## THE UNIVERSITY OF CALGARY

# REINFORCED CONCRETE BEAMS UNDER PURE TORSION 

BY

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Reinforced Concrete Beams Under Pure Torsion", submitted by Wael Mohamed El-Degwy in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Civil Engineering.


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## ABSTRACT

The behavior of reinforced concrete members subjected to pure torsion has been studied in this investigation. Considering both equilibrium and compatibility, skew bending analyses for two rectangular modes were developed to predict the rotations, strains and stresses at all levels of load and to predict the strengths of reinforced concrete members. subjected to pure torsion. Since diagonal cracking, which causes discontinuity in the concrete, and the lateral tension introduced by the reinforcement cause a reduction in the concrete strength, a stress-strain curve for concrete in which the stress has been reduced was used in the analyses. The experimental torsional strengths of 102 beams were compared to the theoretical strengths and the comparison was found to be excellent.

In order to cover the full range between the two rectangular modes, new modes with triangular and trapezoidal compression zones were proposed. Considering only equilibrium, skew bending analyses for the triangular and trapezoidal modes were developed to predict the strength of reinforced concrete members subjected to pure torsion. The effects of cross section aspect ratio, amount of reinforcement, concrete strength and softening of
concrete (reduction in the concrete strength) on the analyses were studied. The results of the triangular and trapezoidal modes were not entirely satisfactory, therefore these modes require further refinement.

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Only symbols which are used repeatedly are defined here. Generally a symbol is also defined when it first appears in the text.
$A_{\ell} \quad=$ cross-sectional area of all longitudinal bars.
$a_{c} \quad=\quad$ area of the compression zone.
$a_{\ell} \quad=\quad$ cross-sectional area of one longitudinal bar.
$a_{s} \quad=\quad$ area of one leg of a closed stirrup.
$a_{\mathbf{x}} \quad=$ area of the compression zone perpendicular to the x axis.
$a_{y} \quad=$ area of the compression zone perpendicular to the $y$ axis.
$a_{z} \quad=$ area of the compression zone perpendicular to the $z$ axis.
$b$ shorter overall dimension of rectangular cross-section.
$\mathrm{b}_{1} \quad=$ shorter center-to-center dimension of a closed. rectangular stirrup.
$b_{2} \quad=$ shorter center-to-center dimension between two longitudinal corner bars.

C $=$ compressive force acting normal to the compression zone.
$d_{1} \quad=$ distance from extreme fiber in compression zone to inner surface of the stirrups at tension side in Mode 1.





$$
\begin{aligned}
\rho_{l} & =\text { longitudinal reinforcement ratio. } \\
& =A_{l} / b h . \\
\rho_{S} \quad & =\text { reinforcement ratio for stirrups. } \\
& =2 a_{S}\left(b_{l}+h_{l}\right) / s b h . \\
\psi & =\text { angle of twist per unit length of beam. }
\end{aligned}
$$

## CHAPTER 1

## INTRODUCTION

### 1.1 General Remarks and Contents

Many components of structures such as curved and spandrel beams are subjected to combined torsion, bending and shear. Designing these components requires an understanding of the behavior of structural concrete under such combined actions. As a contribution to the complete understanding, this thesis will present a rational theory for members subjected to pure torsion.

Following this introduction, Chapter 2 gives a brief review of two rational theories for concrete members under torsion, the Skew Bending Theory and the Space Truss Theory. Skew bending modes using a rectangular compression zone are presented, in Chapter 3 ; both equilibrium and compatibility are considered, and a softened stress-strain relationship for concrete is introduced to the analysis. In Chapter 4, experimental results are compared to the results of these rectangular modes. In the same chapter the effect of softening of concrete on the analysis is investigated.

Skew bending modes using a triangular or trapezoidal compression zone are presented in Chapter 5; only equilibrium is considered. Chapter 6 presents the results of these triangular and trapezoidal modes. In the same chapter the effects of several variables on the analysis are examined. Summary, conclusions and recommendations are presented in Chapter 7.

## 1.2 object. and Scope

The main objectives of this thesis were to:
I. Introduce a softened stress-strain relationship for concrete to the analysis of members by the skew bending rectangular modes and to study its effect on the results of the analysis.
2. Improve upon the analysis of members using the skew bending rectangular modes, mainly with regard to equilibrium of moments, angle of twist and compatibility.
3. Develop analytical expressions based on the skew bending rectangular modes to predict the rotations, strains and stresses at all levels of load and to predict the strengths of reinforced concrete members subjected to pure torsion.
4. Compare the theoretical predictions and the experimental results for beams reported in the Iiterature.
5. Develop new mathematical skew bending failure modes with triangular and trapezoidal compression zones capable of predicting the behavior of symmetrically reinforced concrete members subjected to pure torsion.
6. Study the effect of cross-section aspect ratio, amount of reinforcement, concrete strength and softening of concrete on the results produced by the new failure modes.

## CHAPTER 2

## LITERATURE REVIEW

### 2.1 Introduction

In this chapter, a brief review of two rational theories for concrete members under torsion is presented. First, Lessig's (1958 and 1959) "Skew Bending Theory" is presented in. Section. 2.2 and second, Lampert and Thürlimann's (1968 and 1969) "Space Truss Theory", which was an improvement upon Rausch's (1929) truss theory, is presented in Section 2.3. It is interesting to note that Kuyt (1971) and Elfgren et al (1974) showed that if certain assumptions were made, the skew bending theory and the truss analogy will yield the same equation for the ultimate capacity.

### 2.2 Skew Bending Theory

In 1958 Lessig proposed the skew bending theory, where equilibrium conditions based on the observed failure mechanisms were considered. The failure surface was assumed to be bounded on three sides by a crack that spiralled around the beam at a constant angle to the longitudinal axis and the fourth side had a rectangular compression zone joining the ends of the spiral cracks.

At failure, Lessig assumed that all reinforcement in the tension zone yielded.

Two failure modes were observed by Lessig. In Mode 1, for the case of torsion with predominant bending, the compression zone forms at the top face of the beam, while in Mode 2, for the case of pure torsion or torsion with predominant shear, the compression zone forms at the side face of the beam, Figure 2.1 . The mode having the minimum torsional resistance governs the failure.

McMullen and Warwaruk (1967) amongst others observed a third mode, Mode 3, for beams reinforced with more longitudinal steel in the bottom than in the top and subjected to large torsional moments with small bending moments. The compression zone forms at the bottom face of the beam, Figure 2.1.

Lessig's skew bending theory has been adopted for both reinforced and prestressed concrete members with certain modifications by many others, for instance McMullen and Warwaruk (1967), Goode and Helmy (1968), Collins et al (1968), GangaRao and Zia (1970), Henry and Zia (1971), Woodhead and McMullen (1972), Rangan and Hall (1973), Below et al (1975), Rangan et al (1977) and Ewida and McMullen (1981 and 1982).


Mode 3

Figure 2.1 Skew Bending Failure Modes

Ewida and McMullen used the skew bending theory to develop mathematical models, satisfying both equilibrium and compatibility conditions, for predicting the strength of reinforced concrete members under combined loading. They stated that their models are capable of predicting the strains and rotations at all levels of load for under-reinforced, partially over-reinforced and completely over-reinforced members up to failure.

Under-reinforced beams are beams in which both the longitudinal bars and the stirrups yield before the maximum torque is reached ; partially over-reinforced beams are beams in which either the longitudinal bars or the stirrups, but not both, yield before the maximum torque is reached ; completely over-reinforced beams are beams in which neither the longitudinal bars nor the stirrups yield before the maximum torque is reached.

Unlike Lessig, in their force equilibrium equations they took into consideration the tensile forces in the longitudinal and transverse reinforcement located in the compression zone. In combined loading, the failure surface was assumed by Ewida and McMullen to be bounded on three sides by cracks that spiralled around the beam at different inclinations, Figure 2.2, whereas in pure torsion the inclination of the cracks on all three sides was the same..


Figure 2.2 Mode 1 Failure Surface in Combined Loading after Ewida and McMullen


Figure 2.3 Stress-Strain Relationship for Concrete in Beams under Combined Loading after Ewida and McMullen

The fourth side had a compression zone joining the ends of the cracks.

For pure torsion, they stated that the tension bars in the compression zone and the discontinuity of the concrete due to cracks may reduce the concrete strength to $35 \%$ of its nominal compressive strength, Figure 2.3 . They proposed a variable stress-strain relationship for concrete in members under combined loading such that there is a smooth transition between the case of pure torsion where the reduction factor equals 0.35 and the case of pure flexure where there is no reduction factor.

Ewida and McMullen (1982) found that the sensitivity of the skew bending analysis to the reduction factor increases with an increase in the amount of reinforcement. The author (1980) found that the sensitivity not only increases with an increase in the amount of reinforcement but also increases with an increase in the aspect ratio of the member cross section.

In their mathematical models, Ewida and McMullen neglected the moment due to the forces in the vertical legs of the stirrups in modes 1 and 3 and the moment due to the forces in the horizontal legs of the stirrups in mode 2. Also, in deriving the deformation equations they assumed
that the compressive strain parallel to the direction of the crack is equal to zero.

For a pure torsion failure, Hsu (1968a) presented a skew bending analysis in which the failure surface was idealized in a different manner from Lessig's. He observed that for reinforced concrete members tested under pure torsion and having a height to width ratio equal to or larger than 1.5 , the main cracks are perpendicular to the wider faces of the cross section, while for members of square cross section the main cracks are not perpendicular to the face but are diagonal. This implies that the failure surface for a member of square cross section is a plane perpendicular to a diagonal plane and inclined at $45^{\circ}$ to the longitudinal axis of beam, Figure 2.4. Hsu stated that a failure surface such as this seems to give a minimum torsional resistance for square cross sections. It is interesting to note that Lessig (1958) stated that a failure could occur in which the neutral axis intersected one vertical and one horizontal face of a member, i.e. the compression zone is triangular.

### 2.3 Space Truss Theory

The first attempt to ascertain the influence of steel reinforcement on torsional strength was by Rausch in 1929. After conducting a series of tests on 500 mm square solid


Figure 2.4 Failure Surface and Diagonal Crack of Square Cross Section after Hsu (1968a)
and hollow sections, Lampert and Thürlimann (1968 and 1969) were able to improve upon Rausch's original truss theory with the "Space Truss Theory".

Lampert and Thürlimann's tests showed that the pure torsional strengths of similar solid and hollow sections were identical. Therefore, their truss model was hollow as shown in Figure 2.5 . It consisted of longitudinal reinforcement which was considered to be concentrated into stringers at the corners, and into intermediate shear walls. In the shear walls, the stirrups acted as tension ties and the concrete between the inclined cracks acted as compression diagonals. They stated that the diagonal forces in the shear walls were deflected into the adjacent walls by means of the longitudinal corner bars. Therefore, their location determined the cross sectional dimensions of the model.

The space truss theory has been adopted for both reinforced and prestressed concrete members with certain modifications by many others, for instance Lampert et al (1971), Lampert and Collins (1972), Mitchell and Collins (1974), Onsongo (1978), Collins and Mitchell (1980) and Hsu and Mo (1983).

Satisfying both equilibrium and compatibility conditions, Mitchell and Collins (1974) used the space


Figure 2.5 The Space Truss Model for Pure Torsion
truss theory to develop their "Diagonal Compression Field Theory". The theory was capable of predicting the post-cracking torsional behavior of symmetrically reinforced and prestressed under-reinforced, partially over-reinforced and completely over-reinforced concrete members in pure torsion. The torsional capacity was based on the dimensions of the spalled sections; i.e. the area enclosed by the center line of the stirrups. The author (1980) amongst others found that spalling was observed only after the peak load was reached. Thus the arbitrary reduction of cross sectional area as proposed by Mitchell and Collins is inconsistent with observed behavior.

Collins and Mitchell (1980) used the truss analogy to predict the strength of prestressed and non-prestressed concrete members under combined loading. In their analysis for members in shear they reduced the diagonal compressive strength, reasoning that stresses have to be transmitted across cracked and severely deformed concrete. They introduced this reduction only for members in shear and not for members in torsion. Later Vecchio and Collins (1981), after testing seventeen 890 mm square by 70 mm reinforced concrete panels subjected to pure shear, derived an expression to model the observed stress-strain behavior. They introduced a stress-strain curve for concrete in which
the stress and the strain have been scaled down, Figure 2.6.

Hsu and Mo (1983) used the truss analogy and Vecchio and Collins' (1981) reduced diagonal compressive strength in their theory "Softening of Concrete in Torsional Members" which is applicable to symmetrically under reinforced concrete members in pure torsion. Their theory is the same as Mitchell and Collins' diagonal compression field theory, except that it utilizes the full cross section (not the spalled one) and it takes Vecchio and Collins: reduced diagonal compressive strength (softening of concrete) into consideration.

El-Degwy and McMullen (1985) presented results of thirteen symmetrically prestressed concrete rectangular beams tested under pure torsion. The principal variables studied were aspect ratio and amount of reinforcement.

Three computer program were developed, the first one being for the space truss theory with spalling of the concrete cover after Collins and Mitchell (1980), the second one being for the space truss theory with softening of the concrete after Hsu and Mo (1983), and the third one for the skew bending theory after Ewida (1979) and Ewida and McMullen (1982). The behavior of the beams was compared to the behavior predicted by these three theories.


Figure 2.6 Vecchio and Collins' Stress-strain Curve

El-Degwy and McMullen found that:

1. The space truss theory with softening of concrete gave the best overall prediction of torsional strength.
2. Both the space truss theories and the skew bending theory gave a satisfactory but slightly conservative prediction of torsional strength for lightly reinforced beams.
3. All three theories yielded a prediction of torsional strength that is satisfactory for design purposes for beams that have moderate to heavy reinforcement and an aspect ratio of 2.0 .
4. All three theories gave a high (unsafe) prediction of torsional strength for beams having moderate to heavy reinforcement and an aspect ratio of 1.0 , whereas all three theories gave a low (over-safe) prediction of torsional strength for similar beams having an aspect ratio of 3.0 .
5. Examination of test results available in the literature shows that their findings (2,3 and 4) were true not only for prestressed beams but also for reinforced concrete beams.

## CHAPTER 3

## SKEW BENDING ANALYSIS - RECTANGULAR MODES

### 3.1 Introduction

This chapter describes the skew bending theory for reinforced concrete members subjected to pure torsion. The compression zone is rectangular. Equilibrium and compatibility are considered.

### 3.2 Stress-Strain Curve for Concrete

Mitchell and Collins (1974) and Rangan et al (1977) used a parabolic stress-strain relationship for concrete, Figure 2.3

$$
\begin{equation*}
f_{C}=f_{C}^{\prime}\left[2\left(\epsilon_{C} / \epsilon_{o}\right)-\left(\epsilon_{C} / \epsilon_{o}\right)^{2}\right] \tag{3.1}
\end{equation*}
$$

where $f_{C}^{\prime}=$ compressive strength of concrete,

$$
\epsilon_{o}=\operatorname{strain} \text { at } f_{C}=f_{C}^{\prime} \text {, usually taken as } 0.002
$$

and $\quad f_{C}=$ stress in concrete corresponding to a strain of $\epsilon_{C}$

Ewida and McMullen (1982) used the same relationship except that they introduced a reduction factor of 0.35 that scaled down the stress, Figure 2.3

$$
\begin{equation*}
f_{c}=0.35 f_{c}^{\prime}\left[2\left(\epsilon_{c} / \epsilon_{o}\right)-\left(\epsilon_{c} / \epsilon_{o}\right)^{2}\right] \tag{3.2}
\end{equation*}
$$

Vecchio and Collins (1981) proposed a stress-strain relationship for concrete in which both the stress and the strain were scaled down, Figure 2.6 . The equation for the ascending portion of the curve is

$$
\begin{equation*}
f_{C}=f_{C}^{\prime}\left[2\left(\epsilon_{C} / \epsilon_{o}\right)-\lambda\left(\epsilon_{C} / \epsilon_{o}\right)^{2}\right] \tag{3.3}
\end{equation*}
$$

Equation 3.3 is identical to Equation 3.1 except that an empirical coefficient $\lambda$ has been inserted in the second term. This coefficient was found from tests done at University of Toronto, Vecchio and Collins (1981), to be

$$
\begin{equation*}
\lambda=\sqrt{\left[\left(\epsilon_{\ell}+\epsilon_{\mathrm{s}}+2 \epsilon_{\mathrm{c}}\right) / \epsilon_{\mathrm{c}}\right]-0.3} \tag{3.4}
\end{equation*}
$$

where $\epsilon_{\ell}=$ tensile strain in the longitudinal bars and $\quad \epsilon_{\mathbf{s}}=$ tensile strain in the stirrups.

The peak compressive strength and corresponding strain are

$$
\begin{align*}
& f_{p}=f_{c}^{\prime} / \lambda  \tag{3.5}\\
& \epsilon_{p}=\epsilon_{0} / \lambda \tag{3.6}
\end{align*}
$$

The equation for the descending portion of the curve is

$$
\begin{equation*}
f_{c}=f_{p}\left[1-\left(\frac{\epsilon_{c}-\epsilon}{2 \epsilon_{o}-\epsilon}\right)^{2}\right] \tag{3.7}
\end{equation*}
$$

The average stress coefficient, $k_{1}$, which will be required in subsequent sections of the thesis to determine
the magnitude of $C$, the compression force in the concrete, can be derived as:
for $\epsilon_{C} \leqslant \epsilon_{p}$

$$
\begin{equation*}
k_{1}=\frac{{ }^{\epsilon} c}{\epsilon_{p}}\left(1-\frac{{ }^{\epsilon} c}{3 \epsilon}\right) \tag{3.8}
\end{equation*}
$$

for $\epsilon_{c}>\epsilon_{p}$

$$
\begin{equation*}
\mathrm{k}_{1}=\left(1-\lambda^{\prime}\right)\left(1-\frac{\epsilon_{\mathrm{p}}}{3 \epsilon_{c}}\right)+\lambda^{\prime} \frac{{ }^{\epsilon} \mathrm{c}}{\epsilon_{\mathrm{p}}}\left(1-\frac{\epsilon_{\mathrm{c}}}{3 \epsilon_{\mathrm{p}}}\right) \tag{3.9}
\end{equation*}
$$

where $\quad \lambda^{\prime}=\frac{1}{(2 \lambda-1)^{2}}$

The depth to resultant coefficient, $k_{2}$, which will be required in subsequent sections of the thesis to determine the location of $C$, the compression force in the concrete, can be derived as:
for $\epsilon_{c} \leqslant \epsilon_{p}$

$$
\begin{equation*}
k_{2}=\frac{4-\left(\epsilon_{c} / \epsilon_{p}\right)}{12-4\left(\epsilon_{c} / \epsilon_{p}\right)} \tag{3.10}
\end{equation*}
$$

for $\epsilon_{c}>\epsilon_{p}$

$$
\begin{equation*}
\mathrm{k}_{2}=1-\frac{\frac{1}{2}\left(1-\lambda^{\prime}\right)\left(1-\frac{\epsilon_{\mathrm{p}}^{2}}{6 \epsilon_{\mathrm{C}}}\right)+\lambda^{\prime} \frac{{ }^{\epsilon} \mathrm{c}}{\epsilon_{\mathrm{p}}}\left(\frac{2}{3}-\frac{\epsilon_{\mathrm{c}}}{4 \epsilon_{\mathrm{p}}}\right)}{\left(1-\lambda^{\prime}\right)\left(1-\frac{{ }_{\mathrm{c}}}{3 \epsilon_{\mathrm{c}}}\right)+\lambda^{\prime} \frac{{ }^{\prime} \mathrm{c}}{\epsilon_{\mathrm{p}}}\left(1-\frac{\epsilon_{\mathrm{c}}}{3 \epsilon_{\mathrm{p}}}\right)} \tag{3.11}
\end{equation*}
$$

The coefficients $k_{1}$ and $k_{2}$ have been tabulated in Tables 3.1 and 3.2 respectively as functions of $1 / \lambda$ and $\epsilon_{c}$ assuming $\epsilon_{o}=0.002$.

