THE UNIVERSITY OF CALGARY

Transversal Theory

by

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Abstract

Let A be a set in the plane $\mathbb{E}^2$. Given two points $\bar{a}$ and $\bar{b}$ in A, if $t\bar{a} + (1-t)\bar{b}$ is in A for every $t \in [0,1]$, then we call X a *convex set.* In particular, a set is convex if the line segment joining any two points in the set is also contained in the set. Let $\mathfrak{A} = \{A_i : i \in I\}$ be a family of convex sets. A line that meets every member of $\mathfrak{A}$ is called a *common transversal.* The focus of this work involves an examination of the conditions which must be imposed upon a family of convex sets in order to ensure that a common transversal exists. The intention is to present major results in the study of common transversals and outline several beautiful proofs.

Acknowledgements

I would like to thank Chris Foster for being an excellent study companion. I would like to thank Joanne Longworth for all her kind help in typesetting this manuscript. I would like to thank Robin Pearce for her proof reading and for her support.

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Dedication

For my mother.

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Chapter 1

Introduction

When people ask me what I do as a graduate student, I sometimes tell them that I play with quarters and dental floss. They look at me quizzically and I then proceed to explain in greater detail what it is I do. However, what it really boils down to is a roll of quarters and a box of dental floss.

This introduction is meant to be an informal overview of what is to be discussed in the text. The text discusses some beautiful and pivotol results in various areas dealing with the study of transversals. These results are drawn from research that has been conducted in the last thirty years. It is by no means offered as a comprehensive synopsis of the study of transversals. On the contrary, it is intended as a gentle starting point from which other strains of research can be sought out.

As with most forms of discrete geometry, it is very easy to understand the general problem from which various other problems and generalizations have sprouted. However, solutions to any of these problems are rarely forthcoming. Many proofs are quite involved and require a very sophisticated approach to yield a solution. Nonetheless, the problems are very tangible in the sense that we may draw a diagram of a problem to improve our intuition and from such diagrams we see how to produce or refute the desired result.

This section introduces the general problem in a tangible way. Instead of using formal definitions of convex set and transversal, we first approach the problem intuitively. Picture a flat surface, a table top say, but this tabletop extends infinitely in all directions. Next picture a roll of quarters. Sometimes this roll is finite, sometimes it is infinite. Usually, the roll is infinite unless otherwise specified. We shall start with a finite roll. Now, place the quarters on the table top so that each one is lying flat and no two quarters are overlapping. Next, take a piece of dental floss, making sure to hold it taut, see if you can get it to touch all of the quarters. If you can, we say that the quarters are lying in a good way and if not then they are lying in a bad way. When we have infinitely many quarters then the dental floss is infinitely long, but still taut.

The question that we now ask is: what conditions do we need to impose so that the roll of quarters, after being placed on the table, is lying in a good way? If you play with the quarters and the dental floss on the table then it quickly becomes apparent that the quarters cannot lie just anywhere. In fact, they need to be close in some sense. How do we make sure that the quarters are close enough?

One way to do this is to require that any five of the quarters be lying in a good way. In other words, you can touch any five quarters with a taut piece of dental floss. Certainly this will force the quarters to be close in some sense, but is it enough to ensure that the entire roll of quarters is lying in a good way? Yes! Even if the roll of quarters is infinite; as long as any five can be touched by a taut piece of dental floss, the entire family can.

An obvious question arises now: is five the best possible such number? More precisely, if any four (three or two) quarters are touched by a taut piece of dental floss, can the entire family be touched by a taut piece of dental floss? The answer, in the case of the finite roll of quarters and the infinite roll of quarters, is no.

We now formalize these concepts. A circle is a quarter. When two circles are disjoint we require that the quarters not overlap or even touch at a point. A family of disjoint circles is a roll of quarters that has been placed on the table so that the quarters are lying flat and are not overlapping or even touching. A transversal is a taut piece of dental floss that touches all of the quarters in the roll and an n-transversal is a transversal that meets n of the quarters. The plane is the table top. The family is called T when the quarters are lying in a good way and $T(n)$ when any n of the quarters are lying in a good way.

Given a family of n circles, what is the smallest k such that if the family is $T(k)$ then it is also T? This problem asks what is the smallest k such that if any k circles in the family are touched by a line then the whole family of circles is touched by a line? It was posed by Hadwiger in 1955 and has been the source of much fruitful research in mathematics. In 1958, Grünbaum conjectured, incorrectly, that $k = 4$.

As was mentioned earlier, $k = 5$ and it would seem that our work here is done. On the contrary there are many questions to examine yet. For example, what if we replace circles by squares? So, we play the game with match books instead of quarters. As long as the edges of the matchbooks are parallel then $k = 4$ suffices. What about line segments? So, we play the game with match sticks instead of quarters. Here, if the match sticks are parallel then $k = 3$ suffices.

Other lines of research that can be conducted include looking at families of different sizes of circles, squares and line segments or combinations thereof. Can the problem be generalized to higher dimensional objects such as spheres, cones and cylinders? Are there other transversal properties like T and $T(n)$ that can be studied?

These questions are examined and some solutions are given. Other questions are still open for further research. Before we can proceed to answer these questions we need formal definitions. We introduce these definitions as we proceed, but it may be invaluable for the readers to remind themselves that we are dealing with nothing more than quarters and dental floss.

Chapter 2

Helly's Theorem

We begin with a little history as provided by Danzer, Grünbaum and Klee in [7]. Eduard Helly, born June 1, 1884, in Vienna, is the founder of this particular area of study in Geometry. He received his Ph.D. in 1907 from the University of Vienna under W. Wirtinger. After publishing a few important articles in functional analysis, he made the crucial geometric discovery, which we discuss shortly, in 1913. Helly served in the army in 1914 and was wounded by the Russians. He was interned in Siberia with T. Rado and did not return to Austria until 1920. After having held several distinguished positions at the University of Vienna and working as a consultant for various economic institutions, in 1938 Helly, along with his wife and son, moved to the United States of America. Helly held positions at several post-secondary educational institutions until his death in 1943.

Let C be a set in $\mathbb{E}^n$. Given two points $\bar{a}$ and $\bar{b}$ in C, if $t\bar{a} + (1-t)\bar{b}$ is in C for every $t \in [0,1]$, then we call C a *convex set.* In particular, a set is convex if the line segment joining any two points in the set is also contained in the set. This definition is crucial and will be assumed throughout the text.

Theorem 1 Helly's Theorem. *Let $\mathfrak{A}$ be a family of at least $n+1$ compact, convex sets in $\mathbb{E}^n$. If each $n+1$ members of $\mathfrak{A}$ have a point in common then there is a point in common for all members of $\mathfrak{A}$.*

This is Helly's important discovery. Made in 1913, it has launched many important lines of research in geometry. We sketch out a proof of Helly's Theorem for intervals on a line. Given a family of closed intervals $[a_1, b_1], [a_2, b_2], \ldots$ such that any two of them have a point in common, let $a = sup\{a_i : i \geq 1\}$ and $b = inf\{b_i : i \geq 1\}$. First we verify that $a \leq b$. Suppose that $a > b$, then there exists an i and a j such that $a \geq a_i > b_j \geq b$. Consequently, $[a_i, b_i] \cap [a_j, b_j] = \emptyset$ contrary to our assumption.

If $a = b$ then $[a, b]$ is just a point; otherwise, it is an interval. In either case $[a, b] \subseteq [a_1, b_1] \cap [a_2, b_2] \cap \ldots$. Otherwise there exists either an i or a j such that $a < a_i$ or $b_j < b$. More general proofs can be found in other works, we merely seek to introduce Helly's Theorem.

Let $1 \leq m \leq n$, an *m-flat* in $\mathbb{E}^n$ is an m-dimensional affine space embedded in $\mathbb{E}^n$ and called a hyper-surface or a hyper-plane when $m = n - 1$. Let $\mathfrak{A}$ be a family of compact, convex sets in $\mathbb{E}^n$. A *transversal m-flat* of $\mathfrak{A}$ is an m-flat that intersects each member of $\mathfrak{A}$. An *l-transversal m-flat* of $\mathfrak{A}$ is an m-flat that intersects l members of $\mathfrak{A}$. Using this terminology, Helly's Theorem states that if a family of compact, convex sets in $\mathbb{E}^n$ has an $(n+1)$-transversal 0-flat for each $n+1$ members then it has a transversal 0-flat.

Generalizing the problem leads one to ask: under what conditions does a family of compact, convex sets in $\mathbb{E}^n$ have a transversal m-flat? In particular, if the family has an $(n+1)$-transversal m-flat for each $n+1$ members then does it have a transversal m-flat? For a moment, consider the case where $m = 1$. What we are essentially seeking here are conditions that ensure the existence of a line that meets or intersects each member of the family. Is it sufficient that any $n + 1$ of them be met by a line? Questions like this are what we seek to answer and understand in some detail. Specifically, we look at the two dimensional version of this problem where $m = 1$.

We restrict our attention to compact, convex sets in the plane $\mathbb{E}^2$ and straight lines meeting these sets.

In the two dimensional case, a line that meets each member of $\mathfrak{A}$ is called a *transversal* of $\mathfrak{A}$. If there exists a transversal of $\mathfrak{A}$ then $\mathfrak{A}$ satisfies Property T. If there is a transversal of every sub-family consisting of n members of $\mathfrak{A}$, where $n \in \mathbb{Z}^+$, then $\mathfrak{A}$ satisfies Property $T(n)$. In the study of transversals, one tries to determine the necessary conditions that must be imposed on a family $\mathfrak{A}$ to ensure that $\mathfrak{A}$ satisfies Property T. Ideally, one tries to impose as few conditions as possible. Typically one asks: if the family $\mathfrak{A}$ satisfies Property $T(n)$ then does it satisfy property T? If it does then we write $T(n) \Rightarrow T$, otherwise, $T(n) \not\Rightarrow T$. Next, if such an n exists, we may ask if it is the best possible such n? By best possible we mean smallest or minimal with respect to the stated property.

It is precisely these types of problems that we examine in some detail. Afterwards, we examine some related problems that arise in the planar case. Once we have examined these problems we look at higher dimensional generalizations. We examine conditions under which hyper-planes meet families of compact convex sets in $\mathbb{E}^n, n \geq 3$. More precisely, we ask does there exists a minimal l such that if a family has an l-transversal $(n-1)$-flat for each l members then it has a transversal $(n-1)$-flat? We specifically focus on problems in three dimensions and utilize an intuitive development of these problems. Finally we return to problems in $\mathbb{E}^2$ and examine some related transversal properties. All of the problems studied are so called Helly Type Transversal problems. We do not consider other types of transversal problems in this manuscript.

Chapter 3

"Two Counterexamples Concerning Transversals for Convex Subsets of the Plane"

3.1 Introduction

This chapter provides counterexamples for the following conjectures:

Conjecture 1 *For families of disjoint, congruent squares,* $T(5) \Rightarrow T$.

Conjecture 2 *For families of disjoint, congruent, compact, convex sets,* $T(6) \Rightarrow T$.

Conjecture 3 *For families of* n *disjoint line segments,* $T(n-1) \Rightarrow T$.

The first two conjectures had been open for quite some time. The constructions given by Lewis in [22], outlined in this chapter, are standard and widely cited. A counterexample to Conjecture 1 is given, followed by a counterexample to Conjecture 3, which is used to generate a counterexample to Conjecture 2. These particular constructions are widely cited and no work on transversals would be complete without studying these counterexamples.

3.2 The Counterexamples

A close examination of Figure 3.1 reveals that this configuration of six congruent squares is certainly $T(5)$ but not T. Any five of the six squares have a transversal given by one of the lines $l_1, l_2, \ldots, l_6$. We can check that there is no line that intersects all six squares. To do so we begin by translating $l_1, l_2, \ldots, l_6$ so that these six lines pass through the origin. Next, consider the translates of l_1 and l_2, neither of these lines is a transversal for the six squares nor is any line parallel to either of them. Furthermore, any line that lies within the acute angle formed by the translates of l_1 and l_2 is not a transversal nor is any line parallel to it; this fact is apparent from Figure 3.1. In a similar fashion, we continue to rule out lines until we have the desired result; that is, the family is not T. Hence, we see that Conjecture 1 is false.

Observe that in the preceding construction, the squares S_1 and S_6 can be placed as far apart as desired with the same result achieved; of course, the transversals $l_1, \ldots, l_4$ need to be adjusted appropriately as do S_2 and S_5. Consequently one may add additional squares between S_1 and S_2, and between S_5 and S_6 to obtain the following much stronger result:

Theorem 2 *Given any natural number $k, k > 5$, there exists a disjoint family consisting of k congruent squares such that the family has property $T(5)$ but does not have property $T(6)$.*

To construct the counterexample for Conjecture 3, we begin with $n-1$ directed lines, concurrent at O, oriented and labeled as in Figure 3.2. Take a point on R_1 and connect it to a point on Q_1; this is line segment S_1. To obtain line segment $S_i, 2 \leq i \leq n-1$, take a point on R_i between O and $S_{i-1} \cap R_i$ and connect it to a point on Q_i so that S_i does not intersect any of $S_1, S_2, \ldots, S_{i-1}$. Line segment S_n is obtained by joining O to a point on Q_n. In this manner, we obtain a family of n line segments such that any $n-1$ subfamily has a transversal. One of the transversals is given by a line that passes through a point in the region bounded by R_{n-3} and R_{n-2} and through a second point in the region bounded by Q_{n-1} and Q_n. The remaining transversals are given by the lines $aff(R_i)$.

The family will not be T. This can be checked easily by considering any two adjacent directed lines, neither of these lines, nor any line parallel to them, will be a transversal for the entire family. Furthermore, no line lying in the acute angle formed by these two directed lines is a transversal, nor is any line parallel to it. The case $n = 7$ is depicted in Figure 3.3.

Recall the following result (B^2 is the closed unit disc):

Theorem 3 *If* $\mathfrak{A} = \{A_1, A_2, \ldots, A_n\}$ *is a family of compact convex sets in the plane, which does not posses property* T*, then there exists a real number* $\delta, \delta > 0$*, such that the family* $\tilde{\mathfrak{A}} = \{A_1 + \delta B^2, A_2 + \delta B^2, \ldots, A_n + \delta B^2\}$ *also lacks property* T*.*

This is a restatement of the fact that, given a family of compact convex sets, we may expand the family by an arbitrarily small factor without the risk of introducing transversals or removing existing transversals. Applying the preceding theorem to the construction of the counterexample to Conjecture 3 yields the following result:

Theorem 4 *Given any natural number* $n \geq 3$*, there exists a family of* n *congruent rectangles such that the family has property* $T(n-1)$ *but not* T*.*

Theorem 4 is obtained by considering the family of line segments discussed in the counterexample to Conjecture 3 and then expanding the line segments, essentially fattening them. Each of the newly fattened line segments contains a rectangle. It is the existence of this family of rectangles that verifies Theorem 4. This theorem immediately disproves Conjecture 2 by taking $n = 7$.

3.3 Conclusion

Shortly, we discuss the results of Grünbaum on parallelograms with parallel edges. If one considers disjoint translates of a parallelogram then one obtains $T(5) \Rightarrow T$. Furthermore if one deals with a family of parallelograms with parallel edges then one obtains $T(6) \Rightarrow T$. However, as we see here, if one allows for the possibility of rotations then these statements no longer hold. Take away the restriction to translates and one has the counterexample to Conjecture 1. Allow for the parallelograms to have non-parallel edges and one has the counterexample to Conjecture 2. Thus we see how critical these restrictions are. The counterexample to Conjecture 3 is truly fascinating. One would expect that for any family with n members, property $T(n-1)$ is so restrictive, that the family necessarily has property T. As we have seen, this is not the case.

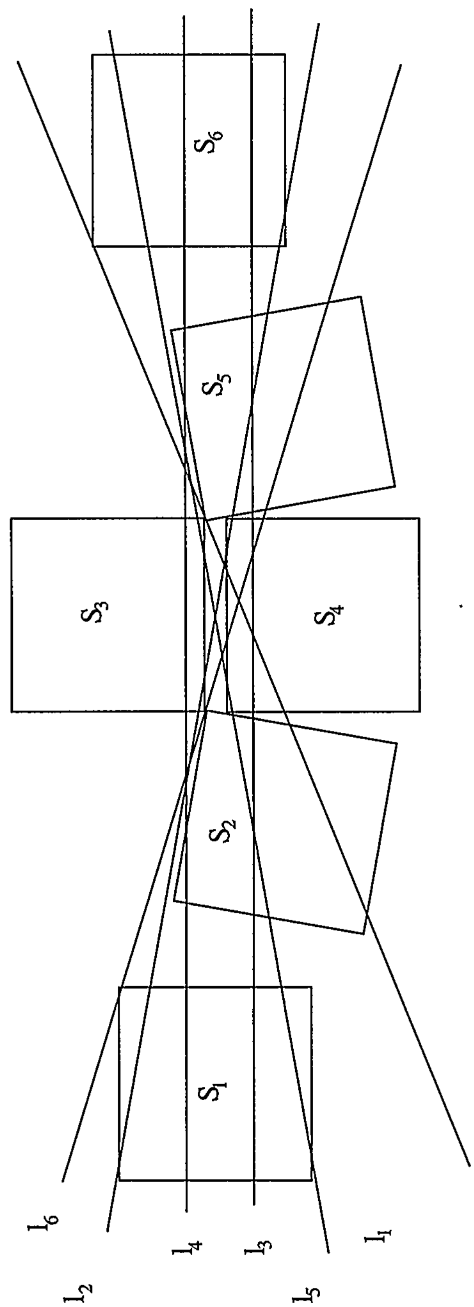


Figure 3.1: A family of six squares that is $T(5)$, but not T. The transversal l_i is the line which meets all of the sets except S_i.

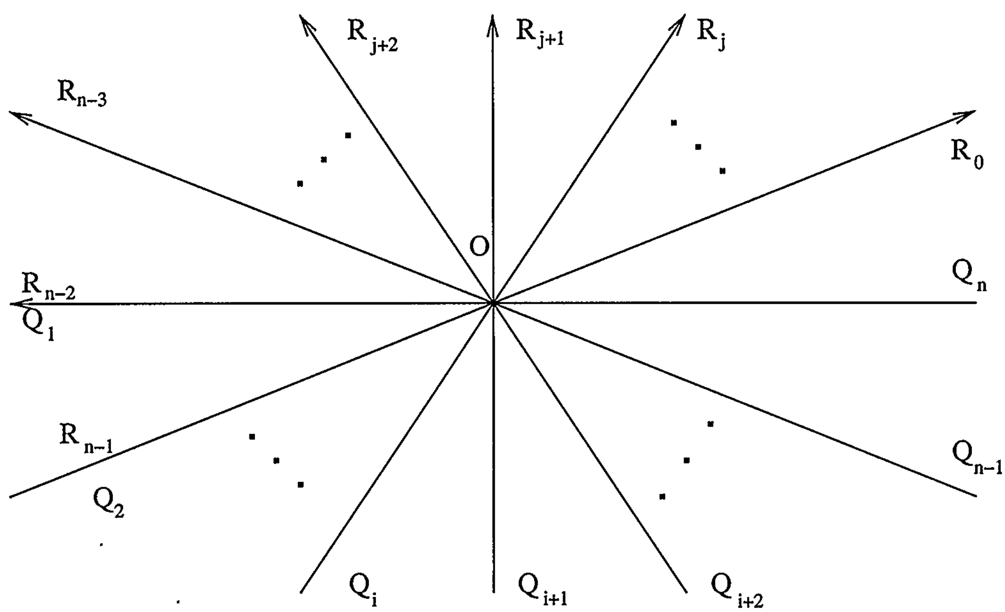


Figure 3.2: Preliminary orientation of the $n-1$ lines for the construction of a counterexample to Conjecture 3.

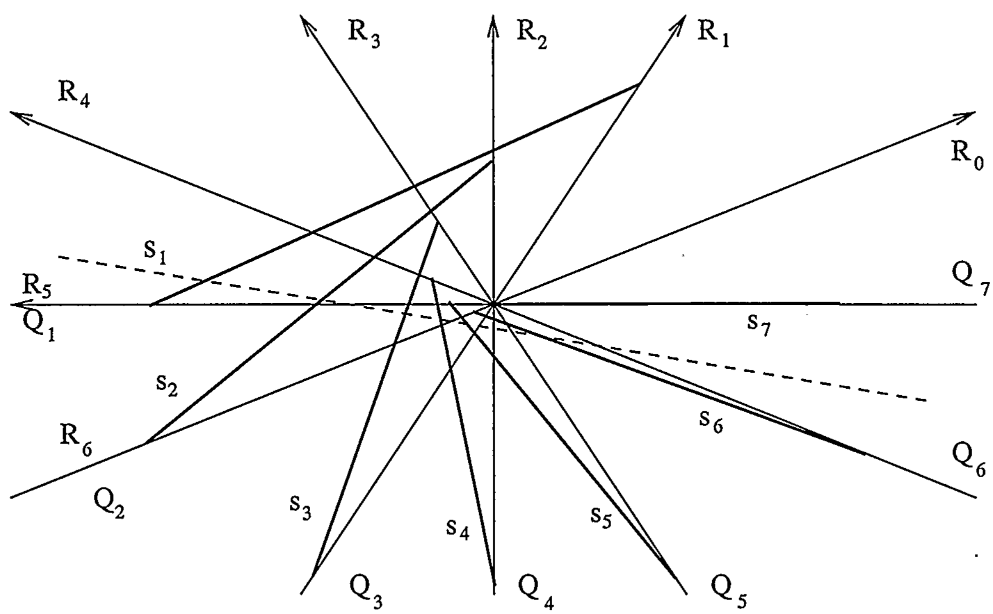


Figure 3.3: Demonstration of Theorem 4 for the case $n = 7$. Observe that $conv(R_i)$ is a transversal that meets all of the line segments except S_{i+1} for $i = 0, 1, 2, 3, 4, 5$. The dashed line indicates a transversal that meets all of the line segments except for S_7. The text describes how to determine that this family has no common transversal.

Chapter 4

"On Common Transversals"

4.1 Introduction

There are two main results discussed in [11]. The first is a very important and widely cited result showing that $T(5)$ implies T for disjoint translates of a parallelogram. The second result is incorrect and we provide a counterexample.

4.2 Theorems

Unless otherwise stated, $\mathfrak{A}$ denotes a family of translates of a parallelogram. Two parallelograms with parallel edges, P_1 and P_2, are *opposed* when P_1 and P_2 are separated by two lines parallel to intersecting edges (cf. Figure 4.1). $\mathfrak{A}$ satisfies property $T'(n)$ if there exist two members of $\mathfrak{A}$, P_1 and P_2, which are opposed such that given any n members of $\mathfrak{A}$, say $A_1, A_2, \ldots, A_n$, there is a line which intersects $P_1, P_2, A_1, A_2, \ldots, A_n$.

Lemma 1 *For any family of parallelograms with parallel edges, $T'(3)$ implies T.*

This is a corollary to Helly's Theorem due to Hadwiger and Debrunner. An English translation of the work Grünbaum cites this result from is unavailable at the time of writing this thesis;therefore we use the result without proof.

Theorem 5 *Let $\mathfrak{A}$ be a family of disjoint translates of a parallelogram. If $\mathfrak{A}$ has property $T(5)$, then it has property T.*

Proof. Without loss of generality, we assume that the members of $\mathfrak{A}$ are squares. The proof for a family of disjoint translates of an arbitrary parallelogram is analogous. The directions determined by the edges of the squares may be assumed to be horizontal and vertical. Further, we assume that there are at least six squares.

Suppose there exist two squares, $P_1, P_2 \in \mathfrak{A}$, such that the squares are opposed (cf. Figure 4.2). Since $\mathfrak{A}$ has property $T(5)$, given any three members of $\mathfrak{A}$ and P_1 and P_2, there is a transversal for these five squares. In particular we observe that the family is $T'(3)$ and by applying Lemma 1, we have that the family is T.

Next, assume that no two squares in the family are opposed. Thus, for any two squares, there exists a horizontal, or vertical, line intersecting the squares, and since they are disjoint there exists a vertical, respectively horizontal, line which separates the pair. Since the family has at least six squares, there are three squares, possibly more, which are separated in pairs by horizontal or vertical lines. We assume the latter is true (the former case is analogous).

Denote by $\mathfrak{A}^*$ a subset of $\mathfrak{A}$ which is maximal with respect to the property that any two members of $\mathfrak{A}^*$ can be separated by a vertical line. If $\mathfrak{A}^* = \mathfrak{A}$ then Theorem 5 follows from well known results on common transversals of sets separated by parallel lines. These results are discussed in a later chapter. Thus we need only consider the situation where $\mathfrak{A}^* \neq \mathfrak{A}$. As a result of the preceding assumptions about $\mathfrak{A}$, it is clear that the problem reduces to the three cases diagramed in Figure 4.3. All other configurations are obtained from these three cases by symmetry. The line in Figure 4.3 is called l and the remaining squares of the family are met by the dashed portion of the line.

By *principal squares*, we mean any of the two, three, or four squares depicted in Figure 4.3. In particular, given any family satisfying the assumptions thus far, there are two or three or four members of the family that conform to one of the the three configurations depicted in Figure 4.3. These squares are the principal squares for the given family.

It is easy to see that the principal squares can be intersected by some line. Such an intersecting line exists as a result of the family being $T(5)$. Let m_1 be the line that intersects the principal squares and forms a minimal angle with l. Let m_2 be the line that intersects the principal squares and forms the maximal angle with l (cf. Figure 4.3). Since the family satisfies $T(5)$, a square, other than one of the principle squares, can be chosen so that a line meets that square and all of the principle squares. However, this square, which we now call S, is arbitrarily chosen, so we may proceed to chose S as far away from the principle squares as we please. It is clear that the transversal for S and the principle squares approaches one of either m_1 or m_2 as S is chosen sufficiently far away from the cluster of principle squares. It follows that one of m_1 or m_2 must intersect all members in the family. If there exists a square S that is not intersected by either m_1 or m_2 then there is a line that intersects S and the principle squares that forms an angle with l less than that formed by m_1 and l or greater than that formed by m_2 and l; and any other possibility would mean that S does not meet the dashed portion of l. Consequently the family satisfies property T.

■

Theorem 6 *For families of disjoint, congruent circles containing at least six members, $T(4)$ implies T.*

This theorem is incorrect. Figure 4.4 provides a counterexample. Inspecting the Figure reveals a family of six circles which satisfies property $T(4)$, but fails to satisfy property T. The counterexample can easily be extended to include as many circles as desired.

4.3 Conclusion

Grünbaum makes two crucial observations regarding Theorem 5. First of all, we note that the restriction to translates in the Theorem is critical. Figure 4.5 demonstrates a family which is certainly $T(5)$, but not $T(6)$. What makes this counterexample possible is the fact that rotations are permitted. The second observation is that $T(5)$ cannot be replaced by $T(4)$, cf. Figure 4.6. Next, Grünbaum makes the following two conjectures:

Conjecture 4 *Let $\mathfrak{A}$ be a family of translates of a parallelogram. If $\mathfrak{A}$ has property $T(5)$, then it has property T.*

Conjecture 5 *Let $\mathfrak{A}$ be a family of disjoint translates of a convex set. If $\mathfrak{A}$ has property $T(5)$, then it has property T.*

In Conjecture 4, the restriction to disjoint parallelograms is dropped, and in Conjecture 5, Grünbaum makes a very sweeping generalization by considering translates of an arbitrary convex set.

Turning to Theorem 6, the counterexample given here can be found in [1] and was produced independently by the author and Chris Foster. The erroneous result is cited as recently as 1993 and is persistent throughout the literature in this area of study.

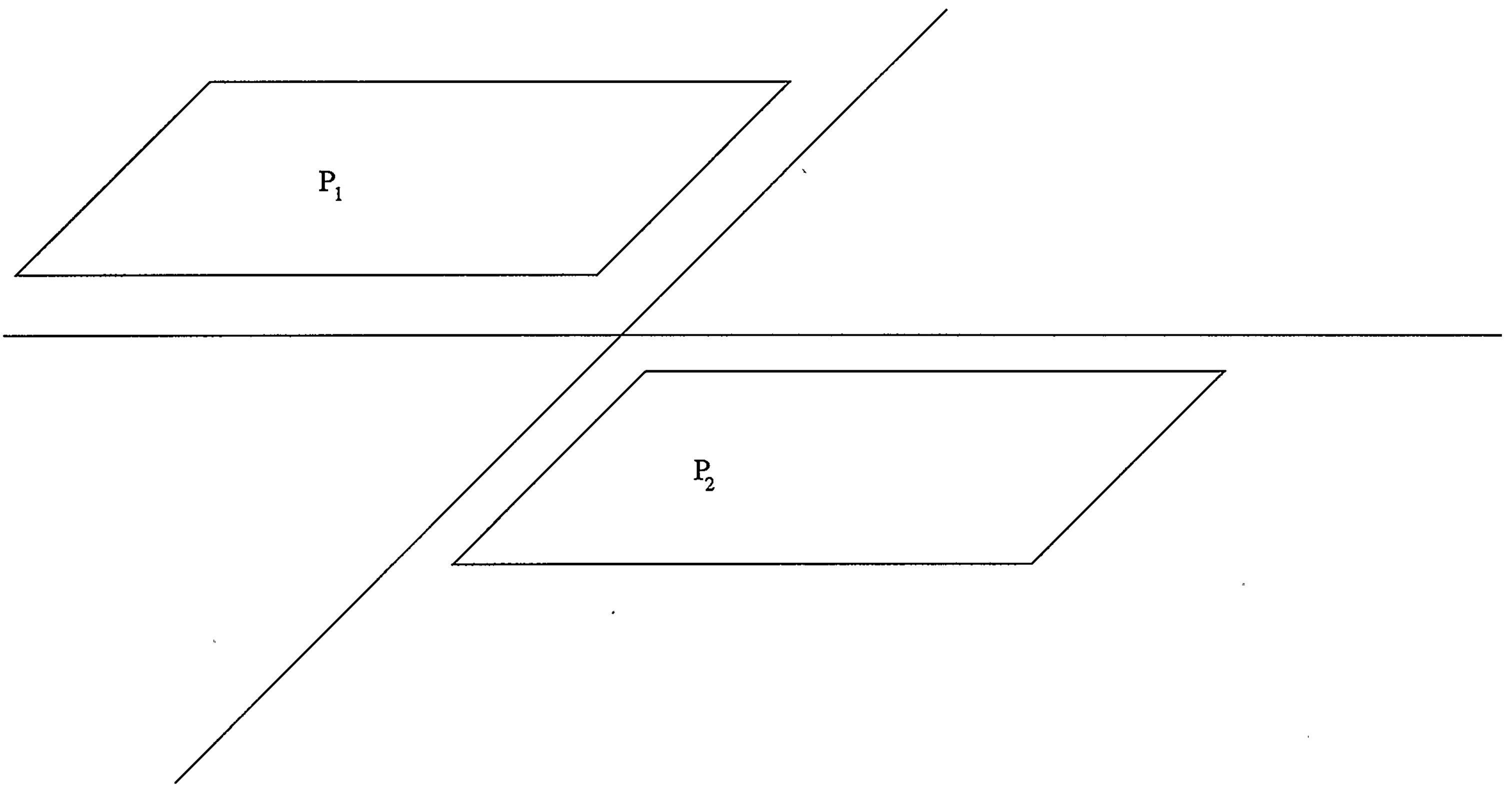


Figure 4.1: P_1 and P_2 are separated by lines parallel to intersecting edges.

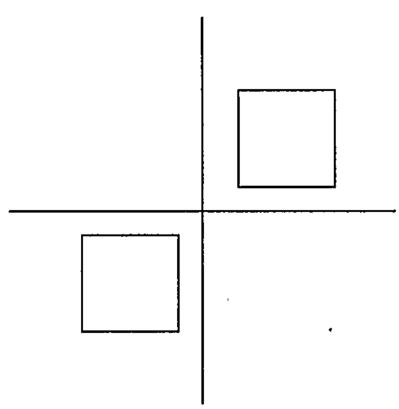


Figure 4.2: Separated Squares

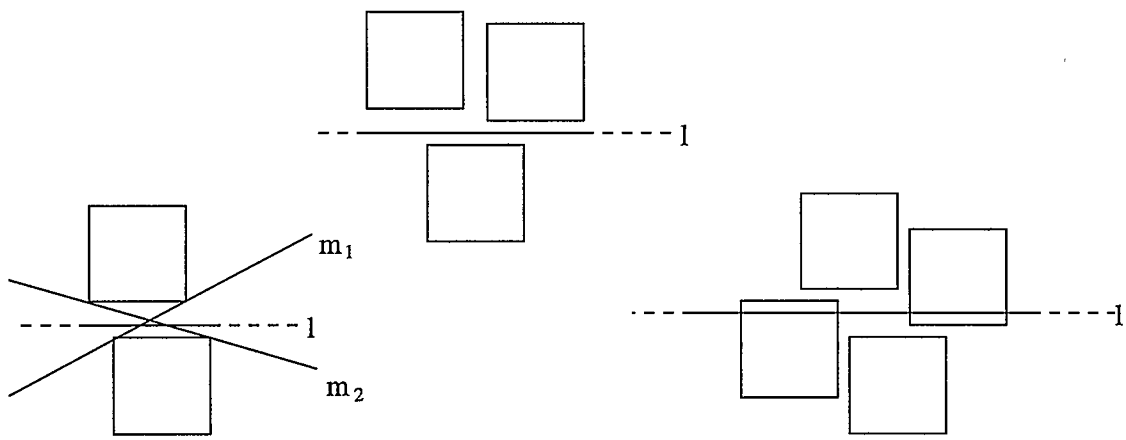


Figure 4.3: Possible arrangements of principal squares. The remaining squares touch the dashed portion of the line.

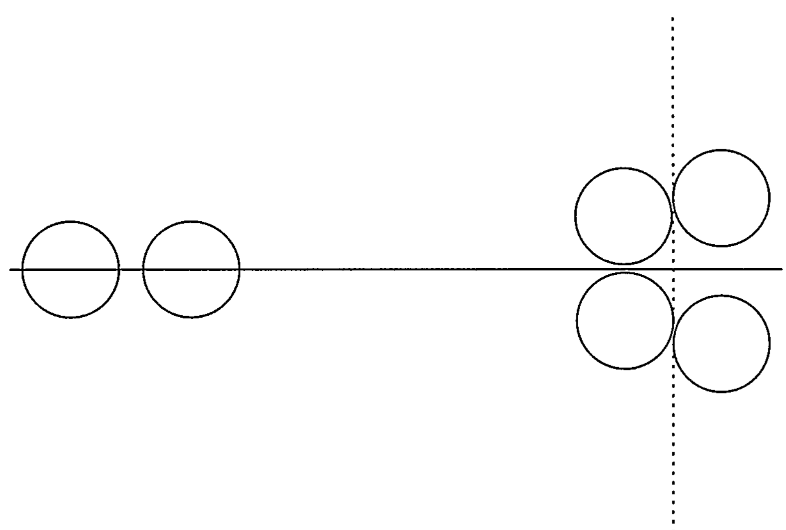


Figure 4.4: Counterexample to Grünbaum's Theorem 6. The arrangement is exaggerated for clarity. A more detailed examination of this counterexample is made later.

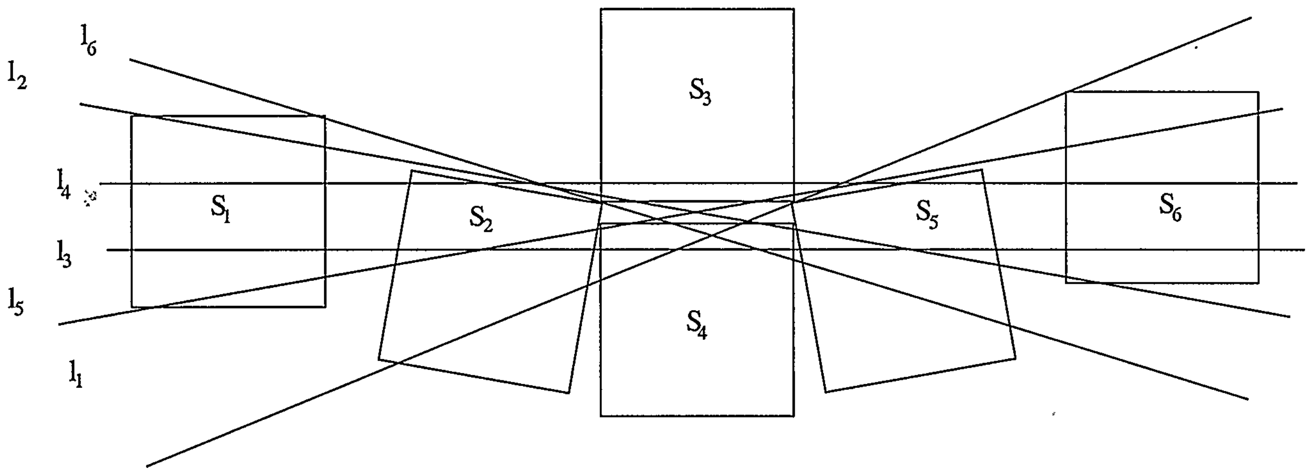


Figure 4.5: In this example, we see what happens when we remove the restriction to translates in Theorem 5. As soon as rotations are allowed, this counterexample arises.

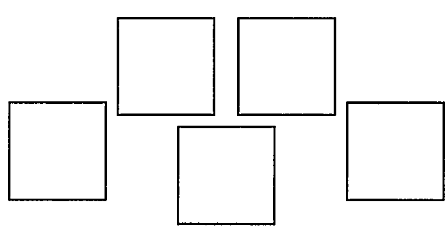


Figure 4.6: $T(4) \not\Rightarrow T$

Chapter 5

"Geometric Permutations for Convex Sets"

5.1 Introduction

This chapter introduces the basic theory of geometric permutations and how these notions relate to the theory of transversals. Fundamental definitions are given and two important theorems are presented. This discussion presents work found in [20].

5.2 Definitions

Let l be a directed line in $\mathbb{E}^2$ and O a point on l. Because l is directed, there is a natural way of viewing points on l as either preceding O or following O. By the *half lines* $l^-(O)$ and $l^+(O)$, we mean the part of l preceding O and following O, respectively. When the point O has been clearly specified, and there is no risk of confusion, we simplify the notation and write l^+, instead of $l^+(O)$, and l^-, instead of $l^-(O)$.

Consider the directed lines l_1 and l_2 intersecting at the point O. The half lines l_1^- and l_2^- precede O and the half lines l_1^+ and l_2^+ follow O. The lines l_1 and l_2 separate the plane into four quadrants: Q_1, Q_2, Q_3 and Q_4. The quadrant Q_1 is bounded by l_1^+ and l_2^+. Next, Q_2 is the quadrant located counterclockwise of Q_1, Q_3 is the quadrant located counterclockwise of Q_2 and the remaining quadrant is Q_4 (cf. Figure 5.1). By an odd quadrant, we mean Q_1 or Q_3 and by an even quadrant, we mean Q_2 or

Q_4.

We say that a set *crosses* a quadrant Q_i if it intersects both of the half lines that bound Q_i, and it *strictly crosses* Q_i if it crosses the quadrant but does not contain O. Let $T \subseteq \mathbb{E}^2$ and T' be a translate of T that contains O. Then T is said to be *odd* with respect to l_1 and l_2 if $T' \subseteq Q_1 \cup Q_3$, and *even* with respect to l_1 and l_2 if $T' \subseteq Q_2 \cup Q_4$. Observe that if a line segment strictly crosses an even quadrant, then it is odd, and if it strictly crosses an odd quadrant, then it is even (cf. Figure 5.2).

Unless otherwise stated, $\mathfrak{A} = \{A_1, \ldots, A_n\}$ denotes a family of n pairwise disjoint, compact, convex sets in the plane, $\mathbb{E}^2$. A straight line meeting each of the sets in $\mathfrak{A}$ is a *common transversal.* All families $\mathfrak{A}$ considered in this chapter have a common transversal.

Let $\mathfrak{A} = \{A_1, \ldots, A_n\}$ be a family of disjoint line segments. Observe that the affine hull of each $A_i \in \mathfrak{A}$ is the line containing A_i. Let $A_i \neq A_j$ in $\mathfrak{A}$. We say that A_i *penetrates* A_j if $aff(A_i) \cap A_j \neq \emptyset$. Observe that if there exist distinct numbers i and j such that $aff(A_i) = aff(A_j)$ then, because the family $\mathfrak{A}$ has a common transversal, $aff(A_x) = aff(A_y)$ for any two members, A_x and A_y, of the family. In other words, all of the line segments lie along a single line. Because this trivial case is of little interest, we rule it out and assume that $aff(A_i) \neq aff(A_j)$ whenever $i \neq j$. Consequently, we have the following result:

$$\text{if } A_i \text{ penetrates } A_j \text{ then } A_j \text{ does not penetrate } A_i, i \neq j. \quad (*)$$

Next, A_i and A_j are *mutually non-penetrating* if neither penetrates the other. Finally, if A_i penetrates A_j and A_j penetrates each $A_k \in \mathfrak{A} \setminus \{A_i, A_j\}$, we say that A_i and

A_j is a *strong pair* and write (A_i, A_j). The family $\{A, B, X_1, X_2, \ldots, X_n\}$ in Figure 5.3 demonstrates (A, B) and mutually non-penetrating sets $X_1, X_2, \ldots, X_n$.

It is clear that a common transversal t of a family $\mathfrak{A}$ meets the sets in a definite order, up to reversal, and therefore determines a permutation, p, and its reverse $-p$. The pair $\tilde{t} = \{p, -p\}$ is called a *geometric permutation* of $\mathfrak{A}$. If our transversal, t, is a directed line then there is no ambiguity regarding the order of the sets in $\mathfrak{A}$. The permutation denoted by p corresponds to the natural ordering of the sets in $\mathfrak{A}$, which results from t meeting the family; with the obvious reverse permutation denoted by $-p$. Strictly speaking the pair consisting of the permutation and its reverse characterizes a G.P. However, when we are dealing with directed common transversals, we simplify the notation by freely interchanging $\tilde{t}$ and p whenever the context is clear.

It is easy to devise families where infinitely many common transversals of the family generate the same geometric permutation. In fact, we can partition the transversals of a family into equivalence classes where each equivalence class consists of transversals which generate the same geometric permutation. Two transversals, t_1 and t_2, of a family $\mathfrak{A}$ are said to be equivalent if and only if they both generate the same geometric permutation. For our purposes, the specific transversal which generates a given geometric permutation is of little consequence. Only the geometric permutation in question is of interest. Thus, we simplify our notation once more and use $\tilde{t}$ to refer to any transversal that generates the geometric permutation $\tilde{t}$ whenever there is no risk of confusion.

If $\tilde{t} = (A_{i_1}, A_{i_2}, \ldots, A_{i_{j-1}}, A_{i_j}, A_{i_{j+1}}, \ldots, A_{i_n})$ is a geometric permutation of $\mathfrak{A}$ then $\tilde{t} \setminus A_{i_j} = (A_{i_1}, A_{i_2}, \ldots, A_{i_{j-1}}, A_{i_{j+1}}, \ldots, A_{i_n})$. Given the geometric permutation

$(A_{i_1}, A_{i_2}, \ldots, A_{i_n})$ of $\mathfrak{A}$, we simplify the notation and denote it by $(i_1, i_2, \ldots, i_n)$. Denote by $\mathcal{P}_{\mathfrak{A}}$ the set of all geometric permutations of $\mathfrak{A}$. We assume throughout that $\mathfrak{A}$ admits a common transversal and hence $\mathcal{P}_{\mathfrak{A}} \neq \emptyset$.

5.3 Theorems

Now that the critical definitions have been introduced, we discuss the two major theorems of the paper. Let $f(n)$ be the maximal integer such that there exists a family, $\mathfrak{A}$, of pairwise disjoint, compact, convex sets in the plane with $|\mathfrak{A}| = n$ and $|\mathcal{P}_{\mathfrak{A}}| = f(n)$. The first theorem provides upper and lower bounds for $f(n)$. It should be noted that the upper bound is very coarse and can be refined by imposing additional restrictions on the family $\mathfrak{A}$. This is precisely what the second theorem achieves by considering a family of disjoint line segments. This result is stated as a corollary.

Theorem 7 $2n - 2 \leq f(n) \leq \begin{pmatrix} n \\ 2 \end{pmatrix}, \forall n \geq 4.$

The geometric construction for the lower bound, given any positive integer n, is straightforward. It involves two congruent discs, A_x and A_y, and $n-2$ parallel line segments, $A_1, A_2, \ldots, A_{n-2}$. Figure 5.5 demonstrates the arrangement of the sets for $n = 5$ and Figure 5.6 indicates the transversals.

Generalizing the preceding construction is easy and yields $2n - 2$ geometric permutations:

$$\begin{array}{ll}
(x, 1, 2, 3, \ldots, n-2, y) & (y, 1, 2, 3, \ldots, n-2, x) \\
(1, x, 2, 3, \ldots, n-2, y) & (1, y, 2, 3, \ldots, n-2, x) \\
(1, 2, x, 3, \ldots, n-2, y) & (1, 2, y, 3, \ldots, n-2, x) \\
\vdots & \vdots \\
(1, 2, 3, \ldots, n-2, x, y) & (1, 2, 3, \ldots, n-2, y, x)
\end{array}$$

Turning to the upper-bound, we note that given any two disjoint, convex, compact sets, A and B, in the plane, $\mathbb{E}^2$, there are at most four lines of support. If neither of the sets is a point then there are exactly four lines of support l_1, l_2, l_3 and l_4. The four lines are chosen so that the lines are tangent to each of A and B as in Figure 5.7. In this instance we write $L(A, B) = \{l_1, \ldots, l_4\}$. If one of the sets is a point, or both of the sets are points, then there are at most two lines, or one line, of support, respectively, and $L(A, B)$ is amended accordingly.

Given a geometric permutation, $\tilde{t}$, of the family of compact, convex sets $\mathfrak{A} = \{A_1, A_2, \ldots, A_n\}$, let t be a transversal that generates $\tilde{t}$. We show that a pair of compact, convex sets in $\mathfrak{A}$, say A_i and A_j, can be chosen so that $t_* \in L(A_i, A_j)$ is a transversal of $\mathfrak{A}$ and t is equivalent to t_*. If t supports two compact, convex sets in $\mathfrak{A}$ then we are done for $t_* = t$ and A_i and A_j are chosen to be the sets which t supports. Suppose that t supports only one set, say A_i. If we rotate the line t through a sufficiently small angle, θ, so that the line maintains tangential contact with A_i then the newly obtained line is a transversal of $\mathfrak{A}$. There is a maximal angle θ_*, so that if one rotates the line beyond this angle, the line is no longer a transversal of $\mathfrak{A}$. It is clear that at θ_* the line is tangent to some $A_j \neq A_i$ and that this new transversal is equivalent to t. Thus t_* is the line obtained by rotating t through the angle θ_*, $t_* \in L(A_i, A_j)$ and t is equivalent to t_*. Next, suppose that t does not support any element of $\mathfrak{A}$. It is easy to see that there is a transversal parallel to t that supports some set A_i. Simply translate a copy of t, keeping it parallel to t, until it is tangent to some set in $\mathfrak{A}$, this is A_i; the whole time ensuring that the new line intersects the family. This new line generates the same geometric permutation as t and so the problem reduces to the previous case. Hence for every geometric permutation of $\mathfrak{A} = \{A_1, A_2, \ldots, A_n\}$, we have a way of associating it with a set $L(A_i, A_j)$. Since there can be at most $\begin{pmatrix} n \\ 2 \end{pmatrix}$ sets $L(A_i, A_j), i \neq j$, it follows that

$$|\mathcal{P}_{\mathfrak{A}}| \leq \begin{pmatrix} n \\ 2 \end{pmatrix}$$

.

Theorem 8 *Let $\mathfrak{A}$ be a family of n disjoint, closed line segments in the plane. Then $|\mathcal{P}_{\mathfrak{A}}| \leq n$. For $n \geq 3$, there exists a family $\mathfrak{A}$ of n disjoint line segments with $|\mathcal{P}_{\mathfrak{A}}| = n$.*

In order to construct a family $\mathfrak{A}$ of n line segments so that $|\mathcal{P}_{\mathfrak{A}}| = n$, we begin with n lines $l_x, l_z, l_0, l_1, l_2, \ldots, l_{n-3}$, concurrent at O and oriented as in Figure 5.8.

