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ADVANCES IN DESIGN OF FILTERS AND CONTROLS

by

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ABSTRACT

There is a class of practical optimal filtering problems for which algebraic solutions can be obtained. This class is Butterworth signals and is defined as follows. The desired signal is white noise passed through a filter whose poles are the roots of an n^{th} order Butterworth polynomial; and the additive noise is white. The algebraic solution for the optimal filter and for the filter's performance of this class of problem and its graphical interpretation on a Bode plot provides considerable useful insight into the general optimal filtering problem. Suboptimal filters are investigated with respect to: (1) the performance degradation which occurs by using a simplier or reduced-order filter; and (2) the sensitivity of the optimal filter to variations in its parameters.

We consider series compensation for a fixed plant, which has pole-zero-excess of at least 3, and which has a saturation nonlinearity at its input. We present a major simplification in design technique for the traditional approach to series compensation. We extend the preceding results to derive a specific very fast very simple graphical technique for the best system considering the saturation nonlinearity.

By assuming one very specific but realistic fixed plant, we are able to obtain an algebraic solution for the LQ regulator. This specific solution and the regulator performance provided considerable insight into the general solution and the trade off between the speed of response and control effort. A new application for the LQ output regulator is presented, in which the LQ regulator is used as the desired system for the series compensation problem, with a saturation nonlinearity.

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to my father and mother ...

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CHAPTER 1

INTRODUCTION

This thesis is concerned with three main problems. They are the optimal filter problem, the classical series compensation problem, and the linear quadratic (LQ) regulator problem. Although solutions to the optimization problem have been well developed, algebraic solutions are rare. Algebraic solutions are much more useful than numerical solutions in understanding the problem. One of our major concerns is to provide insight to the optimal filter and the LQ regulator. Another goal is to study the achievable performance limitation due to a saturation nonlinearity at the input of a fixed plant, and hence to investigate the "best" control systems possible.

1.1. Basic Problems

The optimal filter problem can be simply stated as: "given the spectral characteristic of an additive combination of random signal and noise, what is the optimal linear operation to separate the signal from the noise subject to the minimum mean-square error criterion?". In 1942, Wiener obtained a solution in frequency domain, and his solution became known as the Wiener filter. In 1961, Kalman approached the same problem in time domain, and his solution is commonly known as the Kalman filter. Note that both Wiener and Kalman filters are minimum mean-square error estimators, but their structures are different from each other.

The classical series compensation problem is to make a complex dynamic system with negative feedback perform satisfactorily, and to meet specifications. The traditional approach is to present several methods such as root locus, Bode plots, Nyquist charts, Nichols charts, etc.

The LQ regulator problem is to minimize the performance index which is a measurement of error and control effort. The technique used to solve the LQ regulator problem are very closely related to the technique used to solve the optimal filter problem. In fact, Kalman cast the problem as a so-called "dual-problem". The LQ regulator is also commonly known as the Kalman regulator.

The problems considered tend to be very computationally intensive. Extensive use is made of the new concept of computer algebra software programs. We use one called MACSYMA [1].

1.2. Objectives of the Thesis

Different approaches to solve the optimal filter and optimal control problems have already been developed in the past four decades. This thesis is not intended to give new approaches to solve the optimization problem. One objective of this thesis is to derive algebraic solutions for the optimal filter problem for a class of signals, and to provide new insight to the Wiener filter and its performance.

In conventional control theory, the traditional approaches to series compensation are trial-and-error procedures, rather lengthy, and somewhat arbitrary. Another objective is to present a major simplification in design techniques for the traditional approach to series compensation. The real problem in control systems design is the trade off between speed of response, and control effort or how hard we drive the fixed plant. Another objective is to derive the algebraic solution of the LQ regulator problem for a very specific but realistic fixed plant. This solution illustrates the fundamental nature of the trade off, and provides insight to the general LQ regulator solution.

In a realistic plant, there is a limit to the achievable performance. This limit involves the third pole in the fixed plant and the saturation nonlinearity ($\pm L$) at its input. The last objective is to design the best system, which minimizes the system error considering a saturation nonlinearity, for a stochastic (random) input using the series compensation approach and the optimal control approach.

1.3. Outline of the Thesis

This chapter gave a brief introduction of the main ideas in this thesis. Chapter 2 presents the algebraic closed-loop solutions for Wiener and Kalman filters, and for their performance for a class of signals. Performance of suboptimal filters are also investigated. Sensitivity analysis of the optimal filter is studied as well in Chapter 2. Chapter 3 presents a very simple very fast graphical technique to the classical series compensation problem with practical specifications for realistic plants. We also extend this technique to derive the best solution, which minimizes the system error considering the saturation nonlinearity at the input of the fixed plant, for a random input. Design examples are given for a real servo motor system to illustrate the application of this technique. Chapter 4 presents the algebraic closed-loop solution of the LQ output regulator for a specific fixed plant. The best design of the LQ output regulator for this plant considering a saturation nonlinearity at its input for a random input is also presented. Chapter 6 indicates possible limitations on the design technique, and recommends what could be further researched.

CHAPTER 2

ALGEBRAIC SOLUTIONS FOR WIENER AND KALMAN FILTERS AND FOR THEIR PERFORMANCE FOR A CLASS OF SIGNALS

2.1. Introduction

Algebraic solutions for Wiener and Kalman filters are rare. Algebraic or explicit closed-form solutions are given for Wiener and Kalman filters, and also for their performance, for a class of problems. This class is Butterworth signals, defined below. Algebraic solutions are much more useful than numerical solutions, in understanding a problem as a filter designer, or in learning the subject as a student. Considerable attention is given to the performance of optimal filters. Also important is the considerable insight that these algebraic solutions and their graphical presentation on a Bode plot provide into the general filter problem: i.e. the simple relationship which exists between the given data, the optimal filter, and the optimal filter's performance. The engineering aspects of the optimal filtering problem are emphasized in this chapter.