The diagonal cracking, which causes discontinuity in the concrete, and the lateral tension introduced by the reinforcement, would obviously cause a reduction in the concrete strength. Ewida and McMullen's reduction factor of 0.35 was introduced as an empirical factor that led to reasonable results, whereas Vecchio and Collins' proposed stress-strain curve accomplishes the same thing and is more general. Therefore Vecchio and Collins' stress-strain curve will be adopted in this theory and the effect of the reduction in the concrete strength will be studied in the next chapter.

### 3.3 Skew Bending Theory

### 3.3.1 Mode 1

The failure surface for Mode 1 is shown in Figure 3.1. It is bounded on three sides, the bottom and the two verticals, by cracks spiraling around at a constant angle $\theta$. The fourth side has a rectangular compression zone joining the ends of the cracks.

Table $3.1 k_{1}$ as a Function of $1 / \lambda$ and $\epsilon_{c} \quad\left(\epsilon_{o}=0.002\right)$

|  | 0.0005 | 0.001 | 0.0015 | 0.002 | 0.0025 | 0.003 | 0.0035 | 0.004 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.10 | 0.865 | 0.922 | 0.922 | 0.899 | 0.861 | 0.809 | 0.744 | 0.667 |
| 0.20 | 0.733 | 0.861 | 0.888 | 0.881 | 0.851 | 0.805 | 0.743 | 0.667 |
| 0.30 | 0.602 | 0.798 | 0.853 | 0.860 | 0.841 | 0.801 | 0.742 | 0.667 |
| 0.40 | 0.495 | 0.733 | 0.815 | 0.839 | 0.829 | 0.796 | 0.741 | 0.667 |
| 0.50 | 0.417 | 0.667 | 0.775 | 0.815 | 0.817 | 0.790 | 0.739 | 0.667 |
| 0.60 | 0.359 | 0.602 | 0.733 | 0.789 | 0.803 | 0.784 | 0.738 | 0.667 |
| 0.70 | 0.315 | 0.544 | 0.689 | 0.761 | 0.787 | 0.777 | 0.736 | 0.667 |
| 0.80 | 0.280 | 0.495 | 0.645 | 0.732 | 0.770 | 0.769 | 0.734 | 0.667 |
| 0.90 | 0.252 | 0.453 | 0.602 | 0.700 | 0.751 | 0.760 | 0.732 | 0.667 |
| 1.00 | 0.229 | 0.417 | 0.563 | 0.667 | 0.729 | 0.750 | 0.729 | 0.667 |

Table $3.2 \mathrm{k}_{2}$ as a Function of $1 / \lambda$ and $\epsilon_{\mathrm{c}} \quad\left(\epsilon_{\mathrm{o}}=0.002\right)$



Figure 3.1 Mode 1 Failure Surface

From the geometry of the failure surface the following equation can be obtained:

$$
\begin{equation*}
\tan \beta=w \tan \theta \tag{3.12}
\end{equation*}
$$

where $\beta=$ the inclination of the compression zone, $w=[b+2 h(1-k)] / b$,
$\mathrm{b}=$ shorter overall dimension of rectangular cross-section,
$h=$ longer overall dimension of rectangular cross-section,
and $k=$ coefficient used to determine the depth of the compression zone.

The direction cosines of the line joining points $P_{1}\left(x_{1}, y_{1}, z_{1}\right)$ and $P_{2}\left(x_{2}, Y_{2}, z_{2}\right)$, Figure 3.1 which is perpendicular to the compression zone are:

$$
\begin{align*}
\ell & =\left(x_{2}-x_{1}\right) / L_{12}=\sin \beta \\
m & =\left(y_{2}-y_{1}\right) / L_{12}=0  \tag{3.13}\\
n & =\left(z_{2}-z_{1}\right) / L_{12}=\cos \beta
\end{align*}
$$

where $\quad L_{12}=$ distance between points $P_{1}$ and $P_{2}$

$$
=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}+\left(z_{2}-z_{1}\right)^{2}}
$$

The direction cosines of the line joining points $P_{3}\left(x_{3}, Y_{3}, z_{3}\right)$ and $P_{4}\left(x_{4}, Y_{4}, z_{4}\right)$, which is parallel to the compression zone are (Figure 3.1):

$$
\begin{align*}
\ell^{\prime} & =\left(x_{4}-x_{3}\right) / L_{34}=-\cos \beta \\
m^{\prime} & =\left(y_{4}-y_{3}\right) / L_{34}=0  \tag{3.14}\\
n^{\prime} & =\left(z_{4}-z_{3}\right) / L_{34}=\sin \beta
\end{align*}
$$

where $\quad L_{34}=$ distance between points $P_{3}$ and $P_{4}$

$$
=\sqrt{\left(x_{4}-x_{3}\right)^{2}+\left(y_{4}-y_{3}\right)^{2}+\left(z_{4}-z_{3}\right)^{2}}
$$

Note that:

$$
\begin{align*}
& \ell^{2}+m^{2}+n^{2}=l \\
& \ell^{\prime 2}+m^{\prime}+n^{\prime} 2=1  \tag{3.15}\\
& \ell \ell^{\prime}+m m^{\prime}+n n^{\prime}=0
\end{align*}
$$

The forces in the longitudinal bars are:

$$
\begin{equation*}
F_{1}=F_{2}=F_{3}=F_{4}=a_{\ell} F_{l} \tag{3.16}
\end{equation*}
$$

where $a_{\ell}=$ cross-sectional area of one longitudinal bar and $\quad f_{\ell}=$ stress in the longitudinal bar.

The forces in the legs of the stirrups are:

$$
\begin{align*}
& \mathrm{F}_{5}=\mathrm{a}_{\mathrm{s}} \mathrm{f}_{\mathrm{s}} \mathrm{~b}_{1} \tan \theta / \mathrm{s} \\
& \mathrm{~F}_{6}=\mathrm{a}_{\mathrm{s}} \mathrm{f}_{\mathrm{s}}\left[0.5\left(\mathrm{~h}+\mathrm{h}_{1}\right)-\mathrm{kh}\right] \tan \theta / \mathrm{s}  \tag{3.17}\\
& \mathrm{~F}_{7}=\mathrm{a}_{\mathrm{s}} \mathrm{f}_{\mathrm{s}} \mathrm{~b}_{1} \tan \beta / \mathrm{s} \\
& \mathrm{~F}_{8}=\mathrm{F}_{6}
\end{align*}
$$

where $a_{s}=$ area of one leg of a closed stirrup,

$$
\mathrm{f}_{\mathrm{s}}=\text { stress in the stirrups, }
$$

$$
\begin{aligned}
\mathrm{b}_{1}= & \text { shorter center-to-center dimension of a closed } \\
& \text { rectangular stirrup, }
\end{aligned}
$$

$h_{1}=$ longer center-to-center dimension of a closed rectangular stirrup
and $s=$ spacing of stirrups.

The following equation can be obtained by considering the equilibrium of forces acting normal to the compression plane:

$$
\begin{equation*}
C=F_{x} l+F_{y} m+F_{z} n \tag{3.18}
\end{equation*}
$$

where $C=$ the compressive force acting normal to the compression zone
$=k_{I} f_{C}^{\prime} \operatorname{khbsec} \beta / \lambda$
$\mathrm{F}_{\mathrm{x}}=$ the sum of the forces in the direction of the x axis
$=F_{5}+F_{7}$
$=a_{s}{ }_{s} b_{1} \tan \theta(1+w) / s$
$\mathrm{F}_{\mathrm{Y}}=$ the sum of the forces in the direction of the y axis
$=F_{6}-F_{8}$
$=0$
and $\quad F_{z}=$ the sum of the forces in the direction of the z axis

$$
\begin{align*}
& =F_{1}+F_{2}+F_{3}+F_{4} \\
& =4 a_{\ell} f_{\ell} \tag{3.22}
\end{align*}
$$

Substituting Equations 3.12, 3.13, 3.19, 3.20, 3.21 and 3.22 into Equation 3.18 gives

$$
\begin{equation*}
k=\frac{4 a_{\ell} f_{\ell}+a_{S} f_{s} b_{1} \tan ^{2} \theta\left(w+w^{2}\right) / s}{k_{l} f_{c}^{\prime} h b\left(l+w^{2} \tan ^{2} \theta\right) / \lambda} \tag{3.23}
\end{equation*}
$$

The torsional moment is obtained from the equilibrium of external and internal moments about line $P_{3} P_{4}$ which is parallel to the neutral axis and passes through the point of application of the compressive stress resultant.

$$
\begin{equation*}
T n^{\prime}=M_{x} \ell^{\prime}+M_{y} m^{\prime}+M_{z} n^{\prime} \tag{3.24}
\end{equation*}
$$

where $T=$ external torque,

$$
\begin{align*}
M_{x}= & \text { internal moments about } x \text { axis } \\
= & -2 a_{\ell} f_{\ell}\left(\frac{h+h_{2}}{2}-k_{2} k h\right)+2 a_{\ell} f_{\ell}\left(k_{2} k h-\frac{h-h_{2}}{2}\right) \\
& +a_{S} f_{S}\left(\frac{h+h_{l}}{2}-k h\right) \frac{\tan \theta}{s}[b+h(l-k)] \tan \theta  \tag{3.25}\\
M_{y}= & \text { internal moments about } y \text { axis } \\
= & 0 \tag{3.26}
\end{align*}
$$

$$
M_{z}=\text { internal moments about } z \text { axis }
$$

$$
\begin{align*}
= & a_{s} f_{s} \frac{b_{1}}{s} \tan \theta\left(\frac{h+h_{1}}{2}-k_{2} k h\right)+2 a_{s} f_{s}\left(\frac{h+h_{1}}{2}-k h\right) \frac{b_{1} \tan \theta}{2 s} \\
& -a_{s} f_{s} \frac{b_{1}}{s} \tan \beta\left(k_{2} k h-\frac{h-h_{1}}{2}\right) \tag{3.27}
\end{align*}
$$

and $\quad h_{2}=$ longer center-to-center dimension between two longitudinal corner bars.

Substituting Equations 3.12, 3.14, 3.25, 3.26, and 3.27 into Equation 3.24 gives:

$$
\begin{equation*}
T=\sum_{i=1}^{8} T_{i} \tag{3.28}
\end{equation*}
$$

where $\quad T_{1}=a_{\ell} f_{\ell}\left[h\left(1-2 k_{2} k\right)+h_{2}\right] / 2 w \tan \theta$
$T_{2}=T_{1}$
$T_{3}=a_{\ell} f_{\ell}\left[h\left(I-2 k_{2} k\right)-h_{2}\right] / 2 w \tan \theta$
$\mathrm{T}_{4}=\mathrm{T}_{3}$
$T_{5}=\mathrm{a}_{\mathrm{s}} \mathrm{f}_{\mathrm{s}} \mathrm{b}_{1} \tan \theta\left[\mathrm{~h}\left(1-2 \mathrm{k}_{2} \mathrm{k}\right)+\mathrm{h}_{\mathrm{l}}\right] / 2 \mathrm{~s}$
$T_{6}=a_{s}{ }^{f} \tan \theta\left[h(1-2 k)+h_{l}\right]\left\{b_{1}-[b+h(l-k)] / w\right\} / 4 s$
$T_{7}=a_{s}{ }_{s}{ } b_{1} w \tan \theta\left[h\left(1-2 k_{2} k\right)-h_{1}\right] / 2 s$
and $\quad T_{8}=T_{6}$

To determine the value of $\theta$ corresponding to the minimum value of $T$, Equation 3.28 is differentiated with respect to the crack inclination $\theta$, equated to zero and solved for $\theta$

$$
\begin{equation*}
\theta=\tan ^{-1} \sqrt{\frac{4 a^{\prime} \ell^{s} \ell^{s}}{a_{s} f_{s} b_{1} w} \frac{g_{1}}{\sum_{i=2}^{5} g_{i}}} \tag{3.29}
\end{equation*}
$$

where $\quad g_{1}=h\left(1-2 k_{2} k\right)$

$$
g_{2}=g_{1}+h_{1}
$$

$$
g_{3}=h(1-2 k)+h_{1}
$$

$$
g_{4}=-g_{3}[b+h(l-k)] /\left(b_{1} w\right)
$$

and $\quad g_{5}=w\left(g_{1}-h_{1}\right)$

The following compatibility equations can be derived from the Mohr's circle of strain shown in Figure 3.2

$$
\begin{align*}
& \gamma_{l S}=2\left(\epsilon_{l}+\epsilon_{\mathrm{Cr}}\right) \tan \theta  \tag{3.30}\\
& \gamma_{l S}=2\left(\epsilon_{\mathrm{S}}+\epsilon_{\mathrm{Cr}}\right) / \tan \theta \tag{3.31}
\end{align*}
$$

where $\quad \gamma_{\ell S}=$ shearing strain
and $\quad \epsilon_{\mathrm{cr}}=$ compressive strain parallel to the direction of the cracks.

The following equations can be obtained from Equations 3.30 and 3.31

$$
\begin{align*}
& \epsilon_{\ell}=\frac{\epsilon_{S}+\epsilon_{\mathrm{Cr}}}{\tan ^{2} \theta}-\epsilon_{\mathrm{Cr}}  \tag{3.32}\\
& \epsilon_{\mathrm{Cr}}=0.5 \gamma_{\ell \mathrm{S}} \tan \theta-\epsilon_{\mathrm{S}} \tag{3.33}
\end{align*}
$$

The strain normal to the compression plane ( $\beta$-plane)


Figure 3.2 Mohr's Circle of Strain
at the tension side may be written as:

$$
\begin{equation*}
\epsilon_{\beta t}=\epsilon_{\ell} \cos ^{2} \beta+\epsilon_{s} \sin ^{2} \beta+\gamma_{\ell s} \sin \beta \cos \beta \tag{3.34}
\end{equation*}
$$

The strain normal to the compression plane at the compression side may be written as:

$$
\begin{equation*}
\epsilon_{\beta C}=\gamma_{l S} \sin \beta \cos \beta-\epsilon_{l} \cos ^{2} \beta-\epsilon_{s} \sin ^{2} \beta \tag{3.35}
\end{equation*}
$$

Assuming linear strain distribution normal to the compression plane, the following equations may be written:

$$
\begin{equation*}
\epsilon_{\beta C}=\frac{d_{1}-h(1-k)}{k h} \epsilon_{c e} \tag{3.36}
\end{equation*}
$$

$$
\begin{equation*}
\epsilon_{c e}=\frac{\mathrm{kh}}{\mathrm{~d}_{1}-\mathrm{kh}} \epsilon_{\beta t} \tag{3.37}
\end{equation*}
$$

where $d_{l}=$ distance from extreme fiber in compression zone to inner surface of the stirrups at tension side in Mode 1
and $\quad \epsilon_{c e}=$ compressive strain in concrete at the extreme fiber of the compression zone normal to the compression plane.

Substituting Equations 3.34, 3.35 and 3.37 into Equation 3.36 gives

$$
\begin{equation*}
\gamma_{\ell S}=\frac{1+u}{1-u}\left(\frac{\epsilon_{\ell}}{\tan \beta}+\epsilon_{S} \tan \beta\right) \tag{3.38}
\end{equation*}
$$

where $u=\left[d_{1}-h(1-k)\right] /\left(d_{1}-k h\right)$

The twist of a beam can be visualized according to the skew bending model as a rotation about a longitudinal axis passing through the point of application of the compression stress resultant, Ewida and McMullen (1982). The twist of a beam may be expressed in terms of the shearing strain $\gamma_{l s}$ as:

$$
\begin{equation*}
\psi=\gamma_{l s} / d_{r} \tag{3.39}
\end{equation*}
$$

where $\quad d_{r}=d_{1}-k_{2} k h$

Another expression for the twist of $a$ beam can be derived from the compatibility of warping displacements of a thin walled tube element, Onsongo (1978), as:

$$
\begin{equation*}
\psi=\gamma_{\ell s}\left(b_{1}+h_{1}\right) / b_{1} h_{1} \tag{3.40}
\end{equation*}
$$

Using Equation 3.39 Ewida and McMullen got good correlation with their experimental results. In 1986 Hsu and Mo, in their comments on El-Degwy and McMullen's (1985) paper, stated that Equation 3.39 underestimates the value of the angle of twist.

In the next chapter results using Equations 3.39 and 3.40 will be presented and evaluated.

### 3.3.2 Mode 2

The failure surface for Mode 2 is shown in Figure 3.3. It is bounded on three sides, one of the verticals and the two horizontals, by cracks spiraling around at a constant angle $\theta$. The fourth side has a rectangular compression zone joining the ends of the cracks.