Choose a point P_0^4 on l_0 following O and connect it to a point P_z^4 on l_z preceding O. Notationally, the resulting line segment, A_4, passes through points, which we now label, $P_1^4, P_2^4, \ldots, P_{n-3}^4$, following O, on $l_1, l_2, \ldots, l_{n-3}$, respectively, and through the point, which we now label, P_x^4 on l_x preceding O. To construct line segment $A_k, k \geq 5$, we chose a point P_{k-4}^k on l_{k-4} following P_{k-4}^{k-1} and connect it to a point P_{k-5}^k on l_{k-5} preceding O so that the resulting line segment does not intersect any of $A_4, \ldots, A_{k-1}$. Notationally, A_k passes through the points, which we now label, $P^k_{k-3}, P^k_{k-2}, P^k_{k-1}, \dots P^k_{n-3}$ following O on $l_{k-3}, l_{k-2}, l_{k-1}, \dots, l_{n-3}$, respectively, and through the points, which we now label, $P^k_x, P^k_z, P^k_0, P^k_1, P^k_2, \dots, P^k_{k-5}$ preceding O on $l_x, l_z, l_0, l_1, l_2, \dots, l_{k-5}$, respectively. To construct line segment A_3, we first choose a point P^3_{n-3} on l_{n-3} preceding O. If $n = 3$ then connect P^3_{n-3} to a point on l_z following O. If $n = 4$ then connect P^3_{n-3} to a point on l_0 following O. If $n \geq 5$ then connect P^3_{n-3} to a point P^3_{n-4} on l_{n-4} between O and P^4_{n-4} so that the points $P^3_x, P^3_z, P^3_0, P^3_1, P^3_2, \dots, P^3_{n-5}$ where A_3 intersects each of the respective lines $l_x, l_z, l_0, l_1, l_2, \dots, l_{n-5}$ all follow O. Choose a point P^2_{n-3} on l_{n-3} following P^3_{n-3} and connect it to a point P^2_x between O and P^4_x on l_x; this is line segment A_2. Finally, line segment A_1 is obtained by connecting a point P^1_x between P^2_x and P^4_x to a point P^1_z between O and P^3_z. See Figure 5.9 for an example of six line segments giving rise to six geometric permutations.

Before we can show that $|\mathcal{P}_{\mathfrak{A}}| \leq n$ for any family $\mathfrak{A}$ of n disjoint closed line segments, we need to develop a few minor results. We state these results as lemmas.

Lemma 2 *Given a family of line segments, $\mathfrak{A}$, and $A, B, C, D \in \mathfrak{A}$. If $(A, B), (A, C)$ and (D, B) then $B = C$ and $A = D$.*

Proof. Since (A, B), A penetrates B and B penetrates all line segments $X \in \mathfrak{A} \setminus \{A, B\}$. Since (A, C), A penetrates C and C penetrates all line segments $X \in \mathfrak{A} \setminus \{A, C\}$. Thus if $B \neq C$ then $C \notin \{A, B\}$ so B penetrates C, but $B \notin \{A, C\}$ so C penetrates B, but by (*) this is a contradiction indicating $B = C$. From (A, B) we get that B penetrates all $X \in \mathfrak{A} \setminus \{A, B\}$ and if $A \neq D$ then B penetrates D, but (D, B) means that D penetrates B contradicting (*) which gives $A = D$. ■

Lemma 3 *Given any family of line segments, there are at most three different strong* *pairs.*

Proof. Suppose that there exists a family $\mathfrak{A}$ where (A_i, B_i), for $i = 1, 2, 3, 4$ and that $(A_i, B_i) \neq (A_j, B_j), i \neq j$. If $A_i = A_j, i \neq j$, then by Lemma 2 $B_i = B_j$ which would mean $(A_i, B_i) = (A_j, B_j), i \neq j$, a contradiction. Thus $A_i \neq A_j$ whenever $i \neq j$. A similar argument gives $B_i \neq B_j, i \neq j$. Now A_1 penetrates B_1 and B_1 penetrates all $X \in \mathfrak{A} \setminus \{A_1, B_1\}$ and A_2 penetrates B_2 and B_2 penetrates all $X \in \mathfrak{A} \setminus \{A_2, B_2\}$. If $A_1 \neq B_2$ and $A_2 \neq B_1$ then B_1 penetrates B_2 and B_2 penetrates B_1. This contradiction shows that $A_1 = B_2$ or $A_2 = B_1$. Without loss of generality, we assume the latter. Observe that (A_1, A_2) implies that A_2 penetrates A_3. Applying the previous argument to (A_2, B_2) and (A_3, B_3) yields either $A_2 = B_3$ or $A_3 = B_2$, but the fact that A_2 penetrates A_3 rules out (A_3, A_2) and so $A_3 = B_2$ is the only possibility. Similarly we obtain $A_4 = B_3$. Thus we have (A_1, A_2) and (A_3, A_4), whence A_2 penetrates A_4 and A_4 penetrates A_2. This final contradiction proves Lemma 3. ■

Lemma 4 *If $\mathfrak{A}$ is a family of disjoint line segments in the plane with $|\mathfrak{A}| \geq 3$ then there are at most three different sets, say B_1, B_2, B_3, in $\mathfrak{A}$ such that:*

$$|\{\tilde{p} \setminus B_i : \tilde{p} \in \mathcal{P}_{\mathfrak{A}}\}| < |\mathcal{P}_{\mathfrak{A}}| - 1$$

.

Before we proceed to prove this lemma, we need to make the following observation.

Observation 1 *Let $\mathfrak{A} = \{A_1, A_2, \ldots, A_k, \ldots, A_m, \ldots, A_{n-1}, X\}$ be n line segments with directed lines l_1 and l_2, meeting at O, generating the two geometric permutations:*

$$\tilde{p_1} = (A_1, A_2, \ldots, A_{k-1}, X, A_k, \ldots, A_m, \ldots, A_{n-1})$$

$$\tilde{p_2} = (A_1, A_2, \ldots, A_k, \ldots, A_m, X, A_{m+1}, \ldots, A_{n-1})$$

CASE I. If $k < m$ then

(1) X penetrates A_j for $j \leq k-1$ and for $j \geq m+1$;

(2) A_j penetrates X for all j, $k \leq j \leq m$ with the exception of at most one;

(3) X strictly crosses an odd quadrant of l_1, l_2.

CASE II. If $k = m$ then

(1) If X is even with respect to l_1 and l_2 then X penetrates A_i for $i \neq m$;

(2) If X is not even with respect to l_1 and l_2 then A_m strictly crosses an odd quadrant and A_m penetrates A_i for any $A_i \neq A_m$ and X penetrates A_m;

(3) If A_m does not penetrate X then A_m penetrates A_i for any $i, 1 \leq i \leq n-1, i \neq m$.

We explore this observation intuitively through the use of diagrams. In the first case, the situation may be depicted by Figure 5.10. In the diagram, we have two directed transversals, l_1 and l_2, meeting the line segments in the orders p_1 and p_2, respectively. A careful examination of all possible arrangements of the line segments reveals that this picture is indeed representative of what is occurring in this situation. Segment A^* is unique in the sense that its orientation may vary so that it penetrates X or so that it does not penetrate X. Thus, it is the exceptional line segment in

Case I (2). Clearly, Case I (1) and Case I (3) are satisfied as well. In the second case, the assumption generates three possible arrangements, depicted in Figure 5.11, and the various implications are again clear from the diagrams.

Proof. *Lemma 4.* Suppose that $A \in \mathfrak{A}$ such that $|\{\tilde{p} \setminus A : \tilde{p} \in \mathcal{P}_{\mathfrak{A}}\}| < |\mathcal{P}_{\mathfrak{A}}| - 1$. This occurs when one of two possible conditions is satisfied. The first possibility is that, after removing A from all of the geometric permutations, what had been three distinct geometric permutations have now collapsed into one geometric permutation. The other possibility is that, after removing A from all of the geometric permutations, two pairs of previously distinct geometric permutations, that is all four are pairwise distinct, have collapsed in such a way that each pair is now one geometric permutation. Formally we have two conditions:

Condition 1. There are at least three different geometric permutations $\tilde{p_1}, \tilde{p_2}, \tilde{p_3} \in \mathcal{P}_{\mathfrak{A}}$ such that $\tilde{p_1} \setminus \{A\}, \tilde{p_2} \setminus \{A\}, \tilde{p_3} \setminus \{A\} \in \mathcal{P}_{\mathfrak{A}\setminus\{A\}}$ all form the same geometric permutation in $\mathcal{P}_{\mathfrak{A}\setminus\{A\}}$. In particular $\tilde{p_1} \setminus \{A\} = \tilde{p_2} \setminus \{A\} = \tilde{p_3} \setminus \{A\}$ in $\mathcal{P}_{\mathfrak{A}\setminus\{A\}}$. Here, three geometric permutations have become one.

Condition 2. There exist two distinct geometric permutations $\tilde{p^1}, \tilde{p^2} \in \mathcal{P}_{\mathfrak{A}\setminus\{A\}}$ and there exist for each $i = 1, 2$ distinct geometric permutations $\tilde{p^i_1}, \tilde{p^i_2} \in \mathcal{P}_{\mathfrak{A}}$ such that $\tilde{p^i} = \tilde{p^i_1} \setminus \{A\} = \tilde{p^i_2} \setminus \{A\}$ for each $i = 1, 2$. Here, four geometric permutations have become two.

Thus, to complete the proof, we need to show that there are at most three members of $\mathfrak{A}$ that satisfy either Condition 1 or Condition 2. Suppose that $X \in \mathfrak{A}$ satisfies Condition 1. So, there are three geometric permutations $\tilde{p_1}, \tilde{p_2}, \tilde{p_3} \in \mathcal{P}_{\mathfrak{A}}$

that, after removing X, collapse into one geometric permutation. More precisely, if $k \leq l < m$ then we write:

$$\tilde{p_1} = (A_1, A_2, \dots, A_{k-1}, X, A_k, \dots, A_{n-1})$$

$$\tilde{p_2} = (A_1, A_2, \dots, A_l, X, A_{l+1}, \dots, A_{n-1})$$

$$\tilde{p_3} = (A_1, A_2, \dots, A_m, X, A_{m+1}, \dots, A_{n-1}).$$

Clearly, $\tilde{p_1} \setminus \{X\} = \tilde{p_2} \setminus \{X\} = \tilde{p_3} \setminus \{X\}$ in $\mathcal{P}_{\mathfrak{A} \setminus \{A\}}$. Let l_1, l_2 and l_3 be the directed lines that generate the geometric permutations $\tilde{p_1}, \tilde{p_2}$ and $\tilde{p_3}$ respectively. We now apply Observation Case I to l_1 and l_3 to conclude that at least one of A_k and A_m penetrates X. Without loss of generality, we assume that A_k penetrates X. Now, if $l + 1 < m$ then by applying Observation Case I to l_2 and l_3 we obtain the contradiction X penetrates A_k. So we may assume $l + 1 = m$ and we refer to Observation Case II applied to l_2 and l_3. If X is even with respect to l_2 and l_3 then, by (1), X penetrates A_k, a contradiction. Hence, we may assume that X is not even with respect to l_2 and l_3, whence, by (2), (X, A_m). Thus, we have shown that if X satisfies Condition 1 then there exists some $B \in \mathfrak{A}$ such that (X, B). A similar argument, shows that if X satisfies Condition 2 then there exists some $B \in \mathfrak{A}$ such that (X, B). Regardless of which condition is satisfied by X, we see that (X, B) necessarily follows for some $B \in \mathfrak{A}$. Since, by Lemma 3, there are at most three strong pairs, the desired result follows immediately. ■

With Lemma 4 in hand we can now turn to the proof of the upper bound of Theorem 8. Let $|\mathfrak{A}| = n$. If n is 1 or 2 then it is easy to see that $|\mathcal{P}_{\mathfrak{A}}| = 1 \leq n$ in both cases. Next, there are only six permutations that can be obtained by arranging three objects and half of these permutations are simply reversals of the other half, there can only be at most three distinct geometric permutations. Thus, if $n = 3$, it is clear that $|\mathcal{P}_{\mathfrak{A}}| \leq n$. So assume $n \geq 4$ and we proceed inductively. Since $|\mathfrak{A}| \geq 4$, we can apply Lemma 4. So there is some $B \in \mathfrak{A}$ such that $|\{\tilde{p} \setminus B : \tilde{p} \in \mathcal{P}_{\mathfrak{A}}\}| \geq |\mathcal{P}_{\mathfrak{A}}| - 1$. It is clear that $\{\tilde{p} \setminus B : \tilde{p} \in \mathcal{P}_{\mathfrak{A}}\} \subseteq \mathcal{P}_{\mathfrak{A} \setminus \{B\}}$ so $|\{\tilde{p} \setminus B : \tilde{p} \in \mathcal{P}_{\mathfrak{A}}\}| \leq |\mathcal{P}_{\mathfrak{A} \setminus \{B\}}|$. Because $|\mathfrak{A} \setminus \{B\}| = n - 1$, we can apply the inductive hypothesis to conclude that $|\mathcal{P}_{\mathfrak{A} \setminus \{B\}}| \leq n-1$. Thus, $|\mathcal{P}_{\mathfrak{A}}| \leq |\{\tilde{p} \setminus B : \tilde{p} \in \mathcal{P}_{\mathfrak{A}}\}| + 1 \leq |\mathcal{P}_{\mathfrak{A} \setminus \{B\}}| + 1 \leq (n-1) + 1 = n$. Thus, $|\mathcal{P}_{\mathfrak{A}}| \leq n$ completing the proof of Theorem 8.

Corollary 1 *For families of disjoint line segments* $f(n) = n, \forall n \in \mathbb{N}$.

5.4 Conclusion

This chapter has provided an intensive introduction to the study of geometric permutations. The results discussed here are interesting and the ideas developed prove to be crucial in later chapters.

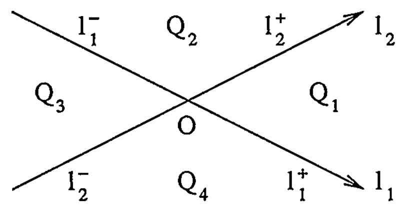


Figure 5.1: An example of two directed lines, l_1 and l_2, intersecting at O, giving rise to the half lines $l_1^-, l_1^+, l_2^-, l_2^+$ and the quadrants Q_1, Q_2, Q_3, Q_4.

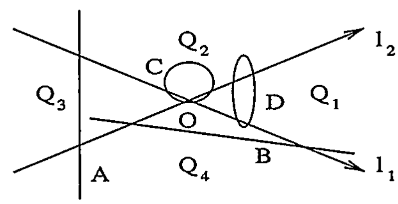


Figure 5.2: Examples of sets that cross quadrants, strictly cross quadrants and sets that are even and sets that are odd: C crosses Q_2; A, B and D strictly cross Q_3, Q_4 and Q_1 respectively; A is even; B is odd.

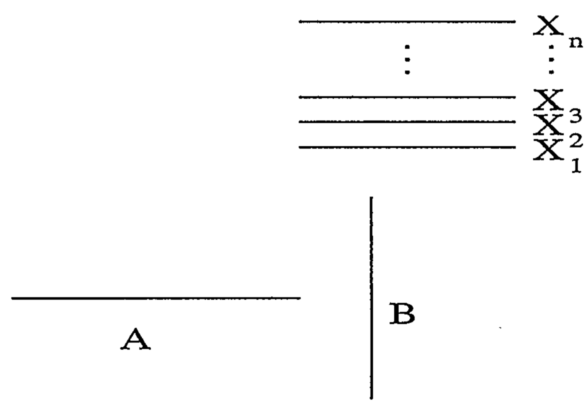


Figure 5.3: A family $\{A, B, X_1, X_2, \dots, X_n\}$ where (A, B) is valid and $X_1, X_2, \dots, X_n$ are mutually non-penetrating.

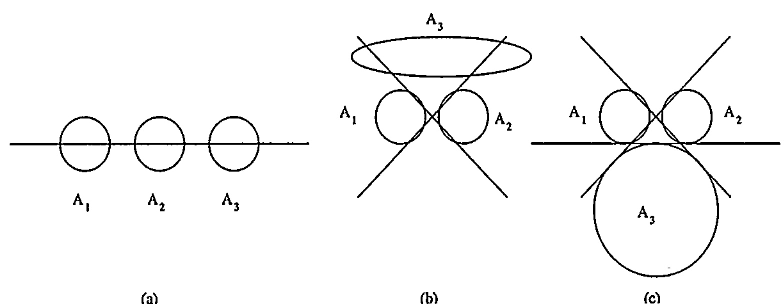


Figure 5.4: Examples of different sets of geometric permutations, $\mathcal{P}_{\mathfrak{A}}$, arising from different families, $\mathfrak{A} = \{A_1, A_2, A_3\}$, of disjoint, convex sets. (a) $\mathcal{P}_{\mathfrak{A}} = \{(1, 2, 3)\}$. (b) $\mathcal{P}_{\mathfrak{A}} = \{(1, 2, 3), (2, 1, 3)\}$. (c) $\mathcal{P}_{\mathfrak{A}} = \{(1, 2, 3), (2, 1, 3), (1, 3, 2)\}$.

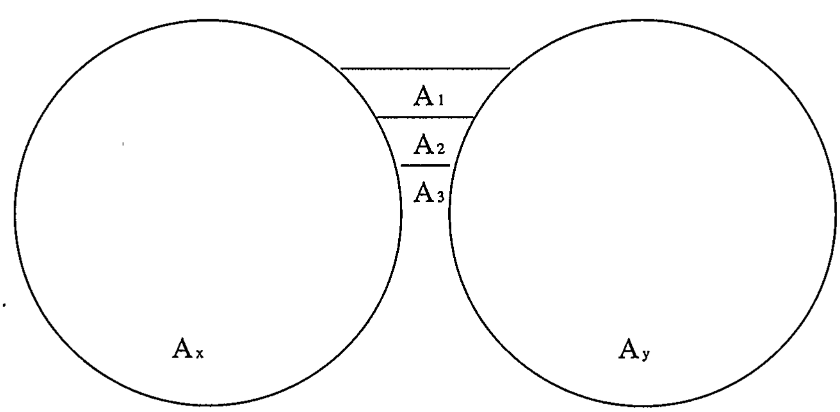


Figure 5.5: An example demonstrating $f(5) \geq 8$.

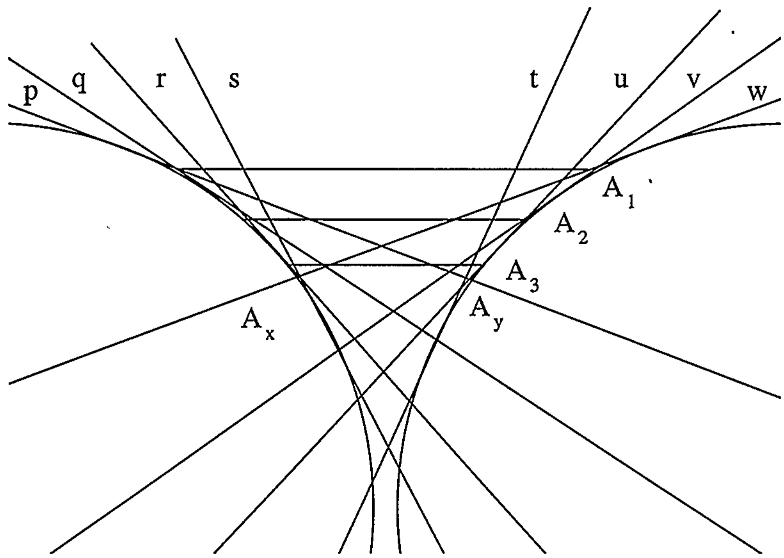


Figure 5.6: A closer view of Figure 5.5 with the positions of the eight transversals indicated. The transversals and their corresponding geometric permutations are $p = (x, 1, 2, 3, y)$, $q = (1, x, 2, 3, y)$, $r = (1, 2, x, 3, y)$, $s = (1, 2, 3, x, y)$, $t = (1, 2, 3, y, x)$, $u = (1, 2, y, 3, x)$, $v = (1, y, 2, 3, x)$, $w = (y, 1, 2, 3, x)$.

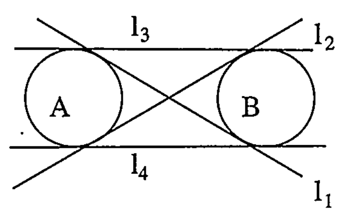


Figure 5.7: l_1 and l_2 separate and support A and B.

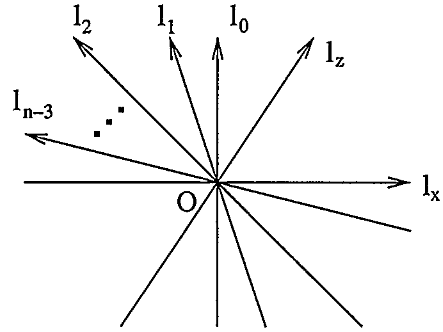


Figure 5.8: Orientation of Transversals.

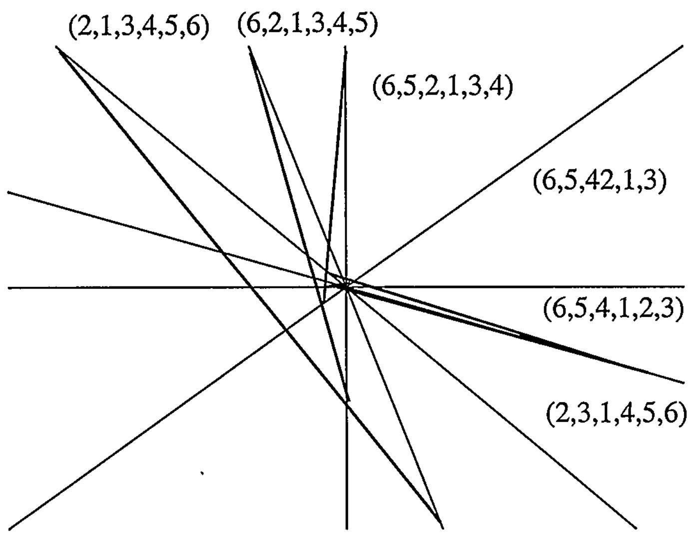


Figure 5.9: An example of six line segments yielding six geometric permutations.

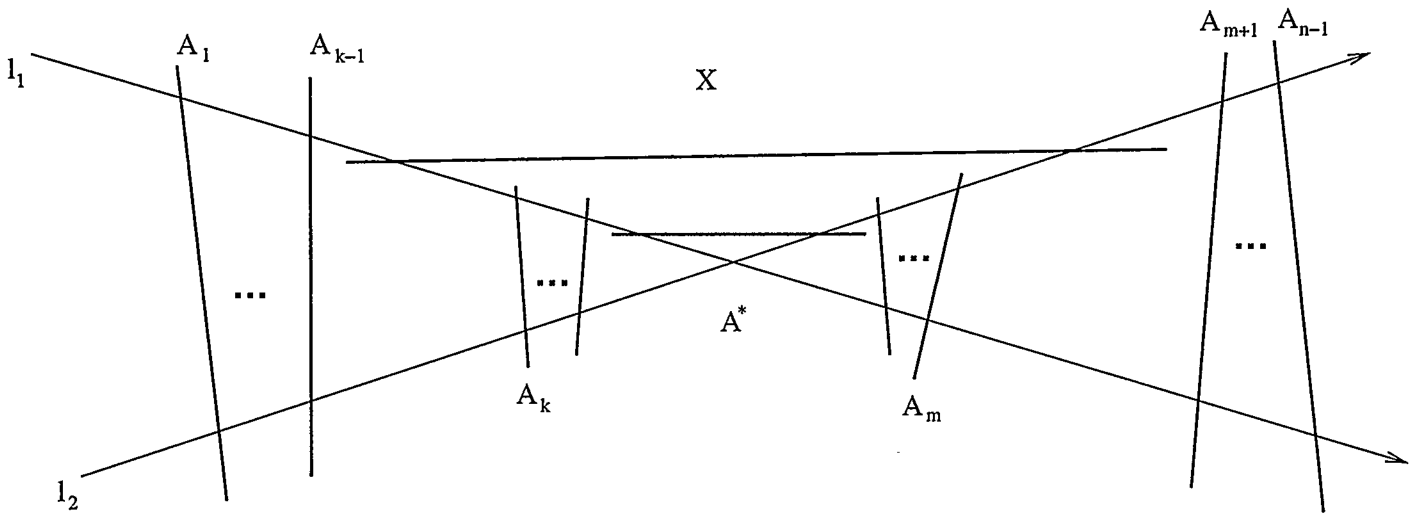


Figure 5.10: An illustration for the observation. In the diagram we have two directed transversals, l_1 and l_2, meeting the line segments in the orders p_1 and p_2, respectively. A careful examination of all possible arrangements of the line segments reveals that this picture is indeed representative of what is occurring in this situation. Segment A^* is unique in the sense that its orientation may vary so that it penetrates X or so that it does not penetrate X. Thus, it is the exceptional line segment in Case I (2). Clearly, Case I (1) and Case I (3) are satisfied as well.

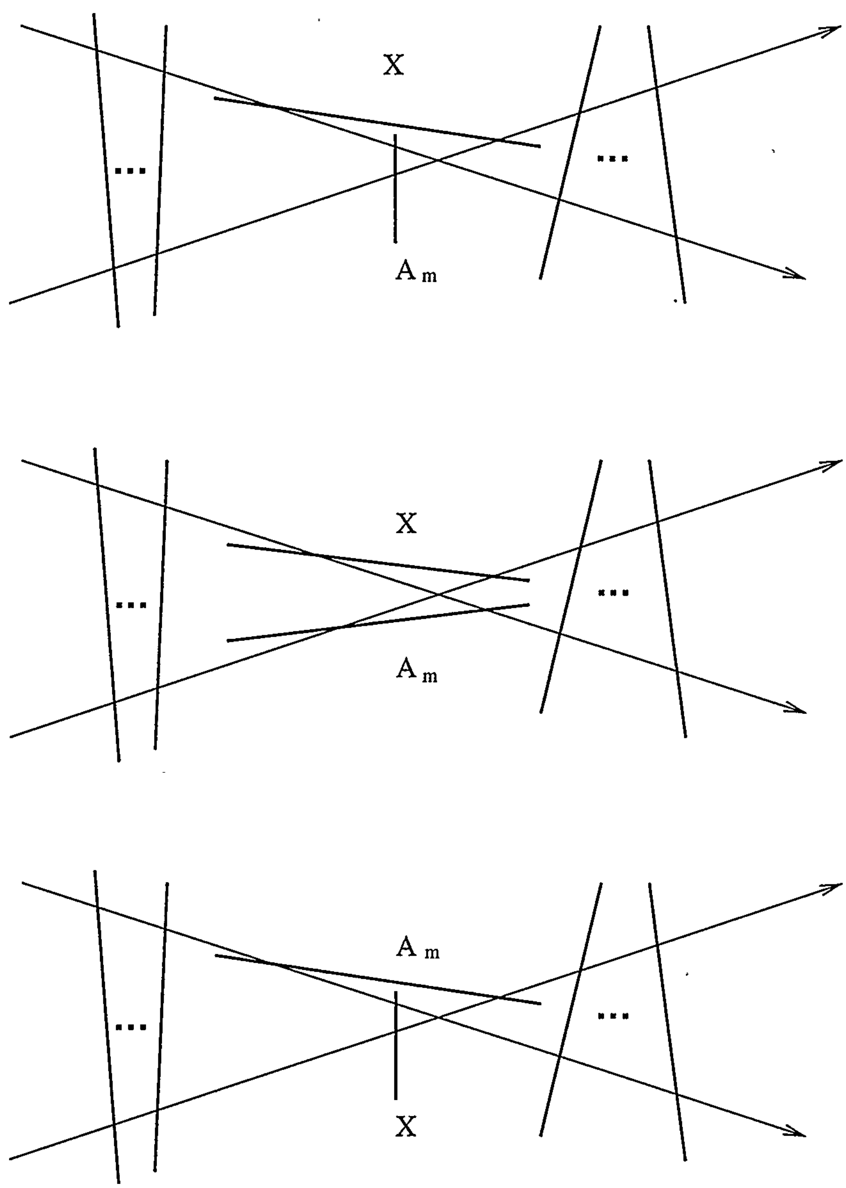


Figure 5.11: An illustration for the observation.

Chapter 6

"The Maximum Number of Ways to Stab n Convex Non-intersecting Sets in the Plane is $2n-2$"

6.1 Introduction

Previously, we showed the construction of a family of n sets that possessed $2n-2$ Geometric Permutations [20]. We now show that this is the maximum number of Geometric Permutations that any family, of n disjoint, convex sets in the plane, can have. The discussion is based on the work in [9].

6.2 Results

Let $\mathfrak{A}$ be a family of n pairwise disjoint, compact, convex sets in the plane. As we have seen previously, a directed line l that meets all of the members of $\mathfrak{A}$ induces a linear ordering of these members in a natural way. We denote the members of $\mathfrak{A}$ by $1, 2, \ldots, n$ and denote the ordering induced by a directed line transversal by $(i_1, i_2, \ldots, i_n)$. We say that $i \in \mathfrak{A}$ is *left tangent* to l if it lies in the closed half plane to the left of l, where left is determined by standing above the directed line and facing in the direction the line is oriented (cf. Figure 6.1). Now, for a critical observation, we note that two disjoint sets have at most two common left tangents (cf. Figure 6.2).

For every $\alpha \in [0, 2\pi)$ we define $l(\alpha)$ as the unique directed line that satisfies:

(i) α is the angle between the positive x-axis and $l(\alpha)$.

(ii) No set in $\mathfrak{A}$ is contained in the open half plane to the left of $l(\alpha)$.

(iii) At least one set of $\mathfrak{A}$ is contained in the closed half plane to the left of $l(\alpha)$.

We observe that, of all the directed transversals of $\mathfrak{A}$ that form an angle α with the positive x-axis, $l(\alpha)$ is the right most parallel transversal. By this we mean that no other directed transversal, parallel to $l(\alpha)$, is contained in the open half plane to the right of $l(\alpha)$ (cf. Figure 6.3). Now, a line $l(\alpha)$ is said to be an *extreme line* if it is left tangent to at least two sets in $\mathfrak{A}$ (cf. Figure 6.4).

Thus far, the discussion has been limited to directed line transversals. However, the goal is to discuss undirected transversals and the geometric permutations they induce. In order to do this, we first show that every undirected transversal can be moved continuously to an extreme line which generates the same geometric permutation.

Lemma 5 *Every undirected transversal of $\mathfrak{A}$ can be moved continuously to an extreme line without ever changing the induced geometric permutation.*

Proof. Let t be an undirected transversal. The transversal t may be directed in one of two directions, let α_0 and $\alpha_1 = \alpha_0 + \pi$ be the angles t makes with the positive x-axis in these respective directions. Observe that t can be translated continuously to $l_0 = l(\alpha_0)$ or $l_1 = l(\alpha_1)$ in such a way that the induced geometric permutation does not change. Now, let i_0 be the set contained in the closed half plane to the left of l_0 and i_1 be the set contained in the closed half plane to the left of l_1. The sets i_0 and i_1 are assumed to be sets that are uniquely left tangent to l_0 and l_1, respectively. If they are not unique, that is to say some set other than i_0 is left tangent to l_0 or some set other than i_1 is left tangent to l_1, then in either case we immediately have an extreme line, given by l_0 or l_1, and we are done.

First, suppose $i_0 \neq i_1$ and l_0 meets i_0 preceding i_1; consequently, l_1 meets i_1 preceding i_0 (cf. Figure 6.5). Rotate l_0 and l_1 clockwise, keeping them parallel to each other and tangent to i_0 and i_1 respectively, until either l_0 is tangent to some set other than i_0 or l_1 is tangent to some set other than i_1. Let l_0^* and l_1^* represent the lines l_0 and l_1 respectively after having been rotated in the fashion just described (cf. Figure 6.5). Lines l_0^* and l_1^* are transversals of $\mathfrak{A}$; they induce the same geometric permutation as t and satisfy one of the following four cases:

CASE I. Line l_0^* is left tangent to i_1 and therefore is an extreme line.

CASE II. Line l_1^* is left tangent to i_0 and therefore is an extreme line.

CASE III. Line l_0^* is tangent to some set i^* other than i_0 and i_1. Since l_1^* is to the left of l_0^* and intersects all of the sets in $\mathfrak{A}$, l_0^* is left tangent to i^*. (cf. Figure 6.5). Thus, l_0^* is an extreme line.

CASE IV. Line l_1^* is tangent to some set i^* other than i_0 and i_1. Since l_0^* is to the left of l_1^* and intersects all of the sets in $\mathfrak{A}$, l_1^* is left tangent to i^*. Thus, l_1^* is an extreme line.

Analogous arguments can be made in the case $i_0 \neq i_1$ and l_0 meets i_1 preceding i_0, as well as in the case $i_0 = i_1$. ■

Recall that two transversals are equivalent if they generate the same geometric permutation. This lemma shows that any transversal of $\mathfrak{A}$ is equivalent to some transversal which is an extreme line. Hence, it is sufficient to determine the maximum number of such extreme lines, as the upper bound on the number of extreme lines is the same as the upper bound on the number of geometric permutations.

Let $i(\alpha)$ be the member of $\mathfrak{A}$ contained in the closed half plane to the left of $l(\alpha)$. In the event that no unique member exists then leave $i(\alpha)$ undefined. Clearly, $i(\alpha)$ is defined except for possibly a discrete number of angles α.

Shortly we describe a method to generate a cyclic sequence of integers $C(\mathfrak{A}) = i_1 i_2 \dots i_m$, which is called a *cycle* of $\mathfrak{A}$ if:

(i) $i_j \neq i_{j+1}$, for $1 \leq j \leq m$ and $i_{m+1} = i_1$.

(ii) the circle of angles can be partitioned into m intervals $[\alpha_j, \alpha_{j+1})$ for $1 \leq j \leq m$ and $\alpha_{m+1} = \alpha_1$, such that $i(\alpha) = i_j$ for all $\alpha \in [\alpha_j, \alpha_{j+1})$.

Consider $l(\alpha)$ as α ranges from 0 to 2π and the corresponding sets $i(\alpha)$. It is easy to see that the set $i(\alpha)$ is defined and remains constant on some interval (α_1, α_2) where $0 \leq \alpha_1 < \alpha_2$; we choose $i_1 = i(\alpha)$ for $\alpha_1 < \alpha < \alpha_2$. Next, it is easy to see that the set $i(\alpha)$ is defined and remains constant on some interval (α_2, α_3) where $\alpha_1 < \alpha_2 < \alpha_3$; we choose $i_2 = i(\alpha)$ for $\alpha_2 < \alpha < \alpha_3$. Continuing in this way we generate the desired sequence. Furthermore, it is clear that $i(\alpha)$ changes every time the angle α yields an extreme line $l(\alpha)$. Hence, a new entry is added to the sequence whenever $l(\alpha)$ becomes an extreme line as α ranges from 0 to 2π. In particular, the *length of the cycle* m is the number of extreme lines. Thus, all that remains to be shown is that $m \leq 2n - 2$.

For the following lemma, a *scattered sub-cycle* of $C(\mathfrak{A})$ is a cyclic sequence obtained from $C(\mathfrak{A})$ by removing some of its members. The remaining integers appear in the same order as they did in $C(\mathfrak{A})$. For example 123, is a scattered sub-cycle of 14253

Lemma 6 *The cycle $C(\mathfrak{A})$ contains no scattered sub-cycle of the form abab, with $a \neq b$.*

Proof. Suppose there is a sub-cycle of the form *abab*. There exist angles $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ such that $i(\alpha_1) = a$, $i(\alpha_2) = b$, $i(\alpha_3) = a$, $i(\alpha_4) = b$. This implies that each of the intervals $(\alpha_1, \alpha_2), (\alpha_2, \alpha_3), (\alpha_3, \alpha_4)$ and (α_4, α_1) contains an angle for which there exists a common left tangent of a and b. However, as was previously noted, only two such left tangents can exist. ■

This lemma shows that if one removes all the members of $C(\mathfrak{A})$ except for a and b then all that will remain is a cycle of the form: $aa \ldots abb \ldots b$. A sequence that satisfies Lemma 6 and has the property that no two consecutive integers are the same is called an $(n, 2)$-*cycle*, where n indicates the maximum number of distinct integers appearing in the sequence. Since $C(\mathfrak{A})$ is clearly an $(n, 2)$-cycle, the following lemma completes the proof.

Lemma 7 *If $i_1 i_2 \ldots i_m$ is an $(n, 2)$-cycle then $m \leq 2n - 2$.*

Proof. First, there is always an integer a that occurs exactly once. For if not, then let i_j and i_k be two consecutive appearances of a such that $k - j$ modulo m is a minimum. However, by Lemma 6, any integer c that occurs in the circular interval $i_j, i_{j+1}, \ldots, i_{k-1}$ cannot occur outside of this interval. In particular, the two instances of a must lie within this circular interval; a contradiction.

The lemma holds trivially in the case $n = 2$. So, assume that the lemma holds for $(n-1, 2)$-cycles. Suppose that $i_1 i_2 \ldots i_m$ is an $(n, 2)$-cycle. From the preceding argument we can find an element a that occurs exactly once in this cycle. Remove a from the cycle. If the predecessor and the successor of a are identical, then remove one of them. Clearly, we are left with an $(n-1,2)$-cycle of cyclic length at least $m-2$ and at most length $m-1$. By the inductive hypothesis we have either $m-1 \leq 2(n-1)-2 = 2n-4$ or $m-2 \leq 2(n-1)-2 = 2n-4$. In either case we obtain $m \leq 2n-2$. ■

Finally, we state the major result of the paper which now follows immediately from the preceding lemmas.

Theorem 9 *For $n \geq 4$, the maximum number of geometric permutations realized by n convex, closed and pairwise non-intersecting sets in the plane is $2n-2$. For $n = 1, 2, 3$ the maximums are $1, 1, 3$ respectively.*

6.3 Conclusion

To conclude, we note that the only place in the preceding argument where compactness is needed is in the definition and construction of the lines $l(\alpha)$. However, with a few minor modifications to the preceding argument, we may dispense with boundedness all together. Thus, we obtain the same results for a family of closed, convex sets whose members need not be compact. On a final note, the $(n,2)$-cycles discussed here are called Davenport-Schinzel cycles and a great deal of study has been devoted to them. In this case, they are employed to prove a beautiful result in the study of transversals.

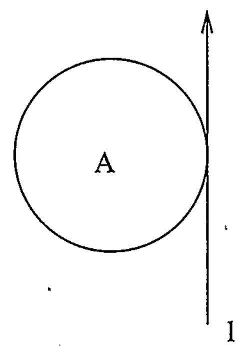


Figure 6.1: A is left tangent to l.

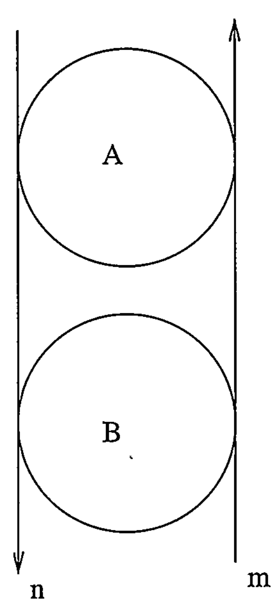


Figure 6.2: Sets A and B have at most two left tangents, namely m and n.

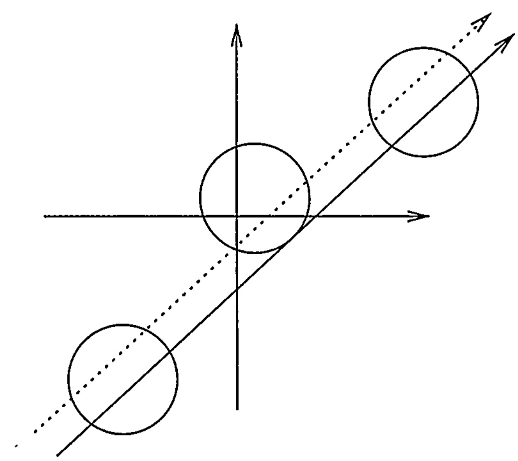


Figure 6.3: No other directed transversal, parallel to $l(\alpha)$, is contained in the open half plane to the right of $l(\alpha)$.

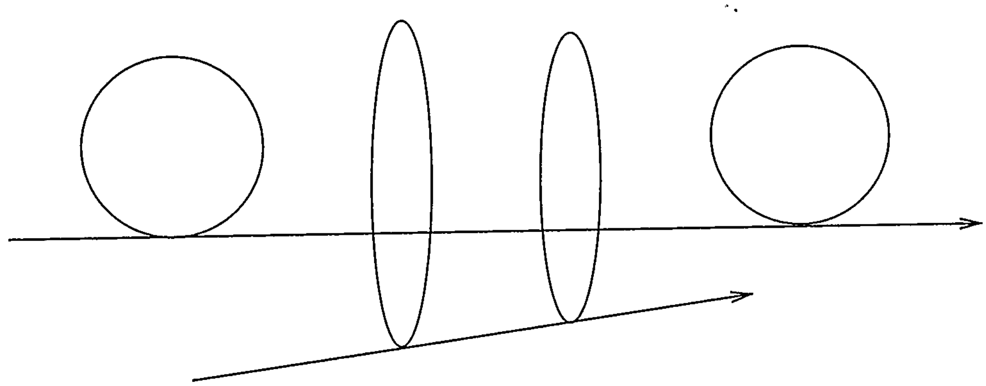


Figure 6.4: Extreme Line.

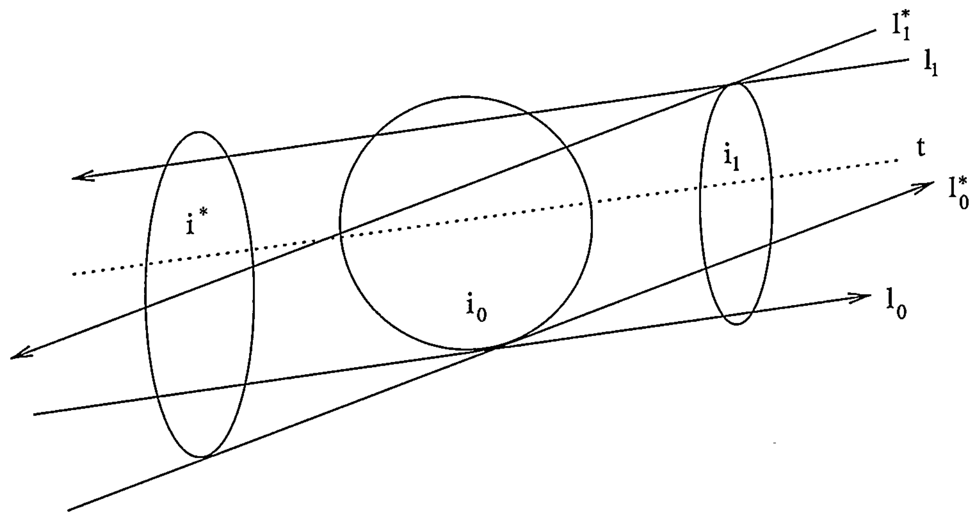


Figure 6.5: An illustration for the proof of Lemma 5.

Chapter 7

"Proof of Grünbaum's Conjecture on Common Transversals for Translates"

7.1 Introduction

Given a family of disjoint translates of a compact, convex set in the plane which satisfies property $T(5)$ then the family satisfies property T. Grünbaum conjectured this in 1958, and in this chapter we examine Tverberg's proof (cf. [23]) of what has come to be known as Grünbaum's Conjecture. The proof is a reductio ad absurdum where one begins by assuming a general counterexample and then by deducing several facts regarding the counterexample, one reduces the complexity of the structure under scrutiny, until finally the resulting structure is easily shown not to exist. The proof technique employed also demonstrates the integration of computers with mathematics to aid in a tedious computational process.

7.2 Preliminaries

Given a compact, convex set K, we indicate the translate of K that results from the translation vector c being applied to K by $K+c$. In this chapter, $\mathfrak{A} = \{K+c_i : i \in I\}$ is a family of disjoint translates of a compact convex set K in $\mathbb{E}^2$ where K contains the origin. Since K contains the origin, c_i is thought of as a point in $K + c_i$. Let $C = \{c_i : i \in I\}$ denote the set of all these points.

The *convex hull* of a set S, denoted $conv(S)$, is the smallest convex set containing S. For example, if S is the set of two points x and y in the plane $\mathbb{E}^2$ then the line segment joining these two points would be their convex hull. See Figure 7.1 for more examples. Given S, a set of points in $\mathbb{E}^2$, the points are said to be *convexly independent* if no point x lies in the interior of $conv(S \setminus \{x\})$. In Figure 7.1 (a), (b) and (d) the points are convexly independent, but in (c) the points are not, as one of the points lies in the interior of the convex hull of the remaining points.

In this chapter, a *direction* D is a line through the origin. Let S be a point set in the plane $\mathbb{E}^2$. We define the *K-height* of S in the direction D to be the quotient of the length of the orthogonal projection of $conv(S)$ on D by the length of the projection of K on the same line (cf. Figure 7.2).

Consider the family of translates of the circle K shown in Figure 7.3. One can see that in the direction perpendicular to the transversal, the point set of centers cannot have K-height greater than 1. So, effectively the K-height gives us a measure of how spread out the family of translates is in a particular direction. In order for a transversal to exist in a particular direction, the family must be sufficiently close, in the sense that the K-height must be less than or equal to 1 in the orthogonal direction. To summarize, if the K-height is greater than 1 then no transversal can exist in the orthogonal direction. If it is less than or equal to 1 then a transversal exists with equality typically indicative of a unique transversal and strict inequality typically indicative of transversals in an open set of neighboring directions.

The following claims are easy to verify and are offered without proof. These claims refine our intuition regarding what the K-height in fact indicates about a family with a transversal. These important results are used throughout this chapter.

Claim 1 *If the family $\mathfrak{A}$ satisfies property T then, in the direction orthogonal to a transversal, the K-height of C is less than or equal to* 1.

Claim 2 *If the family $\mathfrak{A}$ fails to have property T then in all directions, the K-height of C is greater than* 1.

Claim 3 *If the family $\mathfrak{A}$ satisfies Property $T(n)$ for some positive integer n then, for any set $\{c_{i_1}, c_{i_2}, \ldots, c_{i_n}\}$ where $i_1 < i_2 < \ldots < i_n$, the K-height of that set is less than or equal to* 1 *in some direction $D = D(i_1, i_2, \ldots, i_n)$.*

We describe now Hadwiger's Shrinking Process. Observe that in Figure 7.4 the family in (a) has more than one transversal whereas the family in (b) only has one transversal. The goal of Hadwiger's Shrinking Process is to take a family, where any, say, three members have a transversal, and shrink all of the members uniformly so that any three members continue to have a transversal, but some three members are separated and supported by their respective transversal as in Figure 7.4(b). For example, suppose the the family $\mathfrak{A}$ satisfies $T(3)$, let $\lambda \in [0,1]$ such that it is minimal with respect to the property that $\lambda\mathfrak{A} = \{\lambda A : A \in \mathfrak{A}\}$ satisfies $T(3)$. It is clear that there are three members of $\lambda\mathfrak{A}$ that are separated and supported by their respective transversal as in Figure 7.4(b). This process can be carried out for any general property, not just $T(3)$, as we see shortly.

Given a set A, the *affine hull* of A, $aff(A)$, is the smallest affine set which contains A. For example, given two points x and y in $\mathbb{E}^2$, $aff(\{x, y\})$ is the line passing through x and y, and for a line segment s in $\mathbb{E}^2$, $aff(s)$ is the line containing s. Given three non-collinear points x, y and z in the plane $\mathbb{E}^2$, $d(x, y, z)$ denotes the (minimal) distance from x to $aff(\{y, z\})$.

Let P be a convex n-gon with no two sides parallel. Given a side s of P, we assign to s the vertex v of P which has maximal distance from $aff(s)$. We call v the *opposite vertex* of s (cf. Figure 7.5). Label the vertices of the n-gon clockwise $1, 2, \ldots, n$ and then starting with the first side following vertex 1 clockwise, label the sides clockwise $j(1), j(2), \ldots, j(n)$. Let j_i be the opposite vertex of $j(i)$, then the sequence $j_1, j_2, \ldots, j_n$ is called the *shape sequence* of the n-gon (cf. Figure 7.5).

Let P and P' be two convex n-gons. Let f_1 be a map between the vertices of P and P' and f_2 be a map between the edges of P and P'. Then, $f = (f_1, f_2)$ is called a map between P and P'. We say f is a bijection between P and P' if and only if f_1 and f_2 are bijections.

Let f be a bijection between P and P'. Suppose that whenever v is the vertex opposite to the side s in P, $f_1(v)$ is the vertex opposite to the side $f_2(s)$ in P', in this case we say that *opposition is preserved* by f (cf. Figure 7.6). Next, suppose that whenever v is a vertex incident with the side s in P, $f_1(v)$ is the vertex incident with with the side $f_2(s)$ in P', in this case we say that *incidence is preserved* by f. An n-gon P' has the same *shape* as P if there exists a bijection, g, between P and P' such that incidence and opposition are preserved by g.