Butterworth signals of order n are defined to be the output when zero mean white noise passed through a low-pass filter whose poles are the roots of an n^{th} order Butterworth polynomial. Butterworth polynomials $B_n(s)$, and Butterworth filters $H(s) = 1/B_n(s)$, are well known [2]. Table 2.1 gives the coefficients β_k of Butterworth polynomials for n = 1 to 4 in a form suitable for computer algebra. The basic equations for Butterworth polynomials and coefficients are summarized below

$$B_n(s) = \beta_0 + \beta_1 s + \cdots + \beta_{n-1} s^{n-1} + \beta_n s^n \tag{2.1a}$$

$$B_n(s)\, B_n(-s) = 1 + (-s^2)^n \tag{2.1b}$$

where for $k = 0, 1, \cdots, n$

$$\beta_o = 1, \quad \beta_{n-k} = \beta_k \tag{2.1c}$$

$$\beta_{k+1} = \beta_k \cos(k\pi / 2n) \,/\, \sin((k+1)\pi / 2n) \,. \tag{2.1d}$$

Fig. 2.1a presents the poles of Butterworth signals for n = 1 to 4. For the optimal filter problem considered in this chapter, the filter input is a desired Butterworth signal and an additive undesired zero mean white noise.

Some recent books on the now classical problem of optimal filtering include Kailath [3] and Brown [4]. Butterworth signals occur in various applications, and have been studied by Van Trees [5] in connection with communication systems, and by Lindsay [6] in connection with phase-locked-loops.

Butterworth signals are easily obtained in the laboratory. Pass the output of a white noise generator through an active filter[1] with Butterworth poles of n = 1, 2, 3, 4 or 8. The most important characteristic of Butterworth signals is that algebraic solutions, for the optimal filter and its performance, can be obtained for any order of signal, n. For general signals, algebraic solutions for Wiener filters are limited to first or second order signals, and algebraic solutions for Kalman filters are limited to a first order signal. Note that for general random signals, the optimal filter and its

1. Hewlett Packard HP3722A, and Rockland 1000F and 1042F.

performance are very similar to that for a "corresponding" Butterworth signal of the same order.

The solution to the causal Wiener filter problem, shown in Fig. 2.1b, is well-known [3]-[4]. The given data are the power spectral densities (PSD) of the signal and the noise, $S_{ss}(s)$ and $S_{nn}(s)$. The PSD of the filter input is $S_{rr}(s) = S_{ss}(s) + S_{nn}(s)$. The optimal filter which minimizes the expectation $E\{\cdot\}$ of the error-squared, $\sigma_e{}^2 = E\{e^2(t)\} = min.$, is given by

$$W(s) = \frac{1}{S^{+}_{rr}(s)} \left[\frac{S^{+}_{ss}(s)}{S^{-}_{rr}(s)} \right]_{+} \tag{2.2}$$

where

$S^{+}_{xx}(s)$ = the left-half s-plane (LHP) poles and zeros, and square root of the constant of the PSD $S_{xx}(s)$, the factored spectrum. $S^{-}_{xx}(s) = [\,S^{+}_{xx}(s)\,]^{*}$. $[\cdot]^{+}[\cdot]^{-} = [\cdot]$, where $[\cdot]^{+}$ is known as superscript plus.

$[\cdot]_{+}$ = the LHP poles of the partial fraction expansion of $[\cdot]$, the physical realizability operator. $[\cdot]_{+} + [\cdot]_{-} = [\cdot]$, where $[\cdot]_{+}$ is known as subscript plus.

The optimal filter is easier to interpret in terms of G(s) than W(s), see Fig. 2.1b.

$$G(s) = \frac{W(s)}{1 - W(s)} . \tag{2.3}$$

The performance of the filter is given by the PSD of the error, which consists of the error due to the signal, plus the error due to the noise,

$$S_{ee}(s) = S_{es}(s) + S_{en}(s)\ , \tag{2.4a}$$

$$\overset{+}{S}_{es}(s) = [\,1 - W(s)\,]\ \overset{+}{S}_{ss}(s)\ , \tag{2.4b}$$

$$\overset{+}{S}_{en}(s) = W(s)\ \overset{+}{S}_{nn}(s)\ . \tag{2.4c}$$

Rms values for these errors are calculated as shown in Appendix E of Newton [7].

When the noise is white, $S_{nn}(s) = b^2$, a considerable simplification occurs, which is not well-known, and which we call the white noise theorem. These are the Yovits-Jackson formulas as extended by Snyder [8].

$$G(s) = [\ S_{ss}(s)/b^2 + 1\ ]^{+} - 1, \tag{2.5}$$

$$W(s) = G(s)/[\ 1 + G(s)\ ]\ .$$

The error of the optimal filter is given by

$$\sigma_e{}^2 = E\{e^2(t)\} = b^2 \lim_{s\to\infty} sW(s)\ . \tag{2.6}$$

2.2. Wiener Filters

An algebraic solution is derived for Wiener filters and for their performance, for Butterworth signals of any order n, $1 \le n < \infty$. The principal results of this section are given eqns. (2.8a), (2.8b), (2.9b), (2.10), (2.11), and (2.13), and by Fig. 2.2.

Notation is simplified by using: (1) normalized frequency $\lambda = s/\nu$, where ν is the dominant break frequency; and (2) generalized n^{th} order Butterworth polynomials, which are defined by

$$B_n(a, s)\, B_n(a, -s) \equiv a^{2n} + (-s^2)^n \ , \tag{2.7}$$

where a is real constant. Compare this with (2.1b).