Mode 2 can be handled in a manner similar to that used for Mode 1. Equations for Mode 2 will be the same as those for Mode 1 except that $w, u, d_{1}, b, b_{1}, b_{2}, h, h_{1}$ and $h_{2}$ will be changed to $v, r, d_{2}, h, h_{1}, h_{2}, b, b_{1}$ and $b_{2}$ respectively; only the necessary equations will be presented here.

From the geometry of the failure surface:

$$
\begin{equation*}
\tan \beta=v \tan \theta \tag{3.41}
\end{equation*}
$$

where $v=[h+2 b(1-k)] / h$

The coefficient $k$, used to determine the depth of the compression zone can be obtained from:

$$
\begin{equation*}
k=\frac{4 a_{\ell} f_{\ell}+a_{s} \mathrm{f}_{\mathrm{s}} \mathrm{~h}_{1} \tan ^{2} \theta\left(\mathrm{v}+\mathrm{v}^{2}\right) / \mathrm{s}}{\mathrm{k}_{1} \mathrm{f}_{\mathrm{c}}^{1} \mathrm{bh}\left(I+\mathrm{v}^{2} \tan ^{2} \theta\right) / \lambda} \tag{3.42}
\end{equation*}
$$

The torsional moment is obtained as:

$$
\begin{equation*}
T=\sum_{i=1}^{8} T_{i} \tag{3.43}
\end{equation*}
$$



Figure 3.3 Mode 2 Failure Surface
where $\quad T_{1}=a_{\ell} f_{\ell}\left[b\left(I-2 k_{2} k\right)+b_{2}\right] / 2 v \tan \theta$

$$
\begin{aligned}
& \mathrm{T}_{2}=\mathrm{T}_{1} \\
& \mathrm{~T}_{3}=\mathrm{a}_{\ell} \mathrm{f}_{\ell}\left[b\left(1-2 \mathrm{k}_{2} \mathrm{k}\right)-\mathrm{b}_{2}\right] / 2 \mathrm{v} \tan \theta \\
& \mathrm{~T}_{4}=\mathrm{T}_{3} \\
& \mathrm{~T}_{5}=\mathrm{a}_{\mathrm{s}} \mathrm{f}_{\mathrm{s}} \mathrm{~h}_{1} \tan \theta\left[b\left(1-2 \mathrm{k}_{2} k\right)+\mathrm{b}_{1}\right] / 2 \mathrm{~s} \\
& \mathrm{~T}_{6}=\mathrm{a}_{\mathrm{s}} \mathrm{f}_{\mathrm{s}} \tan \theta\left[b(1-2 k)+\mathrm{b}_{1}\right]\left\{\mathrm{h}_{1}-[\mathrm{h}+\mathrm{b}(1-\mathrm{k})] / v\right\} / 4 \mathrm{~s} \\
& \mathrm{~T}_{7}=\mathrm{a}_{\mathrm{s}} \mathrm{f}_{\mathrm{s}} \mathrm{~h}_{1} v \tan \theta\left[b\left(1-2 \mathrm{k}_{2} \mathrm{k}\right)-\mathrm{b}_{1}\right] / 2 \mathrm{~s}
\end{aligned}
$$

and

$$
T_{8}=T_{6}
$$

The crack inclination is obtained as:

$$
\begin{equation*}
\theta=\tan ^{-1} \sqrt{\frac{4 a_{\ell} f_{\ell} s}{a_{s} f_{s} h_{l} v} \sum_{i=2}^{g_{l}}} \tag{3.44}
\end{equation*}
$$

where

$$
\begin{aligned}
& g_{1}=b\left(1-2 k_{2} k\right) \\
& g_{2}=g_{1}+b_{1} \\
& g_{3}=b(1-2 k)+b_{1} \\
& g_{4}=-g_{3}[h+b(1-k)] /\left(h_{1} v\right) \\
& g_{5}=v\left(g_{1}-b_{1}\right)
\end{aligned}
$$

and

The tensile strain in the longitudinal bars is:

$$
\begin{equation*}
\epsilon_{\ell}=\frac{\epsilon_{\mathbf{S}^{+\epsilon}} \mathbf{C r}}{\tan ^{2} \theta}-\epsilon \operatorname{cr} \tag{3.45}
\end{equation*}
$$

The compressive strain parallel to the direction of the cracks is given by:

$$
\begin{equation*}
\epsilon_{C r}=0.5 \gamma_{l S} \tan \theta-\epsilon_{S} \tag{3.46}
\end{equation*}
$$

The strain normal to the compression plane ( $\beta$-plane) at the tension side is:

$$
\begin{equation*}
\epsilon_{\beta t}=\epsilon_{\ell} \cos ^{2} \beta+\epsilon_{s} \sin ^{2} \beta+\gamma_{\ell s} \sin \beta \cos \beta \tag{3.47}
\end{equation*}
$$

The compressive strain in concrete at the extreme fiber of the compression zone normal to the compression plane is obtained as:

$$
\begin{equation*}
\epsilon_{c e}=\frac{\mathrm{kb}}{\mathrm{~d}_{2}-\mathrm{kb}} \epsilon_{\beta t} \tag{3.48}
\end{equation*}
$$

The shearing strain can be obtained from:

$$
\begin{equation*}
\gamma_{l S}=\frac{l+r}{I-r}\left(\frac{\epsilon_{\ell}}{\tan \beta}+\epsilon_{S} \tan \beta\right) \tag{3.49}
\end{equation*}
$$

where $r=\left[d_{2}-b(1-k)\right] /\left(d_{2}-k b\right)$

The angle of twist is:

$$
\begin{equation*}
\psi=\gamma_{l s} / d_{r} \tag{3.50}
\end{equation*}
$$

where $\quad d_{r}=d_{2}-k_{2} k b$
or $\quad \psi=\gamma_{l S}\left(b_{l}+h_{l}\right) /\left(b_{l} h_{l}\right)$

### 3.4 Solution Technique

The stress-strain, geometric, equilibrium and compatibility relationships which have been derived provide enough information to predict the torsional response of a beam if the properties of the section are known. An iterative procedure can be used as follows:

1. Input the beam data.
2. Select strain in the stirrups $\epsilon_{S}$.
3. Assume the angle of crack $\theta$, the depth coefficient $k$ and the strain parallel to the direction of the crack $\epsilon_{\text {cr }}$.
4. Calculate the strain in the longitudinal bars $\epsilon_{\ell}$, Equation 3.32 .
5. Calculate the inclination of the compression zone $\beta$, Equation 3.12 .
6. Calculate the shearing strain $\gamma_{\ell s}$, Equation 3.38.
7. Calculate the strain parallel to the direction of the crack $\epsilon_{\text {cr }}$, Equation 3.33.
8. Calculate the residual of $\epsilon_{c r}$; if it is unacceptable, go back to step 4 using a new value for $\epsilon_{\text {cr }}$.
9. Calculate the coefficient $\lambda$, Equation 3.4, where $\epsilon_{c}$ is taken as $\epsilon_{\text {cr }}$.
10. Calculate the strain normal to the compression plane at the tension side $\epsilon_{\beta t}$, Equation 3.34.
11. Calculate the strain in concrete at the extreme fiber of the compression zone normal to the compression plane $\epsilon_{c e}$, Equation 3.37.
12. Calculate the average stress coefficient $k_{1}$, Equation 3.8 or 3.9 , where $\epsilon_{c}$ is taken as $\epsilon_{c e}$.
13. Calculate the depth coefficient k, Equation 3.23.
14. Calculate the residual of $k$; if it is unacceptable, go back to step 4 using a new value for $k$.
15. Calculate the depth to resultant coefficient $k_{2}$, Equation 3.10 or 3.11 , where $\epsilon_{c}$ is taken as $\epsilon_{c e}$.
16. Calculate angle of crack $\theta$, Equation 3.29.
17. Calculate the residual of $\theta$; if it is unacceptable, go back to step 4 using a new value for $\theta$.
18. Calculate angle of twist per unit length $\psi$, Equations 3.39 and 3.40.
19. Calculate the corresponding torque resistance $T$, Equation 3.28.
20. Repeat steps 2 to 19 for a number of values of strain in the stirrups to get the complete torsional response in Mode 1.
21. Using Mode 2 equations, repeat steps 2 to 19 for a number of values of strain in the stirrups to get the complete torsional response in Mode 2.
22. Select the mode giving the lowest maximum torque.

According to the foregoing iterative procedure, a computer program capable of predicting the behavior of rectangular reinforced concrete beams under pure torsion has been developed.

## CHAPTER 4

## COMPARISON OF ANALYSIS AND TEST RESULTS

### 4.1 Introduction

In this chapter, the experimental torsional strengths of 102 beams reported in the literature are compared to the theoretical strengths predicted by the iterative procedure described in Chapter 3. Torque-twist and torque-strain curves are also presented.

The reduced concrete strength defined by Equations 3.3 and 3.7, Section 3.2, is used in the various calculations of torque in Sections 4.2, 4.3 and 4.4. Its effect on the analysis is explicitly presented in Section 4.5 .

### 4.2 Torsional Strengths

In the iterative procedure, strains in the stirrups are incremented and the corresponding torque is calculated by satisfying both equilibrium and compatibility. The theoretical torsional strength (maximum torque) is the torque at which the slope of the torque-twist curve is zero. Two modes of failure (Mode 1 and Mode 2) have been checked. The theoretical torsional strength of a beam is
defined as the lowest maximum torque of the two strengths computed according to these two modes of failure. Generally, a rectangular beam that has a width less than its depth and that is tested in pure torsion will fail in Mode 2.

The experimental and theoretical torsional strengths of the 102 beams are compared in Table 4.1. Of these beams, 36 have been excluded from the analysis, the reason for exclusion in each case being given in the table. Excessive stirrup spacing (excess. spac.) is defined as the stirrup spacing, $s$, being greater than $\left(b_{1}+h_{1}\right) / 4$ or 300 mm, CAN3-A23.3-M84 (1984). Completely over-reinforced (over-reinf.) beams are defined as those in which the test results show that neither the longitudinal bars nor the stirrups yielded before the maximum torque was reached. Insufficient reinforcement (insuff. reinf.) is when the calculated post-cracking strength is less than the cracking torque $\mathrm{T}_{\mathrm{Cr}}$.

After the exclusion, the average ratio of experimental strength to theoretical strength for the 66 remaining beams is 1.01 and the standard deviation is 7.8\%. For each reference the average $T_{u}(\exp .) / T_{u}(t h$.$) and the standard$ deviation are also given in Table 4.1. Certainly the comparison of experimental and theoretical results is excellent.

Table 4.l Comparison of Experimental and Theoretical

Torsional Strengths


Table 4.1 (cont.I) Comparison of Experimental and

Theoretical Torsional Strengths

| Ref. | Beam | $\begin{gathered} \mathrm{T}_{\mathrm{u}}(\exp .) \\ \mathrm{kN} \cdot \mathrm{~m} \end{gathered}$ | $\begin{gathered} \mathrm{T}_{\mathrm{u}}(\mathrm{th} .) \\ \mathrm{kN}, \mathrm{~m} \end{gathered}$ | $\frac{T_{u}(\exp .)}{T_{u}\left(t h_{.}\right)}$ | Reason for Exclusion |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\begin{aligned} & \text { O} \\ & \text { o } \\ & 0 \\ & 0 \\ & \underset{-}{2} \\ & \text { ⿹ㅗㄱ } \end{aligned}$ | I 2 | 36.0 | 34.2 | 1.05 |  |
|  | 13 | 45.7 | 47.4 | 0.96 |  |
|  | I4 | 58.1 | 59.1 | 0.98 |  |
|  | I5 | 70.7 | 73.5 | 0.96 |  |
|  | 16 | 76.7 | 90.7 | 0.85 | over-reinf. |
|  | J1 | 21.5 | 20.6 | 1.04 |  |
|  | J2 | 29.2 | 28.6 | 1.02 |  |
|  | J3 | 35.3 | 39.3 | 0.90 | over-reinf. |
|  | J4 | 40.7 | 44.6 | 0.91 | over-reinf. |
|  | G1 | $26.8=\mathrm{T}_{\text {cr }}$ | 24.7 | 1.09 | insuff. reinf. |
|  | G2 | 40.3 | 37.0 | 1.09 |  |
|  | G3 | 49.6 | 50.6 | 0.98 |  |
|  | G4 | 64.9 | 64.1 | 1.01 |  |
|  | G5 | 72.0 | 78.7 | 0.91 | over-reinf. |
|  | G6 | 39.1 | 36.9 | 1.06 |  |
|  | G7 | 52.7 | 52.2 | 1.01 |  |
|  | G8 | 73.5 | 69.2 | 1.06 |  |

Table 4.1 (cont. 2) Comparison of Experimental and

Theoretical Torsional Strengths

| Ref. | Beam | $\mathrm{T}_{\mathrm{u}}(\mathrm{exp}$. $\mathrm{kN.m}$ | $\begin{gathered} \mathrm{T}_{\mathrm{u}}(\mathrm{th},) \\ \mathrm{kN} . \mathrm{m} \end{gathered}$ | $\frac{T_{u}(\text { exp. })}{T_{u}(t h .)}$ | Reason for <br> Exclusion |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | NI | 9.1 | 8.3 | 1.10 |  |
|  | Nla | 9.0 | 8.2 | 1.10 |  |
|  | N2 | 14.5 | 13.5 | 1.07. |  |
|  | N2a | 13.2 | 13.4 | 0.99 |  |
|  | N3 | 12.2 | 11.7 | 1.04 |  |
|  | N4 | 15.7 | 16.1 | 0.98 | over-reinf. |
|  | Kl | 15.4 | 13.7 | 1.12 |  |
|  | K2 | 23.7 | 22.4 | 1.06 |  |
|  | K3 | 28.5 | 28.6 | 1.00 |  |
|  | K4 | 35.0 | 31.7 | 1.10 | over-reinf. |
|  | Cl . | $11.3=\mathrm{T}_{\mathrm{Cr}}$ | 10.4 | 1.09 | insuff. reinf. |
|  | C2 | 15.3 | 17.9 | 0.85 | excess. spac. |
|  | C3 | 20.0 | 25.8 | 0.78 | excess. spac. |
|  | C4 | 25.3 | 34.8 | 0.73 | over-reinf. |
|  | C5 | 29.7 | 42.9 | 0.69 | over-reinf. |
|  | C6 | 34.2 | 50.6 | 0.68 | over-reinf. |

Table 4.1 (cont. 3 ) Comparison of Experimental and

Theoretical Torsional Strengths

| Ref. | Beam | $\begin{gathered} \mathrm{T}_{\mathrm{u}}(\exp .) \\ \mathrm{kN} . \mathrm{m} \end{gathered}$ | $\begin{gathered} \mathrm{T}_{\mathrm{u}}(\mathrm{th} .) \\ \mathrm{kN} . \mathrm{m} \end{gathered}$ | $\frac{T_{u}(\exp .)}{T_{u}(t h .)}$ | Reason for <br> Exclusion |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | T1A | 21.7 | 23.0 | 0.94 | Inconsistent Information |
|  | T1B | 22.9 | 23.1 | 0.99 |  |
|  | TlC | 22.1 | 22.9 | 0.97 |  |
|  | 634 | 24.7 | 23.5 | 1.05 |  |
|  | 644 | 32.2 | - | - |  |
|  | 1034 | 48.1 | 45.5 | 1.06 |  |
|  | 1044 | 60.8 | 59.0 | 1.03 |  |
|  | 1055 | 77.5 | 83.3 | 0.93 |  |
|  | 1244 | 59.8 | 65.2 | 0.92 |  |
|  | 1255 | 80.5 | 91.4 | 0.88 |  |
|  | 1644 | 96.4 | 93.3 | 1.03 |  |
|  | 1655 | 119.6 | 125.1 | 0.96 |  |

Table 4.l. (cont.4) Comparison of Experimental and

Theoretical Torsional Strengths

| Ref. | Beam | $\mathrm{T}_{\mathrm{u}}(\exp$. $\mathrm{kN} . \mathrm{m}$ | $\begin{gathered} \mathrm{T}_{\mathrm{u}}(\mathrm{th} .) \\ \mathrm{kN}, \mathrm{~m} \end{gathered}$ | $\frac{T_{u}(\text { exp. })}{T_{u}(\text { th. })}$ | Reason for Exclusion |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | VSI | $11.6=\mathrm{T}_{\text {cr }}$ | 11.0 | 1.05 | insuff. reinf. |
|  | VS2 | 20.0 | 21.1 | 0.95 |  |
|  | VS3 | 29.2 | 30.1 | 0.97 |  |
|  | VS4 | 35.1 | 38.2 | 0.92 | over-reinf. |
|  | VS5 | 19.2 | 17.7 | 1.08 | excess. spac. |
|  | VS6 | 21.1 | 25.8 | 0.82 | excess. spac. |
|  | Vs7/1 | 22.0 | 36.0 | 0.61 | excess. spac. |
|  | VS7/2 | 22.0 | 35.4 | 0.62 | excess. spac. |
|  | VS8/1 | 29.2 | 42.2 | 0.69 | excess. spac. |
|  | Vs8/2 | 29.7 | 43.0 | 0.69 | excess. spac. |
|  | VS9 | 22.0 | 25.4 | 0.87 |  |
|  | vsio | 34.0 | 43.9 | 0.77 | over-reinf. |
|  | VBI | - | - | - | $=$ VSl0 |
|  | VB2 | 43.0 | 46.8 | 0.92 | over-reinf. |
|  | VB3 | 47.4 | 49.5 | 0.96 |  |
|  | VB4 | 49.6 | 50.6 | 0.98 |  |
|  | VQ1 | 21.6 | 21.2 | 1.02 |  |