Claim 4 *If two convex n-gons, P and P', have the same shape sequence then P and P' have the same shape.*

Proof. Let $i_1, i_2, \ldots, i_n$ be the shape sequence for P and $j_1, j_2, \ldots, j_n$ be the shape sequence for P'. The vertices of both P and P' are labeled, clockwise, $1, \ldots, n$. Let $[m, m+1]$ denote the side joining the vertices m and $m+1$ where $n+1 = 1$. It is clear that i_m is the vertex opposite $[m, m+1]$ in P and that j_m is the vertex opposite $[m, m+1]$ in P' . Let $f_1(m) = m$ and $f_2([m, m+1]) = [m, m+1]$. It is clear that $f = (f_1, f_2)$ is a bijection between P and P' that preserves incidence. Given an arbitrary side of P, say $[m, m+1]$, i_m is the vertex opposite this side. Now, $f_1(i_m) = i_m$ and because the shape sequences for P and P' are the same, $i_m = j_m$, but j_m is the vertex opposite $[m, m+1]$ in P'. Since $f_2([m, m+1]) = [m, m+1]$, it follows that $f_1(i_m)$ is the vertex opposite $f_2([m, m+1])$ and so $f = (f_1, f_2)$ preserves opposition. ■

Claim 5 *If two n-gons, P and P′, have the same shape then after an appropriate relabeling of the vertices, P and P′ have the same shape sequence.*

Proof. If P and P' have the same shape sequence then there is nothing to prove. If the shape sequences are different then, using the opposition and incidence preserving bijection $f = (f_1, f_2)$, relabel vertex $f_1(1)$ in P' as 1 and then continuing clockwise label the remaining vertices $2, 3, \ldots, n$. Now check the shape sequences of P and P'. If they are the same then we are done; otherwise, there cannot exist an opposition and incidence preserving bijection between P and P'. ■

Let N be some positive integer. Let Q be a regular $k - gon$ where $k \leq N$ and k is odd. Distribute the points $q_1, q_2, \ldots, q_{N-k}$ on the sides of Q such that none of the points overlap each other, nor do they overlap any of the vertices of Q. Next choose points $p_1, p_2, \ldots, p_{N-k}$ so that each p_i is near q_i and $conv(Q \cup \{p_1, p_2, \ldots, p_{N-k}\})$ is a convex N-gon, which we call P. If each p_i is chosen sufficiently near q_i then, since k is odd, it follows that only the vertices of Q are opposite the sides of P. Thus the shape of P depends only on the distribution of $q_1, q_2, \ldots, q_{N-k}$. A straightforward induction on N shows that all possible shapes are obtained in this manner.

For our purposes we are only interested in the case $N = 6$. So there are only two choices for $k \leq 6$ and k odd; namely $k = 3$ or $k = 5$. Thus $N - k$ is either 1 or 3 and, consequently, there are only four possible shapes (cf. Figure 7.7).

7.3 The Counterexample

Let $\mathfrak{A} = \{K + c_i : i \in I\}$ be a counterexample to Grünbaum's Conjecture. So, $\mathfrak{A}$ is a family of disjoint translates of a compact convex set, K, where $\mathfrak{A}$ satisfies $T(5)$, but not T. Furthermore, we assume $|\mathfrak{A}| \geq 6$. The goal is to examine the properties that $\mathfrak{A}$ exhibits by virtue of being a counterexample. Based on these properties, $\mathfrak{A}$ is reduced from an infinite family of translates of some general compact, convex set to a family of six translates of a compact, convex, centrally symmetric polygon. Once this is done, the six centers are examined. As was established earlier, six points in the plane can have one of only four shape sequences. The various geometric permutations that arise from each shape sequence are studied. This study reveals that incompatible geometric permutations arise from each shape sequence. From this we conclude that no such counterexample, as the one we have supposed, may exist, whence Grünbaum's Conjecture is established.

Since $\mathfrak{A}$ does not have a transversal, we have already noted that $C = \{c_i : i \in I\}$ must have K-height greater than 1 in all directions. Let D be some arbitrary direction. The orthogonal projection of $conv(C)$ onto D is a line segment whose end points are generated by two distinct elements of C, say c_x and c_y. Since the K-height of C is greater than 1, the K-height of $\{c_x, c_y\}$ is greater than 1. Therefore, in a given direction D, a finite subset C_D of C must have K-height greater than 1.

Furthermore, it is clear that the K-height of C_D is greater than 1 in an open set of directions neighboring D. Ideally, C_D should be non-trivial, so $|C_D| \geq 6$.

Thus, we can cover the circle S^1 with open sets, where each of the open sets is associated with a finite subset of C with K-height greater than 1. Because S^1 is compact, we can choose a finite number of these open sets to cover S^1 and the finite union of the associated finite subsets of C has K-height greater than 1 in all directions. Thus we may assume that $|\mathfrak{A}| = N$ is finite. We write $\mathfrak{A} = \{K + c_1, K + c_2, \ldots, K + c_N\}$ and $C = \{c_1, c_2, \ldots, c_N\}$

Next replace K by $\bar{K} = \frac{1}{2}(K - K)$ (cf. Figure 7.8). Clearly, $\bar{K}$ is a compact, convex set. It is easy to check that K-height and $\bar{K}$-height are the same in all directions. Consequently, none of the existing transversals are altered nor are any new transversals added. The family $\bar{\mathfrak{A}} = \{\bar{K}+c_1, \bar{K}+c_2, \ldots, \bar{K}+c_N\}$ has transversals in exactly the same directions that $\mathfrak{A}$ does. It is clear that if two sets intersect then these sets have common transversals in all directions. Therefore, if there is an intersecting pair of elements in $\bar{\mathfrak{A}}$, the pair have common transversals in all directions. Because $\mathfrak{A}$ and $\bar{\mathfrak{A}}$ have transversals in exactly the same directions, the corresponding pair in $\mathfrak{A}$ has common transversals in all directions. This means the pair, in $\mathfrak{A}$, intersects, which is a contradiction; since, the translates in $\mathfrak{A}$ are disjoint. Hence, the family $\bar{\mathfrak{A}}$ is disjoint. The new family $\bar{\mathfrak{A}}$, consisting of translates of a centrally symmetric, compact, convex set, continues to be a counterexample; that is to say, $\bar{\mathfrak{A}}$ satisfies Property $T(5)$, but not T. Thus, we assume that K is centrally symmetric.

Inscribe K in a centrally symmetric polygon K'. It is clear that we can choose K' so that area of $K' \setminus K$ is so small that the difference in the K-height of K' and the K-height of K is arbitrarily small. It follows that for an appropriate choice of

K', the K-height and K'-height of any subset of C differs by an arbitrarily small amount. So little is this difference, that we continue to have a counterexample if we replace K by K'. Hence, we may assume that K is a polygon. (cf. Figure 7.9).

Thus, $\mathfrak{A} = \{K + c_1, K + c_2, \ldots, K + c_N\}$ is a disjoint family of translates of a centrally symmetric, compact, convex polygon K centered at the origin. $\mathfrak{A}$ satisfies Property $T(5)$ but does not satisfy Property T. The set $C = \{c_1, c_2, \ldots, c_N\}$ may be viewed as the set of centers for the respective members of $\mathfrak{A}$; that is c_i is the center for $K + c_i$. For a sufficiently small $\epsilon > 0$, we may replace K by $(1+\epsilon)K$ and continue to have a counterexample. Thus we have the freedom to move the centers so that no three centers are collinear and no two lines $\mathit{aff}\{c_i, c_j\}$ and $\mathit{aff}\{c_m, c_n\}$ are parallel whenever $\{i, j\} \neq \{m, n\}$. Since C and $\mathit{conv}(C)$ have the same K-height in all directions we may remove any points in C that lie in the interior of $\mathit{conv}(C)$ along with the corresponding translates in $\mathfrak{A}$. If, after discarding these sets, $|\mathfrak{A}| < 6$ then the desired result follows trivially. Hence, we assume $|\mathfrak{A}| \geq 6$ and $\mathfrak{A}$ continues to be a counterexample. So we may assume that the centers, $C = \{c_1, c_2, \ldots, c_N\}$, are convexly independent.

Continuing with the mutilation of K, we now arrange for K to have sides parallel to $\mathit{aff}\{c_i, c_j\}$ for each $i \neq j$. This is achieved by selecting one of the directions and then cutting two arbitrarily small triangles from K, so that the cut is parallel to the selected direction (cf. Figure 7.10). The triangles should be small enough so that the family remains $T(5)$. It may be necessary to replace K by $(1+\epsilon)K$ for a sufficiently small $\epsilon > 0$ prior to the cutting process. The process is repeated in each of the directions, $\mathit{aff}\{c_i, c_j\}$.

Next, to each of the $2\begin{pmatrix} N \\ 2 \end{pmatrix}$ newly formed sides, add an isoscles triangle with height denoted by h_{ij} if the cut was made in a direction parallel to $aff\{c_i, c_j\}$. Opposite triangles have the same height to ensure symmetry (cf. Figure 7.11). K^* is the new polygon formed after all the triangles have been added to the cut sides of K and we let $\mathfrak{A}^* = \{K^* + c_1, K^* + c_2, \dots, K^* + c_N\}$. It is clear that the aforementioned process has not removed from $\mathfrak{A}^*$ any transversals that were present in $\mathfrak{A}$ nor has it added any transversals to $\mathfrak{A}^*$ that were not in $\mathfrak{A}$. Thus $\mathfrak{A}^*$ continues to be a counterexample and we no longer distinguish it from $\mathfrak{A}$. Let l_{ij} denote the length of the projection of K in a direction orthogonal to $aff\{c_i, c_j\}$. It is clear, from the way that the original set K was cut, namely arbitrarily small triangles, the possible replacement of K by $(1+\epsilon)K$ for a sufficiently small $\epsilon > 0$ and the arbitrariness of h_{ij}, that each l_{ij} varies over some interval and can be treated as a free variable. Thus if we consider the following polynomials:

$$d(c_t, c_r, c_s)l_{ij} - d(c_k, c_i, c_j)l_{rs}, |\{r, s, t\}| = |\{i, j, k\}| = 3 \leq |\{i, j, r, s\}| \tag{7.1}$$

we can chose l_{ij} so that none of the polynomials vanish.

Let $\lambda \in [0, 1]$ so that it is maximal with respect to the property that there exists some five element subset of $\{c_1, c_2, \dots, c_N\}$, say $\{c_{i_1}, c_{i_2}, \dots c_{i_5}\}$, which has (λK)-height greater than or equal to 1 in all directions, with equality at least once. By applying Hadwiger's Shrinking Process there exists such a λ. Since the (λK)-height of $\{c_{i_1}, c_{i_2}, \dots c_{i_5}\}$ is 1 in some direction, the family $\{K + c_{i_1}, K + c_{i_2}, \dots, K + c_{i_5}\}$ has a transversal in the orthogonal direction. This transversal is tangent to exactly three of the sets. If it were tangent to less than three of the sets then the transversal could be moved to meet the five sets at interior points which would indicate that (λK)-height is less than 1. If it were tangent to more than three sets then either there are three collinear points among the five points $\{c_{i_1}, c_{i_2}, \dots c_{i_5}\}$ or there are two distinct parallel lines generated by four of the points in $\{c_{i_1}, c_{i_2}, \dots c_{i_5}\}$ (cf. Figure 7.12). Let $\lambda K + c_u, \lambda K + c_v, \lambda K + c_w$ be the three sets to which the transversal is tangent, meeting the sets in the given order. Due to the nature of the construction the transversal separates $\lambda K + c_v$ from $\lambda K + c_u$ and $\lambda K + c_w$ and the following equation holds:

$$d(c_v, c_u, c_w)/\lambda l_{uw} = 1. \tag{7.2}$$

Figure 7.13 illustrates the previous equation with $\lambda = 1$. The transversal is unique to $\{c_{i_1}, c_{i_2}, \dots c_{i_5}\}$; if not then there would be $\{c_x, c_y, c_z\} \subseteq \{c_{i_1}, c_{i_2}, \dots c_{i_5}\}$ such that $\{x, z\} \neq \{u, w\}$ and so that $d(c_y, c_x, c_z)/\lambda l_{xz} = 1$. This would make one of the non-vanishing equations cited above vanish because $d(c_v, c_u, c_w)/\lambda l_{uw} = d(c_y, c_x, c_z)/\lambda l_{xz}$ implies $d(c_v, c_u, c_w)\lambda l_{xz} - d(c_y, c_x, c_z)\lambda l_{uw} = 0$ and since $\lambda \neq 0$ we have $d(c_v, c_u, c_w) l_{xz} - d(c_y, c_x, c_z) l_{uw} = 0$. Hence the transversal is unique.

Thus we have found five elements in $\mathfrak{A}$, $\{K + c_{i_1}, K + c_{i_2}, \dots, K + c_{i_5}\}$, which have a unique transversal. There exists some sixth element in $\mathfrak{A}$ that does not share a transversal with these five elements. That is to say that there is some element $K + c_x \in \mathfrak{A}$ that when added to $\{K + c_{i_1}, K + c_{i_2}, \dots, K + c_{i_5}\}$ would make $\{K + c_{i_1}, K + c_{i_2}, \dots, K + c_{i_5}, K + c_x\}$ satisfy Property $T(5)$, but not $T(6)$. If no such element existed then every set in $\mathfrak{A}$ would share a transversal with $\{K + c_{i_1}, K + c_{i_2}, \dots, K + c_{i_5}\}$. Since there is only one transversal that intersects these five sets, each set in $\mathfrak{A}$ would have to meet this transversal. Consequently the family would satisfy Property T contrary to the previous assumption made about the family. Hence $\{K+c_{i_1}, K+c_{i_2}, \ldots, K+c_{i_5}, K+c_x\}$ does not satisfy Property $T(6)$, but does satisfy Property $T(5)$ and is therefore a counterexample to Grünbaum's Conjecture. In particular we may assume that the counterexample to Grünbaum's Conjecture has cardinality $N = 6$.

7.4 The Contradiction

Now that we have reduced the counterexample to this more manageable family of six translates of a compact, convex, centrally symmetric polygon we try to show that this family does not exist. To aid in this endeavor, we employ the earlier established notions of geometric permutations and shape sequences.

Because we know that our family $\mathfrak{A} = \{K + c_1, K + c_2, \ldots, K + c_6\}$ satisfies property $T(5)$, every subfamily of $\mathfrak{A}$ with 5 members has a transversal. Therefore, any configuration of the members of $\mathfrak{A}$ in the plane elicits $\begin{pmatrix} 6 \\ 5 \end{pmatrix} = 6$ geometric permutations of length 5. Consider all 6-tuples of geometric permutations of length 5. By systematically eliminating each 6-tuple until none is left, we conclude that the counterexample cannot exist which in turn asserts Grünbaum's Conjecture. At worst we are looking at 60^6 *six*-tuples, as there are $(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)/2 = 60$ geometric permutations; we divide by 2 because we do not distinguish a geometric permutation and its reverse. The preceding estimate does not account for repetitions of geometric permutations within a 6-tuple, which is obviously disallowed, so the actual number of 6-tuples that need to be checked is considerably lower. Although there are only finitely many possible 6-tuples of geometric permutation of length 5 the number is still quite high. Tverberg checked each possibility by hand and then rechecked his work with a computer. The remaining discussion is intended merely to serve as an illustration of how the work was carried out and by no means is it offered as an exhaustive approach to the problem.

The question that needs to be answered now is: what exactly was Tverberg checking? How was he able to conduct the elimination process? There were two things that Tverberg checked. First, the position of the members of the counterexample $\mathfrak{A}$ conform to very specific configurations that can be described by shape sequences. These configurations limit the possible geometric permutations that arise. Secondly, certain pairs of geometric permutations are *incompatible*, which means that a family exhibiting one of the geometric permutations cannot exhibit the other. As we see shortly, there are only four possible configurations, for now call them P_1, P_2, P_3 and P_4, that members of $\mathfrak{A}$ may conform to. With each of these configurations, whole families of geometric permutations can be eliminated. Then checking for incompatible geometric permutations, we are left with only two 6-tuples and these are eliminated by purely geometric means.

Now, for a somewhat more detailed explanation of the process. The six centers, $C = \{c_1, c_2, \ldots, c_6\}$, can be thought of as the vertices of a polygon. Because the centers have been assumed to be convexly independent, we know that the polygon is a convex 6-gon. Furthermore, we have assumed that no two lines $aff\{c_i, c_j\}$ and $aff\{c_m, c_n\}$ are parallel whenever $\{i, j\} \neq \{m, n\}$, so no two sides of the 6-gon are parallel. Thus, this hexagon conforms to the specifications described in the Preliminaries section; cf. the sequel to Claim 5. In the earlier discussion it was stated that there are only four possible shapes that a 6-gon can have. In particular there is a bijection between any 6-gon and one of the four 6-gons in Figure 7.7. Therefore, the hexagon of centers corresponds to one of these shapes. With these shapes, only certain geometric permutations arise. For example, a closer inspection of the $5, 5, 1, 1, 3, 3$ shape sequence in Figure 7.7 reveals that geometric permutations of the form $(\ldots, 1, \ldots, 5, \ldots, 2, \ldots)$ are not permissible; that is to say there is no line meeting $K + c_1, \ldots, K + c_6$ in the order (for example) $4, 1, 6, 5, 3, 2$. Suppose that, for a contradiction, the preceding statement is not true, and let l be the line of support for $K + c_1$ and $K + c_2$ which does not separate these sets and which is not a line of support for $conv(\mathfrak{A})$. Because we have assumed the existence of a geometric permutation of the form $(\ldots, 1, \ldots, 5, \ldots, 2, \ldots)$, $K + c_5$ meets l. Now, $K + c_5$ is the member of $\mathfrak{A}$ which is furthest away from l; this is clear for this particular shape sequence and how it was obtained. Consequently, $K + c_3, K + c_4$ and $K + c_6$ all meet l. Hence, l is a transversal for $\mathfrak{A}$ which means that $\mathfrak{A}$ satisfies property T, contrary to the assumption that $\mathfrak{A}$ is a counterexample to Grünbaum's Conjecture.

Two geometric permutations are said to be *incompatible* if they cannot both occur for the same configuration of sets. In other words, given a family of compact, convex, sets and a transversal of this family generating one geometric permutation, a second geometric permutation is incompatible with the first if there does not exist a transversal of the family generating the second geometric permutation.

Let $A, B, X, Y, P \in \mathfrak{A}$, then the following pairs are incompatible:

$$I_1 : ABXY, BAYX$$

$$I_2 : AXBY, AYCX$$

$$I_3 : AXPYB, YABX$$

$$I_4 : AXYZ, AYPZX$$

$$I_5 : AXYPZ, AYZX$$

If the family has a geometric permutation that contains one member of the pair I_j where $j \in \{1, 2, \ldots, 5\}$ then there can be no geometric permutation of the family containing the other member. It should be noted that I_1 is true in general and I_2 follows with some minor restrictions having to be put in place, but $I_3, \ldots, I_5$ follow only in the context of this proof. Figure 7.14 demonstrates why I_1 is an incompatible pair of geometric permutations. Each of the two transversals must meet the sets in the given order. Connecting the respective points on each of the lines generates line segments which overlap. Because we are dealing with a family of disjoint, convex sets, an immediate contradiction arises in all possible cases. After the tedious examination of all 6-tuples, done by hand and verified by computer, two 6-tuples remain:

$$((3, 2, 4, 5, 6)(4, 3, 5, 6, 1)(5, 4, 6, 1, 2)(6, 5, 1, 2, 3)(1, 6, 2, 3, 4)(2, 1, 3, 4, 5))$$

$$((2, 3, 4, 6, 5)(3, 4, 5, 1, 6)(4, 5, 6, 2, 1)(5, 6, 1, 3, 2)(6, 1, 2, 4, 3)(1, 2, 3, 5, 4))$$

and they both occur when the shape sequence is $5, 5, 1, 1, 3, 3$ or $5, 5, 6, 1, 3, 4$. These two 6-tuples can be eliminated through purely geometric means. Because of all the restrictions imposed on the members of $\mathfrak{A}$ and the given geometric permutations, if one tries to draw such a family then the members necessarily overlap which is contrary to the assumption that the members of $\mathfrak{A}$ are disjoint. Hence, there is no possible arrangement for the members of $\mathfrak{A}$ in the plane which has not been eliminated. Therefore, we conclude that no counterexample to Grünbaum's Conjecture exists.

7.5 Conclusion

Verifying Grünbaum's Conjecture has served as one of the finest problems in the study of transversals. This solution to the problem required an extensive, but tedious, checking process that was verified by a computer. Tverberg's proof is quite elegant and serves as a bridge between the world of pencil and paper mathematics and the world of mathematics done using computers. The proof relied on notions from disciplines outside of geometry such as topology and analysis. The solution to this problem shows that all fields of mathematics are closely related and that research in one area cannot be conducted in a vacuum oblivious to other areas of study.

Figure 7.1: Examples of convex hulls of point sets.

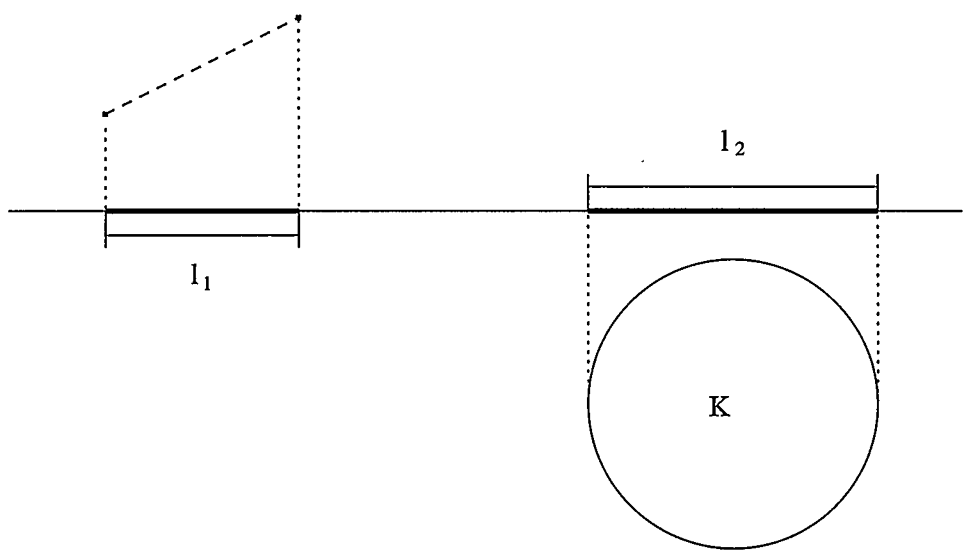


Figure 7.2: The K-height of a set of two points. The K-height is l_1/l_2.

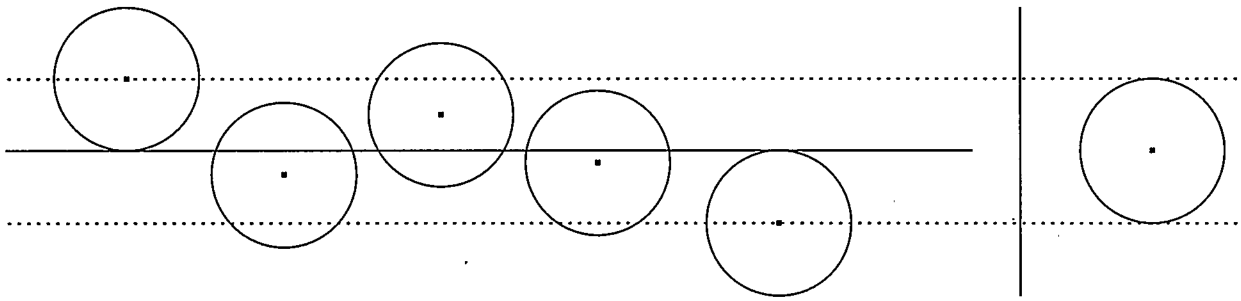


Figure 7.3: The K-height of the set of centers of circles.

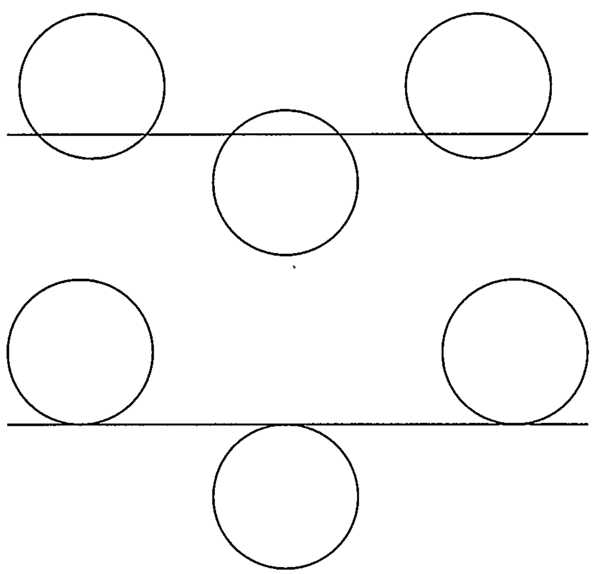


Figure 7.4: An example of a family with more than one transversal and a unique transversal.

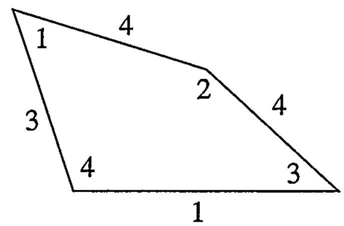


Figure 7.5: An example of a polygon where the opposite vertex for each side is indicated. The vertices are labeled interior to the polygon and each side is labeled with the opposite vertex for that side exterior to the polygon. The shape sequence for this polygon is 4,4,1,3.

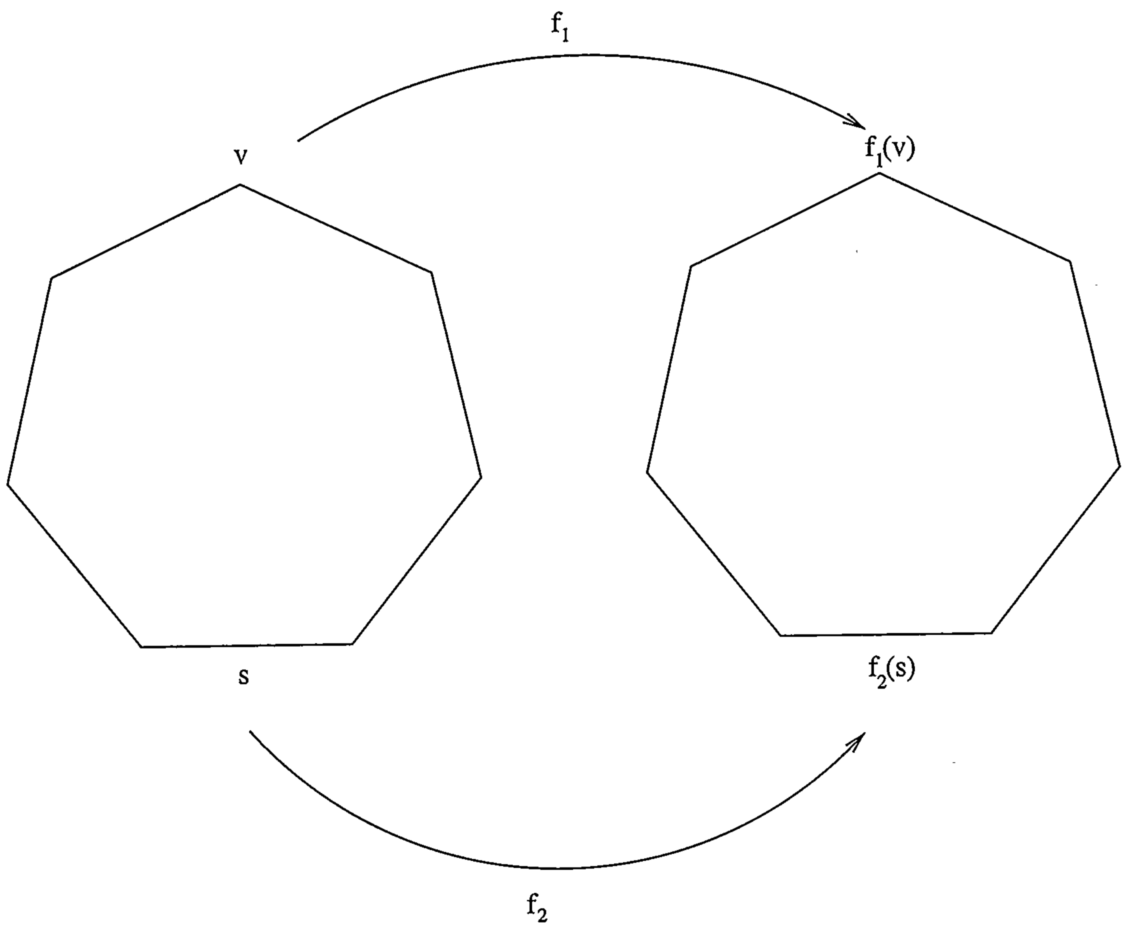


Figure 7.6: Opposition preserving bijection.

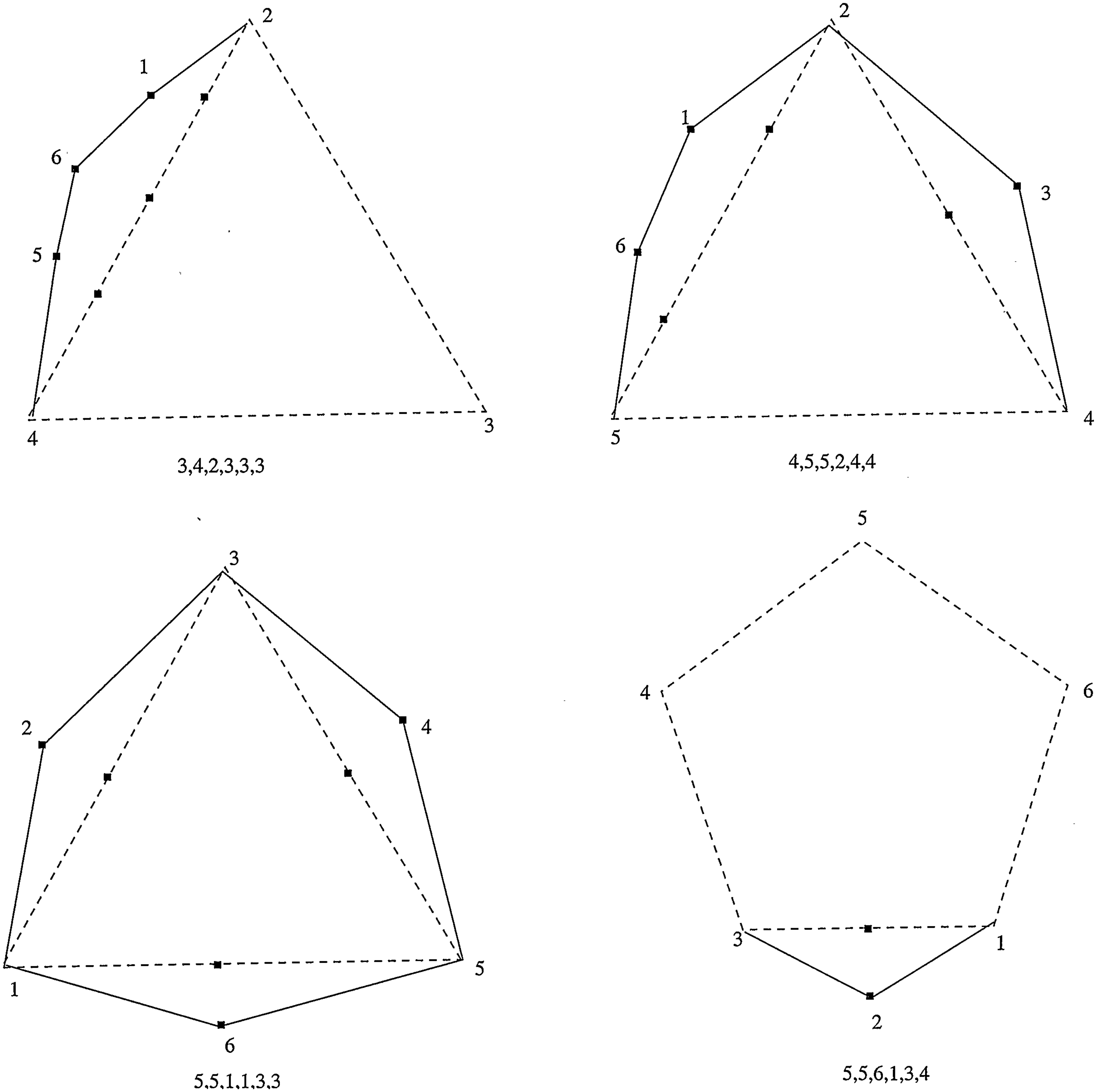


Figure 7.7: Shape Sequences

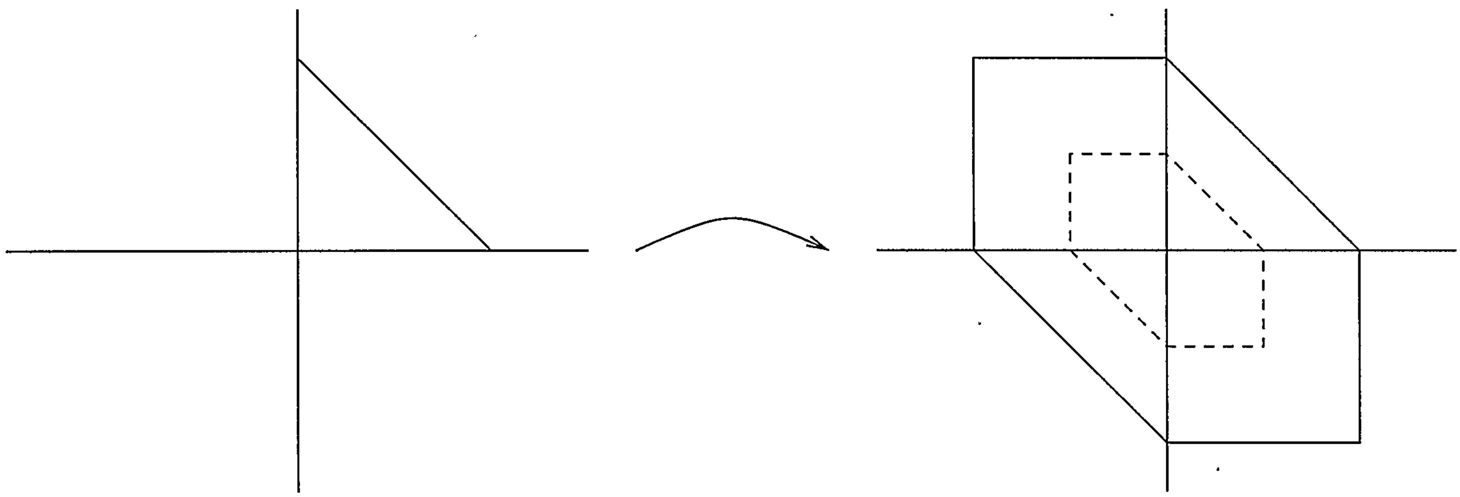


Figure 7.8: $\bar{K} = \frac{1}{2}(K - K)$

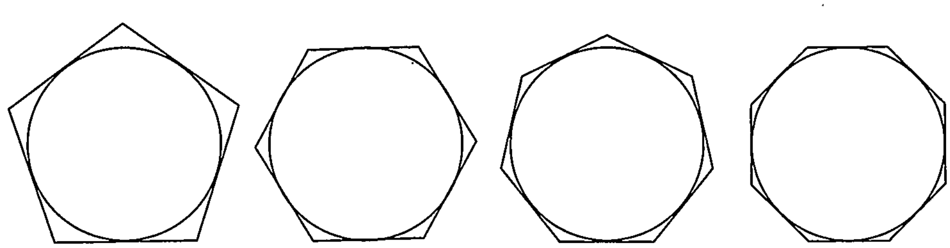


Figure 7.9: Inscribing a compact convex set in a centrally symmetric polygon. K is the circle which has been inscribed in the centrally symmetric polygon K'. Observe that as the number of sides increases the K-height of K' approaches 1.

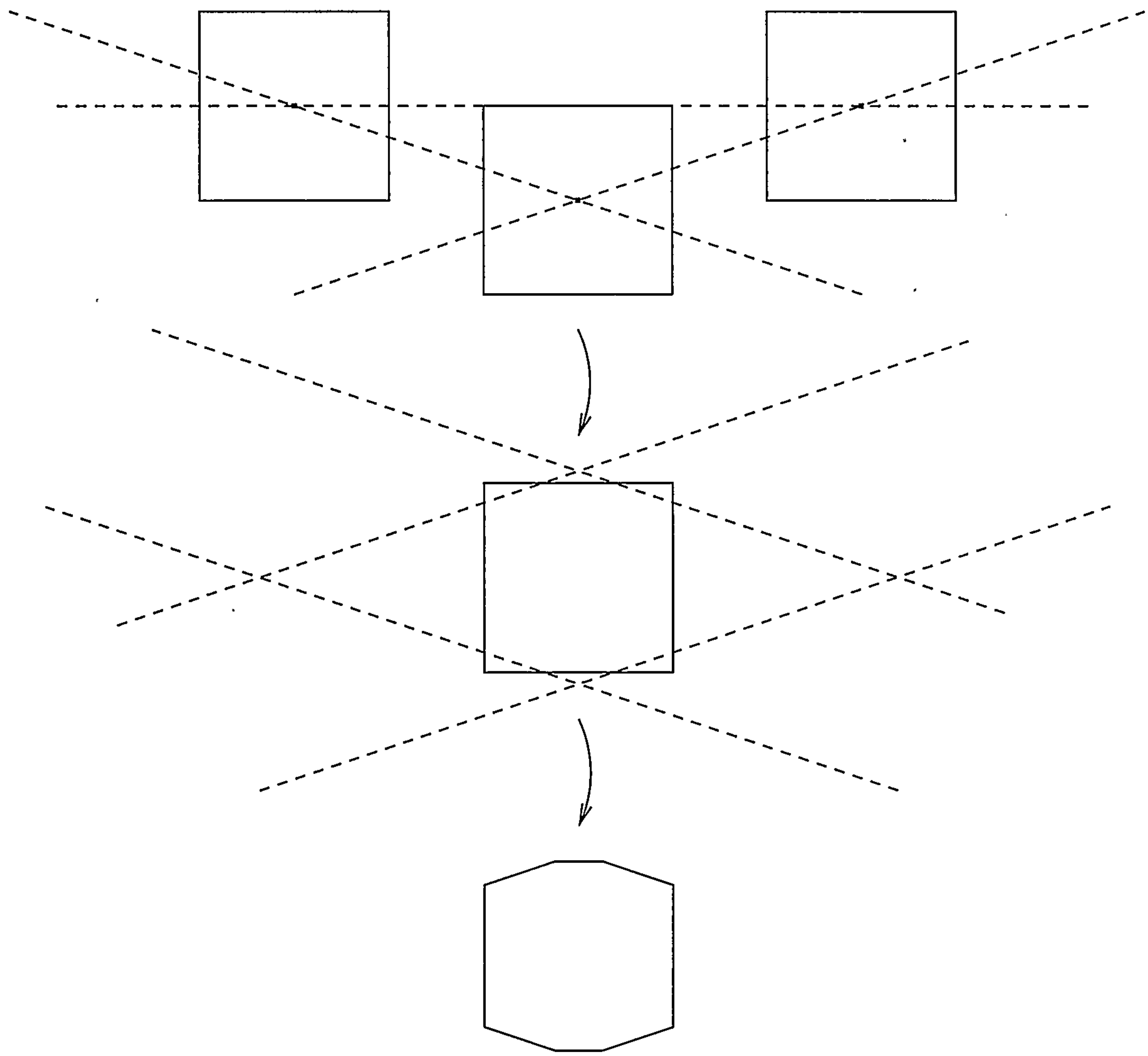


Figure 7.10: Illustration of the cutting process. We do not illustrate the horizontal cuts for simplicity.

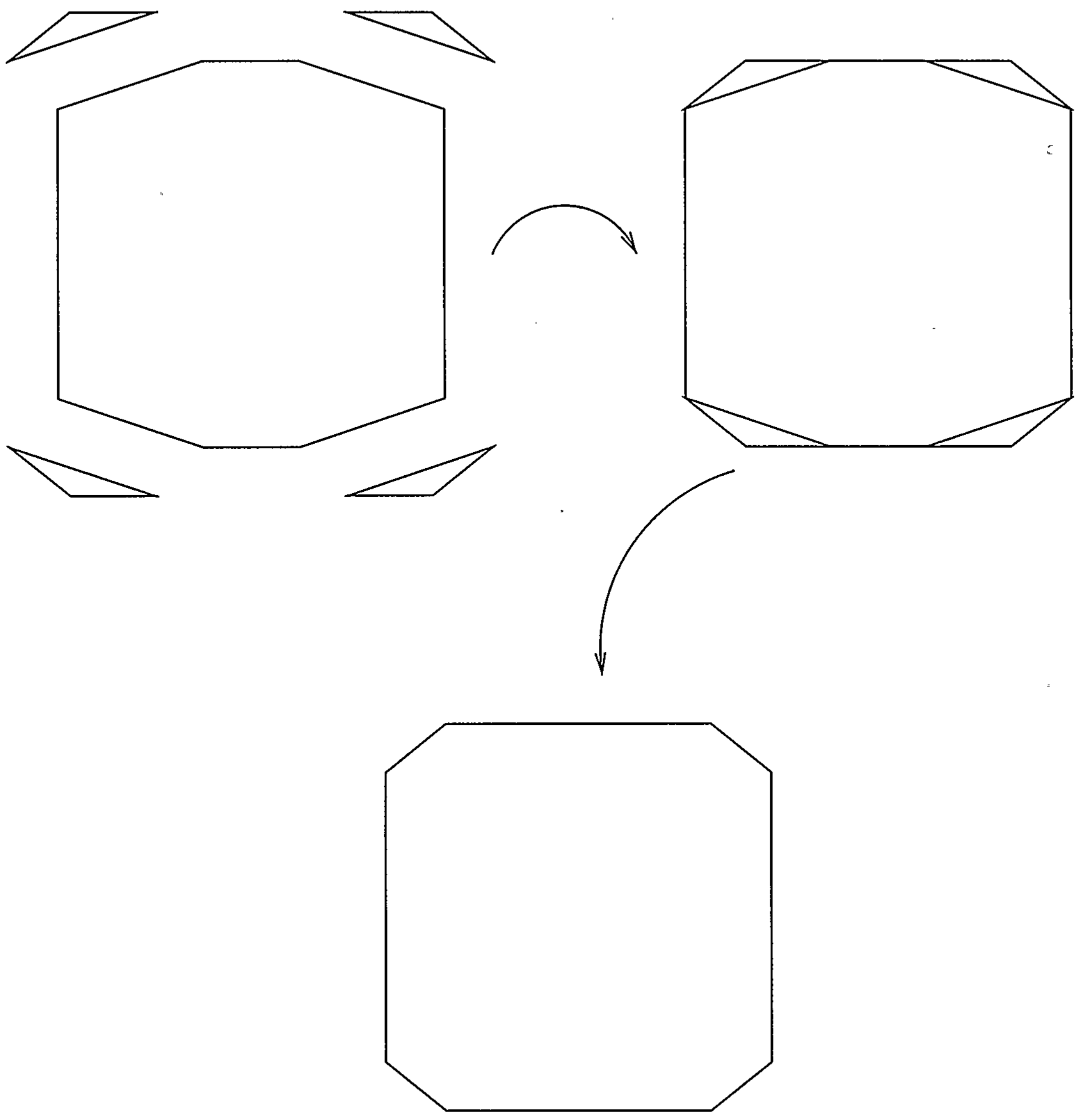


Figure 7.11: Illustration of the gluing process. We do not illustrate the triangles corresponding to the horizontal cuts for simplicity.

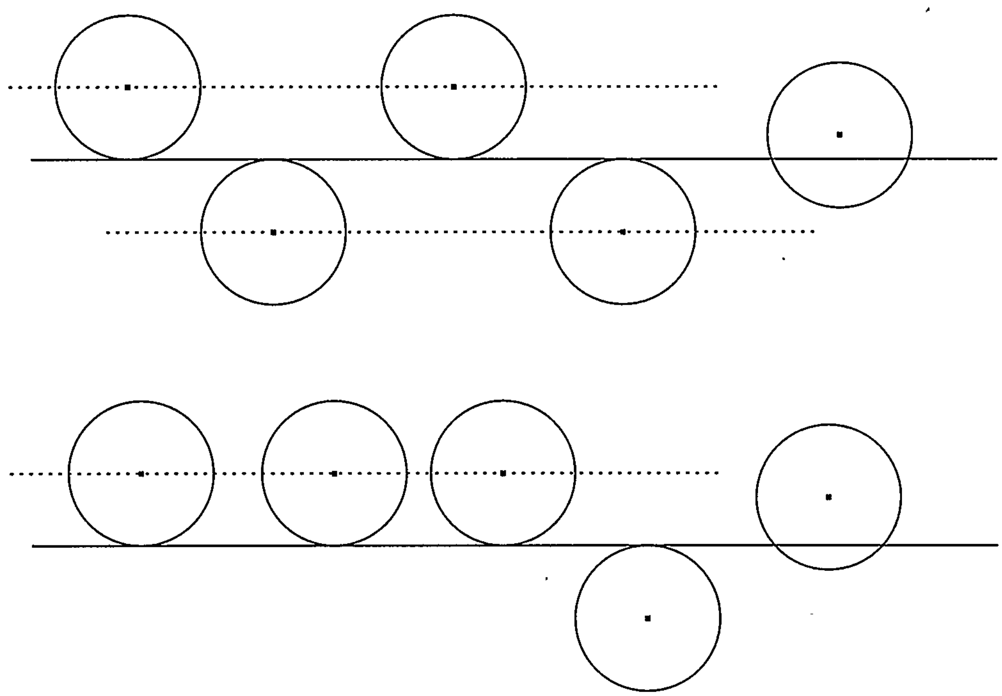


Figure 7.12: Parallel lines and collinear points result if more than three sets are tangent to the transversal. In the above diagram, two distinct parallel lines have been generated by the center points. In the lower diagram, three centers are collinear.

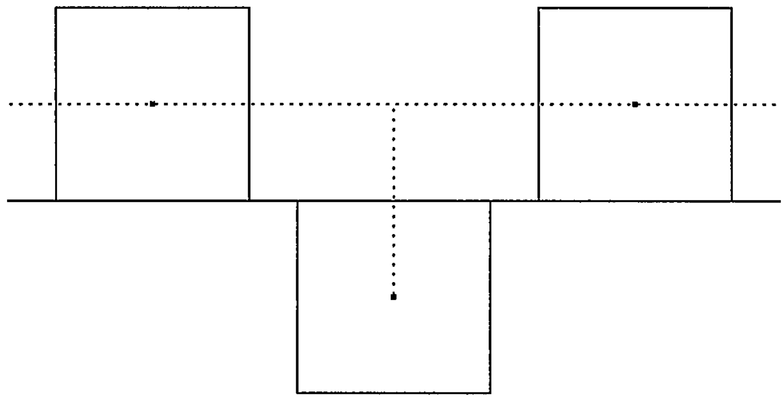
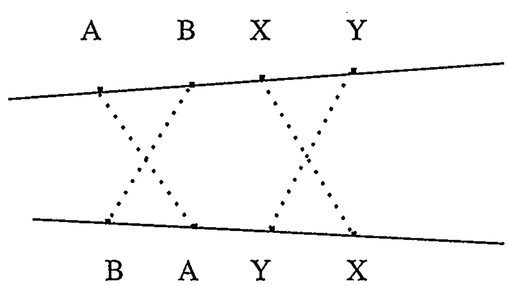


Figure 7.13: Illustration of Equation 7.2 with $\lambda = 1$.



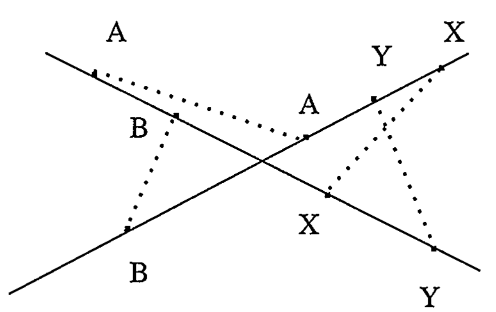


Figure 7.14: Illustration of incompatible pair I_1. Each of the two transversals must meet the sets in the given order. Connecting the respective points on each of the lines generates line segments which overlap. Because we are dealing with a family of disjoint convex sets, an immediate contradiction arises in all possible cases.

Chapter 8

"Common Transversals for Families of Sets"

8.1 Introduction

In this chapter we continue our study of transversals, but from a different perspective. Thus far, our examination has been restricted to the plane. We now broaden our scope and consider the problem of finding common transversals for families of sets in other settings. Recall that this problem of finding common transversals is related to Helly's Theorem. The present discussion relies heavily on this fact and a variation of Helly's Theorem is presented. Furthermore, the proofs, as presented by Grünbaum in [12], are only outlines and rely on results cited elsewhere. In some cases, these results are not available in English. Hence the intent here is to develop an intuitive feel for the problems at hand. This chapter is not intended to be a rigorous study of the problem. We develop the necessary concepts and then apply them loosely to develop our intuition with regards to these types of problems.

8.2 In General

First we present a version of Helly's Theorem that is used throughout this chapter. This particular form of the theorem is somewhat less general than other forms, but is useful for the present discussion. A compact subset C of $\mathbb{E}^n$ is called a *cell* if C is homotopic to a point (cf. Figure 8.1). Let $\mathfrak{C}$ be a family of cells in $\mathbb{E}^n$. Given two integers i and j, where $i \leq j$, if the intersection of any k members of $\mathfrak{C}$ is a cell for each $k \in \{i, i+1, \ldots, j-1, j\}$ then $\mathfrak{C}$ satisfies the *Total Intersection Property from i to j*. In this case we write $\mathfrak{C}$ satisfies property $TIP\{i, \ldots, j\}$.

