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Rational Points on Affinoid Adic Spaces

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Rational Points on Affinoid Adic Spaces" submitted by Tracy Walker in partial fulfillment of the requirements for the degree of Master of Science.

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Abstract

This thesis proves the following result: Let K be a p-adic field (see Definition 1.10.1) with valuation ring K° (see Definition 1.1.13). Let X be an integral affine scheme (see Definition 4.5.2) finitely generated over $\mathrm{Sch}(K^\circ)$ such that the special fibre of X (see Definition 4.9.2) is also an integral affine scheme (see Definition 4.5.10). Let $\mathcal{X}$ denote the formal scheme (see Definition 4.8.1) obtained by completing X along the special fibre of X. Let $\mathfrak{X}$ denote the affinoid adic space over K associated to the formal scheme $\mathcal{X}$ (see Definition 5.5.1). Then there is a canonical bijection between the set of K°-rational points on X and the set of K-rational points on $\mathfrak{X}$.

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To Mom, Dad, Jeff and Shannon.

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Introduction

Rigid analytic spaces and representation theory of p-adic groups are widely researched areas in mathematics, especially in the Langlands program. Michael Harris and Richard Taylor used rigid analytic geometry to solve certain cases of Langlands' conjectures. The category of rigid analytic spaces is a subcategory of the category of adic spaces.

In the thesis, I look at two categories (affine schemes and affinoid adic spaces) and describe a relationship between them. In particular, given an affine scheme I describe how to associate an affinoid adic space to the affine scheme and describe a relationship between these two objects. A lot of research has been done on the category of affine schemes but the category of affinoid adic spaces has only been studied in the last ten years, primarily by Roland Huber. Thus, the machinery to move from the category of affine schemes to the category of affinoid adic spaces will quickly advance the research on affinoid adic spaces since "many of the basic results of the étale cohomology of schemes also hold for the étale cohomology of adic spaces [10]".

The following theorem is the main result of this thesis which, to the best of my knowledge, does not appear in the literature:

Theorem 6.0.5 Let K be a p-adic field (see Definition 1.10.1) with valuation ring K° (see Definition 1.1.13). Let X be an integral affine scheme (see Definition 4.5.2) finitely generated over $\mathrm{Sch}(K^\circ)$ such that the special fibre of X (see Definition 4.9.2) is also an integral affine scheme (see Definition 4.5.10). Let $\mathcal{X}$ denote the formal scheme (see Definition 4.8.1) obtained by completing X along the special fibre of X.

Let $\mathfrak{X}$ denote the affinoid adic space over K associated to the formal scheme $\mathcal{X}$ (see Definition 5.5.1). Then there is a canonical bijection

$$\mathrm{Hom}_{\mathrm{Sch}(K^\circ)}(\mathrm{Sch}(K^\circ), X) \equiv \mathrm{Hom}_{\mathrm{Adic}(K,K^\circ)}(\mathrm{Adic}(K, K^\circ), \mathfrak{X}).$$

The above geometric theorem is a consequence of the following algebraic theorem:

Theorem 3.0.8 Let K be a p-adic field with valuation ring K° and residue field κ. Let A be an integral domain and let $\sigma : K^\circ \longrightarrow A$ be a ring homomorphism such that $A \otimes_{K^\circ} \kappa$ is an integral domain. Let $\hat{A}$ be the completion of A with respect to I (see Definition 2.1.1) where I is the kernel of $\rho_A : A \longrightarrow A \otimes_{K^\circ} \kappa$ defined by $\rho_A(a) = a \otimes 1$. Then $(\hat{A} \otimes_{K^\circ} K, \hat{A})$ is an affinoid ring (see Definition 2.8.1) and the map

$$\begin{aligned} \mathrm{Hom}_{(K,K^\circ)}\left((\hat{A} \otimes_{K^\circ} K, \hat{A}), (K, K^\circ)\right) &\longrightarrow \mathrm{Hom}_{K^\circ}(A, K^\circ) \\ (\varphi^\triangleright, \varphi^+) &\longmapsto \varphi^+ \circ \alpha \end{aligned}$$

is bijective, where $\alpha : A \longrightarrow \hat{A}$ is the unique morphism guaranteed by the universal property of inverse limits (see Proposition A.2.2).

Notice that the geometric result requires an extra condition - that X is finitely generated over $\mathrm{Sch}(K^\circ)$. Thus, Theorem 3.0.8 implies Theorem 6.0.5 but the converse does not necessarily hold.

Since these results are fairly technical, a large proportion of this thesis will be devoted to defining the various terms appearing in the theorems above. Specifically, p-adic fields, adic rings, f-adic rings, Tate rings and affinoid rings, affine schemes, formal schemes and affinoid adic spaces will be discussed.

In this thesis we use the term 'ring' to indicate a commutative ring with unity and 'ring homomorphism' to indicate a ring homomorphism of commutative rings such that the identity is mapped to the identity.

Part I

Algebra

Chapter 1

p-adic Fields

For each prime number p, the field $\mathbb{Q}_p$ is constructed by completing the rational numbers with respect to the p-adic valuation v_p. A p-adic field is, by definition, a finite field extension of $\mathbb{Q}_p$. This chapter is devoted to creating a general understanding of the p-adic fields (as found in Gouvêa [7]).

1.1 Valuations

Definition 1.1.1. *A* totally ordered abelian group *is a non-empty set* Γ *with binary operations* $\cdot$ *and* $<$ *such that* $\{\Gamma, \cdot\}$ *is an abelian group and* $\{\Gamma, <\}$ *is a totally ordered set such that for all* $x, y, z \in \Gamma$, $x < y$ *implies* $xz < yz$.

Example 1.1.2. *The set of positive real numbers is a totally ordered abelian group (with multiplication).*

Definition 1.1.3. *Let* A *be a ring and* Γ *be a totally ordered abelian group. Let* $\Gamma_0 := \{0\} \cup \Gamma$. *Set* $0 < \gamma$, $\gamma \cdot 0 = 0$, *and* $0 \cdot \gamma = 0$ *for all* $\gamma \in \Gamma$. *Since* Γ *is totally ordered,* Γ_0 *is totally ordered. A* valuation *on a ring* A *is a function* $v : A \longrightarrow \Gamma_0$ *that satisfies the following conditions*

i) $v(0) = 0$

ii) $v(1) = 1$

iii) $v(xy) = v(x)v(y)$ *for all* $x, y \in A$

iv) $v(x+y) \leq v(x) + v(y)$ *for all* $x, y \in A$.

If, in addition, v *satisfies the following condition*

v) $v(x+y) \leq \max\{v(x), v(y)\}$ *for all* $x, y \in A$

then v *is said to be* non-archimedean; *otherwise,* v *is called* archimedean. *Note that the conditions listed above are not independent: condition* (v) *implies condition* (iv) *and condition* (iii) *implies condition* (ii).

The pair (A, v) *is called a* valued ring. *The group generated by the image of* v *in* Γ *is called the* valuation group *of* v *and is denoted by* Γ_v. *If the valuation group is finitely generated then the rank of* v *is defined to be the rank of the valuation group. If* Γ_v *is not finitely generated, then the rank of* v *is said to be infinite.*

Remark 1.1.4. *Valuations can also be defined additively. In other words, let* Γ *be an ordered abelian group written additively, and let* $\Gamma \cup \{\infty\} =: \Gamma_\infty$. *Set* $\infty > \gamma$, $\gamma + \infty = \infty$, *and* $\infty + \gamma = \infty$ *for all* $\gamma \in \Gamma$. *Since* Γ *is totally ordered,* Γ_∞ *is totally ordered. The function* $ord : A \longrightarrow \Gamma_\infty$ *is an additive valuation if*

i) $ord(0) = \infty$

ii) $ord(1) = 0$

iii) $ord(xy) = ord(x) + ord(y)$

iv) $ord(x+y) \geq ord(x)ord(y)$

and is non-archimedean if

v) $ord(x+y) \geq min\{ord(x), ord(y)\}$.

Example 1.1.5.

1. *Let* A *be a ring,* u *be a rank-1 multiplicative valuation of* A, *and* γ *be a generator for* Γ_u. *For each* $a \in A$ *with* $u(a) \neq 0$, *let* $n(a)$ *be the unique integer* n *such*

that $u(a) = \gamma^n$. *Define* $v : A \to \mathbb{N} \cup \{\infty\}$ *by* $v(a) = n(a)$ *if* $u(a) \neq 0$ *and* $v(a) = \infty$ *otherwise. Then* v *is an additive valuation.*

2. *Conversely, let* v *be an additive rank-1 valuation of* A *and let* g *be a generator for* Γ_v. *For each* $a \in A$ *with* $v(a) \neq \infty$ *let* $m(a)$ *be the unique integer* m *such that* $v(a) = mg$. *Define* $w : A \to \mathbb{N} \cup \{0\}$ *by* $w(a) = m(a)$ *if* $v(a) \neq \infty$ *and* $w(a) = 0$ *otherwise. Then* w *is a multiplicative valuation.*

Remark 1.1.6. *In this thesis all valuations are multiplicative and non-archimedean unless otherwise noted.*

Example 1.1.7.

1. *Let* A *be a ring. Then*

$$v_0(x) = \begin{cases} 1 & \text{if } x \neq 0_A \\ 0 & \text{if } x = 0_A \end{cases}$$

is a non-archimedean rank zero valuation called the trivial valuation.

2. *Let* $A = \mathbb{R}$. *Then*

$$v_\infty(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

is an archimedean valuation. Note that the rank of v_∞ *is infinite since the valuation group is* $\mathbb{R}^{>0}$ *which is not finitely generated.*

3. *Let* $A = \mathbb{Z}$ *and* $p \in \mathbb{Z}$ *be prime. Then*

$$ord_p(x) = \begin{cases} n & \text{where } x = mp^n \text{ and } p \nmid m \\ \infty & \text{if } x = 0 \end{cases}$$

is a non-archimedean additive valuation.

4. *Let $A = \mathbb{Z}$ and p be prime. Then*

$$v_p(x) = \begin{cases} p^{-ord_p(x)} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is a non-archimedean valuation called the p-adic valuation. *The p-adic valuation on $\mathbb{Z}$ can be extended to a p-adic valuation on $\mathbb{Q}$ by defining $v_p\left(\frac{a}{b}\right) = \frac{v_p(a)}{v_p(b)}$.*

5. *Let A be a ring and let I be a prime ideal of A. Then*

$$v_I(x) = e^{-\max\{n\in\mathbb{N} \mid x\in I^n\}}$$

is a non-archimedean valuation called the I-adic valuation. *If the base e is replaced with any real number greater than 1, the resulting valuation is equivalent to the I-adic valuation (see Definition 1.2.3). If $A = \mathbb{Z}$ and $I = (p)$, take $e = p$ then $v_I = v_p$. Now $ord_I(x) = \max\{n \in \mathbb{N} \mid x \in I^n\}$ is called the* I-adic additive valuation.

6. *Let $A = \mathbb{Z}$ and let p be prime. Then*

$$v_{p,0}(x) = \begin{cases} 1 & p \nmid x \\ 0 & p \mid x \end{cases}$$

is a non-archimedean valuation. Notice that $v_{p,0} \neq v_p$ as shown in Example 1.1.10.

Remark 1.1.8.

1. *If v is a rank zero valuation, then v is not necessarily the trivial valuation. To see this, fix a prime p. If $v = v_{p,0}$ (see Example 1.1.7(6)), then $\Gamma_v = \{1\}$ so v is rank zero, however, v is not the trivial valuation.*

2. *Note that if v_I is a valuation then I is a prime ideal of A. To see this suppose v_I is a valuation and consider part (iii) of the Definition 1.1.3. Suppose $x, y \in A$ and $xy \in I$. Then by Definition 1.1.7 (5), $v_I(x) \leq 1$, $v_I(y) \leq 1$, and $v_I(xy) < 1$. Thus, $v_I(x) < 1$ or $v_I(y) < 1$ which implies $v_I(x) \in I$ or $v_I \in I$, so I is prime.*

Definition 1.1.9. *The subset $v^{-1}(0)$ of A is called the* support *of v and is denoted by* supp(v).

Example 1.1.10.

1. $\mathrm{supp}(\mathrm{v_p}) = (0)$
2. $\mathrm{supp}(\mathrm{v_{p,0}}) = (\mathrm{p})$
3. $\mathrm{supp}(\mathrm{v_0}) = (0)$
4. $\mathrm{supp}(\mathrm{v_\infty}) = (0)$
5. $\mathrm{supp}(\mathrm{v_I}) = (0)$

Proposition 1.1.11. *The support of v is a prime ideal of A.*

Proof. Suppose $s \in \mathrm{supp(v)}$ and $a \in A$. Then $v(sa) = v(s)v(a) = 0(v(a)) = 0$. Now, $sa \in \mathrm{supp(v)}$. Similarly $as \in \mathrm{supp(v)}$. Now if $s, t \in \mathrm{supp(v)}$, then $v(s+t) \leq v(s) + v(t) = 0$. Therefore, supp(v) is an ideal. To show that it is prime let $ab \in \mathrm{supp(v)}$. Then $v(ab) = v(a)v(b) = 0$. So $v(a) = 0$ or $v(b) = 0$. Therefore, $a \in \mathrm{supp(v)}$ or $b \in \mathrm{supp(v)}$, and therefore supp(v) is a prime ideal. $\square$

Proposition 1.1.12. *Let K be a field and v be a non-archimedean valuation on K. Then $K^\circ = \{x \in K \mid v(x) \leq 1\}$ is a subring of K and $\mathfrak{p}_K = \{x \in K \mid v(x) < 1\}$ is a maximal ideal of K°.*

Proof. By definition $v(0) = 0$ and $v(1) = 1$. Therefore, $0, 1 \in K^\circ$. Suppose $x, y \in K^\circ$. Then $v(x+y) \leq \max\{v(x), v(y)\} \leq 1$. Thus, $x+y \in K^\circ$. Now $v(xy) = v(x)v(y)$ and by Definition 1.1.1 $v(x)v(y) \leq 1$. Therefore, $xy \in K^\circ$. Hence, K° is a subring of K. Following the same arguments, $\mathfrak{p}_K$ is a subring of K°. Now let $x \in K^\circ$ and $y \in \mathfrak{p}_K$. Then $v(xy) = v(x)v(y)$ and by Definition 1.1.1 $v(x)v(y) < v(x) \leq 1$. Hence, $\mathfrak{p}_K$ is an ideal of K°. To show $\mathfrak{p}_K$ is maximal consider the quotient ring $K^\circ/\mathfrak{p}_K$. Consider $x + \mathfrak{p}_K \in K^\circ/\mathfrak{p}_K$. Then $x + \mathfrak{p}_K \neq 0$ if and only if $v(x) = 1$. Let $x + \mathfrak{p}_K \neq 0$. Since $x \in K^\circ$, $x \in K$, so x^{-1} exists since K is a field. Since $v(xx^{-1}) = 1$ implies $v(x^{-1}) = 1$, $x^{-1} \in K^\circ$. Thus, $x^{-1} + \mathfrak{p}_K \in K^\circ/\mathfrak{p}_K$ and $x^{-1} + \mathfrak{p}_K \neq 0$. Therefore, $K^\circ/\mathfrak{p}_K$ is a field so $\mathfrak{p}_K$ is maximal. □

Definition 1.1.13. *Let K be a field and v be a non-archimedean valuation on K. Then the subring $K_K^\circ = \{x \in K \mid v(x) \leq 1\}$ is called the* valuation ring *of v. The maximal ideal $\mathfrak{p}_K = \{x \in K \mid v(x) < 1\}$ is called the* valuation ideal *of v, and the quotient field $\kappa_K = K^\circ/\mathfrak{p}_K$ is called the* residue field *of v.*

Example 1.1.14. *Let $K = \mathbb{Q}$. If v is the trivial valuation, then $K_K^\circ = \mathbb{Q}$ and $\kappa_K = \mathbb{Q}$.*

1.2 Equivalent Valuations

Recall that a base for a topology is defined as follows: Let X be a set with a topology $\mathcal{T}$. Let $\mathcal{B}$ be a collection of open sets and let $x \in X$. Let $\mathcal{T}_x := \{U \in \mathcal{T} \mid x \in U\}$ and $\mathcal{B}_x := \{V \in \mathcal{B} \mid x \in V\}$. Then $\mathcal{B}$ is a *base* for the topology if for all $x \in X$ and for all $U \in \mathcal{T}_x$ there exists $V \in \mathcal{B}_x$ such that $V \subseteq U$.

Let A be a ring equipped with a topology. Recall that A is a *topological ring* if $f : A \to A$ defined by $f(x) = a + bx$ is continuous for all $x \in A$ and for all $a, b \in A$.

Definition 1.2.1. *Let $v : A \to \Gamma \cup \{0\}$ be a valuation. The* topology induced on A by v *is the coarsest topology on A making A a topological ring and such that $v^{-1}\{x \in \Gamma \cup \{0\} \mid x < \gamma\}$ is open in A for each $\gamma \in \Gamma$.*

Example 1.2.2. *Let A be a ring and I an ideal of A. Notice that if $I = (0)$, the I-adic valuation (see Example 1.1.7 (5)) on A is the trivial valuation. The topology for A induced by the trivial valuation (see Example 1.1.7 (1)) is the discrete topology.*

Definition 1.2.3. *Let A be a ring. If v and w are valuations on A, then v and w are* equivalent *if they define the same topology on A.*

Theorem 1.2.4 (Ostrowski). *Every non-trivial valuation on $\mathbb{Q}$ with $\Gamma = \mathbb{R}^{>0}$ is equivalent to one of the valuations v_p, where either p is a prime number or $p = \infty$.*

Proof. [7, Theorem 3.1.3] Let v be a non-trivial valuation on $\mathbb{Q}$. We divide the proof into two cases: when the valuation is archimedean and when the valuation is non-archimedean.

Case 1: Suppose v is archimedean. We want to show that v is equivalent to v_∞. Let n_0 be the least positive integer for which $v(n_0) > 1$. Such an integer exists since v is archimedean. Find $\alpha \in \mathbb{R}^{\geq 0}$ such that $v(n_0) = n_0^\alpha$. By Gouvêa [7, Lemma 3.1.2], the valuations v_1 and v_2 are equivalent if there exists a positive real number α such that for every $x \in \mathbb{Q}$ we have $v_1(x) = v_2^\alpha(x)$. In other words, it will be shown that $v(n) = n^\alpha$ for any $n \in \mathbb{Z}^{\geq 0}$. This is clearly true for $n = n_0$. If $n \neq n_0$ write n in base n_0 notation: $n = a_0 + a_1 n_0 + a_2 n_0^2 + \cdots + a_k n_0^k$ where $0 \leq a_i \leq n_0 - 1$ and $a_k \neq 0$.

Now $n_0^k \le n < n_0^{k+1}$ implies $k = \left\lfloor \frac{\log n}{\log n_0} \right\rfloor$. Then

$$\begin{aligned} v(n) &= v(a_0 + a_1 n_0 + a_2 n_0^2 + \cdots + a_k n_0^k) \\ &\le v(a_0) + v(a_1) n_0^\alpha + v(a_2) n_0^{2\alpha} + \cdots + v(a_k) n_0^{k\alpha}. \end{aligned}$$

Since n_0 is the smallest integer such that $v(n_0) > 1$, $v(a_i) \le 1$ for all i. We now have

$$\begin{aligned} v(n) &\le 1 + n_0^\alpha + n_0^{2\alpha} + \cdots + n_0^{k\alpha} \\ &= n_0^{k\alpha}(1 + n_0^{-\alpha} + n_0^{-2\alpha} + \cdots + n_0^{-k\alpha}) \\ &\le n_0^{k\alpha} \sum_{i=0}^{\infty} n_0^{-i\alpha} \\ &= n_0^{k\alpha} \left(\frac{n_0^\alpha}{n_0^\alpha - 1} \right). \end{aligned}$$

Let $C = \frac{n_0^\alpha}{n_0^\alpha - 1} > 0$. Then $v(n) \le C n_0^{k\alpha} \le C n^\alpha$. This is true for every $n \in \mathbb{Z}^{\ge 0}$. Thus $v(n^N) \le C n^{N\alpha}$ which implies $v(n) \le \sqrt[N]{C} n^\alpha$. Now, as $N \to \infty$, $\sqrt[N]{C} \to 1$. Thus, $v(n) \le n^\alpha$.

To obtain the other inequality consider again $n = a_0 + a_1 n_0 + a_2 n_0^2 + \cdots + a_k n_0^k$. Now

$$\begin{aligned} n_0^{(k+1)\alpha} &= v(n_0^{k+1}) \\ &= v(n + n_0^{k+1} - n) \\ &\le v(n) + v(n_0^{k+1} - n). \end{aligned}$$

Thus,

$$\begin{aligned} v(n) &\ge n_0^{(k+1)\alpha} - v(n_0^{k+1} - n) \\ &\ge n_0^{(k+1)\alpha} - (n_0^{k+1} - n)^\alpha \\ &= n_0^{(k+1)\alpha} - \left(1 - \left(1 - \frac{1}{n_0}\right)^\alpha\right). \end{aligned}$$

We now pull the same trick as before. Let $C' = 1 - \left(1 - \frac{1}{n_0}\right)^{\alpha}$. Then $C'n_0^{(k+1)\alpha} \geq C'n^{\alpha}$ which is true for all n. Thus $v(n^N) \geq C'n^{N\alpha}$, which implies $v(n) \geq \sqrt[N]{C'}n^{\alpha}$. Now, as $N \to \infty$, $\sqrt[N]{C} \to 1$. Thus, $v(n) \geq n^{\alpha}$. Now, $v(n) = n^{\alpha}$ as required. Therefore, v is equivalent to v_{∞}.

Now suppose v is non-archimedean. Then $v(n) \leq 1$ for all $n \in \mathbb{Z}$. Since v is non-trivial, there exists a smallest integer n_0 such that $v(n_0) < 1$. We claim that n_0 is prime. Suppose $n_0 = ab$ where $a, b < n_0$. Then $v(a) = v(b) = 1$ since n_0 is the smallest integer such that $v(n_0) < 1$. This is a contradiction. Hence, n_0 is prime. Set $p = n_0$. We will show that v is equivalent to v_p. Suppose $p \not| n$ where $n \in \mathbb{Z}$. Then $n = rp + s$ and $0 < s < p$. By the minimality of p, $v(s) = 1$. Now, $v(rp) < 1$ since v is non-archimedean and $v(p) < 1$. Now since all triangles are isosceles when v is non-archimedean [7, Corollary 2.3.4], $v(n) = 1$. Given any $m \in \mathbb{Z}$, write $m = p^u n'$ where $p \not| n'$. Then $v(m) = (v(p))^u v(n') = (v(p))^u$. Thus, v is equivalent to the p-adic valuation as claimed. $\square$

1.3 p-adic Expansion

Let p be a prime number and let a be an arbitrary but fixed integer. Using the Quotient Remainder Theorem consider

$$\begin{aligned}
a &= q_0 p + r_0, & 0 \leq r_0 < p \\
q_0 &= q_1 p + r_1, & 0 \leq r_1 < p \\
&\vdots & \\
q_n &= q_{n+1} p + r_{n+1}, & 0 \leq r_{n+1} < p.
\end{aligned}$$

In other words, consider the sequences

- $q_0,\ q_1,\ q_2, \ldots$ defined by $q_{n+1} = (q_n \text{ div } p)$, $q_0 = a \text{ div } p$
- $r_0,\ r_1,\ r_2, \ldots$ defined by $r_{n+1} = (q_n \bmod p)$, $r_0 = a \bmod p$.

Definition 1.3.1. *Given an integer a, the formal Laurent series* $\sum_{k=0}^{\infty} r_k p^k$ *where* r_k *is defined above is called the p-*adic expansion *of a.*

Lemma 1.3.2. *Given any* $a \in \mathbb{Z}$, *the p-adic expansion as defined in Definition 1.3.1 converges to a with respect to* v_p.

Proof. We will show that for all $a \in \mathbb{Z}$ and for all $\epsilon \in \mathbb{Q}^+$ there exists an $N \in \mathbb{N}$ such that $v_p\left(a - \sum_{k=0}^{n} r_k p^k\right) < \epsilon$ for all $n > N$. Let $a \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then

$$\begin{aligned} a &= q_0 p + r_0 \\ &= q_1 p^2 + r_1 p + r_0 \\ &= \ldots \\ &= q_m p^{m+1} + \sum_{k=0}^{m} r_k p^k. \end{aligned}$$

Fix $\epsilon > 0$. Let $N \in \mathbb{N}$ be such that $q_n \in \{0, 1\}$ and $p^{-n} < \epsilon$ for all $n > N$. Then $a = \sum_{k=0}^{m} r_k p^k$ and hence $v_p(a - \sum_{k=0}^{m} r_k p^k) = v_p(0) = 0 < \epsilon$. □

Remark 1.3.3. *With the above arugument, we can now write* $x = \sum_{k=0}^{m} r_k p^k$. *Note that this equality in* $\mathbb{Q}$ *refers to the toppology for* $\mathbb{Q}$ *induced by* v_p.

Lemma 1.3.4. *Any rational number a can be represented as a formal Laurent series* $\sum_{k=-i}^{\infty} r_k p^k$ *where* $0 \leq r_k < p$.

