0.101\dot{2}3\dot{2}(+2)$	3	2	
$5\dot{2}$		$.5\dot{2}$	$0.1022443355\dot{7}688XX99xxt\dot{T}(+8)$	12	8	
$5\dot{3}$		$.5\dot{3}$	$0.1122446335577 88XXT99xxt\dot{t}FF$ $88Affssaa\dot{V}(+8)$	13	8	
$5\dot{4}$		$.5\dot{4}$	$0.101\dot{2}2244\dot{4}(+4)$	5	4	
$5\dot{6}$		$.5\dot{6}$	$0.10\dot{2}\dot{2}(+2)$	2	2	
$5\dot{7}$		$.5\dot{7}$	$0.11\dot{2}\dot{2}(+2)$	2	2	
$7\dot{4}$		$.7\dot{4}$	$0.101232\dot{4}546\dot{7}(+4)$	5	4	
$\dot{1}\dot{2}$		$.\dot{1}\dot{2}$	$0.\dot{1}\dot{0}(+1)$	2	1	
$\dot{1}\dot{4}$		$.\dot{1}\dot{4}$	$0.1011212223\dot{2}44446\dot{6}(+4)$	7	4	
$\dot{1}\dot{6}$		$.\dot{1}\dot{6}$	$0.102132\dot{4}4\dot{5}(+2)$	3	2	
$4.\dot{3}$		$4.\dot{3}$	$0.\dot{1}24\dot{3}(+4)$	4	4	(f)
$4.\dot{7}$		$4.\dot{7}$	$0.1\dot{2}(+2)$	1	2	

Table 7.4. G-sequences of infinite recurring octal games.

$\underset{\sim}{d}_1$ \ $\underset{\sim}{d}_2$	1	2	3	4	5	6	7	$\underset{\sim}{4}.\underset{\sim}{d}_1$
0	*	$0\dot{2}$	$0\dot{2}$	$0\dot{4}$	$0\dot{5}$	$0\dot{2}$	$0\dot{2}$	*
1	*	$1\dot{2}$	$0\dot{2}$	$1\dot{4}$	*	$1\dot{6}$	$1\dot{7}$	*
2	*	$\dot{2}$	$\dot{2}$	$2\dot{4}$	$2\dot{5}$	$\dot{2}$	$\dot{2}$	$\dot{2}$
3	*	$3\dot{2}$	$\dot{2}$	$3\dot{4}$	*	$3\dot{2}$	$\dot{2}$	$4.\dot{3}$
4	*	$0\dot{2}$	$1\dot{7}$	$0\dot{2}$	$1\dot{7}$	$0\dot{2}$	$1\dot{7}$	$\dot{2}$
5	*	$5\dot{2}$	$5\dot{3}$	$5\dot{4}$	*	$5\dot{6}$	$5\dot{7}$	*
6	?	$\dot{2}$	$\dot{2}$	$\dot{2}$	$\dot{2}$	$\dot{2}$	$\dot{2}$	$\dot{2}$
7	*	$3\dot{2}$	$\dot{2}$	$7\dot{4}$	*	$3\dot{2}$	$\dot{2}$	$4.\dot{7}$

Table 7.5. Games of the form $.\underset{\sim}{d}_1\dot{\underset{\sim}{d}}_2$, $\underset{\sim}{4}.\dot{\underset{\sim}{d}}_1$.

$\underset{\sim}{d}_1$ \ $\underset{\sim}{d}_2$	0	1	2	3	4	5	6	7
0	*	*	$0\dot{2}$	$0\dot{2}$	$0\dot{4}$	*	$0\dot{2}$	$0\dot{2}$
1	*	*	$\dot{1}\dot{2}$	$0\dot{2}$	$\dot{1}\dot{4}$	*	$\dot{1}\dot{6}$	$1\dot{7}$
2	*	*	$\dot{2}$	$\dot{2}$	*	*	$\dot{2}$	$\dot{2}$
3	*	*	$3\dot{2}$	$\dot{2}$	$3\dot{4}$	*	$3\dot{2}$	$\dot{2}$
4	$0\dot{2}$	$1\dot{7}$	$0\dot{2}$	$1\dot{7}$	$0\dot{2}$	$1\dot{7}$	$0\dot{2}$	$1\dot{7}$
5	*	*	$5\dot{6}$	$5\dot{7}$	$5\dot{4}$	*	$5\dot{6}$	$5\dot{7}$
6	$\dot{2}$	$\dot{2}$	$\dot{2}$	$\dot{2}$	$\dot{2}$	$\dot{2}$	$\dot{2}$	$\dot{2}$
7	*	*	$3\dot{2}$	$\dot{2}$	$7\dot{4}$	*	$3\dot{2}$	$\dot{2}$

Table 7.6. Games of the form $.\dot{\underset{\sim}{d}}_1\dot{\underset{\sim}{d}}_2$.