Theorem 10 Helly's Theorem. *If $\mathfrak{C}$ is a family of cells in $\mathbb{E}^n$ that satisfies $TIP\{2, \ldots, n\}$ and the intersection of any $n+1$ members is not empty, then the intersection of all members of $\mathfrak{C}$ is not empty.*

Let $\mathfrak{H} = \{H_i : 0 \leq i \leq m\}$ be a family of parallel hyperplanes in $\mathbb{E}^n$ where the hyperplane $H_i \in \mathfrak{H}$ lies between H_{i-1} and H_{i+1} for all $1 \leq i \leq m-1$. A family $\mathfrak{K} = \{K_i : 1 \leq i \leq m\}$ of subsets of $\mathbb{E}^n$ is said to be *separated by parallel hyperplanes* or simply *separated* if there exists a family of parallel hyperplanes $\mathfrak{H} = \{H_i : 0 \leq i \leq m\}$ such that K_i is contained in the open region of $\mathbb{E}^n$ bounded by H_{i-1} and H_i for all $1 \leq i \leq m$. Figure 8.2 illustrates this definition in $\mathbb{E}^2$ and Figure 8.3 illustrates this definition in $\mathbb{E}^3$. In this case we also say that $\mathfrak{K}$ is *separated* by $\mathfrak{H}$ or $\mathfrak{H}$ *separates* $\mathfrak{K}$.

Let $\mathfrak{K} = \{K_i : 1 \leq i \leq m\}$ be a family of sets separated by the parallel hyperplanes $\mathfrak{H} = \{H_i : 0 \leq i \leq m\}$. Without loss of generality, we may assume that the hyperplanes are parallel to some axis in $\mathbb{E}^n$. Consider $K_i \in \mathfrak{K}$ and suppose that l is a line that intersects K_i, H_0 and H_m. The points $x_0 = l \cap H_0$ and $x_m = l \cap H_m$ describe the line l uniquely when given as the ordered pair (x_0, x_m). In particular, any line which intersects K_i, H_0 and H_m can be described uniquely by such a pair of points. Furthermore, the pair of points (x_0, x_m) may be viewed as a single point in $\mathbb{E}^{2n-2}$. The reason for this is as follows: regardless of which line generates the points, x_0 and x_m are always in the plane H_0 and H_m respectively.

Since H_0 and H_m are parallel to one of the axes in $\mathbb{E}^n$, one of the coordinates of x_0 and one of the coordinates of x_m is redundant. For example, consider the line (x_0, x_m) in $\mathbb{E}^n$. Write $x_0 = (a_1, a_2, \ldots, a_*, \ldots, a_n)$ and $x_m = (b_1, b_2, \ldots, b_*, \ldots, b_n)$ where a_* and b_* are the coordinates corresponding to the H_0 plane and the H_m plane respectively. Thus the line (x_0, x_m) in $\mathbb{E}^n$ can be uniquely identified with the point $(a_1, a_2, \ldots, a_{*-1}, a_{*+1}, \ldots, a_n, b_1, b_2, \ldots, b_{*-1}, b_{*-1}, \ldots, b_n)$ in $\mathbb{E}^{2n-2}$. Thus, for each $K_i \in \mathfrak{K}$ there exists a set $C_i \in \mathbb{E}^{2n-2}$ such that each point in C_i is uniquely identified with a line that intersects K_i, H_0 and H_m, and for each such line there is a point in C_i which is uniquely identified with it. We call C_i the *C-set* of K_i.

We use Figure 8.4 to illustrate the preceding concepts. In this discussion $n = 2$, so $\mathbb{E}^n = \mathbb{E}^2$ and $\mathbb{E}^{2n-2} = \mathbb{E}^2$. However, to reduce confusion, we continue to write $\mathbb{E}^n$ and $\mathbb{E}^{2n-2}$ to distinguish between these two spaces, but the reader should understand that $\mathbb{E}^n = \mathbb{E}^2$ and $\mathbb{E}^{2n-2} = \mathbb{E}^2$. In the figure, the set $K \in \mathbb{E}^n$ is illustrated in the top and the set $C \in \mathbb{E}^{2n-2}$, which is the C-set of K, is illustrated in the bottom. Now, $K \in \mathbb{E}^n = \mathbb{E}^2$ is the line segment from $(0, -1)$ to $(0, 1)$ and is bounded by the parallel hyperplanes, which in $\mathbb{E}^2$ are just lines, $H_0 = \{(-1, y) : y \in \Re\}$ and $H_m = \{(1, y) : y \in \Re\}$. We now describe a means by which to obtain $C \in \mathbb{E}^{2n-2} = \mathbb{E}^2$. Select a point on K, call it z, and a line that intersects z, H_0 and H_m, call it l. Let $z_0 = l \cap H_0$, $z_m = l \cap H_m$ and write $z_0 = (x_0, y_0)$, $z_m = (x_m, y_m)$. Observe that, if one pivots l about z then a unit increase (decrease) of z_0 along H_0 results in a unit decrease (increase) of z_m along H_m. In other words, fix x_0 and x_m and if one increases (decreases) y_0 by a given amount, then y_m decreases (increases) by the same amount. In particular, this pivoting process generates pairs (y_0, y_m) which lie along a line with negative slope, call it L. It is clear that $L \in \mathbb{E}^{2n-2}$ and every point along L corresponds uniquely to some line through z, H_0 and H_m and every such line corresponds uniquely to some point along L. Therefore, L is the C-set of z. Write $L = C_z$ and note that throughout this discussion z was an arbitrary point in K, so for each $z \in K$ we obtain a C-set C_z. Thus, $C = \bigcup_{z \in K} C_z$, and it is clear that this is a union of all lines with slope -1 and abscissa coordinate in $[-2, 2]$ (cf. Figure 8.4).

Let $\mathfrak{K} = \{K_i : 1 \leq i \leq m\}$ be a family of sets separated by the parallel hyperplanes $\mathfrak{H} = \{H_i : 0 \leq i \leq m\}$, let C_i be the C-set corresponding to K_i and write $\mathfrak{C} = \{C_i : 1 \leq i \leq m\}$ for the family of C-sets corresponding to $\mathfrak{K}$.

Lemma 8 *If the intersection of any n members of $\mathfrak{C}$ is not empty then the family $\mathfrak{K}$ satisfies property $T(n)$.*

Proof. Without loss of generality we choose $K_1, \ldots, K_n$ as an arbitrary subfamily of $\mathfrak{K}$. By assumption, there is a point in $C_1 \cap C_2 \cap \ldots \cap C_n$. Corresponding to this point, there is a line which intersects K_i for each $i = 1, 2 \ldots n$. ■

At this juncture, we discuss an application of Helly's Theorem as given in Theorem 10. This discussion takes place under ideal assumptions which, unfortunately, cannot be made. However, the discussion serves as a useful illustration of how to apply Theorem 10 to C-sets to generate results about transversals for families of convex, compact sets in $\mathbb{E}^n$.

Suppose for a moment that each member of $\mathfrak{C}$ is a cell and that the intersection of any $2, 3, \ldots, 2n-2$ members of $\mathfrak{C}$ is a cell. A straightforward application of Theorem 10 indicates that if the intersection of any $(2n-2)+1 = 2n-1$ members of $\mathfrak{C}$ is not empty then the intersection of all members of $\mathfrak{C}$ is not empty. If the intersection of any $2n-1$ members of $\mathfrak{C}$ is not empty then, by the preceding lemma, $\mathfrak{K}$ satisfies property $T(2n-1)$. If the intersection of all members of $\mathfrak{C}$ is not empty then $\mathfrak{K}$ satisfies property T. Therefore, in this particular situation, property $T(2n-1)$ implies property T. However, as can easily be seen in Figure 8.4, the sets $C_i \in \mathfrak{C}$ need not be bounded and hence are not necessarily cells. Fortunately, the following theorem is valid:

Theorem 11 *If $\mathfrak{K}$ is a separated family of compact, convex subsets of $\mathbb{E}^n$ such that the family of C-sets corresponding to $\mathfrak{K}$, $\mathfrak{C}$, satisfies $TIP\{3,4,\ldots,2n-2\}$, then $T(2n-1)$ implies T.*

In this paper, Grünbaum does not show how to obtain this result directly, but rather cites another paper from which results can be drawn that make proving this theorem possible. The idea behind the proof is that even if the members of $\mathfrak{C}$ are not cells, we are able to cut a sufficiently small piece from each $C_i \in \mathfrak{C}$ to make it a cell (cf. Figure 8.5). Next, it can be shown that the intersection of two members of $\mathfrak{C}$ is a cell. The Appendix shows a computational approach to determine the intersection of two C-sets corresponding to two perpendicular line segments. Finally, under the given assumption we may apply Helly's Theorem to this family of C-sets that have been cut and the result follows. An immediate consequence of this theorem is the following:

Corollary 2 *If $\mathfrak{K}$ is a family of compact convex sets in $\mathbb{E}^n$, whose members are contained in distinct parallel hyperplanes, then $T(2n-1)$ implies T.*

In the plane $\mathbb{E}^2$, this is simply the well known result $T(3)$ implies T for a family of parallel line segments. This corollary gives us higher dimensional analogues of this result. The situation in $\mathbb{E}^3$ is illustrated in Figure 8.6.

Let $K \in \mathbb{E}^n$ be a compact, convex set that contains the origin. A set $\bar{K}$ is *similar* to K if there exists an $x \in \mathbb{E}^n$ and $\lambda > 0$ such that $\bar{K} = x + \lambda K$ (cf. Figure 8.7). Let $\mathfrak{K} = \{x_i + \lambda_i K : i \in I\}$ be a family of sets similar to K. The family $\mathfrak{K}$ is called *ρ-thin* for $\rho \geq 1$ if $(x_i + \rho\lambda_i K) \cap (x_j + \rho\lambda_j K) = \emptyset$ whenever $i \neq j$. The notion of ρ-thinness provides a means by which to describe how far apart any two members in the family are (cf. Figure 8.8). The larger the value of ρ, the more spread out the family is.

Corollary 3 *For 2-thin families of closed spheres in $\mathbb{E}^n$, $T(2n-1)$ implies T.*

As before, Grünbaum does not give a proof of this corollary, but cites another paper from which results can be drawn to complete the proof. Intuitively, the result is clear. Because the family is 2-thin, this somehow ensures that the members of the family are sufficiently far apart, but not too far apart, since the family is $T(2n-1)$, forcing the family to have a common transversal.

8.3 In $\mathbb{E}^3$

Let $\mathfrak{K}$ be a family of convex sets in $\mathbb{E}^3$. The family $\mathfrak{K}$ is called *k-simple* if, whenever the straight lines l_0 and l_1 intersect any k members of $\mathfrak{K}$, say $K_1, \ldots, K_k$, there exists a continuous family of straight lines $l(t), 0 \leq t \leq 1$, such that $l(t) \cap K_i \neq \emptyset$ for all $t \in [0,1]$ and for all $i \in \{1, 2, \ldots, k\}$ and $l(0) = l_0$ and $l(1) = l_1$. Illustrated in Figure 8.9 is a 3-simple family.

Lemma 9 *Let $\mathfrak{K} = \{K_i : 1 \leq i \leq m\}$ be a family of sets separated by the parallel hyperplanes $\mathfrak{H} = \{H_i : 0 \leq i \leq m\}$. The family $\mathfrak{K}$ is k-simple if and only if the intersection of any k sets in $\mathfrak{C} = \{C_i : 1 \leq i \leq m\}$ is path connected.*

Proof. Suppose that $\mathfrak{K}$ is k-simple. Without loss of generality, we consider the intersection $C_1 \cap C_2 \cap \ldots \cap C_k$ of k arbitrarily chosen elements in $\mathfrak{C}$. Suppose $x, y \in C_1 \cap C_2 \cap \ldots \cap C_k$ then there exist lines l_x and l_y that meets each of $K_1, K_2, \ldots, K_k$. By k-simplicity, there exits a continuous family of lines $l(t), 0 \leq t \leq 1$, that intersects each $K_i, i = 1, \ldots, k$ for all $t \in [0,1]$ and $i = 1, \ldots, k$ and $l(0) = l_x, l(1) = l_y$. Let $\overline{l(t)}$ be the unique point in $\mathbb{E}^{2n-2}$, $n = 3$, that corresponds to the line $l(t)$ for all $t \in [0,1]$. Clearly $\overline{l(t)} \in C_1 \cap C_2 \cap \ldots \cap C_k$ for all $t \in [0,1]$. Thus, $L(t) = \overline{l(t)}, t \in [0,1]$ is a path from $L(0) = \overline{l(0)} = \overline{l_x} = x$ to $L(1) = \overline{l(1)} = \overline{l_y} = y$ entirely in $C_1 \cap C_2 \cap \ldots \cap C_k$. For the converse, assume l_0 and l_1 meet any k members in $\mathfrak{K}$. Consider k arbitrarily chosen members of $\mathfrak{K}$, without loss of generality, we may call them $K_1, K_2, \ldots, K_k$. Now, l_0 and l_1 will correspond to two points in $C_1 \cap C_2 \cap \ldots C_k$, say 0 and 1, which is path connected. The lines corresponding to the points on the path yield the desired family. ■

Lemma 10 *In $\mathbb{E}^3$, let $\mathfrak{K} = \{K_i : 1 \leq i \leq m\}$ be a family of sets separated by the parallel hyperplanes $\mathfrak{H} = \{H_i : 0 \leq i \leq m\}$. Let A be the subset of H_0 consisting of points through which pass straight lines intersecting all the members of $\mathfrak{K}$. If A is connected, then it is simply connected.*

Proof. Assume that A is not simply connected. Let x be a point of a bounded component A^* of the complement of A in H_0. Essentially, what we are doing is assuming that that A has a "hole" in it and we attempt to derive a contradiction. We can think of A^* as the hole and the point x lies in it (cf. Figure 8.10). Now, let B_i denote the cone with vertex x and generated by K_i. Let $D_i = B_i \cap H_m$ and $D = \{D_i : i = 1, 2, \ldots, m\}$. Recall that the members of $\mathfrak{K}$ are convex, hence the members of D are also convex. Furthermore, the intersection of any two members of D is convex, so if the intersection of any three members is not empty then $\bigcap_{i=1}^{m} D_i \neq \emptyset$ by Helly's Theorem. However, the line passing through x and some point in the intersection of all the members of D meets each member of K which implies that $x \in A$. This is a contradiction, so there exist integers p, q, r such that $1 \leq p \leq q < r \leq m$ and $D_p \cap D_q \cap D_r = \emptyset$. If $D_i \cap D_j = \emptyset$ for some i and j then there exists a plane passing through x which separates K_i and K_j (cf. Figure 8.11). This can only happen if x is in an unbounded component of the complement of A, which it is not. Hence, we arrive at a contradiction that indicates: $D_i \cap D_j \neq \emptyset$ for all i and j. Consequently, in H_m, $D_p \cup D_q \cup D_r$ has a bounded component which we call D^*. Let E be the ellipse of maximal area inscribed in D^*. Next, let P_q and P_r denote the planes passing through x separating E from B_q and B_r, respectively, and let $H(P_q)$ and $H(P_r)$ be the closed half spaces, bounded by P_q and P_r respectively, that do not contain E. If $F = H_0 \cap H(P_q) \cap H(P_r)$ then it is clear that $x \in F$, F is unbounded, F is connected and $F \cap A$ is empty. But this contradicts the assumption that A^* is a bounded component in the complement of A. This final contradiction asserts the validity of the lemma. ■

Theorem 12 *If $\mathfrak{K}$ is a* 4*-simple separated family of compact convex subsets of* $\mathbb{E}^3$*, then* $T(5)$ *implies* T*.*

Proof. Apply Lemma 10 and Theorem 11 with $n = 3$. ■

8.4 Conclusion

Prior to this chapter, the discussion of transversals was limited to the plane. The results presented here indicate that generalization of previously examined results are possible. However, the problems become increasingly difficult and even generalizing results from $\mathbb{E}^2$ to $\mathbb{E}^3$ are extremely difficult. It should be pointed out that Grünbaum discusses a generalization of the problem to projective space, but we do not consider such generalizations in this text.

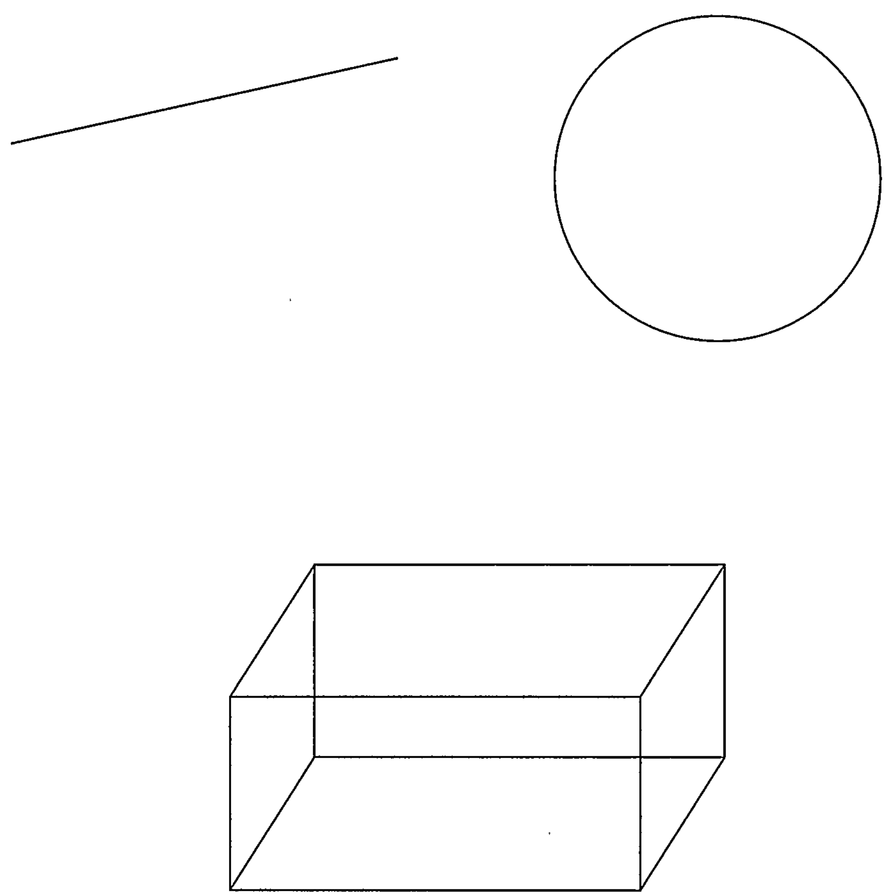


Figure 8.1: Examples of sets homotopic to points in $\mathbb{E}^2$ and $\mathbb{E}^3$.

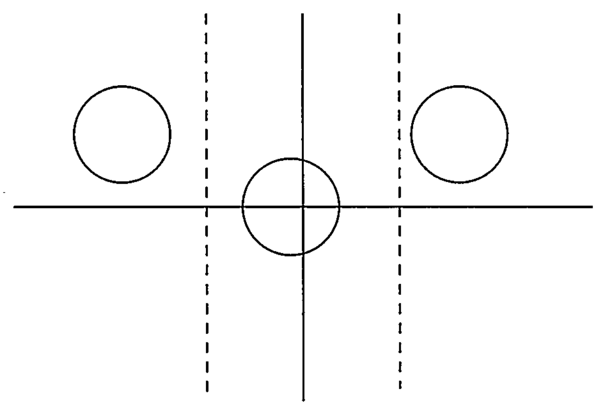


Figure 8.2: Example of separated sets in $\mathbb{E}^2$.

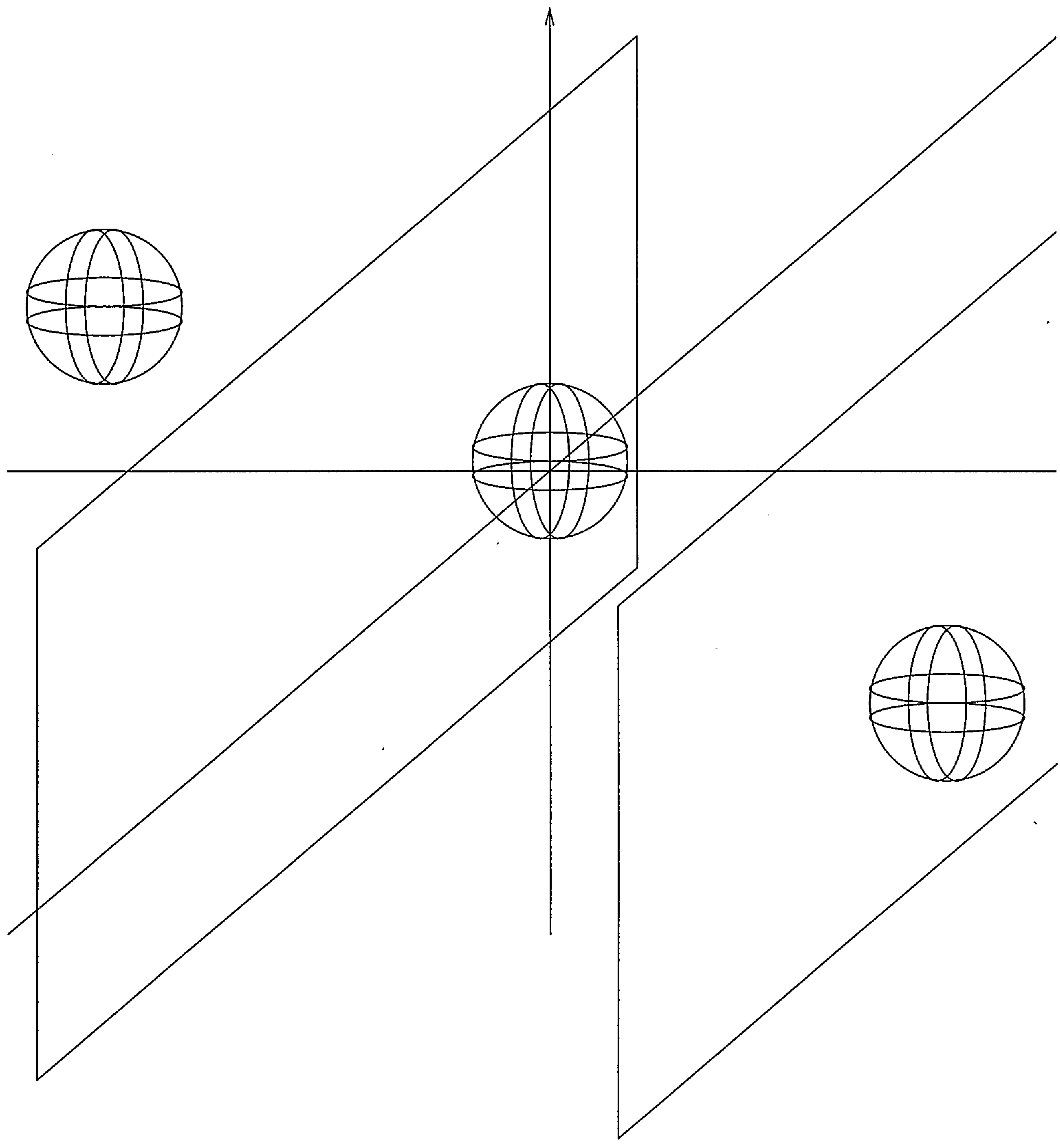


Figure 8.3: Example of separated sets in $\mathbb{E}^3$.

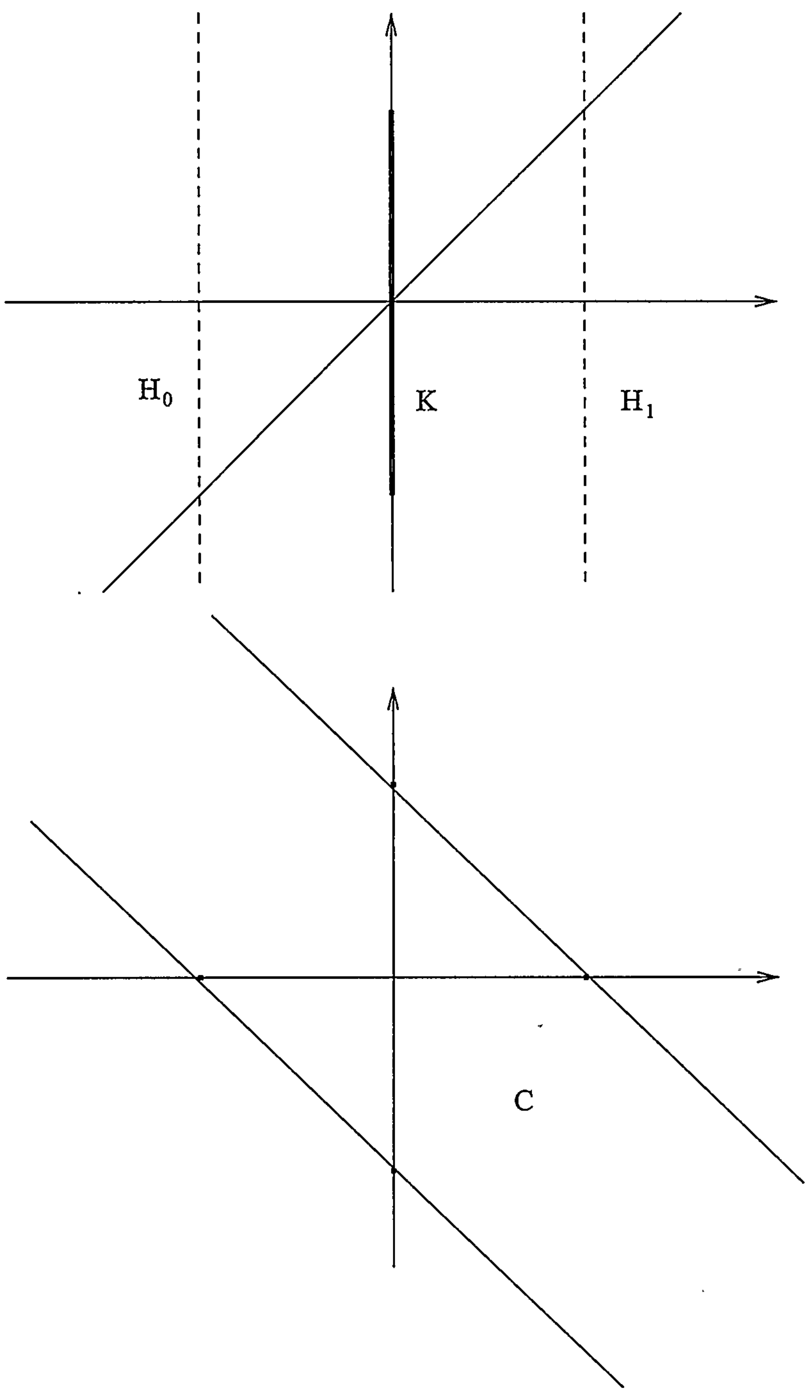


Figure 8.4: Example of a set of points C in $\mathbb{E}^2$ corresponding to all the lines which pass through K, H_0 and H_m in $\mathbb{E}^2$.

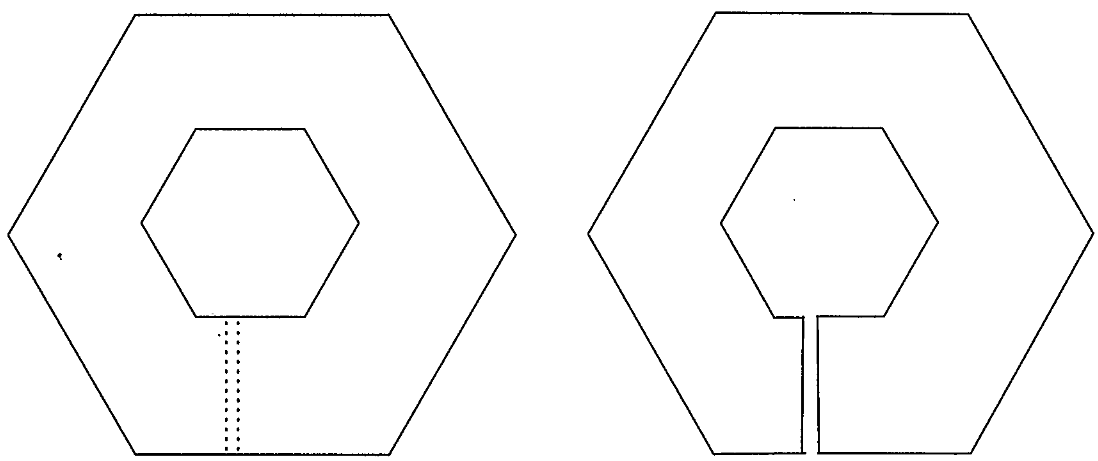


Figure 8.5: Cutting a small piece from the set on the left generates the cell on the right.

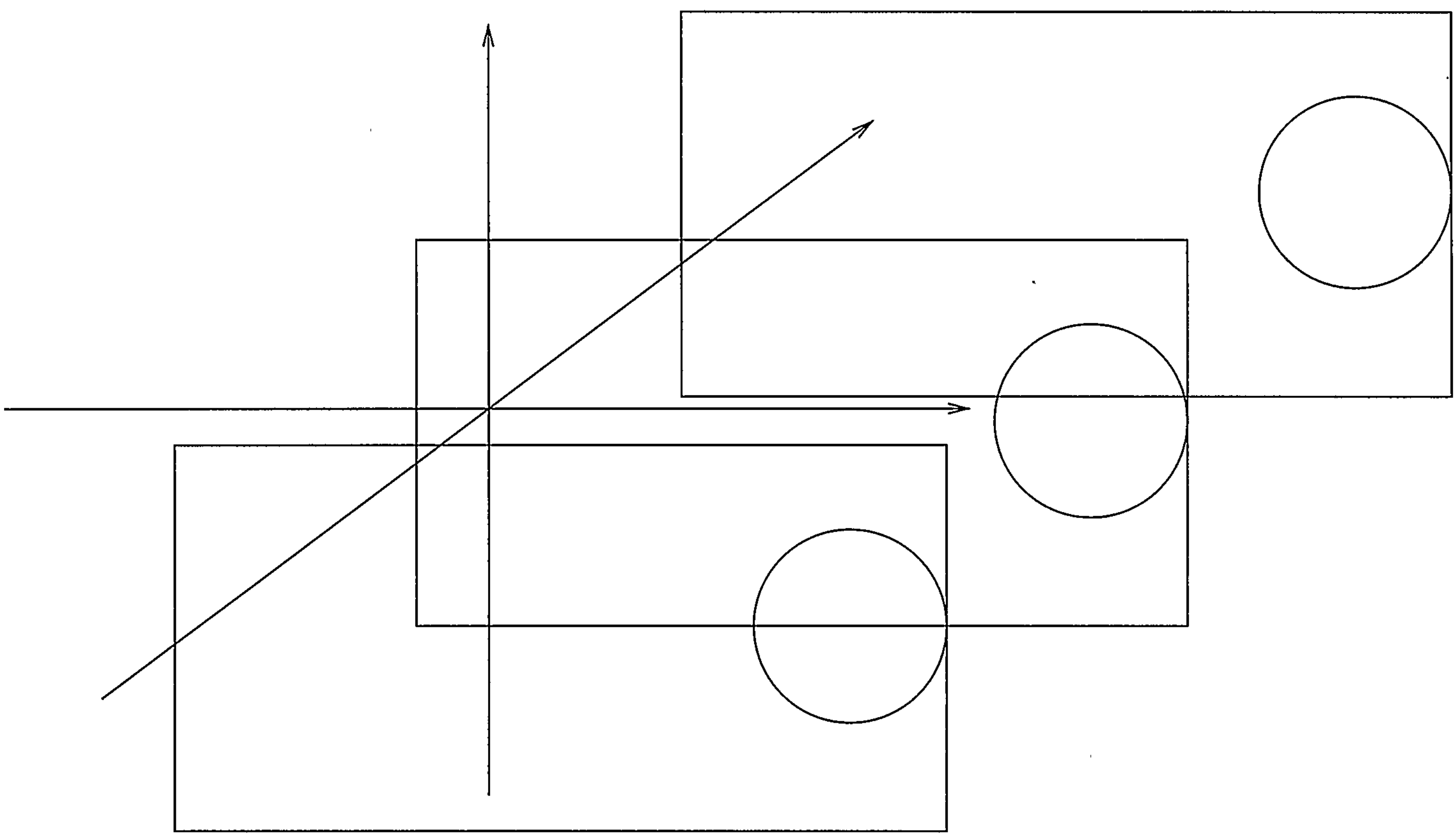


Figure 8.6: An illustration of Corollary 2 in $\mathbb{E}^3$.

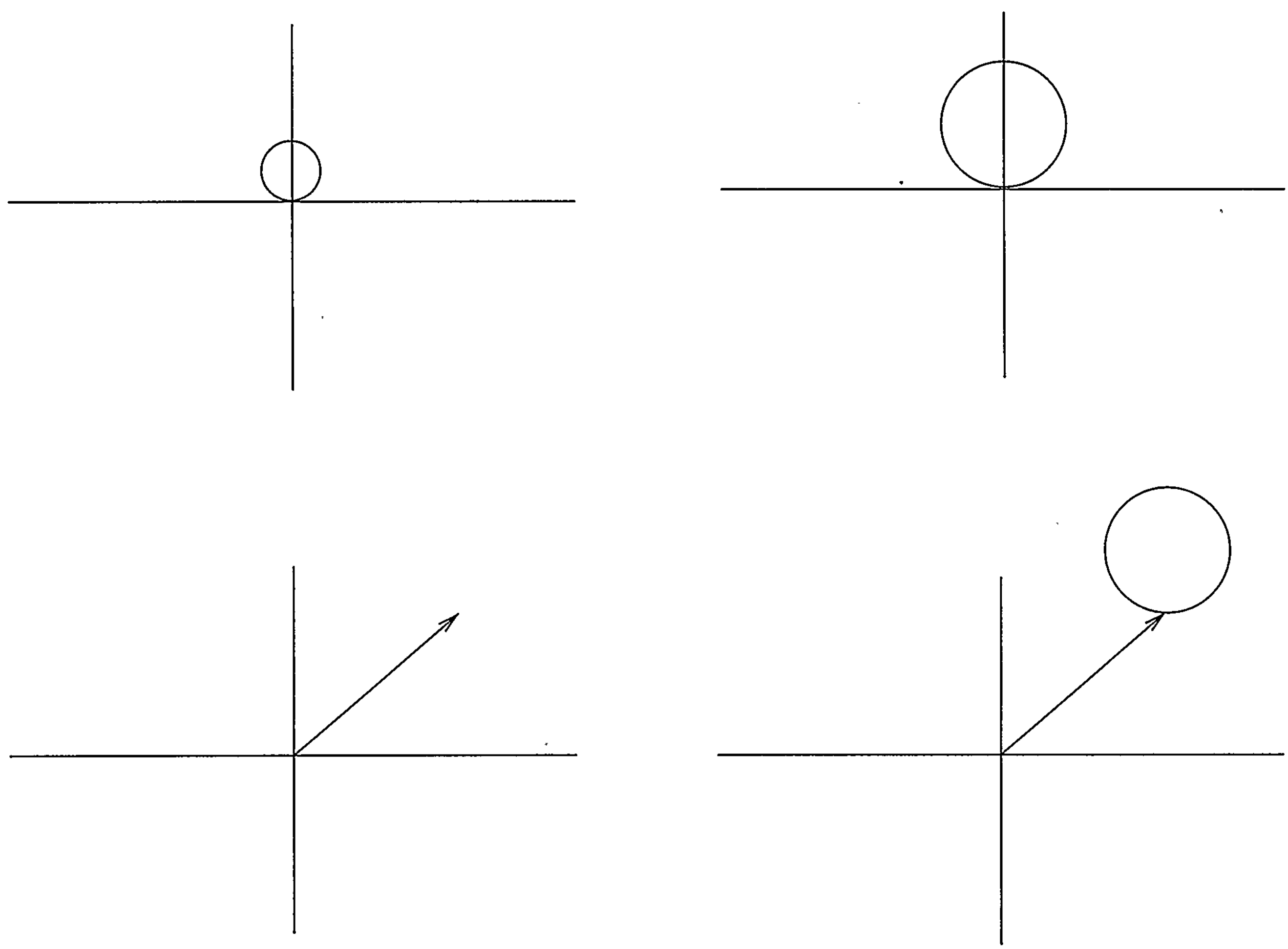


Figure 8.7: Similar sets.

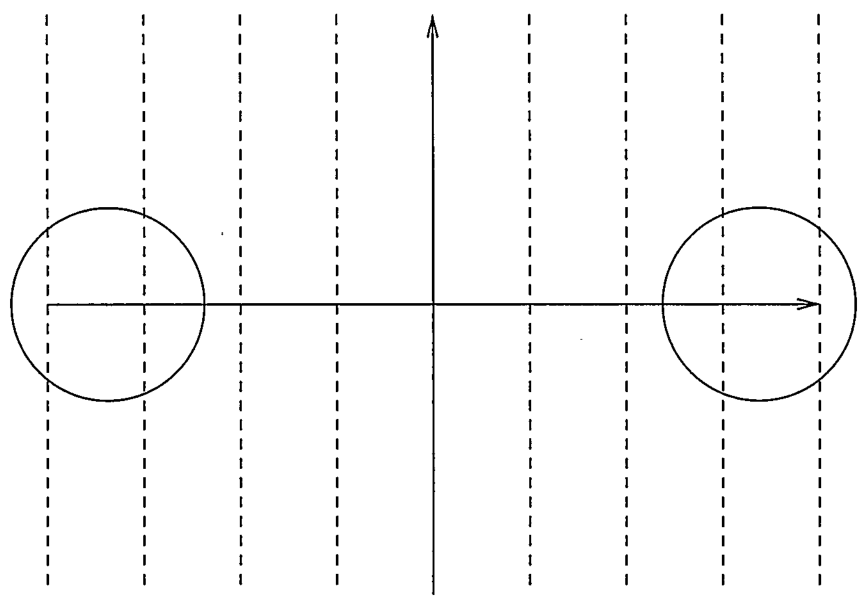


Figure 8.8: These two circles are 2-thin. After doubling their radii their intersection will remain empty.

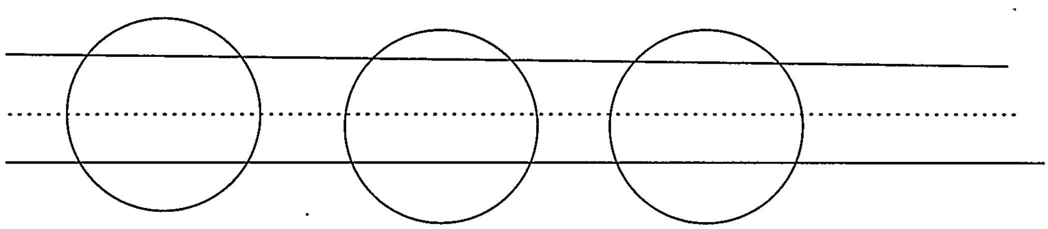


Figure 8.9: A 3-simple family.

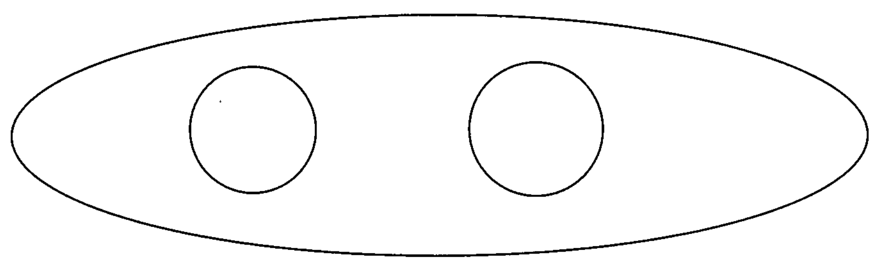


Figure 8.10: Illustration of a set that is NOT simply connected with bounded components in the complement.

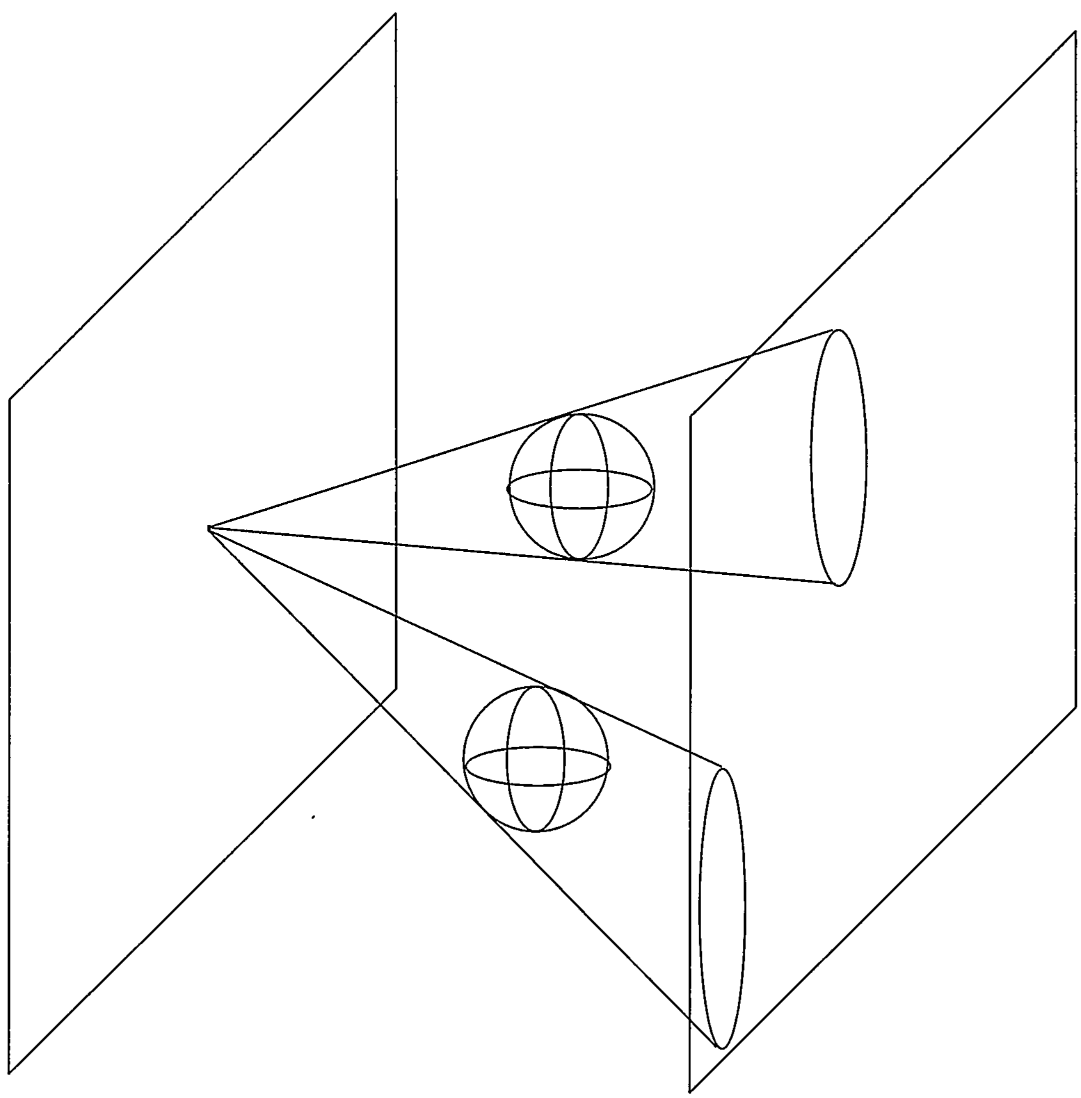


Figure 8.11: $D_i \cap D_j = \emptyset \Rightarrow \exists$ a plane through x which separates K_i and K_j.

Chapter 9

"Thin Sets and Common Transversals"

9.1 Introduction

So far, in our study of transversals, we have looked at lines meeting convex sets. An interesting way to generalize the problem might be to examine planes and hyperplanes meeting convex sets. Here we ask: what conditions must be imposed on a family of compact, convex sets, so that it is met by some hyperplane? This is what is examined in this paper [16].

A convex set in $\mathbb{E}^n$, $n \geq 2$ is called a *thin set* if its dimension is equal to $n-1$. For example, lines and line segments are thin sets in $\mathbb{E}^2$. Suppose $\mathfrak{A}$ is a family of compact convex sets in $\mathbb{E}^n$. A *transversal m-flat of* $\mathfrak{A}$ is an m dimensional affine space that intersects each member of $\mathfrak{A}$. For example, if $\mathfrak{A}$ is a family of compact, convex sets in $\mathbb{E}^2$ that has a transversal 1-flat then that is the same as saying there is a transversal or a line that meets all of the members of $\mathfrak{A}$ (cf. Figure 9.1). A family of compact, convex sets in $\mathbb{E}^3$ that has a transversal 2-flat is a family where each member is met by a two dimensional affine space which is just a plane (cf. Figure 9.2).

Suppose that $\mathfrak{A}$ is a family of compact, convex sets, in $\mathbb{E}^n$, which can be linearly ordered in such a way that each subfamily of $\mathfrak{A}$ with k members has a transversal 1-flat that meets the k members in the specified order. In this instance, we say that $\mathfrak{A}$ satisfies Property $O(k)$. If $\mathfrak{A}$ is a family of compact, convex sets that has a transversal m-flat then we say that $\mathfrak{A}$ satisfies Property T_m. Now, it is clear that a family satisfies Property T_1 if and only if it satisfies Property T.

9.2 Discussion

Theorem 13 *For any finite family of disjoint, compact, convex sets in $\mathbb{E}^n$, if the family satisfies Property $O(3)$ then it satisfies Property T_{n-1}.*

Proof.

Suppose that $\mathfrak{A} = \{A_1, A_2, \ldots A_t\}$ is a family of disjoint compact convex sets that is ordered in accordance with Property $O(3)$. Figure 9.3 illustrates what happens when $A_j \cap conv(A_i \cup A_k) = \emptyset$ in $\mathbb{E}^2$, $i \leq j \leq k$. It is clear from the figure that no line meets the three sets A_i, A_j, A_k. Since any three sets A_i, A_j, A_k in $\mathfrak{A}$, where $i \leq j \leq k$, are met by a line, it follows that $A_j \cap conv(A_i \cup A_k) \neq \emptyset$.

We now employ a procedure that is common in the study of transversals and that has been demonstrated before. For each i, where $1 \leq i \leq t$, we fix a point $x_i \in A_i$ and contract A_i about the point x_i by a factor of λ, where $0 \leq \lambda \leq 1$. In particular, $A_i^\lambda = \{x_i + \lambda(x - x_i) : x \in A_i\}$ is the set A_i after it has been contracted by λ about x_i. Let $\mathfrak{A}^\lambda = \{A_1^\lambda, A_2^\lambda, \ldots, A_t^\lambda\}$.

Let $\beta = inf\{\lambda : 0 \leq \lambda \leq 1$ and $A_j^\lambda \cap conv(A_i^\lambda \cup A_k^\lambda) \neq \emptyset$ for any i, j, k where $1 \leq i < j < k \leq t\}$. We note that there exist integers x, y, z where $1 \leq x < y < z \leq t$ such that $A_y^\lambda \cap conv(A_x^\lambda \cup A_z^\lambda) = \emptyset$ for all λ where $0 \leq \lambda < \beta \leq 1$. To see this, observe that the members of $\mathfrak{A}$ are compact, convex sets from which it follows that $A_i \in \mathfrak{A}$ is closed for each i, $1 \leq i \leq t$. So, given A_i, A_j, A_k in $\mathfrak{A}$, there is a maximal $\lambda_{i,j,k}$ so that $A_j^\lambda \cap conv(A_i^\lambda \cup A_k^\lambda) = \emptyset$ for all λ where $0 \leq \lambda < \lambda_{i,j,k} \leq 1$. In other words, we contract the closed sets until they are separated and supported by a hyperplane. In $\mathbb{E}^2$, this is referred to as the Hadwiger Shrinking Process and in higher dimensions the existence of a separating and supporting hyperplane is ensured by results in Grünbaum's classic work on convex polytopes. Clearly, $\beta = max\{\lambda_{i,j,k} : 1 \leq i < j < k \leq t\}$. Thus, there exist integers x, y, z where $1 \leq x < y < z \leq t$ such that $A_y^\lambda \cap conv(A_x^\lambda \cup A_z^\lambda) = \emptyset$ for all λ where $0 \leq \lambda < \beta \leq 1$. And when $\lambda = 0$ we obtain $1 \leq x < y < z \leq t$ such that $A_y^\lambda \cap conv(A_x^\lambda \cup A_z^\lambda) \neq \emptyset$ and $A_y^\lambda)$ is separated from $A_x^\lambda)$ and $A_z^\lambda)$.