Proof. Suppose $a\frac{1}{y}$ where $y \in \mathbb{Z}$ and $p \not| \, y$. Then $y = \sum_{k=0}^{\infty} s_k p^k$ by Remark 1.3.3 and $s_0 \neq 0$ so $v_p\left(\sum_{k=0}^{\infty} s_k p^k\right) = 1$. Define $\sum_{k=0}^{\infty} t_k p^k$ by

$$t_0 = s_0^{-1} \pmod{p}$$

which gives

$$s_0 t_0 = 1 + q_0 p$$

for a unique $q_0 \in \mathbb{Z}$; and for $n \geq 1$

$$t_n = -s_0^{-1}\left(\sum_{j=0}^{n-1} s_{n-j} t_j + q_{n-1}\right)$$

which gives

$$-1 + \sum_{k=0}^{n}\left(\sum_{i+j=n} s_i t_j\right) p^k = q_n p^{n+1}$$

for some unique $q_n \in \mathbb{Z}$. Then

$$\left(\sum_{k=0}^{n} s_k p^k\right)\left(\sum_{k=0}^{n} t_k p^k\right) = \sum_{k=0}^{n}\left(\sum_{i+j=k} s_i t_j\right) p^k = 1 + q_n p^{n+1}.$$

Now,

$$\begin{aligned}
&\lim_{n\to\infty} v_p\left(\frac{1}{\sum_{k=0}^{n} s_k p^k} - \sum_{k=0}^{n} t_k p^k\right) \\
&= \lim_{n\to\infty} v_p\left(\frac{1}{\sum_{k=0}^{n} s_k p^k}\left(1 - \left(\sum_{k=0}^{n} s_k p^k\right)\left(\sum_{k=0}^{n} t_k p^k\right)\right)\right) \\
&= \lim_{n\to\infty} v_p\left(\frac{1}{\sum_{k=0}^{n} s_k p^k}\right) v_p\left(1 - \left(\sum_{k=0}^{n} s_k p^k\right)\left(\sum_{k=0}^{n} t_k p^k\right)\right) \\
&= \lim_{n\to\infty} v_p\left(1 - \left(\sum_{k=0}^{n} s_k p^k\right)\left(\sum_{k=0}^{n} t_k p^k\right)\right) \\
&= \lim_{n\to\infty} v_p(1 - 1 + q_n p^{n+1}) \\
&= 0.
\end{aligned}$$

Thus, we can now write $\frac{1}{y} = \sum_{k=0}^{\infty} t_k p^k$ as defined above. Therefore, if $a = p^l \frac{x}{y} \in \mathbb{Q}$ where $l \in \mathbb{Z}$, then $x = \sum_{k=0}^{n} r_k p^k$ and $y = \sum_{k=0}^{n} t_k p^k$ by Remark 1.3.3. Therefore,

$$a = p^l \sum_{k=0}^{n} r_k p^k \sum_{k=0}^{n} t_k p^k = \sum_{k \in \mathbb{Z}}^{\infty} a_k p^k$$

where $\sum_{k}^{\infty} t_k p^k$ is defined above and $a_k = p^l \sum_{i+j=k} r_i t_j$. □

1.4 Completions of $\mathbb{Q}$

It is well known that the real numbers are formed by completing the rational numbers with respect to v_∞. This is done by adding limit points of Cauchy sequences to the rational numbers. The valuation v_∞ extends to $\mathbb{R}$, the topological ring $\mathbb{R}$ is complete with respect to this valuation, and $\mathbb{Q}$ is dense in $\mathbb{R}$ with respect to v_∞. However, $\mathbb{R}$ is not the only complete topological field containing $\mathbb{Q}$ as a dense subfield.

Definition 1.4.1. *Let v be a non-trivial valuation on $\mathbb{Q}$ with $\Gamma = \mathbb{R}^{>0}$ (see Theorem 1.2.4). Let $\mathcal{C}_v(\mathbb{Q})$ denote the set of all Cauchy sequences in $\mathbb{Q}$ with respect to v. Let $\mathcal{N}_v(\mathbb{Q})$ denote the set of null sequences in $\mathbb{Q}$ with respect to v (i.e. $(a_n)_{n\in\mathbb{N}} \in \mathcal{N}_v(\mathbb{Q})$ implies $\lim_{n\in\mathbb{N}} v(a_n) = 0$ in $\mathbb{R}$).*

Lemma 1.4.2. *If $a \in \mathcal{C}_v(\mathbb{Q})$, then*

1. *there is some $M_a \in \mathbb{R}^{>0}$ such that $v(a_n) < M_a$ for all $n \in \mathbb{N}$*
2. *$\lim_{n\to\infty} v(a_n)$ exists.*

Proof. Since a is Cauchy in $\mathbb{Q}$, $\{v(a_n) \mid n \in \mathbb{N}\}$ is Cauchy in $\mathbb{R}^{\geq 0}$, so $\{v(a_n) \mid n \in \mathbb{N}\}$ is Cauchy in $\mathbb{R}$. Since Cauchy sequences in $\mathbb{R}$ are bounded [12, Lemma 10.10], $v(a_n)$ is bounded and since Cauchy sequence are convergent [12, Theorem 10.11], $v(a_n)$ is convergent. □

Proposition 1.4.3. *The set $\mathcal{C}_v(\mathbb{Q})$ is a ring and $\mathcal{N}_v(\mathbb{Q})$ is a maximal ideal of $\mathcal{C}_v(\mathbb{Q})$.*

Proof. Pick $a = (a_n)_{n\in\mathbb{N}}, b = (b_n)_{n\in\mathbb{N}} \in \mathcal{C}_v(\mathbb{Q})$. Let $c = (c_n)_{n\in\mathbb{N}}$ where $c_n = a_n b_n$. Fix $\epsilon > 0$. Let M_a and M_b be as in Lemma 1.4.2 (1). Take N_1 such that if $n, m > N_1$, then $v(a_n - a_m) < \frac{\epsilon}{2M_b}$ and N_2 such that if $n, m > N_1$, then $v(b_n - b_m) < \frac{\epsilon}{2M_a}$. Let $N = \max\{N_1, N_2\}$. Suppose $n, m > N$. Then

$$\begin{aligned} v(c_n - c_m) &= v(a_n b_n + a_n b_m - a_n b_m - a_m b_m) \\ &\leq v(a_n)v(b_n - b_m) + v(b_m)v(a_n - a_m) \\ &< M_a \cdot \frac{\epsilon}{2M_a} + M_b \cdot \frac{\epsilon}{2M_b} \\ &= \epsilon. \end{aligned}$$

Therefore, if $a, b \in \mathcal{C}_v(\mathbb{Q})$, then $ab \in \mathcal{C}_v(\mathbb{Q})$. Now choose N_3 such that $v(a_n - a_m) < \frac{\epsilon}{2}$ for all $n, m > N_3$ and N_4 such that $v(b_n - b_m) < \frac{\epsilon}{2}$ for all $n, m > N_4$. Let $N' = \max\{N_3, N_4\}$. Let $c_n = a_n + b_n$ and suppose $n, m > N'$. Then

$$\begin{aligned} v(c_n - c_m) &= v(a_n + b_n - a_m - b_m) \\ &\leq v(a_n - a_m) + v(b_n - b_m) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Therefore, $\mathcal{C}_v(\mathbb{Q})$ is closed under addition. Clearly $(0)_{n\in\mathbb{N}}$ is in $\mathcal{C}_v(\mathbb{Q})$. Therefore, $\mathcal{C}_v(\mathbb{Q})$ is a ring.

To show $\mathcal{N}_v(\mathbb{Q})$ is a subring, notice that $(0)_{n\in\mathbb{N}} \in \mathcal{N}_v(\mathbb{Q})$. Let $a = (a_n)_{n\in\mathbb{N}}, b =$

$(b_n)_{n\in\mathbb{N}} \in \mathcal{N}_v(\mathbb{Q})$. Let $c = (c_n)_{n\in\mathbb{N}}$ where $c_n = a_n + b_n$. Then

$$\begin{aligned}\lim_{n\to\infty} v(c_n) &= \lim_{n\to\infty} v(a_n + b_n) \\ &\le \lim_{n\to\infty} v(a_n) + \lim_{n\to\infty} v(b_n) \\ &= 0.\end{aligned}$$

Thus, $\mathcal{N}_v(\mathbb{Q})$ is closed under addition. To show $\mathcal{N}_v(\mathbb{Q})$ is closed under multiplication suppose $c_n = a_n b_n$. Then

$$\begin{aligned}\lim_{n\to\infty} v(c_n) &= \lim_{n\to\infty} v(a_n b_n) \\ &\le \lim_{n\to\infty} v(a_n) \lim_{n\to\infty} v(b_n) \\ &= 0.\end{aligned}$$

Thus, $\mathcal{N}_v(\mathbb{Q})$ is a subring. To show that it is an ideal suppose $a = (a_n)_{n\in\mathbb{N}} \in \mathcal{N}_v(\mathbb{Q})$, $b = (b_n)_{n\in\mathbb{N}} \in \mathcal{C}_v(\mathbb{Q})$, and $c = (c_n)_{n\in\mathbb{N}}$ where $c_n = a_n b_n$. Let M_b be as in Lemma 1.4.2 (1). Then

$$\begin{aligned}\lim_{n\to\infty} v(c_n) &= \lim_{n\to\infty} v(a_n b_n) \\ &= \lim_{n\to\infty} v(a_n) v(b_n) \\ &< \lim_{n\to\infty} v(a_n) M_b \\ &= 0.\end{aligned}$$

Thus, $\mathcal{N}_v(\mathbb{Q})$ is an ideal. Now, to show $\mathcal{N}_v(\mathbb{Q})$ is maximal suppose $\mathcal{N}_v(\mathbb{Q}) \subset I \subseteq \mathcal{C}_v(\mathbb{Q})$ and let $c = (c_n)_{n\in\mathbb{N}} \in I$, $c \notin \mathcal{N}_v(\mathbb{Q})$. Let $L = \lim_{n\in\mathbb{N}} v(c_n)$. This limit exists by Lemma 1.4.2 (2) and is not equal to zero since $c \notin \mathcal{N}_v(\mathbb{Q})$. Therefore, there exists $N \in \mathbb{N}$ such that for all $n > N$, $0 \neq L - \delta < v(c_n) < L + \delta$. Thus, for $n > N$,

$v(c_n^{-1}) < (L-\delta)^{-1} := m^{-1} < 1$. Now define $b = (b_n)_{n\in\mathbb{N}}$ by

$$b_n = \begin{cases} 0 & \text{if } c_n = 0 \\ c_n^{-1} & \text{if } c_n \neq 0 \end{cases}$$

and $a = (a_n)_{n\in\mathbb{N}}$ by

$$a_n = \begin{cases} 1 & \text{if } c_n = 0 \\ 0 & \text{if } c_n \neq 0. \end{cases}$$

Now, $a \in \mathcal{N}_v(\mathbb{Q}) \subset \mathcal{C}_v(\mathbb{Q})$. Let $n, m > N$. Then

$$\begin{aligned} v(b_n - b_m) &= v(c_n^{-1} - c_m^{-1}) \\ &= v(c_n - c_m)v((c_n c_m)^{-1}) \\ &< \epsilon \cdot m^{-1} \\ &< \epsilon. \end{aligned}$$

Therefore $b \in \mathcal{C}_v(\mathbb{Q})$ and $(1)_{n\in\mathbb{N}} = cb + a \in I \Rightarrow I = \mathcal{C}_v(\mathbb{Q})$, Hence, $\mathcal{N}_v(\mathbb{Q})$ is maximal. $\square$

Definition 1.4.4. *Let $\mathbb{Q}_v$ denote the quotient field $\mathcal{C}_v(\mathbb{Q})/\mathcal{N}_v(\mathbb{Q})$. If $a = (a_n)_{n\in\mathbb{N}} \in \mathbb{Q}_v$, let $x = [a]$, the equivalence class of a. Define $w_v : \mathbb{Q}_v \to \mathbb{R}^{\geq 0}$ by $w_v(x) = \lim_{n\to\infty} v(a_n)$.*

Lemma 1.4.5. *The function w_v defined above is a well-defined valuation.*

Proof. Suppose $x = [a] = [b]$. Then

$$\begin{aligned} w_v(x-x) = 0 &\Leftrightarrow \lim_{n\to\infty} v(a_n - b_n) = 0 \\ &\Leftrightarrow \lim_{n\to\infty} v(a_n) - \lim_{n\to\infty} v(b_n) = 0 \\ &\Leftrightarrow w_v(x) - w_v(x) = 0. \end{aligned}$$

Therefore, w_v is well-defined.

Now we will show that w_v is a valuation. Let $\lambda \in \mathcal{C}_v(\mathbb{Q})/\mathcal{N}_v(\mathbb{Q})$. Then $\lambda = 0$ if and only if $(x_n)_{n\in\mathbb{N}}$ representing λ is in $\mathcal{N}_v(\mathbb{Q})$. Thus, $w_v(\lambda) = 0$ if and only if $\lim_{n\to\infty} v(x_n) = 0$. Let λ represent $(x_n)_{n\in\mathbb{N}}$ and μ represent $(y_n)_{n\in\mathbb{N}}$ in $\mathcal{C}_v(\mathbb{Q})/\mathcal{N}_v(\mathbb{Q})$. Then $\lambda\mu$ is represented by $(x_n y_n)_{n\in\mathbb{N}}$. For each n we have $v(x_n y_n) = v(x_n)v(y_n)$. Taking the limit gives $w_v(\lambda\mu) = w_v(\lambda)w_v(\mu)$. Also for each n, $v(x_n + y_n) \leq v(x_n) + v(y_n)$. Thus, $w_v(\lambda+\mu) \leq w_v(\lambda) + w_v(\mu)$. Thus, w_v is a valuation. $\square$.

Remark 1.4.6. *The field $\mathbb{Q}_{v_\infty}$ is exactly $\mathbb{R}$ by definition.*

Proposition 1.4.7. *The field $\mathbb{Q}_v$ is a complete topological field.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$ be Cauchy in $\mathbb{Q}_v$ with respect to v. In other words, for all $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n, m > N$, $v(a_n - a_m) < \epsilon$. Let $a_n = [(\alpha_{n,i})_{i\in\mathbb{N}}]$ where $\alpha_{n,i} \in \mathcal{C}_v(\mathbb{Q})$ and $b = [(\beta_n)_{n\in\mathbb{N}}]$ where $\beta_n = \alpha_{n,n}$. Now, fix $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} n, k > N \quad &\Rightarrow \quad v(a_n - a_k) < \epsilon \\ &\Leftrightarrow \quad v(\alpha_{n,k} - \alpha_{k,k}) < \epsilon \\ &\Leftrightarrow \quad v(\alpha_{n,k} - \beta_k) < \epsilon \\ &\Leftrightarrow \quad v(a_n - b) < \epsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} a_n = b$ and hence, $\mathbb{Q}_v$ is complete. $\square$

Proposition 1.4.8. *The ring homomorphism $\iota : \mathbb{Q} \to \mathbb{Q}_v$ defined by inclusion is continuous.*

Proof. By Remark 1.4.6 $\mathbb{Q}_{v_\infty} = \mathbb{R}$ and therefore $\iota : \mathbb{Q} \to \mathbb{R}$ is continuous. Now suppose $v = v_p$ and consider $v_p \circ \iota : \mathbb{Q} \to \mathbb{Q}_p \to \mathbb{R}^{\geq 0}$. Now, $v_p \circ \iota$ is continuous (since the valuation group is $\mathbb{R}^{\geq 0} \subset \mathbb{R}$) and v_p is continuous. Hence, ι is continuous. □

Proposition 1.4.9. *The rationals are dense in $\mathbb{Q}_v$.*

Proof. Fix $\epsilon > 0$ and let $x = [a] \in \mathbb{Q}_v$. Then there is an $N \in \mathbb{N}$ such that for all $n, m > N$, $v(a_n - a_m) < \frac{\epsilon}{2}$. Consider the constant sequence $y = (x_N)_{n \in \mathbb{N}}$. Now $w(a_n - y) = \lim_{n\to\infty} v(a_n - y)$. If $n > N$, then $v(a_n - y) = v(a_n - a_N) < \frac{\epsilon}{2}$ and therefore, $\lim_{n\to\infty} v(a_n - y) \leq \frac{\epsilon}{2} < \epsilon$. Thus, $y \in B(x, \epsilon)$ and $\mathbb{Q}$ is dense in $\mathbb{Q}_v$. □

1.5 p-adic Fields

For each prime p a field $\mathbb{Q}_p$ will be constructed that is complete with respect to the p-adic valuation v_p.

Definition 1.5.1. *The* p-adic field *$\mathbb{Q}_p$ is defined to be $\mathbb{Q}_{v_p}$. Extend v_p to $\mathbb{Q}_p$ by setting $v_p := w_{v_p}$ (see Definition 1.4.4). The ring $\mathbb{Z}_p := \{x \in \mathbb{Q}_p \mid v_p(x) \leq 1\}$ is called* the ring of p-adic integers. *Both $\mathbb{Z}_p$ and $\mathbb{Q}_p$ are equipped with the topology induced by v_p (see Definition 1.2.1).*

Proposition 1.5.2. *The valuation ring, valuation ideal, and residue field of $\mathbb{Q}_{v_p}$ as defined in Definition 1.1.13 are given by: $K^\circ = \mathbb{Z}_p$, $\mathfrak{p}_K = p\mathbb{Z}_p$ and $\kappa_K = \mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{F}_p$.*

Proof. The valuation ring is $\mathbb{Z}_p$ by definition. Now, $\mathfrak{p}_K = \{\lambda \in \mathbb{Q}_p \mid v_p(\lambda) < 1\} = \{\lambda \in \mathbb{Q}_p \mid ord(\lambda) > 0\} \Rightarrow p \mid \lambda$. Thus, $\mathfrak{p}_K = p\mathbb{Z}_p$. It follows that $\kappa_K = \mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{F}_p$. □

Proposition 1.5.3. *The set $p\mathbb{Z}_p$ is both open and closed in $\mathbb{Z}_p$.*

Proof. Consider $\left[0, \frac{1}{p}\right]$ which is closed in $\mathbb{R}^{\geq 0}$. Then

$$\begin{aligned} v_p^{-1}\left(\left[0, \frac{1}{p}\right]\right) &= \left\{x \in \mathbb{Z}_p \mid v_p(x) \leq \frac{1}{p}\right\} \\ &= \{x \in \mathbb{Z}_p \mid ord_p(x) \geq p\} \\ &= p\mathbb{Z}_p. \end{aligned}$$

Since v_p is continuous, $p\mathbb{Z}_p$ is closed in $\mathbb{Z}_p$.

Let $0 < \epsilon < \frac{1}{p}$ and consider $\left[0, \frac{1}{p} + \epsilon\right)$ which is open in $\mathbb{R}^{\geq 0}$. Then

$$\begin{aligned} v_p^{-1}\left(\left[0, \frac{1}{p} + \epsilon\right)\right] &= \left\{x \in \mathbb{Z}_p \mid v_p(x) \leq \frac{1}{p} + \epsilon\right\} \\ &= \left\{x \in \mathbb{Z}_p \mid v_p(x) \leq \frac{1}{p}\right\} \\ &= p\mathbb{Z}_p. \end{aligned}$$

Thus, $p\mathbb{Z}_p$ is also open in $\mathbb{Z}_p$. □

Proposition 1.5.4. *The ring $\mathbb{Z}_p$ is an open subring of $\mathbb{Q}_p$.*

Proof.

$$\begin{aligned} v_p^{-1}\left([0, 1 + \epsilon)\right] &= \{x \in \mathbb{Z}_p \mid v_p(x) \leq 1 + \epsilon\} \\ &= \{x \in \mathbb{Z}_p \mid v_p(x) \leq 1\} \\ &= \mathbb{Z}_p. \end{aligned}$$

□

Working with elements of $\mathbb{Q}_p$ as equivalence classes of Cauchy sequences is difficult. Using the p-adic expansion from Section 1.3 gives us a nice way of viewing the elements in this field.

Lemma 1.5.5. *Every $x \in \mathbb{Q}_p$ can be written in the form $x = \sum_{k\in\mathbb{Z}}^{\infty} r_k p^k$ with $0 \leq r_k \leq p-1$.*

Proof. Let $x = [(a_n)_{n\in\mathbb{N}}] \in \mathbb{Q}_p$. Where $a_n \in \mathbb{Q}$. Then by Lemma 1.3.4, $a_n = \sum_{k=-i}^{\infty} r_{n,k}p^k$. Define $b := \sum_{k=-i}^{\infty} r_{k,k}p^k$. Now, fix $\epsilon > 0$. Since $[(a_n)_{n\in\mathbb{N}}]$ is Cauchy, there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} n,k > N \quad &\Rightarrow \quad v(a_n - a_k) < \epsilon \\ &\Leftrightarrow \quad v(\alpha_{n,k} - \alpha_{k,k}) < \epsilon \\ &\Leftrightarrow \quad v(a_n - b) < \epsilon. \end{aligned}$$

Therefore, $\lim_{n\to\infty} a_n = b$. Thus, x can be represented in the desired form. □

Example 1.5.6. *In $\mathbb{Q}_p$, for any prime p, we have*

$$-1 = \sum_{i=0}^{\infty}(p-1)p^i.$$

Moreover, if $x = \sum_{i=0}^{\infty} a_i p^i$ then

$$-x = p - a_0 + \sum_{i=1}^{\infty}(p-(1+a_i))p^i.$$

Proof. First, notice that

$$\begin{aligned} 1 + \sum_{i=0}^{n}(p-1)p^i &= \underbrace{1+p-1}+\sum_{i=1}^{n}(p-1)p^i \\ &= \underbrace{p+(p-1)p}+\sum_{i=2}^{n}(p-1)p^i \\ &= \underbrace{p^2+(p-1)p^2}+1+\sum_{i=3}^{n}(p-1)p^i \\ &= \cdots \\ &= p^{n+1}. \end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} v_p\left(1+\sum_{i=0}^{n}(p-1)p^i\right) &= \lim_{n\to\infty} v_p(p^{n+1}) \\ &= 0\end{aligned}$$

The proof of the second equation is similar to the proof of the first:

$$\left(\sum_{i=0}^{n} a_i p^i\right) + p - a_0 + \sum_{i=1}^{n}(p-(1+a_i))p^i = p^{n+1}$$

so

$$\begin{aligned}\lim_{n\to\infty} v_p\left(\left(\sum_{i=0}^{n} a_i p^i\right) + p - a_0 + \sum_{i=1}^{n}(p-(1+a_i))p^i\right) &= \lim_{n\to\infty} v_p(p^{n+1}) \\ &= 0.\end{aligned}$$

□

Remark 1.5.7. *It should be noted that adding, subtracting, and dividing p-adic numbers results in carrying forward as demonstrated in the above proof.*

Corollary 1.5.8. *The ring $\mathbb{Z}_p$ is complete.*

1.6 Hensel's Lemma

Lemma 1.6.1 (Hensel's Lemma). *Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in \mathbb{Z}_p[x]$. Suppose that there exists a p-adic integer $\alpha_1 \in \mathbb{Z}_p$ such that*

$$f(\alpha_1) \equiv 0 \pmod{p\mathbb{Z}_p}$$

and

$$f'(\alpha_1) \not\equiv 0 \pmod{p\mathbb{Z}_p}.$$

Then there exists a p-adic integer $\alpha \in \mathbb{Z}_p$ such that $\alpha \equiv \alpha_1 \pmod{p\mathbb{Z}_p}$ and $f(\alpha) = 0$.

Example 1.6.2. *Let* $f(x) = x^2 + 1 \in \mathbb{Z}_5[x]$. *Then*

$$f(2) = 2^2 + 1 = 5 \equiv 0 \pmod{5\mathbb{Z}_5}$$

and

$$f'(2) = 4 \not\equiv 0 \pmod{5\mathbb{Z}_5}.$$

Thus, by Hensel's Lemma, there exists a p-adic integer $\alpha \in \mathbb{Z}_5$ *such that* $\alpha \equiv 2 \pmod{5\mathbb{Z}_5}$ *and* $f(\alpha) = 0$. *Thus,* $\sqrt{-1} \in \mathbb{Q}_5$.

Lemma 1.6.3 (Hensel's Lemma, 2nd version). *Let* $f(x) \in \mathbb{Z}_p[x]$, *and assume that there exist polynomials* $g_1(x)$ *and* $h_1(x)$ *in* $\mathbb{Z}_p[x]$ *such that*

1. $g_1(x)$ *is monic*
2. $g_1(x)$ *and* $h_1(x)$ *are relatively prime modulo* p
3. $f(x) \equiv g_1(x)h_1(x) \pmod{p}$.

Then there exist polynomials $g(x), h(x) \in \mathbb{Z}_p[x]$ *such that*

1. $g(x)$ *is monic*
2. $g(x) \equiv g_1(x) \pmod{p}$ *and* $h(x) \equiv h_1(x) \pmod{p}$
3. $f(x) = g(x)h(x)$.

Proof. See Gouvêa [7, Section 3.4] for a proof of either version of Hensel's Lemma. □

Proposition 1.6.4. *The second version of Hensel's lemma (see Lemma 1.6.3) implies the first version (see Lemma 1.6.1).*

Proof. Let $f(x)$ be a monic polynomial. Let $g_1(x) = x - \alpha_1$ and let $h_1(x)$ be relatively prime to $g_1(x)$ such that $f(x) \equiv g_1(x)h_1(x) \pmod{p}$. Then α_1 is not a double root. Thus $f'(\alpha_1) \not\equiv 0 \pmod{p\mathbb{Z}_p}$ and therefore by the second version of Hensel's Lemma there exists polynomials $g(x), h(x) \in \mathbb{Z}_p$ such that $g(x) = 1 - \alpha$ and $f(x) = g(x)h(x)$. This is Lemma 1.6.1. □

Remark 1.6.5. *The converse of Proposition 1.6.4 is also true.*

1.7 Extension of v_p

Let $K : \mathbb{Q}_p$ be a field extension. Since $(\mathbb{Q}_p, v_p)$ is a valued field one wants to consider valuations $v : K \to \mathbb{R}^{\geq 0}$ such that these valuations extend the p-adic valuations on $\mathbb{Q}_p$.

Lemma 1.7.1. *There is at most one valuation on K extending the p-adic valuation on $\mathbb{Q}_p$.*

Proof. Suppose v and w are valuations on K which extend the p-adic valuation. Then v and w are equivalent [7, Corollary 5.3.2]. Thus, by Gouvêa [7, Lemma3.1.2] there is a positive real number α such that $v(x) = (w(x))^\alpha$ for every $x \in K$. But $v(x) = w(x)$ whenever $x \in \mathbb{Q}_p$ since both valuations extend the p-adic valuation. Compute both valuations at $x = p$. Then $v(p) = \frac{1}{p} = (w(p))^\alpha \Rightarrow \alpha = 1$. Thus, v and w are equal. □

1.8 Norm Function

Let $K : \mathbb{Q}_p$ be a finite extension. Lemma 1.7.1 tells us that there can be at most one valuation on K extending p-adic valuation on $\mathbb{Q}_p$. However, the existence of such a valuation has not yet been established. This section defines the norm function and uses this function to construct the desired valuation.

Definition 1.8.1. *Let $K : F$ be a finite extension. There exists a function $N_{K:F} : K \longrightarrow F$ called the* norm *from K to F. Think of K as a finite-dimensional F-vector space. Take $\alpha \in K$, and consider the F-linear map $\beta : K \longrightarrow K$ given by $x \mapsto \alpha x$. Since β is linear, it corresponds to a matrix. Then $N_{K:F}(\alpha)$ is defined to be the determinant of this matrix.*

Proposition 1.8.2. *Let $K : F$ be a finite extension of degree n.*

a) Suppose $\alpha \in K$ and let $m_{\alpha,F} = x^d - a_{d-1}x^{d-1} + \cdots + (-1)^d a_0$ be the minimal polynomial of α over F. Then $N_{K:F}(\alpha) = a_0^{n/d}$.

b) If $K : F$ is normal, then $N_{K:F}(\alpha) = \prod_{\sigma \in Gal(K:F)} \sigma(\alpha)$.