The given data for this problem are the factored PSD's of the n^{th} order Butterworth signal and of the white noise,

$$\overset{+}{S}_{ss}(\lambda) = \frac{bd}{B_n(1, \lambda)} \ , \quad \overset{+}{S}_{nn}(\lambda) = b \ , \tag{2.8a}$$

where the "signal-to-noise ratio" is defined as

$$\text{SNR} = d = [\, S_{ss}(0) / S_{nn} \,]^{1/2} \ , \tag{2.8b}$$

an easily identifiable characteristic of the given data. The factored PSD of the filter input is given by

$$\overset{+}{S}_{rr}(\lambda) = \frac{bB_n(\alpha, \lambda)}{B_n(1, \lambda)} \tag{2.9a}$$

$$\alpha \equiv (d^2 + 1)^{1/2n} \tag{2.9b}$$

where α is approximately equal to the crossover frequency ω_c, where $|\overset{+}{S}_{ss}(j\omega_c)| = |\overset{+}{S}_{nn}|$ an obviously critical point, which on Bode plot is $\alpha = d^{1/n}$. See Fig. 2.2.

The Wiener filter, which is derived below, is given by

$$W(\lambda) = \frac{B_n(\alpha, \lambda) - B_n(1, \lambda)}{B_n(\alpha, \lambda)} \ . \tag{2.10}$$

The performance of the Wiener filter is given by the rms value of the error divided by the rms value of the signal, or the rms error-to-signal ratio (ESR),

$$\mathrm{ESR} = \frac{\sigma_e}{\sigma_s} = \frac{[\,2n\,(\alpha - 1)\,]^{1/2}}{d} \quad . \tag{2.11}$$

This is plotted in Fig. 2.3 for Butterworth signals with n = 1, 2, 3, 4, and 8.

Derivation of (2.9)-(2.11). The white noise theorem (2.5)-(2.6) and generalized Butterworth polynomial (2.7) are used.

Given: $\overset{+}{S}_{ss}(\lambda) = \dfrac{bd}{B_n(1, \lambda)}$, $\overset{+}{S}_{nn}(\lambda) = b$, $\lambda = s / \nu$

Then:

$$G(\lambda) = \Big[\, S_{ss}(\lambda) / b^2 + 1 \,\Big]^+ - 1$$

$$= \left[\frac{d^2}{1 + (-\lambda^2)^n} + 1 \right]^+ - 1 , \qquad \alpha^{2n} = d^2 + 1$$

$$= \frac{\Big[\, \alpha^{2n} + (-\lambda^2)^n \,\Big]^+}{B_n(1, \lambda)} - 1 = \frac{B_n(\alpha, \lambda) - B_n(1, \lambda)}{B_n(1, \lambda)} = \frac{N}{D}$$

$$W(\lambda) = \frac{G(\lambda)}{1 + G(\lambda)} = \frac{N}{D + N} = \frac{B_n(\alpha, \lambda) - B_n(1, \lambda)}{B_n(\alpha, \lambda)}$$

$$\sigma_e{}^2 = E\{e^2(t)\} = b^2 \lim_{\lambda \to \infty} \nu\lambda W(\lambda) = b^2 \nu \beta_1 (\alpha - 1)$$

$$\sigma_s{}^2 = E\{s^2(t)\} = \int_{-\infty}^{\infty} S_{ss}(f)\, df = \int_{-\infty}^{\infty} \frac{b^2 d^2}{1 + (2\pi f / \nu)^{2n}}\, df \ , \qquad x = 2\pi f / \nu$$

$$= \frac{b^2 d^2 \nu}{\pi} \int_0^\infty \frac{1}{1 + x^{2n}}\, dx = \frac{b^2 d^2 \nu}{\pi} \frac{\pi}{2n\ sin(\pi / 2n)} .$$

From (2.1d),

$$\beta_1 = 1 / \sin(\pi / 2n)$$

$$\sigma_s{}^2 = b^2 d^2 \nu \beta_1 / 2n$$

$$\mathrm{ESR} = [\ \sigma_e{}^2 / \sigma_s{}^2\ ]^{1/2} = \frac{[\ 2n(\alpha - 1)\ ]^{1/2}}{d} .$$

Three points should be noted.

(1) The corresponding PSD for this given data are:

$$S_{ss}(\lambda) = \frac{b^2 d^2}{1 + (-\lambda^2)^n} , \quad S_{nn}(\lambda) = b^2 , \quad \lambda = s / \nu . \tag{2.12}$$

(2) The optimal open-loop system (Fig. 2.1b) is

$$G(\lambda) = \frac{B_n(\alpha, \lambda) - B_n(1, \lambda)}{B_n(1, \lambda)} . \tag{2.13}$$

Note that the poles of the $G(\lambda)$ are the poles of $S^{+}_{ss}(\lambda)$.

(3) The optimal filter's performance can be approximated by its asymptotes as follows,

$$\mathrm{ESR} = \frac{\sigma_e}{\sigma_s} \approx \begin{cases} 1 & , d \ll 1 \\ \dfrac{(2n)^{1/2}}{d^{1 - 1/2n}} & , d \gg 1 \end{cases} . \tag{2.14}$$

It is useful to consider a specific example. Consider a third order Butterworth signal, n = 3. In this case we have

$$\overset{+}{S}_{ss}(\lambda) = \frac{bd}{1 + 2\lambda + 2\lambda^2 + \lambda^3} , \quad \overset{+}{S}_{nn}(\lambda) = b , \quad \lambda = s / \nu , \tag{2.15}$$

$$S_{ss}(\lambda) = \frac{b^2 d^2}{1 - \lambda^6} , \quad S_{nn}(\lambda) = b^2 ,$$

$$\alpha = (d^2 + 1)^{1/6} , \quad d = \mathrm{SNR}$$

$$W(\lambda) = \frac{(\alpha^3 - 1) + 2(\alpha^2 - 1)\lambda + 2(\alpha - 1)\lambda^2}{\alpha^3 + 2\alpha^2\lambda + 2\alpha\lambda^2 + \lambda^3} ,$$

$$G(\lambda) = \frac{(\alpha^3 - 1) + 2(\alpha^2 - 1)\lambda + 2(\alpha - 1)\lambda^2}{1 + 2\lambda + 2\lambda^2 + \lambda^3} ,$$

$$\mathrm{ESR} = \frac{\sigma_e}{\sigma_s} = \frac{[\, 6((d^2 + 1)^{1/6} - 1) \,]^{1/2}}{d} ,$$

$$\approx \begin{cases} 1 & , d << 1 \\ \dfrac{\sqrt{6}}{d^{5/6}} & , d >> 1 \end{cases} .$$

For $d \geq 16$ (good performance), $\alpha \approx d^{1/3}$, and

$$G(\lambda) \approx \frac{d \; B_2(1, \sqrt{2}\lambda / \alpha)}{B_3(1, \lambda)} .$$

A Bode plot of the given data $\overset{+}{S}_{ss}(\lambda)$ and $\overset{+}{S}_{nn}(\lambda)$ normalized by the white noise $\overset{+}{S}_{nn}(\lambda) = b$, and of the optimal $W(\lambda)$ and $G(\lambda)$ is given in Fig. 2.2. Notice the simple graphical relationship between the normalized factored given data, $\overset{+}{S}_{ss}(\lambda) / b$, and the optimal filters $W(\lambda)$ and $G(\lambda)$. This is very important.