Table 4.1 (cont.5) Comparison of Experimental and

Theoretical Torsional Strengths

| Ref. | Beam | $\begin{gathered} \mathrm{T}_{\mathrm{u}}(\exp .) \\ \mathrm{kN} . \mathrm{m} \end{gathered}$ | $\begin{aligned} & \mathrm{T}_{\mathrm{u}}(\mathrm{th} .) \\ & \mathrm{kN} . \mathrm{m} \end{aligned}$ | $\frac{T_{u}(\text { exp. })}{T_{u}(t h .)}$ | Reason for Exclusion |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | VQ2 | - | - | - | $=\mathrm{VS} 2$ |
|  | VQ3 | 20.4 | 21.4 | 0.95 |  |
|  | VQ4 | 31.2 | 39.4 | 0.79 | over-reinf. |
|  | VQ5 | - | - | - | $=\mathrm{VS} 4$ |
|  | VQ6 | 36.0 | 37.2 | 0.97 | over-reinf. |
|  | VQ9. | 22.4 | 21.0 | 1.07 |  |
|  | VAl | 38.3 | 45.6 | 0.84 | over-reinf. |
|  | VA2 | 37.8 | 43.5 | 0.87 | over-reinf. |
|  | VA3 | 34.5 | 37.6 | 0.92 | over-reinf. |
|  | vul | 24.5 | 25.2 | 0.97 |  |
|  | vU2 | 31.0 | 31.3 | 0.99 |  |
|  | vU3 | 31.7 | 33.3 | 0.95 | over-reinf. |
|  | vU4 | 26.5 | 28.2 | 0.94 |  |
|  | VM1 | 14.2 | 12.7 | 1.12 |  |
|  | VM2 | 40.0 | 37.7 | 1.06 |  |
|  | VM3 | 103.0 | 98.9 | 1.04 |  |
|  | VM4 | 285.0 | 296.3 | 0.96 |  |

Table 4.1 (cont.6) Comparison of Experimental and

Theoretical Torsional Strengths

| Ref. | Beam | $\mathrm{T}_{\mathrm{u}}(\exp$. $\mathrm{kN} . \mathrm{m}$ | $\begin{gathered} \mathrm{T}_{\mathrm{u}}(\mathrm{th} .) \\ \mathrm{kN} . \mathrm{m} \end{gathered}$ | $\frac{T_{u}(\text { exp. })}{T_{u}(t h .)}$ | Reason for Exclusion |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | Al | 13.1 | 11.3 | 1.16 |  |
|  | AlR | 12.5 | 11.2 | 1.12 |  |
|  | A2 | 22.6 | 20.7 | 1.09 |  |
|  | A3 | 27.8 | 28.3 | 0.98 |  |
|  | A4 | 34.5 | 38.4 | 0.90 |  |
|  | Bl | 12.8 | 10.4 | 1.23 |  |
|  | BlR | 12.3 | 10.4 | 1.18 |  |
|  | B2 | 20.8 | 18.9 | 1.10 |  |
|  | B3 | 25.3 | 25.3 | 1.00 |  |
|  | B4 | 31.8 | 33.2 | 0.96 |  |

Table 4.1 (cont.7) Comparison of Experimental and Theoretical Torsional Strengths

| Reference | No. of <br> Total <br> Beams | No. of <br> Eligible <br> Beams | $\frac{T_{u} \text { (exp.) }}{T_{u} \text { (th.) }}$ | Standard <br> Deviation <br> $\%$ |
| :---: | :---: | :---: | :---: | :---: |
| Hsu (1968b) | 49 | 30 | 1.01 | 7.1 |
| Cameron (1971) | 12 | 11 | 0.98 | 5.6 |
| Leonhardt and <br> Schelling (1974) <br> McMullen and <br> Rangan (1978) | 10 | 15 | 0.99 | 6.0 |
| Total | 102 | 10.07 | 10.2 |  |

If completely over-reinforced beams are not excluded, the average ratio of experimental strength to theoretical strength for (in this case) 89 beams would be 0.97 with a standard deviation of $10.7 \%$ which is not quite as good correlation" as when the completely over-reinforced beams are excluded.

### 4.3 Torque-Twist Curves

The torque-twist curves for beams having a square, A3, and a rectangular, B3, cross-section tested by McMullen and Rangan (1978) are presented in Figures 4.1 and 4.2 respectively. Each figure includes one experimental as well as two theoretical curves.

For the theoretical torque-twist curves the angle of twist is calculated twice for a specific torque, first using Equation 3.39 (Mode 1) or Equation 3.50 (Mode 2) and secondly using Equation 3.40 (Mode l) or Equation 3.51 (Mode 2).

It is apparent that the torque-twist curves calculated using Equation 3.40 or 3.51 for angle of twist are closer to the corresponding experimental curves than the curves calculated using Equation 3.39 or 3.50 for angle of twist.


Figure 4.1 Experimental and Theoretical Torque-Twist Curves for Beam A3 Tested by McMullen and Rangan (1978)


Figure 4.2 Experimental and Theoretical Torque-Twist Curves for Beam B3 Tested by McMullen and Rangan (1978)

### 4.4 Torque-Strain Curves

Figure 4.3 presents the torque versus longitudinal steel strain curve predicted by the theory presented in Chapter 3 along with the experimental curve for McMullen and Rangan's (1978) A3 beam. Figure 4.4 presents the torque versus stirrup strain curves for the same beam. Figures 4.5 and 4.6 present similar curves for McMullen and Rangan's B3 beam. It can be noted from Figures 4.3, 4.4 and 4.5 that the theoretical predictions fit the experimental curves quite well. In Figure 4.6 the correlation between the theoretical and the experimental curves is, in the initial stages, not as good as in the previous figures because the experimental strain increases at an irregular rate ; however, in the final stages, correlation is quite good.

### 4.5 Effect of the Reduction in the Concrete Strength

The torque-twist curves for two beams tested at the University of Calgary, McMullen and Rangan (1978), are presented in Figures 4.7 and 4.8. Each figure includes one experimental as well as two theoretical curves. For the two theoretical curves the angle of twist is calculated using Equation 3.40 or 3.51 and the torque is calculated twice, first using the full strength of concrete defined by



Figure 4.4 Experimental and Theoretical Torque-Stirrup Strain Curves for Beam A3 Tested by McMullen and Rangan (1978)


Figure 4.5 Experimental and Theoretical Torque-Longitudinal Bar Strain


Figure 4.6 Experimental and Theoretical Torque-Stirrup Strain Curves


Figure 4.7 Torque-Twist Curves for Beam A3 (McMullen and Rangan)


Figure 4.8 Torque-Twist Curves for Beam B3 (McMullen and Rangan)

Equation 3.1 and secondly using the reduced concrete strength defined by Equations 3.3 and 3.7.

It is apparent that the torque-twist curves calculated using the reduced concrete strength are closer to the corresponding experimental curves than the curves calculated using the full strength of concrete. They are closer in all aspects, first the maximum torque, second the descending branches of the curves and third the angles of twist at maximum torque. In conclusion, the reduced concrete strength provides good predictions for the experimental results.

## CHAPTER 5

SKEW BENDING ANALYSIS - TRIANGULAR AND TRAPEZOIDAL MODES

### 5.1 Introduction

In Chapter 3 the analysis presented was for skew bending modes with a rectangular compression zone. This chapter describes modes with triangular and trapezoidal compression zones.

### 5.2 Trianqular and Trapezoidal Modes

To study the shape of the failure surface visually, Hsu (1968a) used a large diamond saw to cut reinforced concrete beams perpendicular to their longitudinal axis. They had various height to width ratios and had been tested under pure torsion. He observed that for beams having a height to width ratio equal to or larger than 1.5 , the main cracks seen on the wider faces of the cross sections penetrated perpendicularly into the beams. For beams of square cross section the main cracks were not perpendicular to the face but were diagonal.

This implies that the failure surface for a member of square cross section is a plane perpendicular to a diagonal plane and inclined $45^{\circ}$ to the longitudinal axis of the
beam, Figure 2.4. Hsu stated that a failure surface such as this seems to give a minimum torsional resistance for square cross sections. It is interesting to note that Lessig (1958), who first proposed the skew bending theory, stated that a failure could occur in which the neutral axis intersected one vertical and one horizontal face of a member, i.e. the compression zone is triangular.

Hsu (1968a) mentioned that beams with square cross sections require extensive additional investigation. The available literature shows that nobody tried to analyse a skew bending failure mode for reinforced concrete beams with square or rectangular cross section under pure torsion having a triangular compression zone. The author is the first to try to do so.

According to the analyses and results presented in Chapters 3 and 4 for Modes 1 and 2 with rectangular compression zones, the mode giving the lowest maximum torque was selected. Generally Mode 2 governs for beams having a rectangular cross section. For symmetrically reinforced beams having a square cross section, Mode 1 and Mode 2 give exactly the same results. Hsu and Lessig have inferred that a failure surface with a triangular compression zone could occur and perhaps this would give a
minimum torsional resistance for beams having a square cross section.

Mode 1 has its rectangular compression zone on the top side joining the ends of the spiral crack, Figure 5.la, while Mode 2 has its rectangular compression zone on one of the vertical sides joining the ends of the spiral crack, Figure 5.li. A failure surface with a triangular compression zone on a corner is the proposed triangular mode of failure which lies between Mode 1 and Mode 2, Figures 5.ld, e and $f$.

In order to cover the full range between the two rectangular modes (Modes 1 and 2), two failure surfaces with trapezoidal compression zones are needed. These are the proposed trapezoidal modes of failure shown in Figures 5.1b and h. The transitional modes between the trapezoidal and triangular modes are shown in Figures 5.1c and g.

For each mode shown in Figure 5.1 there are three views, Front View (F.V.), Top View (T.V.) and Side View (S.V.). More detailed figures were presented in Chapter 3 for the two rectangular modes (Modes 1 and 2). In the following sections more detailed figures will be presented for the proposed triangular and trapezoidal modes. Also,

in the following sections the proposed modes of failure will be analysed.

### 5.3 Trianqular Mode

The failure surface for the triangular mode is shown in Figure 5.2. It is bounded by a crack that spirals around the member at a constant angle $\theta$ and has a triangular compression zone across one corner of the beam joining the ends of the spiral crack.

From the geometry of the failure surface the following equations can be obtained:

$$
\begin{equation*}
\tan \beta_{2}=w_{12} \tan \theta \tag{5.1}
\end{equation*}
$$

where $\beta_{2}=$ the angle between line iii-ii and the $x$ axis (Figure 5.2, Top View),
$w_{12}=\frac{b\left(2-k_{b}\right)+h\left(2-k_{h}\right)}{k_{b} b}$,
$\mathrm{b}=$ shorter overall dimension of rectangular cross-section,
$k_{b}=$ coefficient used to determine the base length of the compression zone,
$h$ = longer overall dimension of rectangular cross-section,
and $k_{h}=$ coefficient used to determine the height of the compression zone.


Figure 5.2 Triangular Mode (1-2)

$$
\begin{equation*}
\tan \beta_{1}=\tan \beta_{2}-\left(\mathrm{k}_{\mathrm{h}} \mathrm{~h} / \mathrm{k}_{\mathrm{b}} \mathrm{~b}\right) \tan \beta_{3} \tag{5.2}
\end{equation*}
$$

where $\beta_{1}=$ the angle between line $i-i i$ and the $x$ axis (Figure 5.2, Top View),
and $\quad \beta_{3}=$ the angle between line i-iii and the $y$ axis (Figure 5.2, Front View).

$$
\begin{equation*}
\tan \beta_{4}=\tan a \tan \beta_{2} \tag{5.3}
\end{equation*}
$$

where $\beta_{4}=$ the angle between line $i i-i i i$ and the $y$ axis (Figure 5.2, Front View),
and $\quad a=\tan ^{-1}\left(k_{b} b / k_{h} h\right)$
The direction cosines of a line perpendicular to the compression zone are:

$$
\begin{align*}
\ell & =a_{x} / a_{c} \\
m & =a_{y} / a_{c}  \tag{5.5}\\
n & =a_{z} / a_{c}
\end{align*}
$$

where

$$
\begin{align*}
& a_{x}=0.5 k_{b} b \tan \beta_{1} k_{h} h \\
& a_{y}=0.5 k_{b} b \tan \beta_{3} k_{h} h \\
& a_{z}=0.5 k_{b} b k_{h} h \\
& a_{c}=\sqrt{a_{x}^{2}+a_{y}^{2}+a_{z}^{2}} \tag{5.6}
\end{align*}
$$

and $\quad \ell^{2}+m^{2}+n^{2}=1$.

The direction cosines of the line iv-v which is parallel to the neutral axis and passes through the point of application of the compressive stress resultant are:

$$
\begin{aligned}
& \ell^{\prime}=\frac{x_{v}-x_{i v}}{L_{i v-v}} \\
& m^{\prime}=\frac{y_{v}-y_{i v}}{L_{i v-v}} \\
& n^{\prime}=\frac{z_{v}-z_{i v}}{L_{i v-v}}
\end{aligned}
$$

where $x_{i v}=b^{\prime}$

$$
\begin{array}{ll}
x_{v} & =-b^{\prime} \\
y_{i v} & =h^{\prime} \\
y_{v} & =-h^{\prime} \\
z_{i v} & =-b^{\prime} \tan \beta_{2} \\
z_{v} & =b^{\prime} \tan \beta_{2}
\end{array}
$$

$$
L_{i v-v}=\sqrt{\left(x_{v}-x_{i v}\right)^{2}+\left(y_{v}-y_{i v}\right)^{2}+\left(z_{v}-z_{i v}\right)^{2}}
$$

$$
\ell^{\prime 2}+m^{\prime 2}+n^{\prime 2}=1
$$

$$
b^{\prime} \quad=0.5 k_{2} k_{b} b
$$

$$
h^{\prime}=0.5 k_{2} k_{h} h
$$

and $k_{2}=$ the depth to resultant coefficient.

$$
l l^{\prime}+m m^{\prime}+n n^{\prime}=0
$$

The forces in the longitudinal bars are:

$$
F_{1}=F_{2}=F_{3}=F_{4}=a_{\ell}{ }^{f} \ell Y
$$

where $a_{\ell}=$ cross-sectional area of one longitudinal bar, and $\quad f_{l y}=$ yield stress of the longitudinal bars.

The forces in the legs of the stirrups are:

$$
\begin{aligned}
& F_{5}=a_{s} f_{s y}\left(b_{1} / s\right) \cdot \tan \theta \\
& F_{6}=a_{s}{ }^{f} s_{Y}\left(h_{1} / s\right) \tan \theta
\end{aligned}
$$

$$
F_{7}=a_{s}{ }^{f} \frac{b\left(0.5-k_{b}\right)+0.5 b_{1}}{s} \tan \theta \text { "for } k_{b}<0.5\left[1+\left(b_{1} / b\right)\right] "
$$

$$
\text { or } \mathrm{F}_{7}=0 \quad \text { "for } k_{b} \geqslant 0.5\left[1+\left(b_{1} / b\right)\right] "
$$

$$
F_{8}=a_{s} f_{S Y} \frac{b\left(k_{b}-0.5\right)+0.5 b_{1}}{s} \tan \beta_{1} \text { "for } k_{b}<0.5\left[1+\left(b_{1} / b\right)\right] "
$$

$$
\text { or } F_{8}=a_{s} f_{S Y}\left(b_{1} / s\right) \tan \beta_{1} \quad \text {, for } k_{b} \geqslant 0.5\left[1+\left(b_{1} / b\right)\right] \text { " }
$$

$$
F_{g}=a_{s}{ }^{f} \frac{h\left(k_{h}-0.5\right)+0.5 h_{1}}{s} \tan \beta_{3} \quad \text { "for } k_{h}<0.5\left[1+\left(h_{1} / h\right)\right] "
$$

$$
\text { or } F_{g}=a_{s} f_{s y}\left(h_{1} / s\right) \tan \beta_{3} . \quad \text { for } k_{h} \geqslant 0.5\left[1+\left(h_{1} / h\right)\right] "
$$

$$
\mathrm{F}_{10}=\mathrm{a}_{\mathrm{s}} \mathrm{f}_{\mathrm{sy}} \frac{\mathrm{~h}\left(0.5-\mathrm{k}_{\mathrm{h}}\right)+0.5 \mathrm{~h}_{1}}{\mathrm{~s}} \tan \theta \text { "for } \mathrm{k}_{\mathrm{h}}<0.5\left[1+\left(\mathrm{h}_{1} / \mathrm{h}\right)\right] "
$$

```
or \(F_{10}=0\)
"for \(k_{h} \geqslant 0.5\left[1+\left(h_{1} / h\right)\right] "\)
```

```
where \(a_{s}=\) area of one leg of a closed stirrup,
    \(f_{s y}=y i e l d\) stress of the stirrups,
    \(b_{1}=\) shorter center-to-center dimension of
    a closed rectangular stirrup,
    \(h_{1}=\) longer center-to-center dimension of a closed
    rectangular stirrup
and \(s=\) spacing of stirrup.
```

The following equation can be obtained by considering the equilibrium of forces acting normal to the compression plane:

$$
C=F
$$

where $C=$ the compressive force acting normal to the compression zone

$$
\begin{equation*}
=k_{1} f_{c}^{\prime} a_{c} / \lambda \tag{5.8}
\end{equation*}
$$

F = the tensile force acting normal to the compression zone

$$
\begin{equation*}
=F_{x} \ell+F_{y^{\prime}} m+F_{z} n \tag{5.9}
\end{equation*}
$$