Proof. A sketch of the proof is found in Christie [2, Prop. 2.4]. □

Example 1.8.3. *Let $F = \mathbb{Q}_5$ and $K = \mathbb{Q}_5(\sqrt{2})$. Compute $N_{K:F}(a + b\sqrt{2})$. There are two automorphisms in the Galois group: $\sigma(a+b\sqrt{2}) = a+b\sqrt{2}$ and $\sigma(a+b\sqrt{2}) = a - b\sqrt{2}$. Thus $N_{K:F}(a + b\sqrt{2}) = (a + b\sqrt{2})(a - b\sqrt{2}) = a^2 - 2b^2$.*

Lemma 1.8.4. *Norms are multiplicative: $N_{K:F}(\alpha\beta) = N_{K:F}(\alpha)N_{K:F}(\beta)$.*

Proof. Using Definition 1.8.1, a norm is a determinant and determinants are multiplicative. □

Remark 1.8.5. *Norms are not additive. Let* $F = \mathbb{Q}_5$ *and* $K = \mathbb{Q}_5(\sqrt{2})$. *From Example 1.8.3* $N_{K:F}(a + b\sqrt{2}) = a^2 - 2b^2$. *Similarly,* $N_{K:F}(a - b\sqrt{2}) = a^2 - 2b^2$. *However,* $N_{K:F}(2a) = 2a \neq 2a^2 - 4b^2$.

Theorem 1.8.6. *Let* $K : \mathbb{Q}_p$ *be a finite extension of degree* n. *The function* $v_K : K \longrightarrow \mathbb{R}^{\geq 0}$ *defined by*

$$v_K(x) = \sqrt[n]{v_p(N_{K:\mathbb{Q}_p}(x))}$$

is a non-archimedean valuation on K *which extends the* p*-adic valuation on* $\mathbb{Q}_p$.

Proof. Check the four criterion for a valuation (see Definition 1.1.3).

1. $v_K(0) = 0$ and $v_K(1) = 1$ are clear.

2. Since $N_{K:\mathbb{Q}_p}(xy) = N_{K:\mathbb{Q}_p}(x)N_{K:\mathbb{Q}_p}(y)$ by Lemma 1.8.4, and v_p is multiplicative, it follows that $v_K(xy) = v_K(x)v_K(y)$.

3. If $x \in \mathbb{Q}_p$, then $N_{K:\mathbb{Q}_p}(x) = x^n$ by Proposition 1.8.2(a), so that $v_K(x) = \sqrt[n]{(v_p(x))^n} = v_p(x)$.

4. To show $v_K(x+y) \leq \max\{v_K(x), v_K(y)\}$, show that $v_K(x+1) \leq \max\{v_K(x), 1\}$. Observe that $v_K(x) \leq 1$ will happen exactly when $v_p(N_{K:\mathbb{Q}_p}(x)) \leq 1$. Thus, it needs to be shown that

$$v_p(N_{K:\mathbb{Q}_p}(x)) \leq 1 \Rightarrow v_K(N_{K:\mathbb{Q}_p}(x-1)) \leq 1$$

or

$$N_{K:\mathbb{Q}_p}(x) \in \mathbb{Z}_p \Rightarrow N_{K:\mathbb{Q}_p}(x-1) \in \mathbb{Z}_p$$

which is true from Lemma 1.8.8. Thus $v_K(x) \leq 1 \Rightarrow v_K(x-1) \leq 1$ and

$$\begin{aligned} v_K(x) \leq 1 &\Rightarrow v_K(-x) \leq 1 \\ &\Rightarrow v_K(-x-1) \leq 1 \\ &\Rightarrow v_K(x+1) \leq 1. \end{aligned}$$

Case 1: if $v_K(x) \leq 1$, then $\max\{v_K(x), 1\} = 1$, thus $v_K(x+1) \leq \max\{v_K(x), 1\}$.

Case 2: if $v_K(x) > 1$, then $v_K\left(\frac{1}{x}\right) < 1$, which yields

$$v_K\left(\frac{x+1}{x}\right) = v_K\left(1 + \frac{1}{x}\right) \leq 1$$

which says $v_K(x+1) \leq v_K(x) = \max\{v_K(x), 1\}$.

$\square$

Lemma 1.8.7. *If $f(x) = x^n + \cdots + a_1 x + a_0$ is a monic irreducible polynomial with coefficients in $\mathbb{Q}_p$ and $a_0 \in \mathbb{Z}_p$, then $a_i \in \mathbb{Z}_p$ for all $1 \leq i \leq n-1$.*

Proof. [7, Lemma 5.3.6] Assume that $a_0 \in \mathbb{Z}_p$ but some $a_i \notin \mathbb{Z}_p$. Choose m to be the smallest exponent such that $p^m a_i \in \mathbb{Z}_p$ for every $i \geq 1$. Set $g(x) = p^m f(x) = b_n x^n + b_{n-1} x^{n-1} + \ldots b_1 x + b_0$, so that $b_i = p^m a_i$. Since $f(x)$ is monic, $b_n = p^m$ is divisible by p; since $a_0 \in \mathbb{Z}_p$, $b_0 = p^m a_0$ is also divisible by p. Let k be the smallest i such that b_i is not divisible by p. Then

$$g(x) \equiv (b_n x^{n-k} + \cdots + b_k) x^k (\text{mod } p)$$

and the two factors are relatively prime modulo p. By the second version of Hensel's Lemma (see Proposition 1.6.3), it follows that $g(x) = p^m f(x)$ is reducible, and therefore so is $f(x)$ itself. This contradicts the assumption. Hence, $a_i \in \mathbb{Z}_p$ for all $1 \leq i \leq n-1$. $\square$

Lemma 1.8.8. *Let $K : \mathbb{Q}_p$ be a finite extension of degree n and let $\alpha \in K$. Then $N_{K:\mathbb{Q}_p}(\alpha) \in \mathbb{Z}_p \Rightarrow N_{K:\mathbb{Q}_p}(\alpha - 1) \in \mathbb{Z}_p$.*

Proof. Since the formula for the norm does not depend on the choice of field containing α [7, Proposition 5.3.4], we can assume that $K = \mathbb{Q}_p(\alpha)$, the smallest field containing α. Let $m_{\alpha,\mathbb{Q}_p} = x^n - a_{n-1}x^{n-1} + \cdots + (-1)^{n-1}a_1 x + (-1)^n a_0$ be the minimal polynomial for α. Then the minimal polynomial for $\alpha - 1$ is $m_{\alpha-1,\mathbb{Q}_p} = x^n - (a_{n-1} + n)x^{n-1} + \cdots + (-1)^n(1 + a_{n-1} + \cdots + a_1 + a_0)$ (since $m_{\alpha-1,\mathbb{Q}_p}(\alpha - 1) = 0$ and the degree is right). Using the second definition of the norm (see Proposition 1.8.2 (a)), $N_{K:\mathbb{Q}_p}(x) = a_0$ and $N_{K:\mathbb{Q}_p}(x-1) = 1+a_{n-1}+\cdots+a_1+a_0$. Now $m_{\alpha,\mathbb{Q}_p}$ is an irreducible polynomial and $a_0 \in \mathbb{Z}_p$, so $1 + a_{n-1} + \cdots + a_1 + a_0 \in \mathbb{Z}_p$ by Lemma 1.8.7. □

Given any finite extension $K : \mathbb{Q}_p$ of degree n it has been shown that there exists a unique valuation on K which extends the p-adic valuation on $\mathbb{Q}_p$. This additive valuation is called the *p-adic valuation on K*. For any $x \in K^\times$ this valuation is defined by

$$ord_K(x) = \frac{1}{n} v_p(N_{K:\mathbb{Q}_p}(x)).$$

1.9 Finite Extensions of $\mathbb{Q}_p$

Definition 1.9.1. *Let $K : \mathbb{Q}_p$ be a finite extension of degree n, and let $e = e(K : \mathbb{Q}_p)$ be the unique smallest positive integer defined by*

$$ord_K(K^\times) = \frac{1}{e}\mathbb{Z}.$$

The integer e is called the ramification index *of K over $\mathbb{Q}_p$. The extension $K : \mathbb{Q}_p$ is* unramified *if $e = 1$,* ramified *if $e > 1$, and* totally ramified *if $e = n$. An element $\pi \in K$ is a* uniformizer *if $ord_K(\pi) = \frac{1}{e}$.*

Remark 1.9.2. *It is easy to see that $1 \leq e \leq n$. In fact e divides n.*

Example 1.9.3. *Consider the fields $\mathbb{Q}_5(\sqrt{2})$ and $\mathbb{Q}_5(\sqrt{5})$. In the following two examples we will determine the ramification index, type of extension, valuation ring, valuation ideal, and residue field of these two fields.*

1. *Let $K = \mathbb{Q}_5(\sqrt{2})$. Then $n = 2$. Thus, $1 \leq e \leq 2$. To determine the exact value of e, the image of ord_K must be calculated. The Galois group $Gal(K : \mathbb{Q}_5) = \{1, \sigma\}$ where $\sigma(x + \sqrt{2}y) = x - \sqrt{2}y$. Thus by Proposition 1.8.2 (b), $N_{K:\mathbb{Q}_5}(x+\sqrt{2}) = x^2 - 2y^2$. Therefore, $ord_K = \frac{1}{2}ord_5(x^2 - 2y^2)$. Write $x = 5^n u$, $y = 5^m v$, where $m, n \in \mathbb{Z}$, $u, v \in \mathbb{Z}_5^*$. Then $x^2 - 2y^2 = 5^{2n}u^2 - 2(5^{2m}v^2)$. Now, $ord_5(x^2 - 2y^2) \geq min\{2n, 2m\}$ with equality if $n \neq m$. If $n = m$ then $x^2 - 2y^2 = 5^{2n}(u^2 - 2v^2) \Rightarrow ord_5(x^2 - 2y^2) = 2n + ord_5(u^2 - 2v^2)$. If $ord_5(u^2 - 2v^2) > 0$ then $u^2 - 2v^2 \equiv 0(mod\ 5)$.*

   - *If $u \equiv 0$, then $v \equiv 0$ which is not possible since $u, v \in \mathbb{Z}_5^*$.*
   - *If $u \equiv 1, -1$, then $2v^2 \equiv 1$ which is a contradiction since $v^2 \equiv 3$ has no solutions* (mod 5).
   - *If $u \equiv 2, -2$, then $2v^2 \equiv -1$ which is a contradiction since $v^2 \equiv 2$ has no solutions* (mod 5).

   Thus, $ord_5(u^2 - 2v^2) = 0 \Rightarrow ord_5(x^2 - 2y^2) = 2n \in 2\mathbb{Z}$, and $e = 1$. Therefore, $\mathbb{Q}_5(\sqrt{2}) : \mathbb{Q}_5$ is an unramified extension.

Now, $ord_K(x+\sqrt{2}y) = 1 \Longleftrightarrow ord_5(x^2-2y^2) = 2$. *Let* $\pi = x+\sqrt{2}y = 5 = p$.

$$\begin{aligned} K^\circ &= \{z \in K \mid ord_K \geq 0\} \\ &= \{x+\sqrt{2}y \in K \mid ord_5(x^2-2y^2) \geq 0\} \\ &= \{x+\sqrt{2}y \in K \mid x, y \in \mathbb{Z}_5\} \\ &= \mathbb{Z}_5[\sqrt{2}] \end{aligned}$$

$$\begin{aligned} \mathfrak{p}_K &= \{z \in K \mid ord_K > 0\} \\ &= \{x+\sqrt{2}y \in K \mid ord_5(x^2-2y^2) > 0\} \\ &= \{x+\sqrt{2}y \in K \mid x, y \in 5\mathbb{Z}_5\} \\ &= 5\mathbb{Z}_5[\sqrt{2}] \end{aligned}$$

$$\begin{aligned} \kappa_K = K^\circ/\mathfrak{p}_K &= \mathbb{Z}_5[\sqrt{2}]/5\mathbb{Z}_5[\sqrt{2}] \\ &\cong \mathbb{F}_5[\sqrt{2}] \\ &= \mathbb{F}_5(\sqrt{2}) \\ &\cong \mathbb{F}_{25} \end{aligned}$$

2. *Let* $K = \mathbb{Q}_5(\sqrt{5})$ *so* $n = 2$ *and* $1 \leq e \leq 2$. *As above, it is easy to calculate the norm:* $N_{K:\mathbb{Q}_5}(x+\sqrt{5}y) = x^2 - 5y^2$. *To calculate the ramification index, it is sufficient to find* $x+\sqrt{5}y \in K$ *such that* $ord_5(x^2-5y^2) = 1$. *Let* $x = 0, y = 1$. *Then* $ord_K(\sqrt{5}) = \frac{1}{2}v_5(-1) = \frac{1}{2}$. *Thus,* $im(ord_K) = \frac{1}{2}\mathbb{Z}$, $e = 2$, *and* $\mathbb{Q}(\sqrt{5}) : \mathbb{Q}_5$ *is totally ramified. A uniformizer,* $\pi = x+\sqrt{5}y = \sqrt{5}$, *was calculated when*

finding the ramification index.

$$\begin{aligned}
K^\circ &= \{z \in K \mid ord_K \geq 0\} \\
&= \{x + \sqrt{5}y \in K \mid ord_5(x^2 - 5y^2) \geq 0\} \\
&= \mathbb{Z}_5[\sqrt{5}]
\end{aligned}$$

$$\begin{aligned}
\mathfrak{p}_K &= \{z \in K \mid ord_K > 0\} \\
&= \{x + \sqrt{5}y \in K \mid ord_5(x^2 - 5y^2) > 0\} \\
&= 5\mathbb{Z}_5 + \sqrt{5}\mathbb{Z}_5
\end{aligned}$$

$$\begin{aligned}
\kappa_K = K^\circ/\mathfrak{p}_K &= \mathbb{Z}_5[\sqrt{5}]/(5\mathbb{Z}_5 + \sqrt{5}\mathbb{Z}_5) \\
&\cong \mathbb{Z}_5/5\mathbb{Z}_5 \\
&\cong \mathbb{F}_5
\end{aligned}$$

Remark 1.9.4. *Notice that the uniformizer of K is not unique. In the unramified case, p is usually chosen as the uniformizer.*

1.10 p-adic Fields and Hensel's Lemma Revisited

Definition 1.10.1. *A field K is a* p-adic field *if K is a finite extension of $\mathbb{Q}_p$.*

Proposition 1.10.2. *Let K° be the valuation ring of K, let $\mathfrak{p}_K$ be its maximal ideal, and let κ_K be the residue field. Fix a uniformizer π in K. Then*

1. *the ideal $\mathfrak{p}_K \subset K^\circ$ is principal and π is a generator*
2. *any element $x \in K$ can be written in the form $x = u\pi^{ord_K(x)}$, where $u \in (K^\circ)^\times$ is a unit*

3. *the residue field* κ_K *is a finite extension of* $\mathbb{F}_p$ *whose degree is less than or equal to the degree* $[K : \mathbb{Q}_p]$

4. *any element of* K° *is the root of a monic polynomial with coefficients in* $\mathbb{Z}_p$

5. *if* $x \in K$ *is the root of a monic polynomial with coefficients in* $\mathbb{Z}_p$*, then* $x \in K^\circ$*.*

Proof.

1)

$$\begin{aligned} x \in \mathfrak{p}_K &\Rightarrow v_K(x) < 1 \\ &\Rightarrow ord_K(x) > 0 \\ &\Rightarrow ord_K(x) \geq \frac{1}{e} \\ &\Rightarrow ord_K(\pi^{-1}x) \geq 0 \\ &\Rightarrow v_K(\pi^{-1}x) \leq 1 \\ &\Rightarrow \pi^{-1}x \in K^\circ \\ &\Rightarrow x \in \pi K^\circ \end{aligned}$$

2)

$$\begin{aligned} x \in K &\Rightarrow v_K(x) < \infty \\ &\Rightarrow ord_K(x) > -\infty \\ &\Rightarrow x \in K^\circ \left[\frac{1}{\pi}\right] \end{aligned}$$

3) The result follows from Proposition 1.10.3.

4) If α is the root of such a polynomial then its norm (up to a sign) is a power of the constant term, which is in $\mathbb{Z}_p$. Hence $v_K(\alpha) = \sqrt[n]{v_p(N_{K:\mathbb{Q}_p}(\alpha))} \leq 1$.

5) Follows from Lemma 1.8.7 and Hensel's Lemma. $\square$

Proposition 1.10.3. *Let $K : \mathbb{Q}_p$ be a field extension and let κ_K be the residue field of K. Let $f = f(K : \mathbb{Q}_p)$ be as defined in Definition 1.9.1. Then $[\kappa_K : \mathbb{F}_p] = f$, so that $\kappa_K = \mathbb{F}_{p^f}$ is the finite field with p^f elements.*

The proof to this proposition is quite long and the reader is referred to Gouvêa [7, Prop. 5.4.6]. The proposition was stated to illustrate that the degree $[K : \mathbb{Q}_p] = n$ of a finite extension breaks up into factors e and $f := \frac{n}{e}$, where e measures the change in the image of the additive p-adic valuation *ord* and $f = [\kappa : \mathbb{F}_p]$ measures the change in the residue field.

Lemma 1.10.4 (Hensel's Lemma, general version). *Let K be a p-adic field and let π be a uniformizer. Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial whose coefficients are in the valuation ring $K^\circ = \{x \in K \mid v_p(x) \leq 1\}$. Suppose that there exists $\alpha \in K^\circ$ such that $f(\alpha) \equiv 0 \pmod{\pi}$ and $f'(\alpha) \not\equiv 0 \pmod{\pi}$. Then there exists $\alpha_0 \in K^\circ$ such that $\alpha \equiv \alpha_0 \pmod{\pi}$ and $f(\alpha_0) = 0$.*

Notice that Lemma 1.6.1 is a specific case of Lemma 1.10.4. If $K = \mathbb{Q}_p$, then $K^\circ = \mathbb{Z}_p$ by Definition 1.5.1. Let $\pi = p$. Lemma 1.6.1 is now obtained from Lemma 1.10.4 and thus a general version of Lemma 1.6.3 could be obtained in a similar manner.

Chapter 2

Classes of Rings

This chapter will introduce the category of affinoid rings with an emphasis on Tate rings. To define Tate rings, adic and f-adic rings are introduced. The completion of rings is discussed to provide examples of such rings.

2.1 Completion of Rings

Let A be a ring and let I and J be a ideals of A. Recall that the multiplication of two ideals is defined as follows: for $I \lhd A$ and $J \lhd A$,

$$I \cdot J := \left\{ \sum_i^n a_i b_i \mid a_i \in I, b_i \in J, n \in \mathbb{N} \right\}.$$

Then we have natural homomorphisms

$$A/I \leftarrow A/I^2 \leftarrow A/I^3 \leftarrow \cdots$$

which make $(A/I^n)_{n \in \mathbb{N}}$ into an inverse system of rings (see Definition A.2.1). Note that each natural homomorphism $A/I^n \leftarrow A/I^{n+1}$ is surjective and is defined by $a + I^{n+1} \mapsto a + I^n$.

Let $v : A \to \Gamma \cup \{0\}$ be a function defined by $v_I(x) = e^{-\max\{n \in \mathbb{N}^\times \mid x \in I^n\}}$. The topology on A induced by v is the coarsest topology on A making A a topological ring and such that $v^{-1}\{x \in \Gamma \cup \{0\} \mid x < \gamma\}$ is open in A for each $\gamma \in \Gamma$. Recall from Chapter 1, if I is a prime ideal, then v is a valuation.

Definition 2.1.1. *Let A be a ring and I be an ideal of A and make $(A/I^n)_{n\in\mathbb{N}}$ into an inverse system of rings as above. The set*

$$\hat{A} := \varprojlim_{n\in\mathbb{N}} A/I^n$$

(see Definition A.2.1) is called the completion of A with respect to I. *Let $\hat{I} := \{\alpha \in \hat{A} \mid \alpha_1 = 0\}$ where $\alpha = (\alpha_1, \alpha_2, \alpha_3 \ldots)$ and $\alpha_n \in A/I^n$.*

Lemma 2.1.2. *The set $\hat{A}$ is a ring and $\hat{I}$ is an ideal of $\hat{A}$.*

Proof. Clearly $0 = (0, 0, \ldots)$ and $1 = (1, 1, \ldots)$ are in $\hat{A}$. Let $\alpha = (\alpha_1, \alpha_2, \ldots)$, $\beta = (\beta_1, \beta_2, \ldots) \in \hat{A}$. Then $\alpha\beta = (\alpha_1\beta_1, \alpha_2\beta_2, \ldots)$ and $\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots)$ are in $\hat{A}$ since $A/I^n \leftarrow A/I^{n+1}$ is a homomorphism for all n. Thus, $\hat{A}$ is a ring. The set $\hat{I}$ is clearly a subring of $\hat{A}$ and since multiplication is computed component wise, $a\hat{I} \subseteq \hat{I}$ and $\hat{I}a \subseteq \hat{I}$ for all $a \in \hat{A}$. Therefore $\hat{I}$ is an ideal of $\hat{A}$. □

Lemma 2.1.3. *The ideal $\hat{I}^k$ consists of those $\alpha = (\alpha_1, \alpha_2, \ldots) \in \hat{A}$ such that $\alpha_k = 0$.*

Proof. If $\alpha \in \hat{I}^k$. Without loss of generality, take $\alpha = \beta_1\beta_2 \cdots \beta_k$ where $\beta_m \in \hat{I}$. Let $\alpha_2 = \prod_{i=0}^{k} \beta_{2,i}$. Choose a representative a_2 from the equivalence class α_2 and $b_{2,i}$ from $\beta_{2,i}$. Then $a_2 = \prod_{i=0}^{k} b_{2,i}$ and $b_{2,i} \in I$ (since $\beta_{1,i} = 0$). Thus, $\prod_{i=0}^{k} \beta_{2,i} \in I^k$ which implies $a_2 \in I^k$. Hence, $\alpha_2 = 0$. □

Proposition 2.1.4. *There exists a bijection between $\hat{A}/\hat{I}$ and A/I. Thus, I is a prime ideal in A if and only if $\hat{I}$ is a prime ideal in $\hat{A}$.*

Proof. Consider the following diagram

$$\begin{tikzcd} \hat{A} \arrow[r, "\hat{q}"] \arrow[dr, "\mu"] & \hat{A}/\hat{I} \arrow[d, dashed, "\varphi"] \\ A \arrow[u, "\sigma"] \arrow[r, "q"] & A/I \end{tikzcd}$$

such that $\varphi(\alpha+\hat{I}) = \mu(\alpha)+I$ for $\alpha \in \hat{A}$. Then the diagram commutes. Since $\alpha \in \hat{A}$ if and only if $\alpha_1 \in I$, φ is bijective. $\square$

Corollary 2.1.5. *There exists a bijection between $\hat{A}/\hat{I}^k$ and A/I^k.*

Example 2.1.6.

1. *Let $\mathcal{I} = \{p^n\mathbb{Z} \mid n \in \mathbb{N}\}$ be the partially ordered set with respect to $\mu_{m,n}$: $\mathbb{Z}/p^m\mathbb{Z} \to \mathbb{Z}/p^n\mathbb{Z}$ defined by $k+p^m\mathbb{Z} \mapsto k+p^n\mathbb{Z}$ when $n \leq m$ (with the regular order on $\mathbb{N}$; i.e., $n \leq m \Leftrightarrow m-n \geq 0$). An element of $\varprojlim_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ is then a sequence $(a_n)_{n\geq 1}$ such that $a_n \in \mathbb{Z}/p^n\mathbb{Z}$ and if $n \leq m$ then $a_n \equiv a_m (\mathrm{mod}\ p^n)$. Every natural number m defines such a sequence and can therefore be regarded as a p-adic integer. For example, 35 as a 2-adic integer would be written as the sequence $\{1, 3, 3, 3, 3, 35, 35, \ldots\}$. In fact, the ring $\varprojlim_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ has already appeared; it is exactly the ring of p-adic integers (see Definition 1.5.1).*

   Let $\alpha = (\alpha_n)_{n\in\mathbb{N}} \in \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$. For each α_n, choose a representative $a_n \in \mathbb{Z}$. Without loss of generality choose $0 \leq a_n < p$ and consider the base p expansion (see Definition 1.3.1) of a_n (i.e. $a_n = \sum_{k=0}^{\infty} r_{n,k}p^k = \sum_{k=0}^{n-1} r_{n,k}p^k +$ higher order terms). Define $b_n := r_{n+1,n}$.

   Now consider

$$\begin{aligned} \varphi : \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} &\to \mathbb{Z}_p \\ \alpha = (\alpha_n)_{n\in\mathbb{N}} &\mapsto \beta = \sum_{n=0}^{\infty} b_n p^n \end{aligned}$$

where b_n is defined above. Also consider

$$\begin{aligned}\psi : \mathbb{Z}_p &\rightarrow \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z} \\ \beta = \sum_{n=0}^{\infty} b_n p^n &\mapsto \alpha = (\alpha_n)_{n\in\mathbb{N}}\end{aligned}$$

where $\alpha_n = a_n + p^n\mathbb{Z}$ and $a_n = \sum_{k=0}^{n-1} b_k p^k$.

Now φ is surjective since for any $\sum_{n=0}^{\infty} b_n p^n \in \mathbb{Z}_p$ we can define a sequence $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ in $\varprojlim_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ by $\alpha_n = a_n + p^n\mathbb{Z}$ and $a_n = \sum_{k=0}^{n-1} b_k p^k$. The map φ is injective since it has an inverse, namely ψ. To illustrate this, let $\alpha \in \varprojlim_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$. Then

$$\begin{aligned}(\psi \circ \varphi)(\alpha) &= \psi\left(\sum_{n=0}^{\infty} b_n p^n\right) \\ &= \alpha.\end{aligned}$$

Thus, φ is bijective (and ψ is its inverse). To show that these rings are isomorphic as topological rings, we will show ψ is continuous. Consider $\psi^{-1}(\widehat{p\mathbb{Z}})$. Now, $\alpha_1 = 0 \Rightarrow a_1 \in (p)$ so choose $a_1 = 0$. Hence, $b_0 = 0$ and therefore, $\psi^{-1}(\widehat{p\mathbb{Z}}) = \alpha(\widehat{p\mathbb{Z}}) = p\mathbb{Z}_p$. Thus, ψ is continuous.

We now have two constructions of the p-adic integers: the inverse limit construction and the algebraic construction illustrated in Chapter 1.

2. *Let $\widehat{\mathbb{Z}_p[T]}$ denote the completion of $\mathbb{Z}_p[T]$ with respect to the ideal (p); thus, $\widehat{\mathbb{Z}_p[T]} = \varprojlim_{n\in\mathbb{N}} \mathbb{Z}_p[T]/(p^n)\mathbb{Z}_p[T]$. Then $\widehat{\mathbb{Z}_p[T]} = \mathbb{Z}_p\{T\}$, the ring of convergent power series. The verification here is identical to the argument above.*

Proposition 2.1.7. *Let A be a ring and I an ideal of A. For each n we have a natural map $A \rightarrow A/I^n$ so by the universal property of inverse limits (see Proposition A.2.2)* *we obtain a homomorphism $A \longrightarrow \hat{A}$. Then $\hat{A}$ is complete, the ring homomorphism $\sigma : A \longrightarrow \hat{A}$ is continuous (where A is given the I-adic topology and $\hat{A}$ is given the $\hat{I}$-adic topology), and A is dense in $\hat{A}$.*

Proof. Since $\sigma : A \longrightarrow \hat{A}$ is unique, $v_I = v_{\hat{I}} \circ \sigma$. Thus, σ is continuous since v_I and $v_{\hat{I}}$ are continuous.