The performance of the optimal filter is important. First, for low signal-to-noise ratios, $\mathrm{SNR} = d << 1$, the filter's maximum gain for any order is $W(0) = (\alpha^n - 1)/\alpha^n \approx d^2/2$, which rapidly approaches zero. Second, there should be some lower bound on acceptable performance of the filter. A priori, a limit of $\mathrm{ESR} \leq 0.25$ would appear reasonable. Table 2.2 presents the values of $\mathrm{SNR} = d$, for ESR of 0.25 and 0.50, for Butterworth filters of orders n = 1, 2, 3, 4, and 8. Note that most of the literature refers to d^2, but that d is more directly related to the optimal system and its performance.

A simulation study aids in the interpretation of the filters performance as measured by its ESR. Consider a first order Butterworth signal n = 1. Let $d = 10$, $\nu = 2\pi\,31.62\ rad/s$, $b = 3.162\times10^{-4}$ for $f \leq 25.60\ kHz$. The given data are

$$S^{+}_{ss}(s) = \frac{bd}{1 + s/\nu}\,, \quad S^{+}_{nn}(s) = b\ ; \tag{2.16}$$

and the optimal filter is,

$$W(s) = \frac{(\alpha - 1)/\alpha}{1 + s/\alpha\nu}\ .$$

Fig. 2.4a presents the noise $n(t)$; and Fig. 2.4b presents the signal $s(t)$ and the filter output $y(t)$. These figures are presented because, they do not exist in the literature, and they give the engineer a physical feeling for the problem. The author was surprised at how acceptable an $\mathrm{ESR} = 0.5$ was. The theoretical ESR is 0.4254, and the measured ESR is 0.4105. In this rather pessimistic example the optimal filter performs surprisingly well. On the basis of this we will assume that an ESR of 0.5 is acceptable in many cases.

As a rule of thumb we can summarize the results of Table 2.2 as follows. For reasonable performance

$$\begin{aligned} n &= 1 \quad d \geq 7 \\ n &\geq 2 \quad d \geq 4 \end{aligned} \tag{2.17}$$

Most examples in textbooks have simplified the numbers for tutorial reasons, and this results in a SNR = d between 1 and 2 which results in a filter with a poor performance.

An *engineering interpretation* of these results for optimal filters, see Fig. 2.2, is as follows. We are considering low-pass signals and white noise, and spectra refers to the factored PSD. The frequency where the spectra of the signal and noise are equal, crossover frequency (α), is obviously important. The behavior of the optimum filter in this region is examined in Sec. 2.4 on suboptimal filters. For frequencies where the spectra of the signal is much greater than the spectra of the noise, the optimal filter does the best it can to pass the signal, which is $W(s) \approx 1$ and $G(s) \approx S_{ss}^{+}(s)/b$. For frequencies where the spectra of the noise is greater than the spectra of the signal, the optimal filter cuts off as $1/s$ or with a -1 slope. In this case the optimal filter does the minimum it can do to make the error due to noise finite. It does the minimum because the pole-zero-excess (PZE) of the filter translates into a phase lag, and this acts as a pure time delay which increases the error due to the signal rapidly.

Results for a low-pass signal in colored noise, which represent a significant extension, have been presented in [9-10]. The basic difference is that error due to the noise, $S^{+}_{en}(s) = W(s) S^{+}_{nn}(s)$, cuts off as $1/s$ or with a -1 slope, instead of $W(s)$.

2.3. Kalman Filters

Algebraic solutions are presented for Kalman filters for Butterworth signals of orders, $1 \leq n \leq 4$. This involves the solution of an algebraic Riccati equation. Note that, after solving a first order numerical optimal state estimation problem (Kalman filter) example 8.2-1, Sage and White [11] state that: "As is readily apparent, the effect involved in solving this simple Riccati equation is such as to suggest that the analytical solution of a Riccati equation of higher order than the first would be prohibitive." Wiener and Kalman filters are different solutions to the same problem. The performance for Kalman filters is the same as for the corresponding Wiener filters.

It is assumed that the given data has been experimently measured with an appropriate signal processing computer system[2]. In this situation we usually do not know the appropriate physical variables. Therefore phase variables are assumed. If the physical variables are known, the conversion of these results is routine, equate the Kalman $G(\lambda)$ to the Wiener $G(\lambda)$.

The solution to the Kalman filter problem is well-known. In this chapter we follow the notation of Brown [4] and Sage and Melsa [12]. The Kalman filter problem and its conventional solution is shown in Fig. 2.5. Note that the optimal filter is the given data with: (1) a unity feedback loop added; and (2) the given g replaced by the

2. HP 5423A Digital Signal Analyzer or a B&K 2032 Dual Channel Signal Analyzer.

optimal gain vector $\underline{k}$.

The message model is:

$$\dot{\underline{x}}(t) = \underline{F}\,\underline{x}(t) + \underline{g}\,w(t) \tag{2.18}$$

$$y(t) = \underline{h}^T \underline{x}(t)\ .$$

The observation model is:

$$z(t) = y(t) + v(t)\ .$$

The priori statistics for the zero-mean inputs are

$$E[\,w(t)w(\tau)\,] = Q\,\delta(t-\tau)$$

$$E[\,v(t)v(\tau)\,] = R\,\delta(t-\tau)$$

$$E[\,w(t)v(\tau)\,] = 0\ .$$

The problem is to find the optimal minimum-error-variance filter and gain algorithm, i.e. Kalman filter.