$k_{1}=$ average stress coefficient,
$\mathrm{f}_{\mathrm{c}}^{\prime}=$ compressive strength of concrete,
$\lambda=$ softening coefficient,

$$
\begin{aligned}
\mathrm{F}_{\mathrm{x}}= & \text { the sum of the tensile forces in the direction } \\
& \text { of } \mathrm{x} \text { axis } \\
= & \mathrm{F}_{5}-\mathrm{F}_{7}+\mathrm{F}_{8} \\
\mathrm{~F}_{\mathrm{Y}}= & \text { the sum of the tensile forces in the direction } \\
& \text { of } \mathrm{y} \text { axis } \\
= & F_{6}+\mathrm{F}_{9}-\mathrm{F}_{10}
\end{aligned}
$$

and $\quad F_{z}=$ the sum of the tensile forces in the direction of $z$ axis
$=F_{1}+F_{2}+F_{3}+F_{4}$

$$
=4 a_{\ell}{ }^{f} \ell Y
$$

The torsional moment is obtained from the equilibrium of external and internal moments about line iv-v which is parallel to the neutral axis and passes through the point of application of the compressive stress resultant

$$
\begin{equation*}
T n^{\prime}=M_{x} \ell^{\prime}+M_{y^{\prime}} m^{\prime}+M_{z} n^{\prime} \tag{5.10}
\end{equation*}
$$

where $T=$ external moment,

$$
\begin{aligned}
& \mathrm{M}_{\mathrm{X}}=\text { internal moments about } \mathrm{x} \text { axis } \\
& =-2 a_{\ell} f_{\ell y}\left[\left(h+h_{2}\right) / 2-h^{\prime}\right]+2 a_{\ell} f_{\ell y^{\prime}}\left[h^{\prime}-\left(h-h_{2}\right) / 2\right] \\
& +a_{s}{ }^{f}{ }_{s y}\left(h_{1} / s\right) \tan \theta\left\{\left[\left(k_{2} k_{h} \tan \beta_{3}\right) / 2\right]\right. \\
& \left.+\left[1-\left(k_{2} / 2\right)\right] k_{b} b \tan \beta_{I}-\left(1-k_{b}\right) b \tan \theta-(h / 2) \tan \theta\right\} \\
& -a_{s}{ }^{f} s y\left\{\left[h\left(2 k_{h}-1\right)+h_{I}\right] / 2 s\right\} \tan \beta_{3}\left\{\left[\left(k_{h} h \tan \beta_{3}\right) / 2\right]\right. \\
& \left.+\left[\left(k_{2} k_{b} b \tan \beta_{1}\right) / 2\right]-\left[\left(k_{2} k_{h} h \tan \beta_{3}\right) / 2\right]\right\} \\
& +a_{s} f_{s y}\left\{\left[h\left(1-2 k_{h}\right)+h_{1}\right] / 2 s\right\} \tan \theta\left\{k_{h} h \tan \beta_{3}\right.
\end{aligned}
$$

$$
\begin{align*}
& +\left[\left(k_{2} k_{b} b \tan \beta_{1}\right) / 2\right]-(h / 2)\left(1-k_{h}\right) \\
& \left.-\left[\left(k_{2} k_{h} h \tan \beta_{3}\right) / 2\right]\right\} \tag{5.11}
\end{align*}
$$

$$
\begin{align*}
M_{y}= & \text { internal moments about } y \text { axis } \\
= & -2 a_{\ell} f_{\ell y}\left[\left(b+b_{2}\right) / 2-b^{\prime}\right]+2 a_{\ell} f_{\ell y}\left[b^{\prime}-\left(b-b_{2}\right) / 2\right] \\
& -a_{s} f_{S y}\left(b_{1} / s\right) \tan \theta\left\{\left(k_{2} k_{h} h \tan \beta_{3}\right) / 2\right. \\
& \left.+\left[1-\left(k_{2} / 2\right)\right] k_{b} b \tan \beta_{1}-\left(1-k_{b}\right) b \tan \theta-h \tan \theta-(b / 2) \tan \theta\right\} \\
& +a_{s} f_{s y}\left\{\left[b\left(1-2 k_{b}\right)+b_{1}\right] / 2 s\right\} \tan \theta\left\{\left(k_{2} k_{h} h \tan \beta_{3}\right) / 2\right. \\
& \left.+\left(1-k_{2} / 2\right) k_{b} b \tan \beta_{1}-\left[\left(1-k_{b}\right) b \tan \theta\right] / 2\right\} \\
& -a_{s} f_{s y}\left\{\left[b\left(2 k_{b}-1\right)+b_{1}\right] / 2 s\right\} \tan \beta_{1}\left[\left(k_{b} b \tan \beta_{1}\right) / 2\right. \\
& \left.-\left(k_{2} k_{b} b \tan \beta_{1}\right) / 2+\left(k_{2} k_{h} h \tan \beta_{3}\right) / 2\right] \tag{5.12}
\end{align*}
$$

$M_{z}=$ internal moments about $z$ axis

$$
=a_{s} f_{s y}\left(b_{1} / s\right) \tan \theta\left[\left(h+h_{1}\right) / 2-h^{\prime}\right]
$$

$$
+a_{s} f_{s y}\left(h_{1} / s\right) \tan \theta\left[\left(b+b_{1}\right) / 2-b^{\prime}\right]
$$

$$
+a_{s}{ }^{f} s y\left\{\left[b\left(1-2 k_{b}\right)+b_{1}\right] / 2 s\right\} \tan \theta\left[h^{\prime}-\left(h-h_{1}\right) / 2\right]
$$

$$
-a_{s} f_{s y}\left\{\left[b\left(2 k_{b}-1\right)+b_{1}\right] / 2 s\right\} \tan \beta_{1}\left[h^{\prime}-\left(h-h_{1}\right) / 2\right]
$$

$$
-a_{s} f_{s y}\left\{\left[h\left(2 k_{h}-1\right)+h_{1}\right] / 2 s\right\} \tan \beta_{3}\left[b^{\prime}-\left(b-b_{1}\right) / 2\right]
$$

$$
\begin{equation*}
+a_{s} f_{s y}\left\{\left[h\left(1-2 k_{h}\right)+h_{l}\right] / 2 s\right\} \tan \theta\left[b^{\prime}-\left(b-b_{1}\right) / 2\right] \tag{5.13}
\end{equation*}
$$

$$
\begin{aligned}
\mathrm{h}_{2}= & \text { longer center-to-center dimension between two } \\
& \text { longitudinal corner bars }
\end{aligned}
$$

and $\quad b_{2}=$ shorter center-to-center dimension between two longitudinal corner bars.

Substituting Equations 5.11, 5.12 and 5.13 into Equation 5.10 gives

$$
\begin{equation*}
T=\sum_{i=1}^{10} T_{i} \tag{5.14}
\end{equation*}
$$

$$
\begin{aligned}
& \text { where } T_{1}=-a_{\ell} f_{\ell y^{\prime}}\left[\left(h+h_{2}\right) / 2-h^{\prime}\right]\left(\ell^{\prime} / n^{\prime}\right) \\
& +a_{\ell} f_{\ell y^{\prime}}\left[b^{\prime}-\left(b-b_{2}\right) / 2\right]\left(m^{\prime} / n^{\prime}\right) \\
& T_{2}=-a_{\ell} f_{\ell y^{\prime}}\left[\left(h+h_{2}\right) / 2-h^{\prime}\right]\left(\ell^{\prime} / n^{\prime}\right) \\
& -a_{\ell} f_{\ell y}\left[\left(b+b_{2}\right) / 2-b^{\prime}\right]\left(m^{\prime} / n^{\prime}\right) \\
& T_{3}={ }^{=} a_{\ell} f_{\ell y}\left[h^{\prime}-\left(h-h_{2}\right) / 2\right]\left(\ell^{\prime} / n^{\prime}\right) \\
& -a_{\ell} f_{\ell y}\left[\left(b+b_{2}\right) / 2-b^{\prime}\right]\left(m^{\prime} / n^{\prime}\right) \\
& T_{4}=a_{l} f_{l y}\left[h^{\prime}-\left(h-h_{2}\right) / 2\right]\left(l^{\prime} / n^{\prime}\right) \\
& +a_{\ell} f_{\ell y^{\prime}}\left[b^{\prime}-\left(b-b_{2}\right) / 2\right]\left(m^{\prime} / n^{\prime}\right) \\
& T_{5}=-a_{s} f_{s y}\left(b_{1} / s\right) \tan \theta\left\{\left[\left(1-0.5 k_{2}\right) k_{b} w_{12}\right.\right. \\
& \left.\left.-\left(1.5-k_{b}\right)\right] b-h\right\} \tan \theta-\left(1-k_{2}\right) k_{h} h \tan \beta_{3}\left(m^{\prime} / n^{\prime}\right) \\
& +a_{s} f_{S y}\left(b_{1} / s\right) \tan \theta\left[\left(h+h_{1}\right) / 2-h^{\prime}\right] \\
& \mathrm{T}_{6}=\mathrm{a}_{\mathrm{s}}{ }^{\mathrm{f}} \mathrm{Sy}\left(\mathrm{~h}_{1} / \mathrm{s}\right) \tan \theta\left\{\left[1-\left(\mathrm{k}_{2} / 2\right)\right] \mathrm{k}_{\mathrm{b}}{ }^{\mathrm{bw}} 12\right. \\
& \left.-\left(1-k_{b}\right) b-(h / 2)\right\} \tan \theta-\left(1-k_{2}\right) k_{h} h \tan \beta_{3}\left(l^{\prime} / n^{\prime}\right) \\
& +a_{s}{ }^{f} s y\left(h_{1} / s\right) \tan \theta\left[\left(b+b_{1}\right) / 2-b^{\prime}\right] \\
& T_{7}=a_{s} f_{s y}\left\{\left[b\left(1-2 k_{b}\right)+b_{1}\right] / 2 s\right\} \tan \theta\left\{\left[\left(1-0.5 k_{2}\right) \mathrm{k}_{\mathrm{b}} \mathrm{w}_{12}\right.\right. \\
& \left.\left.-0.5\left(1-k_{b}\right)\right] b \tan \theta-\left(1-k_{2}\right) k_{h} h \tan \beta_{3}\right\}\left(m^{\prime} / n^{\prime}\right) \\
& +a_{s}{ }^{f} s y\left\{\left[b\left(1-2 k_{b}\right)+b_{1}\right] / 2 s\right\} \tan \theta\left[h^{\prime}-\left(h-h_{1}\right) / 2\right] \\
& \text { "for } k_{b}<0.5\left[1+\left(b_{1} / b\right)\right] " \\
& \text { or } \\
& T_{7}=0 \\
& \text { "for } k_{b} \geqslant 0.5\left[1+\left(b_{1} / b\right)\right] "
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{T}_{8}= & -\mathrm{a}_{\mathrm{s}^{\mathrm{f}}} \mathrm{sy}\left\{\left[\mathrm{~b}\left(2 \mathrm{k}_{\mathrm{b}}-1\right)+\mathrm{b}_{1}\right] / 2 \mathrm{~s}\right\}\left[\mathrm{w}_{12} \tan \theta\right. \\
& \left.-\left(\tan \beta_{3} / \tan a\right)\right]\left\{\left[\left(1-\mathrm{k}_{2}\right) / 2\right] \mathrm{k}_{\mathrm{b}} \mathrm{bw} 12 \tan \theta\right. \\
& \left.+\left(\mathrm{k}_{2}-0.5\right) \mathrm{k}_{\mathrm{h}} \tan \beta_{3}\right\}\left(\mathrm{m}^{\prime} / \mathrm{n}^{\prime}\right) \\
& -\mathrm{a}_{\mathrm{s}^{\mathrm{f}}} \mathrm{sy}\left\{\left[\mathrm{~b}\left(2 \mathrm{k}_{\mathrm{b}}-1\right)+\mathrm{b}_{1}\right] / 2 \mathrm{~s}\right\}\left[\mathrm{w}_{12} \tan \theta\right. \\
& \left.-\left(\tan \beta_{3} / \tan a\right)\right]\left[\mathrm{h}^{\prime}-\left(\mathrm{h}-\mathrm{h}_{1}\right) / 2\right]
\end{aligned}
$$

"for $k_{b}<0.5\left[1+\left(b_{1} / b\right)\right] "$
or

$$
\begin{aligned}
T_{8}= & -a_{s} f_{s y}\left(b_{1} / s\right)\left[w_{12} \tan \theta\right. \\
& -\left(\tan \beta_{3} / \tan a\right)\left\{\left[\left(1-k_{2}\right) / 2\right] k_{b} b w_{12} \tan \theta\right. \\
& \left.+\left(k_{2}-0.5\right) k_{h} h \tan \beta_{3}\right\}\left(m^{\prime} / n^{\prime}\right)-a_{s} f_{s y}\left(b_{1} / s\right)\left[w_{12} \tan \theta\right. \\
& \left.-\left(\tan \beta_{3} / \tan a\right)\right]\left[h^{\prime}-\left(h-h_{1}\right) / 2\right] \\
& \text { "for } k_{b} \geqslant 0.5\left[1+\left(b_{1} / b\right)\right] " \\
T_{9}= & -a_{s} f_{s y}\left\{\left[h\left(2 k_{h}-1\right)+h_{1}\right] / 2 s\right\} \tan \beta_{3}\left[0.5 k_{2} k_{b} w_{12} \tan \theta\right. \\
& \left.-\left(k_{2}-0.5\right) k_{h} h \tan \beta_{3}\right]\left(l^{\prime} / n^{\prime}\right) \\
& -a_{s^{\prime}} f_{s y}\left\{\left[h\left(2 k_{h}-1\right)+h_{1}\right] / 2 s\right\} \tan \beta_{3}\left[b^{\prime}-\left(b-b_{1}\right) / 2\right] \\
& \text { "for } k_{h}<0.5\left[1+\left(h_{1} / h\right)\right] "
\end{aligned}
$$

or

$$
\begin{aligned}
T_{9}= & -a_{s^{f}}{ }_{s y}\left(h_{1} / s\right) \tan \beta_{3}\left[0.5 k_{2} k_{b} b w_{12} \tan \theta\right. \\
& \left.-\left(k_{2}-0.5\right) k_{h} h \tan \beta_{3}\right]\left(\ell^{\prime} / n^{\prime}\right) \\
& -a_{S^{\prime}}{ }_{s y}\left(h_{1} / s\right) \tan \beta_{3}\left[b^{\prime}-\left(b-b_{1}\right) / 2\right] \\
& \quad \text { for } k_{h} \geqslant 0.5\left[1+\left(h_{1} / h\right)\right] "
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{T}_{I 0}= & \mathrm{a}_{\mathrm{s}} \mathrm{f}_{\mathrm{sy}}\left\{\left[\mathrm{~h}\left(1-2 \mathrm{k}_{\mathrm{h}}\right)+\mathrm{h}_{\mathrm{l}}\right] / 2 \mathrm{~s}\right\} \tan \theta\left\{0 . 5 \left[\mathrm{k}_{2} \mathrm{k}_{\mathrm{b}} \mathrm{bw}_{12}\right.\right. \\
& \left.\left.-\mathrm{h}\left(1-\mathrm{k}_{\mathrm{h}}\right)\right] \tan \theta+\left(1-\mathrm{k}_{2}\right) \mathrm{k}_{\mathrm{h}} \mathrm{~h} \tan \beta_{3}\right\}\left(\ell^{\prime} / \mathrm{n}^{\prime}\right) \\
& +\mathrm{a}_{\mathrm{s}^{\prime}{ }_{\mathrm{sy}}\left\{\left[\mathrm{~h}\left(1-2 \mathrm{k}_{\mathrm{h}}\right)+\mathrm{h}_{1}\right] / 2 \mathrm{~s}\right\} \tan \theta\left[\mathrm{b}^{\prime}-\left(\mathrm{b}-\mathrm{b}_{1}\right) / 2\right]} \\
& \text { "for } \mathrm{k}_{\mathrm{h}}<0.5\left[1+\left(\mathrm{h}_{1} / \mathrm{h}\right)\right] "
\end{aligned}
$$

or $\quad T_{10}=0$
"for $k_{h} \geqslant 0.5\left[1+\left(h_{1} / h\right)\right]$ ".

The triangular mode (1-2) was developed as a transitional mode from the rectangular mode $l$ to the other rectangular mode 2. Another triangular mode (2-1), Figure 5.3, was developed as a transitional mode from the rectangular mode 2 to the other rectangular mode 1 . It was found that either triangular mode, (1-2) or (2-1), can be used, as both will give identical results for any given value of $a$.

### 5.4 Solution Technigue for the Trianquiar Mode

The geometric and equilibrium relationships which have been derived provide enough information to predict the torsional response of a beam if the properties of the section are known.

The following iterative procedure can be used:

1. Input the beam data.
2. Initiate the angle $a$.
3. Assume a range for the angle of crack $\theta$.
4. Assume the coefficient $k_{b}$.
5. Assume a range for the angle $\beta_{3}$.
6. Calculate the coefficient $k_{h}$, Equation 5.4.
7. Calculate the angles $\beta_{2}$ and $\beta_{4}$, Equations 5.1 and 5.3 respectively.


Front View
Side View


Figure 5.3 Triangular Mode (2-1)
8. Calculate the angle $\beta_{1}$, Equation 5.2.
9. Calculate the area of the compression zone $a_{c}$, Equation 5.6.
10. Find the minimum value of $a_{c}$ for the range of $\beta_{3}$ (do. loop between steps 8 and 10).
11. Calculate the forces $C$ and $F$, Equations 5.8 and 5.9 respectively.
12. Check if. $C=F$; if not change the coefficient $k_{b}$ and go back to step 5.
13. Calculate the corresponding resisting torque $T$, Equation 5.14.
14. Find the minimum value of $T$ for the range of $\theta$ (do loop between steps 4 and 14).
15. Repeat steps 3 to 14 for a number of values of angle a to get the full torsional response for the triangular mode.

In accordance with the foregoing iterative procedure, a computer program capable of predicting the behavior of rectangular reinforced concrete beams under pure torsion has been developed for the triangular mode.

### 5.5 Special Trianqular Mode

As a check for the Triangular Mode, a special. triangular mode (45-S) for square cross sections, Figure 5.4, with a equal to $45^{\circ}$ was developed and the



Front View


Side View

$$
\begin{aligned}
& \beta_{1}=\beta_{3} \\
& B_{2}=\beta_{4} \\
& f_{1}=f_{3} \\
& f_{5}=f_{6} \\
& f_{7}=f_{10} \\
& f_{8}=f_{9}
\end{aligned}
$$

Figure 5.4 Spectal Triangular Mode (45-S)
results agreed with the results given by the general triangular mode for this special case.

### 5.6 Trapezoidal Mode (1-2)

To cover the transition range from the rectangular mode (Mode l) to the triangular mode, a trapezoidal mode is needed. The failure surface for the trapezoidal mode is shown in Figure 5.5. It is bounded by a crack that spirals around three sides of the member at a constant angle $\theta$ and has a trapezoidal compression zone across the top (fourth) side of the beam joining the ends of the spiral crack.