Let $(a_n)_{n \in \mathbb{N}}$ be Cauchy in $\hat{A}$ with respect to $v_{\hat{I}}$. In other words, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n, m > N$, $v_{\hat{I}}(a_n - a_m) < \epsilon$. Let $a_n = (\alpha_{n,k})_{k \in \mathbb{N}}$ where $\alpha_{n,k} \in A/I^k$. Let $b = (\beta_0, \beta_1, \dots) \in \hat{A}$ where $\beta_n = \alpha'_{n,n}$ and $\alpha'_{n,n} = \alpha_{n,n}$ under the bijection from Corollary 2.1.5. Fix $\epsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$\begin{aligned} n, k > N \quad &\Rightarrow \quad v_{\hat{I}}(\alpha_{n,k} - \alpha_{k,k}) < \epsilon \\ &\Leftrightarrow \quad v_{\hat{I}}(\alpha_{n,k} - \beta_k) < \epsilon \\ &\Leftrightarrow \quad v_{\hat{I}}(a_n - b) < \epsilon. \end{aligned}$$

Therefore $\lim_{n \to \infty} a_n = b$ and hence, $\hat{A}$ is complete.

Since A is a ring equipped with the I-adic topology, $\{I^n \mid n \in \mathbb{N}\}$ is a base at $0 \in A$. Therefore $\hat{A} = \bigcup_{n=0}^{\infty} (A + I^n)$. Now

$$\begin{aligned} x \in \hat{A} \quad &\Leftrightarrow \quad x \in \bigcup_{n=0}^{\infty} (A + I^n) \\ &\Leftrightarrow \quad x = y + z, \; y \in A, \; z \in I^n \\ &\Leftrightarrow \quad y = x - z \in (x + I^n) \cap A \\ &\Leftrightarrow \quad x \in \bar{A}. \end{aligned}$$

Therefore A is dense in $\hat{A}$. □

Remark 2.1.8. *If a ring A is complete it may not be of the form described above. For example, the real numbers are complete but not of this form.*

Proposition 2.1.9. *Let A be a ring, K be a p-adic field. Let $\sigma : K^\circ \to A$ be a ring homomorphism and let π denote the image of a uniformizer (see Definition 1.9.1) of K under σ. Let $\varphi : A \to \hat{A}$ be the canonical map (see Proposition A.2.2). Define $\hat{\pi} := (\varphi \circ \sigma)(\pi)$. Then $\hat{A} \otimes_{K^\circ} K = \hat{A}_{\hat{\pi}}$ (where $\hat{A} \otimes_{K^\circ} K$ is the pushout of $\hat{A} \leftarrow A \leftarrow K^\circ \to K$).*

$$\begin{tikzcd}
 & & B \\
\hat{A} \arrow[r, "\lambda_{\hat{\pi}}"] \arrow[urr, bend left, "f"] & \hat{A}_{\hat{\pi}} \arrow[ur, dotted, "\exists\,!\,\theta"] & \\
A \arrow[u, "\varphi"] & & \\
K^\circ \arrow[u, "\sigma"] \arrow[r, "\lambda_\pi"'] & K^\circ_\pi \arrow[uu, "\psi"'] \arrow[uuur, bend right, "g"'] &
\end{tikzcd}$$

Proof. Recall that K°_π is the ring K° localized at π and λ is the localization map(see Definition B.0.3). Notice that $K = K^\circ_\pi$. Let $\alpha \in \hat{A}$ and $\frac{k}{\pi^n} \in K^\circ_\pi$. Define $\psi(\frac{k}{\pi^n}) = \frac{(\varphi\circ\sigma)(k)}{(\varphi\circ\sigma)(\pi^n)} = \frac{(\varphi\circ\sigma)(k)}{\hat{\pi}^n}$. Then $(\psi\circ\lambda_\pi)(k) = (\lambda_{\hat{\pi}}\circ\varphi\circ\sigma)(k)$. Suppose $g\circ\lambda_\pi = f\circ\varphi\circ\sigma$. Then $(f\circ\varphi\circ\sigma)(k) = g(\frac{k}{1})$. Now define $\theta : \hat{A}_{\hat{\pi}} \to B$ by $\frac{\alpha}{\hat{\pi}^n} \mapsto \frac{f(\alpha)}{f(\hat{\pi})^n}$. Then $(\theta\circ\lambda_{\hat{\pi}})(\alpha) = f(\alpha)$ and

$$\begin{aligned}
(\theta \circ \psi)\left(\frac{k}{\pi^n}\right) &= \theta\left(\frac{(\varphi\circ\sigma)(k)}{\hat{\pi}^n}\right) \\
&= \frac{(f\circ\varphi\circ\sigma)(k)}{f(\hat{\pi}^n)} \\
&= \frac{(f\circ\varphi\circ\sigma)(k)}{(f\circ\varphi\circ\sigma)(\pi^n)} \\
&= g\left(\frac{k}{\pi^n}\right).
\end{aligned}$$

Thus, θ exists. To show the uniqueness of θ, suppose $\theta' : \hat{A}_{\hat{\pi}} \to B$ such that $\theta'\circ\lambda_{\hat{\pi}} = f$

and $\theta' \circ \psi = g$. Then

$$\begin{aligned} \theta'\left(\frac{\alpha}{\hat{\pi}^n}\right) &= (\theta' \circ \lambda_{\hat{\pi}})\left(\frac{\alpha}{\hat{\pi}^n}\right) \\ &= f\left(\frac{\alpha}{\hat{\pi}^n}\right) \\ &= \left(\frac{f(\alpha)}{f(\hat{\pi}^n)}\right) \\ &= \theta\left(\frac{\alpha}{\hat{\pi}^n}\right). \end{aligned}$$

Therefore $\theta' = \theta$ and θ is unique. $\square$

2.2 Adic Rings

Definition 2.2.1. *Let A be a topological ring. An ideal I of A is called an* ideal of definition *if $\{I^n \mid n \in \mathbb{N}\}$ is a base for the topology (see Section 1.2) at $0 \in A$. A topological ring A is said to be* adic *if A contains an ideal of definition. If A and B are adic rings and $f : A \to B$ a ring homomorphism such that given an ideal of definition I of A, $f(I)$ is an ideal of definition of B, then f is called an* adic ring homomorphism.

Lemma 2.2.2. *The composition of adic ring homomorphisms is adic.*

Proof. Let A, B and C be adic rings with adic ring homomorphisms $f : A \to B$ and $g : B \to C$. Let I be an ideal of definition of A. Since f is adic, $f(I)$ is an ideal of definition of B. Since g is adic, $g(f(I))$ is an ideal of definition of C. Therefore $g \circ f$ is adic. $\square$

Example 2.2.3.

1. *Let $A = \mathbb{Z}_p$ and consider $I = p\mathbb{Z}_p$. Then I^n is a base at 0*

$$\begin{aligned} x \in v_p^{-1}\left(\left[0, \frac{1}{p^n}\right]\right) &\Leftrightarrow v_p(x) \leq \frac{1}{p^n} \\ &\Leftrightarrow ord_p(x) \geq n \\ &\Leftrightarrow x \in p^n\mathbb{Z}_p \\ &\Leftrightarrow x \in I^n. \end{aligned}$$

Thus, $p\mathbb{Z}_p$ is an ideal of definition for $\mathbb{Z}_p$ and hence, $\mathbb{Z}_p$ is adic.

2. *The ring $\mathbb{Q}_p$ is not adic since the only prime ideal is (0) which cannot be an ideal of definition.*

3. *Consider the ring $A = \mathbb{Z}_p\{T\}$ (see Example 2.1.6(2)) with ideal $I = p\mathbb{Z}_p\{T\}$. Let $f = \sum_{n=0}^{\infty} a_n p^n \in A$. Then I^m is a base for the topology at 0 if $I^m = v_I^{-1}\left(\left[0, \frac{1}{p^m}\right]\right)$. Without loss of generality, take the base of the valuation to be p (see Example 1.1.7 (5)).*

$$\begin{aligned} f \in v_I^{-1}\left(\left[0, \frac{1}{p^m}\right]\right) &\Leftrightarrow v_I(f) \in [0, \frac{1}{p^m}] \\ &\Leftrightarrow v_I(a_n) \leq \frac{1}{p^m}, \ \forall\, n \in \mathbb{N} \\ &\Leftrightarrow a_n \in p^m\mathbb{Z}_p, \ \forall\, n \in \mathbb{N} \\ &\Leftrightarrow f \in p^m\mathbb{Z}_p\{T\} = I^m. \end{aligned}$$

Thus, $\mathbb{Z}_p\{T\}$ is adic and $I = p\mathbb{Z}_p\{T\}$ is an ideal of definition.

4. *The ring $\mathbb{Q}_p\{T\}$ is an adic ring and $p\mathbb{Q}_p\{T\}$ is an ideal of definition.*

5. *The ring $\mathbb{Z}_p[[T]]$ is adic and $p\mathbb{Z}_p[[T]]$ is an ideal of definition.*

Proposition 2.2.4. *An adic ring homomorphism is continuous.*

Proof. Let A and B be adic rings with adic ring homomorphism $f : A \to B$. Suppose I and $J = f(I)B$ are ideals of definition of A and B respectively. Then f is continuous at 0 if for all $N \in \mathbb{N}$ there exists $M \in \mathbb{N}$ such that $v_I(a) < e^{-M}$ which implies $v_J(f(a)) < e^{-N}$, or equivalently, for all $N \in \mathbb{N}$ there exists $M \in \mathbb{N}$ such that $a \in I^M \Rightarrow f(a) \in J^N$. Take $M := N$. Then $a \in I^M \Rightarrow f(a) \in f(I^M) = f(I)^M \subseteq J^M$. Thus, f is continuous at $0 \in A$. To show that f is continuous at $x \in A$, let U be an open neighbourhood of x. Since I is an ideal of definition there exists $K \in \mathbb{N}$ such that $x \in x + I^K \subseteq U \Rightarrow f(x) \in f(x + I^K) = f(x) + f(I)^K \subseteq f(x) + J^K$. Therefore f is continuous.

□

Example 2.2.5.

1. *The identity map is adic.*

2. *Define $\varphi : \mathbb{Q}_p\{T\} \to \mathbb{Q}_p\{T\}$ by $\varphi(T) = pT$ and φ is $\mathbb{Q}_p$-linear. Then φ is a ring homomorphism and*

$$\varphi(\mathbb{Z}_p\{T\}) = \left\{ \sum_{n\in\mathbb{N}} b_n T^n \in \mathbb{Q}_p\{T\} \mid b_n \in p^n\mathbb{Z}_p \text{ and } \frac{b_n}{p^n} \to 0 \right\} \subseteq \mathbb{Z}_p\{T\}.$$

   Thus, φ is continuous, and an adic ring homomorphism. More generally, $\varphi_k : \mathbb{Q}_p\{T\} \to \mathbb{Q}_p\{T\}$ is an adic homomorphism where $\varphi_k(T) = p^kT$ and φ_k is $\mathbb{Q}_p$-linear.

2.3 f-adic Rings

Definition 2.3.1. *A topological ring A is said to be* f-adic *if there exists an open adic subring $A_0 \subseteq A$ such that A_0 has a finitely generated ideal of definition. If A is f-adic then any open adic subring of A is called a* ring of definition *of A. Let A, B be f-adic rings. If $g : A \to B$ such that $g(A_0) \subseteq B_0$ and $g|_{A_0} : A_0 \to B_0$ is adic, then g is said to be an f-adic homomorphism.*

Example 2.3.2.

1. *Any adic ring which is also noetherian (i.e. every ideal is finitely generated) is f-adic by taking $A_0 = A$.*
2. *Let $A = \mathbb{Z}_p$, $A_0 = \mathbb{Z}_p$, and $I = p\mathbb{Z}_p$. Then A is f-adic.*
3. *Let $A = \mathbb{Q}_p$, $A_0 = \mathbb{Z}_p$, and $I = p\mathbb{Z}_p$. Then A is f-adic, but not adic.*
4. *Let $A = \mathbb{Q}_p\{T\}$ be the ring of convergent power series equipped with the topology defined by $v(\sum_{n\in\mathbb{N}} a_n T^n) = \sup_{n\in\mathbb{N}} v_p(a_n)$. Then A is an f-adic ring with ring of definition $A_0 = \mathbb{Z}_p\{T\}$ and ideal of definition (for A_0) given by $I = p\mathbb{Z}_p\{T\}$.*

Proposition 2.3.3. *Let A and B be f-adic rings. If $g : A \to B$ is an f-adic homomorphism, then g is continuous.*

Proof. Let A, B be f-adic rings and let $g : A \to B$ be an f-adic homomorphism. Let $U \subseteq B$ be open in B. Since B is a topological ring, without loss of generality, take $U = g(I)B_0$ where I is an ideal of definition of A_0. Now, $g|_{A_0}$ is continuous, since $g|_{A_0}$ is adic. Therefore, $A_0 \xrightarrow{g} g(I)B_0 \xhookrightarrow{\iota} B$ is continuous which implies $g : A \to B$ is continuous. ☐

Example 2.3.4. *The adic ring homomorphisms described in Example 2.2.5 are also f-adic.*

2.4 Power-bounded Elements

Definition 2.4.1. *Let A be an f-adic ring. An element $a \in A$ is* power-bounded *if $a \in B$ where B is a ring of definition of A (see Definition 2.3.1). Define A° to be the set of all power-bounded elements of A. In other words, $a \in A^\circ$ if $a \in \bigcup B$, the union of all rings of definition B of A.*

Example 2.4.2.

1. *Let $A = \mathbb{Q}_p$. Then $A^\circ = \mathbb{Z}_p$.*
2. *Let $A = \mathbb{Z}_p$. Then $A^\circ = \mathbb{Z}_p$.*
3. *Let $A = k((T))$ with ideal of definition $Tk[[T]]$. Then $A^\circ = k[[T]]$.*

Corollary 2.4.3. *A° is a subring of A.*

Proof. Clearly $0, 1 \in A^\circ$ since 0 and 1 are in all rings of definition. Now consider $b, c \in A^\circ$. If b, c are both in the same ring of definition, B, then bc and $b + c$ are in A°. Suppose $b \in B$ and $c \in C$ where B and C are rings of definition of A and $B \neq C$. Then B has an ideal of definition I and C has an ideal of definition J. Now, $B + C$ is a ring of definition of A since $I + J$ is an ideal of definition. Since $B + C$ is the smallest ring containing both B and C, bc and $b+c$ are in $B+C$. Thus $B+C \subseteq A^\circ$ implies $b, c \in A^\circ$. □

2.5 Topologically Nilpotent Elements

Definition 2.5.1. *Let A be an f-adic ring with ring of definition A_0. Then $a \in A$ is* topologically nilpotent *if for all ideals of definition I of A_0, and for all $m \in \mathbb{N}^\times$, there exists some $N \in \mathbb{N}^\times$ such that $a^k \in I^m$ for all $k \geq N$. Define $A^{\circ\circ}$ to be the set of all topologically nilpotent elements of A.*

Example 2.5.2.

1. *For every prime, $p \in \mathbb{Q}_p$ is topologically nilpotent.*

2. *The unity $1_A \in A$ is not topologically nilpotent.*

Proposition 2.5.3. *Let A be f-adic. Let I and J be ideals of definition of A. Pick $a \in A$. Then for all $m \in \mathbb{N}^\times$ there exists some $N \in \mathbb{N}^\times$ such that $a^k \in I^m$ for all $k \geq N$ if and only if there exists some $M \in \mathbb{N}^\times$ such that $a^k \in J^m$ for all $k \geq M$. Thus, to see if $a \in A$ is topologically nilpotent, it suffices to check one ideal of definition.*

Proof. Let A be an f-adic ring and $a \in A$ be topologically nilpotent with respect to I. Consider the ideal of definition $J \neq I$. Then there exists $m \in \mathbb{N}$ such that $I^m \subseteq J$. Since $a \in A$ is topologically nilpotent with respect to I there exists some $N \in \mathbb{N}^\times$ such that $a^k \in I^m$ for all $k \geq N$. Thus $a^k \in I^m \subseteq J$ for all $k \geq N$. Therefore, $a \in A$ is topologically nilpotent with respect to J. $\square$

Corollary 2.5.4. *Let A be an f-adic ring. Then $a \in A^{\circ\circ}$ if and only if there exists an ideal of definition I such that for all $m \in \mathbb{N}^\times$ there exists some $N \in \mathbb{N}^\times$ such that $a^k \in I^m$ for all $k \geq N$.*

Proposition 2.5.5. *The set of topologically nilpotent elements is an ideal in A_0.*

Proof. Let A be an adic ring, I be an ideal of definition of A, and $a, b \in A^{\circ\circ}$. Then for all $k \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $a^n \in I^k$ for all $n \geq N$ and there exists $M \in \mathbb{N}$ such that $b^m \in I^k$ for all $m \geq M$. Let $N + M = K$. Then

$$(a+b)^K = \sum_{i+0}^{K} \binom{K}{i} a^i b^{K-i}.$$

Now, $0 \leq i \leq N$ implies $-N \leq -i \leq 0$ which implies $0 \leq N-i \leq N$ so $M \leq K-i \leq K$. Thus $b^{K-i} \in I^k$. Similarly $N \leq i \leq K \Rightarrow a^i \in I^k$. Thus $a+b \in A^{\circ\circ}$. Also, if $a \in A$ and $b \in A^{\circ\circ}$, then there exists $M \in \mathbb{N}$ such that $b^m \in I^k$ for all $m \geq M$. Now, $(ab)^m = a^m b^m \in I^k$. Thus, $A^{\circ\circ}$ is an ideal in A_0. □

2.6 Tate Rings

Definition 2.6.1. *An f-adic ring A is a* Tate ring *if there exists a topologically nilpotent unit in A.*

Example 2.6.2.

1. *The field $\mathbb{Q}_p$ is a Tate ring since $\mathbb{Q}_p$ is f-adic and $p \in \mathbb{Q}_p$ is a topologically nilpotent unit.*

2. *The ring $\mathbb{Z}_p$ is not a Tate ring. Let $x \in (\mathbb{Z}_p)^{\circ\circ}$. Then $x \in \mathbb{Z}_p^*$ if $\lim_{n \in \mathbb{N}} v(x^n) = 0$. But $\lim_{n \in \mathbb{N}} v(y^n) = 1$ for all $y \in \mathbb{Z}_p^*$. Thus, there are no units in $\mathbb{Z}_p$ that are topologically nilpotent.*

Proposition 2.6.3. *Let A be a topological ring, fix $s \in A$, and consider the localization A_s (see Definition B.0.3). Equip A_s with the coarsest topology such that*

$\lambda : A \longrightarrow A_s$ is continuous. Then A_s is a Tate ring. Conversely, if A is a Tate ring then $A = B_s$ for some topological ring B and for some $s \in B$ where B_s is equipped with the adic topology generated by $\{(s^n) \mid n \in \mathbb{N}\}$.

Proof. To show that A_s is a Tate ring we need to show A_s is f-adic and has a topologically nilpotent unit. To show that A_s is f-adic we need to find an open subring A_0 of A with an ideal of definition. We claim that $B = \lambda((s)^0) = \lambda(A) = \left\{\frac{a}{1} \mid a \in A\right\}$ is a ring of definition. It is clear that B is both open and is a subring of A_s. We take $I = \lambda(s) = \left\{\frac{sa}{1} \mid a \in A\right\} = \frac{s}{1}B$ which is clearly an ideal of B, and $I^n = (\lambda(s))^n = \lambda((s)^n)$ since λ is a ring homomorphism, and hence I is a fundamental system of neighbourhoods. Therefore, A_s is f-adic. We claim that $t := \frac{s}{1} \in A_s$ is a topologically nilpotent unit. Now, t is a unit since $\left(\frac{s}{1}\right)\left(\frac{1}{s}\right) = \frac{1}{1} = 1_{A_s}$. Consider $\{t^n \mid n \geq 1\} \subset I = tB$. Then $\left\{t^{n+k} \mid n \geq 1\right\} \subset I^k$ which implies t is topologically nilpotent.

Conversely, suppose A is a Tate ring. Choose a ring of definition B of A. Then the result follows from Lemma 2.6.4 and Lemma 2.6.5. ☐

Lemma 2.6.4. *Let A be a Tate ring and let B be a ring of definition of A. Then B contains a topologically nilpotent unit of A.*

Proof. Since A is a Tate ring, by definition, there exists a topologically nilpotent unit $t \in A$. Then for all $U \subseteq A$ open, $0 \in U$, there exists $N \in \mathbb{N}$ such that $\{t^n \mid n \geq N\} \subseteq U$. Since B is a ring of definition of A it is an open neighbourhood of 0 in A. Set $U = B$ and the result follows. ☐

Lemma 2.6.5. *Let A be a Tate ring, let B be a ring of definition of A, and let $s \in B$ be a topologically nilpotent unit of A. Then $A \cong B_s$ and sB is an ideal of definition* *of* B.

Proof. Let A be a Tate ring, let B be a ring of definition of A, and let $s \in B$ be a topologically nilpotent unit of A. Then $A = A_s$ since s is a unit. Let $\varphi : B_s \to A_s$. Define $\varphi\left(\frac{b}{s^n}\right) = \frac{b}{s^n}$.

$$\begin{tikzcd} A \arrow[r, "\lambda_s"] & A_s = A \\ B \arrow[u, hook] \arrow[r, "\lambda_s'"] & B_s \arrow[u, "\varphi"] \end{tikzcd}$$

Now,

$$\begin{aligned} \varphi\left(\frac{b}{s^n}\right) = \varphi\left(\frac{b'}{s^{n'}}\right) &\Leftrightarrow \frac{b}{s^n} = \frac{b'}{s^{n'}} \\ &\Leftrightarrow s^m(bs^{n'} - b's^n) = 0 \text{ for some } m \in \mathbb{N} \\ &\Leftrightarrow bs^{n'} - b's^n = 0 \text{ since } s \text{ is a unit} \\ &\Leftrightarrow bs^{n'} = b's^n \\ &\Leftrightarrow bs^{-n} = b's^{-n'}. \end{aligned}$$

Thus, φ is one-to-one. To show that φ is onto, pick $a \in A$. We need to find $b \in B$, $n \in \mathbb{N}$ such that $\varphi\left(\frac{b}{s^n}\right) = a$. In other words $bs^{-n} = a \Leftrightarrow b = s^n a$. Consider the map $f_a : A \to A$ where $x \mapsto ax$. This is a continuous map. Thus, $f_a^{-1}(B) = \{x \in A \mid ax \in B\}$ is an open neighbourhood of 0 in A. Since $s \in B$ and s is a topologically nilpotent unit, there exists $N \in \mathbb{N}$ such that $\{s^n \mid n \geq N\} \subseteq f_a^{-1}(B)$. In particular, $as^n \in B$ and thus φ is onto and hence an isomorphism.

To show sB is an ideal of definition of B, let $U \subseteq A$ be open and $0 \in U$. Then there exists $n \in \mathbb{N}$ such that $0 \in I^n \subseteq U$. Since s is topologically nilpotent, $s^m \in I^n$ for some $m \in \mathbb{N}$. Thus, $s^m B \subseteq I^m B = I^n B \subseteq U$. So, $(sB)^m = s^m B \subseteq U$ for some $m \in \mathbb{N}$. Therefore sB is an ideal of definition for B. $\square$

Example 2.6.6.

1. *The ring* $\mathbb{Q}_p\{T\}$ *is Tate and* $\mathbb{Q}_p\{T\} = (\mathbb{Z}_p\{T\})_p$.

2. *The field* $\mathbb{Q}_p$ *is Tate and* $\mathbb{Q}_p = (\mathbb{Z}_p)_p$.

2.7 Integral Elements

Definition 2.7.1. *Let* A *and* B *be rings with* $A \subseteq B$. *Then* $a \in A$ *is* integral *over* B *if and only if there exists* $f \in B[x]$ *such that* $f \neq 0$, f *is monic, and* $f(a) = 0$. A *is integral over* B *if and only if for all* $a \in A$, a *is integral over* B. *If every element of* A *that is integral over* B *belongs to* B, *then* B *is* integrally closed *in* A. *A ring* A *is said to be a* subring of integral elements *over an f-adic ring* B *if*

- A *is a subring of* B
- A *is open in* B
- A *is integrally closed in* B
- $A \subseteq B^\circ$.

Example 2.7.2. *Let* $B = \mathbb{Q}_p$ *and* $A = \mathbb{Z}_p$. *Then*

- A *is a subring of* B *by Definition 1.5.1.*
- A *is open in* B *by Proposition 1.5.4.*
- *For each* $a \in A$, *let* $f_a(x) = x - a \in B[x]$. *Then* $f_a(a) = 0$. *Thus* A *is integrally closed in* B.
- $A \subseteq B^\circ$ *by Example 2.4.2(1).*

Thus, the ring $\mathbb{Z}_p$ *is a subring of integral elements in* $\mathbb{Q}_p$.

2.8 Affinoid Rings

Definition 2.8.1. *An* affinoid ring *is a pair $A = (A^{\triangleright}, A^{+})$ where $A^{\triangleright}$ is an f-adic ring and A^{+} is a subring of integral elements over $A^{\triangleright}$ (see Definition 2.7.1). Let A and B be affinoid rings. A morphism $g = (g^{\triangleright}, g^{+}) : A \longrightarrow B$ is called an* affinoid ring homomorphism *if $g^{\triangleright} : A^{\triangleright} \longrightarrow B^{\triangleright}$ is f-adic, $g^{\triangleright}(A^{+}) \subseteq B^{+}$, and $g^{+} : A^{+} \longrightarrow B^{+}$ is defined by $g^{+}(a) = g^{\triangleright}(a)$ for each $a \in A^{+}$.*

Remark 2.8.2. *By Lemma 2.3.3, $g^{\triangleright}$ is continuous.*

Example 2.8.3.