Previously, when carrying out the aforementioned shrinking process, we observed that the process does not alter the properties a family possesses. For example, we have seen results in $\mathbb{E}^2$ where a family continues to exhibit critical properties, such as $T(k)$, after the shrinking process has been carried out. However, as a result of the shrinking process, a unique transversal has been obtained for some subfamily of the original family. Consequently, in order for $T(k)$ to hold, the entire family must meet the unique transversal and we conclude that the family satisfies property T. In this case, even after the shrinking process has been carried out, the family $\mathfrak{A}^\beta$ continues to satisfy Property $O(3)$ because of the way β was chosen. To conclude the proof, we show that A_x^β, A_y^β and A_z^β meet some hyperplane, that the remaining members of $\mathfrak{A}^\beta$ must also meet. We do so by considering two cases.

CASE I. $aff(A_y^\beta \cup conv(A_x^\beta \cup A_z^\beta)) \neq \mathbb{E}^n$.

In this case, $conv(A_x^\beta \cup A_z^\beta)$ is contained in some hyperplane, call it H. Now, $aff(conv(A_x^\beta \cup A_z^\beta) = aff(A_x^\beta \cup A_z^\beta)$ and so $aff(A_x^\beta \cup A_z^\beta)$ is contained in H. Since, $\mathfrak{A}^\beta$ satisfies $O(3)$ all members of $\mathfrak{A}^\beta$ meet $aff(A_x^\beta \cup A_z^\beta)$ which is contained in H. Consequently, $\mathfrak{A}^\beta$ has an transversal $(n-1)$-flat. As $A_i^\beta \subseteq A_i$, for each $i, 1 \leq i \leq t$, it follows that $\mathfrak{A}$ satisfies Property T_{n-1}.

CASE II. $aff(A_y^\beta \cup conv(A_x^\beta \cup A_z^\beta)) = \mathbb{E}^n$.

As discussed earlier, there exists a hyperplane H that separates A_y^β from A_x^β and A_z^β. It is clear that H supports A_x^β, A_y^β and A_z^β. Let H^+ be the closed half space determined by H containing A_y^β. Let H^- be the closed half space determined by H containing A_x^β and A_z^β. Figure 9.4 illustrates these concepts in $\mathbb{E}^2$.

We claim that H intersects A_w^β, for any $w \in \{1, 2, \ldots t\} \setminus \{x, y, z\}$. We examine the case $1 \leq w < x$. The other cases $x < w < y$, $y < w < z$ and $z < w < t$ are entirely analogous. Since, $A_x^\beta \cap conv(A_w^\beta \cup A_y^\beta) \neq \emptyset$ we obtain $A_w^\beta \cap H^- \neq \emptyset$. Similarly, $A_y^\beta \cap conv(A_w^\beta \cup A_z^\beta) \neq \emptyset$ yields $A_w^\beta \cap H^+ \neq \emptyset$. Thus, $A_w^\beta \cap H \neq \emptyset$. Therefore, $\mathfrak{A}^\beta$ is met by an transversal $(n-1)$-flat. As $A_i^\beta \subseteq A_i$, for each $i, 1 \leq i \leq t$ it follows that $\mathfrak{A}$ satisfies Property T_{n-1}.

■

Corollary 4 Hadwiger's Theorem. *If a finite family of disjoint compact convex sets in $\mathbb{E}^2$ can be linearly ordered in such a way that each subfamily consisting of three members admits a transversal intersecting the members in the specified order, then the family satisfies Property T.*

Proof. Apply Theorem 13 with $n = 2$. ■

Theorem 14 *Let $\mathfrak{A}$ be a finite family of compact, convex, thin sets in $\mathbb{E}^n$. If any three members admit a transversal 1-flat and if for any two members of $\mathfrak{A}$, say A_1 and A_2, we have $A_1 \cap aff(A_2) = \emptyset = A_2 \cap aff(A_1)$ then $\mathfrak{A}$ admits a transversal $(n-1)$-flat.*

Proof. Observe that, if it can be shown that $\mathfrak{A}$ satisfies $O(3)$ then by applying Theorem 13 the result follows immediately. Let $\mathfrak{A} = \{A_1, A_2, \ldots A_t\}$ and we induce an ordering on $\mathfrak{A}$. Before we do this, however, we make a critical observation. Given a subset of $\mathfrak{A}$, say $\mathfrak{A}'$, and some $A \in \mathfrak{A}'$, a *division* $X \cup Y$ of $\mathfrak{A}' \setminus \{A\}$ is obtained by letting all the members of $\mathfrak{A}' \setminus \{A\}$ that lie on one side of the hyperplane $\mathit{aff}(A)$ be in X and the rest be in Y, whence $X \cup Y = \mathfrak{A}' \setminus \{A\}$.

Observe that this division is made possible by two assumptions. The first is that the members of $\mathfrak{A}$ are thin sets. Hence, given $A_i \in \mathfrak{A}$, there is a hyperplane that contains A_i. Secondly, no other member of $\mathfrak{A}$ intersects that same hyperplane because of the assumption that for any two members of $\mathfrak{A}$, say A_1 and A_2, we have $A_1 \cap \mathit{aff}(A_2) = \emptyset = A_2 \cap \mathit{aff}(A_1)$.

Now, define $A_1 < A_2$ and write $\mathfrak{A}_m = \{A_1, A_2, \ldots A_m\}, m \geq 2$. We assume that $\mathfrak{A}_m$ has been linearly ordered, so that for each $i, 1 \leq i \leq m$ if $X \cup Y = \mathfrak{A}_m \setminus \{A_i\}$ is the division described above then it satisfies the following condition: *All of the members of X are smaller than A_i and lie on one side of the hyperplane $\mathit{aff}(A_i)$, which we call A_i^- and all the members of Y are larger than A_i and lie on the other side of the hyperplane $\mathit{aff}(A_i)$, which we call A_i^+.* It is easy to see that this condition produces an ordering of $\mathfrak{A}_m$ that is transitive and that under this ordering $\mathfrak{A}_m$ is $O(3)$.

Now, given A_{m+1} and $1 \leq i \leq m$, define $A_{m+1} < A_i$ if $A_{m+1} \subset A_i^-$ and $A_{m+1} > A_i$ if $A_{m+1} \subset A_i^+$ and let $\mathfrak{A}_{m+1} = \mathfrak{A}_m \cup \{A_{m+1}\}$. If $A_i, A_j, A_k \in \mathfrak{A}_{m+1}$ and $A_i < A_j$ and $A_j < A_k$, but $A_k \leq A_i$ then A_k lies in a region of $\mathbb{E}^n$ bounded by $\mathit{aff}(A_i)$ and $\mathit{aff}(A_j)$ that does not intersect either A_i nor A_j (Figure 9.5). However, this cannot occur because A_i would be bounded away from A_i and A_j in such a way that no line could meet all three sets, contrary to the assumptions of the theorem. Thus, $A_i < A_j$ and $A_j < A_k$ imply $A_i < A_k$ and so the ordering is transitive, which means $\mathfrak{A}_{m+1}$ is linearly ordered. Furthermore, any transversal 1-flat that meets any three sets of $\mathfrak{A}_{m+1}$, say A_i, A_j, A_k where $A_i < A_j < A_k$, must meet them in the specified order because A_i and A_j lie on different sides of $aff(A_j)$, whence $\mathfrak{A}_{m+1}$ is $O(3)$. Hence, by induction $\mathfrak{A}$ satisfies Property $O(3)$. ■

Corollary 5 Santaló's Theorem. *If every three members of a finite family of parallel line segments in the plane admit a transversal then the family admits a transversal.*

Proof. Apply Theorem 14 with $n = 2$ ■

Observe that we can make an even stronger statement in $\mathbb{E}^2$ than Santaló's Theorem. Recall that two compact convex sets, A_1 and A_2, are mutually non-penetrating if $A_1 \cap aff(A_2) = \emptyset = A_2 \cap aff(A_1)$. By dropping the requirement that the segments be parallel and ask that they only be mutually non-penetrating, we have a stronger form of Santaló's Theorem.

9.3 Conclusion

In this chapter, well known and important results in the plane have been generalized to higher dimensions. The techniques employed were generalizations of already well known techniques that have been used to prove the planar cases of these important results. The higher dimensional version of Hadwiger's and Santalo's Theorems are interesting and exciting. Even more interesting is that the planar case of Santalo's Theorem, as proved here, allows us to drop the requirement that the segments be parallel and ask that they only be mutually non-penetrating.

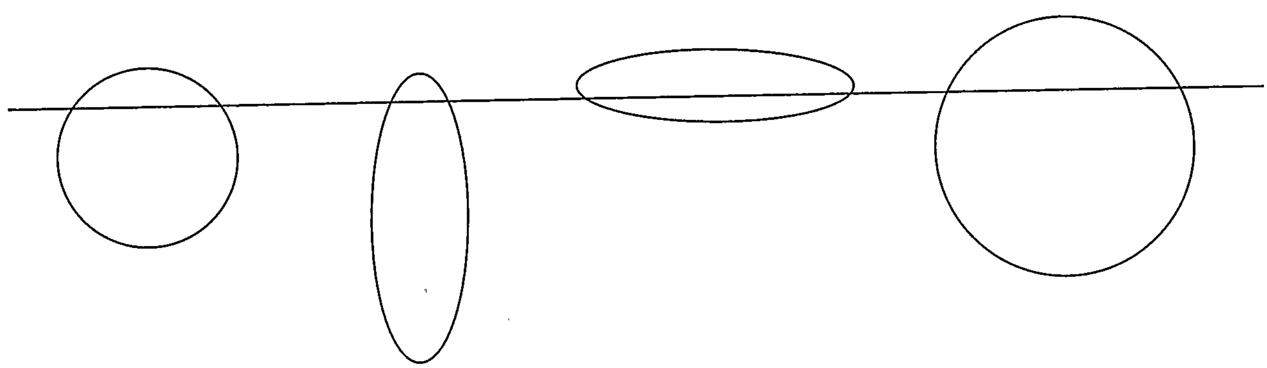


Figure 9.1: A family of sets in $\mathbb{E}^2$ that has a 1-transversal flat.

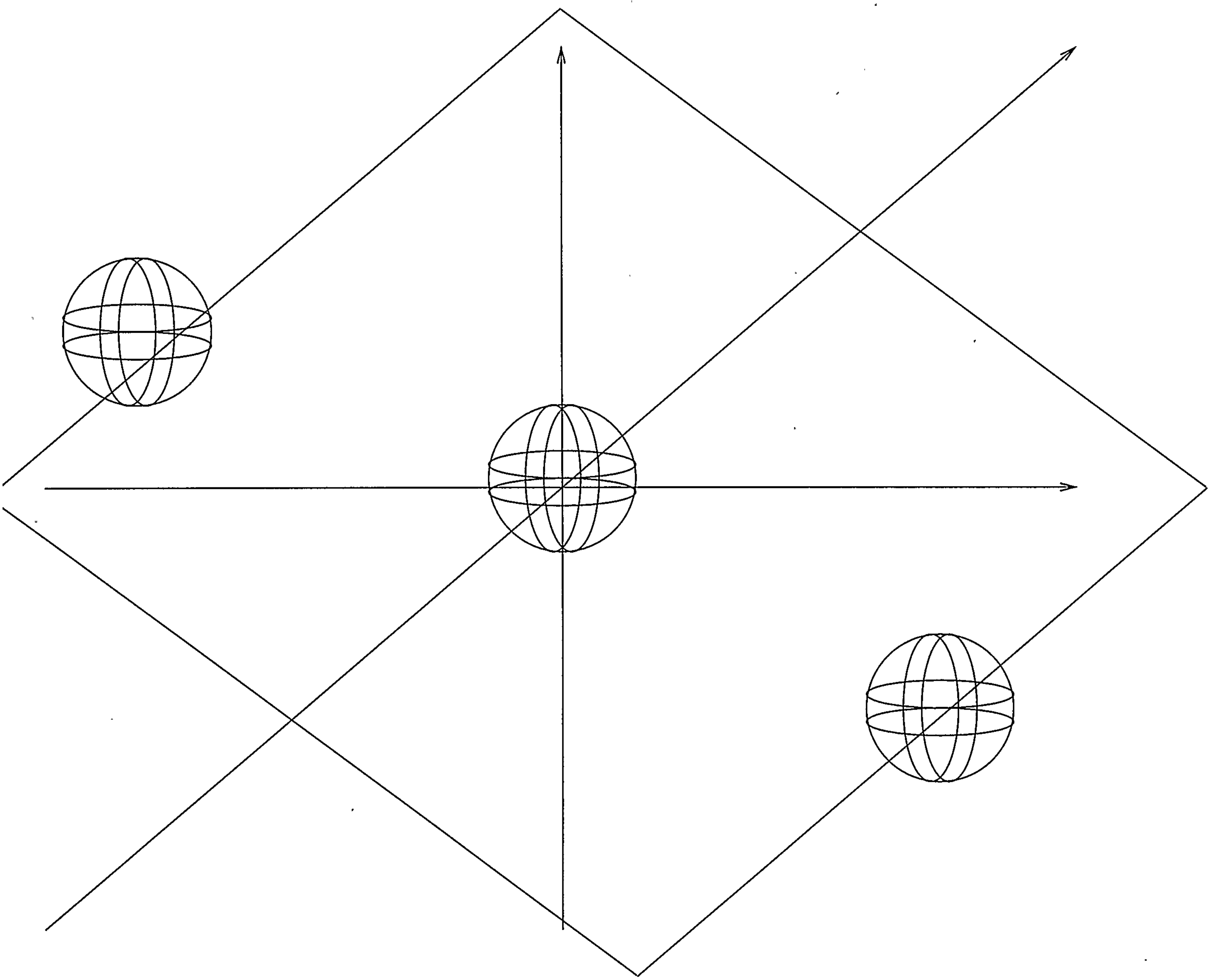


Figure 9.2: A family of sets in $\mathbb{E}^3$ that has a transversal 2-flat.

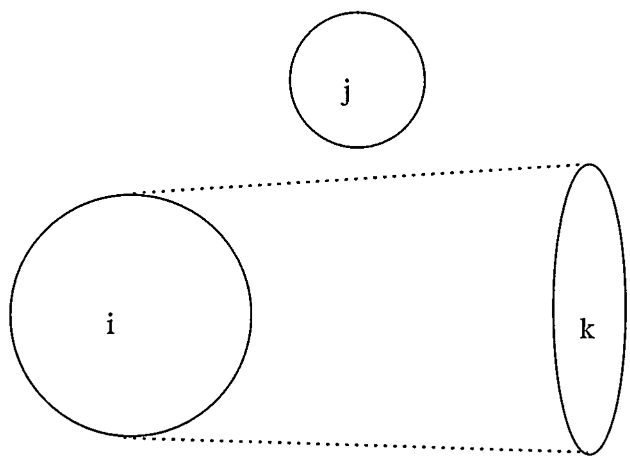


Figure 9.3: An illustration of what happens when $A_j \cap conv(A_i \cup A_k) = \emptyset$ in $\mathbb{E}^2$.

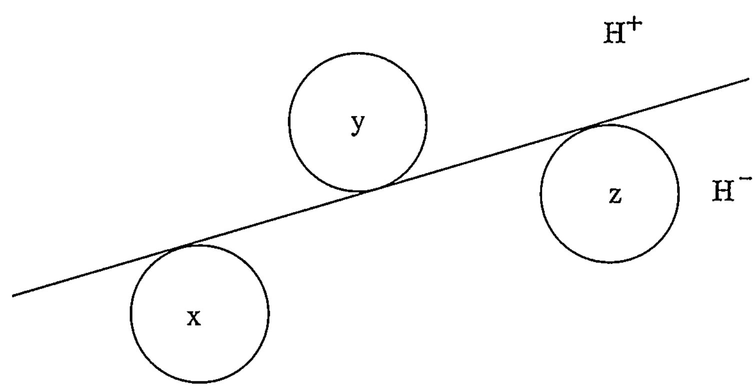


Figure 9.4: An illustration for Case II of Theorem 13. There exists a hyperplane H that separates A_y^β from A_x^β and A_z^β. It is clear that H supports A_x^β, A_y^β and A_z^β. Let H^+ be the closed half space determined by H containing A_y^β. Let H^- be the closed half space determined by H containing A_x^β and A_z^β.

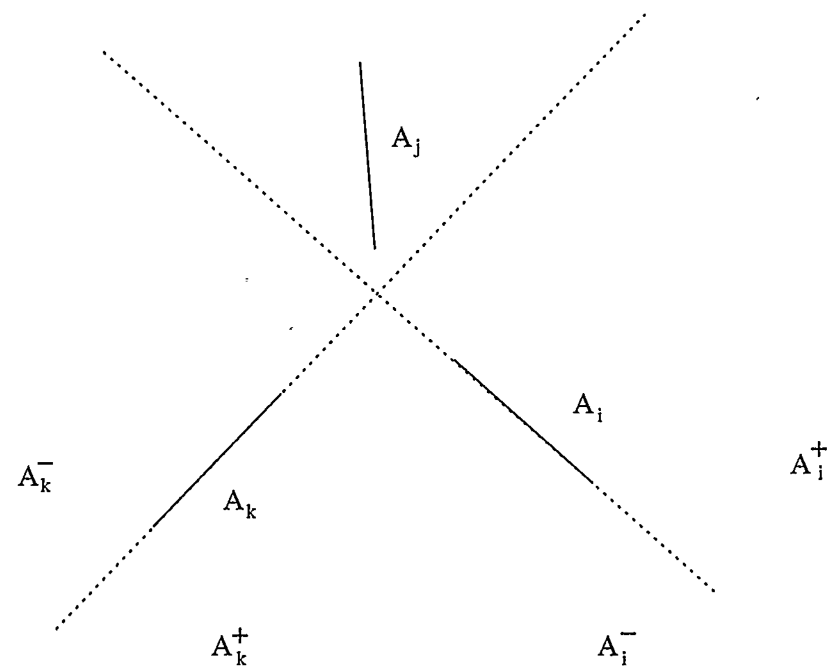


Figure 9.5: In the plane: $A_i, A_j, A_k \in \mathfrak{A}_{m+1}$ and $A_i < A_j$ and $A_j < A_k$, but $A_k \leq A_i$. Then, A_k lies in a region of $\mathbb{E}^2$ bounded by $aff(A_i)$ and $aff(A_j)$ that does not intersect either A_i nor A_j.

Chapter 10

"On the Helly Number for Hyperplane Transversals to Unit Balls."

10.1 Introduction

This article [1] provides a survey of major results in the study of transversals. In particular, three, important results are mentioned and proofs are given. We discuss each result in turn.

10.2 Discussion

Theorem 15 *For each integer $n \geq 6$, there exists a family of n pairwise disjoint unit discs in $\mathbb{E}^2$ such that any four have a common transversal, but some five do not.*

In other words, this result indicates that property $T(4)$ does not imply property T. This is an important result, because in [11], Grünbaum claims to have proved that $T(4)$ does imply T, for circles. This erroneous result has been cited and appealed to over the last forty years without question. The paper currently being discussed is the only work, known to the author, that has attempted to correct this result. Figure 10.1 shows an example of a family which proves Theorem 15 for the case $n = 6$. The other cases are easy to extrapolate.

Now, we introduce an interesting generalization of the notion of pairwise disjoint in the plane. We say that a family of compact, convex sets in $\mathbb{E}^d$, $1 \leq k \leq d$, is $(k-1)$-*separated* if no $k+1$ of the sets has a transversal $(k-1)$-flat. So, 0-separated is the same as pairwise disjoint. In the case of 1-separated, this is the same as the requirement that no three sets are met by a line. Recall that if a family satisfies Property T_n then the family is met by a transversal n flat. If a family satisfies Property $T_n(k)$ then every subfamily consisting of k members has a transversal n-flat which we call the *k-transversal n-flat* for those particular k members of the family.

Theorem 16 *If there exists a collection of n, $(d-2)$-separated unit balls in $\mathbb{E}^d$ for which Property $T_{d-1}(k)$ holds, but Property $T_{d-1}(k+1)$ does not hold, then there is a family of $n+1$, $(d-1)$-separated unit balls in $\mathbb{E}^{d+1}$ for which Property $T_d(k+1)$ holds, but Property $T_d(k+2)$ does not hold.*

Proof. Let $\mathfrak{A} = \{G_1, \ldots, G_n\}$ be a family of $(d-2)$-separated unit balls in $\mathbb{E}^d$ for which Property $T_{d-1}(k)$ holds, but Property $T_{d-1}(k+1)$ does not hold. Without loss of generality, we may assume that any k members of the family are met in the interior by some hyperplane. This assumption follows from the fact that we may enlarge the members of the family without damaging any of the existing conditions and properties the family exhibits. We now construct the required family $\mathfrak{A}' = \{D_1, \ldots, D_n, D_{n+1}\}$ of $(d-1)$-separated unit balls in $\mathbb{E}^{d+1}$ for which Property $T_d(k+1)$ holds, but Property $T_d(k+2)$ does not hold. Embed $\mathbb{E}^d$ in $\mathbb{E}^{d+1}$ and denote the hyperplane $\mathbb{E}^d$ in $\mathbb{E}^{d+1}$ by H_0. If $(x_{i1}, x_{i2}, \ldots, x_{id})$ is the center of G_i in $\mathbb{E}^d$ then let D_i be the unit ball with center $(x_{i1}, x_{i2}, \ldots, x_{id}, y_i)$ and D_{n+1} be the unit ball with center $(x_{n1}, x_{n2}, \ldots, x_{nd}, y_{n+1})$ where each y_i is yet to be determined. Figure 10.2

demonstrates these notions when $d = 3$. From the diagram, it is easy to see that G_i is simply the projection of D_i onto H_0. Observe that both D_n and D_{n+1} project onto G_n in H_0.

First, we choose $y_1, y_2, \ldots, y_n, y_{n+1}$ so that $y_1 < y_2 < \ldots < y_n < y_{n+1}$. Second, we show that $y_1, y_2, \ldots, y_n, y_{n+1}$ can be chosen so that $\mathfrak{A}'$ is $(d-1)$-separated. Without loss of generality, let $D_1, \ldots, D_{d+1}$ be an arbitrary subfamily of $\mathfrak{A}'$. It is clear that y_{d+1} can be chosen, sufficiently large, so that it lies above all hyperplanes meeting $D_1, \ldots, D_{d+1}$. For if not then a standard compactness argument shows that there exists a limiting hyperplane, say H, that meets $D_1, \ldots, D_{d+1}$ such that $H \cap H_0$ is a transversal $(d-2)$-flat of $G_1, \ldots, G_{d+1}$; contrary to the assumption that $\mathfrak{A}$ is $(d-2)$-separated. Hence, we may choose $y_1, y_2, \ldots, y_n, y_{n+1}$ so that $\mathfrak{A}'$ is $(d-1)$-separated.

Third the family $\mathfrak{A}'$ satisfies Property $T_d(k+1)$. Given any $k+1$, members of $\mathfrak{A}'$ the property follows trivially if D_n and D_{n+1} are among them. Simply project these members of $\mathfrak{A}'$ onto their corresponding members of $\mathfrak{A}$ in H_0. Then, by Property $T_{d-1}(k)$ there exists a k-transversal $(d-1)$-flat which meets G_n and the other $k-1$ members of $\mathfrak{A}$ that are projections of members of $\mathfrak{A}'$. Finally, take the $(k+1)$-transversal d-flat that contains the aforementioned k-transversal $(d-1)$-flat and we are done. Next, without loss of generality, suppose that $D_1, \ldots, D_{k+1}$ is an arbitrary subfamily of $\mathfrak{A}$ that contains at most one of D_n or D_{n+1}. By Property $T_{d-1}(k)$, there exists a k-transversal $(d-1)$-flat which meets $G_1, \ldots, G_k$ and we can find a hyperplane, say H, in $\mathbb{E}^{d+1}$ containing H and meeting $D_1, \ldots, D_k$. Now, by tilting H appropriately and choosing y_{k+1} sufficiently large, H will also meet D_{k+1} yielding the desired result.

Finally, the family does not satisfy Property $T_d(k+2)$. Without loss of generality, we choose $G_1, \dots, G_{k+1}$ having no transversal $(d-1)$-flat. If y_{n+1} is chosen sufficiently large then $D_1, \dots, D_{k+1}, D_{n+1}$ has no transversal d-flat. The proof of this is similar to the preceding argument showing $\mathfrak{A}'$ is $(d-1)$-separated. ∎

The Helly number is the smallest integer k, so that if a family in $\mathbb{E}^d$ satisfies Property $T_{d-1}(k)$ then it also satisfies Property T_{d-1}. The preceding two Theorems yield the following corollary.

Corollary 6 *The Helly number for hyperplane transversals to families of $d-2$ or more $(d-2)$-separated unit balls in $\mathbb{E}^d$ is at least $d+3$.*

Theorem 17 Danzer's Theorem. *Given $n \geq 5$ pairwise disjoint unit discs in the plane, if any five of the discs have a common transversal then the whole family has a common transversal.*

Recall that Grünbaum conjectured and Tverberg verified that Property $T(5)$ implies Property T for a family of disjoint translates. Theorem 17 is simply a special case of this result. The authors claim to have derived a proof which is independent of Danzer's proof. Essentially, the authors mimic Tverberg's proof of the Grünbaum Conjecture from [23].

They assume that a counterexample exists and show, as Tverberg did, that there is a reduction to a family consisting of six circles, the centers of the circles are convexly independent and no three centers are collinear. Recall that, after Tverberg completed this part of the proof, he showed that such a family cannot exist. This was achieved through a somewhat tedious computational process where all possible combinations of geometric permutations the family could exhibit were checked and eliminated. The authors do the same thing, but by citing a previous work done by one of the authors on geometric permutations, the number of cases that need to be checked is reduced drastically.

10.3 Conclusion

One final note regarding this paper is that an interesting conjecture is made in the introductory section. The authors conjectured that: *for every $d > 2$ there exists an integer k_d such that for families of $(d-2)$-separated families of unit balls in $\mathbb{E}^d$ Property $T_{d-1}(k_d)$ implies Property T_{d-1}.* There is no indication of how to produce a proof for this conjecture, but the authors cite recent work in this area that suggests such a conjecture is plausible.

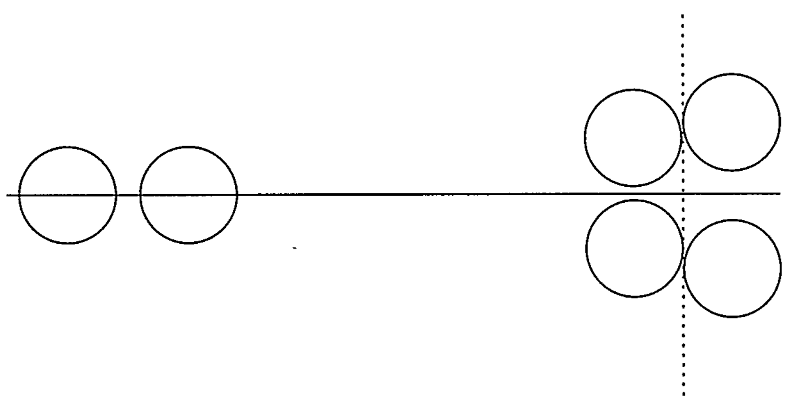


Figure 10.1: An example of Theorem 15 for the case $n = 6$. The centers are $(0,0), (3,0)(10, 1+\epsilon^2), (10, -1-\epsilon^2), (12, 1+\epsilon), (12, -1-\epsilon)$. If we choose $\epsilon > 0$ sufficiently small then the example works. In the diagram, the choice of ϵ has been exaggerated for clarity.

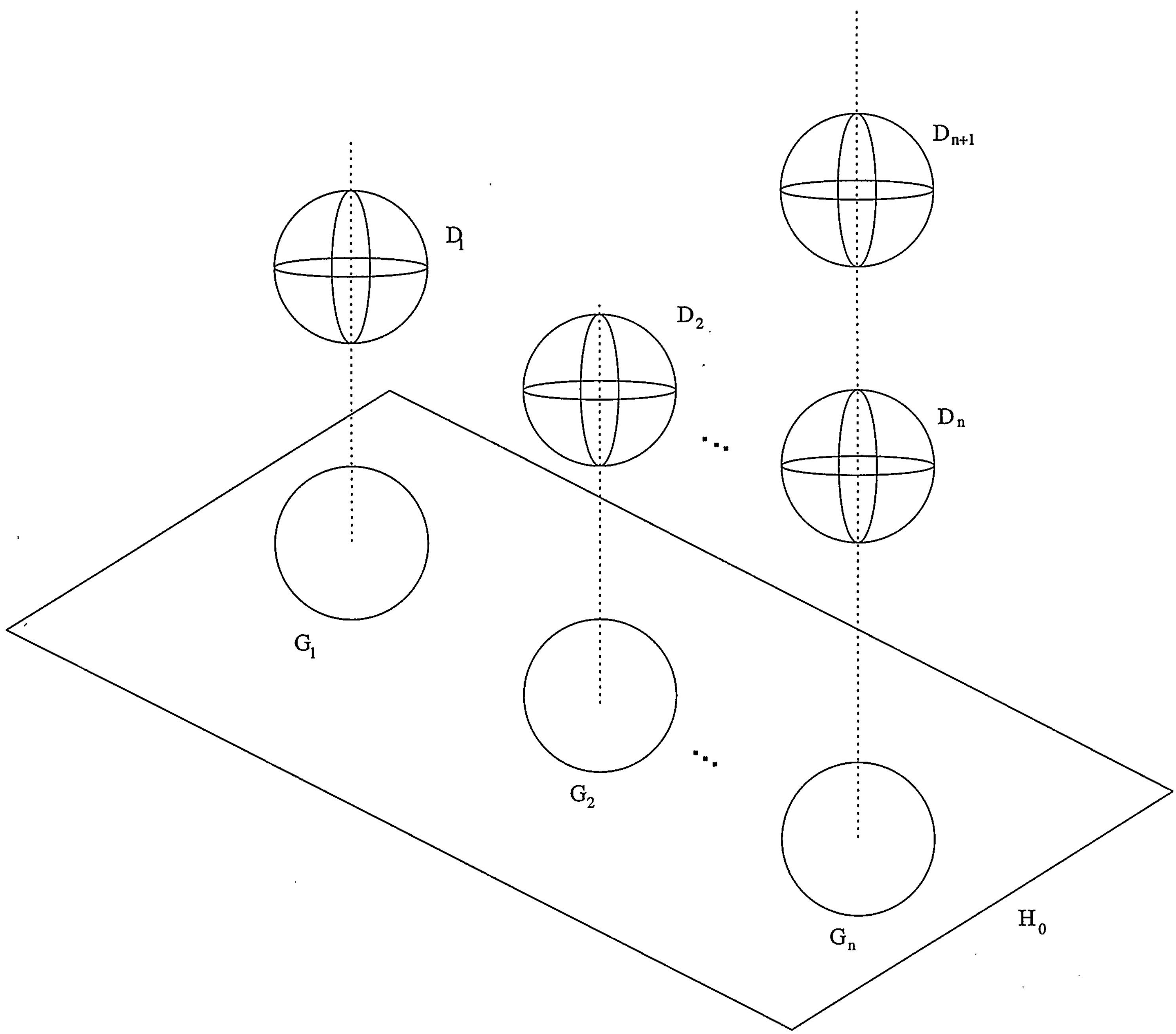


Figure 10.2: An illustration for Theorem 16. For each i, G_i is simply the projection of D_i onto H_0.

Chapter 11

"Cutting Families of Convex Sets"

11.1 Introduction

In this chapter, we introduce a new transversal property and discuss a few results related to this property. The discussion is based on material found in [18]. A family $\mathfrak{A}$, of sets in the plane $\mathbb{E}^2$, has property $T-k, k \geq 0$, if there exists a straight line intersecting all but at most k members of $\mathfrak{A}$. The main result of this chapter shows the existence of some integer k, such that if one has a family of pairwise disjoint translates exhibiting property $T(3)$ then it also has property $T-k$. It should be noted, in advance, that the k considered in this paper is universal for all families of pairwise disjoint translates. In this chapter, we outline the major proofs. We discuss other transversal properties in greater detail later; our goal here is to gain a familiarity with this particular transversal property.

11.2 Discussion

Lemma 11 *Let $\{A_1, B, A_2\}$ be a family of rectangles satisfying the following conditions:*

(L1) the edges are of length no greater than r and parallel to the coordinate axes;

(L2) the distance between each two rectangles of the family is greater than r;

(L3) the horizontal axis h intersects the three rectangles in the order A_1, B, A_2 and separates $A_1 \cup A_2$ from B (cf Figure 11.1).

If D is any rectangle satisfying (L1) and the family $\{A_1, B, A_2, D\}$ satisfies (L2) and also has property $T(3)$ then D intersects h.

Proof. Referring to Figure 11.1, suppose that D lies strictly above h. If D does not intersect the vertical strip generated by extending the edges of B, which are perpendicular to h, then one of $\{A_1, B, D\}$ or $\{A_2, B, D\}$ fails to have a transversal, contrary to the assumption that $\{A_1, B, A_2, D\}$ is $T(3)$; or D is within a distance of r to one of the other three sets, contrary to the assumption that $\{A_1, B, A_2, D\}$ satisfies (L2).

If D lies above the line generated by extending the upper edge of A_2 then $\{A_2, B, D\}$ fails to have a transversal. Hence, D intersects Y, the rectangular region generated by extending the edges of B, which are perpendicular to h, and the upper edge of A_2. Consequently, the distance between D and B is less than r, contrary to the assumption that $\{A_1, B, A_2, D\}$ satisfies (L2). Thus, D cannot lie strictly above h. An analogous argument shows that D cannot lie strictly below h and so we have that D intersects h. ■

Theorem 18 *Let $\mathfrak{A}$ be a family of compact convex sets in the plane and suppose that each member of $\mathfrak{A}$ has a diameter no greater than $r > 0$. If $\mathfrak{A}$ has property $T(3)$ then there exist three discs of radius $3r$ such that there is a common transversal for all members of $\mathfrak{A}$ which do not intersect any of the discs.*

Proof. Let $\mathfrak{A} = \{C(\gamma) : \gamma \in \Gamma\}$ be a family of compact convex sets where each set has diameter no greater than $r > 0$ and the family satisfies the property $T(3)$. For each $\gamma \in \Gamma$, choose a point $x(\gamma) \in C(\gamma)$ and contract $C(\gamma)$ about the point $x(\gamma)$ by a factor of $\lambda \in [0,1]$. Let λ_0 be the minimum value of λ, such that for the contracted family, $\mathfrak{A}' = \{C'(\gamma) : \gamma \in \Gamma\}$, there is a common transversal for every three members $C'(\gamma_1), C'(\gamma_2), C'(\gamma_3)$, whenever $dist(C(\gamma_i), C(\gamma_k)) > (1+\sqrt{2})r, 1 \leq i \neq j \leq 3$. First, consider the case where no three members of $\mathfrak{A}'$ are mutually separated by a distance of $(1+\sqrt{2})r$. In this case, there are two further possibilities: no members are separated by a distance of at least $(1+\sqrt{2})r$ or there is some pair separated by a distance of at least $(1+\sqrt{2})r$.

Theorem 19 Jung's Theorem. *If a compact, convex set has diameter less than r it is contained in a circle of radius no greater than $r/\sqrt{3}$.*

If all members of $\mathfrak{A}'$ are within a distance of $(1+\sqrt{2})r$ of each other then choose any two members of $\mathfrak{A}'$ and appealing to Jung's Theorem yields that each of these two sets is contained in a disc of radius no greater than $r/\sqrt{3}$. The remaining members of $\mathfrak{A}'$ must be within a distance of $(1+\sqrt{2})r$ of at least one of these discs and so must be contained in at least one of two discs of radius $(1+\sqrt{2}+1/\sqrt{3})r < 3r$. Thus, in this case the theorem is trivially satisfied. Next, suppose that there is some pair separated by a distance of at least $(1+\sqrt{2})r$. It is easy to see that each of these sets will be contained in a square with edge length at most r. In turn, this square can be inscribed in a circle of radius at most $r/\sqrt{2}$ and the remaining sets in the family will be at most a distance of $(1+\sqrt{2})r$ from each of these circles.

All that is left is to examine the case where there are at least three members of $\mathfrak{A}'$ mutually separated by a distance of $(1+\sqrt{2})r$. By the above described shrinking process (Hadwiger's Shrinking Process), there exist three members of $\mathfrak{A}'$, call them

$C'(\gamma_1)$, $C'(\gamma_2)$ and $C'(\gamma_3)$, such that $C'(\gamma_2)$ is separated from $C'(\gamma_1)$ and $C'(\gamma_3)$ by a line h and h is tangent to $C'(\gamma_1)$, $C'(\gamma_2)$ and $C'(\gamma_3)$. Let A_1 be the smallest rectangle, with one pair of edges parallel to h, containing $C'(\gamma_1)$. Let B be the smallest rectangle, with one pair of edges parallel to h, containing $C'(\gamma_2)$. Let A_2 be the smallest rectangle, with one pair of edges parallel to h, containing $C'(\gamma_3)$.

It is clear that A_1, B and A_2 correspond to Lemma 11. Thus, any member of $\mathfrak{A}'$ that does not intersect one of the circles of radius $3r$ centered about $C'(\gamma_1)$, $C'(\gamma_2)$ and $C'(\gamma_3)$ meets the line h as a result of a straightforward application of Lemma 11. ■

Corollary 7 *Given $r > 0, \beta > 0$ and $n > 0$, there is some positive integer $l = l(r, \beta, n)$ such that $T(3)$ implies $T - l$ for any family $\mathfrak{A}$ of compact convex sets satisfying:*

(C1) each member of $\mathfrak{A}$ has diameter no greater than r;

(C2) for every n-membered subfamily of $\mathfrak{A}$, say $\{A_1, A_2, \dots, A_n\} \subseteq \mathfrak{A}$, the area $\bigcup_{i=1}^{n} A_i$ is no smaller than βn.

The corollary is an immediate consequence of Theorem 18.

Theorem 20 *There exists a positive integer k such that for any family of pairwise disjoint translates of a compact, convex set $T(3)$ implies $T - k$.*

Proof. Apply the corollary with $r = \sqrt{2}, \beta = \frac{1}{2}, n = 1$. ■

11.3 Conclusion

This completes our examination of the transversal property $T - k$. It should be noted that other literature places an upper bound of 128 on k. In this paper an upper bound of $48\pi r^2\beta^{-1} + (n-1)$ is placed, where r is the largest diameter of any member of the given family.

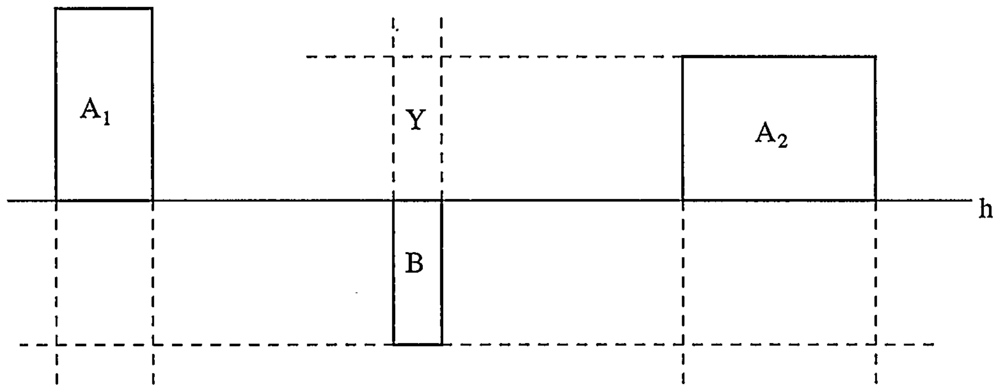


Figure 11.1: Illustration for Lemma 11.

Chapter 12

“An upper bound for families of linearly related plane convex sets.”

12.1 Introduction

In this chapter we explore a different type of problem related to transversals as described in [5]. Given a family $\mathfrak{A}$ of pairwise disjoint, compact, convex sets in $\mathbb{E}^2$, we recall that $\mathfrak{A}$ satisfies Property $T(n)$ provided that every subfamily of $\mathfrak{A}$, consisting of n members, has a transversal. It satisfies Property T if there exists a common transversal that intersects the entire family. If for any n members of the family $\mathfrak{A}$ there is a common transversal that meets no other members of the family then $\mathfrak{A}$ satisfies Property $G(n)$. The family $\mathfrak{A}$ satisfies Property $I(n)$ provided that any line meets at most n members of $\mathfrak{A}$. In this chapter we explore the Property $G(n)$ and discuss a theorem that shows that any family of compact, convex sets that satisfies the property $G(n), n \geq 3$, has cardinality at most $n + 46$.

12.2 The Result

Lemma 12 *Let J be a system of intervals on a line such that no point is covered by more than k members of J. Then $J = J_1 \cup J_2 \cup \ldots \cup J_k$ where $J_i \cap J_k = \emptyset$ and each J_i consists of pairwise disjoint intervals.*

The proof of this lemma is due to Hajós and Wiener. We use this result without proof.

Lemma 13 *Let $\mathfrak{A}^*$ be a family of compact, convex sets in $\mathbb{E}^2$ such that the members of $\mathfrak{A}^*$ are separated by parallel lines. Then, if the family satisfies Property $T(3)$, it satisfies Property T.*

This is equivalent to asking that the family be separated by one dimensional hyperplanes, namely lines, and the results in [12] apply.

Lemma 14 *Let $\mathfrak{A}$ be a family of pairwise disjoint compact convex sets of $\mathbb{E}^2$ where $|\mathfrak{A}| \geq 9$. Then, if the family satisfies Property $G(3)$ then it satisfies Property $I(7)$.*

Proof.

Suppose that the preceding statement is not true. Let p be a line that meets eight members of $\mathfrak{A}$, say $A_1, A_2, \dots, A_8$, in the given order up to reversal. We now introduce some notation:

$$H' = conv(\bigcup_{i=1}^{4} A_i)$$

$$H'' = conv(\bigcup_{j=5}^{8} A_j)$$

$$H^* = conv(\bigcup_{k=4}^{7} A_k)$$

$$p' = conv(p \cap (A_1 \cup A_4))$$

$$p'' = conv(p \cap (A_5 \cup A_8))$$

$$p^* = conv(p \cap (A_4 \cup A_7))$$

We make the following observations. First, H' is the convex hull of the first four members of $A_1, A_2, \dots, A_8$ and H'' is the convex hull of the latter four, while H^* is the convex hull of four specially selected members. Next, p' is a line segment lying along p that runs from A_1 to A_4, p'' is a line segment lying along p that runs from A_5 to A_8, and p^* is a line segment lying along p that runs from A_4 to A_7. Finally, it is clear that $p' \cap p'' = \emptyset$. (cf. Figure 12.1).

Claim 6 *There exists $A_i \subset int(H')$ for some $i \in \{1,2,3,4\}$ and there exists $A_j \subset int(H'')$ for some $j \in \{5,6,7,8\}$.*

Suppose that $A_i \not\subset int(H')$ for each $i \in \{1,2,3,4\}$. It is clear that $G(3)$ yields $A_i \cap bd(H')$ is connected for each $i = 1,2,3,4$ (cf. Figure 12.2). As a result, we obtain an orientation of the closed convex curve $bd(H')$ so that the sets A_1, A_2, A_3 and A_4 meet $bd(H')$ in one of the following cyclic orders: A_1, A_2, A_3, A_4, A_1, called $CO1$, or A_1, A_2, A_4, A_3, A_1, called $CO2$. Other orders are obtained through symmetry and some orders are not possible as demonstrated in Figure 12.3.

Suppose that $A_4 \subset int(H^*)$. Immediately we have that $A_4 \subset int(conv(A_t \cup A_m))$ for some $t < m$ where $t, m \in \{5,6,7\}$; an intuitive demonstration of this fact is given in Figure 12.4. From this we conclude that, any line which meets A_4, but meets neither A_t nor A_m, separates A_t and A_m. Now, in the case of $CO1$, because the family satisfies Property $G(3)$, we obtain a line q that meets A_2, A_4 and A_8 but does not meet any of the other sets in $\{A_1, A_2, \dots, A_8\}$. In particular q separates A_t and A_m, whence $q \cap p \in conv(p \cap (A_t \cup A_m)) \subseteq p''$. However, it is easy to check that our choice of q separates A_1 and A_3, whence $q \cap p \in conv(p \cap (A_1 \cup A_3)) \subseteq p'$. Since, p' and p'' are disjoint, we arrive at a contradiction. A similar argument for $CO2$ can be made by considering a line q which only meets A_1, A_4 and A_8. In this instance $q \cap p \in conv(p \cap (A_t \cup A_m)) \subseteq p''$ and $q \cap p \in conv(p \cap (A_2 \cup A_3)) \subseteq p'$ which is again a contradiction since $p' \cap p'' = \emptyset$. Thus, $A_4 \not\subset int(H^*)$.

Suppose that $A_t \subset int(H^*)$ for some $t \in \{5, 6, 7\}$. In the case of $CO1$, if we choose a line r that only meets A_1, A_3 and A_t, we immediately have $r \cap p \in p' \setminus A_4$ because r separates A_2 and A_4 and $r \cap p \in p^* \setminus A_4$ because $A_t \subset int(H^*)$. So the line r intersects the line p twice, a contradiction. An identical contradiction arises in the case of $CO2$ by considering the line r that intersects the sets,. and only the sets, A_2, A_3 and A_t. Hence, $A_t \not\subset int(H^*)$ for all $t \in \{5, 6, 7\}$.

So, $A_k \cap bd(H^*) \neq \emptyset$ for all $k \in \{4, 5, 6, 7\}$ and as before, we obtain an orientation of $bd(H^*)$ so that the sets A_4, A_5, A_6, A_7 meet the boundary in on of the two cyclic orders: A_4, A_5, A_6, A_7, A_1, $(CO3)$, and A_4, A_5, A_7, A_6, A_4, $(CO4)$. If $CO1$ and $CO3$ hold then a line meeting only A_2, A_4, A_6 intersects p twice since it will separate A_1 from A_3 and A_5 from A_7. Similarly, if $CO1$ and $CO4$ hold then a line meeting only A_2, A_4, A_7 intersects p twice; if $CO2$ and $CO3$ hold then a line meeting only A_1, A_4, A_6 intersects p twice; if $CO2$ and $CO4$ hold then a line meeting only A_1, A_4, A_7 intersects p twice. All of these contradictions indicate that $A_i \subset int(H')$ for some $i \in \{1, 2, 3, 4\}$. An analogous argument shows that $A_j \subset int(H'')$ for some $j \in \{5, 6, 7, 8\}$ Therefore, Claim 6 holds.

Let A_i and A_j be the sets in Claim 6 and let $A_q \in \mathfrak{A} \setminus \{A_1, A_2, \ldots, A_8\}$. Since the family satisfies Property $G(3)$, there is a line t which meets only the sets A_i, A_j, A_q. It is immediately clear that t intersects p' and p'' which is a contradiction, since p' and p'' are disjoint. Therefore, $\mathfrak{A}$ satisfies property $I(7)$.

■

Theorem 21 *Let $\mathfrak{A}$ be a family of pairwise disjoint, compact, convex sets with the*

Property $G(n), n \geq 3$. *Then* $|\mathfrak{A}| \leq n+46$.

Proof.

We proceed inductively. Let $n = 3$. For each $A \in \mathfrak{A}$, let $\mu(A)$ denote the orthogonal projection of A onto the fixed line μ and let $J = \{\mu(A) : A \in \mathfrak{A}\}$. First, observe that because the members of $\mathfrak{A}$ are convex, the members of J form a system of intervals. By Lemma 14, $\mathfrak{A}$ satisfies Property $I(7)$ whence no point on μ is covered by more than seven members of J. Thus, by Lemma 12, $J = J_1 \cup J_2 \cup \ldots \cup J_7$, where $J_i \cap J_k = \emptyset$, and each J_i consists of pairwise disjoint intervals. Second, the authors cite other work, which is found in the following chapter, showing that, under the assumption given $|\mathfrak{A}| < \infty$. Consequently, a straightforward geometric argument yields $\mu(A) = \mu(B)$ implies $A = B$, from which it is follows that $|\mathfrak{A}| = |J|$.

Clearly there is an integer k, $1 \leq k \leq 7$, such that $|J_k| \geq |J|/7 = |\mathfrak{A}|/7$. Let $F_k = \{A \in \mathfrak{A} : \mu(A) \in J_k\}$. It is easy to check that $|F_k| = |J_k|$. Since the members of F_k are separated by parallel lines we may apply Lemma 13 to show that F_k admits a common transversal. Furthermore, by Lemma 14, $|F_k| \leq 7$, otherwise Property $I(7)$ would fail. So, $|\mathfrak{A}|/7 \leq |J_k| = |F_k| \leq 7$ and we immediately have $|\mathfrak{A}| \leq 49 = 3 + 46 = n + 46, n = 3$. So, the base case holds.