From the geometry of the failure surface the following equations can be obtained:

$$
\begin{equation*}
\tan \beta_{2}=w_{12} \tan \theta \tag{5.15}
\end{equation*}
$$

where $\quad \beta_{2}=$ the angle between line iv-iii and the $x$ axis (Figure 5.5, Top View),
$w_{12}=\left[b+h\left(2-k_{h r}-k_{h l}\right)\right] / b$,
$b$ = shorter overall dimension of rectangular cross-section,
$h$ longer overall dimension of rectangular cross-section,
$k_{h r}=$ coefficient used to determine the height of the right side of the compression zone,


Figure 5.5 Trapezoldal Mode (1-2)
and $\quad k_{h \ell}=$ coefficient used to determine the height of the left side of the compression zone.

$$
\begin{equation*}
\tan \beta_{1}=\tan \beta_{2}-\left\{\left[\left(\mathrm{k}_{\mathrm{h} \ell}-\mathrm{k}_{\mathrm{hr}}\right) \mathrm{h}\right] / \mathrm{b}\right\} \tan \beta_{3} \tag{5.16}
\end{equation*}
$$

where $\beta_{1}=$ the angle between line $i-i i$ and the $x$ axis (Figure 5.5, Top View),
and $\quad \beta_{3}=$ the angle between lines i-iv and ii-iii and the y axis (Figure 5.5, Front View).

$$
\begin{equation*}
\tan \beta_{4}=\tan a \tan \beta_{2} \tag{5.17}
\end{equation*}
$$

where $\beta_{4}=$ the angle between line iii-iv and the $y$ axis (Figure 5.5, Front View)
and $\quad a=\tan ^{-1}\left\{b /\left[\left(k_{h \ell^{-}} \mathrm{k}_{\mathrm{hr}}\right) \mathrm{h}\right]\right\}$
The direction cosines of a line perpendicular to the compression zone are:

$$
\begin{align*}
\ell & =a_{x} / a_{c} \\
m & =a_{y} / a_{c}  \tag{5.19}\\
n & =a_{z} / a_{c}
\end{align*}
$$

where $\quad a_{x}=0.5 b \tan \beta_{1}\left(k_{h r}+k_{h \ell}\right) h$

$$
a_{\mathrm{y}}=0.5 \mathrm{btan} \beta_{3}\left(\mathrm{k}_{\mathrm{hr}}+\mathrm{k}_{\mathrm{hl}}\right) \mathrm{h}
$$

$$
a_{z}=0.5 b\left(k_{h r}+k_{h l}\right) h
$$

$$
\begin{equation*}
a_{c}=\sqrt{a_{x}^{2}+a_{y}^{2}+a_{z}^{2}} \tag{5.20}
\end{equation*}
$$

and

$$
\ell^{2}+m^{2}+n^{2}=1
$$

The direction cosines of the line $v$-vi which is parallel to the neutral axis and passes through the point of application of the compressive stress resultant are:

$$
\begin{align*}
& \ell^{\prime}=\frac{x_{v i}-x_{v}}{L_{v-v i}} \\
& m^{\prime}=\frac{y_{v i}-y_{v}}{L_{v-v i}}  \tag{5.21}\\
& n^{\prime}=\frac{z_{v i}-z_{v}}{L_{v-v i}}
\end{align*}
$$

where

$$
x_{v i}=-b^{\prime}
$$

$$
x_{v}=b-b, \quad \text { for } k_{2} k_{b} b \geqslant b "
$$

or $\quad x_{v}=k_{2} k_{b} b-b^{\prime}$
"for $k_{2} k_{b} b<b "$

$$
y_{v i}=h^{\prime}-k_{2} k_{h} h
$$

$$
y_{v}=\left[1-\left(b^{\prime} / b\right)\right]\left(k_{h \ell}-k_{h r}\right) h \quad \text { for } k_{2} k_{b} b \geqslant b "
$$

or

$$
\begin{aligned}
& y_{v}=h^{\prime} \quad \text { "for } \\
& z_{v i}=b^{\prime} \tan \beta_{1}-z^{\prime}+k_{2} k_{h} h \tan \beta_{3} \\
& z_{v}=-\left(b-b^{\prime}\right) \tan \beta_{1}-z^{\prime}+k_{h l} h\left[k_{2}-\left(1 / k_{b}\right)\right] \tan \beta_{3}
\end{aligned}
$$

$$
\text { "for } k_{2} k_{b} b<b "
$$

$$
\text { "for } k_{2} k_{b} b \geqslant b "
$$

or $\quad z_{v}=-\left(b-b^{\prime}\right) \tan \beta_{1}-z^{\prime}+\left(1-k_{2} k_{b}\right) b \tan \beta_{1} \quad$ for $k_{2} k_{b} b<b "$

$$
\begin{aligned}
L_{v-v i} & =\sqrt{\left(x_{v i}-x_{v}\right)^{2}+\left(y_{v i}-y_{v}\right)^{2}+\left(z_{v i}-z_{v}\right)^{2}} \\
\ell^{\prime 2} & +m^{\prime}{ }^{2}+n^{\prime 2}=1
\end{aligned}
$$

$$
\begin{aligned}
b^{\prime} & =\left(b / 2 k_{h \ell}\right)\left[k_{h r}+k_{2}\left(k_{h \ell}-k_{h r}\right)\right] \\
h^{\prime} & =k_{2} k_{h \ell} h-b^{\prime}\left(k_{h \ell} k_{h r}\right) h / b \\
z^{\prime} & =\left[k_{2}-\left(b^{\prime} / k_{b} b\right)\right] k_{h \ell} h \tan \beta_{3} \\
k_{2} & =\text { the depth to resultant coefficient }
\end{aligned}
$$

$$
\begin{equation*}
\text { and } \quad k_{b}=k_{h l} /\left(k_{h l}-k_{h r}\right) \tag{5.22}
\end{equation*}
$$

Note that:

$$
l^{\prime}+m m^{\prime}+n n^{\prime}=0
$$

The forces in the longitudinal bars are:

$$
F_{1}=F_{2}=F_{3}=F_{4}=a_{\ell} f_{\ell y}
$$

where $a_{\ell}=$ cross-sectional area of one longitudinal bar, and $\quad f_{l y}=y i e l d$ stress of the longitudinal bars.

The forces in the legs of the stirrups are:

$$
F_{5}=a_{s} f_{S y}\left(b_{1} / s\right) \tan \theta
$$

$$
\mathrm{F}_{6}=a_{s} f_{s y}\left\{\left[h\left(0.5-k_{h r}\right)+\left(h_{l} / 2\right)\right] / s\right\} \tan \theta
$$

$$
\text { "for } \mathrm{k}_{\mathrm{hr}}>0.5\left[1-\left(\mathrm{h}_{1} / \mathrm{h}\right)\right] "
$$

or

$$
F_{6}=a_{s} f_{s y}\left(h_{1} / s\right) \tan \theta \quad \text { "for } k_{h r} \leqslant 0.5\left[1-\left(h_{1} / h\right)\right] "
$$

$$
F_{7}=a_{s} f_{s y}\left(b_{1} / s\right) \tan \beta_{1}
$$

$$
F_{8}=a_{s} f_{s y}\left\{\left[h\left(k_{h l}-0.5\right)+\left(h_{1} / 2\right)\right] / s\right\} \tan \beta_{3}
$$

$$
\text { "for } \mathrm{k}_{\mathrm{hl}}<0.5\left[1+\left(\mathrm{h}_{1} / \mathrm{h}\right)\right] "
$$

$$
\text { or } \quad F_{8}=a_{s} f_{s y}\left(h_{l} / s\right) \tan \beta_{3} \quad \text { for } k_{h l} \geqslant 0.5\left[1+\left(h_{l} / h\right)\right] "
$$

$$
\text { or } \quad F_{9}=0
$$

$$
\begin{aligned}
& \mathrm{F}_{9}=a_{S} f_{S Y}\left\{\left[h\left(0.5-k_{h \ell}\right)+\left(h_{1} / 2\right)\right] / s\right\} \tan \theta \\
& \text { "for } k_{h \ell}<0.5\left[1+\left(h_{1} / h\right)\right] " \\
& F_{9}=0 \text { "for } k_{h \ell} \geqslant 0.5\left[1+\left(h_{1} / h\right)\right] "
\end{aligned}
$$

where $a_{s}=$ area of one leg of a closed stirrup,

$$
f_{s y}=\text { yield stress of the stirrups, }
$$

$$
b_{1}=\text { shorter center-to-center dimension of } a
$$ closed rectangular stirrup,

$h_{I}=$ longer center-to-center dimension of a closed rectangular stirrup
and $\quad s \quad=$ spacing of stirrup.

The following equation can be obtained by considering the equilibrium of forces acting normal to the compression plane:
$C=F$

Where $C=$ the compressive force acting normal to the compression zone
$=k_{I} f_{c}{ }_{c}{ }_{c} / \lambda$
$F=$ the tensile force acting normal to the compression zone
$=F_{x} \ell+F_{y}{ }^{m}+F_{z} n$
$k_{l}=$ average stress coefficient,
$f_{C}^{\prime}=$ compressive strength of concrete,
$\lambda=$ softening coefficient,

$$
\begin{aligned}
F_{x}= & \text { the sum of the tensile forces in the } \\
& \text { direction of } x \text { axis } \\
= & F_{5}+F_{7} \\
F_{Y}= & \text { the sum of the tensile forces in the } \\
& \text { direction of } y \text { axis } \\
= & F_{6}+F_{8}-F_{9}
\end{aligned}
$$

and

$$
\begin{aligned}
F_{z}= & \text { the sum of the tensile forces in the } \\
& \text { direction of } z \text { axis } \\
= & F_{1}+F_{2}+F_{3}+F_{4} \\
= & 4 a_{l} f_{l y}
\end{aligned}
$$

The torsional moment is obtained from the equilibrium of external and internal moments about line v -vi which is parallel to the neutral axis and passes through the point of application of the compressive stress resultant

$$
\begin{equation*}
T n^{\prime}=M_{x} l^{\prime}+M_{y} m^{\prime}+M_{z} n^{\prime} \tag{5.25}
\end{equation*}
$$

where $T=$ external moment,

$$
\begin{align*}
& M_{x}=\text { internal moments about } \mathrm{x} \text { axis } \\
& =-2 a_{\ell}{ }_{\ell Y}\left[\left(h+h_{2}\right) / 2-h^{\prime}\right]+2 a_{\ell} f_{\ell y}\left[h^{\prime}-\left(h-h_{2}\right) / 2\right] \\
& +\mathrm{a}_{\mathrm{S}} \mathrm{f}_{\mathrm{Sy}}\left\{\left[\mathrm{~h}\left(0.5-\mathrm{k}_{\mathrm{hr}}\right)+\left(\mathrm{h}_{\mathrm{I}} / 2\right)\right] / \mathrm{s}\right\} \tan \theta\left[\left(\mathrm{b}-\mathrm{b}^{\prime}\right) \tan \beta_{1}\right. \\
& \left.+\mathrm{z}^{\prime}-\mathrm{k}_{\mathrm{hr}} \mathrm{htan}_{3}-0.5 \mathrm{~h}\left(1-\mathrm{k}_{\mathrm{hr}}\right) \tan \theta\right] \\
& -a_{s} f_{s y}\left\{\left[h\left(k_{h \ell}-0.5\right)+\left(h_{I} / 2\right)\right] / s\right\} \tan \beta_{3}\left(0.5 k_{h \ell} h \tan \beta_{3}\right. \\
& \left.+b^{\prime} \tan \beta_{1}-z^{\prime}\right) \\
& +\mathrm{a}_{\mathrm{s}} \mathrm{f}_{\mathrm{sy}}\left\{\left[\mathrm{~h}\left(0.5-\mathrm{k}_{\mathrm{h} \mathrm{\ell}}\right)+\left(\mathrm{h}_{\mathrm{l}} / 2\right)\right] / \mathrm{s}\right\} \tan \theta\left[\mathrm{k}_{\mathrm{h} \ell} \mathrm{htan} \beta_{3}\right. \\
& \left.+\mathrm{b}^{\prime} \tan \beta_{1}-0.5 \mathrm{~h}\left(1-\mathrm{k}_{\mathrm{h} \ell}\right) \tan \theta-\mathrm{z}^{\prime}\right] \tag{5.26}
\end{align*}
$$

$$
\begin{align*}
M_{Y}= & \text { internal moments about } y \text { axis } \\
= & -2 a_{\ell} f_{\ell Y}\left[\left(b+b_{2}\right) / 2-b^{\prime}\right]+2 a_{\ell} f_{\ell y^{\prime}}\left[b^{\prime}-\left(b-b_{2}\right) / 2\right] \\
& -a_{s^{\prime}} f_{S Y}\left(b_{1} / s\right) \tan \theta\left[z^{\prime}+\left(b-b^{\prime}\right) \tan \beta_{1}-k_{h r} h \tan \beta_{3}\right. \\
& \left.-h\left(1-k_{h r}\right) \tan \theta-(b / 2) \tan \theta\right] \\
& -a_{s^{\prime}} f_{S Y}\left(b_{1} / s\right) \tan \beta_{1}\left[z^{\prime}+(b / 2) \tan \beta_{1}-b^{\prime} \tan \beta_{1}\right] \tag{5.27}
\end{align*}
$$

$M_{z}=$ internal moments about $z$ axis
$=a_{s} f_{s y}\left(b_{1} / s\right) \tan \theta\left[\left(h+h_{1}\right) / 2-h^{\prime}\right]$
$-a_{S} f_{S Y}\left(b_{1} / s\right) \tan \beta_{1}\left[h^{\prime}-\left(h-h_{1}\right) / 2\right]$
$+\mathrm{a}_{\mathrm{s}} \mathrm{f}_{\mathrm{sy}}\left\{\left[\mathrm{h}\left(0.5-\mathrm{k}_{\mathrm{hr}}\right)+\left(\mathrm{h}_{1} / 2\right)\right] / \mathrm{s}\right\} \tan \theta\left[\left(\mathrm{b}+\mathrm{b}_{1}\right) / 2-\mathrm{b}^{\prime}\right]$
$-a_{s} f_{s y}\left\{\left[h\left(k_{h \ell}-0.5\right)+\left(h_{1} / 2\right)\right] / s\right\} \tan \beta_{3}\left[b^{\prime}-\left(b-b_{1}\right) / 2\right]$
$+a_{s} f_{s y}\left\{\left[h\left(0.5-k_{h \ell}\right)+\left(h_{l} / 2\right)\right] / s\right\} \tan \theta\left[b^{\prime}-\left(b-b_{l}\right) / 2\right]$
$h_{2}=$ longer center-to-center dimension between two longitudinal corner bars
and $\quad b_{2}=$ shorter center-to-center dimension between two longitudinal corner bars.

Substituting Equations 5.26, 5.27 and 5.28 into Equation 5.25 gives:

$$
\begin{equation*}
T=\sum_{i=1}^{9} T_{i} \tag{5.29}
\end{equation*}
$$

where

$$
\begin{aligned}
T_{I}= & -a_{\ell} f_{\ell y^{\prime}}\left[\left(h+h_{2}\right) / 2-h^{\prime}\right]\left(\ell^{\prime} / n^{\prime}\right) \\
& +a_{\ell}{ }^{f} \ell Y\left[b^{\prime}-\left(b-b_{2}\right) / 2\right]\left(m^{\prime} / n^{\prime}\right) \\
T_{2}= & -a_{\ell} f_{\ell y^{\prime}}\left[\left(h+h_{2}\right) / 2-h^{\prime}\right]\left(\ell^{\prime} / n^{\prime}\right) \\
& -a_{\ell} f_{\ell Y}\left[\left(b+b_{2}\right) / 2-b^{\prime}\right]\left(m^{\prime} / n^{\prime}\right)
\end{aligned}
$$

$$
\begin{aligned}
& T_{3}=a_{\ell} f_{\ell y^{[ }}\left[h^{\prime}-\left(h-h_{2}\right) / 2\right]\left(\ell^{\prime} / n^{\prime}\right) \\
& -a_{\ell} f_{\ell Y}\left[\left(b+b_{2}\right) / 2-b^{\prime}\right]\left(m^{\prime} / n^{\prime}\right) \\
& T_{4}=a_{\ell} f_{\ell y^{\prime}}\left[h^{\prime}-\left(h-h_{2}\right) / 2\right]\left(l^{\prime} / n^{\prime}\right) \\
& +a_{\ell} f_{\ell y}\left[b^{\prime}-\left(b-b_{2}\right) / 2\right]\left(m^{\prime} / n^{\prime}\right) \\
& T_{5}=-a_{s} f_{s y}\left(b_{1} / s\right) \tan \theta\left[\left(b-b^{\prime}\right) w_{12}-h\left(1-k_{h r}\right)-0.5 b\right] \tan \theta \\
& -\left\{\left[\left(b-b^{\prime}\right) / \tan a\right]+k_{h r} h\right\} \tan \beta_{3}+z^{\prime} \quad\left(m^{\prime} / n^{\prime}\right) \\
& +a_{s} f_{s y}\left(b_{1} / s\right) \tan \theta\left[\left(h+h_{1}\right) / 2-h^{\prime}\right] \\
& T_{6}=a_{s} f_{s y}\left\{\left[h\left(1-2 k_{h r}\right)+h_{1}\right] / 2 s\right\} \tan \theta\left[\left(b-b^{\prime}\right) w_{12}\right. \\
& \left.-0.5 h\left(1-k_{h r}\right)\right] \tan \theta-\left\{\left[\left(b-b^{\prime}\right) / \tan a\right]\right. \\
& \left.+k_{h r} h\right\} \tan \beta_{3}+z^{\prime}\left(\ell^{\prime} / n^{\prime}\right) \\
& +a_{s} f_{s y}\left\{\left[h\left(1-2 k_{h r}\right)+h_{I}\right] / 2 s\right\} \tan \theta\left[\left(b+b_{I}\right) / 2-b^{\prime}\right] \\
& \text { "for } k_{h r}>0.5\left[1-\left(h_{l} / h\right)\right] "
\end{aligned}
$$

or

$$
\begin{aligned}
& T_{6}=a_{s}{ }^{f}{ }_{s y}\left(h_{1} / s\right) \tan \theta\left[\left(b-b^{\prime}\right) w_{12}\right. \\
& \left.-0.5 \mathrm{~h}\left(1-\mathrm{k}_{\mathrm{hr}}\right)\right] \tan \theta-\left\{\left[\left(\mathrm{b}-\mathrm{b}^{\prime}\right) / \tan a\right]\right. \\
& \left.+\mathrm{k}_{\mathrm{hr}}\right\} \tan \beta_{3}+z^{\prime}\left(\ell^{\prime} / \mathrm{n}^{\prime}\right) \\
& +a_{s} f_{s y}\left(h_{1} / s\right) \tan \theta\left[\left(b+b_{1}\right) / 2-b^{\prime}\right] \\
& \text { "for } \mathrm{k}_{\mathrm{hr}} \leqslant 0.5\left[1-\left(\mathrm{h}_{1} / \mathrm{h}\right)\right] " \\
& T_{7}=-\mathrm{a}_{\mathrm{s}} \mathrm{f}_{\mathrm{sy}}\left(\mathrm{~b}_{1} / \mathrm{s}\right)\left[\mathrm{w}_{12} \tan \theta-\left(\tan \beta_{3} / \tan a\right)\right][(0.5 b \\
& \left.-\mathrm{b}^{\prime}\right) \mathrm{w}_{12} \tan \theta-\left(0.5 \mathrm{~b}-\mathrm{b}^{\prime}\right)\left(\tan \beta_{3} / \tan a\right) \\
& \left.+z^{\prime}\right]\left(m^{\prime} / n^{\prime}\right) \\
& -a_{s} f_{S Y}\left(b_{1} / s\right)\left[w_{12} \tan \theta-\left(\tan \beta_{3} / \tan a\right)\right]\left[h^{\prime}\right. \\
& \left.-\left(h-h_{1}\right) / 2\right]
\end{aligned}
$$

or

$$
\begin{aligned}
& T_{8}=-a_{S} f_{S Y}\left(h_{1} / s\right) \tan \beta_{3}\left\{b^{\prime} w_{12} \tan \theta\right. \\
&\left.+\left[0.5 k_{h \ell} h-\left(b^{\prime} / \tan a\right)\right] \tan \beta_{3}-z^{\prime}\right\}\left(\ell^{\prime} / n^{\prime}\right) \\
&-a_{S_{S Y}}{ }_{S Y}\left(h_{1} / s\right) \tan \beta_{3}\left[b^{\prime}-\left(b-b_{1}\right) / 2\right] \\
& \quad \text { for } k_{h l} \geqslant 0.5\left[1+\left(h_{1} / h^{\prime}\right)\right] "
\end{aligned}
$$

and

$$
\text { or } \quad T_{9}=0
$$

5.7 Trapezoidal Mode (2-1)

To cover the transition range from the rectangular mode (Mode 2) to the triangular mode, trapezoidal mode (2-1) was developed. Its failure surface is shown in Figure 5.6. It is bounded by a crack that spirals around three sides of the member at a constant angle $\theta$ and has a trapezoidal compression zone across one of the vertical (fourth) sides of the beam joining the ends of the spiral crack.