The well-known solution to this problem is as follows.

(1) Solve the algebraic Riccati equation for the symmetric matrices $\underline{P}$ which satisfy

$$\underline{F}\,\underline{P} + \underline{P}\,\underline{F}^T - \underline{P}\,\underline{h}\,R^{-1}\underline{h}^T\underline{P} + \underline{g}\,Q\,\underline{g}^T = 0\ . \tag{2.19}$$

(2) Select the one matrix $\underline{P}$ which is positive definite.

(3) The optimal gain vector is

$$\underline{k} = \underline{P}\,\underline{h}\,R^{-1}\ . \tag{2.20}$$

(4) The optimal filter, see Fig. 2.5, is

$$\dot{\hat{\underline{x}}}(t) = \underline{F}\,\hat{\underline{x}}(t) + \underline{k}\,e(t) \tag{2.21}$$

$$\hat{y}(t) = \underline{h}^T\,\hat{\underline{x}}(t)$$

$$e(t) = z(t) - \hat{y}(t)\,.$$

Expressed mathematically, this problem is a subset of a larger problem which has been studied by Gohberg, Lancaster and Rodman [13] among others. They would express this subset as follows. "Our primary concern is the analysis of real symmetric solutions X of the Riccati equation of the form

$$XDX - XA - A^*X - C = 0.$$

This equation will be considered under the assumptions that A, C, D are n×n matrices with real entries with C hermitian, and D positive definite."

For Butterworth signals of order 1 through 4, the Kalman filter solution is presented below.

(1) The Kalman filter for first order Butterworth signals, n = 1, is as follows.

The given data is

$$\dot{x}(t) = -\nu x(t) + \nu w(t) \tag{2.22}$$

$$y(t) = x(t)$$

$$z(t) = y(t) + v(t)$$

$$E[\,w(t)w(\tau)\,] = b^2 d^2\,\delta(t-\tau)\,,\quad d^2 = \alpha^2 - 1$$

$$E[\, v(t) v(\tau)\,] = b^2\, \delta(t - \tau)\ .$$

The optimal gain is

$$k = (\alpha - 1)\nu\ ,$$

and the error covariance is

$$P = b^2\nu(\alpha - 1)\ .$$

(2) The Kalman filter for second order Butterworth signals, n = 2, is as follows.

The given data is

$$\underline{\dot{x}}(t) = \begin{bmatrix} 0 & 1 \\ -\nu^2 & -\sqrt{2}\nu \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ \nu^2 \end{bmatrix} w(t) \tag{2.23}$$

$$y(t) = [\,1\ 0\,]\, \underline{x}(t)$$

$$z(t) = y(t) + v(t)$$

$$E[\, w(t) w(\tau)\,] = b^2 d^2\, \delta(t - \tau), \quad d^2 = \alpha^4 - 1\ .$$

$$E[\, v(t) v(\tau)\,] = b^2\, \delta(t - \tau)\ .$$

The optimal gain vector is

$$\underline{k}^T = [\, \sqrt{2}(\alpha - 1)\nu, \quad (\alpha - 1)^2\nu^2\,]\ ,$$

and the error covariance matrix is

$$\underline{P} = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$$

$$p_{11} = b^2 v\ \sqrt{2}(\alpha - 1)$$

$$p_{12} = b^2 v^2\ \ (\alpha - 1)^2$$

$$p_{22} = b^2 v^3 \sqrt{2}(\alpha - 1)(\alpha^2 - \alpha + 1)\ .$$

(3) The Kalman filter for third order Butterworth signals, n = 3, is as follows.

The given data is

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -v^3 & -2v^2 & -2v \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 0 \\ v^3 \end{bmatrix} w(t) \tag{2.24}$$

$$y(t) = \ [\ 1\ 0\ 0\ ]\ \underline{x}(t)$$

$$z(t) = y(t) + v(t)$$

$$E[\ w(t)\, w(\tau)\ ] = b^2 d^2\ \delta(t - \tau), \quad d^2 = \alpha^6 - 1\ .$$

$$E[\ v(t)\, v(\tau)\ ] = b^2\ \delta(t - \tau)\ .$$

The optimal gain vector is

$$\underline{k} = \begin{bmatrix} 2(\alpha - 1)v \\ 2(\alpha - 1)^2 v^2 \\ (\alpha - 1)(\alpha^2 - 3\alpha + 1)v^3 \end{bmatrix},$$

and the error covariance matrix is

$$\underline{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{bmatrix}$$

$$p_{11} = b^2 \nu \; 2(\alpha - 1)$$

$$p_{12} = b^2 \nu^2 2(\alpha - 1)^2$$

$$p_{13} = b^2 \nu^3 \; (\alpha - 1)[\, \alpha^2 - 3\alpha + 1\,]$$

$$p_{22} = b^2 \nu^3 \; (\alpha - 1)[\, 3\alpha^2 - 5\alpha + 3\,]$$

$$p_{23} = b^2 \nu^4 2(\alpha - 1)^4$$

$$p_{33} = b^2 \nu^5 2(\alpha - 1)[\, \alpha^4 - 3\alpha^3 + 5\alpha^2 - 3\alpha + 1\,] \, .$$

(4) The Kalman filter for fourth order Butterworth signals, n = 4, is as follows.