1. *The pair $(\mathbb{Q}_p, \mathbb{Z}_p)$ is an affinoid ring.*

2. *The pair $(\mathbb{Q}_p\{T\}, \mathbb{Z}_p\{T\})$ is an affinoid ring.*

3. *Let $A = (\mathbb{Q}, \mathbb{Q})$ be equipped with the p-adic topology and let $B = (\mathbb{Q}, \mathbb{Q})$ be equipped with the discrete topology. Then the identity map $id : A \longrightarrow B$ is not continuous and hence not affinoid. To see this notice that $\{0\}$ is open in $B^{\triangleright}$ but there is no power of (p) such that $(p) \subseteq \{0\}$.*

Proposition 2.8.4. *Let A be an integral domain and let π be a prime element in A. Let $\hat{A}$ denote the completion of A with respect to πA (see Definition 2.1.1). Let $\hat{\pi}$ denote the image of π in $\hat{A}$ under the unique ring homomorphism $A \longrightarrow \hat{A}$. Then $\hat{A}_{\hat{\pi}}$ is a Tate ring, and $(\hat{A}_{\hat{\pi}}, \hat{A})$ is an affinoid ring.*

Proof. Let $R = \hat{A}$ and $s = \hat{pi}$. Then $R_s = \hat{A}_{\hat{\pi}}$ and $I = sR$ is an ideal of definition. Now, the ring R_s is Tate by Proposition 2.6.3 (note that R is a topological ring with the topology given by v_I). Thus, R is f-adic by Definition 2.6.1. Since multiplication in R is done component wise, R is an integral domain since A is an integral domain and s is prime in R since π is prime in A. Thus, R is integrally closed in R_s by Lemma 2.8.5, R is clearly a subring of R_s, and R is open in R_s since

$$\begin{aligned} R &= \{r \in R_s \mid v_I(r) \leq 1\} \\ &= v_I^{-1}([0,1]) \\ &= v_I^{-1}([0, 1+\epsilon)) \text{ for some } \epsilon > 0. \end{aligned}$$

To show that $R \subseteq (R_s)^{\circ}$, it is sufficient to show that R is a ring of definition of R_s. Thus, it remains to be shown that R has a finitely generated ideal of definition (see Definition 2.2.1). Without loss of generality, take s to be the base in the I-adic valuation (see Example 1.1.7 (5)). Then $\{s^n \mid n \in \mathbb{N}\}$ is a base for the topology on R so (s) is an ideal of definition of R. Therefore, (R_s, R) is affinoid. □

Lemma 2.8.5. *If A is an integral domain and s is a prime element of A, then A is integrally closed in A_s.*

Proof. We must show that if $b \in A_s$ is integral over A then $b \in A$. Write $b = \frac{a}{s^n}$ with $a \in A$ and $n \in \mathbb{N}$. If $n = 0$, then $b \in A$ trivially.

Therefore, to begin, suppose $b = \frac{a}{s^n}$ is integral over A with $n = 1$. Let $f = \sum_{i=0}^{d} a_i T^i$ be a monic polynomial such that $f(b) = 0$. Then $\sum_{i=0}^{d} a_i \frac{a^i}{s^i} = 0$ in A_s and since A is an integral domain, $\sum_{i=0}^{d} a_i a^i s^{d-i} = 0$ in A. Thus,

$$a_0 s^d + a_1 a s^{d-1} + a_2 a^2 d^{d-2} \cdots + a_{d-1} a^{d-1} s + a^d a_d = 0.$$

Since f is monic, $a_d = 1$, from which it follows that $a^d \in sA$. Since s is prime, this implies $a \in sA$. Writing $a = sa_1$ we have $\frac{a}{s} = \frac{sa_1}{s} = \frac{a_1}{1}$, so $b \in A$.

Next, suppose $b = \frac{a}{s^n}$ is integral over A with $n > 1$. Since $\frac{s}{1} \in A_s$ is integral over A and since the set of elements of A_s which are integral over A is a subring of A_s [4, 15.3, Corollary 19] then $\frac{s^{n-1}}{1}\frac{a}{s^n} = \frac{a}{s}$ is integral over A. As above, it follows that $a = sa_1$, so $b = \frac{a_1}{s^{n-1}}$. Again, it follows that $\frac{a_1}{s}$ is integral over A, so $a_1 = sa_2$, whence $b = \frac{a_2}{s^{n-2}}$. Continuing in this manner have $b = \frac{a_n}{1}$ showing that $b \in A$. $\square$

Chapter 3

The Main Algebraic Result

The previous chapters have provided all of the tools necessary to prove the main algebraic result - Theorem 3.0.8. This result will be interpreted geometrically in Chapter 6 as the main geometric result - Theorem 6.0.5.

Definition 3.0.6. *Let A, B and R be rings. Let $\sigma : R \to A$ and $\tau : R \to B$. Then*

$$\mathrm{Hom}_R(A, B) := \{\varphi : A \to B \mid \varphi \circ \sigma = \tau\}.$$

$$\begin{tikzcd}
A \arrow[r, "\varphi"] & B \\
R \arrow[u, "\sigma"] \arrow[ur, "\tau"'] &
\end{tikzcd}$$

Lemma 3.0.7. *Let A, B and R be rings and let $\alpha : A \to \hat{A}$, $\beta : B \to \hat{B}$ be the unique maps guaranteed by the universal property of inverse limits (see Proposition A.2.2), where $\hat{A}$ is the completion of A with respect to an ideal I of A and $\hat{B}$ is the completion*

$$\begin{tikzcd}
& & \hat{A} \arrow[dll, "\alpha_n"'] \arrow[dl, "\alpha_m"] \arrow[rrr, dotted, "\hat{\varphi}"] & & & \hat{B} \arrow[dll, "\beta_n"'] \arrow[dl, "\beta_m"] \\
A/I^n & A/I^m \arrow[l, "\delta_{mn}"'] & & B/J^n & B/J^m \arrow[l, "\mu_{mn}"'] & \\
& & A \arrow[ull, "\gamma_n"] \arrow[ul, "\gamma_m"'] \arrow[uu, "\alpha"'] \arrow[rrr, "\varphi"] & & & B \arrow[ull, "\nu_n"] \arrow[ul, "\nu_m"'] \arrow[uu, "\beta"'] \\
& & R \arrow[u, "\sigma"] \arrow[urrr, "\tau"'] & & &
\end{tikzcd}$$

Figure 3.1: Extension of ring homomorphisms in Hom-sets

of B with respect to an ideal J of B. Let $\sigma : R \to A$ and $\tau : R \to B$. Then for all $\varphi \in \mathrm{Hom}_R(A, B)$ such that $\varphi(I) \subseteq J$ there exists a unique $\hat{\varphi} \in \mathrm{Hom}_R(\hat{A}, \hat{B})$ such that $\hat{\varphi} \circ \alpha \circ \sigma = \beta \circ \tau$.

Proof. Let the hypotheses of Lemma 3.0.7 be satisfied and consider the commuting diagram in Figure 3.1. Since $\varphi(I) \subseteq J$, there is a map $\omega_n : A/I^n \to B/J^n$ for all $n \in \mathbb{N}$ defined by $\omega_n(a + I^n) = \varphi(a) + J^n$. Then

$$\begin{aligned} (\nu_n \circ \varphi)(a) &= \varphi(a) + J^n \\ &= \omega_n(a + I^n) \\ &= (\omega_n \circ \gamma_n)(a). \end{aligned}$$

Now define $\hat{\varphi}((a_n + I^n)_{n\in\mathbb{N}}) := (\varphi(a_n) + J^n)_{n\in\mathbb{N}}$. The ring homomorphism $\hat{\varphi}$ is well defined since if $(a_n + I^n)_{n\in\mathbb{N}} = (b_n + I^n)_{n\in\mathbb{N}}$, then

$$\begin{aligned} \hat{\varphi}((a_n + I^n)_{n\in\mathbb{N}} - (b_n + I^n)_{n\in\mathbb{N}}) = 0 &\Leftrightarrow \hat{\varphi}((a_n - b_n + I^n)_{n\in\mathbb{N}} = 0 \\ &\Leftrightarrow (\varphi(a_n - b_n) + J^n)_{n\in\mathbb{N}} = 0 \\ &\Leftrightarrow (\varphi(a_n) + J^n)_{n\in\mathbb{N}} - (\varphi(b_n) + J^n)_{n\in\mathbb{N}} = 0 \\ &\Leftrightarrow \hat{\varphi}((a_n + I^n)_{n\in\mathbb{N}}) - \hat{\varphi}((b_n + I^n)_{n\in\mathbb{N}}) = 0 \end{aligned}$$

Let $r \in R$. Then

$$\begin{aligned} (\beta \circ \tau)(r) &= (\tau(r) + J^n)_{n\in\mathbb{N}} \\ &= (\varphi(\sigma(r)) + J^n)_{n\in\mathbb{N}} \\ &= \hat{\varphi}((\sigma(r) + I^n)_{n\in\mathbb{N}}) \\ &= (\hat{\varphi} \circ \alpha \circ \sigma)(r). \end{aligned}$$

To show that $\hat{\varphi}$ is unique, suppose $\hat{\varphi}' : \hat{A} \to \hat{B}$ such that $\hat{\varphi}' \circ \alpha = \beta \circ \varphi$. If $a \in A$, then

$$\hat{\varphi}'(\alpha(a)) = (\varphi(a) + J^n)_{n \in \mathbb{N}} = \hat{\varphi}(\alpha(a)).$$

Thus, $\hat{\varphi}' = \hat{\varphi}$. □

Theorem 3.0.8. *Let K be a p-adic field with valuation ring K° and residue field κ (see Definition 1.1.13). Let A be an integral domain and let $\sigma : K^\circ \to A$ be a ring homomorphism such that $A \otimes_{K^\circ} \kappa$ is an integral domain. Let $\hat{A}$ be the completion of A with respect to I (see Definition 2.1.1) where I is the kernel of $\rho_A : A \to A \otimes_{K^\circ} \kappa$ defined by $\rho_A(a) = a \otimes 1$. Then $(\hat{A} \otimes_{K^\circ} K, \hat{A})$ is an affinoid ring and the map*

$$\begin{aligned} \mathrm{Hom}_{(K,K^\circ)}\Big((\hat{A} \otimes_{K^\circ} K, \hat{A}), (K, K^\circ)\Big) &\to \mathrm{Hom}_{K^\circ}(A, K^\circ) \\ (\varphi^\triangleright, \varphi^+) &\mapsto \varphi^+ \circ \alpha \end{aligned}$$

is bijective, where $\alpha : A \to \hat{A}$ is the unique morphism guaranteed by the universal property of inverse limits.

Proof. Let $\varphi \in \mathrm{Hom}_{K^\circ}(A, K^\circ)$. Then by Definition 3.0.6 $\varphi : A \to K^\circ$ and $\varphi \circ \sigma = id_{K^\circ}$. Now, the kernel of $\rho_A : A \to A \otimes_{K^\circ} \kappa$ is $\sigma(\pi)A$ where π is a uniformizer of K.

$$\begin{tikzcd}
\hat{A} \otimes_{K^\circ} \kappa & \hat{A} \arrow[l] \arrow[r] & \hat{A} \otimes_{K^\circ} K \arrow[dd, bend left] \\
A \otimes_{K^\circ} \kappa \arrow[u] & A \arrow[l] \arrow[r] \arrow[u, "\alpha"] \arrow[d, bend left] & A \otimes_{K^\circ} K \arrow[u] \\
\kappa \arrow[u] & K^\circ \arrow[l] \arrow[r] \arrow[u, "\sigma"] & K \arrow[u]
\end{tikzcd}$$

Figure 3.2: Algebraic Theorem

Since K° is complete with respect to π, $\widehat{K^\circ} = K^\circ$. Recall that by Proposition 2.1.9, $\hat{A} \otimes_{K^\circ} K = \hat{A}_{\hat{\pi}}$. Thus, by Proposition 2.8.4, $(\hat{A} \otimes_{K^\circ} K, \hat{A})$ is an affinoid ring.

Since, $\varphi(\sigma(\pi)A) = \pi\varphi(A) \subset \pi K^\circ$, by Lemma 3.0.7 there is a unique $\hat{\varphi} : \hat{A} \to K^\circ$ such that $\hat{\varphi} \circ \alpha = \varphi$. Let $\hat{I} = \{\alpha \in \hat{A} \mid \alpha_1 = 0\}$; then $\hat{I}$ is an ideal of definition in $\hat{A}$ and $\hat{\varphi}(\hat{I}) = \varphi(I) = (\pi)$. Therefore, $\hat{\varphi}$ is adic.

$$\begin{tikzcd}
\hat{A}_{\hat{\pi}} = \hat{A} \otimes_{K^\circ} K \arrow[r, dotted, "\phi^{\rhd}"] & K \\
\hat{A} \arrow[u, "\eta_{\hat{A}}"] \arrow[r, dotted, "\hat{\varphi}"] & K^\circ \arrow[u, "\eta_K"'] \\
A \arrow[u, "\alpha"] \arrow[r, "\varphi"] & K^\circ \arrow[u, equal] \\
K^\circ \arrow[u, "\sigma"] \arrow[ur, "id_K^\circ"'] &
\end{tikzcd}$$

To extend $\hat{\varphi}$ to a ring homomorphism $\phi^{\rhd} : \hat{A}_{\hat{\pi}} \to K$, observe that

$$K = K^\circ \otimes_{K^\circ} K = K^\circ_{\pi}.$$

Thus, the homomorphism $\phi^{\rhd}$ only needs to be defined on $\frac{1}{\hat{\pi}}$. Define $\phi^{\rhd}\left(\frac{1}{\hat{\pi}}\right) = \frac{1}{\pi}$. To show $\phi^{\rhd}$ is f-adic, notice that $\hat{A}$ is a ring of definition of $\hat{A}_{\hat{\pi}}$, K° is a ring of definition of K, and $\phi^{\rhd}(\hat{A}) \subseteq K^\circ$. Now, $\phi^{\rhd}|_{\hat{A}} = \hat{\varphi}$ which is adic. Therefore, $\phi^{\rhd}$ is f-adic. Let $\phi^+ := \phi^{\rhd}|_{\hat{A}} = \hat{\varphi} : \hat{A} \to K^\circ$. Then ϕ^+ is adic. Thus, $(\phi^{\rhd}, \phi^+) \in \mathrm{Hom}_{(K,K^\circ)}\left((\hat{A} \otimes_{K^\circ} K, \hat{A}), (K, K^\circ)\right)$ and $(\phi^{\rhd}, \phi^+) \mapsto \phi^+ \circ \theta$ by construction. Thus, $\mathrm{Hom}_{(K,K^\circ)}\left((\hat{A} \otimes_{K^\circ} K, \hat{A}), (K, K^\circ)\right) \to \mathrm{Hom}_{K^\circ}(A, K^\circ)$ is bijective. $\square$

Proposition 3.0.9. *There exists a bijection between* $\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[T], \mathbb{Z}_p)$ *and* $\mathbb{Z}_p$.

Proof. Let $\sigma : \mathbb{Z}_p \to \mathbb{Z}_p[T]$. Define a map $e_T : \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[T], \mathbb{Z}_p) \to \mathbb{Z}_p$ by $\varphi \mapsto \varphi(T)$. To show e_T is surjective let $z \in \mathbb{Z}_p$. Since $\varphi \circ \sigma = id_{\mathbb{Z}_p}$, there exists $\varphi \in$

$\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[T], \mathbb{Z}_p)$ such that $\varphi(T) = z$. Thus, e_T is surjective. To show e_T is injective let $\varphi, \psi \in \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[T], \mathbb{Z}_p)$ and suppose $e_t(\varphi) = e_T(\psi)$. Thus, $\varphi(T) = \psi(T)$. Now let $\sum_{n=0}^{\infty} a_n T^n \in \mathbb{Z}_p[T]$. Then

$$\varphi\left(\sum_{n=0}^{\infty} a_n T^n\right) = \sum_{n=0}^{\infty} \varphi(a_n)\varphi(T)^n = \sum_{n=0}^{\infty} a_n \psi(T)^n = \psi\left(\sum_{n=0}^{\infty} a_n T^n\right).$$

Thus, $\varphi = \psi$ and e_T is injective. $\square$

Example 3.0.10. *Let $K = \mathbb{Q}_p$ and $A = \mathbb{Z}_p[T]$. By Definition 1.5.1, $K^\circ = \mathbb{Z}_p$. Thus, there is a ring homomorphism $\sigma : \mathbb{Z}_p \to \mathbb{Z}_p[T]$ defined by $\sigma(x) = x$. Then by Example 2.1.6 (2), $\hat{A} = \mathbb{Z}_p\{T\}$. By Remark 1.9.4 $\pi = p$ and thus by Example B.0.5 (4) $\hat{A}_{\hat{\pi}} = \mathbb{Q}_p\{T\}$. Then*

$$\mathrm{Hom}_{(\mathbb{Q}_p,\mathbb{Z}_p)}((\mathbb{Q}_p\{T\}, \mathbb{Z}_p\{T\}), (\mathbb{Q}_p, \mathbb{Z}_p)) \equiv \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[T], \mathbb{Z}_p) \equiv \mathbb{Z}_p.$$

$$\begin{tikzcd}
\mathbb{F}_p[T] & \mathbb{Z}_p\{T\} \arrow[l, "\rho_{\hat{A}}"'] \arrow[r, "\eta_{\hat{A}}"] & \mathbb{Q}_p\{T\} \arrow[dd, bend left] \\
\mathbb{F}_p[T] \arrow[u] & \mathbb{Z}_p[T] \arrow[l, "\rho_A"'] \arrow[r, "\eta_A"] \arrow[u, "\alpha"] \arrow[d, bend left] & \mathbb{Q}_p[T] \arrow[u] \\
\mathbb{F}_p \arrow[u] & \mathbb{Z}_p \arrow[l, "\rho_K"'] \arrow[r, "\eta_K"] \arrow[u, "\sigma"] & \mathbb{Q}_p \arrow[u]
\end{tikzcd}$$

Figure 3.3: Algebraic Theorem for Example 3.0.10

Part II

Geometry

Chapter 4

Category of Schemes

The following chapter introduces the basics of scheme theory such as found in Chapter 1 of *The Geometry of schemes* by David Eisenbud and Joe Harris. By studying the spectrum of a ring, assigning the Zariski topology, and attaching a sheaf to this topological ring, one can construct affine schemes. A scheme is an object that is locally an affine scheme.

4.1 The Set of Prime Ideals

Definition 4.1.1. *Let A be a commutative ring with unity. Then the* spectrum *of A is the set of prime ideals of A and is denoted by* $|\operatorname{Spec}(A)|$.

Remark 4.1.2. *Note that A itself is not a prime ideal, and the zero ideal* (0) *is prime if and only if A is an integral domain.*

Example 4.1.3.

1. *Let K be a field. Then* $|\operatorname{Spec}(K)| = \{(0)\}$.
2. *Let $A = \mathbb{C}[T]$. Then the spectrum of A is* $\{(T-a) \mid a \in \mathbb{C}\} \cup \{(0)\}$.
3. *The spectrum of the integers is* $\{(p) \mid p \text{ prime}\} \cup \{(0)\}$.
4. *The spectrum of the p-adic integers is* $\{(0), (\pi)\}$.

Proposition 4.1.4. *Let A and B be rings and let $\varphi : A \to B$ be a ring homomorphism. If I is a prime ideal of B, then $\varphi^{-1}(I)$ is a prime ideal in A.*

Proof. Let A and B be rings and let I be a prime ideal of B. Let $x \in A$. Then $\varphi(x) \in \varphi(A) \subseteq B$. Since I is an ideal, $\varphi(a)\varphi(x) = \varphi(ax) \in I \Rightarrow ax \in \varphi^{-1}(I)$ for all $a \in \varphi^{-1}(I)$. Similarly, $\varphi(x)\varphi(a) \Rightarrow xa \in \varphi^{-1}(I)$. Therefore, $\varphi^{-1}(I)$ is an ideal of A. Now, let $xy \in \varphi^{-1}(I)$ for $x, y \in A$. Then $\varphi(xy) \in I \Rightarrow \varphi(x)\varphi(y) \in I \Rightarrow \varphi(x)$ or $\varphi(y) \in I$. Without loss of generality, suppose $\varphi(x) \in I$. Then $x \in \varphi^{-1}(I)$, and hence, φ^{-1} is a prime ideal of A. □

Remark 4.1.5. *It is worth noting that, while the inverse image of a prime ideal is a prime ideal, the inverse image of a maximal ideal is not, in general, a maximal ideal. To see this, consider the inclusion map $\iota : \mathbb{Z} \to \mathbb{Q}$. The ideal $I = (0)$ is maximal in $\mathbb{Q}$ but $\iota^{-1}(I) = (0)$ is not maximal in $\mathbb{Z}$.*

Definition 4.1.6. *Let A be a ring. A subset $V(I) \subseteq |\operatorname{Spec}(A)|$ is* closed *in $|\operatorname{Spec}(A)|$ if there exists an ideal I of A such that $V(I) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathfrak{p} \supseteq I\}$. We define open sets $U(I)$ as simply the complement of the closed set $V(I)$ (i.e. $U(I) = \{\mathfrak{p} \in |\operatorname{Spec}(A)| \mid \mathfrak{p} \not\supseteq I\}$).*

Example 4.1.7.

1. *Let $A = \mathbb{C}[x]$ and $I = (x^2)$. Then*

$$V(I) = \{\mathfrak{p} \in |\operatorname{Spec}(\mathbb{C}[x])| \mid \mathfrak{p} \supseteq (x^2)\} = \{(x)\}$$

2. *Let A be a ring and let I be a maximal ideal. Then $V(I) = \{I\}$.*

3. *Let $A = K^{\circ}$. Then $V((\pi)) = \{(\pi)\}$.*

Lemma 4.1.8. *Let A be a ring and $X = |\operatorname{Spec}(A)|$. Then for any ideal I of A there exists a bijection between $V(I)$ and $|\operatorname{Spec}(A/I)|$.*

Proof. The fourth isomorphism theorem for rings [4, 7.3 Theorem 8 (3)], when applied to commutative rings with unity, states that if A is a ring and B is an ideal of A then a ideal I of B is an ideal of A if and only if B/I is an ideal of A/I. Thus,

$$\begin{aligned} \mathfrak{p} \text{ is prime in } A/I &\Leftrightarrow \mathfrak{p} \text{ is prime in } A \text{ and } I \subseteq \mathfrak{p} \\ &\Leftrightarrow \mathfrak{p} \in V(I). \end{aligned}$$

□

Definition 4.1.9. *Let A be a ring and let $X = |\operatorname{Spec}(A)|$. For each $s \in A$, define*

$$X_s = \{\mathfrak{p} \in |\operatorname{Spec}(A)| \mid s \notin \mathfrak{p}\}.$$

Any set of this from is called a distinguished open set. *We will also use the symbol $|\operatorname{Spec}(A)|_s$ to denote this set.*

Remark 4.1.10. *Although $|\operatorname{Spec}(A)|_s \neq |\operatorname{Spec}(A_s)|$ (see Definition B.0.3), there is a canonical bijection between these sets defined by $\mathfrak{p} \in |\operatorname{Spec}(A)|_s \mapsto \lambda_s(\mathfrak{p})A_s \in |\operatorname{Spec}(A_s)|$.*

Definition 4.1.11. *Let A be a ring and let I be an ideal of A. Then $\sqrt{I}$ denotes the set $\{a \in A \mid a^k \in I\}$ for some $k \in \mathbb{N}$ called the* radial *of I.*

Lemma 4.1.12. *Let A be a ring and let I be an ideal of A. Then $\sqrt{I}$ is an ideal and $I \subseteq \sqrt{I}$.*

Proof. Let $i \in \sqrt{I}$ and $a \in A$. Then $i^k \in I$ for some $k \in \mathbb{N}$. Since I is an ideal $(ai)^k = a^k i^k \in I$, so $ai \in \sqrt{I}$. Now, $(a+i)^k = \sum_{n=0}^{k} \binom{k}{n} a^k i^{k-n} \in I$ (since each term is in I and I is an ideal), so $a+i \in \sqrt{I}$. Thus, $\sqrt{I}$ is an ideal. The fact that $I \subseteq \sqrt{I}$ is obvious by the definition. □

Lemma 4.1.13. *Let A be a ring, $X = |\operatorname{Spec}(A)|$. Then $X_s \subseteq X_t$ if and only if $(s) \subseteq \sqrt{(t)}$.*

Proof. Let $S = (s)$ and $T = (t)$. Then

$$\begin{aligned} X_s \subseteq X_t &\Rightarrow U(S) \subseteq U(T) \\ &\Rightarrow V(S) \supseteq V(T) \\ &\Rightarrow \bigcap_{\mathfrak{p} \in V(S)} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p} \in V(T)} \mathfrak{p} \\ &\Rightarrow \sqrt{S} \subseteq \sqrt{T} \\ &\Rightarrow S \subseteq \sqrt{T}. \end{aligned}$$

Conversely, suppose $(s) \in \sqrt{(t)}$. Then $s^k = tr$ for some $r \in A$ and some $k \in \mathbb{N}$. Now,

$$\begin{aligned} X_s &= \{\mathfrak{p} \in |\operatorname{Spec}(A)| \mid s \notin \mathfrak{p}\} \\ &= \{\mathfrak{p} \in |\operatorname{Spec}(A)| \mid s^k \notin \mathfrak{p}\} \text{ since } \mathfrak{p} \text{ is prime} \\ &= X_{s^k} \\ &= X_{tr} \qquad (4.1) \\ &\subseteq X_t. \end{aligned}$$

□

Corollary 4.1.14. *Let A be a ring and let $X = |\operatorname{Spec}(A)|$. If $X_s \subseteq X_t$, then there exists $r \in A$ such that $X_s = X_{tr}$.*

Proof. A direct result from Lemma 4.1.13, Equation 4.1. □

4.2 Zariski Topology

Proposition 4.2.1. *Let A be a ring. The sets $V(I)$, as I runs over all ideals of A, define a topology $\mathcal{Z}$ on* $|\operatorname{Spec}(A)|$.