$$
\begin{aligned}
& T_{9}=a_{s} f_{s y}\left\{\left[h\left(1-2 k_{h 1}\right)+h_{1}\right] / 2 s\right\} \tan \theta\left\{\left[b^{\prime} w_{12}\right.\right. \\
& \left.-0.5 \mathrm{~h}\left(1-\mathrm{k}_{\mathrm{h} \ell}\right)\right] \tan \theta+\left[\mathrm{k}_{\mathrm{h} \ell} \mathrm{~h}\right. \\
& \left.\left.-\left(b^{\prime} / \tan a\right)\right] \tan \beta_{3}-z^{\prime}\right\}\left(\ell^{\prime} / n^{\prime}\right) \\
& +a_{s}{ }^{f} \mathrm{sy}\left\{\left[\mathrm{~h}\left(1-2 \mathrm{k}_{\mathrm{h} \ell}{ }^{\prime}\right)+\mathrm{h}_{\mathrm{l}}\right] / 2 \mathrm{~s}\right\} \tan \theta\left[\mathrm{b}^{\prime}-\left(\mathrm{b}-\mathrm{b}_{\mathrm{j}}\right) / 2\right] \\
& \text { "for } k_{h l}<0.5\left[1+\left(h_{1} / h\right)\right] " \\
& \text { "for } k_{h l} \geqslant 0.5\left[1+\left(h_{1} / h\right)\right] "
\end{aligned}
$$

$$
\begin{aligned}
& \mathrm{T}_{8}=-\mathrm{a}_{\mathrm{S}} \mathrm{f}_{\mathrm{sy}}\left\{\left[\mathrm { h } \left(2 \mathrm{k}_{\left.\left.\left.\mathrm{h} \ell^{-1}\right)+\mathrm{h}_{1}\right] / 2 \mathrm{~s}\right\} \tan \beta_{3}\left\{\mathrm{~b}^{\prime} \mathrm{w}_{12} \tan \theta\right.}\right.\right.\right. \\
& \left.+\left[0.5 \mathrm{k}_{\mathrm{h} \ell} \mathrm{~h}-\left(\mathrm{b}^{\prime} / \tan a\right)\right] \tan \beta_{3}-\mathrm{z}^{\prime}\right\}\left(\ell^{\prime} / \mathrm{n}^{\prime}\right) \\
& -a_{s} f_{s y}\left\{\left[h\left(2 k_{h \ell}-1\right)+h_{1}\right] / 2 s\right\} \tan \beta_{3}\left[b^{\prime}-\left(b-b_{1}\right) / 2\right] \\
& \text { "for } k_{h \ell}<0.5\left[1+\left(h_{1} / h\right)\right] "
\end{aligned}
$$



Figure 5.6 Trapezoidal Mode (2-1)

The trapezoidal mode (2-1) can be handled in a manner similar to that used for the trapezoidal mode (1-2). For square cross-sections, the trapezoidal mode (2-1) leads to the same results as the trapezoidal mode (1-2) for the angle complementary to the angle $a ; e . g$. the torque computed by mode (2-1) for angle $a=10^{\circ}$ is equal to the torque computed by mode (1-2) for angle $a=80^{\circ}$.

### 5.8 Solution Technique for the Trapezoidal Modes

The geometric and the equilibrium relationships which have been derived provide enough information to predict the torsional response of a beam if the properties of the section are known.

The following iterative procedure, similar to that. used for the triangular mode, can be used:

1. Input the beam data.
2. Initiate the angle $a$.
3. Assume a range for the angle of crack $\theta$.
4. Assume the coefficient $k_{h l}$.
5. Assume a range for the angle $\boldsymbol{\beta}_{3}$.
6. Calculate the coefficient $\mathrm{k}_{\mathrm{hr}}$, Equation 5.18.
7. Calculate the coefficient $\mathrm{k}_{\mathrm{b}}$, Equation 5.22.
8. Calculate the angles $\beta_{2}$ and $\beta_{4}$, Equations 5.15 and 5.17 respectively.
9. Calculate the angle $\beta_{1}$, Equation 5.16.

10: Calculate the area of the compression zone $a_{c}$, Equation 5.20.
11. Find the minimum value of $a_{c}$ for the range of $\beta_{3}$ (do loop between steps 9 and ll).
12. Calculate the forces $C$ and $F$, Equations 5.23 and 5.24 respectively.
13. Check if $C=F$; if not change the coefficient $k_{h \ell}$ and go back to step 5.
14. Calculate the corresponding resisting torque $T$, Equation 5.29.
15. Find the minimum value of $T$ for the range of $\theta$ (do loop between steps 4 and 15).
16. Repeat steps 3 to 15 for a number of values of angle $a$ to get the full torsional response for the trapezoidal mode (1-2).

In accordance with the foregoing iterative procedure, a computer program capable of predicting the behavior of rectangular reinforced concrete beams under pure torsion has been developed for the trapezoidal mode (1-2). The values of the angle a used in the computer program vary from the value corresponding to the rectangular mode (Mode 1), i.e. $a=90^{\circ}$ and $k_{b}=\infty$, to the value corresponding to the triangular mode. The value of $a$ corresponding to the triangular mode is the value that
results in $k_{b}$ being equal to 1.0 .

Using the trapezoidal mode (2-1) equations, another computer program has been developed. The values of the angle $a$ used in this program vary from the value corresponding to the rectangular mode (Mode 2), i.e. $a=0^{0}$ and $k_{h}=\infty$, to the value corresponding to the triangular mode. The value of $a$ corresponding to.the triangular mode is the value that results in $k_{h}$ being equal to 1.0 .

### 5.9 The coefficients $\underline{k}_{\underline{1}}+\underline{k}_{2}$ and $\underline{\lambda}$

For the stress-strain relationship of concrete presented by Equation 3.1

$$
\begin{equation*}
f_{c}=f_{C}^{\prime}\left[2\left(\epsilon_{C} / \epsilon_{o}\right)-\left(\epsilon_{C} / \epsilon_{o}\right)^{2}\right] \tag{3.1}
\end{equation*}
$$

and for a rectangular compression zone such as in Modes 1 and 2, the average stress coefficient, $k_{l r}$ can be derived as:

$$
\begin{equation*}
\mathrm{k}_{\mathrm{lr}}=\frac{\epsilon_{\mathrm{c}}}{\epsilon_{\mathrm{o}}}\left(1-\frac{\epsilon_{\mathrm{c}}}{3 \epsilon_{\mathrm{o}}}\right) \tag{5.30}
\end{equation*}
$$

and the depth to resultant coefficient, $k_{2 r}$ can be derived as:

$$
\begin{equation*}
k_{2 r}=\frac{4-\left(\epsilon_{c} / \epsilon_{o}\right)}{12-4\left(\epsilon_{c} / \epsilon_{o}\right)} \tag{5.31}
\end{equation*}
$$

where $\quad f_{c}^{\prime}=$ compressive strength of concrete,

$$
\text { and } \quad \begin{aligned}
\epsilon_{\mathrm{O}} & =\text { strain at } \mathrm{f}_{\mathrm{C}}=\mathrm{f}_{\mathrm{C}}^{\prime} \text { usually taken as } 0.002 \\
\mathrm{f}_{\mathrm{C}} & =\text { stress in concrete corresponding to a strain } \\
& \text {, of } \epsilon_{\mathrm{C}}
\end{aligned}
$$

For the same stress-strain relationship for concrete (Equation 3.1) but for a triangular compression zone such as in triangular modes $1-2,2-1$ and $45-S$, the average stress coefficient, $k_{l t}$ can be derived as:

$$
\begin{equation*}
k_{l t}=\frac{2}{3} \frac{\epsilon_{C}}{\epsilon_{o}}\left(1-\frac{\epsilon_{C}}{4 \epsilon_{o}}\right) \tag{5.32}
\end{equation*}
$$

and the depth to resultant coefficient, $k_{2 t}$ can be derived as:

$$
\begin{equation*}
k_{2 t}=\frac{10-2\left(\epsilon_{\mathrm{c}} / \dot{\epsilon}_{\mathrm{o}}\right)}{20-5\left(\epsilon_{\mathrm{c}} / \epsilon_{\mathrm{o}}\right)} \tag{5.33}
\end{equation*}
$$

The average stress coefficient, $\mathrm{k}_{\mathrm{lz}}$ corresponding to the trapezoidal compression zone of Mode (1-2) is assumed to be:

$$
\begin{equation*}
k_{l z}=k_{l t} \frac{k_{h \ell}-k_{h r}}{k_{h \ell}}+k_{l r} \frac{k_{h r}}{k_{h \ell}} \tag{5.34}
\end{equation*}
$$

where $\quad k_{l z}=k_{l r}$
"for $k_{h r}=k_{h \ell}$ "
and $\quad k_{l z}=k_{l t}$
"for $k_{h r}=0 \quad "$

The depth to resultant coefficient, $k_{2 z}$ corresponding to the trapezoidal compression zone of Mode (1-2) is assumed to be:

$$
\begin{equation*}
k_{2 z}=k_{2 t} \frac{k_{h \ell}-k_{h r}}{k_{h \ell}}+k_{2 r} \frac{k_{h r}}{k_{h \ell}} \tag{5.35}
\end{equation*}
$$

where $\quad k_{2 z}=k_{2 r}$

$$
\begin{aligned}
& \text { "for } k_{h r}=k_{h \ell} " \\
& \text { "for } k_{h r}=0 \quad "
\end{aligned}
$$

Equations similar to 5.34 and 5.35 can be assumed for the trapezoidal compression zone of Mode (2-1).

Rüsh (1960) stated that the shape of the cross section has a decisive effect on the value of ultimate strain as shown in Figure 5.7. The extreme fiber concrete strain, ${ }^{\epsilon}$ u, varies from 0.003 to 0.0035 for a rectangular cross section and it varies from 0.0038 to 0.0048 for a triangular cross section. Consequently conservative values for $\epsilon$ may be assumed as 0.003 for a rectangular compression zone and 0.004 for a triangular compression zone. Therefore the coefficients $k_{1 r}$ and $k_{2 r}$ are. calculated for $\epsilon_{C}=0.003$ from Equations 5.30 and 5.31 to be 0.750 and 0.417 respectively and the coefficients $k_{\text {lt }}$ and $k_{2 t}$ are calculated for $\epsilon_{c}=0.004$ from Equations 5.32 and 5.33 to be 0.667 and 0.600 respectively.

The average stress coefficient and the depth to

resultant coefficient corresponding to a trapezoidal compression zone, $k_{l z}$ and $k_{2 z}$ respectively, could be derived as have been done for rectangular (Equations 5.30 and 5.31) and triangular (Equations 5.32 and 5.33) compression zones. In this case an assumption must be made regarding the variation of the extreme fiber concrete strain, $\epsilon_{u}$, from 0.003 for a rectangular compression zone to 0.004 for a triangular compression zone.

For modes having a rectangular compression zone (Modes 1 and 2) both equilibrium and compatibility were considered and the softening coefficient $\lambda$ was calculated from Equation 3.4. For the triangular and the trapezoidal modes, compatibility was not considered. (only equilibrium) and therefore the softening coefficient $\lambda$ is assumed to vary linearly with respect to the angle $a$ and it can be expressed as:

$$
\begin{equation*}
\lambda=\lambda_{1}+\left(\lambda_{2}-\lambda_{1}\right) \frac{90^{\circ}-a}{90^{\circ}} \tag{5.36}
\end{equation*}
$$

where $\quad \lambda_{l}=$ softening coefficient for Mode 1 ,
$\lambda_{2}=$ softening coefficient for Mode 2 ,
$\lambda=\lambda_{1}$
"for $a=90^{\circ}$
and $\quad \lambda=\lambda_{2}$
"for $a=0^{\circ}$ ".

## CHAPTER 6

## RESULTS OF TRIANGULAR AND TRAPEZOIDAL MODES

### 6.1 Introduction

In this chapter, the results obtained from the triangular and trapezoidal modes will be discussed. The effects of the aspect ratio, amount of reinforcement, concrete strength and softening of concrete (reduction in the concrete strength) on the analyses are presented in this chapter.

### 6.2 Effect of Aspect Ratio

To study the effect of the aspect ratio on the analyses, two beams (I and 2) are considered having aspect ratios of 1.0 and 2.0 respectively. They have $m=A_{l} f_{\ell Y} S / 2 a_{S}\left(b_{l}+h_{l}\right) f_{S Y}=1.0$ and ${ }^{f_{l Y}}=f_{S Y}=300 \mathrm{MPa}$. Details are shown in Table 6.1. In the iterative procedures described in Chapter 5 for the trapezoidal and triangular modes, the angle $a$ is incremented and the corresponding torque is calculated.

The torque-alpha curve for the beam having a square cross section (Beam 1) is shown in Figure 6.1. At the starting point, the angle a was equal to $90^{\circ}$, which corresponds to the rectangular mode 1 , then its value was

Table 6.1 Details of Beams 1-6

| Beam | b <br> mm | h <br> mm | $\mathrm{b}_{1}$ <br> mm | $\mathrm{h}_{1}$ <br> mm | $\begin{array}{r} f_{c}^{\prime} \\ \mathrm{MPa} \end{array}$ | Reinforcement |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  |  |  |  | Long. | $\begin{gathered} \mathrm{A}_{\ell} \\ \mathrm{mm}^{2} \end{gathered}$ | $\begin{aligned} & \rho_{\ell} \\ & \% \end{aligned}$ | $\begin{array}{\|c} \text { Sti- } \\ \text { rupps } \end{array}$ | $\begin{gathered} \mathrm{a}_{\mathrm{s}} \\ \mathrm{~mm}^{2} \end{gathered}$ | mm | $\begin{aligned} & \rho_{s} \\ & \% \end{aligned}$ |
| 1 | 300 | 300 | 257 | 257 | 40 | 8\#4 | 1032 | 1.15 | \# 4 | 129 | 129 | 1.15 |
| 2 | 212 | 424 | 169 | 381 | 40 | 8》 ${ }^{\text {4 }}$ | 1032 | 1.15 | \#4 | 129 | 138 | 1.15 |
| 3 | 300 | 300 | 257 | 257 | 40 | 4\#\#5,4\#\#4 | 1316 | 1.46 | 栍 | 129 | 101 | 1.46 |
| 4 | 212 | 424 | 169 | 381 | 40 | 4\#\#5,4\#4 | 1316 | 1.46 | \#4 | 129 | 108 | 1.46 |
| 5 | 300 | 300 | 257 | 257 | 25 | 8\#4 | 1032 | 1.15 | \#4 | 129 | 129 | 1.15 |
| 6 | 212 | 424 | 169 | 381 | 25 | 8\#4 | 1032 | 1.15 | \#4 | 129 | 138 | 1.15 |



Figure 6.1 Torque-Alpha Curve for Beam 1
decreased in decrements of $1^{\circ}$ till it reached $45^{\circ}$. A torque-alpha curve for a beam having a square cross section with the same reinforcement on each face has an axis of symmetry, i.e. the values of the torque are exactly the same for the angle $a$ and its complementary angle. Thus, Figure 6.1 actually represents half a torque-alpha curve.

For Beam 1 the triangular mode with $a=45^{\circ}$ gives, as shown in Figure 6.1, the minimum value for the torque. This implies that the failure surface for a beam having a square cross section is bounded by a crack that spirals around the beam at a constant angle and has a $45^{\circ}$ triangular compression zone across one corner joining the ends of the crack. This supports Hsu's statement mentioned in Chapters 2 and 5, that a failure surface such as this seems to give a minimum torsional resistance for square cross sections.

The torque-alpha curve for the beam having a rectangular cross section (Beam 2) is shown in Figure 6.2. Starting with the trapezoidal mode l-2, the angle a was decremented and the corresponding torque was calculated. The first value for $a$ in this mode was $90^{\circ}$, which corresponds to the rectangular mode 1 , and the last value before the mode changed to the triangular mode was $69^{\circ}$. Then using the trapezoidal mode $2-1$ the first value


Figure 6.2 Torque-Alpha Curve for Beam 2
for $a$ was $0^{\circ}$, which corresponds to the rectangular mode 2 , and the last value before the mode changed to the triangular mode was $15^{\circ}$. The triangular mode covered the remaining range of $a$ from $16^{\circ}$ to $68^{\circ}$. Unlike the torque-alpha curve for the square cross section shown in Figure 6.1, the torque-alpha curve for a rectangular cross section shown in Figure 6.2 has no axis of symmetry.