The given data is

$$\dot{\underline{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\nu^4 & -\beta_1 \nu^3 & -\beta_2 \nu^2 & -\beta_1 \nu \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \nu^4 \end{bmatrix} w(t) \qquad (2.25)$$

$$y(t) = [\, 1\; 0\; 0\; 0\,]\, \underline{x}(t)$$

$$z(t) = y(t) + v(t)$$

$$E[\, w(t) w(\tau)\,] = b^2 d^2 \, \delta(t - \tau), \quad d^2 = \alpha^8 - 1 \, .$$

$$E[\, v(t) v(\tau)\,] = b^2 \, \delta(t - \tau) \, .$$

The optimal gain vector is

$$\underline{k} = \begin{bmatrix} (\alpha - 1)\beta_1 v \\ (\alpha - 1)^2 \beta_2 v^2 \\ (\alpha - 1)(\alpha^2 - (\sqrt{2} + 1)\alpha + 1)\beta_1 v^3 \\ (\alpha - 1)^2(\alpha^2 - 2(\sqrt{2} + 1)\alpha + 1)v^4 \end{bmatrix},$$

and the error covariance matrix is

$$\underline{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{12} & p_{22} & p_{23} & p_{24} \\ p_{13} & p_{23} & p_{33} & p_{34} \\ p_{14} & p_{24} & p_{34} & p_{44} \end{bmatrix}$$

$$p_{11} = b^2 v\, \beta_1 (\alpha - 1)$$

$$p_{12} = b^2 v^2 \beta_2 (\alpha - 1)^2$$

$$p_{13} = b^2 v^3 \beta_1 (\alpha - 1)[\,\alpha^2 - (\sqrt{2} + 1)\alpha + 1\,]$$

$$p_{14} = b^2 v^4 \quad (\alpha - 1)^2 [\,\alpha^2 - 2(\sqrt{2} + 1)\alpha + 1\,]$$

$$p_{22} = b^2 v^3 \beta_1 (\sqrt{2} + 1)(\alpha - 1)[\,\alpha^2 - (2\sqrt{2} - 1)\alpha + 1\,]$$

$$p_{23} = b^2 v^4 \quad (2\sqrt{2} + 3)(\alpha - 1)^4$$

$$p_{24} = b^2 v^5 \beta_1 (\alpha - 1)[\,\alpha^4 - (2\sqrt{2} + 3)\alpha^3 + (3\sqrt{2} + 5)\alpha^2 - (2\sqrt{2} + 3)\alpha + 1\,]$$

$$p_{33} = b^2 v^5 \beta_1 (\sqrt{2} + 1)\,(\alpha - 1)[\,\alpha^4 - (2\sqrt{2} + 1)\alpha^3 + (2\sqrt{2} + 3)\alpha^2 - (2\sqrt{2} + 1)\alpha + 1\,]$$

$$p_{34} = b^2 v^6 \quad (\sqrt{2} + 2)(\alpha - 1)^2 [\,\alpha^2 - (\sqrt{2} + 1)\alpha + 1\,]^2$$

$$p_{44} = b^2 v^7 \beta_1 (\alpha - 1)[\,\alpha^6 - (2\sqrt{2} + 3)\alpha^5 + (6\sqrt{2} + 9)\alpha^4 - (8\sqrt{2} + 13)\alpha^3 + (6\sqrt{2} + 9)\alpha^2 - (2\sqrt{2} + 3)\alpha + 1\,]\,.$$

Using MACSYMA [1], it is easy to verify that the error covariance matrix $\underline{P}$: (a) is positive definite, (b) satisfies the Riccati equation (2.19), and (c) yields $\underline{k}^T$. MACSYMA can directly solve for n = 2 and 3 using "algsys", but not for n = 4. The n = 4 case was solved as follows. The optimal gain vector was determined from the Wiener $G(\lambda)$. With $\underline{k}$ known, the Riccati equation becomes linear, where by symmetry 4 of the 10 elements are known. This solution is clearly still not easy.

Fig. 2.6 compares the block diagrams for the Wiener and Kalman filters for n = 3 in terms of normalized frequency λ. The optimal gain vector supplies the zeros required in the optimal filter. This figure is very useful in visualizing the problem and the solution.

2.4. Suboptimal Filters

The results of this section require the evaluation of (2.4) to obtain mean-square values as given in Appendix E of Newton [7], which was accomplished using MACSYMA. The first question is: "How much performance degradation will occur by using a simplier or reduced-order filter?" The optimal filter $W(\lambda)$ has $n-1$ zeros and n poles which are very close together, separated by approximately $\sqrt{2}$. Is this little "wiggle" in $W(\lambda)$, which is exaggerated in Fig. 2.2, necessary? As suboptimal filter 1, we consider a first order filter with the correct asymptotes, which is shown in Fig. 2.7a,

$$W_1(\lambda) = \frac{k}{1 + k\lambda / c}\,, \quad k = (\alpha^n - 1)/\alpha^n\,, \quad c = \beta_1(\alpha - 1)\,. \tag{2.26}$$

Fig. 2.8 presents the rms error of the suboptimal filter 1 divided by the rms error of the optimal filter, as a function of SNR = d, for Butterworth signals of orders n = 2, 3 and 4. Recall that for $d \leq 1$, the optimal filter rapidly approaches zero. For reasonable values of d, equation (2.17), the optimal filter performs significantly better. The small "wiggle" in $W(\lambda)$ is necessary. Note Fig. 2.2, and that this is easier to see in terms of $G(\lambda)$ than $W(\lambda)$.

The second question is: "how sensitive is the optimal filter to incorrect values of its parameters?" We consider two cases. As suboptimal filter 2, we consider the correct form of the open-loop filter with an incorrect high frequency asymptote, x, as shown in Fig. 2.7b. In an optimal filter, $x = 1$.

$$G_2(\lambda) = \frac{B_n(\alpha, \lambda / x) - B_n(1, \lambda / x)}{B_n(1, \lambda / x)} \tag{2.27}$$

$$= \frac{(\alpha^n - 1) + \cdots + \beta_1(\alpha - 1)(\lambda / x)^{n-1}}{1 + \cdots + (\lambda / x)^n}\,.$$

Fig. 2.9 presents the rms error of the suboptimal filter 2, divided by the rms error of the optimal filter, as a function of normalized break frequency x, for Butterworth signals of orders n = 1 and 2.