Now, assume that for any family $\mathfrak{A}^*$ of pairwise disjoint, compact, convex sets with the Property $G(n)$, $|\mathfrak{A}^*| \leq n+46$. Next, suppose that $\mathfrak{A}$ is a family of pairwise disjoint, compact, convex sets with the Property $G(n+1)$. Let $A \in \mathfrak{A}$, we claim that $\mathfrak{A} \setminus \{A\}$ has Property $G(n)$. Given $\{A_1, A_2, \ldots, A_n\} \subseteq \mathfrak{A} \setminus \{A\}$, we observe that $\{A, A_1, A_2, \ldots, A_n\} \subseteq \mathfrak{A}$ has a transversal, l, that meets all of these $n+1$ sets and only these sets in $\mathfrak{A}$, because $\mathfrak{A}$ satisfies Property $G(n+1)$. Clearly, l is a transversal that meets all of the sets $\{A_1, A_2, \ldots, A_n\} \subseteq \mathfrak{A} \setminus \{A\}$ and only these sets in $\mathfrak{A} \setminus \{A\}$ and so $\mathfrak{A} \setminus \{A\}$ satisfies Property $G(n)$. By the induction hypothesis $|\mathfrak{A}| - 1 = |\mathfrak{A} \setminus \{A\}| \leq n + 46$, whence $|\mathfrak{A}| \leq (n+1) + 46$. ∎

12.3 Conclusion

This chapter briefly introduced the notions of Property $G(n)$ and Property $I(n)$. The methods utilized in the proofs in this chapter are standard and commonly employed when dealing with these properties. The final theorem is interesting and it is important, because whenever one is dealing with a family that satisfies Property $G(n)$, one immediately knows that the family is finite and an upper bound is known.

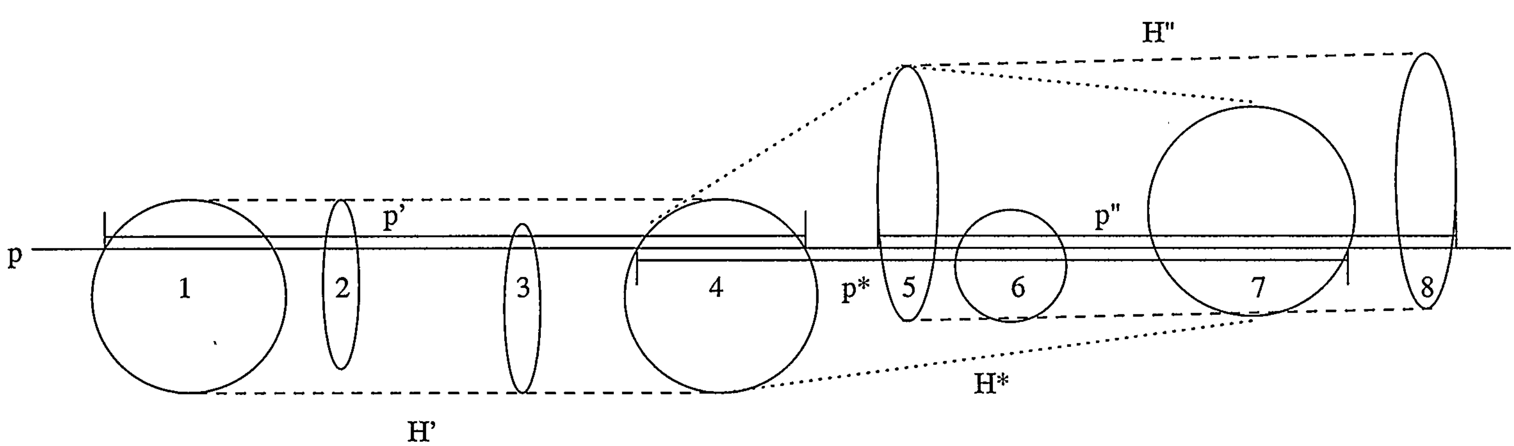


Figure 12.1: A schematic representation of H', H'', H^*, p', p'' and p^*. The family shown here need not satisfy any properties of Lemma 14.

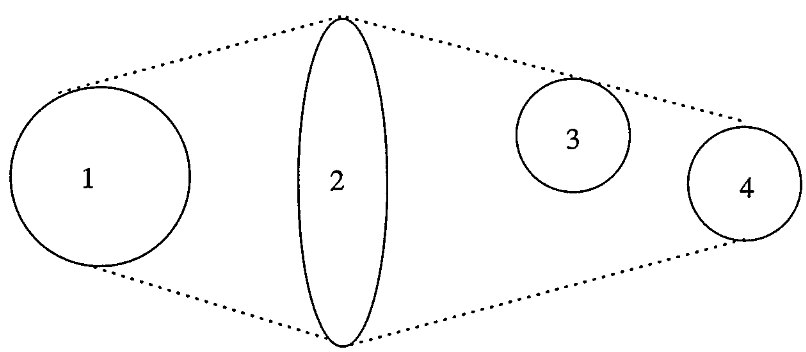


Figure 12.2: An example of why $A_i \cap bd(H')$ is connected for each $i = 1, 2, 3, 4$. Here $A_2 \cap bd(H')$ is not connected. There is no line that meets the sets A_1, A_3, A_4 that does not meet A_2. Hence this family fails to exhibit Property $G(3)$.

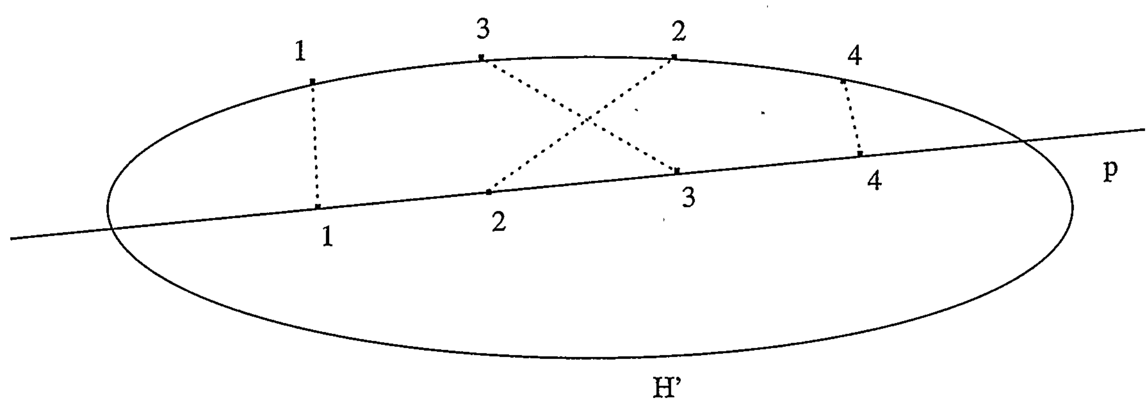


Figure 12.3: An example demonstrating that the sets A_1, A_2, A_3, A_4 cannot meet $bd(H')$ in the cyclic order A_1, A_3, A_2, A_4, A_1. The sets meet p in the order $1, 2, 3, 4$. If the sets meet $bd(H')$ in the cyclic order $1, 3, 2, 4, 1$ then the line segment joining 2 on $bd(H')$ to 2 on p intersects the line segment joining 3 on $bd(H')$ to 3 on p. Since the sets are convex, the point of intersection lies in both 2 and 3, contradicting the assumption that the sets are disjoint.

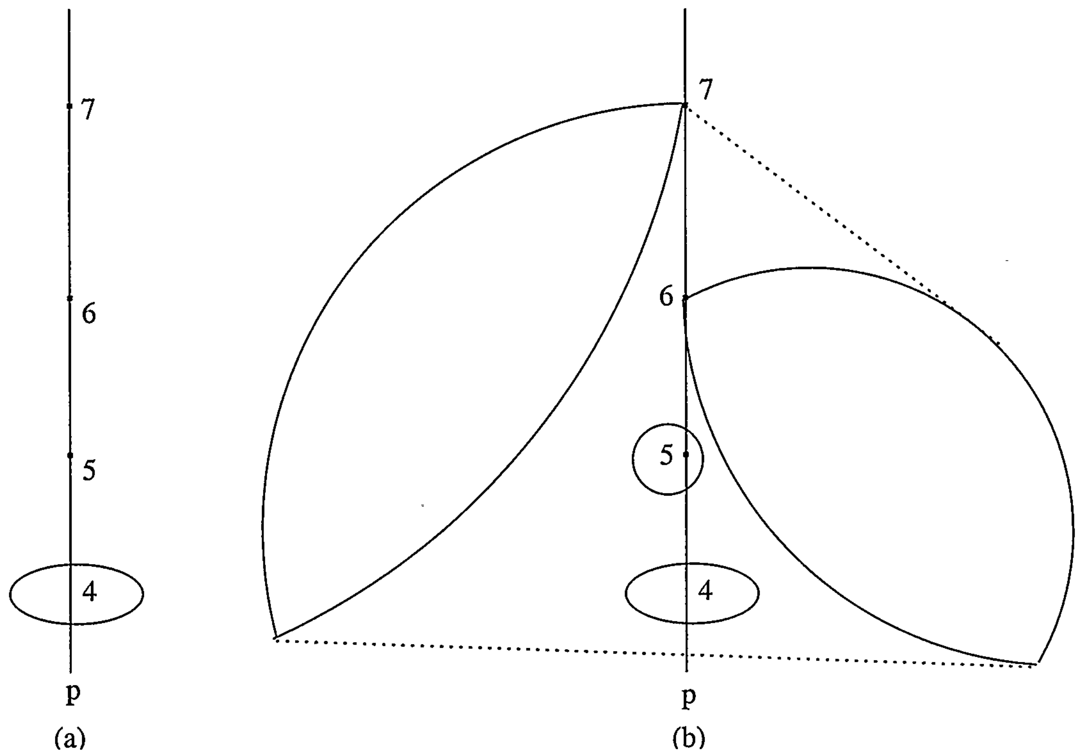


Figure 12.4: An intuitive explanation of why $A_4 \subset int(H^*) \Rightarrow A_4 \subset int(conv(A_t \cup A_m))$ for some $t < m$ where $t, m \in \{5, 6, 7\}$. The sets A_4, A_5, A_6, A_7 meet p in that order (a). Thus, if $A_4 \subset int(H^*)$ then a configuration similar to (b) arises and the result is immediately apparent.

Chapter 13

"Linearly Related Plane Convex Sets."

13.1 Introduction

In this chapter, based on [6], we continue our exploration of other transversal properties. To begin, we summarize the properties of interest. Let $\mathfrak{A}$ be a family of compact convex sets in $\mathbb{E}^2$ and n be a positive integer.

Definition 1 *Property $G(n)$. For any n members of $\mathfrak{A}$, there is a line meeting exactly these n sets. See Figure 13.1.*

Definition 2 *Property $H(n)$. The boundary of the convex hull of any subset $\mathfrak{A}'$ of $\mathfrak{A}$ meets at most n sets of $\mathfrak{A}'$. See Figure 13.2.*

Definition 3 *Property $I(n)$. Any line meets at most n members of $\mathfrak{A}$.*

Definition 4 *Property I. Any line meets only a finite number of members of $\mathfrak{A}$.*

We first observe that if a family satisfies Property $G(n)$ then it satisfies Property $T(n)$, so $G(n)$ is a much stronger condition than $T(n)$. Thus far, when discussing the Property $T(n)$, we have imposed stringent conditions on the family so that it may satisfy property T. In particular the discussion has been limited to line segments, parallelograms and translates of a compact, convex sets. The present discussion yields a remarkable result: if a family satisfies Property $G(n)$ then the family is finite, irrespective of the members of the family.

Note that if a family satisfies Property $I(n)$ then it satisfies Property I and that if a family satisfies Property $H(n)$ then the convex hull of any sub-family of the family is the convex hull of at most n members of the sub-family. These observations, although trivial, are easily overlooked and are used frequently in what follows without further mention.

13.2 $G(n) \Rightarrow H(n+2)$

Theorem 22 *Let $\mathfrak{A}$ be a family of pairwise disjoint, compact, convex sets in $\mathbb{E}^2$ that satisfies Property $G(n)$. Then $\mathfrak{A}$ satisfies Property $H(n+2)$.*

Proof. Suppose that $\mathfrak{A}$ does not satisfy Property $H(n+2)$. Therefore, there is a subset $\mathfrak{A}'$ of $\mathfrak{A}$ such that the boundary of the convex hull of $\mathfrak{A}'$ meets at least $n+3$ members of $\mathfrak{A}'$, say $A_1, A_2, \ldots, A_{n+3}$. Let $H = conv(\mathfrak{A}')$ and we use the standard notation δH to denote the boundary of H. Observe that $A_i \cap \delta H$ is connected for each $i, 1 \leq i \leq n+3$. For, if there is some $j, 1 \leq j \leq n+3$, such that $A_j \cap \delta H$ is not connected then $(intH) \setminus A_j$ has two components from which we can draw the two sets A_x and A_y in $\mathfrak{A}'$ so that they lie in different components of $(intH) \setminus A_j$. Furthermore, it is clear from the way A_x and A_y have been chosen, that any line that meets A_x and A_y must also meet A_j. Figure 13.3 helps illustrate the situation described. Now, choose A_x, A_y and $n-2$ members from $\mathfrak{A}'$ that are distinct from A_x, A_y and A_j. By $G(n)$ there is a line that meets only the n members of $\mathfrak{A}'$ we have just chosen. In particular, the line does not meet A_j. However, this line intersects A_x and A_y, whence it intersects A_j and we arrive at a contradiction. Therefore, $A_i \cap \delta H$ is connected for each $i, 1 \leq i \leq n+3$.

Now, δH is a closed convex curve. Thus, without loss of generality, we may orient δH and label the sets $A_1, A_2, \ldots, A_{n+3}$ so that δH meets the sets in the cyclic order $A_1, A_2, \ldots, A_{n+3}, A_1$. By Property $G(n)$, there exist two lines p and q such that p meets only the sets $A_1, A_3, A_5, A_7, A_8, A_9, \ldots, A_{n+3}$ and q meets only the sets $A_2, A_4, A_6, A_7, A_8, A_9, \ldots, A_{n+3}$. More precisely, p meets all of the sets $A_1, A_2, \ldots, A_{n+3}$ except A_2, A_4, A_6 and q meets all of the sets $A_1, A_2, \ldots, A_{n+3}$ except A_1, A_3, A_5. Because A_1 and A_3 are disjoint and p intersects both of these sets, there is a segment of p that lies between A_1 and A_3 which we call $\tilde{p}$. Formally, $\tilde{p} = conv(p \cap (A_1 \cup A_3)) \setminus (A_1 \cup A_3)$.

Because $A_2 \cap (p \cup A_1 \cup A_3) = \emptyset$, A_2 is contained in a region R of H bounded by A_1, A_3 and $\tilde{p}$ (cf. Figure 13.4). Due to the cyclic ordering of $A_1, A_2, \ldots, A_{n+3}$, it follows that $R \cap \delta H$ meets only the sets A_1, A_2, A_3, and these sets are the only sets contained entirely in R. Therefore, it follows that if $A_j \cap R \neq \emptyset, j = 4, 5, \ldots, n+3$ then $A_j \cap \tilde{p} \neq \emptyset$. The preceding statement yields $(A_4 \cup A_6) \not\subset H \setminus R$ implies $\tilde{p}$, and consequently p, meets $(A_4 \cup A_6)$ which contradicts the choice of p. So, $(A_4 \cup A_6) \subset H \setminus R$. If $q \cap \tilde{p} = \emptyset$ then q would be a line that meets A_2, but not $\tilde{p} \cup A_1 \cup A_3$. Clearly, such a line cannot intersect $H \setminus R$ which means the q cannot meet $(A_4 \cup A_6) \subset H \setminus R$. Therefore $q \cap \tilde{p} \neq \emptyset$ and a completely symmetric argument yields $p \cap \tilde{q} \neq \emptyset$ where $\tilde{q} = conv(q \cap (A_4 \cup A_6)) \setminus (A_4 \cup A_6)$. Thus, $\tilde{p} \cap \tilde{q} \neq \emptyset$, which can only occur if δH meets the sets A_1, A_3, A_4, A_6 in one of the cyclic orders A_1, A_4, A_3, A_6, A_1 or A_1, A_3, A_6, A_4, A_1. Since it cannot, due to the original cyclic ordering imposed on $A_1, A_2, \ldots, A_{n+3}$ and δH, the theorem follows immediately. ■

13.3 $G(n) \Rightarrow I(n+8)$

Again, $\mathfrak{A}$ is a family of pairwise disjoint, compact, convex sets in $\mathbb{E}^2$ and we assume that it satisfies Property $G(n)$. To facilitate the discussion we introduce some notation. Given two distinct members of $\mathfrak{A}$, say A_i and A_j, let $H_{i,j} = conv(A_i \cup A_j)$. We stipulate that at most one member of $\mathfrak{A}$ is a point. If we allow more than one member of $\mathfrak{A}$ to be a point, say A_i and A_j are two such points in $\mathbb{E}^2$, then $H_{i,j}$ is a line segment contained in a line t, say. Now, by $G(n)$ any $n-2$ members of $\mathfrak{A}$ distinct from A_i and A_j will meet t, consequently $|\mathfrak{A}| \leq n$. Recall that the major result in this discussion is that a family, which satisfies $G(n)$, is finite; in this case, where more than one member of the family is a point, the result follows trivially.

Because $H_{i,j}$ is not a line segment, this follows from the preceding stipulation that at most one member of $\mathfrak{A}$ is a point, we obtain distinct lines of support for A_i and A_j which are also lines of support for $H_{i,j}$. We label these lines $t_{i,j}$ and $t_{j,i}$ and write $t^*_{i,j} = t_{i,j} \cap H_{i,j}$, $t^*_{j,i} = t_{j,i} \cap H_{i,j}$ (cf. Figure 13.5). Observe that if some member of $\mathfrak{A} \setminus \{A_i, A_j\}$, say A_k, meet both $t^*_{i,j}$ and $t^*_{j,i}$ then any line which intersects both A_i and A_j must intersect A_k. If $|\mathfrak{A}| > n$ then choose $n-2$ members of $\mathfrak{A} \setminus \{A_i, A_j, A_k\}$. By Property $G(n)$, there is a line which intersect these $n-2$ sets and A_i, A_j, but does not intersect any other member of $\mathfrak{A}$. However,this line meets A_i and A_j, so it also meet A_k, which is a contradiction. Therefore, we reduce to the trivial case where $|\mathfrak{A}| \leq n$. Henceforth, we assume that no set meets both $t^*_{i,j}$ and $t^*_{j,i}$.

Let $\mathfrak{A}$ be a family of pairwise disjoint, compact, convex sets in $\mathbb{E}^2$ and assume that it satisfies Property $G(n)$. Let us assume p is a line that meets the sets $A_1, A_2, \ldots, A_{n+3}$, where $A_i \in \mathfrak{A}$ and $1 \leq i \leq n+3$, in the order given. Let

$\tilde{p} = conv(p \cap (A_1 \cup A_{n+3}))$. Since the family satisfies Property $G(n)$, we may apply Theorem 22 to conclude that the family satisfies Property $H(n+2)$. Thus, $A_j \subset int(conv(A_1 \cup A_2 \cup \ldots \cup A_{n+3}))$, for some $j, 1 \leq j \leq n+3$.

Lemma 15 *Let q be a line such that $q \cap A_j \neq \emptyset$ and $q \cap A_i = \emptyset$ for $1 \leq i \neq j \leq n+3$. Then $q \cap \tilde{p} \neq \emptyset$.*

Proof. Observe that the fashion in which p meets the sets $A_1, A_2, \ldots, A_{n+3}$ generates an obvious linear ordering. If $A_j = A_1$ then as we have seen before $A_j \in H(A_m, A_n)$ for some m and n and q separates A_m and A_n. So, immediately we have the desired result. A similar argument applies if $A_j = A_{n+3}$.

Thus, A_j lies between A_1 and A_{n+3} in the sense that $A_j \cap H_{1,n+3} \neq \emptyset$, otherwise the line p could not intersect the sets in the order given. If $A_j \subset H_{1,n+3}$ then clearly q separates A_1 and A_{n+3}. Consequently, p meets q at a point in $H_{1,n+3}$, and $q \cap \tilde{p} \neq \emptyset$. If $A_j \not\subset H_{1,n+3}$ then it must meet one of the line segments $t^*_{1,n+3}$ or $t^*_{n+3,1}$, but not both. It must meet at least one, since $A_j \cap H_{1,n+3} \neq \emptyset$, but it cannot meet both as was discussed earlier. Without loss of generality, assume $A_j \cap t^*_{1,n+3} \neq \emptyset$. Because $A_j \subset int(conv(A_1 \cup A_2 \cup \ldots \cup A_{n+3}))$, there exist i and k, $1 \leq i \neq j \neq k \leq n+3$, such that A_i and A_k both meet $t^*_{1,n+3}$ and $A_j \setminus H_{1,n+3} \subset H_{i,k}$. Thus, $A_j \subset H_{1,n+3} \cup H_{i,j}$ and q separates A_x and A_y, $y \in \{1, i\}$ and $x \in \{j, n+3\}$. Schematically, this situation is represented in Figure 13.6. In the diagram, we see that the critical aspect of this proof lies in the fact that A_i and A_k meet $t^*_{1,n+3}$. As a result, the only way q can meet A_j is if it separates the sets A_1, A_i, A_k and A_{n+3} in the previously described fashion. Consequently, p meets q in a point in $A_j \subset H_{1,n+3} \cup H_{i,j}$, so $q \cap \tilde{p} \neq \emptyset$. ■

Theorem 23 *If $\mathfrak{A}$ is a family of pairwise disjoint, compact, convex sets in $\mathbb{E}^2$ that* *satisfies Property $G(n)$ and $|\mathfrak{A}| > n+9$ then the family satisfies Property $I(n+8)$.*

Proof. Before proceeding, we note that the result is straight forward, but somewhat tedious to prove in the case of $|\mathfrak{A}| = n+9$ and the result makes no sense if $|\mathfrak{A}| < n+9$.

We first show that the theorem holds for $n = 3$ and then proceed inductively. Suppose that $\mathfrak{A}$ is a family of pairwise disjoint, compact, convex sets in $\mathbb{E}^2$ that satisfies Property $G(3)$ and $|\mathfrak{A}| > 12$, but the family does not satisfy Property $I(11)$. So, there is a line p that meets (at least) twelve members of $\mathfrak{A}$ which we call $A_1, \dots, A_{12}$. Clearly, p generates a linear ordering of those sets and without loss of generality we assume that p meets the sets in the stated order. Because the family satisfies Property $G(3)$, by Theorem 22, the family satisfies Property $H(5)$. Hence, there is an $A_i \in \{A_1, \dots, A_6\}$ and an $A_j \in \{A_7, \dots, A_{12}\}$ such that $A_i \subset int(conv(A_1 \cup \dots \cup A_6))$ and $A_j \subset int(conv(A_7 \cup \dots \cup A_{12}))$. Because $|\mathfrak{A}| > 12$, there is a $A \in \mathfrak{A} \setminus \{A_1, \dots, A_{12}\}$. Now, by appealing to Property $G(3)$, we obtain a line, q, that meets the sets A, A_i, A_j but does not meet any other set of $\mathfrak{A}$. This means that q satisfies the conditions of Lemma 15 with respect to the collection $A_1, \dots, A_6$, so q meets p at a point in $\tilde{p} = conv(p \cap (A_1 \cup \dots \cup A_6))$. However, q also satisfies the conditions of Lemma 15 with respect to the collection $A_7, \dots, A_{12}$, so q meets p at a point in $\tilde{\tilde{p}} = conv(p \cap (A_7 \cup \dots \cup A_{12}))$. Since $\tilde{p} \cap \tilde{\tilde{p}} = \emptyset$, we arrive at a contradiction as the preceding statements indicate that line q, which is certainly distinct from p, meets the line p at two distinct points.

Suppose that for any family $\mathfrak{A}'$ of pairwise disjoint, compact, convex sets in $\mathbb{E}^2$ that satisfies Property $G(n)$ and $|\mathfrak{A}'| > n+9$, the family also satisfies Property $I(n+8)$. Let $\mathfrak{A}$ be a family of pairwise disjoint compact convex sets in $\mathbb{E}^2$ that satisfies Property $G(n+1)$ and $|\mathfrak{A}| > (n+1)+9 = n+10$. Choose an element $A \in \mathfrak{A}$. Next choose n members of $\mathfrak{A} \setminus \{A\}$, call them $A_1, \ldots, A_n$. By appealing to Property $G(n+1)$, we obtain a line t that intersects the sets $A, A_1, \ldots, A_n$ and only these sets in $\mathfrak{A}$. Clearly, t intersects the sets $A_1, \ldots, A_n$ and only these sets in $\mathfrak{A} \setminus \{A\}$. As $A_1, \ldots, A_n$ were arbitrarily chosen in $\mathfrak{A} \setminus \{A\}$, we see that any n members of $\mathfrak{A} \setminus \{A\}$ have a line which intersects them, and only them, in $\mathfrak{A} \setminus \{A\}$. Hence, $\mathfrak{A} \setminus \{A\}$ satisfies Property $G(n)$ and $|\mathfrak{A} \setminus \{A\}| > n+10-1 = n+9$, so by the induction hypothesis $\mathfrak{A} \setminus \{A\}$ satisfies Property $I(n+8)$. Therefore, given any line it can meet at most $n+8$ elements in $\mathfrak{A} \setminus \{A\}$ and that same line may possibly meet A as well, which means that any line will meet at most $n+9$ members of $\mathfrak{A}$. In fact, if a line met $n+10$ members of $\mathfrak{A}$ then, after possibly removing A from those $n+10$ sets, we would have $n+9$ members of $\mathfrak{A} \setminus \{A\}$ with a line meeting them; this would contradict the assumption that $\mathfrak{A} \setminus \{A\}$ is $I(n+8)$. In particular, $\mathfrak{A}$ satisfies Property $I(n+9) = I((n+1)+8)$ and the induction is complete. ■

13.4 $H(n) + I \Rightarrow |\mathfrak{A}| < \infty$

Let $\mathfrak{A}$ be a family of pairwise disjoint, compact, convex sets in $\mathbb{E}^2$ that satisfies Property $H(n)$ and Property I. Suppose that $A \in \mathfrak{A}$ and denote by $s(A)$ the set of all supporting lines of A. If $p \in s(A)$ then Q_p denotes the closed half plane bounded by p that contains A, and R_p denotes the closed half plane bounded by p that does not contain A (cf. Figure 13.7). Observe that, in the space of lines $\mathbb{E}^2$, the set $s(A)$ is connected, path connected in fact, and there is an obvious way to describe what it means for two lines in $s(A)$ to be sufficiently close. Furthermore, observe that Q_p

and R_p vary continuously with p in the sense that if the line t is sufficiently close to p in $s(A)$ then Q_t is arbitrarily close to Q_p and R_t is arbitrarily close to R_p in $\mathbb{E}^2$ (cf. Figure 13.8). Let $s^*(A) = \{p \in s(A) : R_p$ contains an infinite number of sets of $\mathfrak{A}\}$ and $s^o(A) = s(A) \setminus s^*(A)$ (cf. Figure 13.9).

Lemma 16 *For $A \in \mathfrak{A}$, either $s(A) = s^*(A)$ or $s(A) = s^o(A)$.*

Proof. We first show that $s^*(A)$ and $s^o(A)$ are open in $s(A)$.

Let $p \in s^*(A)$ and write $F_p^* = \{A \in \mathfrak{A} : A \subset int(R_p)\}$ and $H_p^* = conv(\bigcup_{A \in F_p^*} A)$. Clearly, R_p contains infinitely many members of $\mathfrak{A}$. Property I ensures that at most only a finite number of those members meet p so the remainder must lie in the interior of R_p. In particular, H_p^* contains infinitely many members of $\mathfrak{A}$. However, Property $H(n)$ ensures that H_p^* is the convex hull of at most n members of $\mathfrak{A}$, which indicates that H_p^* is a closed and bounded convex set in $int(R_p)$. So, $p \cap H_p^* = \emptyset$, but more importantly, $t \cap H_p^* = \emptyset$ for all t sufficiently close to p. Since R_t tends to R_p as t tends to p, $H_p^* \subset R_p$ implies $H_p^* \subset R_t$. Since H_p^* contains infinitely many members of $\mathfrak{A}$, then so does R_t. Consequently, $t \in s^*(A)$ for all t sufficiently close to p. Thus we have shown that, for an arbitrary element $p \in s^*(A)$, we can find an open neighborhood of points in $s(A)$ about p that is contained in $s^*(A)$. Hence, $s^*(A)$ is open $s(A)$.

Next, let $p \in s^o(A)$ and write $F_p^o = \{A \in \mathfrak{A} : A \subset int(Q_p)\}$ and $H_p^o = conv(\bigcup_{A \in F_p^o} A)$. Clearly R_p contains only finitely many members of $\mathfrak{A}$, so Q_p must contain the rest. Property I ensures that at most only finitely many members of $\mathfrak{A}$ meet p, whence $int(Q_p)$ contains infinitely many members of $\mathfrak{A}$. In particular, H_p^o contains infinitely many members of $\mathfrak{A}$. Because of Property $H(n)$, H_p^o is a closed and bounded convex set. Again, lines t, chosen sufficiently close to p in $s(A)$, yield half planes Q_t that contain H_p^o in their interior. Since, H_p^o contains all but a finite number of members of $\mathfrak{A}$, $t \in s^o(A)$ for all t sufficiently close to p. Thus, we have shown that for an arbitrary element $p \in s^o(A)$, we can find an open neighborhood of points in $s(A)$ about p that is contained in $s^o(A)$. Hence, $s^0(A)$ is open in $s(A)$.

Finally, we note that $s(A) = s^*(A) \cup s^o(A)$, $s^*(A) \cap s^o(A) = \emptyset$ and $s^*(A)$, $s^o(A)$ are both open in and contained in $s(A)$. Thus, if $s^*(A) \neq \emptyset$ and $s^o(A) \neq \emptyset$ then $s(A)$ is not a connected set which is a contradiction. Hence, either $s^o(A) = \emptyset$ or $s(A) = \emptyset$ and the lemma follows immediately. ∎

Theorem 24 *Let $\mathfrak{A}$ be a family of pairwise disjoint compact convex sets in $\mathbb{E}^2$ that satisfies Property $H(n)$ and Property I. Then $|\mathfrak{A}|$ is finite.*

Proof. Let $H = conv(\mathfrak{A})$. By appealing to property $H(n)$, there are at most n sets that meet the boundary of H. Let A and B be two such sets. As A and B are disjoint, there is a line t which strictly separates them. Let P_A be the closed half plane bounded by t which contains A and P_B be the closed half plane bounded by t which contains B.

Since A is convex and $t \cap A = \emptyset$, there is a line $p \in s(A)$ that is parallel to t such that $P_B \subset R_p$. Because we have chosen A so that it meets the boundary of H, there is a line $s \in s(A)$ that supports H. Now, $H \subset Q_s$ and because of Property I, there can be at most finitely many members of $\mathfrak{A}$ that meet s and consequently there are not infinitely many members of $\mathfrak{A}$ in R_s. Hence, $s(A) = s^o(A)$, which means that R_p contains only finitely many members of $\mathfrak{A}$ and, in turn, P_B contains only finitely many members of $\mathfrak{A}$. An analogous argument shows that P_A contains only finitely many members of $\mathfrak{A}$. Since $\mathbb{E}^2 = P_A \cup P_B$, the result follows. ■

13.5 $G(n) \Rightarrow |\mathfrak{A}| < \infty$

Corollary 8 *If $\mathfrak{A}$ is a family of pairwise disjoint, compact, convex, sets in $\mathbb{E}^2$ that satisfies Property $G(n)$ then $|\mathfrak{A}|$ is finite.*

Proof. $G(n) \Rightarrow H(n+2) + I(n+8) \Rightarrow H(k) + I$, where $k = n+2 \Rightarrow |\mathfrak{A}| < \infty$.

■

13.6 Conclusion

The main result of this paper made no major assumptions about the family, other than it satisfies Property $G(n)$, with the major consequence being that the family must be finite. Other results in the study of transversals have been restrictive in the sense that the families were composed of translates, or parallelograms. However, here we have been free to consider any arbitrary family. It is interesting to note that neither Property $H(n)$ (cf. Figure 13.11) nor Property I (cf. Figure 13.11) are sufficient to ensure that the family is finite.

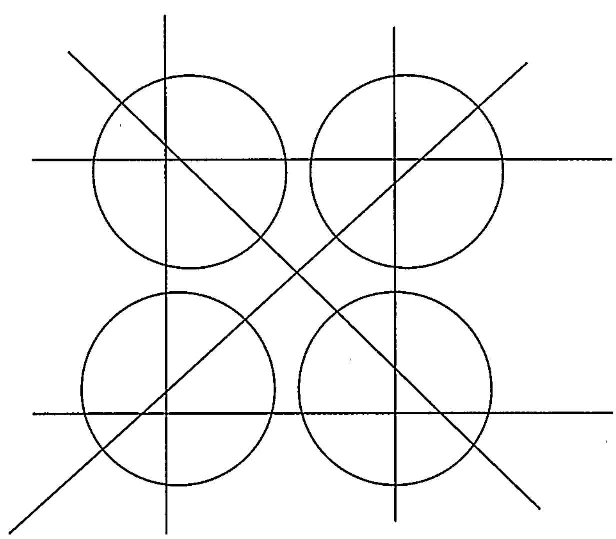


Figure 13.1: Property $G(2)$.

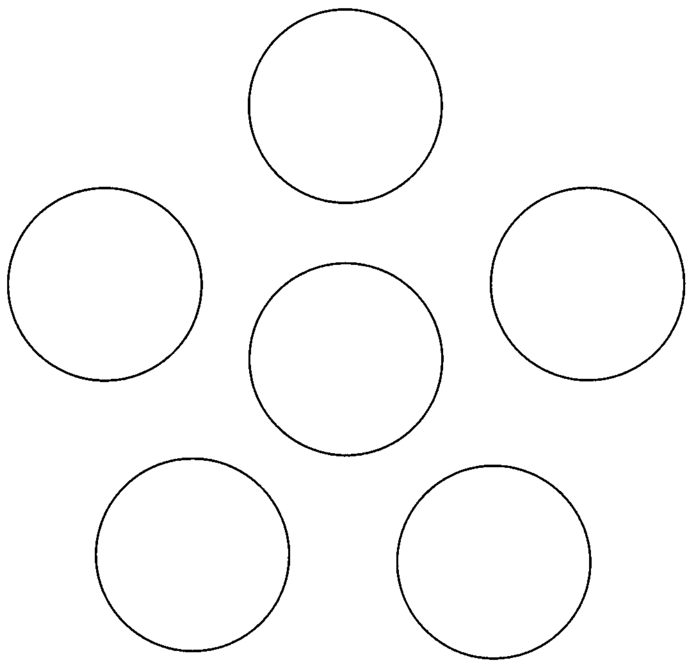


Figure 13.2: Property $G(2)$ and $H(5)$.

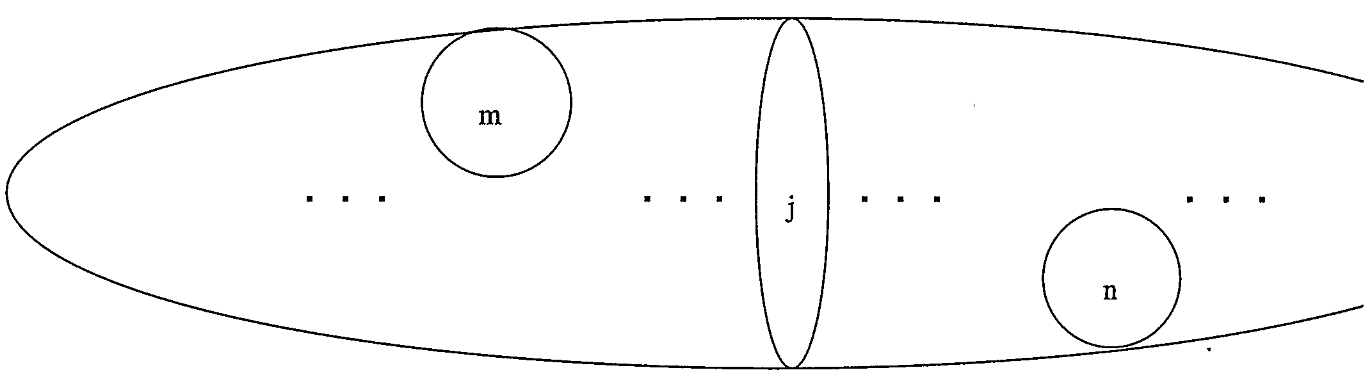


Figure 13.3: An illustration for Theorem 22. The set $A_j \cap \delta H$ is not connected and any line that meets both A_m and A_n must also pierce A_j.

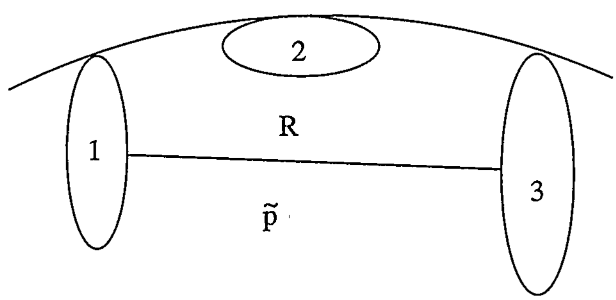


Figure 13.4: The region R.

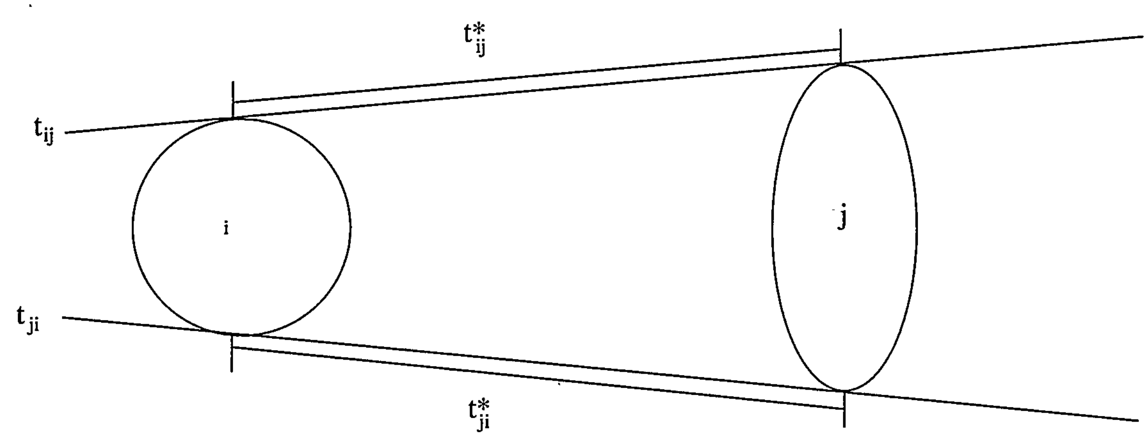


Figure 13.5: We obtain distinct lines of support for A_i and A_j, which are also lines of support for H_{ij}. We label these lines t_{ij} and t_{ji} and write $t^*_{ij} = t_{ij} \cap H_{ij}$, $t^*_{ji} = t_{ji} \cap H_{ij}$.

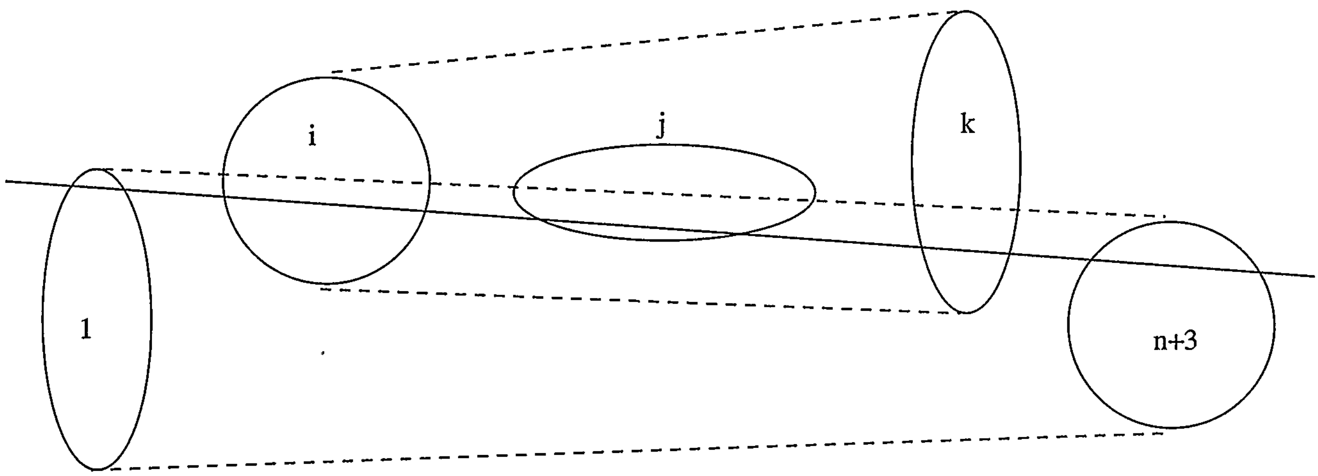


Figure 13.6: An illustration for Lemma 15. We see that A_i and A_k meet $t^*_{1,n+3}$. As a result, the only way q can meet A_j is if it separates the sets A_1, and A_{n+3} and if it separates the sets A_i and A_k. Hence $q \cap \tilde{p} \neq \emptyset$.

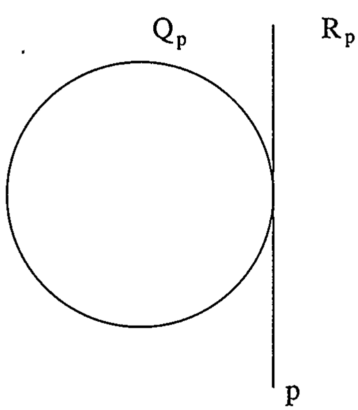


Figure 13.7: An illustration of Q_p and R_p.

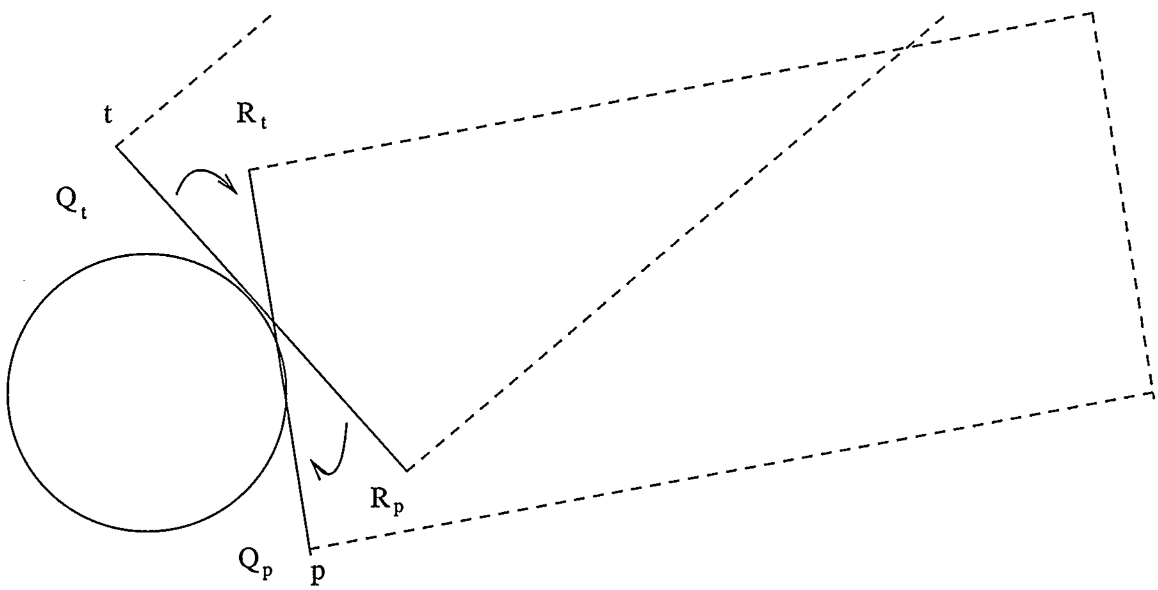


Figure 13.8: An illustration of how Q_t is arbitrarily close to Q_p and how R_t is arbitrarily close to R_p in $\mathbb{E}^2$ provided t is sufficiently close to p in $s(A)$.

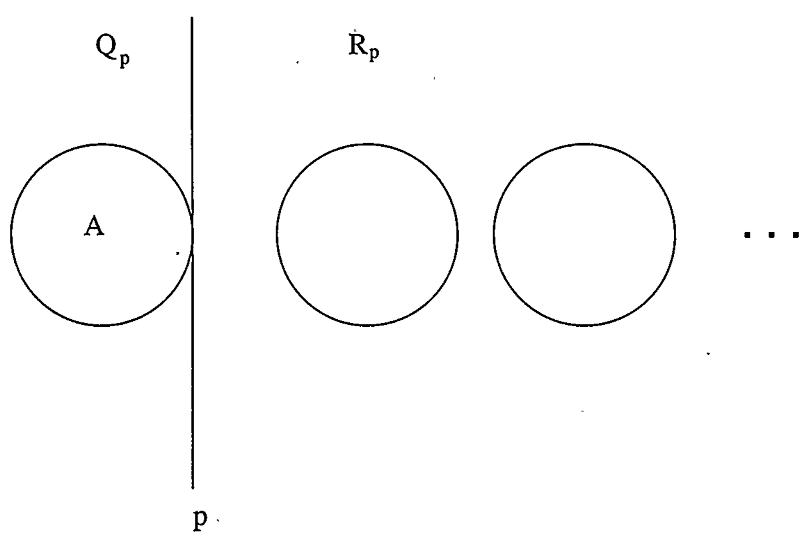


Figure 13.9: An illustration of a line p in $s^*(A)$.

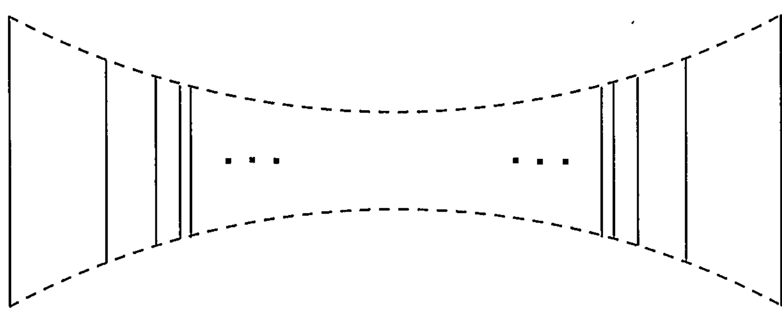


Figure 13.10: Property $H(2)$ alone does not ensure $|\mathfrak{A}|$ is finite.

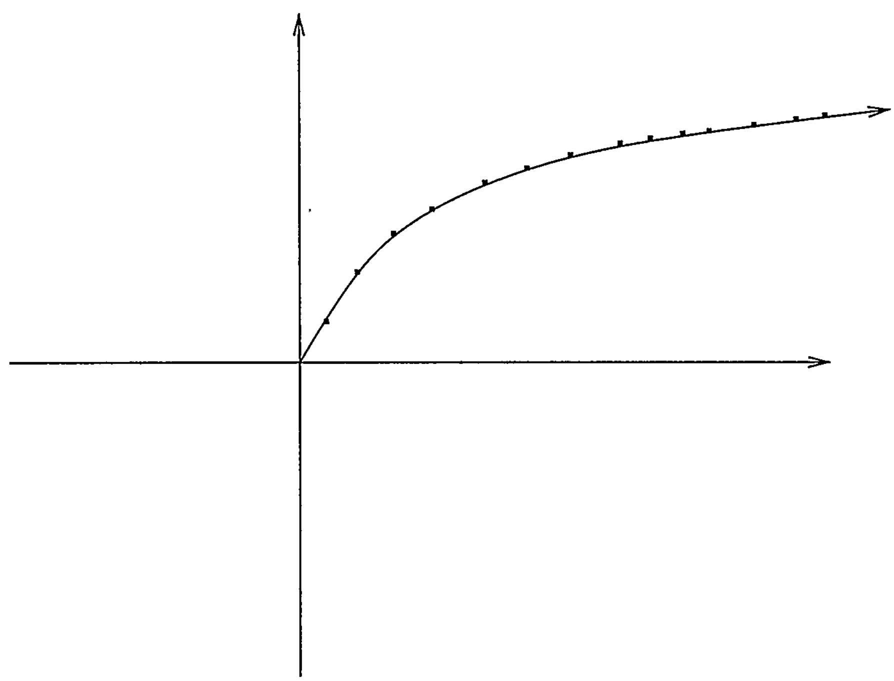


Figure 13.11: Property I alone does not ensure $|\mathfrak{A}|$ is finite.

Chapter 14

"On the $(n-2)$ Transversals of n Convex Subsets of the Plane "

14.1 Introduction

This chapter continues the exploration of other transversal properties. Let $\mathfrak{A}$ be a family of pairwise disjoint, compact, convex sets and suppose that $\emptyset \neq \mathfrak{A}' \subseteq \mathfrak{A}$. We write $A' = conv(\mathfrak{A}')$. Recall that a family $\mathfrak{A}$ may satisfy the following properties:

Definition 5 *Property $G(n)$. For any n members of $\mathfrak{A}$ there is a line meeting exactly these n sets.*

Definition 6 *Property $H(n)$. The boundary of the convex hull of any subset $\mathfrak{A}'$ of $\mathfrak{A}$ meets at most n sets of $\mathfrak{A}'$.*

In addition to these previously discussed properties, the family may satisfy:

Definition 7 *Property $J(n)$. There exist n sets in $\mathfrak{A}$ so that these sets are met by a line.*

Let $|\mathfrak{A}| = n \geq 3$. Observe that, in this case, $J(n)$ and $T(n)$ are equivalent. In the discussion of [22], it was demonstrated that $T(n-1)$ need not imply $T(n)$. Consequently, in this situation, $T(n-1)$ need not imply $J(n)$ as well. However, it is natural to ask if $T(n-2)$ implies $T(n-1)$ or $J(n-1)$. The goal of this chapter is to provide an answer to these two questions by discussing the results of [3].