Proof. Let $X = |\operatorname{Spec}(A)|$. To show that the sets $V(I)$ define a topology on $|\operatorname{Spec}(A)|$, we begin by showing $\emptyset \in \mathcal{Z}$ and $X \in \mathcal{Z}$

$$V((0)) = \{\mathfrak{p} \in X \mid \mathfrak{p} \supseteq (0)\} = X$$

$$V(A) = \{\mathfrak{p} \in X \mid \mathfrak{p} \supseteq A\} = \emptyset.$$

To show that $\mathcal{Z}$ is closed under arbitrary intersection consider $\bigcap_{\alpha\in J} V(I_\alpha)$. Now,

$$\begin{aligned} \mathfrak{p} \in \bigcap_{\alpha\in J} V(I_\alpha) &\Leftrightarrow \mathfrak{p} \supseteq I_\alpha \text{ for all } \alpha \\ &\Leftrightarrow \mathfrak{p} \supseteq \bigcup_{\alpha\in J} I_\alpha \\ &\Leftrightarrow \mathfrak{p} \in V\left(\bigcup_{\alpha\in J} I_\alpha\right). \end{aligned}$$

Thus, $\mathcal{Z}$ is closed under arbitrary intersection so it only remains to be shown that $\mathcal{Z}$ is closed under finite union. Consider $V(I) \cup V(J)$ where I and J are ideals in A. Since $IJ \subset I$ and $JI \subseteq J$ it is clear that $V(I) \cup V(J) \subseteq V(IJ)$. To prove $V(IJ) \subseteq V(I) \cup V(J)$, let $\mathfrak{p} \in V(IJ)$ and suppose $J \not\subseteq \mathfrak{p}$. Then we want to prove $I \subseteq \mathfrak{p}$. Let $f \in J$ such that $f \notin \mathfrak{p}$ and let $g \in I$. Then $fg \in \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal and $f \notin \mathfrak{p}$, we conclude that $g \in \mathfrak{p}$. This means $I \subseteq \mathfrak{p}$ since g was arbitrary. Hence $\mathfrak{p} \in V(I)$ as required. Thus, $\mathcal{Z}$ is indeed a topology on $|\operatorname{Spec}(A)|$. □

Definition 4.2.2. *Let A be a ring. The topology $\mathcal{Z}$ on* $|\operatorname{Spec}(A)|$ *is called the* Zariski Topology. *Write* $\operatorname{Spec}(A)$ *for the topological space formed by equipping* $|\operatorname{Spec}(A)|$ *with $\mathcal{Z}$.*

Proposition 4.2.3. *The distinguished open sets form a base for the Zariski topology.*

Proof. By Section 1.2, it suffices to show that for all points $\mathfrak{p}$ in an open subset $U \subseteq A$ there is a distinguished open set X_f such that $\mathfrak{p} \in X_f \subseteq U$. Let $S \subseteq A$ and $\mathfrak{p} \in \mathrm{Spec}(A)$. Suppose $\mathfrak{p} \in U(S)$. Then $S \not\subseteq \mathfrak{p}$, so let $f \in S, f \notin \mathfrak{p}$. Thus, $\mathfrak{p} \in X_f$. To show that $X_f \subseteq U(S)$, let $\mathfrak{q} \in X_f$. Then $f \notin \mathfrak{q}$ so that $S \not\subseteq \mathfrak{q}$. Hence $\mathfrak{q} \in U(S)$. Thus for every $\mathfrak{p} \in U(S)$, there is some X_f such that $\mathfrak{p} \in X_f \subseteq U(S)$ and hence, the distinguished open sets form a base for the topology. $\square$

Lemma 4.2.4. *A singleton* $\{\mathfrak{p}\}$ *in* $\mathrm{Spec}(A)$ *is closed if and only if* $\mathfrak{p}$ *is a maximal ideal of* A.

Proof. Suppose $\mathfrak{p}$ is a maximal ideal of A. Then there are no ideals I of A such that $\mathfrak{p} \subset I$ and especially no such prime ideals. Thus, $V(\mathfrak{p}) = \{\mathfrak{p}\}$ so $\mathfrak{p}$ is closed. Now suppose $\mathfrak{p} \in X$ is a closed point. Then there exists some ideal I of A such that $V(I) = \{\mathfrak{p}\}$ so $I \subseteq \mathfrak{p}$. Now suppose J is an ideal of A such that $\mathfrak{p} \subset J \subseteq A$. Assume $J \neq A$. By Zorn's Lemma, every proper ideal is contained in some maximal ideal. Therefore, let $\mathfrak{m}$ be a maximal ideal such that $J \subseteq \mathfrak{m}$. Since every maximal ideal is prime, $\mathfrak{m} \in V(I)$. This contradicts $V(I) = \{\mathfrak{p}\}$, and hence $J = A$ and $\mathfrak{p}$ is maximal. $\square$

Definition 4.2.5. *Let* $\varphi : A \to B$ *be a ring homomorphism. Let*

$$\begin{aligned} \mathrm{Spec}(\varphi) : \mathrm{Spec}(B) &\to \mathrm{Spec}(A) \\ J &\mapsto \varphi^{-1}(J). \end{aligned}$$

Recall that $\varphi^{-1}(J)$ *is a prime ideal by Proposition 4.1.4.*

Proposition 4.2.6. *If* $\varphi : A \to B$ *is a ring homomorphism, then* $\mathrm{Spec}(\varphi)$*, as defined above, is continuous.*

Proof. Consider the open subset $X_s \subseteq \mathrm{Spec}(A)$. Then,

$$\begin{aligned}
(\mathrm{Spec}(\varphi))^{-1}(X_s) &= \{\mathfrak{b} \in \mathrm{Spec}(B) \mid \mathrm{Spec}(\varphi)(\mathfrak{b}) \in X_s\} \\
&= \{\mathfrak{b} \in \mathrm{Spec}(B) \mid \varphi^{-1}(\mathfrak{b}) \in X_s\} \\
&= \{\mathfrak{b} \in \mathrm{Spec}(B) \mid s \notin \varphi^{-1}(\mathfrak{b})\} \\
&= \{\mathfrak{b} \in \mathrm{Spec}(B) \mid \varphi(s) \notin \mathfrak{b}\} \\
&= Y_{\varphi(s)} \subseteq \mathrm{Spec}(B).
\end{aligned}$$

□

Remark 4.2.7. *The functor* Spec *is a contravarient functor.*

4.3 Local Ringed Spaces

The following section introduces the category of local ringed spaces and assumes the reader has some background knowledge on sheaves of rings. If needed see Appendix C for an introduction to the study of sheaves. For further details see Eisenbud and Harris [5] or Hartshorne [8].

Definition 4.3.1. *A* local ringed space *is a pair* $(X, \mathcal{O}_X)$ *where* X *is a topological space and* $\mathcal{O}_X$ *is a sheaf on* X *such that for each* $x \in X$ *the stalk* $\mathcal{O}_{X,x}$ *(see Definition C.0.21) is a local ring. The unique maximal ideal in* $\mathcal{O}_{X,x}$ *is denoted* $\mathfrak{m}_{X,x}$*. A morphism of local ringed spaces from* $(X, \mathcal{O}_X)$ *to* $(Y, \mathcal{O}_Y)$ *is a pair* $(f, f^{\#})$ *consisting of a continuous morphism* $f : X \to Y$ *and a morphism* $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ *of sheaves* *of rings on* Y *such that if* $x \in X$ *and* $y = f(x)$, *then* $(f^{\#}_{x,y})^{-1}(\mathfrak{m}_{X,x}) = \mathfrak{m}_{Y,y}$, *where* $f^{\#}_{x,y}$ *denotes the composition* $\mathcal{O}_{Y,y} \to (f_*\mathcal{O}_X)_y \to \mathcal{O}_{X,x}$ *(where these maps result from the universal property of direct limits (see Proposition A.1.3)).*

Example 4.3.2. *The pair* $(\mathrm{Spec}(A), \mathcal{O}_A)$ *is a local ringed space where* $\mathcal{O}_A$ *is the sheaf defined in Definition C.0.23.*

Remark 4.3.3. *Local ringed spaces form a category.*

Definition 4.3.4. *Let* A *and* B *be rings with ring homomorphism* $\varphi : A \to B$. *Let* $\mathrm{Sch}(A)$ *denote the local ringed space* $(\mathrm{Spec}(A), \mathcal{O}_A)$. *Define* $\mathrm{Sch}(\varphi) : \mathrm{Sch}(B) \to \mathrm{Sch}(A)$ *by* $\mathrm{Sch}(\varphi) := (f, f^{\#})$ *such that* $f := \mathrm{Spec}(\varphi)$ *from Definition 4.2.5 and* $f^{\#}(\mathrm{Spec}(A)_s) := \varphi_s(A_s)$ *as in Definition B.0.6.*

Remark 4.3.5. *Let* $\varphi : A \to B$, $X = \mathrm{Spec}(B)$, *and* $Y = \mathrm{Spec}(A)$. *To see that* $\mathrm{Sch}(\varphi)$ *is well-defined let* $Y_s \subseteq Y_t$ *and consider the following commutative diagram*

$$\begin{tikzcd} \mathcal{O}_Y(Y_s) \arrow[r] & f_*\mathcal{O}_X(Y_s) \\ \mathcal{O}_Y(Y_t) \arrow[r] \arrow[u] & f_*\mathcal{O}_X(Y_t) \arrow[u] \end{tikzcd}$$

Then $f_*\mathcal{O}_X(Y_s) = \mathcal{O}_X(f^{-1}(Y_s)) = \mathcal{O}_X(X_{\varphi(s)}) = B_{\varphi(s)}$. *Thus, we have*

$$\begin{tikzcd} A_s \arrow[r] & B_{\varphi(s)} \\ A_t \arrow[r] \arrow[u] & B_{\varphi(t)} \arrow[u] \end{tikzcd}$$

Now by Corollary 4.1.14 there exists $r \in A$ *such that* $A_s = A_{tr}$. *Thus, the above diagram commutes by Lemma B.0.7. Hence* $\mathrm{Sch}(\varphi)$ *is well-defined by Proposition C.0.18 (2).*

Lemma 4.3.6. *If* $\varphi : A \to B$ *and* $\psi : B \to C$*, then* $\mathrm{Sch}(\psi \circ \varphi) = \mathrm{Sch}(\varphi) \circ \mathrm{Sch}(\psi)$.

Proof. Let

$$\begin{array}{ll} (f, f^{\#}) : \mathrm{Sch}(B) \to \mathrm{Sch}(A) & \text{induced by } \varphi \\ (g, g^{\#}) : \mathrm{Sch}(C) \to \mathrm{Sch}(B) & \text{induced by } \psi \\ (h, h^{\#}) : \mathrm{Sch}(C) \to \mathrm{Sch}(A) & \text{induced by } \psi \circ \varphi. \end{array}$$

Let $\mathfrak{p} \in \mathrm{Spec}(C)$. Then

$$\begin{aligned} h(\mathfrak{p}) &= (\mathrm{Spec}(\psi \circ \varphi))(\mathfrak{p}) \\ &= (\psi \circ \varphi)^{-1}(\mathfrak{p}) \\ &= (\varphi^{-1}(\psi^{-1}(\mathfrak{p})) \\ &= (\mathrm{Spec}(\varphi)(\mathrm{Spec}(\psi))(\mathfrak{p}) \\ &= (\mathrm{Spec}(\varphi) \circ \mathrm{Spec}(\psi))(\mathfrak{p}) \\ &= (f \circ g)(\mathfrak{p}) \end{aligned}$$

and

$$\begin{aligned} h^{\#}(\mathrm{Spec}(C)_s) &= (\psi \circ \varphi)_s && \text{By Definition 4.3.4} \\ &= \psi_{\varphi(s)} \circ \varphi_s && \text{By Lemma B.0.8} \\ &= (g^{\#} \circ f^{\#})(\mathrm{Spec}(C)_s) && \text{By Definition 4.3.4.} \end{aligned}$$

□

4.4 Residue Fields

Definition 4.4.1. *Let* $(X, \mathcal{O}_X)$ *be a local ringed space and let* $x \in X$*. If* $X = \mathrm{Spec}(A)$ *then the residue field* $\kappa(x)$ *is the quotient field of* $A/\mathfrak{p}$*, where* $\mathfrak{p}$ *is the prime ideal corresponding to* x*. If* $f \in A$ *define* $f(x)$ *to be the image of* f *via the morphisms* $A \to A/\mathfrak{p} \to \kappa(x)$.

Remark 4.4.2. *Let* $X = \mathrm{Sch}(A)$ *(see Definition 4.3.4). Then* $\mathcal{O}_X = \mathcal{O}_A$ *and* $\mathcal{O}_{X,x} = A_{\mathfrak{p}}$ *where* $x = \mathfrak{p}$ *and* $\mathfrak{m}_{X,x} = \mathfrak{p}A\mathfrak{p}$*. Thus, the residue field is* $qf\,(A/\mathfrak{p})$ *which is* $(A/\mathfrak{p})_{(0)}$ *[see Appendix B.0.5 (5)], which is also* $A\mathfrak{p}/\mathfrak{p}A\mathfrak{p}$*, which can be written as* $\mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$*. This last expression can be used to define the residue field* $\kappa(x)$ *for* $x \in X$ *when* X *is an arbitrary local ringed space.*

Example 4.4.3. *Let* K *be a p-adic field and let* $X = \mathrm{Spec}(K^\circ)$*. Then* $X = \{x, y\}$ *where* $x = (0)$ *and* $y = (\pi)$*. Then*

$$\kappa(x) = qf\,(K^\circ/(0)) = qf(K^\circ) = K$$

$$\kappa(y) = qf\,(K^\circ/(\pi)) = qf(\kappa) = \kappa.$$

Recall that by Example 4.1.7 (3), (π) *is closed in* $\mathrm{Spec}(K^\circ)$*. Thus, this example illustrates that the residue field defined in Chapter 1 is the residue field of a closed point in* $\mathrm{Spec}(K^\circ)$*.*

Remark 4.4.4. *The Zariski topology can also be thought of in terms of residue fields. For each subset* $S \subseteq A$*,*

$$V(S) = \{x \in \mathrm{Spec}(A) \mid f(x) = 0 \textit{ for all } f \in S\}.$$

Likewise, if $X = \mathrm{Spec}(A)$ *then, for each* $f \in A$*,*

$$X_f = \{x \in \mathrm{Spec}(A) \mid f(x) \neq 0\}.$$

4.5 Affine Schemes

Definition 4.5.1. *Recall the definition of a stalk from Definition C.0.21. For any local ringed space* $(X, \mathcal{O}_X)$ *define*

$$\begin{aligned} \pi_{X,x} : \mathcal{O}_X(X) &\to \mathcal{O}_{X,x} \\ s &\mapsto [X, s] \end{aligned}$$

where $[X, s] \sim [U, t]$ *if there exists* $V \subseteq U \subseteq X$*, open in* X*, with* $x \in V, u \in \mathcal{O}_X(V)$ *such that* $\mathcal{O}_X(V \subseteq X)(s) = \mathcal{O}_X(V \subseteq U)(t)$*. If* $f \in \mathcal{O}_X(X)$*, then*

$$\begin{aligned} X_f &= \{x \in X \mid f(x) \neq 0\} \\ &= \{x \in X \mid \pi_{X,x}(f) \notin \mathfrak{m}_{X,x}\} \\ &= \{x \in X \mid f \notin \pi_{X,x}^{-1}(\mathfrak{m}_{X,x})\}. \end{aligned}$$

Definition 4.5.2. *An affine scheme is a local ringed space* $(X, \mathcal{O}_X)$ *such that*

1. *the ring of sections* $\mathcal{O}_X(X_f)$ *(see C.0.9) equals the localization* $\mathcal{O}_X(X)_f$ *for all* $f \in \mathcal{O}_X(X)$
2. *the morphism* $X \to \mathrm{Sch}(\mathcal{O}_X(X))$ *defined by* $x \mapsto \pi_{X,x}^{-1}(\mathfrak{m}_{X,x})$ *is a homeomorphism where* $\pi_{X,x} : \mathcal{O}_X(X) \to \mathcal{O}_{X,x}$ *and* $\mathfrak{m}_{X,x}$ *is the maximal ideal of* $\mathcal{O}_{X,x}$*.*

Definition 4.5.3. *Let* X *and* S *be affine schemes. The pair* (X, a) *is called an* affine S-scheme *(also called an* affine scheme over S*) if* $a : X \to S$*. A morphism of affine* S*-schemes* (X, a) *and* (X, b) *is a morphism of schemes* $f : X \to Y$ *such that* $b \circ f = a$*.*

Proposition 4.5.4. *If* $(X, \mathcal{O}_X)$ *is an affine scheme there exists an isomorphism in the category of local ringed spaces* $(X, \mathcal{O}_X) \to \mathrm{Sch}(\mathcal{O}_X(X))$*.*

Proof. We want to define an isomorphism in the category of local ringed spaces. Let $f : X \longrightarrow \mathrm{Spec}(\mathcal{O}_X(X))$ be defined by $x \mapsto \pi_{X,x}^{-1}(\mathfrak{m}_{X,x})$. This is a homeomorphism by Definition 4.5.2 (2). Now let $A = \mathcal{O}_X(X)$ and let $s \in A$. Then

$$\begin{aligned}
f^{-1}(\mathrm{Spec}(A_s)) &= \{x \in X \mid f(x) \in \mathrm{Spec}(A_s)\} \\
&= \{x \in X \mid \pi_{X,x}^{-1}(\mathfrak{m}_{X,x}) \in \mathrm{Spec}(A_s)\} \\
&= \{x \in X \mid s \notin \pi_{X,x}^{-1}(\mathfrak{m}_{X,x})\} \\
&= \{x \in X \mid \pi_{X,x}(s) \notin (\mathfrak{m}_{X,x})\} \\
&= \{x \in X \mid s(x) \neq 0\} \\
&= X_s.
\end{aligned}$$

Now, $f^{\#}(\mathrm{Spec}(A_s)) : \mathcal{O}_A(\mathrm{Spec}(A_s)) \longrightarrow (f_*\mathcal{O}_X)(\mathrm{Spec}(A_s)) = \mathcal{O}_X(f^{-1}(\mathrm{Spec}(A_s)))$ can be viewed as $f^{\#}(\mathrm{Spec}(A_s)) : A_s \longrightarrow \mathcal{O}_X(X_s)$ or $f^{\#}(\mathrm{Spec}(A_s)) : A_s \longrightarrow A_s$. Therefore we can define $f^{\#}(\mathrm{Spec}(A_s)) = id_{A_s}$. Hence $(f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow \mathrm{Sch}(\mathcal{O}_X(X))$ is an isomorphism. □

Proposition 4.5.5. *Let $(f, f^{\#}) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ be a morphism of affine schemes. Let $\varphi = f^{\#}(Y)$. Then $(f, f^{\#}) = \mathrm{Sch}(\varphi)$.*

$$\begin{tikzcd}
(X, \mathcal{O}_X) \arrow[r, "{(f,f^{\#})}"] \arrow[d, "\xi"'] & (Y, \mathcal{O}_Y) \arrow[d, "\psi"] \\
\mathrm{Sch}(\mathcal{O}_X(X)) \arrow[r, "\mathrm{Sch}(\varphi)"] & \mathrm{Sch}(\mathcal{O}_Y(Y))
\end{tikzcd} \tag{4.2}$$

Proof. Let the morphisms ξ and ψ be viewed as the isomorphism shown to exist in Proposition 4.5.4. Therefore we have $(f, f^{\#}) = \mathrm{Sch}(\varphi)$ if Diagram 4.2 commutes. First consider the morphisms on sheaves. Recall that the stalk $(f_*\mathcal{O}_X)_y$ is a direct

$$\begin{array}{ccccc} \mathcal{O}_{Y,y} & \longrightarrow & (f_*\mathcal{O}_X)_y & \longrightarrow & \mathcal{O}_{X,x} \\ \uparrow & & \uparrow & \nearrow & \\ \mathcal{O}_Y & \longrightarrow & (f_*\mathcal{O}_X) = \mathcal{O}_X & & \end{array}$$

Figure 4.1: Sheaves and Stalks

limit of rings taken over the direct system of open sets containing y (see Definition C.0.21). Since Y is one such set, by the definition of direct limits there exists a ring homomorphism $\rho_Y : (f_*\mathcal{O}_X)(Y) \to (f_*\mathcal{O}_X)_y$. Given $\nu_Y : (f_*\mathcal{O}_X)(Y) \to \mathcal{O}_{X,x}$ there is a unique map $\theta : (f_*\mathcal{O}_X)_y \to \mathcal{O}_{X,x}$ by the universal property of direct limits (see Proposition A.1.3). Thus, the triangle on the right in Figure 4.1 commutes. A similar argument shows the square on the left commutes. Therefore, the entire diagram commutes and $(f, f^\#) = \mathrm{Sch}(\varphi)$ on the level of sheaves. Now on the level of sets, Diagram 4.2 gives

$$\begin{array}{ccc} x & \longmapsto & f(x) =: y \\ \downarrow & & \downarrow \\ \mathfrak{p} := \pi_{X,x}^{-1}(\mathfrak{m}_{X,x}) & \longmapsto & \pi_{Y,y}^{-1}(\mathfrak{m}_{Y,y}) =: \mathfrak{q} \end{array}$$

To show that $\mathfrak{p} \mapsto \mathfrak{q}$ recall that $\mathfrak{m}_{X,x}$ is the maximal ideal in $\mathcal{O}_{X,x}$ and $\mathfrak{m}_{Y,y}$ is the maximal ideal in $\mathcal{O}_{Y,y}$. Since the morphisms on sheaves commute in Figure 4.1, we have $\mathfrak{p} \mapsto \mathfrak{q}$. Thus, $(f, f^\#) = \mathrm{Sch}(\varphi)$. $\square$

Definition 4.5.6. *Let ASchemes denote the category consisting of affine schemes and morphisms of local ringed spaces as objects and morphisms respectively.*

Theorem 4.5.7. *The category of affine schemes is equivalent to the category of commutative rings.*

Proof. Let A be a ring and $(X, \mathcal{O}_X)$ be an affine scheme with ring homomorphism $\varphi : A \to \mathcal{O}_X(X)$. Let Sch : *Rings* $\to$ *ASchemes* and Γ : *ASchemes* $\to$ *Rings* be defined by

- $\mathrm{Sch}(A) = (\mathrm{Spec}(A), \mathcal{O}_A)$
- $\mathrm{Sch}(\varphi) = (\mathrm{Spec}(\varphi), f^{\#})$ where $f^{\#}(Y_s) = \varphi_s$ (see Definition 4.3.4)
- $\Gamma((X, \mathcal{O}_X)) = \mathcal{O}_X(X)$ (see Definition C.0.23)
- $\Gamma((f, f^{\#})) = \varphi$.

Now,

- $(\Gamma \circ \mathrm{Sch})(A) = \Gamma(\mathrm{Spec}(A), \mathcal{O}_A) = \mathcal{O}_A(\mathrm{Spec}(A)) = A$ by Definition C.0.23
- $(\Gamma \circ \mathrm{Sch})(\varphi) = \Gamma(f, f^{\#}) = \varphi$ by Proposition 4.5.5
- $(\mathrm{Sch} \circ \Gamma)(X, \mathcal{O}_X) = \mathrm{Sch}(\mathcal{O}_X(X)) = (X, \mathcal{O}_X)$ by Proposition 4.5.4
- $(\mathrm{Sch} \circ \Gamma)(f, f^{\#}) = \mathrm{Sch}(\varphi) = (f, f^{\#})$ by Definition 4.3.4.

Thus, $\Gamma \circ \mathrm{Sch} = id_{ASchemes}$ and $\mathrm{Sch} \circ \Gamma = id_{Ring}$. Therefore, the category of affine schemes is equivalent to the category of rings. $\square$

Corollary 4.5.8. *Let A,B and C be rings and let $X = \mathrm{Sch}(B)$, $Y = \mathrm{Sch}(A)$ and $S = \mathrm{Sch}(C)$. Then*

$$\mathrm{Hom}_C(A, B) \equiv \mathrm{Hom}_S(X, Y).$$

Proof. Let $\varphi \in \mathrm{Hom}_C(A, B)$. By Theorem 4.5.7 we have the following

$$\begin{tikzcd}
A \arrow[rr, "\varphi"] & & B & & & Y & & X \arrow[ll, "(f,f^{\#})"'] \arrow[dl] \\
& C \arrow[ul] \arrow[ur] & & \arrow[r, bend left, "\mathrm{Sch}"] & \arrow[l, bend left, "\Gamma"] & & S \arrow[from=ul] &
\end{tikzcd}$$

Thus, $(f, f^{\#}) \in \mathrm{Hom}_S(X, Y)$. $\square$

Proposition 4.5.9. *If $S = \mathrm{Sch}(\mathbb{Z}_p)$ and $X = \mathrm{Sch}(\mathbb{Z}_p[T])$, then there exists a bijection between $\mathrm{Hom}_S(S, X)$ and $\mathbb{Z}_p$.*

Proof. By Corollary 4.5.8 $\mathrm{Hom}_S(S, X) \equiv \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p[T], \mathbb{Z}_p)$ which is equivalent to $\mathbb{Z}_p$ by Proposition 3.0.9. $\square$

Definition 4.5.10. *An* affine integral scheme *is an affine scheme $(X, \mathcal{O}_X)$ such that $\mathcal{O}_X(X)$ is an integral domain.*

Example 4.5.11. *Let $A = \mathbb{Z}_p[T]$. The affine scheme* $(\mathrm{Spec}(A), \mathcal{O}_A)$ *is an integral affine scheme over $\mathbb{Z}_p$.*

4.6 Schemes

In this paper all examples and results deal with affine schemes, however, for completeness the more general notion of a scheme will be defined.

Definition 4.6.1. *A* scheme *is a local ringed space $(X, \mathcal{O}_X)$ in which every point has an open neighbourhood U such that the topological space U together with the restricted sheaf $\mathcal{O}_X|_U$ (see Definition C.0.15) is an affine scheme. Let X and S be* *a schemes. The pair (X, a) is called an S-*scheme *(also called a* scheme over S*) if $a : X \to S$. A morphism of S-schemes (X, a) and (X, b) is a morphism of schemes $f : X \to Y$ such that $b \circ f = a$.*

4.7 Varieties

Definition 4.7.1. *Let $X = \mathrm{Sch}(A)$ be an affine scheme and let I be an ideal of A. A* variety *is $X(I) := \mathrm{Sch}(A/I)$ equipped with a morphism $f : X(I) \to X$ defined as $\mathrm{Sch}(A \to A/I)$.*

Remark 4.7.2. *Note that by Lemma 4.1.8, $X(I) = V(I)$.*

Example 4.7.3.

1. *If $X = \mathrm{Sch}(K[x])$, then $X(I) = \mathrm{Sch}(K[x]/(x)) = \{(0)\}$ is a variety.*
2. *If $X = \mathrm{Sch}(\mathbb{Z}_{(p)})$, then $X(I) = \mathrm{Sch}(\mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}) = \{(0)\}$ is a variety.*
3. *If $X = \mathrm{Sch}(K^\circ)$, then $X(I) = \mathrm{Sch}(\kappa)$ is a variety.*

4.8 Formal Schemes

Formal Schemes are used in a very limited way in the main Geometric Theorem. The sheaf is not needed in the main results but is stated here for completeness.

Definition 4.8.1. *Let $X(I)$ be a variety. The* completion of X along the variety *$X(I)$ is the set of open prime ideals in* $\mathrm{Spec}(\hat{A})$ *where $\hat{A}$ is the completion of A with respect to I (see Definition 2.1.1). This set is denoted by* $|\mathrm{Spf}(A)|$*. If equipped with the subspace topology, the topological space is denoted by* $\mathrm{Spf}(A)$*. A sheaf on*

$\mathcal{X}$ *is* $\mathcal{O}_{\mathcal{X}} = \varprojlim_{n \in \mathbb{N}} \mathcal{O}_X/\mathcal{I}_X^n$ *where* $\mathcal{I}_X = \mathcal{O}_X(U \subseteq X)(I)\mathcal{O}_X(U)$. *The local ringed space* $\mathrm{Schf}(A) = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ *is a* formal scheme.

Remark 4.8.2. *Affine formal schemes are not affine schemes as illustrated in the following example.*

Example 4.8.3. *Consider the affine formal scheme* $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ *obtained by completing* $k[x]$ *along* $k[x]/(x)$. *Then* $|\mathrm{Spf}(k[x])| = \{(0)\}$ *and* $\mathcal{O}_{\mathcal{X}}(k[x]) = K[[x]]$. *If this were an affine scheme then* $|\operatorname{Spec}(\mathcal{O}_{\mathcal{X}}(k[x]))| = |\mathrm{Spf}(k[x])|$. *However,*

$$|\operatorname{Spec}(K[[x]])| = \{(0), (x)\} \neq |\mathrm{Spf}(k[x])|.$$

Thus, this is not an affine scheme.