As shown in Figure 6.2, the rectangular mode 2 gives a lower value of torsional resistance than the rectangular mode 1 but the minimum value for the torsional resistance is still given by the triangular mode with $a=45^{\circ}$. This implies that the failure surface for Beam 2, which has a rectangular cross section with an aspect ratio equal to 2.0 , is similar to the failure surface for Beam 1 which has a square cross section.

This result is unexpected as for beams having a rectangular cross section a failure surface such as this has not been suggested previously while it has been suggested for beams having a square cross section.

### 6.3 Effect of Amount of Reinforcement

To examine the effect of the amount of reinforcement on the analyses, two more beams (3 and 4) are considered, Table 6.1. They also have $m=1.0, f_{\ell Y}=f_{s y}=300 \mathrm{MPa}$ and aspect ratios of 1.0 and 2.0 respectively.

The torque-alpha curves for the beams having a square cross section (Beams 1 and 3) are shown in Figure 6.3.

As mentioned in the previous section, the triangular mode with $a=45^{\circ}$ gives the minimum torsional resistance for Beam 1. The torque at $a=90^{\circ}$ is $37.6 \mathrm{kN} . \mathrm{m}$ and it is $34.3 \mathrm{kN} . \mathrm{m}$ at $a=45^{\circ}$. Beam l has $\rho_{\ell}=\rho_{S}=1.15 \%$. For Beam 3, which has more reinforcement than Beam 1 ( $\rho_{l}=\rho_{S}=$ 1.46\%), the rectangular mode gives the minimum torsional resistance. The torque at $a=90^{\circ}$ is $46.7 \mathrm{kN} . \mathrm{m}$ and it is $46.9 \mathrm{kN} . \mathrm{m}$ at $a=45^{\circ}$.

The torque-alpha curves for the beams having a rectangular cross section (Beams 2 and 4) are shown in Figure 6.4.

As mentioned in the previous section the triangular mode with $a=45^{\circ}$ gives the minimum torsional resistance for Beam 2 which has $\rho_{l}=\rho_{S}=1.15 \%$. The torque at $a=90^{\circ}$ is $35.5 \mathrm{kN} . \mathrm{m}$, at $a=0^{\circ}$ is $34.2 \mathrm{kN} . \mathrm{m}$ and it is 29.7 kN.m at $a=45^{\circ}$. For Beam 4 which has more reinforcement than Beam $2\left(\rho_{\ell}=\rho_{S}=1.46 \%\right)$, the triangular mode still gives the minimum torsional resistance but at $a=44^{\circ}$. The torque at $a=90^{\circ}$ is $44.5 \mathrm{kN} . \mathrm{m}$, at $a=0^{\circ}$ is $42.8 \mathrm{kN} . \mathrm{m}$ and it is $41.4 \mathrm{kN} . \mathrm{m}$ at $a=44^{\circ}$.


Figure 6.3 Torque-Alpha Curves for Beams 1 and 3


Figure 6.4 Torque-Alpha Curves for Beams 2 and 4

### 6.4 Effect of Concrete Strength

To study the effect of concrete strength on the analyses, two more beams (5 and 6) are considered, Table 6.1. Beams 5 and 6 are identical to Beams 1 and 2 respectively except that for Beams 5 and $6 f_{C}^{\prime}$ is equal to 25 MPa whereas it is 40 MPa for Beams 1 and 2.

The torque-alpha curves for the beams having a square cross section (Beams 1 and 5) are shown in Figure 6.5. For Beam 5 with $f_{C}^{\prime}=25 \mathrm{MPa}$, the rectangular mode $\left(a=90^{\circ}\right)$ gives $T=35.6 \mathrm{kN} . \mathrm{m}$, which is, as expected, less than the torque ( $37.6 \mathrm{kN} . \mathrm{m}$ ) given by the rectangular mode for Beam 1 with $f_{C}^{\prime}=40 \mathrm{MPa}$. The $45^{\circ}$ triangular mode gives $T=38.8 \mathrm{kN} . \mathrm{m}$ for Beam 5 . which is unexpectedly higher than the torque ( $34.3 \mathrm{kN} . \mathrm{m}$ ) given by the $45^{\circ}$ triangular mode for Beam 1.

The torque-alpha curves for the beams having a rectangular cross section (Beams 2 and 6) are shown in Figure 6.6. For Beam 6, having $f_{c}^{\prime}=25 \mathrm{MPa}$, the torque at $a=90^{\circ}$ is $34.2 \mathrm{kN} . \mathrm{m}$, at $a=0^{\circ}$ is $32.3 \mathrm{kN} . \mathrm{m}$ and it is $34.3 \mathrm{kN} . \mathrm{m}$ at $a=43^{\circ}$; i.e. the rectangular mode 2 $\left(a=0^{\circ}\right.$ ) gives the minimum torsional resistance for Beam 6. As mentioned before, the triangular mode with $a=45^{\circ}$ gives the minimum torsional resistance ( $29.7 \mathrm{kN} . \mathrm{m}$ ) for Beam 2 having $f_{C}^{\prime}=40 \mathrm{MPa}$.


Figure 6.5 Torque-Alpha Curves for Beams 1 and 5


Figure 6.6 Torque-Alpha Curves for Beams 2 and 6

Like the triangular mode for beams having a square cross section (Beams 1 and 5), Figure 6.5, the triangular mode gives unexpected results for the beams having a rectangular cross section (Beams 2 and 6), Figure 6.6.

It is apparent from both Figures 6.5 and 6.6 that when the concrete strength for identical beams has been decreased, the torsional resistance calculated by the rectangular modes has been expectedly and reasonably decreased, while the torsional resistance calculated by the triangular mode has been unexpectedly and unreasonably increased.

### 6.5 Effect of Softening of Concrete

To examine the sensitivity of the analyses to the softening coefficient, all six beams are considered.

For Figures 6.1 through 6.6 the softening coefficient, $\lambda$, used at each value of $a$ was calculated by Equation 5.36 using $\lambda_{1}$ and $\lambda_{2}$ for the rectangular modes 1 and 2 respectively.

Figure 6.7 through 6.12 represent torque-alpha curves for Beams 1 through 6 . with three different values of $1 / \lambda$ for each value of $a$. The curves using $\lambda_{1}$ and $\lambda_{2}$ in calculating $\lambda$ are presented by the symbol + and values of $1 / \lambda$ are shown on the figures. Results are also shown on


Figure 6.7 Torque-A1pha Curyes for Beam 1


Figure 6.8 Torque-Alpha Curves for Beam 2


Figure 6.2 Torque-Alpha Curves for Beam 3


Figure 6.10 Torque-Alpha Curves for Beam 4


Figure 6.11 Torque-Alpha Curves for Beam 5


Figure 6.12 Torque-Alpha Curves for Beam 6

Figures 6.7 through 6.12 for values of $1 / \lambda 0.05$ higher and 0.05 lower than the values used to calculate the points represented by the symbol +. The symbol * is used for the higher values of $1 / \lambda$ while the symbol o is used for the lower values.

It is clear from Figures 6.7 through 6.12 that, for beams having both square (1, 3 and 5) and rectangular (2, 4 and 6) cross sections, the rectangular modes give expected results while the triangular modes give unexpected ones. When the value of $1 / \lambda$ is increased, the torsional resistance calculated by the rectangular modes is increased, while the torsional resistance calculated by the triangular mode is decreased. Exactly the opposite occurs when the value of $1 / \lambda$ is decreased.

### 6.6 Discussion and Evaluation of Results

To show the difference between the rectangular and the $45^{\circ}$ triangular modes, the torques are calculated for Beams 1 and 5 (square cross section) for just one value of the angle of crack, $\theta=35^{\circ}$. The trapezoidal mode with. the angle a equal to $90^{\circ}$, which corresponds to the rectangular mode 1 , and the triangular mode with a equal to $45^{\circ}$ are used to calculate the torque as shown in Tables 6.2 and 6.3 respectively.

Table 6.2 Torque for Rectangular Mode, $a=90^{\circ}$


Table 6.3 Torque for Triangular Mode, $a=45^{\circ}$


As has been done in the solution techniques (Sections 5.4 and 5.8), the coefficient $k$ used to determine the dimension of the compression zone has been assumed and the compressive and tensile forces, $C$ and $F$ respectively, have been calculated. If the force $F$ does not equal the force $C$ then a new value of $k$ has been assumed and the procedure is continued until the forces $F$ and $C$ are equal. The corresponding resisting torque $T$ has then been calculated.

It is apparent from Table 6.2 that for both beams having $a=90^{\circ}$ the tensile force $F$ (calculated from Equation 5.24) decreases slightly with the increase in the value of the coefficient $k$ as there are minor changes in the values of the direction cosines $\ell$ and $n$ and in the tensile force component $F_{x}$. The compressive force $C$ (calculated from Equation 5.23) increases significantly with the increase in the value of the coefficient $k$ as they are proportionate.

For both beams having $a=45^{\circ}$. (Table 6.3), both the force $F$ (calculated from Equation 5.9) and the force $C$ (calculated from Equation 5.8) increase significantly with the increase in the value of the coefficient $k$. The values of the direction cosine $n$ and the force components $F_{x}$ and $F_{y}$ increase with the increase in $k$ and consequently the force $F$ increases. Like beams having $a=90^{\circ}$, the force $C$
for beams having $a=45^{\circ}$ increases with the increase in $k$ as they are proportionate.

In Tables 6.2 and 6.3 each of the torque values were calculated for just one value of the angle of crack, $\theta$. A range of $\theta$ should be assumed in order to get the minimum value of the torque at a specific angle $a$ as has been done in the solution techniques. From Table 6.2 for the rectangular mode, the value of the torque for Beam 1 with $f_{C}^{\prime}=40 \mathrm{MPa}$ is $37.63 \mathrm{kN} . \mathrm{m}$ which is higher than the torque ( $35.75 \mathrm{kN} . \mathrm{m}$ ) for Beam 5 with $\mathrm{f}_{\mathrm{C}}^{\prime}=25 \mathrm{MPa}$. This trend has been previously noticed when the minimum value of the torque was calculated as shown in Section 6.4, Figure 6.5. From Table 6.3 for the triangular mode, the value of the torque for Beam lis $34.99 \mathrm{kN} . \mathrm{m}$ which is less than the torque ( $38.88 \mathrm{kN} . \mathrm{m}$ ) for Beam 5. This trend also has been noted when the minimum value of the torque was calculated as shown in section 6.4, Figure 6.5.

The direction cosines $\ell, m$ and $n$ are the direction cosines perpendicular to the compression zone. It is clear from Tables 6.2 and 6.3 that the values of the direction cosines varied with the variation in the value of the coefficient $k$ used to determine the dimension of the compression zone. The variation in the direction cosines for the rectangular mode $\left(a=90^{\circ}\right.$, Table 6.2) is less than
the variation for the triangular mode ( $a=45^{\circ}$, Table 6.3). This variation in the direction cosines is the primary reason for the change in the tensile force $F$, with the increase in the value of the coefficient $k$, from slightly decreasing for the rectangular mode to significantly increasing for the triangular mode.

As mentioned before, the forces $F$ and $C$ must be equal, i.e. the forces acting normal to the compression plane must be in equilibrium if the corresponding resisting torque is to be calculated. In Figure 6.8 for Beam 2 the torque-alpha curve with the higher values of $1 / \lambda$ is not completed as equilibrium of the forces $F$ and $C$ could not be attained, i.e. the $F-k$ and $C-k$ curves did not intersect.

As the direction cosines are a function of the shape of the failure surface, Equations 5.5 and 5.19, the proposed failure surfaces for the triangular and trapezoidal modes need to be modified so that they will give results as reasonable as the results obtained from the failure surfaces for rectangular modes. Although the proposed failure surfaces with triangular and trapezoidal compression zones which are shown in Figures 5.1 through 5.6 are a logical progression from the failure surface of rectangular mode 1 to the failure surface of rectangular mode 2 , they do not produce results that are reasonable, and hence will require further modification.

For this reason, comparisons with experimental results would at this time be, meaningless and have not been done.

### 6.7 Hsu's Failure Surface

Hsu (1968a) predicted the failure surface shown in Figure 2.4 for square cross sections and it is reproduced in Figure 6.13.

The following equation can be obtained by considering the equilibrium of forces acting normal to the compression plane

$$
\begin{equation*}
k=\sqrt{\frac{8\left(a_{\ell} f_{\ell y^{+}}+a_{s} f_{s y} b_{l} / s\right)}{3 k_{l} f_{c}^{\prime} b^{2} / \lambda}} \tag{6.1}
\end{equation*}
$$

The torsional moment is obtained from the equilibrium of external and internal moments as:

$$
\begin{equation*}
T=2\left(1-k_{2} k\right) b\left(a_{\ell} f_{\ell Y}+a_{S} f_{S Y} b_{1} / s\right) \tag{6.2}
\end{equation*}
$$

For Beam 1 with $f_{c}^{\prime}=40 \mathrm{MPa}$, Hsu's triangular mode gives $T=52.3 \mathrm{kN} . \mathrm{m}$ which is higher than the torque, $T=46.6 \mathrm{kN} . \mathrm{m}$, given by Hsu's triangular mode for Beam 5 with $f_{c}^{\prime}=25 \mathrm{MPa}$. Thus, when the concrete strength for otherwise identical beams is decreased, the torsional resistance calculated is also decreased, which is reasonable. However, the results calculated do not agree with Hsu's statement that "a failure surface such as this


Figure 6.13 Hsu's Triangular Mode
seems to give a minimum torsional resistance for square cross sections". The rectangular mode gives $T=37.6 \mathrm{kN} . \mathrm{m}$ for Beam 1 and $T=35.6 \mathrm{kN}$. m for Beam 5 (Section 6.4) which are less than the $52.3 \mathrm{kN} . \mathrm{m}$ and $46.6 \mathrm{kN} . \mathrm{m}$ for Beams 1 and 5 respectively calculated using Hsu's triangular mode.

CHAPTER 7

## SUMMARY, CONCLUSIONS AND RECOMMENDATIONS

### 7.1 Summary

The behavior of reinforced concrete members subjected to pure torsion has been studied in this investigation. Theoretical analyses were developed to predict the torsional strengths, rotations, strains and stresses at all levels of load for reinforced concrete members reported in the literature. Failure mechanisms were proposed and consequently theoretical analyses were developed to predict the torsional strengths for reinforced concrete members under pure torsion.

### 7.2 Conclusions

Based on the findings of this investigation, the following conclusions are drawn.

1. The analyses developed for skew bending rectangular modes satisfactorily predict the torsional behavior for symmetrically reinforced structural concrete members subjected to pure torsion.
2. The comparison of experimental and theoretical maximum torque is excellent as the average ratio
of experimental strength to theoretical strength for 66 beams reported in the literature is 1.01 with a standard deviation of $7.8 \%$.
3. For skew bending rectangular modes, the theoretical torque-twist curves calculated using the equation for twist derived from compatibility of a thin walled tube element are closer to the corresponding experimental curves than the theoretical curves calculated using the equation for twist based on the skew bending model.
4. The strains in the longitudinal steel and the strains in the stirrups at all levels of load are satisfactorily predicted by the analysis developed for skew bending rectangular modes.
5. The softened stress-strain relationship used in the analysis developed for skew bending rectangular modes provides good predictions for the experimental torque-twist curves.
6. When the value of $a$ is taken as $90^{\circ}$, the analysis developed for the proposed skew bending trapezoidal mode l-2 gives a torque value equal to the torque given by the rectangular mode 1 . Also, when the value of $a$ is taken as $0^{\circ}$, the analysis developed for the proposed skew bending trapezoidal mode 2-1 g̣ives a torque value equal
to the torque given by the rectangular mode 2 .
7. When $a$ is taken equal to the angle corresponding to transition from trapezoidal mode l-2 to the triangular mode, the results given by the trapezoidal mode 1-2 and the triangular mode are identical ; similarly, when $a$ is taken equal to the angle corresponding to transition from trapezoidal mode $2-1$ to the triangular mode, the results given by the trapezoidal mode $2-1$ and the triangular mode are identical.
8. The special triangular mode 45-S for the square cross section with angle a equal to $45^{\circ}$ gives a torque value equal to the torque given by the general triangular mode for this special case. The general triangular mode covers the range between the last value of angle a for the trapezoidal mode l-2 and the last value of angle $a$ for the trapezoidal mode 2-1.
9. The torque-alpha curve for a beam having a square cross section with the same reinforcement on each face has an axis of symmetry where the values of the torque are exactly the same for the angle $a$ and its complementary angle. For a beam having a rectangular cross section the torque- alpha curve has no axis of symmetry.
10. The triangular mode with a equal to $45^{\circ}$ gives the
minimum torsional resistance for a lightly reinforced beam (Beam l) having a square cross section. However, if the amount of reinforcement is increased, the rectangular mode gives the minimum torsional resistance for that beam (Beam 3). This implies that the shape of the failure surface is influenced by the amount of reinforcement.
11. The proposed triangular mode gives the minimum torsional resistance for a beam (Beam 2) having a rectangular cross section. A failure surface with a triangular compression zone has not been suggested previously for beams having a rectangular cross section although it has been suggested for beams having a square cross section.
12. When the concrete strength in square and rectangular cross sections is decreased, the torsional resistance calculated by the, rectangular modes is also decreased whereas the torsional resistance calculated by the triangular mode increases. This implies that the proposed failure surface for the triangular mode requires further refinement.
13. For both square and rectangular cross sections,
when the value of $1 / \lambda$ is increased (where $\lambda$ is the softening coefficient > l) the torsional resistance calculated by the rectangular modes is increased whereas the torsional resistance calculated by the triangular mode is decreased. Again, this implies that the proposed failure surface for the triangular mode requires further refinement.
14. Although the proposed failure mechanisms did not lead to satisfactory results, they are good pioneer steps on the rough road to solution of the mystery of the failure surface for beams having a square cross section and subjected to pure torsion.

### 7.3 Recommendations

Some areas recommended for further study are:

1. The analyses developed to predict the torsional strengths, rotations, strains and stresses at all levels of load for reinforced concrete members under pure torsion should be extended to prestressed concrete members as well.
2. The stress-strain curve for concrete under combined loading requires investigation in order to extend the analyses to reinforced and
prestressed concrete members under such loading.
3. The failure mechanisms with triangular and trapezoidal compression zones require more investigation to satisfy equilibrium conditions, then both equilibrium and compatibility conditions should be satisfied as has been done for rectangular modes.

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