14.2 Discussion

Let $\mathfrak{A} = \{A_1, A_2, \ldots, A_n\}$ be a family of n pairwise disjoint, compact, convex sets in $\mathbb{E}^2$. We assume $n \geq 4$, since $n = 3$ yields trivial results. We write $B_{i,j} = conv(A_i, A_j)$. If $\mathfrak{A}$ has Property $G(n-2)$, then given A_i and A_j in $\mathfrak{A}$, where $A_i \neq A_j$, we obtain a transversal $L_{i,j}$ of $\mathfrak{A} \setminus \{A_i, A_j\}$.

Lemma 17 *For any positive integer k, there exists a smallest integer $g(k)$ such that if $\mathfrak{A}^*$ is a family of compact convex sets in $\mathbb{E}^2$, such that no three members of $\mathfrak{A}^*$ satisfy property $H(2)$ and $|\mathfrak{A}^*| > g(k)$ then there are k sets in $\mathfrak{A}^*$, say, $A_1, A_2, \ldots, A_k$ so that $A_i \cap bd(conv(A_1 \cup A_2 \cup \ldots \cup A_k)) \neq \emptyset$ for each $i = 1, 2 \ldots, k$.*

We omit a proof of this Lemma which can be found in [4]. Instead, we note that this is a generalization of a theorem for points in the plane. In the simpler version, we ask: what is the fewest number of points $g(k)$ in the plane, no three collinear, required to ensure that there is a convex k-gon among those points? In the case of $k = 3$ it is clear that $g(3) = 3$. However, less obvious is $g(4) = 5$. Examining Figure 14.1 reveals that four points can be arranged so that the convex hull of those four points is only a triangle. However, the stipulation that no three points are collinear ensures that any fifth point must lie in one of the twelve regions indicated in Figure 14.1. By examining each case we obtain the desired result. The Property $H(2)$ in Lemma 17 is similar to the stipulation that no three points are collinear and $A_i \cap bd(conv(A_1 \cup A_2 \cup \ldots \cup A_k)) \neq \emptyset$ is similar to the notion of a convex k-gon but the "vertices" are convex sets instead of points. The existence of the desired number $g(k)$ is ensured by the cited article and we proceed without any further remarks.

Lemma 18 *If $\mathfrak{A}$ has the Property $G(n-2)$, then $\mathfrak{A}$ has Property $H(5)$.*

Proof. Suppose that $\mathfrak{A}$ does not satisfy $H(5)$. So, there is a subset of $\mathfrak{A}$, say $\mathfrak{A}' = \{A_1, A_2, \ldots, A_6\}$, such that $A_i \cap bd(A') \neq \emptyset$, where $A' = conv(\mathfrak{A}')$. If $A_j \cap bd(A')$ is not connected then we observe that $(intA') \setminus A_j$ has two components from which we can choose two sets, A_x and A_y, in A' so that they lie in different components of $(intA') \setminus A_j$. Since $L_{1,j}$ meets A_x and A_y and any line that meets A_x and A_y clearly meets A_j, $L_{1,j}$ meets A_j; a contradiction. Hence, $A_j \cap bd(A')$ is connected. Thus, we may assume that the sets $A_1, A_2, \ldots, A_6$ meet the closed convex curve $bd(A')$ in the cyclic order $A_1, A_2, \ldots, A_6, A_1$.

If $L_{1,3}$ does not separate A_1 and A_3 then A_1 and A_3 must lie on the same side of $L_{1,3}$. Due to the way the sets $A_1, A_2, \ldots, A_6$ meet the closed convex curve $bd(A')$, $A_2 \cap bd(A')$ lies on the same side of $L_{1,3}$ as A_1 and A_3. Thus, in order for A_2 to meet $L_{1,3}$ it must cross $int(conv(A_1 \cup A_3))$. In particular, if t_1 and t_2 are the two lines that support A_1 and A_3 that also support $conv(A_1 \cup A_3)$ and $t_1^* = t_1 \cap conv(A_1 \cup A_3)$ and $t_2^* = t_2 \cap conv(A_1 \cup A_3)$ then $A_2 \cap t_1^* \neq \emptyset$ and $A_2 \cap t_2^* \neq \emptyset$. Recall from the discussion of [6] that this cannot happen.

So, $L_{1,3}$ separates A_1 and A_3, and $L_{4,6}$ separates A_4 and A_6. Thus, the lines $L_{1,3}$ and $L_{4,6}$ meet at a point $q = L_{1,3} \cap L_{4,6}$ and $q \in B_{1,3} \setminus \{A_1 \cup A_3\}$, $q \in B_{4,6} \setminus \{A_4 \cup A_6\}$. This implies that $bd(A')$ meets the sets A_1, A_3, A_4, A_6 in the cyclic order A_1, A_6, A_3, A_4, A_1 (cf. Figure 14.2). This contradicts the original cyclic ordering and the lemma holds. ■

Lemma 19 *If $\mathfrak{A}'$ is a subset of $\mathfrak{A}$ where any three members of $\mathfrak{A}'$ satisfy Property $H(2)$ then $\mathfrak{A}'$ satisfies Property $H(2)$.*

Proof. If $\mathfrak{A}'$ does not satisfy $H(2)$ then we obtain at least three sets that meet the boundary of their respective convex hull which is a contradiction. ■

Lemma 20 *If $\mathfrak{A}$ has the Property $G(n-2)$, then no eight element subset of $\mathfrak{A}$ has Property $H(2)$.*

Proof. Let $\{A_1, A_2, \ldots, A_8\} \subseteq \mathfrak{A}$ have Property $H(2)$. Figure 14.3 demonstrates what such a set might look like. By $H(2)$, there exist three sets A_1, A_2, A_3 so that $A_2 \subset B_{1,3}$ and $A_i \not\subset B_{1,3}, i = 4,5,6,7,8$. Observe that $L_{1,3}$ strictly separates A_1 and A_3. Next, we develop some notation that is necessary in the following discussion; Figure 14.4 illustrates the notation. First, we obtain lines M and N that support A_1, A_3 and $B_{1,3}$. Let $M^* = M \cap B_{1,3}$, $N' = N \cap B_{1,3}$, $L^*_{1,3}$ be the component of $L_{1,3} \setminus (intB_{1,3})$ that meets M^* and $L'_{1,3}$ be the component of $L_{1,3} \setminus (intB_{1,3})$ that meets N', $p^* = L^*_{1,3} \cap M^*$, $p' = L'_{1,3} \cap N'$, $[p^*, p'] = conv\{p^*, p'\}$. Finally, $F^* = \{A_i : A_i \cap (M^* \cup L^*_{1,3}) \neq \emptyset$ and $4 \leq i \leq 8\}$, $F' = \{A_i : A_i \cap (N' \cup L'_{1,3}) \neq \emptyset$ and $4 \leq i \leq 8\}$.

We make the following observations. First, $L_{1,3} = L^*_{1,3} \cup [p^*, p'] \cup L'_{1,3}$. Next, $G(n-2)$, which ensures that $A_4, \ldots, A_8$ meet $L_{1,3}$, and $A_i \not\subset B_{1,3}, i = 4,5,6,7,8$ allows us to conclude $F^* \cup F' = \{A_4, A_5, \ldots, A_8\}$. Finally, if $F^* \cap F' \neq \emptyset$ then choose $A_i \in F^* \cap F'$. There are four cases to examine. If $A_i \cap M^* \neq \emptyset$ and $A_i \cap N' \neq \emptyset$ then we arrive at the usual contradiction that arises when a set crosses the convex hull of two other sets, A_1 and A_3 in this case. If $A_i \cap L^*_{1,3} \neq \emptyset$ and $A_i \cap L'_{1,3} \neq \emptyset$ then the same contradiction arises. Finally, $A_i \cap M^* \neq \emptyset$ and $A_i \cap L'_{1,3} \neq \emptyset$ cannot occur because of $H(2)$ and $A_i \not\subset B_{1,3}, i = 4,5,6,7,8$. Similarly, $A_i \cap L^*_{1,3} \neq \emptyset$ and $A_i \cap N' \neq \emptyset$ cannot occur. Thus, $F^* \cap F' = \emptyset$.

One of F^* or F' must contain three of the sets $A_4, \ldots, A_8$. Without loss of generality, we assume $\{A_4, A_5, A_6\} \subset F^*$ and $A_5 \subset B_{4,6}$. Clearly, $L_{1,3}$ separates A_1 and A_3, and $L_{4,6}$ separates A_4 and A_6. So, we obtain a point $q = L_{1,3} \cap L_{4,6}$ and it is easy to check that $q \in B_{1,3} \setminus \{A_1 \cup A_3\}$ and $q \in B_{4,6} \setminus \{A_4 \cup A_6\}$. We write $L_{1,3} = L^*_{1,3} \cup [p^*, q] \cup [q, p'] \cup L'_{1,3}$. If $A_4 \cap (L^*_{1,3} \cup [p^*, q]) \neq \emptyset$ and $A_4 \cap (L'_{1,3} \cup [q, p']) \neq \emptyset$ then we arrive at contradictions identical to those described in the preceding paragraph, when it was shown that $F^* \cap F' = \emptyset$. So without loss of generality, we assume $A_4 \cap (L^*_{1,3} \cup [p^*, q]) = \emptyset$. As $F^* \cap F' = \emptyset$ and $L_{1,3}$ meets A_4 we get $A_4 \cap [q, p'] \neq \emptyset$. Since $A_4 \subset F_*$, it meets M^* as well. However this can only occur if $L_{4,6}$ meets A_4, contrary to the choice of $L_{4,6}$. ■

In the proof of the next theorem we use Ramsey Theory. Suppose that a set has a cardinality equal to or greater than the Ramsey Number $R_3(a, b)$. This means that if the 3-tuples of our set are colored red or blue then no matter how the coloring is carried out there exists a subset of our set that is all red and has cardinality a or there exists a subset of our set that is all blue and has cardinality b.

Theorem 25 *If $\mathfrak{A}$ has the property $G(n-2)$, then there is an integer N, independent of n, such that $n < N$.*

Proof. Let $N = R_3(8, g(6) + 1)$. Suppose $|\mathfrak{A}| \geq N$. Color every three element subset red if it is $H(2)$ and blue otherwise. Now, there is a subset $\mathfrak{A}_1 \subset \mathfrak{A}$ such that $|\mathfrak{A}_1| = 8$ and it has the property that every three element subset of $|\mathfrak{A}_1|$ is $H(2)$ (i.e. $\mathfrak{A}_1$ is red) or there is a subset $\mathfrak{A}_2 \subset \mathfrak{A}$ such that $|\mathfrak{A}_2| = g(6) + 1$ so that no three members of $\mathfrak{A}_2$ are $H(2)$ (i.e. $\mathfrak{A}_2$ is blue). In the first case, by Lemma 19, $\mathfrak{A}_1$ is $H(2)$. So, we have eight members of $\mathfrak{A}$ that are $H(2)$ which contradicts Lemma 20. In the second case, by Lemma 17, there are six members of $\mathfrak{A}$ that meet the boundary of their respective convex hull. This last statement contradicts Lemma 18. ■

Corollary 9 *Let $\mathfrak{A}$ be a family of pairwise disjoint, compact, convex sets in $\mathbb{E}^2$ that satisfies Property $G(m)$. Then $|\mathfrak{A}| \leq m+1$ for $m > N$.*

Proof. If $|\mathfrak{A}| > m+1$ then there is $\mathfrak{A}' \subseteq \mathfrak{A}$ such that $|\mathfrak{A}'| = m+2$ and $\mathfrak{A}'$ is $G(m)$. Let $n = m+2$, $|\mathfrak{A}'| = n$ and $\mathfrak{A}'$ is $G(n-2)$. So, by Theorem 25, $N > n = m+2 > N+2$ which is a contradiction. ■

Corollary 10 *Let $\mathfrak{A}$ be a family of $n \geq N$ pairwise disjoint, compact, convex sets in $\mathbb{E}^2$ that satisfies Property $T(n-2)$. Then $\mathfrak{A}$ satisfies Property $J(n-1)$.*

Proof. If $\mathfrak{A}$ is $T(n-2)$, but not $J(n-1)$ then clearly $\mathfrak{A}$ is $G(n-2)$. Let $m = n-2$, so $\mathfrak{A}$ is $G(m)$ and $|\mathfrak{A}| = n = m+2$. However, by Corollary 9, $|\mathfrak{A}| \leq m+1$. So, we arrive at a contradiction. ■

Theorem 26 *For any positive integer $n \geq 4$, there exists a family of n pairwise disjoint compact convex sets in $\mathbb{E}^2$ which satisfies Property $T(n-2)$ but not $T(n-1)$.*

Proof. Refer to the chapter on [22]. Replace Q_n by Q_n^* where $Q_n^* = conv\{a, b\}$ where a is a point on R_{n-4} between O and $S_{n-4} \cap R_{n-4}$ and b is a point on Q_n. Checking as we did in the chapter on Lewis' work the result follows immediately (cf. Figure 14.5). ■

14.3 Conclusion

The questions posed at the beginning of the chapter have now been answered. Corollary 10 shows that $T(n-2)$ does indeed imply $J(n-1)$. However, Theorem 26 shows that $T(n-2)$ does not imply $T(n-1)$.

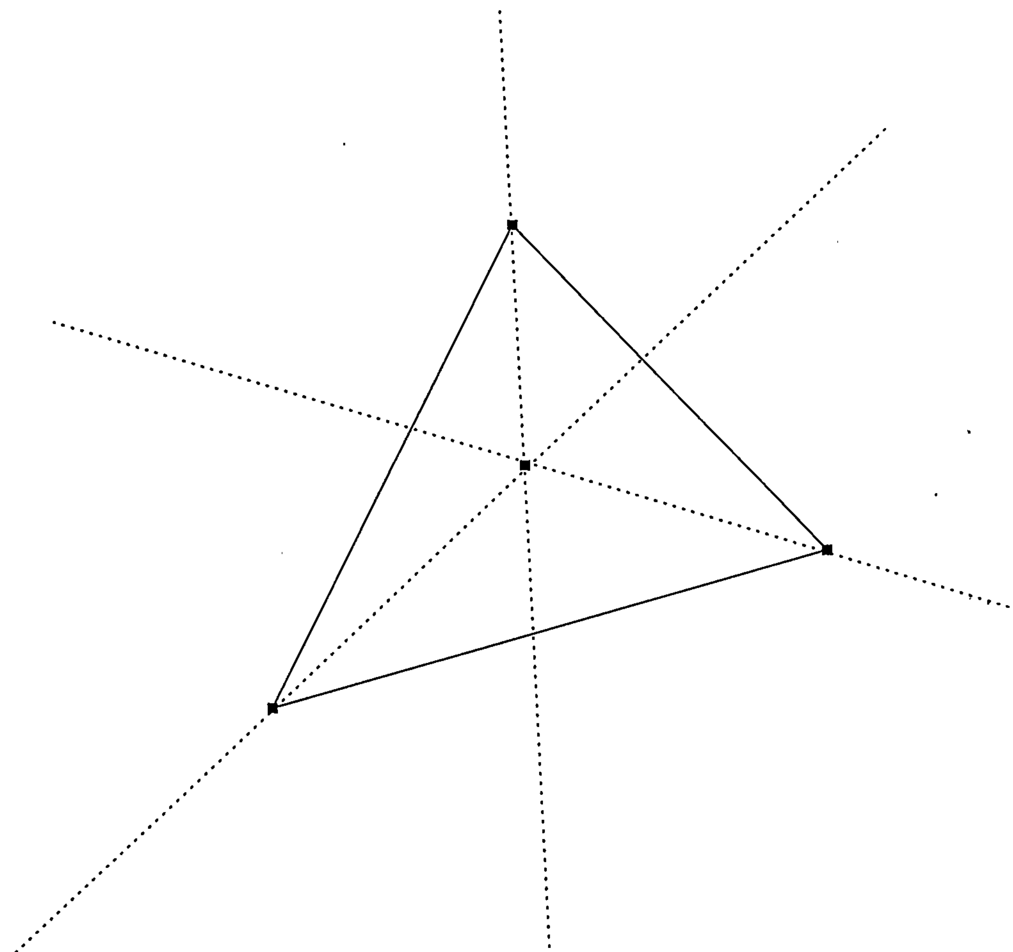


Figure 14.1: An illustration indicating why $g(4) \geq 5$.

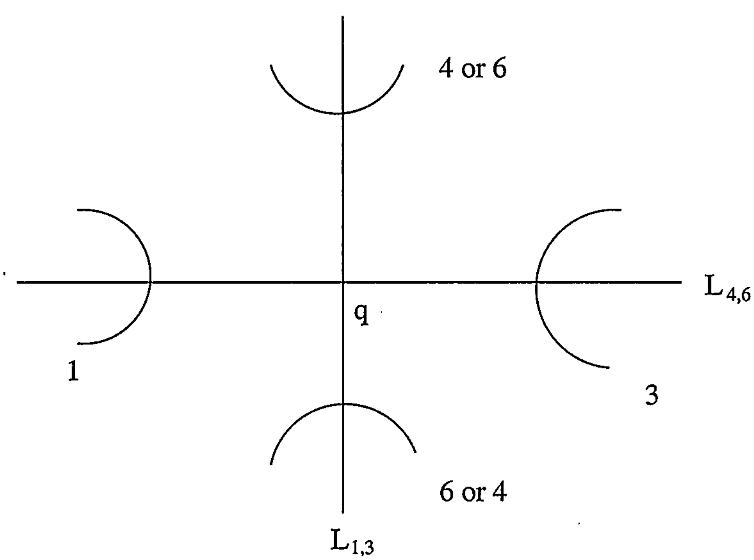


Figure 14.2: An illustration indicating why the cycling ordering is $1, 6, 3, 4$ or $1, 4, 3, 6$ contrary to the original ordering.

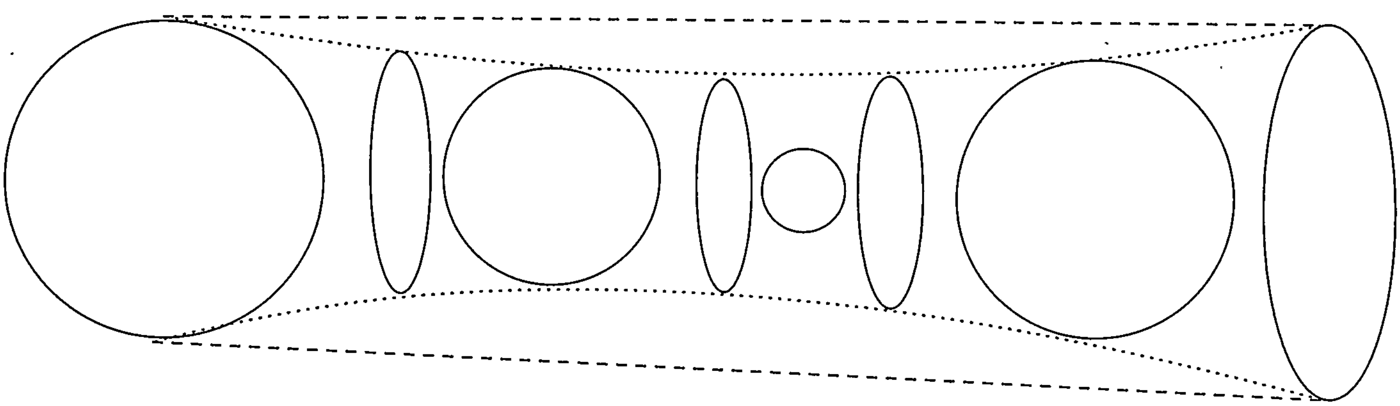


Figure 14.3: Eight convex sets that are $H(2)$.

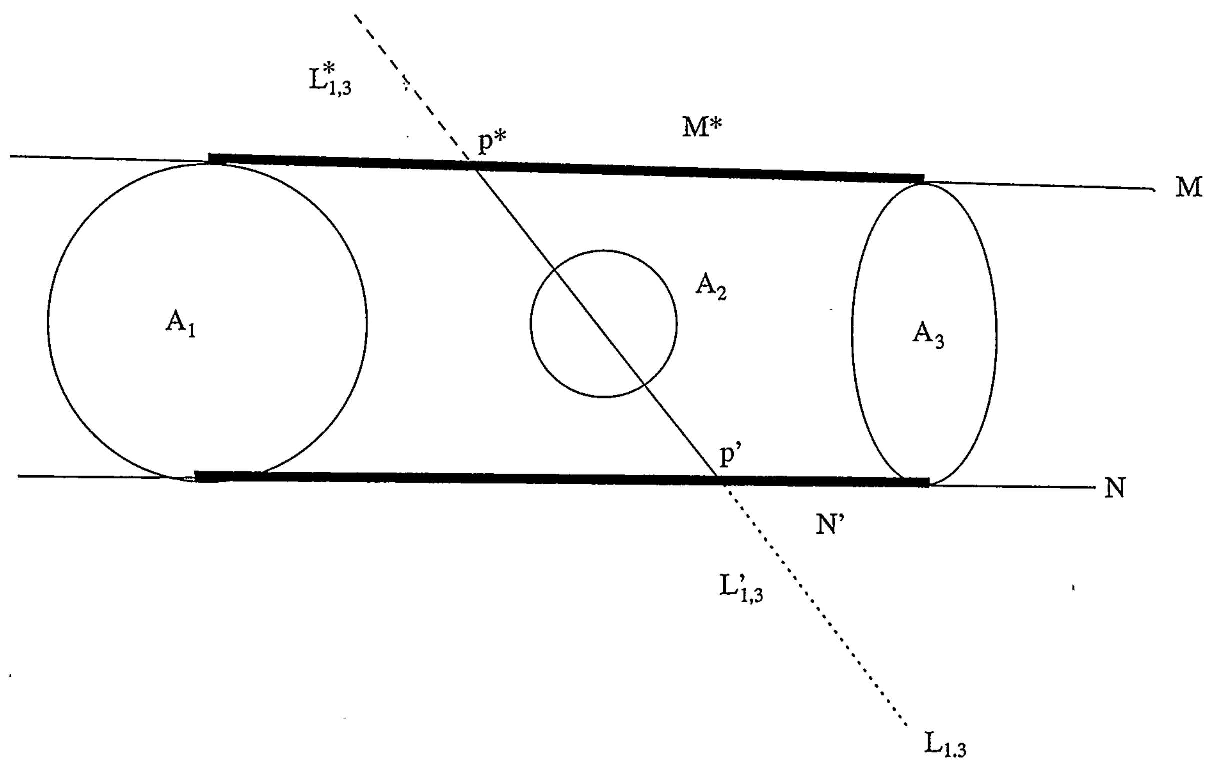


Figure 14.4: An illustration for Lemma 20 indicating the notation used.

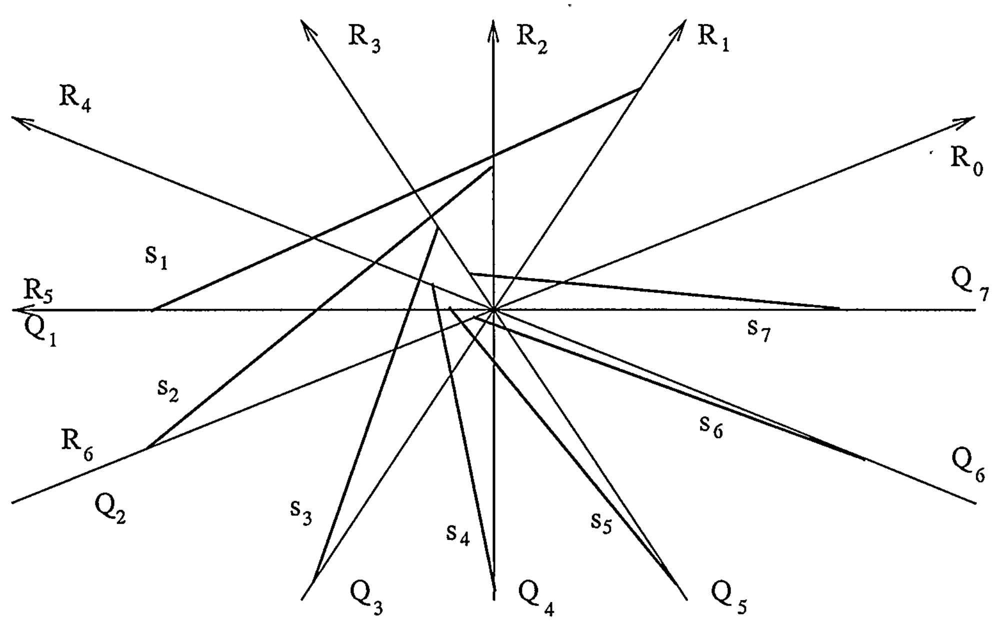


Figure 14.5: An illustration of Theorem 26 for $n = 7$.

Chapter 15

Other Papers

In preparing this manuscript, other papers were studied and, for various reasons, not included. In some cases, the results were repetitions of results already discussed. In other cases, the proofs were incorrect or definitions were lacking or the notation was incomprehensible. We discuss some of these papers now and give reasons for not including them.

Due to errors in the proof of the main result and excessively poor notation, we do not devote a chapter to [17]. Similarly, poor notation made [15] incomprehensible. Since it and [13] are special cases of results discussed in [12], we omit an in depth discussion of these papers.

Several extensions and generalizations of the planar transversal problem have been discussed. One particular generalization is omitted, however, and that is the generalization to the projective plane. A great deal of the discussion in this manuscript has required very little a priori familiarity with concepts in geometry. The goal throughout has been to provide a straight forward intuitive development of the study of transversals. In order to examine the generalization to the projective plane, a familiarity with projective geometry is necessary. Thus, parts of [12] and all of [21] are omitted from the discussion.

A family, in $\mathbb{E}^2$, is said to satisfy *Property* T^n if the family can be partitioned into n or fewer subfamilies, each of which have a transversal. The problem of finding the smallest n such that for each $r \geq 3$, $T(r)$ implies T^n is called a *Gallai-type transversal* *problem.* This manuscript only deals with Helly-type transversal problems and so [8] is omitted.

Chapter 16

Conclusion

Arriving at the end of the discussion, we now look back to the beginning. It all started with a roll of quarters and a piece of dental floss and evolved in many directions. Generalizations of the problem in the plane were examined, analogous problems in higher dimensions were examined, problems dealing with geometric permutations and other transversal properties were looked at as well.

In the study of transversals one tries to determine the necessary conditions that must be imposed on a family $\mathfrak{A}$ to ensure that $\mathfrak{A}$ satisfies Property T. Ideally, one tries to impose as few conditions as possible. As we have seen, many of the results have been restricted to translates of compact convex sets. Ultimately, we seek a result of this nature for nothing more than an arbitrary family of compact convex sets. However, as we have seen, allowing certain rotations makes obtaining such a result nearly impossible. The various avenues of research discussed here are providing very interesting results and much work is still needed.

On a final note, we briefly discuss a useful application of the theory presented here. Most pure mathematicians go about their business of proving theorems with little consideration for how these results may be applied in a real world context. Nonetheless, it is sometimes quite interesting to see how these theoretical matters are applied. In the literature, several applications of the study of transversals are mentioned, but the one discussed now is the most interesting and straightforward one.

In the design of computers, computer boards and electronic circuitry, it is desirable to minimize the distance between certain components to improve computer speed and decrease processing time. The components may be modeled by convex sets, parallelograms for the most part, and the circuitry joining these components may be viewed as transversals. A whole family of components needs to be connected together as well as certain sub-families of components. It is often necessary to connect any n components and all components in the most efficient way possible, by straight lines of circuits. The results from the study of transversals allow computer component designers to know whether their designs are even possible before attempting to design such circuitry.

To conclude, we note that the results discussed here are simple, but beautiful. Most of the works discussed require no specialized knowledge. No intensive theorems that require years of study were used, just simple intuitive reasoning. This is the inherent beauty of geometry. The problems are easy to understand and almost deceptively simple. However, the solutions are creative and require a great deal of ingenuity. This ingenuity is witnessed here, as some of the most beautiful results in the study of transversals have been examined.

Chapter 17

Appendix

We briefly outline a method for obtaining the intersection of two C-sets. The work described here is the result of an attempt to improve the author's intuition regarding Grünbaums reasoning. The end result is a few nice diagrams of the intersection of two C-sets.

We begin with two line segments in the plane (cf. Figure 17.1). The line segments are perpendicular to each other and are of length two. One lies along the x-axis from 0 to 2; we call it l_2. The other is centered at -1 and runs from $(-1, 1)$ to $(-1, -1)$; we call it l_1. The line segments are bounded on either side by two lines, H_0 and H_1, parallel to the y-axis passing through -3 and 3, respectively.

Our goal is to determine the intersection of the C-sets for these two line segments where H_0 and H_1 are the "parallel hyperplanes". That intersection is simply all of the lines that intersect both line segments. The text describes a means to determine the C-set of a single line segment. We modify that approach here.

First, partition one of the line segments, say l_2, into k evenly spaced disjoint, intervals. For the end point of each interval do the following: pivot a line about the end point so that the line passes through all points on l_1. Clearly, this generates closed intervals on H_0 and H_1 for each end point on l_2. Essentially, H_0 becomes the x-axis in the space where the C-sets are situated and H_1 becomes the y-axis in the space where the C-sets are situated. For a unit increase along the H_0 axis we obtain a unit decrease along the H_1 axis. So, for each endpoint on l_2, we are generating lines in the new space with negative slope and passing through the origin. In fact, we are generating line segments, because the intervals along H_0 and H_1, generated for each end point on l_2, are closed and bounded. The union of all of these line segments produces an approximation of the intersection of the C-sets for the given line segments.

We derive equations for these line segments and by refining the partition along l_2 we obtain a more precise approximation for the intersection of these two C-sets. By appealing to the Mathematical and Statistical package Maple, we may plot these line segments to produce an approximate picture of what the intersection of the C-sets for these two line segments looks like. The Maple code and diagrams of approximations for partitions of $10, 100$ and 1000 points (along l_2) are attached. The important thing to observe, in the following diagrams, is that the intersection of the two C-sets is a cell.

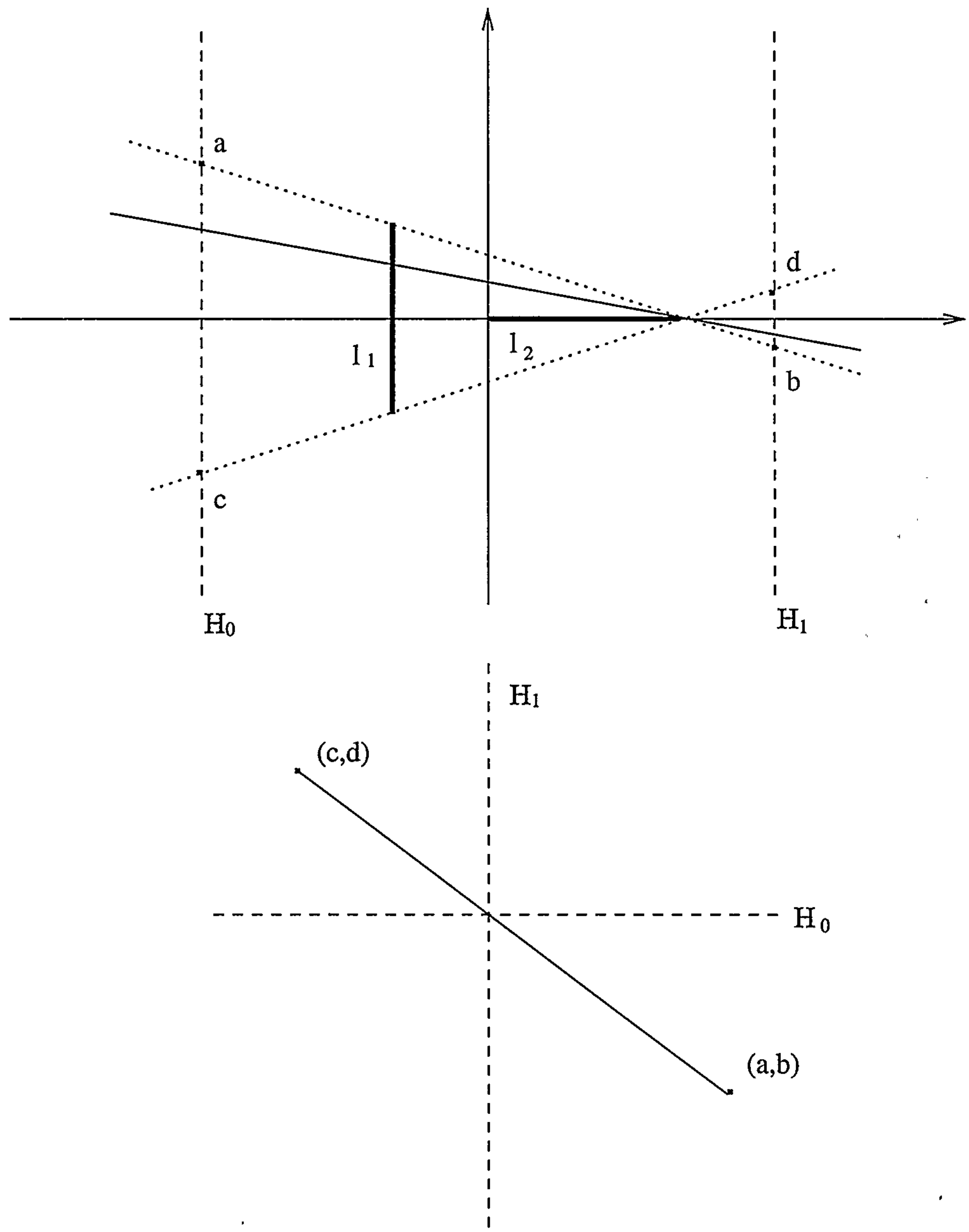


Figure 17.1: A line segment that lies in the intersection of the C-sets of the line segments l_1 and l_2. The solid line passing through the end point of l_2, when pivoted about that end point, between the two dotted lines, sweeps out a closed interval on H_0 and H_1. By interpreting H_0 and H_1 as the axis of a new space where the C-sets are situated we obtain a line segment that lies in the intersection of the C-sets for l_1 and l_2.

```
> k := 10;
```

$$k := 10$$

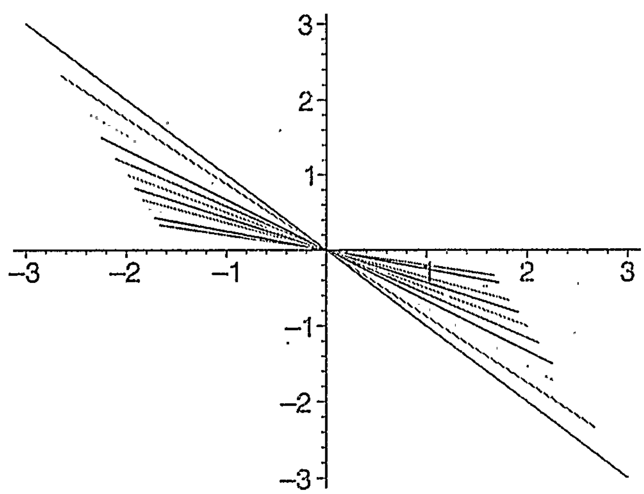
```
> X := [seq((3*k+2*n)/(k+2*n), n=0..k)];
```

$$X := \left[3, \frac{8}{3}, \frac{17}{7}, \frac{9}{4}, \frac{19}{9}, 2, \frac{21}{11}, \frac{11}{6}, \frac{23}{13}, \frac{12}{7}, \frac{5}{3}\right]$$

```
> Y := [seq((-3*k+2*n)/(k+2*n), n=0..k)];
```

$$Y := \left[-3, \frac{-7}{3}, \frac{-13}{7}, \frac{-3}{2}, \frac{-11}{9}, -1, \frac{-9}{11}, \frac{-2}{3}, \frac{-7}{13}, \frac{-3}{7}, \frac{-1}{3}\right]$$

```
> plot([seq([[X[i],Y[i]],[-X[i],-Y[i]]], i=1..k+1)]);
```




```
> k := 100;
```

$$k := 100$$

```
> X := [seq((3*k+2*n)/(k+2*n), n=0..k)];
```

$$X := \left[3, \frac{151}{51}, \frac{38}{13}, \frac{153}{53}, \frac{77}{27}, \frac{31}{11}, \frac{39}{14}, \frac{157}{57}, \frac{79}{29}, \frac{159}{59}, \frac{8}{3}, \frac{161}{61}, \frac{81}{31}, \frac{163}{63}, \frac{41}{16}, \frac{33}{13}, \frac{83}{33}, \frac{167}{67}, \frac{42}{17}, \frac{169}{69}, \frac{17}{7}, \frac{171}{71}, \frac{43}{18}, \frac{173}{73},\right.$$
$$\frac{87}{37}, \frac{7}{3}, \frac{44}{19}, \frac{177}{77}, \frac{89}{39}, \frac{179}{79}, \frac{9}{4}, \frac{181}{81}, \frac{91}{41}, \frac{183}{83}, \frac{46}{21}, \frac{37}{17}, \frac{93}{43}, \frac{187}{87}, \frac{47}{22}, \frac{189}{89}, \frac{19}{9}, \frac{191}{91}, \frac{48}{23}, \frac{193}{93}, \frac{97}{47}, \frac{39}{19}, \frac{49}{24}, \frac{197}{97}, \frac{99}{49},$$
$$\frac{199}{99}, 2, \frac{201}{101}, \frac{101}{51}, \frac{203}{103}, \frac{51}{26}, \frac{41}{21}, \frac{103}{53}, \frac{207}{107}, \frac{52}{27}, \frac{209}{109}, \frac{21}{11}, \frac{211}{111}, \frac{53}{28}, \frac{213}{113}, \frac{107}{57}, \frac{43}{23}, \frac{54}{29}, \frac{217}{117}, \frac{109}{59}, \frac{219}{119}, \frac{11}{6}, \frac{221}{121},$$
$$\frac{111}{61}, \frac{223}{123}, \frac{56}{31}, \frac{9}{5}, \frac{113}{63}, \frac{227}{127}, \frac{57}{32}, \frac{229}{129}, \frac{23}{13}, \frac{231}{131}, \frac{58}{33}, \frac{233}{133}, \frac{117}{67}, \frac{47}{27}, \frac{59}{34}, \frac{237}{137}, \frac{119}{69}, \frac{239}{139}, \frac{12}{7}, \frac{241}{141}, \frac{121}{71}, \frac{243}{143}, \frac{61}{36},$$
$$\left.\frac{49}{29}, \frac{123}{73}, \frac{247}{147}, \frac{62}{37}, \frac{249}{149}, \frac{5}{3}\right]$$

```
> Y := [seq((-3*k+2*n)/(k+2*n), n=0..k)];
```

$$Y := \left[-3, \frac{-149}{51}, \frac{-37}{13}, \frac{-147}{53}, \frac{-73}{27}, \frac{-29}{11}, \frac{-18}{7}, \frac{-143}{57}, \frac{-71}{29}, \frac{-141}{59}, \frac{-7}{3}, \frac{-139}{61}, \frac{-69}{31}, \frac{-137}{63}, \frac{-17}{8}, \frac{-27}{13}, \frac{-67}{33}, \frac{-133}{67}, \frac{-33}{17},\right.$$

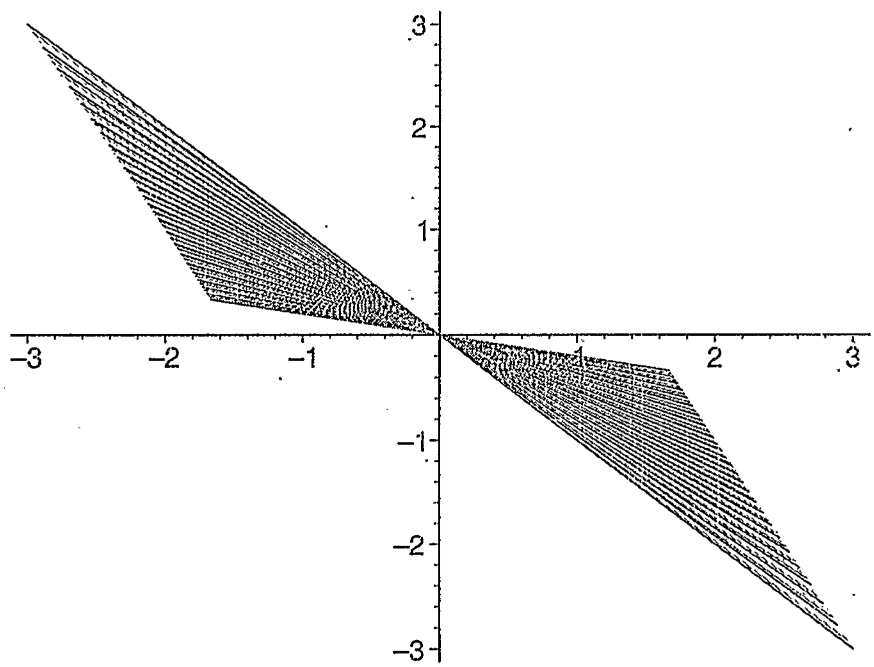
$$\frac{-131}{69}, \frac{-13}{7}, \frac{-129}{71}, \frac{-16}{9}, \frac{-127}{73}, \frac{-63}{37}, \frac{-5}{3}, \frac{-31}{19}, \frac{-123}{77}, \frac{-61}{39}, \frac{-121}{79}, \frac{-3}{2}, \frac{-119}{81}, \frac{-59}{41}, \frac{-117}{83}, \frac{-29}{21}, \frac{-23}{17}, \frac{-57}{43}, \frac{-113}{87}, \frac{-14}{11},$$

$$\frac{-111}{89}, \frac{-11}{9}, \frac{-109}{91}, \frac{-27}{23}, \frac{-107}{93}, \frac{-53}{47}, \frac{-21}{19}, \frac{-13}{12}, \frac{-103}{97}, \frac{-51}{49}, \frac{-101}{99}, -1, \frac{-99}{101}, \frac{-49}{51}, \frac{-97}{103}, \frac{-12}{13}, \frac{-19}{21}, \frac{-47}{53}, \frac{-93}{107}, \frac{-23}{27}, \frac{-91}{109},$$

$$\frac{-9}{11}, \frac{-89}{111}, \frac{-11}{14}, \frac{-87}{113}, \frac{-43}{57}, \frac{-17}{23}, \frac{-21}{29}, \frac{-83}{117}, \frac{-41}{59}, \frac{-81}{119}, \frac{-2}{3}, \frac{-79}{121}, \frac{-39}{61}, \frac{-77}{123}, \frac{-19}{31}, \frac{-3}{5}, \frac{-37}{63}, \frac{-73}{127}, \frac{-9}{16}, \frac{-71}{129}, \frac{-7}{13}, \frac{-69}{131},$$

$$\left.\frac{-17}{33}, \frac{-67}{133}, \frac{-33}{67}, \frac{-13}{27}, \frac{-8}{17}, \frac{-63}{137}, \frac{-31}{69}, \frac{-61}{139}, \frac{-3}{7}, \frac{-59}{141}, \frac{-29}{71}, \frac{-57}{143}, \frac{-7}{18}, \frac{-11}{29}, \frac{-27}{73}, \frac{-53}{147}, \frac{-13}{37}, \frac{-51}{149}, \frac{-1}{3}\right]$$

```
> plot([seq([[X[i],Y[i]],[-X[i],-Y[i]]], i=1..k+1)]);
```




```
> k := 1000;
```

$$k := 1000$$

```
> X := [seq((3*k+2*n)/(k+2*n), n=0..k)];
```

$$X := \left[3, \frac{1501}{501}, \frac{751}{251}, \frac{1503}{503}, \frac{188}{63}, \frac{301}{101}, \frac{753}{253}, \frac{1507}{507}, \frac{377}{127}, \frac{1509}{509}, \frac{151}{51}, \frac{1511}{511}, \frac{189}{64}, \frac{1513}{513}, \frac{757}{257}, \frac{303}{103}, \frac{379}{129},\right.$$

$$\frac{1517}{517}, \frac{759}{259}, \frac{1519}{519}, \frac{38}{13}, \frac{1521}{521}, \frac{761}{261}, \frac{1523}{523}, \frac{381}{131}, \frac{61}{21}, \frac{763}{263}, \frac{1527}{527}, \frac{191}{66}, \frac{1529}{529}, \frac{153}{53}, \frac{1531}{531}, \frac{383}{133}, \frac{1533}{533}, \frac{767}{267},$$

$$\frac{307}{107}, \frac{192}{67}, \frac{1537}{537}, \frac{769}{269}, \frac{1539}{539}, \frac{77}{27}, \frac{1541}{541}, \frac{771}{271}, \frac{1543}{543}, \frac{193}{68}, \frac{309}{109}, \frac{773}{273}, \frac{1547}{547}, \frac{387}{137}, \frac{1549}{549}, \frac{31}{11}, \frac{1551}{551}, \frac{194}{69},$$

$$\frac{1553}{553}, \frac{777}{277}, \frac{311}{111}, \frac{389}{139}, \frac{1557}{557}, \frac{779}{279}, \frac{1559}{559}, \frac{39}{14}, \frac{1561}{561}, \frac{781}{281}, \frac{1563}{563}, \frac{391}{141}, \frac{313}{113}, \frac{783}{283}, \frac{1567}{567}, \frac{196}{71}, \frac{1569}{569}, \frac{157}{57},$$

$\frac{1571}{571}$, $\frac{393}{143}$, $\frac{1573}{573}$, $\frac{787}{287}$, $\frac{63}{23}$, $\frac{197}{72}$, $\frac{1577}{577}$, $\frac{789}{289}$, $\frac{1579}{579}$, $\frac{79}{29}$, $\frac{1581}{581}$, $\frac{791}{291}$, $\frac{1583}{583}$, $\frac{198}{73}$, $\frac{317}{117}$, $\frac{793}{293}$, $\frac{1587}{587}$, $\frac{397}{147}$,

$\frac{1589}{589}$, $\frac{159}{59}$, $\frac{1591}{591}$, $\frac{199}{74}$, $\frac{1593}{593}$, $\frac{797}{297}$, $\frac{319}{119}$, $\frac{399}{149}$, $\frac{1597}{597}$, $\frac{799}{299}$, $\frac{1599}{599}$, $\frac{8}{3}$, $\frac{1601}{601}$, $\frac{801}{301}$, $\frac{1603}{603}$, $\frac{401}{151}$, $\frac{321}{121}$, $\frac{803}{303}$,

$\frac{1607}{607}$, $\frac{201}{76}$, $\frac{1609}{609}$, $\frac{161}{61}$, $\frac{1611}{611}$, $\frac{403}{153}$, $\frac{1613}{613}$, $\frac{807}{307}$, $\frac{323}{123}$, $\frac{202}{77}$, $\frac{1617}{617}$, $\frac{809}{309}$, $\frac{1619}{619}$, $\frac{81}{31}$, $\frac{1621}{621}$, $\frac{811}{311}$, $\frac{1623}{623}$, $\frac{203}{78}$,

$\frac{13}{5}$, $\frac{813}{313}$, $\frac{1627}{627}$, $\frac{407}{157}$, $\frac{1629}{629}$, $\frac{163}{63}$, $\frac{1631}{631}$, $\frac{204}{79}$, $\frac{1633}{633}$, $\frac{817}{317}$, $\frac{327}{127}$, $\frac{409}{159}$, $\frac{1637}{637}$, $\frac{819}{319}$, $\frac{1639}{639}$, $\frac{41}{16}$, $\frac{1641}{641}$, $\frac{821}{321}$,

$\frac{1643}{643}$, $\frac{411}{161}$, $\frac{329}{129}$, $\frac{823}{323}$, $\frac{1647}{647}$, $\frac{206}{81}$, $\frac{1649}{649}$, $\frac{33}{13}$, $\frac{1651}{651}$, $\frac{413}{163}$, $\frac{1653}{653}$, $\frac{827}{327}$, $\frac{331}{131}$, $\frac{207}{82}$, $\frac{1657}{657}$, $\frac{829}{329}$, $\frac{1659}{659}$, $\frac{83}{33}$,

$\frac{1661}{661}$, $\frac{831}{331}$, $\frac{1663}{663}$, $\frac{208}{83}$, $\frac{333}{133}$, $\frac{833}{333}$, $\frac{1667}{667}$, $\frac{417}{167}$, $\frac{1669}{669}$, $\frac{167}{67}$, $\frac{1671}{671}$, $\frac{209}{84}$, $\frac{1673}{673}$, $\frac{837}{337}$, $\frac{67}{27}$, $\frac{419}{169}$, $\frac{1677}{677}$, $\frac{839}{339}$,