As suboptimal filter 3, we consider the correct form of the open-loop filter with an incorrect low-frequency asymptote, x, see Fig. 2.7c.

$$G_3(\lambda) = \frac{B_n(\alpha, \lambda) - B_n(1, \lambda)}{B_n(x, \lambda)} \tag{2.28}$$

$$= \frac{(\alpha^n - 1) + \cdots + \beta_1(\alpha - 1)\lambda^{n-1}}{x^n + \cdots + \lambda^n} .$$

Fig. 2.10 presents the rms error of the suboptimal filter 3, divided by the rms error of the optimal filter, as a function of normalized break frequency x, for Butterworth signals of orders n = 1 and 2.

Note Fig. 2.10, and that the actual $G_3(\lambda)$ may have a much higher gain than that of optimal $G_o(\lambda)$, at low frequencies where the spectra of the signal is larger than the spectra of the noise, with little degradation in performance. This is important in real world applications. For example, consider a data record with a low SNR, it is best to estimate the desired signal optimistically. This is illustrated by suboptimal filter 4, Fig. 2.7d,

$$G_4(\lambda) = \frac{B_n(\alpha, \lambda) - B_n(1, \lambda)}{\lambda\, B_{n-1}(1, \lambda)} \tag{2.29}$$

$$= \frac{(\alpha^n - 1) + \cdots + \beta_1(\alpha - 1)\lambda^{n-1}}{\lambda\,[\,1 + \cdots + \lambda^{n-1}\,]} .$$

Table 2.3 presents, for d = 10, the rms error of the suboptimal filter $G_4(\lambda)$, divided by the rms error of the optimal filter, for Butterworth signals of orders n = 1, 2, and 3.

2.5. Conclusions

Algebraic solutions are given for a useful practical class of problems. Wiener filters are given for Butterworth signals of any order $1 \le n < \infty$. Kalman filters are given for Butterworth signals of orders $1 \le n \le 4$. A graphical approach yields a great deal of insight into the problem. That is, a Bode plot of the normalized factored spectra, $S^{+}_{ss}(\lambda)/b$ and $S^{+}_{nn}(\lambda)/b = 1$, is directly related to the optimal open-loop filter $G(\lambda)$ and to $W(\lambda)$, see Fig. 2.2.

Van Trees has obtained an algebraic solution for the Wiener filter's performance, but not for the Wiener filter. Van Trees also presented numerical solutions for Kalman filters. In this chapter, a much simpler algebraic solution is presented for the optimal filter, and for its performance, and these are shown to be directly related. Compare (2.8)-(2.11) with (153)-(156) of [5]; and compare (2.22)-(2.25) with Figs. 6.42-6.44 of [5].

The performance of the filters is examined in Fig. 2.3. The appropriate criteria for these filters performance is the rms error divided by the rms signal or "error-to-signal ratio" ESR. The "signal-to-noise ratio" SNR = d for reasonable performance is shown to be much larger than usually assumed. The simulation shown in Fig. 2.4 aids in the interpretation of these results. It shows that a rather low ESR = 0.5 yields results which would be acceptable in many applications. Suboptimal filters are also examined. The computer algebra programs used in this thesis are presented in [14].

TABLE 2.1
Butterworth coefficients

n	β_o	β_1	β_2	β_3	β_4
1	1	1			
2	1	$\sqrt{2}$	1		
3	1	2	2	1	
4	1	$\sqrt{2\beta_2}$	$2 + \sqrt{2}$	$\sqrt{2\beta_2}$	1

TABLE 2.2

SNR = d required achieve an ESR of 0.25 and 0.50,
n = 1, 2, 3, 4, and 8.

	ESR	
Order	0.25	0.50
n	d	d
1	30.984	6.928
2	12.888	4.113
3	10.791	3.669
4	9.993	3.488
8	9.013	3.254