Notes to Table 7.4.

(a) Duplicate Nim.

(b) Triplicate Nim. This game is equivalent to Nim in which an exact power of 2 ($2^0 = 1$) may not be taken.

(c) Nim.

(d) Double Duplicate Nim.

(e) This game is equivalent to .3F, the sedecimal game analyzed by J.C. Kenyon which has a period of 6, saltus 3.

(f) Lasker's Nim.

7.5. Arithmetico-periodic sedecimal games.

Table 7.7 contains the G-sequences of those sedecimal games that were discovered to be arithmetico-periodic. The layout of the table is identical to that of Table 7.4. G-values greater than 9 are represented, both in the table and the notes that follow, by the following symbols:

10	11	12	13	14	15	16	17	18	19	20
X	x	T	t	F	f	S	s	A	a	V

Table 7.7. *G*-sequences of sedecimal games.

	Standard Form	*G*-sequence			Period	Saltus	Notes
0A0	.13137F	.11(+1)			2	1	
0B0	.130F	.11022(+1)			2	1	
0C0	.1133777F	.111(+1)			3	1	
11B	.11B	.1110002223	3344455566	6(+4)	11	4	
128	.128	.100110(+2)			5	2	
138	.13A	.110022			2	1	
13C	.13C	.1100122332	445546677(+4)		9	4	
169	.169	.102102132(+3)			9	3	
18C	.18C	.1000222244	4466663333	8888777755	48	16	(9)
18E	.18E	.10002223(+2)			4	2	
18F	.18F	.10102223(+2)			4	2	
194	.194	.1100222244	4466663333	8888777755	48	16	(12)
19B	.19B	.1110002223	3344455566	6(+4)	10	4	
19F	.19F	.11102223(+2)			4	2	
1A0	.1A	.100122(+1)			2	1	
280	.300F	.1012010123	2345343456	7678976789	53	16	(16)
3F	.3F	.120123(+3)			6	3	
890	.10FF	.1012223(+2)			4	2	
9B0	.9B	.1100223344	5566(+4)		7	4	
9C0	.9C	.1002224446	6633388877	7555999xxX	36	16	(20)
9E0	.9E	.100223(+2)			3	2	
A80	.30FE	.1012323(+2)			4	2	
B80	.B8	.1010232345	45676(+4)		7	4	
BA0	.BA	.102(+1)			1	1	
BB	.BB	.1203(+1)			1	1	
BC	.BC	.101232(+2)			3	2	
C9	.17FE	.1122(+2)			3	2	
F8	.F8	.1010232345	45678(+4)		6	4	
FB	.FB	.12304(+1)			1	1	
FC	.FC	.1012324546	7(+4)		5	4	

Notes to Table 7.7.

(9) There are only four irregular values. $G(0) = 0$; $G(1) = 1$; $G(16) = 6$; $G(36) = 9$; otherwise, the G-sequence is

$$\begin{array}{l} \mathtt{0.\dot{0}000\ 2222\ 4444\ 666\bar{4}\ 3333\ 8888} \\ \quad\mathtt{7777\ 5555\ 9991\ xxxX\ tttt\ fff\dot{F}(+16),} \end{array}$$

where the entry $\bar{4}$ means that for $k \geq 1$, $G(16+48k) = 16k-4$.

(12) There are only seven irregular values. $G(0) = 0$; $G(1) = G(2) = 1$; $G(15) = G(16) = 6$; $G(35) = G(36) = 9$; otherwise, the G-sequence is

$$\begin{array}{l} \mathtt{0.\dot{0}000\ 2222\ 4444\ 66\bar{4}\bar{4}\ 3333\ 8888} \\ \quad\mathtt{7777\ 5555\ 9911\ xxXX\ tttt\ ffF\dot{F}(+16),} \end{array}$$

where $\bar{4}$ means that for $k \geq 1$, $G(15+48k) = G(16+48k) = 16k-4$.

(16) There are no irregular values. The G-sequence is

$$\begin{array}{l} \mathtt{\dot{0}10120101232345343456789.} \\ \mathtt{7678989XxXxTXxXxTtFfFftFfF\dot{f}(+16).} \end{array}$$

(20) There are only four irregular values. $G(0) = 0$; $G(1) = 1$; $G(12) = 6$; $G(27) = 9$; otherwise the G-sequence is

$$\begin{array}{l} \mathtt{0.\dot{0}00\ 222\ 444\ 66\bar{4}\ 333\ 888} \\ \quad\mathtt{777\ 555\ 991\ xxX\ ttt\ ff\dot{F}(+16),} \end{array}$$

where $\bar{4}$ means that for $k \geq 1$, $G(12+36k) = 16k-4$.

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