4.9 Special Fibres

Definition 4.9.1. *Let* K *be a* p*-adic field and let* K° *and* κ *be the ring of integers and residue field respectively. Let* $X = \mathrm{Sch}(A)$ *be an affine scheme. The* special fibre *is the pull-back of*

$$X \longrightarrow \mathrm{Sch}(K^\circ) \longleftarrow \mathrm{Sch}(\kappa)\,.$$

Let X_κ *denote the special fibre.*

Example 4.9.2. *If* $K = \mathbb{Q}_p$, $K^\circ = \mathbb{Z}_p$, *and* $\kappa = \mathbb{F}_p$, *then* $\mathrm{Sch}(\mathbb{F}_p[T])$ *is the special fibre along with the canonical morphisms.*

Lemma 4.9.3. *If* $X = \mathrm{Sch}(A)$, *then the special fibre is* $\mathrm{Sch}(A/\sigma(\mathfrak{p}_K)A)$ *with the canonical morphisms.*

$$\begin{tikzcd}
 & & \mathrm{Sch}(B) \arrow[dll, bend right, "\nu"'] \arrow[dl, dotted, "\exists\,!\,\theta"'] \arrow[ddl, bend left, "\psi"] \\
\mathrm{Sch}(A) \arrow[d, "\sigma"] & \mathrm{Sch}(A/\sigma(\mathfrak{p}_K)A) \arrow[l, "f"'] \arrow[d, "g"] & \\
\mathrm{Sch}(K^\circ) & \mathrm{Sch}(\kappa) \arrow[l, "\varphi"] &
\end{tikzcd}$$

Figure 4.2: Special Fibre of $\mathrm{Sch}(A)$

Proof. In Figure 4.2, take global sections of each affine scheme (see Definition C.0.9). Thus, we have

$$\begin{tikzcd}
 & & B \\
A \arrow[r, "f"] \arrow[urr, bend left, "\nu"] & A/\sigma(\mathfrak{p}_K)A \arrow[ur, dotted, "\exists\,!\,\theta"] & \\
K^\circ \arrow[u, "\sigma"] \arrow[r, "\varphi"] & \kappa \arrow[u, "g"] \arrow[uur, bend right, "\psi"'] &
\end{tikzcd}$$

By Proposition C.0.13, it suffices to show that with morphisms f and g defined below, $A/\sigma(\mathfrak{p}_K)A$ is the pushout in the above diagram. Let $a \in A$ and $k \in \kappa$. Define $f(a) = a + \sigma(\mathfrak{p}_K)A$ and $g(k) = \sigma(s) + \sigma(\mathfrak{p}_K)A$ where $k = s + \mathfrak{p}_K$ with $s \in K^\circ$. Then $(f \circ \sigma)(s) = \sigma(s) + \sigma(\mathfrak{p}_K)A$ and $(g \circ \varphi)(s) = g(\varphi(s)) = g(s + \mathfrak{p}_{\mathfrak{K}}) = \sigma(s) + \sigma(\mathfrak{p}_K)A$. Therefore, the diagram commutes. Suppose, $\nu \circ \sigma = \psi \circ \varphi$. Then $(\nu \circ \sigma)(s) = \nu(\sigma(s)) = \psi(\varphi(s)) = \psi(s + \mathfrak{p}_K) = \psi(k)$. Define $\theta(a + \sigma(\mathfrak{p}_K)A) = \nu(a)$. Now, $(\theta \circ f)(a) = \theta(a + \sigma(\mathfrak{p}_K)A) = \nu(a)$ and $(\theta \circ g)(k) = \theta(\sigma(a)) = \sigma(\mathfrak{p}_K)A = \nu(\sigma(a)) = \psi(\varphi(a)) = \psi(k)$.

Now consider $\theta' : A/\sigma(\mathfrak{p}_K)A \to B$ such that $\theta' \circ f = \nu$ and $\theta' \circ g = \psi$. Then $\theta'(a + \sigma(\mathfrak{p}_K)A) = (\theta' \circ f)(a) = \nu(a) = \theta(a + \sigma(\mathfrak{p}_K)A)$ and therefore $\theta' = \theta$ and θ is unique. $\square$

Corollary 4.9.4. *If X is an affine scheme then X_κ is a variety.*

Example 4.9.5. *By Corollary 4.9.4 special fibres, varieties, and formal schemes are all related. Let $K = \mathbb{Q}_p$, $K^\circ = \mathbb{Z}_p$ and $\kappa = \mathbb{F}_p$. Then $X = \mathrm{Sch}(\mathbb{Z}_p[T])$ is an integral affine scheme over K° by Example 4.5.11 and $\mathrm{Sch}(\mathbb{F}_p[T]) = \mathrm{Sch}(\mathbb{Z}_p[t]/p\mathbb{Z}_p[T])$ is the special fibre by Example 4.9.2. The completion of X along the special fibre is $\mathrm{Schf}(\varprojlim_{n\in\mathbb{N}} \mathbb{Z}_p[T]/p\mathbb{Z}_p[T]) = \mathrm{Schf}(\mathbb{Z}_p\{T\})$ by Example 2.1.6 (2).*

Chapter 5

Adic Spaces

This chapter gives an introduction to affinoid adic spaces. For a more thorough understanding of the subject, see Huber [10]. The reader should notice that the construction of affinoid adic spaces parallels the construction of affine schemes.

5.1 The Set of Continuous Valuations

Definition 5.1.1. *Let $A = (A^{\triangleright}, A^{+})$ be an affinoid ring (see Definition 2.8.1). The set of equivalence classes of continuous non-archimedean valuations v on $A^{\triangleright}$ (written multiplicatively) such that $v(a) \leq 1 \ \forall \ a \in A^{+}$ is denoted by $|\mathrm{Spa}(A)|$.*

Example 5.1.2.

By Ostrowski (see Theorem 1.2.4) the non-archimedean valuations on $\mathbb{Q}$ are (up to equivalence) v_0 and v_p where p is prime.

1. *Equip $\mathbb{Q}$ with the discrete topology. Then all valuations are continuous.*
   (a) *Since $v(a) \leq 1 \ \forall \ a \in \mathbb{Z}$ for all v, $|\mathrm{Spa}(\mathbb{Q}, \mathbb{Z})| = \{v_p \mid p \text{ prime}\} \cup \{v_0\}$.*
   (b) *Since $v(a) \leq 1 \ \forall \ a \in \mathbb{Q}$ if and only if v is trivial, $|\mathrm{Spa}(\mathbb{Q}, \mathbb{Q})| = \{v_0\}$.*
2. *Fix a prime p and equip $\mathbb{Q}$ with the p-adic topology. Since $\{0\}$ is not open in $\mathbb{Q}$ with the p-adic topology, v_0 is not continuous. Thus,*
   (a) $|\mathrm{Spa}(\mathbb{Q}, \mathbb{Z})| = \{v_p\}$
   (b) $|\mathrm{Spa}(\mathbb{Q}, \mathbb{Q})| = \emptyset$ *since* $v_p(a) \not\leq 1 \ \forall \ a \in \mathbb{Q}$ $(v_p(p^{-1}) = p > 1)$

(c) $|\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)| = \{v_p\}$.

5.2 Adic Spectrum

Definition 5.2.1. *Let $A = (A^{\triangleright}, A^{+})$ be an affinoid ring. A subset $U \subseteq |\mathrm{Spa}(A)|$ is called* rational *if there exist $a_1, a_2, \ldots, a_n, b \in A^{\triangleright}$ such that the ideal $a_1A^{\triangleright} + \cdots + a_nA^{\triangleright}$ is open in $A^{\triangleright}$ and*

$$U = |\mathrm{Spa}(A)|\left(\frac{a_1, \ldots, a_n}{b}\right) = \{v \in |\mathrm{Spa}(A)| \mid v(a_i) \leq v(b),\ v(b) \neq 0,\ i = 1, \ldots, n\}.$$

The adic spectrum $\mathrm{Spa}(A)$ *is the topological space generated by the rational subsets on $|\mathrm{Spa}(A)|$.*

Example 5.2.2.

1. *Equip $\mathbb{Q}$ with the discrete topology. Then*
   (a) *if l is prime, then $|\mathrm{Spa}(\mathbb{Q}, \mathbb{Z})|\left(\frac{\frac{1}{l}}{1}\right) = \{v_p \mid p \neq l\} \cup \{v_0\}$ (note that since $\mathbb{Q}$ is equipped with the discrete topology $\frac{1}{l}\mathbb{Q}$ is open in $\mathbb{Q}$)*
   (b) *$|\mathrm{Spa}(\mathbb{Q}, \mathbb{Z})|\left(\frac{a}{b}\right) = \{v_p \mid p \not| a\} \cup \{v_0\}$ (note that since $\mathbb{Q}$ is equipped with the discrete topology $\frac{a}{b}\mathbb{Q}$ is open in $\mathbb{Q}$).*
2. *Equip $\mathbb{Q}$ with the p-adic topology. Then $|\mathrm{Spa}(\mathbb{Q}, \mathbb{Z})|\left(\frac{1}{p}\right) = \{v_0\}$ (note that since $\mathbb{Q}$ is a field $\frac{1}{p}\mathbb{Q} = \mathbb{Q}$ is open in $\mathbb{Q}$).*

Lemma 5.2.3. *[9, page 468] If V is a finite subset of $A^{\triangleright}$, then $V \cdot A^{\triangleright}$ is open in $A^{\triangleright}$ if and only if $(A^{\triangleright})^{\circ\circ} \subseteq \sqrt{(V)}$.*

Example 5.2.4. *Let $A = (\mathbb{Q}_p\{T\}, \mathbb{Z}_p\{T\})$. By Lemma 5.2.3, $p\mathbb{Q}_p\{T\}$ is open in $\mathbb{Q}_p\{T\}$ but $T\mathbb{Q}_p\{T\}$ is not open. To see the latter, consider $p \in (\mathbb{Q}_p\{T\})^{\circ\circ}$. There is no natural number n such that $p^n \subseteq (T)$.*

Remark 5.2.5. *Let $aA^{\triangleright}$ be open in $A^{\triangleright}$. It follows that $\frac{a}{b}A^{\triangleright}$ is open in $A^{\triangleright}$ when b is a unit in $A^{\triangleright}$. Thus, the rational subsets $|\mathrm{Spa}(A)|\left(\frac{a}{b}\right)$ and $|\mathrm{Spa}(A)|\left(\frac{\frac{a}{b}}{1}\right)$ are equivalent if b is a unit in $A^{\triangleright}$.*

Definition 5.2.6. *Let $A = (A^{\triangleright}, A^{+})$ and $B = (B^{\triangleright}, B^{+})$ be affinoid rings and $\varphi = (\varphi^{\triangleright}, \varphi^{+}) : A \to B$ an affinoid ring homomorphism. Define $\mathrm{Spa}(\varphi) : \mathrm{Spa}(B) \to \mathrm{Spa}(A)$ by $v \mapsto v \circ \varphi^{\triangleright}$.*

Remark 5.2.7. *The morphism $\mathrm{Spa}(\varphi)$ is well-defined since if $b \in B^{+}$, $(v \circ \varphi^{\triangleright})(b) = v(\varphi^{\triangleright}(b)) \leq 1$ since $\varphi^{\triangleright}(b) \in A^{+}$.*

Proposition 5.2.8. *If $\varphi : A \to B$ is an affinoid ring homomorphism, then $\mathrm{Spa}(\varphi)$ as defined above is continuous.*

Proof. Let $a, b \in A$, $b \neq 0$ and consider the rational subset $|\mathrm{Spa}(A)|\left(\frac{a}{b}\right)$. Then

$$\begin{aligned}
&\mathrm{Spa}(\varphi)^{-1}\left(\mathrm{Spa}(A)\left(\frac{a}{b}\right)\right)\\
&= \left\{v \in \mathrm{Spa}(B) \mid v \circ \varphi^{\triangleright} \in \mathrm{Spa}(A)\left(\frac{a}{b}\right)\right\}\\
&= \{v \in \mathrm{Spa}(B) \mid (v \circ \varphi^{\triangleright})(a) \leq (v \circ \varphi^{\triangleright})(b),\ (v \circ \varphi^{\triangleright})(b) \neq 0\}\\
&= \mathrm{Spa}(B)\left(\frac{\varphi^{\triangleright}(a)}{\varphi^{\triangleright}(b)}\right).
\end{aligned}$$

□

5.3 A Presheaf on Spa(A)

Definition 5.3.1. *[11, page 39] Let $A = (A^{\triangleright}, A^{+})$ be an affinoid ring and let U be a rational subset of $\mathrm{Spa}(A)$. Then*

$$\mathcal{O}_A(U) := A^{\triangleright}\left(\frac{\widehat{a_1, \ldots, a_n}}{b}\right)$$

such that $A^{\triangleright}\left(\frac{a_1,\ldots,a_n}{b}\right)$ *is the localization* $(A^{\triangleright})_b$ *(see Definition B.0.3) equipped with the f-adic topology such that if* B *is a ring of definition of* $A^{\triangleright}$ *and* I *is an ideal of definition of* B *then* $B\left[\frac{a_1}{b},\cdots,\frac{a_n}{b}\right]$ *is a ring of definition of* $(A^{\triangleright})_b$ *and* $I\cdot B\left[\frac{a_1}{b},\cdots,\frac{a_n}{b}\right]$ *is an ideal of definition of* $B\left[\frac{a_1}{b},\cdots,\frac{a_n}{b}\right]$.

Example 5.3.2.

1. *Equip* $\mathbb{Q}$ *with the discrete topology and let* $U = |\mathrm{Spa}(\mathbb{Q},\mathbb{Z})|\left(\frac{a}{b}\right)$. *Then*

$$\mathcal{O}_A(U) = \mathbb{Q}.$$

2. *Equip* $\mathbb{Q}$ *with the p-adic topology and let* $U = |\mathrm{Spa}(\mathbb{Q},\mathbb{Z})|\left(\frac{a}{b}\right)$. *Then*

$$\mathcal{O}_A(U) = \mathbb{Q}_p.$$

Definition 5.3.3. *Let* U *and* V *be rational subsets of* $\mathrm{Spa}(A)$ *such that* $U \subseteq V$. *The restriction map* $\mathcal{O}_A(V) \to \mathcal{O}_A(U)$ *is defined in Huber [10, Lemma 1.5]. Thus, if* W *is an open subset of* $\mathrm{Spa}(A)$, *then a presheaf on* $\mathrm{Spa}(A)$ *is* $\mathcal{O}_A(W) = \varprojlim_{U\subseteq W} \mathcal{O}_A(U)$ *where* U *is rational. For every* $x \in \mathrm{Spa}(A)$ *let* $\mathcal{O}_{A,x} = \varinjlim_{x\in U} \mathcal{O}_A(U)$ *be the* stalk *of* $\mathcal{O}_A$ *at* x.

Remark 5.3.4. *The presheaf* $\mathcal{O}_A$ *is not, in general, a sheaf. See Huber [10, page 520] for an example.*

Proposition 5.3.5 ([10, Theorem 2.2])**.** *Let* $A = (A^{\triangleright}, A^{+})$ *be an affinoid ring such that* $A^{\triangleright}$ *has a noetherian ring of definition or* $A^{\triangleright}$ *is a strongly noetherian Tate ring. Then* $\mathcal{O}_A$ *is a sheaf of complete topological rings on* $\mathrm{Spa}(A)$.

Example 5.3.6. *The ring* $\mathbb{Q}_p$ *has a noetherian ring of definition, namely* $\mathbb{Z}_p$. *Hence, the affinoid ring* $(\mathbb{Q}_p,\mathbb{Z}_p)$ *satisfies the hypotheses of Proposition 5.3.5.*

Remark 5.3.7. *All affinoid rings A discussed in this paper will satisfy the hypotheses of Proposition 5.3.5, and thus, $\mathcal{O}_A$ is always a sheaf.*

5.4 Adic(A)

Definition 5.4.1. *Let (V) be the category of triples $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}, (v_x \mid x \in \mathfrak{X}))$ where $\mathfrak{X}$ is a topological space, $\mathcal{O}_{\mathfrak{X}}$ is a sheaf (see Definition C.0.12) of topological rings on $\mathfrak{X}$, and for every $x \in \mathfrak{X}$, v_x is an equivalence class of valuations of the stalk (see DefinitionC.0.21) $\mathcal{O}_{\mathfrak{X},x}$. The morphisms $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}, (v_x \mid x \in \mathfrak{X})) \to (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}, (v_y \mid y \in \mathfrak{Y}))$ are the pairs $(f, f^{\#})$ where $f = \mathfrak{X} \to \mathfrak{Y}$ is a continuous function and $f^{\#} : \mathcal{O}_{\mathfrak{Y}} \to f_* \mathcal{O}_{\mathfrak{X}}$ such that for every $x \in \mathfrak{X}$, $v_{f(x)} = v_x \circ f^{\#}_x$.*

Definition 5.4.2. *Let A be an affinoid ring satisfying the hypotheses of Proposition 5.3.5. Let* $\mathrm{Adic}(A)$ *be the triple* $(\mathrm{Spa}(A), \mathcal{O}_A, (v_a \mid a \in \mathrm{Spa}(A))) \in (V)$ *where* $\mathrm{Spa}(A)$ *is the topological space defined in Definition 5.2.1 and $\mathcal{O}_A$ is the sheaf of topological rings defined in Definition 5.3.1. Let $\varphi : (\varphi^{\triangleright}, \varphi^{+}) : (A^{\triangleright}, A^{+}) \to (B^{\triangleright}, B^{+})$. Define* $\mathrm{Adic}(\varphi) : \mathrm{Adic}(B) \to \mathrm{Adic}(A)$ *by* $\mathrm{Adic}(\varphi) := (f, f^{\#})$ *where* $f := \mathrm{Spa}(\varphi)$ *as in Definition 5.2.6 and* $f^{\#}(|\mathrm{Spa}(A)|\left(\frac{a}{b}\right)) := \widehat{\varphi^{\triangleright}}_b$ *where $\widehat{\varphi^{\triangleright}}_b$ is the morphism such that the following diagram commutes*

$$\begin{tikzcd}
A^{\triangleright}\left\langle \frac{a}{b} \right\rangle \arrow[r, "\hat{\varphi}^{\triangleright}_b"] & A^{\triangleright}\left\langle \frac{\varphi(a)}{\varphi(b)} \right\rangle \\
A^{\triangleright}_b \arrow[u] \arrow[r, "\varphi^{\triangleright}_b"] & A^{\triangleright}_{\varphi(b)} \arrow[u]
\end{tikzcd}$$

Lemma 5.4.3. *Let $\varphi : (A^{\triangleright}, A^{+}) \to (B^{\triangleright}, B^{+})$ and $\psi : (B^{\triangleright}, B^{+}) \to (C^{\triangleright}, C^{+})$. Then* $\mathrm{Adic}(\psi \circ \varphi) = \mathrm{Adic}(\varphi) \circ \mathrm{Adic}(\psi)$.

Proof. Let

$$(f, f^{\#}) : \mathrm{Adic}(B) \to \mathrm{Adic}(A) \quad \text{induced by } \varphi$$
$$(g, g^{\#}) : \mathrm{Adic}(C) \to \mathrm{Adic}(B) \quad \text{induced by } \psi$$
$$(h, h^{\#}) : \mathrm{Adic}(C) \to \mathrm{Adic}(A) \quad \text{induced by } \psi \circ \varphi.$$

Let $v \in \mathrm{Spa}(C)$. Then

$$\begin{aligned} h(v) &= (\mathrm{Spa}(\psi \circ \varphi))(v) \\ &= v \circ (\psi \circ \varphi)^{\triangleright} \\ &= v \circ \psi^{\triangleright} \circ \varphi^{\triangleright} \\ &= (\mathrm{Spa}(\varphi) \circ \mathrm{Spa}(\psi))(v) \\ &= (f \circ g)(v). \end{aligned}$$

Let U be a rational subset of $\mathrm{Spa}(C)$. Then

$$\begin{aligned} h^{\#}(U) &= \widehat{(\psi \circ \varphi)}_b \\ &= \widehat{\psi}_{\varphi(b)} \circ \widehat{\varphi}_b \\ &= (g^{\#} \circ f^{\#})(U). \end{aligned}$$

Now if $v_{f(x)} = v_x \circ f_x^{\#}$ and $v_{g(x)} = v_x \circ g_x^{\#}$ then

$$\begin{aligned} v_{h(x)} &= v_{(g \circ f)(x)} \\ &= v_{f(x)} \circ g_{f(x)}^{\#} \\ &= v_x \circ f_x^{\#} \circ g_{f(x)}^{\#} \\ &= v_x \circ (f^{\#} \circ g^{\#})_x \\ &= v_x \circ (h^{\#})_x. \end{aligned}$$

Therefore, $\mathrm{Adic}(\psi \circ \varphi) = \mathrm{Adic}(\varphi) \circ \mathrm{Adic}(\psi)$. $\square$

5.5 Affinoid Adic Spaces

Definition 5.5.1. *An* affinoid adic space *is a triple* $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}, (v_x \mid x \in \mathfrak{X}))$ *which is isomorphic to the adic space associated to an affinoid ring as in Definition5.4.2.*

5.6 Adic Spaces

In this paper all examples and results deal with affinoid adic spaces, however, for completeness the general notion of an adic space will be defined.

Definition 5.6.1. *An* Adic Space *is a triple* $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}, (v_x \mid x \in \mathfrak{X}))$ *in which every* $x \in \mathfrak{X}$ *has an open neighbourhood* $U \subseteq \mathfrak{X}$ *such that* $(U, \mathcal{O}|_U, (v_x \mid x \in U))$ *is an affinoid adic space.*

5.7 Some Results in Affinoid Adic Spaces

Definition 5.7.1. *Let* K° *be the valuation ring of a p-adic field* K. *Let* $X = (\mathrm{Spec}(A), \mathcal{O}_A)$ *be an integral affine scheme finitely generated over* $\mathrm{Sch}(K^\circ)$ *such that the special fibre of* X *is also an integral affine scheme. Let* $\mathcal{X}$ *denote the formal scheme obtained by completing* X *along the special fibre of* X. *The affinoid adic space over* K *associated with the formal scheme* $\mathcal{X}$ *is* $\mathrm{Adic}(\hat{A} \otimes_{K^\circ} K, \hat{A})$.

Example 5.7.2. *Let* $K = \mathbb{Q}_p$, *and* $X = \mathrm{Sch}(\mathbb{Z}_p[T])$. *Then* $K^\circ = \mathbb{Z}_p$, $\kappa = \mathbb{F}_p$ *(see Proposition 1.5.2), the special fibre of* X *is* $\mathrm{Sch}(\mathbb{Z}_p[T])$ *(see Example 4.9.2), and* $\mathcal{X} = \mathrm{Schf}(\mathbb{Z}_p\{T\})$ *(see Example 4.9.5). Therefore, the affinoid adic space over* K *associated with the formal scheme* $\mathcal{X}$ *is* $\mathrm{Adic}(\mathbb{Q}_p\{T\}, \mathbb{Z}_p\{T\})$.

Proposition 5.7.3. *Let K be a p-adic field with valuation ring K°. Let $X = (\mathrm{Spec}(A), \mathcal{O}_A)$ be an integral affine scheme finitely generated over* $\mathrm{Sch}(K^\circ)$ *such that the special fibre of X is also an integral affine scheme. Let $\mathcal{X}$ denote the formal scheme obtained by completing X along the special fibre of X. Let $\mathfrak{X}$ denote the affinoid adic space over K associated to the formal scheme $\mathcal{X}$. Then*

$$\mathrm{Hom}_{\mathrm{Adic}(K,K^\circ)}(\mathrm{Adic}(K,K^\circ),\mathfrak{X}) \equiv \mathrm{Hom}_{(K,K^\circ)}\Big((\hat{A}\otimes_{K^\circ} K, \hat{A}), (K, K^\circ)\Big)$$

Proof. Let the hypotheses of the above proposition by satisfied. By Proposition 2.1.9, $\hat{A}\otimes_{K^\circ} K \equiv \hat{A}_{\hat{\pi}}$ and by the proof of Proposition 2.6.3, a ring of definition of $\hat{A}\otimes_{K^\circ} K$ is $\hat{A}$. The ring $\hat{A}$ is noetherian because A is finitely generated over K°. Thus, by Proposition 5.3.5, $\mathfrak{X}$ is an affinoid adic space. By Huber [10, Prop.2.1(i)], the map

$$\begin{aligned}\mathrm{Hom}_{(K,K^\circ)}\Big((\hat{A}\otimes_{K^\circ} K, \hat{A}), (K, K^\circ)\Big) &\rightarrow \mathrm{Hom}_{\mathrm{Adic}(K,K^\circ)}(\mathrm{Adic}(K,K^\circ),\mathfrak{X}) \\ \varphi &\mapsto \mathrm{Adic}(\varphi)\end{aligned}$$

is a bijection. (See Definition 5.4.2 for $\mathrm{Adic}(\varphi)$.) □

Chapter 6

The Main Geometric Result

All of the tools have now been introduced to prove the main geometric result - Theorem 6.0.5. This result uses the categorical equivalence between the category of commutative rings and the category of affine schemes (see Theorem 4.5.7) to restate Theorem 3.0.8 geometrically. Notice that in the geometric theorem, X is finitely generated over $\mathrm{Sch}(K^\circ)$, however in the algebraic theorem A is not necessarily finitely generated over K°. Thus, the geometric theorem is only a consequence of the algebraic theorem.