$\frac{1679}{679}$, $\frac{42}{17}$, $\frac{1681}{681}$, $\frac{841}{341}$, $\frac{1683}{683}$, $\frac{421}{171}$, $\frac{337}{137}$, $\frac{843}{343}$, $\frac{1687}{687}$, $\frac{211}{86}$, $\frac{1689}{689}$, $\frac{169}{69}$, $\frac{1691}{691}$, $\frac{423}{173}$, $\frac{1693}{693}$, $\frac{847}{347}$, $\frac{339}{139}$, $\frac{212}{87}$,

$\frac{1697}{697}$, $\frac{849}{349}$, $\frac{1699}{699}$, $\frac{17}{7}$, $\frac{1701}{701}$, $\frac{851}{351}$, $\frac{1703}{703}$, $\frac{213}{88}$, $\frac{341}{141}$, $\frac{853}{353}$, $\frac{1707}{707}$, $\frac{427}{177}$, $\frac{1709}{709}$, $\frac{171}{71}$, $\frac{1711}{711}$, $\frac{214}{89}$, $\frac{1713}{713}$, $\frac{857}{357}$,

$\frac{343}{143}$, $\frac{429}{179}$, $\frac{1717}{717}$, $\frac{859}{359}$, $\frac{1719}{719}$, $\frac{43}{18}$, $\frac{1721}{721}$, $\frac{861}{361}$, $\frac{1723}{723}$, $\frac{431}{181}$, $\frac{69}{29}$, $\frac{863}{363}$, $\frac{1727}{727}$, $\frac{216}{91}$, $\frac{1729}{729}$, $\frac{173}{73}$, $\frac{1731}{731}$, $\frac{433}{183}$,

$\frac{1733}{733}$, $\frac{867}{367}$, $\frac{347}{147}$, $\frac{217}{92}$, $\frac{1737}{737}$, $\frac{869}{369}$, $\frac{1739}{739}$, $\frac{87}{37}$, $\frac{1741}{741}$, $\frac{871}{371}$, $\frac{1743}{743}$, $\frac{218}{93}$, $\frac{349}{149}$, $\frac{873}{373}$, $\frac{1747}{747}$, $\frac{437}{187}$, $\frac{1749}{749}$, $\frac{7}{3}$,

$\frac{1751}{751}$, $\frac{219}{94}$, $\frac{1753}{753}$, $\frac{877}{377}$, $\frac{351}{151}$, $\frac{439}{189}$, $\frac{1757}{757}$, $\frac{879}{379}$, $\frac{1759}{759}$, $\frac{44}{19}$, $\frac{1761}{761}$, $\frac{881}{381}$, $\frac{1763}{763}$, $\frac{441}{191}$, $\frac{353}{153}$, $\frac{883}{383}$, $\frac{1767}{767}$, $\frac{221}{96}$,

$\frac{1769}{769}$, $\frac{177}{77}$, $\frac{1771}{771}$, $\frac{443}{193}$, $\frac{1773}{773}$, $\frac{887}{387}$, $\frac{71}{31}$, $\frac{222}{97}$, $\frac{1777}{777}$, $\frac{889}{389}$, $\frac{1779}{779}$, $\frac{89}{39}$, $\frac{1781}{781}$, $\frac{891}{391}$, $\frac{1783}{783}$, $\frac{223}{98}$, $\frac{357}{157}$, $\frac{893}{393}$,

$\frac{1787}{787}$, $\frac{447}{197}$, $\frac{1789}{789}$, $\frac{179}{79}$, $\frac{1791}{791}$, $\frac{224}{99}$, $\frac{1793}{793}$, $\frac{897}{397}$, $\frac{359}{159}$, $\frac{449}{199}$, $\frac{1797}{797}$, $\frac{899}{399}$, $\frac{1799}{799}$, $\frac{9}{4}$, $\frac{1801}{801}$, $\frac{901}{401}$, $\frac{1803}{803}$, $\frac{451}{201}$,

$\frac{361}{161}$, $\frac{903}{403}$, $\frac{1807}{807}$, $\frac{226}{101}$, $\frac{1809}{809}$, $\frac{181}{81}$, $\frac{1811}{811}$, $\frac{453}{203}$, $\frac{1813}{813}$, $\frac{907}{407}$, $\frac{363}{163}$, $\frac{227}{102}$, $\frac{1817}{817}$, $\frac{909}{409}$, $\frac{1819}{819}$, $\frac{91}{41}$, $\frac{1821}{821}$, $\frac{911}{411}$,

$\frac{1823}{823}$, $\frac{228}{103}$, $\frac{73}{33}$, $\frac{913}{413}$, $\frac{1827}{827}$, $\frac{457}{207}$, $\frac{1829}{829}$, $\frac{183}{83}$, $\frac{1831}{831}$, $\frac{229}{104}$, $\frac{1833}{833}$, $\frac{917}{417}$, $\frac{367}{167}$, $\frac{459}{209}$, $\frac{1837}{837}$, $\frac{919}{419}$, $\frac{1839}{839}$, $\frac{46}{21}$,

$\frac{1841}{841}$, $\frac{921}{421}$, $\frac{1843}{843}$, $\frac{461}{211}$, $\frac{369}{169}$, $\frac{923}{423}$, $\frac{1847}{847}$, $\frac{231}{106}$, $\frac{1849}{849}$, $\frac{37}{17}$, $\frac{1851}{851}$, $\frac{463}{213}$, $\frac{1853}{853}$, $\frac{927}{427}$, $\frac{371}{171}$, $\frac{232}{107}$, $\frac{1857}{857}$, $\frac{929}{429}$,

$\frac{1859}{859}$, $\frac{93}{43}$, $\frac{1861}{861}$, $\frac{931}{431}$, $\frac{1863}{863}$, $\frac{233}{108}$, $\frac{373}{173}$, $\frac{933}{433}$, $\frac{1867}{867}$, $\frac{467}{217}$, $\frac{1869}{869}$, $\frac{187}{87}$, $\frac{1871}{871}$, $\frac{234}{109}$, $\frac{1873}{873}$, $\frac{937}{437}$, $\frac{15}{7}$, $\frac{469}{219}$,

$\frac{1877}{877}$, $\frac{939}{439}$, $\frac{1879}{879}$, $\frac{47}{22}$, $\frac{1881}{881}$, $\frac{941}{441}$, $\frac{1883}{883}$, $\frac{471}{221}$, $\frac{377}{177}$, $\frac{943}{443}$, $\frac{1887}{887}$, $\frac{236}{111}$, $\frac{1889}{889}$, $\frac{189}{89}$, $\frac{1891}{891}$, $\frac{473}{223}$, $\frac{1893}{893}$, $\frac{947}{447}$,

$\frac{379}{179}, \frac{237}{112}, \frac{897}{897}$

$\frac{379}{179}, \frac{237}{112}, \frac{1897}{897}, \frac{949}{449}, \frac{1899}{899}, \frac{19}{9}, \frac{1901}{901}, \frac{951}{451}, \frac{1903}{903}, \frac{238}{113}, \frac{381}{181}, \frac{953}{453}, \frac{1907}{907}, \frac{477}{227}, \frac{1909}{909}, \frac{191}{91}, \frac{1911}{911}, \frac{239}{114},$

$\frac{1913}{913}, \frac{957}{457}, \frac{383}{183}, \frac{479}{229}, \frac{1917}{917}, \frac{959}{459}, \frac{1919}{919}, \frac{48}{23}, \frac{1921}{921}, \frac{961}{461}, \frac{1923}{923}, \frac{481}{231}, \frac{77}{37}, \frac{963}{463}, \frac{1927}{927}, \frac{241}{116}, \frac{1929}{929}, \frac{193}{93},$

$\frac{1931}{931}, \frac{483}{233}, \frac{1933}{933}, \frac{967}{467}, \frac{387}{187}, \frac{242}{117}, \frac{1937}{937}, \frac{969}{469}, \frac{1939}{939}, \frac{97}{47}, \frac{1941}{941}, \frac{971}{471}, \frac{1943}{943}, \frac{243}{118}, \frac{389}{189}, \frac{973}{473}, \frac{1947}{947}, \frac{487}{237},$

$\frac{1949}{949}, \frac{39}{19}, \frac{1951}{951}, \frac{244}{119}, \frac{1953}{953}, \frac{977}{477}, \frac{391}{191}, \frac{489}{239}, \frac{1957}{957}, \frac{979}{479}, \frac{1959}{959}, \frac{49}{24}, \frac{1961}{961}, \frac{981}{481}, \frac{1963}{963}, \frac{491}{241}, \frac{393}{193}, \frac{983}{483},$

$\frac{1967}{967}, \frac{246}{121}, \frac{1969}{969}, \frac{197}{97}, \frac{1971}{971}, \frac{493}{243}, \frac{1973}{973}, \frac{987}{487}, \frac{79}{39}, \frac{247}{122}, \frac{1977}{977}, \frac{989}{489}, \frac{1979}{979}, \frac{99}{49}, \frac{1981}{981}, \frac{991}{491}, \frac{1983}{983}, \frac{248}{123},$

$\frac{397}{197}, \frac{993}{493}, \frac{1987}{987}, \frac{497}{247}, \frac{1989}{989}, \frac{199}{99}, \frac{1991}{991}, \frac{249}{124}, \frac{1993}{993}, \frac{997}{497}, \frac{399}{199}, \frac{499}{249}, \frac{1997}{997}, \frac{999}{499}, \frac{1999}{999}, 2, \frac{2001}{1001}, \frac{1001}{501},$

$\frac{2003}{1003}, \frac{501}{251}, \frac{401}{201}, \frac{1003}{503}, \frac{2007}{1007}, \frac{251}{126}, \frac{2009}{1009}, \frac{201}{101}, \frac{2011}{1011}, \frac{503}{253}, \frac{2013}{1013}, \frac{1007}{507}, \frac{403}{203}, \frac{252}{127}, \frac{2017}{1017}, \frac{1009}{509}, \frac{2019}{1019},$

$\frac{101}{51}, \frac{2021}{1021}, \frac{1011}{511}, \frac{2023}{1023}, \frac{253}{128}, \frac{81}{41}, \frac{1013}{513}, \frac{2027}{1027}, \frac{507}{257}, \frac{2029}{1029}, \frac{203}{103}, \frac{2031}{1031}, \frac{254}{129}, \frac{2033}{1033}, \frac{1017}{517}, \frac{407}{207}, \frac{509}{259},$

$\frac{2037}{1037}, \frac{1019}{519}, \frac{2039}{1039}, \frac{51}{26}, \frac{2041}{1041}, \frac{1021}{521}, \frac{2043}{1043}, \frac{511}{261}, \frac{409}{209}, \frac{1023}{523}, \frac{2047}{1047}, \frac{256}{131}, \frac{2049}{1049}, \frac{41}{21}, \frac{2051}{1051}, \frac{513}{263}, \frac{2053}{1053},$

$\frac{1027}{527}, \frac{411}{211}, \frac{257}{132}, \frac{2057}{1057}, \frac{1029}{529}, \frac{2059}{1059}, \frac{103}{53}, \frac{2061}{1061}, \frac{1031}{531}, \frac{2063}{1063}, \frac{258}{133}, \frac{413}{213}, \frac{1033}{533}, \frac{2067}{1067}, \frac{517}{267}, \frac{2069}{1069}, \frac{207}{107},$

$\frac{2071}{1071}, \frac{259}{134}, \frac{2073}{1073}, \frac{1037}{537}, \frac{83}{43}, \frac{519}{269}, \frac{2077}{1077}, \frac{1039}{539}, \frac{2079}{1079}, \frac{52}{27}, \frac{2081}{1081}, \frac{1041}{541}, \frac{2083}{1083}, \frac{521}{271}, \frac{417}{217}, \frac{1043}{543}, \frac{2087}{1087},$

$\frac{261}{136}, \frac{2089}{1089}, \frac{209}{109}, \frac{2091}{1091}, \frac{523}{273}, \frac{2093}{1093}, \frac{1047}{547}, \frac{419}{219}, \frac{262}{137}, \frac{2097}{1097}, \frac{1049}{549}, \frac{2099}{1099}, \frac{21}{11}, \frac{2101}{1101}, \frac{1051}{551}, \frac{2103}{1103}, \frac{263}{138},$

$\frac{421}{221}, \frac{1053}{553}, \frac{2107}{1107}, \frac{527}{277}, \frac{2109}{1109}, \frac{211}{111}, \frac{2111}{1111}, \frac{264}{139}, \frac{2113}{1113}, \frac{1057}{557}, \frac{423}{223}, \frac{529}{279}, \frac{2117}{1117}, \frac{1059}{559}, \frac{2119}{1119}, \frac{53}{28}, \frac{2121}{1121},$

$\frac{1061}{561}, \frac{2123}{1123}, \frac{531}{281}, \frac{17}{9}, \frac{1063}{563}, \frac{2127}{1127}, \frac{266}{141}, \frac{2129}{1129}, \frac{213}{113}, \frac{2131}{1131}, \frac{533}{283}, \frac{2133}{1133}, \frac{1067}{567}, \frac{427}{227}, \frac{267}{142}, \frac{2137}{1137}, \frac{1069}{569},$

$\frac{2139}{1139}, \frac{107}{57}, \frac{2141}{1141}, \frac{1071}{571}, \frac{2143}{1143}, \frac{268}{143}, \frac{429}{229}, \frac{1073}{573}, \frac{2147}{1147}, \frac{537}{287}, \frac{2149}{1149}, \frac{43}{23}, \frac{2151}{1151}, \frac{269}{144}, \frac{2153}{1153}, \frac{1077}{577}, \frac{431}{231},$

$\frac{539}{289}, \frac{2157}{1157}, \frac{1079}{579}, \frac{2159}{1159}, \frac{54}{29}, \frac{2161}{1161}, \frac{1081}{581}, \frac{2163}{1163}, \frac{541}{291}, \frac{433}{233}, \frac{1083}{583}, \frac{2167}{1167}, \frac{271}{146}, \frac{2169}{1169}, \frac{217}{117}, \frac{2171}{1171}, \frac{543}{293},$

$\frac{2173}{1173}, \frac{1087}{587}, \frac{87}{47}, \frac{272}{147}, \frac{2177}{1177}, \frac{1089}{589}, \frac{2179}{1179}, \frac{109}{59}, \frac{2181}{1181}, \frac{1091}{591}, \frac{2183}{1183}, \frac{273}{148}, \frac{437}{237}, \frac{1093}{593}, \frac{2187}{1187}, \frac{547}{297}, \frac{2189}{1189},$

$\frac{219}{119}, \frac{2191}{1191}, \frac{274}{149}, \frac{2193}{1193}, \frac{1097}{597}, \frac{439}{239}, \frac{549}{299}, \frac{2197}{1197}, \frac{1099}{599}, \frac{2199}{1199}, \frac{11}{6}, \frac{2201}{1201}, \frac{1101}{601}, \frac{2203}{1203}, \frac{551}{301}, \frac{441}{241}, \frac{1103}{603},$

$\frac{2207}{1207}, \frac{276}{151}, \frac{2209}{1209}, \frac{221}{121}, \frac{2211}{1211}, \frac{553}{303}, \frac{2213}{1213}, \frac{1107}{607}, \frac{443}{243}, \frac{277}{152}, \frac{2217}{1217}, \frac{1109}{609}, \frac{2219}{1219}, \frac{111}{61}, \frac{2221}{1221}, \frac{1111}{611}, \frac{2223}{1223},$

$\frac{278}{153}, \frac{89}{49}, \frac{1113}{613}, \frac{2227}{1227}, \frac{557}{307}, \frac{2229}{1229}, \frac{223}{123}, \frac{2231}{1231}, \frac{279}{154}, \frac{2233}{1233}, \frac{1117}{617}, \frac{447}{247}, \frac{559}{309}, \frac{2237}{1237}, \frac{1119}{619}, \frac{2239}{1239}, \frac{56}{31},$

$\frac{2241}{1241}, \frac{1121}{621}, \frac{2243}{1243}, \frac{561}{311}, \frac{449}{249}, \frac{1123}{623}, \frac{2247}{1247}, \frac{281}{156}, \frac{2249}{1249}, \frac{9}{5}, \frac{2251}{1251}, \frac{563}{313}, \frac{2253}{1253}, \frac{1127}{627}, \frac{451}{251}, \frac{282}{157}, \frac{2257}{1257},$

$\frac{1129}{629}, \frac{2259}{1259}, \frac{113}{63}, \frac{2261}{1261}, \frac{1131}{631}, \frac{2263}{1263}, \frac{283}{158}, \frac{453}{253}, \frac{1133}{633}, \frac{2267}{1267}, \frac{567}{317}, \frac{2269}{1269}, \frac{227}{127}, \frac{2271}{1271}, \frac{284}{159}, \frac{2273}{1273}, \frac{1137}{637},$

$\frac{91}{51}, \frac{569}{319}, \frac{2277}{1277}, \frac{1139}{639}, \frac{2279}{1279}, \frac{57}{32}, \frac{2281}{1281}, \frac{1141}{641}, \frac{2283}{1283}, \frac{571}{321}, \frac{457}{257}, \frac{1143}{643}, \frac{2287}{1287}, \frac{286}{161}, \frac{2289}{1289}, \frac{229}{129}, \frac{2291}{1291},$

$\frac{573}{323}, \frac{2293}{1293}, \frac{1147}{647}, \frac{459}{259}, \frac{287}{162}, \frac{2297}{1297}, \frac{1149}{649}, \frac{2299}{1299}, \frac{23}{13}, \frac{2301}{1301}, \frac{1151}{651}, \frac{2303}{1303}, \frac{288}{163}, \frac{461}{261}, \frac{1153}{653}, \frac{2307}{1307}, \frac{577}{327},$

$\frac{2309}{1309}, \frac{231}{131}, \frac{2311}{1311}, \frac{289}{164}, \frac{2313}{1313}, \frac{1157}{657}, \frac{463}{263}, \frac{579}{329}, \frac{2317}{1317}, \frac{1159}{659}, \frac{2319}{1319}, \frac{58}{33}, \frac{2321}{1321}, \frac{1161}{661}, \frac{2323}{1323}, \frac{581}{331}, \frac{93}{53},$

$\frac{1163}{663}, \frac{2327}{1327}, \frac{291}{166}, \frac{2329}{1329}, \frac{233}{133}, \frac{2331}{1331}, \frac{583}{333}, \frac{2333}{1333}, \frac{1167}{667}, \frac{467}{267}, \frac{292}{167}, \frac{2337}{1337}, \frac{1169}{669}, \frac{2339}{1339}, \frac{117}{67}, \frac{2341}{1341}, \frac{1171}{671},$

$\frac{2343}{1343}, \frac{293}{168}, \frac{469}{269}, \frac{1173}{673}, \frac{2347}{1347}, \frac{587}{337}, \frac{2349}{1349}, \frac{47}{27}, \frac{2351}{1351}, \frac{294}{169}, \frac{2353}{1353}, \frac{1177}{677}, \frac{471}{271}, \frac{589}{339}, \frac{2357}{1357}, \frac{1179}{679}, \frac{2359}{1359},$

$\frac{59}{34}, \frac{2361}{1361}, \frac{1181}{681}, \frac{2363}{1363}, \frac{591}{341}, \frac{473}{273}, \frac{1183}{683}, \frac{2367}{1367}, \frac{296}{171}, \frac{2369}{1369}, \frac{237}{137}, \frac{2371}{1371}, \frac{593}{343}, \frac{2373}{1373}, \frac{1187}{687}, \frac{19}{11}, \frac{297}{172},$

$\frac{2377}{1377}, \frac{1189}{689}, \frac{2379}{1379}, \frac{119}{69}, \frac{2381}{1381}, \frac{1191}{691}, \frac{2383}{1383}, \frac{298}{173}, \frac{477}{277}, \frac{1193}{693}, \frac{2387}{1387}, \frac{597}{347}, \frac{2389}{1389}, \frac{239}{139}, \frac{2391}{1391}, \frac{299}{174}, \frac{2393}{1393},$

$\frac{1197}{697}, \frac{479}{279}, \frac{599}{349}, \frac{2397}{1397}, \frac{1199}{699}, \frac{2399}{1399}, \frac{12}{7}, \frac{2401}{1401}, \frac{1201}{701}, \frac{2403}{1403}, \frac{601}{351}, \frac{481}{281}, \frac{1203}{703}, \frac{2407}{1407}, \frac{301}{176}, \frac{2409}{1409}, \frac{241}{141},$

$\frac{2411}{1411}, \frac{603}{353}, \frac{2413}{1413}, \frac{1207}{707}, \frac{483}{283}, \frac{302}{177}, \frac{2417}{1417}, \frac{1209}{709}, \frac{2419}{1419}, \frac{121}{71}, \frac{2421}{1421}, \frac{1211}{711}, \frac{2423}{1423}, \frac{303}{178}, \frac{97}{57}, \frac{1213}{713}, \frac{2427}{1427},$

$\frac{607}{357}, \frac{2429}{1429}, \frac{243}{143}, \frac{2431}{1431}, \frac{304}{179}, \frac{2433}{1433}, \frac{1217}{717}, \frac{487}{287}, \frac{609}{359}, \frac{2437}{1437}, \frac{1219}{719}, \frac{2439}{1439}, \frac{61}{36}, \frac{2441}{1441}, \frac{1221}{721}, \frac{2443}{1443}, \frac{611}{361},$

$\frac{489}{289}, \frac{1223}{723}, \frac{2447}{1447}, \frac{306}{181}, \frac{2449}{1449}, \frac{49}{29}, \frac{2451}{1451}, \frac{613}{363}, \frac{2453}{1453}, \frac{1227}{727}, \frac{491}{291}, \frac{307}{182}, \frac{2457}{1457}, \frac{1229}{729}, \frac{2459}{1459}, \frac{123}{73}, \frac{2461}{1461},$

$\frac{1231}{731}, \frac{2463}{1463}, \frac{308}{183}, \frac{493}{293}, \frac{1233}{733}, \frac{2467}{1467}, \frac{617}{367}, \frac{2469}{1469}, \frac{247}{147}, \frac{2471}{1471}, \frac{309}{184}, \frac{2473}{1473}, \frac{1237}{737}, \frac{99}{59}, \frac{619}{369}, \frac{2477}{1477}, \frac{1239}{739},$

$\frac{2479}{1479}, \frac{62}{37}, \frac{2481}{1481}, \frac{1241}{741}, \frac{2483}{1483}, \frac{621}{371}, \frac{497}{297}, \frac{1243}{743}, \frac{2487}{1487}, \frac{311}{186}, \frac{2489}{1489}, \frac{249}{149}, \frac{2491}{1491}, \frac{623}{373}, \frac{2493}{1493}, \frac{1247}{747}, \frac{499}{299},$

$\frac{312}{187}, \frac{2497}{1497}, \frac{1249}{749}, \frac{2499}{1499}, \frac{5}{3}]$

```
> Y := [seq((-3*k+2*n)/(k+2*n), n=0..k)];
```

$$Y := \left[-3, \frac{-1499}{501}, \frac{-749}{251}, \frac{-1497}{503}, \frac{-187}{63}, \frac{-299}{101}, \frac{-747}{253}, \frac{-1493}{507}, \frac{-373}{127}, \frac{-1491}{509}, \frac{-149}{51}, \frac{-1489}{511}, \frac{-93}{32}, \frac{-1487}{513}, \frac{-743}{257}, \frac{-297}{103},\right.$$

$$\frac{-371}{129}, \frac{-1483}{517}, \frac{-741}{259}, \frac{-1481}{519}, \frac{-37}{13}, \frac{-1479}{521}, \frac{-739}{261}, \frac{-1477}{523}, \frac{-369}{131}, \frac{-59}{21}, \frac{-737}{263}, \frac{-1473}{527}, \frac{-92}{33}, \frac{-1471}{529}, \frac{-147}{53}, \frac{-1469}{531},$$

$$\frac{-367}{133}, \frac{-1467}{533}, \frac{-733}{267}, \frac{-293}{107}, \frac{-183}{67}, \frac{-1463}{537}, \frac{-731}{269}, \frac{-1461}{539}, \frac{-73}{27}, \frac{-1459}{541}, \frac{-729}{271}, \frac{-1457}{543}, \frac{-91}{34}, \frac{-291}{109}, \frac{-727}{273}, \frac{-1453}{547},$$

$$\frac{-363}{137}, \frac{-1451}{549}, \frac{-29}{11}, \frac{-1449}{551}, \frac{-181}{69}, \frac{-1447}{553}, \frac{-723}{277}, \frac{-289}{111}, \frac{-361}{139}, \frac{-1443}{557}, \frac{-721}{279}, \frac{-1441}{559}, \frac{-18}{7}, \frac{-1439}{561}, \frac{-719}{281}, \frac{-1437}{563},$$

$$\frac{-359}{141}, \frac{-287}{113}, \frac{-717}{283}, \frac{-1433}{567}, \frac{-179}{71}, \frac{-1431}{569}, \frac{-143}{57}, \frac{-1429}{571}, \frac{-357}{143}, \frac{-1427}{573}, \frac{-713}{287}, \frac{-57}{23}, \frac{-89}{36}, \frac{-1423}{577}, \frac{-711}{289}, \frac{-1421}{579}, \frac{-71}{29},$$

$$\frac{-1419}{581}, \frac{-709}{291}, \frac{-1417}{583}, \frac{-177}{73}, \frac{-283}{117}, \frac{-707}{293}, \frac{-1413}{587}, \frac{-353}{147}, \frac{-1411}{589}, \frac{-141}{59}, \frac{-1409}{591}, \frac{-88}{37}, \frac{-1407}{593}, \frac{-703}{297}, \frac{-281}{119}, \frac{-351}{149},$$

$$\frac{-1403}{597}, \frac{-701}{299}, \frac{-1401}{599}, \frac{-7}{3}, \frac{-1399}{601}, \frac{-699}{301}, \frac{-1397}{603}, \frac{-349}{151}, \frac{-279}{121}, \frac{-697}{303}, \frac{-1393}{607}, \frac{-87}{38}, \frac{-1391}{609}, \frac{-139}{61}, \frac{-1389}{611}, \frac{-347}{153},$$

$$\frac{-1387}{613}, \frac{-693}{307}, \frac{-277}{123}, \frac{-173}{77}, \frac{-1383}{617}, \frac{-691}{309}, \frac{-1381}{619}, \frac{-69}{31}, \frac{-1379}{621}, \frac{-689}{311}, \frac{-1377}{623}, \frac{-86}{39}, \frac{-11}{5}, \frac{-687}{313}, \frac{-1373}{627}, \frac{-343}{157},$$

$$\frac{-1371}{629}, \frac{-137}{63}, \frac{-1369}{631}, \frac{-171}{79}, \frac{-1367}{633}, \frac{-683}{317}, \frac{-273}{127}, \frac{-341}{159}, \frac{-1363}{637}, \frac{-681}{319}, \frac{-1361}{639}, \frac{-17}{8}, \frac{-1359}{641}, \frac{-679}{321}, \frac{-1357}{643}, \frac{-339}{161},$$

$$\frac{-271}{129}, \frac{-677}{323}, \frac{-1353}{647}, \frac{-169}{81}, \frac{-1351}{649}, \frac{-27}{13}, \frac{-1349}{651}, \frac{-337}{163}, \frac{-1347}{653}, \frac{-673}{327}, \frac{-269}{131}, \frac{-84}{41}, \frac{-1343}{657}, \frac{-671}{329}, \frac{-1341}{659}, \frac{-67}{33},$$

$$\frac{-1339}{661}, \frac{-669}{331}, \frac{-1337}{663}, \frac{-167}{83}, \frac{-267}{133}, \frac{-667}{333}, \frac{-1333}{667}, \frac{-333}{167}, \frac{-1331}{669}, \frac{-133}{67}, \frac{-1329}{671}, \frac{-83}{42}, \frac{-1327}{673}, \frac{-663}{337}, \frac{-53}{27}, \frac{-331}{169},$$

$$\frac{-1323}{677}, \frac{-661}{339}, \frac{-1321}{679}, \frac{-33}{17}, \frac{-1319}{681}, \frac{-659}{341}, \frac{-1317}{683}, \frac{-329}{171}, \frac{-263}{137}, \frac{-657}{343}, \frac{-1313}{687}, \frac{-82}{43}, \frac{-1311}{689}, \frac{-131}{69}, \frac{-1309}{691}, \frac{-327}{173},$$

$$\frac{-1307}{693}, \frac{-653}{347}, \frac{-261}{139}, \frac{-163}{87}, \frac{-1303}{697}, \frac{-651}{349}, \frac{-1301}{699}, \frac{-13}{7}, \frac{-1299}{701}, \frac{-649}{351}, \frac{-1297}{703}, \frac{-81}{44}, \frac{-259}{141}, \frac{-647}{353}, \frac{-1293}{707}, \frac{-323}{177},$$

$$\frac{-1291}{709}, \frac{-129}{71}, \frac{-1289}{711}, \frac{-161}{89}, \frac{-1287}{713}, \frac{-643}{357}, \frac{-257}{143}, \frac{-321}{179}, \frac{-1283}{717}, \frac{-641}{359}, \frac{-1281}{719}, \frac{-16}{9}, \frac{-1279}{721}, \frac{-639}{361}, \frac{-1277}{723}, \frac{-319}{181},$$

$$\frac{-51}{29}, \frac{-637}{363}, \frac{-1273}{727}, \frac{-159}{91}, \frac{-1271}{729}, \frac{-127}{73}, \frac{-1269}{731}, \frac{-317}{183}, \frac{-1267}{733}, \frac{-633}{367}, \frac{-253}{147}, \frac{-79}{46}, \frac{-1263}{737}, \frac{-631}{369}, \frac{-1261}{739}, \frac{-63}{37},$$

$$\frac{-1259}{741}, \frac{-629}{371}, \frac{-1257}{743}, \frac{-157}{93}, \frac{-251}{149}, \frac{-627}{373}, \frac{-1253}{747}, \frac{-313}{187}, \frac{-1251}{749}, \frac{-5}{3}, \frac{-1249}{751}, \frac{-78}{47}, \frac{-1247}{753}, \frac{-623}{377}, \frac{-249}{151}, \frac{-311}{189},$$

$$\frac{-1243}{757}, \frac{-621}{379}, \frac{-1241}{759}, \frac{-31}{19}, \frac{-1239}{761}, \frac{-619}{381}, \frac{-1237}{763}, \frac{-309}{191}, \frac{-247}{153}, \frac{-617}{383}, \frac{-1233}{767}, \frac{-77}{48}, \frac{-1231}{769}, \frac{-123}{77}, \frac{-1229}{771}, \frac{-307}{193},$$

$$\frac{-1227}{773}, \frac{-613}{387}, \frac{-49}{31}, \frac{-153}{97}, \frac{-1223}{777}, \frac{-611}{389}, \frac{-1221}{779}, \frac{-61}{39}, \frac{-1219}{781}, \frac{-609}{391}, \frac{-1217}{783}, \frac{-76}{49}, \frac{-243}{157}, \frac{-607}{393}, \frac{-1213}{787}, \frac{-303}{197},$$

$$\frac{-1211}{789}, \frac{-121}{79}, \frac{-1209}{791}, \frac{-151}{99}, \frac{-1207}{793}, \frac{-603}{397}, \frac{-241}{159}, \frac{-301}{199}, \frac{-1203}{797}, \frac{-601}{399}, \frac{-1201}{799}, \frac{-3}{2}, \frac{-1199}{801}, \frac{-599}{401}, \frac{-1197}{803}, \frac{-299}{201},$$

$$\frac{-239}{161}, \frac{-597}{403}, \frac{-1193}{807}, \frac{-149}{101}, \frac{-1191}{809}, \frac{-119}{81}, \frac{-1189}{811}, \frac{-297}{203}, \frac{-1187}{813}, \frac{-593}{407}, \frac{-237}{163}, \frac{-74}{51}, \frac{-1183}{817}, \frac{-591}{409}, \frac{-1181}{819}, \frac{-59}{41},$$

$$\frac{-1179}{821}, \frac{-589}{411}, \frac{-1177}{823}, \frac{-147}{103}, \frac{-47}{33}, \frac{-587}{413}, \frac{-1173}{827}, \frac{-293}{207}, \frac{-1171}{829}, \frac{-117}{83}, \frac{-1169}{831}, \frac{-73}{52}, \frac{-1167}{833}, \frac{-583}{417}, \frac{-233}{167}, \frac{-291}{209},$$

$$\frac{-1163}{837}, \frac{-581}{419}, \frac{-1161}{839}, \frac{-29}{21}, \frac{-1159}{841}, \frac{-579}{421}, \frac{-1157}{843}, \frac{-289}{211}, \frac{-231}{169}, \frac{-577}{423}, \frac{-1153}{847}, \frac{-72}{53}, \frac{-1151}{849}, \frac{-23}{17}, \frac{-1149}{851}, \frac{-287}{213},$$

$$\frac{-1147}{853}, \frac{-573}{427}, \frac{-229}{171}, \frac{-143}{107}, \frac{-1143}{857}, \frac{-571}{429}, \frac{-1141}{859}, \frac{-57}{43}, \frac{-1139}{861}, \frac{-569}{431}, \frac{-1137}{863}, \frac{-71}{54}, \frac{-227}{173}, \frac{-567}{433}, \frac{-1133}{867}, \frac{-283}{217},$$

$$\frac{-1131}{869}, \frac{-113}{87}, \frac{-1129}{871}, \frac{-141}{109}, \frac{-1127}{873}, \frac{-563}{437}, \frac{-9}{7}, \frac{-281}{219}, \frac{-1123}{877}, \frac{-561}{439}, \frac{-1121}{879}, \frac{-14}{11}, \frac{-1119}{881}, \frac{-559}{441}, \frac{-1117}{883}, \frac{-279}{221},$$

$$\frac{-223}{177}, \frac{-557}{443}, \frac{-1113}{887}, \frac{-139}{111}, \frac{-1111}{889}, \frac{-111}{89}, \frac{-1109}{891}, \frac{-277}{223}, \frac{-1107}{893}, \frac{-553}{447}, \frac{-221}{179}, \frac{-69}{56}, \frac{-1103}{897}, \frac{-551}{449}, \frac{-1101}{899}, \frac{-11}{9},$$

$$\frac{-1099}{901}, \frac{-549}{451}, \frac{-1097}{903}, \frac{-137}{113}, \frac{-219}{181}, \frac{-547}{453}, \frac{-1093}{907}, \frac{-273}{227}, \frac{-1091}{909}, \frac{-109}{91}, \frac{-1089}{911}, \frac{-68}{57}, \frac{-1087}{913}, \frac{-543}{457}, \frac{-217}{183}, \frac{-271}{229},$$

$$\frac{-1083}{917}, \frac{-541}{459}, \frac{-1081}{919}, \frac{-27}{23}, \frac{-1079}{921}, \frac{-539}{461}, \frac{-1077}{923}, \frac{-269}{231}, \frac{-43}{37}, \frac{-537}{463}, \frac{-1073}{927}, \frac{-67}{58}, \frac{-1071}{929}, \frac{-107}{93}, \frac{-1069}{931}, \frac{-267}{233},$$

$$\frac{-1067}{933}, \frac{-533}{467}, \frac{-213}{187}, \frac{-133}{117}, \frac{-1063}{937}, \frac{-531}{469}, \frac{-1061}{939}, \frac{-53}{47}, \frac{-1059}{941}, \frac{-529}{471}, \frac{-1057}{943}, \frac{-66}{59}, \frac{-211}{189}, \frac{-527}{473}, \frac{-1053}{947}, \frac{-263}{237},$$

$$\frac{-1051}{949}, \frac{-21}{19}, \frac{-1049}{951}, \frac{-131}{119}, \frac{-1047}{953}, \frac{-523}{477}, \frac{-209}{191}, \frac{-261}{239}, \frac{-1043}{957}, \frac{-521}{479}, \frac{-1041}{959}, \frac{-13}{12}, \frac{-1039}{961}, \frac{-519}{481}, \frac{-1037}{963}, \frac{-259}{241},$$

$$\frac{-207}{193}, \frac{-517}{483}, \frac{-1033}{967}, \frac{-129}{121}, \frac{-1031}{969}, \frac{-103}{97}, \frac{-1029}{971}, \frac{-257}{243}, \frac{-1027}{973}, \frac{-513}{487}, \frac{-41}{39}, \frac{-64}{61}, \frac{-1023}{977}, \frac{-511}{489}, \frac{-1021}{979}, \frac{-51}{49},$$

$$\frac{-1019}{981}, \frac{-509}{491}, \frac{-1017}{983}, \frac{-127}{123}, \frac{-203}{197}, \frac{-507}{493}, \frac{-1013}{987}, \frac{-253}{247}, \frac{-1011}{989}, \frac{-101}{99}, \frac{-1009}{991}, \frac{-63}{62}, \frac{-1007}{993}, \frac{-503}{497}, \frac{-201}{199}, \frac{-251}{249},$$

$$\frac{-1003}{997}, \frac{-501}{499}, \frac{-1001}{999}, -1, \frac{-999}{1001}, \frac{-499}{501}, \frac{-997}{1003}, \frac{-249}{251}, \frac{-199}{201}, \frac{-497}{503}, \frac{-993}{1007}, \frac{-62}{63}, \frac{-991}{1009}, \frac{-99}{101}, \frac{-989}{1011}, \frac{-247}{253}, \frac{-987}{1013},$$

$$\frac{-493}{507}, \frac{-197}{203}, \frac{-123}{127}, \frac{-983}{1017}, \frac{-491}{509}, \frac{-981}{1019}, \frac{-49}{51}, \frac{-979}{1021}, \frac{-489}{511}, \frac{-977}{1023}, \frac{-61}{64}, \frac{-39}{41}, \frac{-487}{513}, \frac{-973}{1027}, \frac{-243}{257}, \frac{-971}{1029}, \frac{-97}{103},$$

$$\frac{-969}{1031}, \frac{-121}{129}, \frac{-967}{1033}, \frac{-483}{517}, \frac{-193}{207}, \frac{-241}{259}, \frac{-963}{1037}, \frac{-481}{519}, \frac{-961}{1039}, \frac{-12}{13}, \frac{-959}{1041}, \frac{-479}{521}, \frac{-957}{1043}, \frac{-239}{261}, \frac{-191}{209}, \frac{-477}{523}, \frac{-953}{1047},$$

$$\frac{-119}{131}, \frac{-951}{1049}, \frac{-19}{21}, \frac{-949}{1051}, \frac{-237}{263}, \frac{-947}{1053}, \frac{-473}{527}, \frac{-189}{211}, \frac{-59}{66}, \frac{-943}{1057}, \frac{-471}{529}, \frac{-941}{1059}, \frac{-47}{53}, \frac{-939}{1061}, \frac{-469}{531}, \frac{-937}{1063}, \frac{-117}{133},$$

-187 -467 -933 -233 -931 -93 -929 -58 -927 -463 -37 -231 -923 -461 -921 -23 -919
213' 533' 1067' 267' 1069' 107' 1071' 67' 1073' 537' 43' 269' 1077' 539' 1079' 27' 1081'

-459 -917 -229 -183 -457 -913 -57 -911 -91 -909 -227 -907 -453 -181 -113 -903 -451
541' 1083' 271' 217' 543' 1087' 68' 1089' 109' 1091' 273' 1093' 547' 219' 137' 1097' 549'

-901 -9 -899 -449 -897 -56 -179 -447 -893 -223 -891 -89 -889 -111 -887 -443 -177
1099' 11' 1101' 551' 1103' 69' 221' 553' 1107' 277' 1109' 111' 1111' 139' 1113' 557' 223'

-221 -883 -441 -881 -11 -879 -439 -877 -219 -7 -437 -873 -109 -871 -87 -869 -217
279' 1117' 559' 1119' 14' 1121' 561' 1123' 281' 9' 563' 1127' 141' 1129' 113' 1131' 283'

-867 -433 -173 -54 -863 -431 -861 -43 -859 -429 -857 -107 -171 -427 -853 -213 -851
1133' 567' 227' 71' 1137' 569' 1139' 57' 1141' 571' 1143' 143' 229' 573' 1147' 287' 1149'

-17 -849 -53 -847 -423 -169 -211 -843 -421 -841 -21 -839 -419 -837 -209 -167 -417
23' 1151' 72' 1153' 577' 231' 289' 1157' 579' 1159' 29' 1161' 581' 1163' 291' 233' 583'

-833 -52 -831 -83 -829 -207 -827 -413 -33 -103 -823 -411 -821 -41 -819 -409 -817
1167' 73' 1169' 117' 1171' 293' 1173' 587' 47' 147' 1177' 589' 1179' 59' 1181' 591' 1183'

-51 -163 -407 -813 -203 -811 -81 -809 -101 -807 -403 -161 -201 -803 -401 -801 -2
74' 237' 593' 1187' 297' 1189' 119' 1191' 149' 1193' 597' 239' 299' 1197' 599' 1199' 3'

-799 -399 -797 -199 -159 -397 -793 -99 -791 -79 -789 -197 -787 -393 -157 -49 -783
1201' 601' 1203' 301' 241' 603' 1207' 151' 1209' 121' 1211' 303' 1213' 607' 243' 76' 1217'

-391 -781 -39 -779 -389 -777 -97 -31 -387 -773 -193 -771 -77 -769 -48 -767 -383
609' 1219' 61' 1221' 611' 1223' 153' 49' 613' 1227' 307' 1229' 123' 1231' 77' 1233' 617'

-153 -191 -763 -381 -761 -19 -759 -379 -757 -189 -151 -377 -753 -47 -751 -3 -749
247' 309' 1237' 619' 1239' 31' 1241' 621' 1243' 311' 249' 623' 1247' 78' 1249' 5' 1251'

-187 -747 -373 -149 -93 -743 -371 -741 -37 -739 -369 -737 -46 -147 -367 -733 -183
313' 1253' 627' 251' 157' 1257' 629' 1259' 63' 1261' 631' 1263' 79' 253' 633' 1267' 317'

-731 -73 -729 -91 -727 -363 -29 -181 -723 -361 -721 -9 -719 -359 -717 -179 -143
1269' 127' 1271' 159' 1273' 637' 51' 319' 1277' 639' 1279' 16' 1281' 641' 1283' 321' 257'

-357 -713 -89 -711 -71 -709 -177 -707 -353 -141 -44 -703 -351 -701 -7 -699 -349
643' 1287' 161' 1289' 129' 1291' 323' 1293' 647' 259' 81' 1297' 649' 1299' 13' 1301' 651'

-697 -87 -139 -347 -693 -173 -691 -69 -689 -43 -687 -343 -137 -171 -683 -341 -681
1303' 163' 261' 653' 1307' 327' 1309' 131' 1311' 82' 1313' 657' 263' 329' 1317' 659' 1319'

-17 -679 -339 -677 -169 -27 -337 -673 -42 -671 -67 -669 -167 -667 -333 -133 -83
33' 1321' 661' 1323' 331' 53' 663' 1327' 83' 1329' 133' 1331' 333' 1333' 667' 267' 167'

-663 -331 -661 -33 -659 -329 -657 -41 -131 -327 -653 -163 -651 -13 -649 -81 -647
1337' 669' 1339' 67' 1341' 671' 1343' 84' 269' 673' 1347' 337' 1349' 27' 1351' 169' 1353'

-323 -129 -161 -643 -321 -641 -8 -639 -319 -637 -159 -127 -317 -633 -79 -631 -63
677' 271' 339' 1357' 679' 1359' 17' 1361' 681' 1363' 341' 273' 683' 1367' 171' 1369' 137'

$$\frac{-629}{1371}, \frac{-157}{343}, \frac{-627}{1373}, \frac{-313}{687}, \frac{-5}{11}, \frac{-39}{86}, \frac{-623}{1377}, \frac{-311}{689}, \frac{-621}{1379}, \frac{-31}{69}, \frac{-619}{1381}, \frac{-309}{691}, \frac{-617}{1383}, \frac{-77}{173}, \frac{-123}{277}, \frac{-307}{693}, \frac{-613}{1387},$$

$$\frac{-153}{347}, \frac{-611}{1389}, \frac{-61}{139}, \frac{-609}{1391}, \frac{-38}{87}, \frac{-607}{1393}, \frac{-303}{697}, \frac{-121}{279}, \frac{-151}{349}, \frac{-603}{1397}, \frac{-301}{699}, \frac{-601}{1399}, \frac{-3}{7}, \frac{-599}{1401}, \frac{-299}{701}, \frac{-597}{1403}, \frac{-149}{351},$$

$$\frac{-119}{281}, \frac{-297}{703}, \frac{-593}{1407}, \frac{-37}{88}, \frac{-591}{1409}, \frac{-59}{141}, \frac{-589}{1411}, \frac{-147}{353}, \frac{-587}{1413}, \frac{-293}{707}, \frac{-117}{283}, \frac{-73}{177}, \frac{-583}{1417}, \frac{-291}{709}, \frac{-581}{1419}, \frac{-29}{71}, \frac{-579}{1421},$$

$$\frac{-289}{711}, \frac{-577}{1423}, \frac{-36}{89}, \frac{-23}{57}, \frac{-287}{713}, \frac{-573}{1427}, \frac{-143}{357}, \frac{-571}{1429}, \frac{-57}{143}, \frac{-569}{1431}, \frac{-71}{179}, \frac{-567}{1433}, \frac{-283}{717}, \frac{-113}{287}, \frac{-141}{359}, \frac{-563}{1437}, \frac{-281}{719},$$

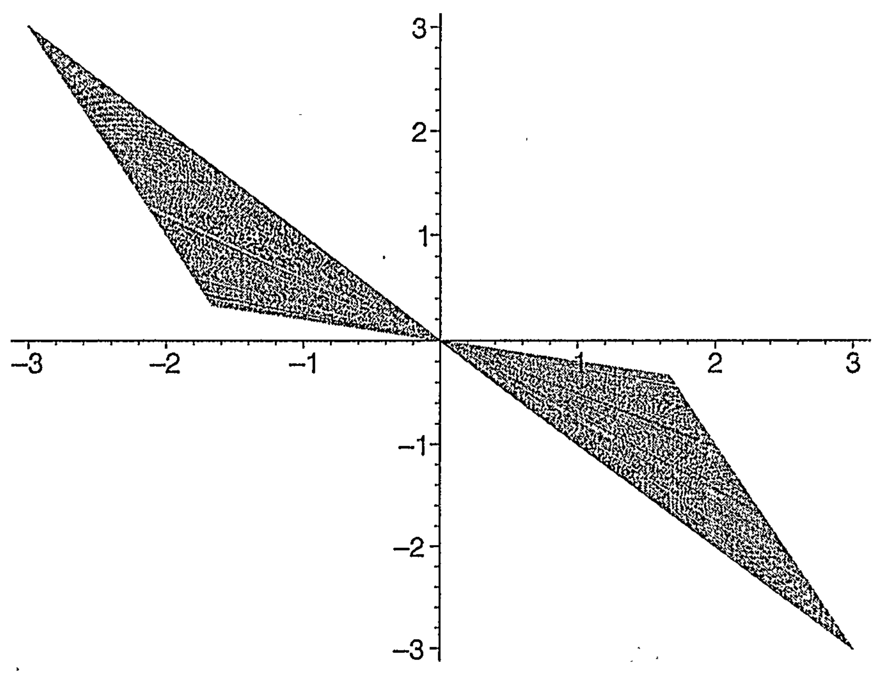
$$\frac{-561}{1439}, \frac{-7}{18}, \frac{-559}{1441}, \frac{-279}{721}, \frac{-557}{1443}, \frac{-139}{361}, \frac{-111}{289}, \frac{-277}{723}, \frac{-553}{1447}, \frac{-69}{181}, \frac{-551}{1449}, \frac{-11}{29}, \frac{-549}{1451}, \frac{-137}{363}, \frac{-547}{1453}, \frac{-273}{727}, \frac{-109}{291},$$

$$\frac{-34}{91}, \frac{-543}{1457}, \frac{-271}{729}, \frac{-541}{1459}, \frac{-27}{73}, \frac{-539}{1461}, \frac{-269}{731}, \frac{-537}{1463}, \frac{-67}{183}, \frac{-107}{293}, \frac{-267}{733}, \frac{-533}{1467}, \frac{-133}{367}, \frac{-531}{1469}, \frac{-53}{147}, \frac{-529}{1471}, \frac{-33}{92},$$

$$\frac{-527}{1473}, \frac{-263}{737}, \frac{-21}{59}, \frac{-131}{369}, \frac{-523}{1477}, \frac{-261}{739}, \frac{-521}{1479}, \frac{-13}{37}, \frac{-519}{1481}, \frac{-259}{741}, \frac{-517}{1483}, \frac{-129}{371}, \frac{-103}{297}, \frac{-257}{743}, \frac{-513}{1487}, \frac{-32}{93}, \frac{-511}{1489},$$

$$\left.\frac{-51}{149}, \frac{-509}{1491}, \frac{-127}{373}, \frac{-507}{1493}, \frac{-253}{747}, \frac{-101}{299}, \frac{-63}{187}, \frac{-503}{1497}, \frac{-251}{749}, \frac{-501}{1499}, \frac{-1}{3}\right]$$

```
> plot([seq([[X[i],Y[i]],[-X[i],-Y[i]]], i=1..k+1)]);
```

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>
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