Definition 6.0.4. *Consider the affinoid ring (K, K°) (see Definition 2.8.1) where K is a p-adic field with valuation ring K°. Let X be an affine scheme over K° (see Definition 4.5.2) and $\mathfrak{X}$ an affinoid adic space (see Definition 5.5.1). Let $s : X \to \mathrm{Sch}(K^\circ)$ be a morphism of affine schemes, then $f \in \mathrm{Hom}_{\mathrm{Sch}(K^\circ)}(\mathrm{Sch}(K^\circ), X)$ if $s \circ f = id_{\mathrm{Sch}(K^\circ)}$.*

$$\begin{tikzcd} \mathrm{Sch}(K^\circ) \arrow[rr, "f"] \arrow[dr, "id"'] & & X \arrow[dl, "s"] \\ & \mathrm{Sch}(K^\circ) & \end{tikzcd}$$

Similarly, if $t : \mathrm{Adic}(\mathfrak{X}) \to \mathrm{Adic}(K, K^\circ)$ is a morphism of affinoid adic spaces, then $g \in \mathrm{Hom}_{\mathrm{Adic}(K,K^\circ)}(\mathrm{Adic}(K, K^\circ), \mathfrak{X})$ if $t \circ g = id_{\mathrm{Adic}(K,K^\circ)}$.

$$\begin{tikzcd} \mathrm{Adic}(K, K^\circ) \arrow[rr, "g"] \arrow[dr, "id"'] & & \mathfrak{X} \arrow[dl, "t"] \\ & \mathrm{Adic}(K, K^\circ) & \end{tikzcd}$$

Theorem 6.0.5. *Let K be a p-adic field with valuation ring K°. Let X be an integral affine scheme finitely generated over* $\mathrm{Sch}(K^\circ)$ *such that the special fibre of X is also an integral affine scheme. Let $\mathcal{X}$ denote the formal scheme obtained by completing X along the special fibre of X. Let $\mathfrak{X}$ denote the affinoid adic space over K associated to the formal scheme $\mathcal{X}$. Then there is a canonical bijection*

$$\mathrm{Hom}_{\mathsf{Sch}(K^\circ)}(\mathrm{Sch}(K^\circ), X) \equiv \mathrm{Hom}_{\mathsf{Adic}(K,K^\circ)}(\mathrm{Adic}(K, K^\circ), \mathfrak{X}).$$

Proof. Let the hypotheses of the Theorem 6.0.5 be satisfied. Then

$$\begin{aligned}
&\mathrm{Hom}_{\mathsf{Adic}(K,K^\circ)}(\mathrm{Adic}(K, K^\circ), \mathfrak{X}) \\
&\quad \equiv \mathrm{Hom}_{(K,K^\circ)}\Big((\hat{A} \otimes_{K^\circ} K, \hat{A}), (K, K^\circ)\Big) && \text{by Proposition 5.7.3} \\
&\quad \equiv \mathrm{Hom}_{K^\circ}(A, K^\circ) && \text{by Theorem 3.0.8} \\
&\quad \equiv \mathrm{Hom}_{\mathsf{Sch}(K^\circ)}(\mathrm{Sch}(K^\circ), X) && \text{by Proposition 4.5.8}
\end{aligned}$$

□

Example 6.0.6. *Consider the geometric interpretation of Example 3.0.10. Let* $X = \mathrm{Sch}(\mathbb{Z}_p[T])$, *and let $K = \mathbb{Q}_p$. Thus, using results from Example 5.7.2, Theorem 6.0.5 gives*

$$\mathrm{Hom}_{\mathsf{Sch}(\mathbb{Z}_p)}(\mathrm{Sch}(\mathbb{Z}_p), \mathrm{Sch}(\mathbb{Z}_p[T])) \equiv$$
$$\mathrm{Hom}_{\mathsf{Adic}(\mathbb{Q}_p,\mathbb{Z}_p)}(\mathrm{Adic}(\mathbb{Q}_p, \mathbb{Z}_p), \mathrm{Adic}(\mathbb{Q}_p\{T\}, \mathbb{Z}_p\{T\}))\,.$$

Recall from Proposition 4.5.9 that $\mathrm{Hom}_{\mathsf{Sch}(\mathbb{Z}_p)}(\mathrm{Sch}(\mathbb{Z}_p), \mathrm{Sch}(\mathbb{Z}_p[T])) \equiv \mathbb{Z}_p$. *Thus,* $\mathrm{Hom}_{\mathsf{Adic}(\mathbb{Q}_p,\mathbb{Z}_p)}(\mathrm{Adic}(\mathbb{Q}_p, \mathbb{Z}_p), \mathrm{Adic}(\mathbb{Q}_p\{T\}, \mathbb{Z}_p\{T\})) \equiv \mathbb{Z}_p$.

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Appendix A

Direct and Inverse Limits

A.1 Direct Limits

Definition A.1.1. *Suppose we have a partially ordered set $\mathcal{I}$ and a set of rings $\{A_i \mid i \in \mathcal{I}\}$ such that*

- *for all $i, j \in \mathcal{I}$ there exists $k \in \mathcal{I}$ such that $i \leq k$ and $j \leq k$*
- *for every pair of indices i, j with $i \leq j$ there is a map $\rho_{ij} : A_i \to A_j$ such that the following hold*
    1. *$\rho_{jk} \circ \rho_{ij} = \rho_{ik}$ whenever $i \leq j \leq k$*
    2. *$\rho_{ii} = id_{A_i}$ for all $i \in \mathcal{I}$.*

Let $B = \coprod_{i \in \mathcal{I}} A_i$ be the disjoint union of all the A_i and define a relation $\sim$ on B by $a \sim b$ if and only if there exists $k \in \mathcal{I}$ with i, $j \leq k$ and $\rho_{ik}(a) = \rho_{jk}(b)$ for $a \in A_i$ and $b \in A_j$. Then $\sim$ is an equivalence relation on B and we define $\varinjlim_{i \in \mathcal{I}} A_i = B/\sim$, the direct limit *of $\{A_i\}_{i \in \mathcal{I}}$. Let $\bar{a}$ denoted the equivalence class of a in $\varinjlim_{i \in \mathcal{I}} A_i$, and define $\rho_i : A_i \to \varinjlim_{i \in \mathcal{I}} A_i$ by $\rho_i(a) = \bar{a}$. Then the diagram*

$$\begin{tikzcd} & \varinjlim_{i \in \mathcal{I}} A_i \\ A_i \arrow[ur, "\rho_i"] \arrow[r, "\rho_{ij}"] & A_j \arrow[u, "\rho_j"'] \end{tikzcd}$$

commutes. If $\alpha, \beta \in \varinjlim_{i \in \mathcal{I}} A_i$ where $\alpha = \bar{a}, a \in A_i$ and $\beta = \bar{b}, b \in A_j$, then $\alpha + \beta = \bar{a} + \bar{b} = \overline{\rho_{ik}(a) + \rho_{jk}(b)}$ for some $k \in \mathcal{I}, k \geq i, k \geq j$.

$$
\begin{tikzcd}
 & \varinjlim_{i \in \mathcal{I}} A_i \arrow[dd, dotted, bend left=60, "\exists\,!\,\theta"] \\
A_i \arrow[ur, "\rho_i"] \arrow[r, "\rho_{ij}"] \arrow[dr, "\eta_i"'] & A_j \arrow[u, "\rho_j"'] \arrow[d, "\eta_j"] \\
 & B
\end{tikzcd}
$$

Figure A.1: Universal Property of Direct Limits

Example A.1.2. *Let K be a field and let $\mathcal{I} = \{L \mid L : K$ finite $\}$. Define a partial order on $\mathcal{I}$ by $L_1 \leq L_2 \Leftrightarrow L_2 : L_1 : K$. Then $\mathcal{I}$ is directed since $L_1 : K$ finite $\Rightarrow$ $L_1 : K$ algebraic, finite $\Rightarrow L_1 = K(\alpha_1, \ldots, \alpha_n)$ for some $\alpha_1, \ldots, \alpha_n$ algebraic over K. Similarly, $L_2 = K(\beta_1, \ldots, \beta_n)$ for some $\beta_1, \ldots, \beta_n$ algebraic over K. Let $L = K(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$, then $L : K$ is finite, $L : L_1 : K$, and $L : L_2 : K$. Define the rings $A_L = L$ and for $L_1 \leq L_2$ the maps $\rho_{L_1 L_2} : A_{L_1} \to A_{L_2}$ to be inclusion. Then $\varinjlim_{L:K \text{finite}} L = \overline{K}$ where $\overline{K}$ is the algebraic closure of K.*

Proposition A.1.3 (Universal Property). *Let $\mathcal{I}$, $\{A_i \mid i \in \mathcal{I}\}$, $\{\rho_{ij} \mid i, j \in \mathcal{I}, i \leq j\}$ satisfy the conditions of a direct limit. Let B be a commutative ring with unity equipped with maps $\eta_i : A_i \to B$ for all i, such that $\eta_j \circ \rho_{ij} = \eta_i$ for all $i \leq j$. Then there exists a unique map $\theta : \varinjlim_{i \in \mathcal{I}} A_i \to B$ such that $\eta_i = \theta \circ \rho_i$.*

A.2 Inverse Limits

The inverse limit is defined similarly to that of the direct limit. Essentially all of the maps used in direct limits are reversed to define the inverse limit.

Definition A.2.1. *Let $\mathcal{I}$ be a partially order set and suppose for every pair of indices i, j with $i \leq j$ there is a map $\mu_{ji} : A_j \to A_i$ such that the following hold*

1. $\mu_{ji} \circ \mu_{kj} = \mu_{ki}$ *whenever* $i \leq j \leq k$
2. $\mu_{ii} = id_{A_i}$ *for all* $i \in \mathcal{I}$.

Such a set is called an inverse system. The inverse limit *of the system $\{A_i\}_{i\in\mathcal{I}}$ is the subset of elements $(a_i)_{i\in\mathcal{I}}$ in the direct product $\prod_{i\in I} A_i$ such that $\mu_{ji}(a_i) = a_i$ whenever $i \leq j$. The inverse limit is denoted $\varprojlim_{i\in\mathcal{I}} A_i$. For each $i \in \mathcal{I}$ let $\mu_i : \varprojlim_{i\in I} A_i \to A_i$ be the projection onto its i^{th} component. Then the following diagram*

$$
\begin{tikzcd}
 & \varprojlim_{i\in\mathcal{I}} A_i \arrow[dl, "\mu_i"'] \arrow[d, "\mu_j"] \\
A_i & A_j \arrow[l, "\mu_{ji}"']
\end{tikzcd}
$$

commutes.

Proposition A.2.2 (Universal Property). *Let $\mathcal{I}$, $\{A_i \mid i \in I\}$, $\{\mu_{ij} \mid i, j \in \mathcal{I}, i \leq j\}$ satisfy the conditions of a inverse limit. Let B be a commutative ring with unity equipped with maps $\nu_i : B \to A_i$ for all i such that $\mu_{ji} \circ \nu_j = \nu_i$ for all $i \leq j$. Then there exists a unique map $\theta : B \to \varprojlim_{i\in\mathcal{I}} A_i$ such that $\nu_i = \mu_i \circ \theta$.*

$$
\begin{tikzcd}
 & \varprojlim_{i\in\mathcal{I}} A_i \arrow[dl, "\mu_i"'] \arrow[d, "\mu_j"] \\
A_i & A_j \arrow[l, "\mu_{ji}"'] \\
 & B \arrow[ul, "\nu_i"] \arrow[u, "\nu_j"'] \arrow[uu, dotted, bend right, "\exists\,!\,\theta"']
\end{tikzcd}
$$

Figure A.2: Universal Property of Inverse Limits

Appendix B

Localization

Definition B.0.3. *Let A be a ring and let $M \subseteq A$ be a multiplicative set ($1_A \in M$ and $m_1, m_2 \in M \Rightarrow m_1 m_2 \in M$). Consider $A \times M = \{(a, m) \mid a \in A, m \in M\}$ with an equivalence relation $(a, m) \equiv (b, n) \Leftrightarrow k(an - bm) = 0$ for some $k \in M$. The equivalence class (a, m) is denoted by $\frac{a}{m}$. Then $A_M = \left\{\frac{a}{m} \mid a \in A, m \in M\right\}$ is called the* localization *of A at M and comes equipped with a map $\lambda_M : A \to A_M$ defined by $a \mapsto \frac{a}{1}$.*

Remark B.0.4.

1. *If A is an integral domain and $M = A^{\times}$, then A_M is the quotient field of A.*

2. *Let A be a ring and $\mathfrak{p}$ a prime ideal of A. Then the complement of $\mathfrak{p}$ is a multiplicative set, but we notate the localization as $A_{\mathfrak{p}}$.*

Example B.0.5.

1. *Let $A = \mathbb{Z}$ and $M = \mathbb{Z}^{\times}$. Then $\frac{a}{m} = \frac{b}{n} \Leftrightarrow k(an - bm) = 0$ for some $k \in M \Leftrightarrow$ $an = bm$. Then $A_M = \mathbb{Q}$.*

2. *Let $A = \mathbb{C}[t]$, and $M = \{t^n \mid n \in \mathbb{N}\}$. Then $A_M = \mathbb{C}[t]_t = \left\{\frac{f}{t^n} \mid f \in \mathbb{C}[[t]]\right\}$.*

3. *Let $A = \mathbb{C}[t]$, and $M = \{a \in A \mid a \notin \mathfrak{p}\}$. Then $A_M = A_{\mathfrak{p}} = \mathbb{C}[t]_{(t)} =$ $\left\{\frac{f}{g} \in \mathbb{C}(t) \mid g \notin (t)\right\} = \left\{\frac{f}{g} \in \mathbb{C}(t) \mid t \not| g\right\}$*

4. *Let $A = \mathbb{Z}_p$, and $M = \{p^n \mid n \in \mathbb{N}\}$. Then $A_M = (\mathbb{Z}_p)_p = \mathbb{Q}_p$.*

5. *Let A be a ring. Then $A_{(0)} = \{\frac{a}{s} \mid s \neq 0\}$, the quotient field of A.*

Definition B.0.6. *Let $\varphi : A \to B$ be a ring homomorphism. Let $s \in A$, $t = \varphi(s)$, and A_s be the ring localized at the multiplicative set generated by s. Then $\varphi_s : A_s \to B_t$ is defined by $\frac{a}{s^n} \mapsto \frac{\varphi(a)}{\varphi(s)^n}$.*

Lemma B.0.7. *Let A and B be rings and $\varphi : A \to B$. Let $r, t \in A$. Then the following diagram commutes.*

$$\begin{tikzcd}
A_t \arrow[r, "\varphi_t"] \arrow[d, "\lambda_r"'] & B_{\varphi(t)} \arrow[d, "\lambda_{\varphi(r)}"] \\
A_{tr} \arrow[r, "\varphi_{tr}"] & B_{\varphi(tr)}
\end{tikzcd}$$

Proof. Let $\frac{a}{t^n} \in A_t$. Then

$$(\lambda_{\varphi(r)} \circ \varphi_t)\left(\frac{a}{t^n}\right) = \lambda_{\varphi(r)}\left(\frac{\varphi(a)}{\varphi(t)^n}\right) = \frac{\varphi(a)}{1 \cdot \varphi(t)^n} = \frac{\varphi(a)}{\varphi(t)^n}$$

and

$$(\lambda_r \circ \varphi_{tr})\left(\frac{a}{t^n}\right) = \varphi_{tr}\left(\frac{a}{1 \cdot t^n}\right) = \frac{\varphi(a)}{\varphi(t)^n}.$$

Thus, the diagram commutes. □

Corollary B.0.8. *Let $\varphi : A \to B$ and $\psi : B \to C$. If $s \in A$, then $(\psi \circ \varphi)_s = \psi_{\varphi(s)} \circ \varphi_s$.*

Proof. Consider the following diagram.

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "\lambda_s"] & B \arrow[r, "\psi"] \arrow[d, "\lambda_{\varphi(s)}"] & C \arrow[d, "\lambda_{\psi(\varphi(s))}"] \\
A_s \arrow[r, "\varphi_s"'] & B_{\varphi(s)} \arrow[r, "\psi_{\varphi(s)}"'] & C_{\psi(\varphi(s))}
\end{tikzcd}$$

Take $t = 1$ in Lemma B.0.7 and the two outside squares in the above diagram commutes. Hence, the entire diagram commutes and the result follows. □

Appendix C

Sheaves of Rings

Definition C.0.9. *Let X be a topological space. A* presheaf *assigns a ring $\mathcal{F}(U)$ to each open $U \subseteq X$ and a ring homomorphism $\mathcal{F}(U \subseteq V) : \mathcal{F}(V) \to \mathcal{F}(U)$ called the* restriction homomorphism *to every pair of nested open subsets $U \subseteq V$. The restriction homomorphism must satisfy the following conditions*

1. $\mathcal{F}(U = U) = id_{\mathcal{F}(U)}$
2. $\mathcal{F}(U \subseteq W) = \mathcal{F}(U \subseteq V) \circ \mathcal{F}(V \subseteq W)$ *for all* $U \subseteq V \subseteq W$.

The ring $\mathcal{F}(U)$ is called the ring of sections, *elements of $\mathcal{F}(U)$ are called the* sections of $\mathcal{F}$ over U, *and elements of $\mathcal{F}(X)$ are called* global sections.

Remark C.0.10. *A presheaf is a functor from the category of open subsets of X to the category of rings.*

Example C.0.11. *[3]*

1. *Consider the topological spaces $\mathbb{R}$ and $\mathbb{C}$ (with the usual topology). For all open $U \subset \mathbb{R}$ we can define a presheaf of sets by taking $\mathcal{F}(U)$ to be the set of continuous functions from U to $\mathbb{C}$. Here the restriction maps are the usual restriction of a function from one set to a smaller set contained inside it. The presheaf takes its values in the category of commutative rings by defining point wise addition and multiplication on the functions.*

2. *Give $\mathbb{R}$ the usual topology. For all $U \subset \mathbb{R}$ define a presheaf on $\mathbb{R}$ by taking $\mathcal{G}(U)$ to be the ring of constant functions on U with values in $\mathbb{R}$.*

Definition C.0.12 (Sheaf Axiom). *A presheaf is called* a sheaf *if it satisfies one extra condition, the* sheaf axiom*: for each $\{f_i \in \mathcal{F}(U_i) \mid i \in \mathcal{I}\}$ such that $\mathcal{F}(U_i \cap U_j \hookrightarrow U_i)(f_i) = \mathcal{F}(U_i \cap U_j \hookrightarrow U_j)(f_j)$ for all $i, j \in \mathcal{I}$ there exists a unique $f \in \mathcal{F}(\bigcup_{i\in I} U_i)$ such that $\mathcal{F}(U_k \hookrightarrow \bigcup_{i\in I} U_i)(f) = f_k$ for all $k \in \mathcal{I}$.*

Proposition C.0.13. *A sheaf is a functor which takes pullbacks to pushouts.*

Proof. A sketch of the proof in the simplest case: Let U and V be sets. Clearly $U \cup V$ is the pushout of $U \hookleftarrow U \cap V \hookrightarrow V$ with inclusion maps. Now by the sheaf axiom,

$$\begin{tikzcd} \mathcal{F}(V) \arrow[d] & \mathcal{F}(U \cup V) \arrow[l] \arrow[d] \\ \mathcal{F}(U \cap V) & \mathcal{F}(U) \arrow[l] \end{tikzcd}$$

$\mathcal{F}(U \cup V)$ is the pullback in the above diagram. ◻

Example C.0.14. *[3]*

1. *The presheaf in Example C.0.11 (1) is a sheaf. Let U and V be open in $\mathbb{R}$. If f is a continuous function on $U \cap V$, then f can clearly be extended to a continuous function on $U \cup V$.*

2. *The presheaf in Example C.0.11 (2) is not a sheaf. Suppose U can be written as the disjoint union of two open subsets V and W. Look at the section $s \in \mathcal{G}(V)$ that takes on the constant value 0, and at the section $t \in \mathcal{G}(W)$ that takes on the constant value 1. Because the intersection $V \cap W$ is empty, the sections s and*

t restrict to the same (trivial) function on their intersection. However, there is no way to patch these two sections together to define a compatible constant function on the entire space U.

Definition C.0.15. *If $Z \subseteq X$, the restriction of a sheaf $\mathcal{F}$ on X to Z is simply $(\mathcal{F}|_Z)(U) := \mathcal{F}(U)$ where U is an open subset of Z. If U and V are both open subsets of Z such that $U \subseteq V \subseteq Z$, then $\mathcal{F}|_Z(U \subseteq V) := \mathcal{F}(U \subseteq V)$.*

Definition C.0.16. *A morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves on a space X is defined simply to be a collection of maps $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for every inclusion $U \subseteq V$ the diagram*

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\varphi(V)"] \arrow[d, "\mathcal{F}(U\subseteq V)"'] & \mathcal{G}(V) \arrow[d, "\mathcal{G}(U\subseteq V)"] \\ \mathcal{F}(U) \arrow[r, "\varphi(U)"] & \mathcal{G}(U) \end{tikzcd}$$

commutes. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on X, the same definition defines a morphism of sheaves.

Definition C.0.17. *Let $\mathcal{B}$ be a base for a topology. A $\mathcal{B}$-*sheaf *is a collection $\{\mathcal{F}(U) \mid U \in \mathcal{B}\}$ of rings equipped with a collection*

$$\{\mathcal{F}(U \subseteq V) : \mathcal{F}(V) \to \mathcal{F}(U) \mid U, V \in \mathcal{B}\}$$

of maps between the rings such that

1. *$\mathcal{F}(U \subseteq W) = \mathcal{F}(U \subseteq V) \circ \mathcal{F}(V \subseteq W)$ for all $U \subseteq V \subseteq W \in \mathcal{B}$*
2. *$\mathcal{F}(U = U) = id_{\mathcal{F}(U)}$ for all $U \in \mathcal{B}$*
3. *for each $\{f_i \in \mathcal{F}(U_i) \mid i \in \mathcal{I}\}$ such that $V \subseteq (U_i \cap U_j)$, $V, U_i \in \mathcal{B}$ implies $\mathcal{F}(V \hookrightarrow U_i)(f_i) = \mathcal{F}(V \hookrightarrow U_j)(f_j)$ there exists a unique $f \in \mathcal{F}(\bigcup_{i \in I} U_i)$ such that $\mathcal{F}(U_k \hookrightarrow \bigcup_{i \in I} U_i)(f) = f_k$ for all $k \in \mathcal{I}$.*

Proposition C.0.18. *[5, Prop. I-12] Let $\mathcal{B}$ be a base of open sets for X. Then*

1. *Every $\mathcal{B}$-sheaf on X extends uniquely to a sheaf on X*

2. *Given sheaves $\mathcal{F}$ and $\mathcal{G}$ on X and a collection of maps*

$$\bar{\varphi}(U) : \mathcal{F}(U) \to \mathcal{G}(U)$$

for all $U \in \mathcal{B}$ commuting with restrictions, there is a unique morphism φ : $\mathcal{F} \to \mathcal{G}$ of sheaves such that $\varphi(U) = \bar{\varphi}(U)$ for all $U \in \mathcal{B}$.

Definition C.0.19. *Let $\alpha : X \to Y$ be a continuous map on topological spaces and let $\mathcal{F}$ be a presheaf on X. The* pushforward *$\alpha_*\mathcal{F}$ of $\mathcal{F}$ by α is defined to be the presheaf on Y given by $\alpha_*\mathcal{F}(V) := \mathcal{F}(\alpha^{-1}(V))$ for any open $V \subseteq Y$.*

Proposition C.0.20. *The pushforward of a sheaf of rings is again a sheaf of rings.*

Proof. Let $\mathcal{G} := \alpha_*\mathcal{F}$. Let U be open in X. Note that if $\iota : U \to V$ is the inclusion map then $\mathcal{F}(V) \to \mathcal{F}(U)$ is a restriction map and $\alpha^{-1} \circ \iota : \alpha^{-1}(V) \to \alpha^{-1}(U)$. Therefore, $\mathcal{G}(\iota) := \mathcal{F}(\alpha^{-1} \circ \iota) : \mathcal{F}(\alpha^{-1}(V)) \to \mathcal{F}(\alpha^{-1}(U))$. Hence, $\mathcal{G}$ is a presheaf.

To show that $\mathcal{G}$ satisfies the sheaf condition, let $X \supseteq U = \bigcup_{i\in I} U_i$, $f_i \in \mathcal{F}(U_i)$, and $f_j \in \mathcal{F}(U_j)$. Since $\mathcal{F}$ is a sheaf, if $\mathcal{F}(U_i \cap U_j \hookrightarrow U_i)(f_i) = \mathcal{F}(U_i \cap U_j \hookrightarrow U_j)(f_j)$, then there exists a unique $f \in \mathcal{F}(\bigcup_{i\in I} U_i)$ such that $\mathcal{F}(U_k \hookrightarrow \bigcup_{i\in I} U_i)(f) = f_k$ for all $k \in \mathcal{I}$. Now let $Y \supseteq V = \bigcup_{i\in I} V_i$, $g_i \in \alpha_*\mathcal{F}(V_i)$, and $f_j \in \alpha_*\mathcal{F}(V_j)$. Then

$$
\begin{aligned}
&\alpha_*\mathcal{F}(V_i \cap V_j \hookrightarrow V_i)(g_i) = \alpha_*\mathcal{F}(V_i \cap V_j \hookrightarrow V_j)(g_j)\\
&\Rightarrow\ \mathcal{F}(\alpha^{-1}(V_i) \cap \alpha^{-1}(V_j) \hookrightarrow \alpha^{-1}(V_i))(g_i) = \mathcal{F}(\alpha^{-1}(V_i) \cap \alpha^{-1}(V_j) \hookrightarrow \alpha^{-1}(V_j))(g_j)\\
&\Rightarrow\ \exists\,!\ g \in \mathcal{F}(\bigcup_{i\in\mathcal{I}} \alpha^{-1}(V_i) \text{ such that } \mathcal{F}(\alpha^{-1}(V_i) \hookrightarrow \alpha^{-1}(\bigcup_{i\in\mathcal{I}} V_i))(g) = g_i\\
&\Rightarrow\ \exists\,!\ g \in \alpha_*\mathcal{F}(V) \text{ such that } \alpha_*\mathcal{F}(V_i \hookrightarrow V)(g) = g_i.
\end{aligned}
$$

Since $\bigcup_{i\in\mathcal{I}} \alpha^{-1}(V_i) = \alpha^{-1}(\bigcup_{i\in\mathcal{I}} V_i)$, the pushforward satisfies the sheaf axiom and is therefore a sheaf. $\square$

Definition C.0.21. *If $\mathcal{F}$ is a presheaf on X and $x \in X$, then the* stalk *of $\mathcal{F}$ at x is* $\mathcal{F}_x = \varinjlim_{x\in U,\ open} \mathcal{F}(U)$.

Remark C.0.22. *Note that a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves of X induces a morphism $\phi_x : \mathcal{F}_x \to \mathcal{G}_x$ on the stalks for any x in X.*

Definition C.0.23. *Let $X = \operatorname{Spec}(A)$ and consider the distinguished open sets X_s of X (see Definition 4.1.9). Define $\mathcal{O}_X(X_s) := A_s$ (see Definition B.0.3). Then $\mathcal{O}_X$ is a $\mathcal{B}$-sheaf (see Definition C.0.17) and by Proposition C.0.18 $\mathcal{O}_X$ extends uniquely to a sheaf on X. If $U \subseteq X$ and U is open, then*

$$\mathcal{O}_X(U) := \varprojlim_{X_s\subseteq U} \mathcal{O}_X(X_s)$$

and if $Y_s \subseteq Y_t$, the following diagram commutes.

$$\begin{array}{ccc}
\mathcal{O}_Y(Y_s) & \longrightarrow & (f_*\mathcal{O}_X)(Y_s) = \mathcal{O}_X(f^{-1}(Y_s)) = \mathcal{O}_X(X_{\varphi(s)}) \\
\uparrow & & \uparrow \\
\mathcal{O}_Y(Y_t) & \longrightarrow & (f_*\mathcal{O}_X)(Y_t) = \mathcal{O}_X(f^{-1}(Y_t)) = \mathcal{O}_X(X_{\varphi(t)})
\end{array}$$