TABLE 2.3
Increase in rms error for suboptimal filter $G_4(\lambda)$

n	Increase in ESR
1	1.0246
2	1.0685
3	1.0940

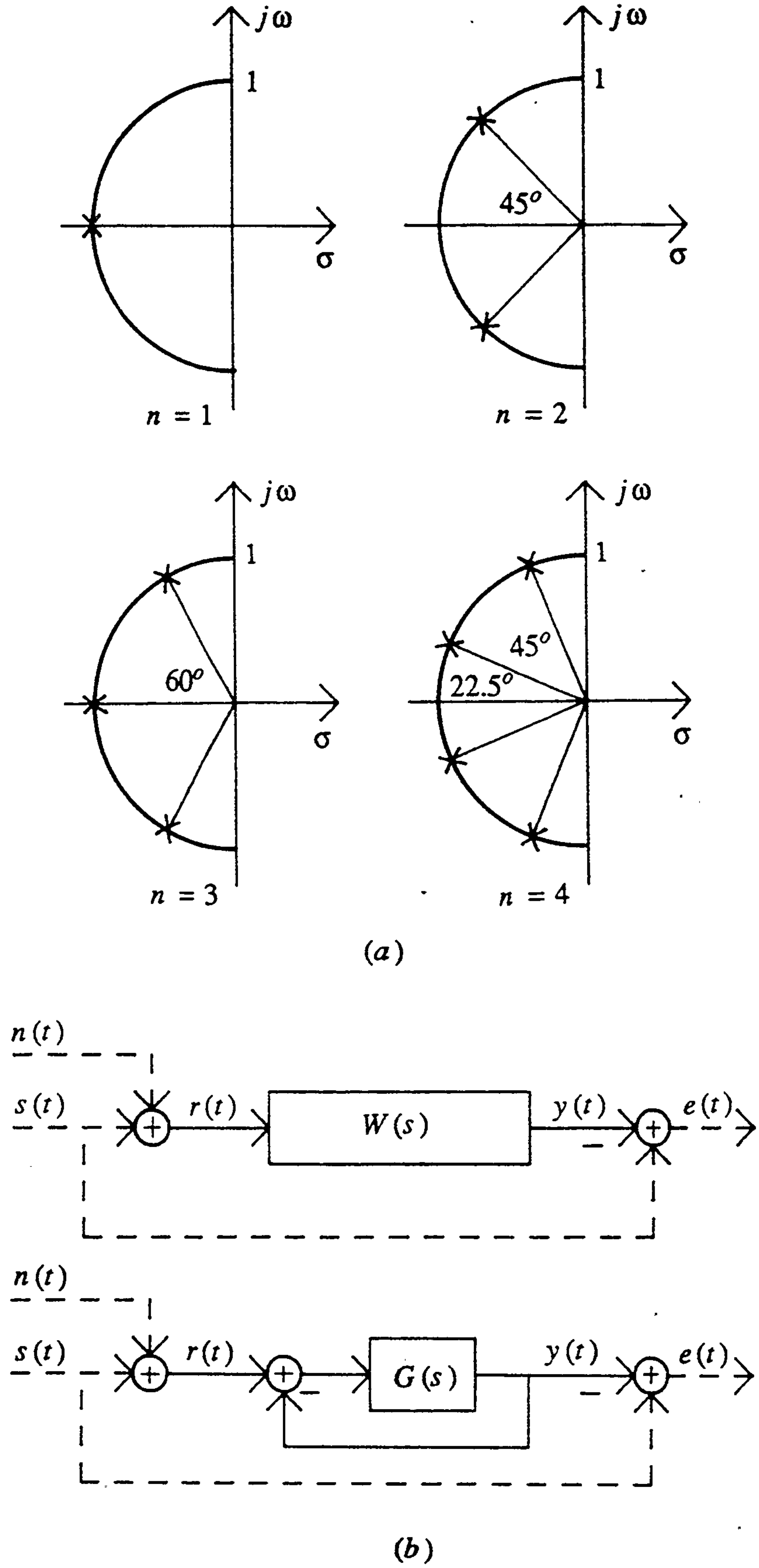


Fig. 2.1 Terminology. (a) Poles of Butterworth signals. (b) The Wiener filter problem, the optimal filter W(s), and the Wiener open-loop filter G(s).

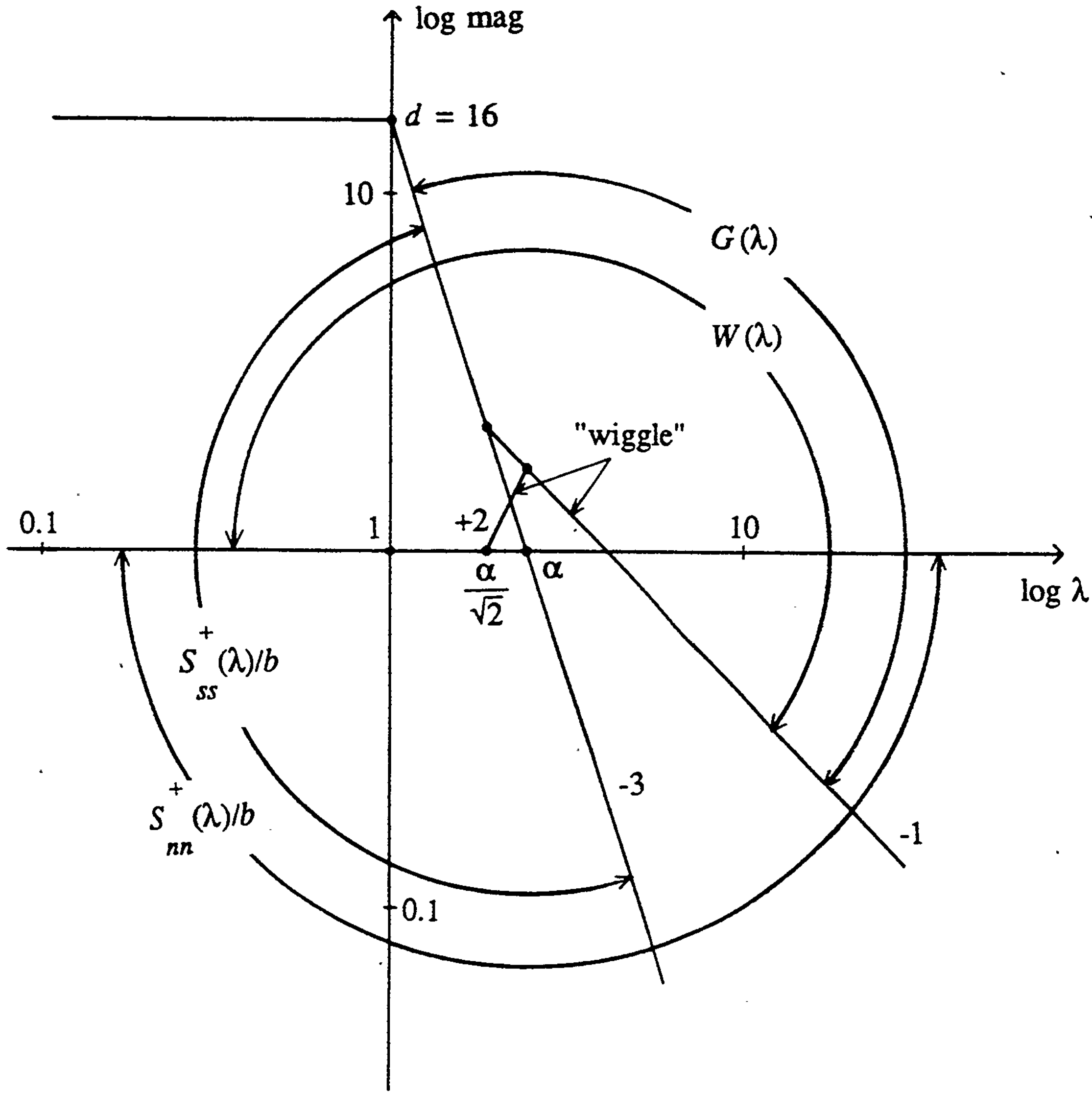


Fig. 2.2 Bode plots for an optimal filter, with n = 3 and d = 16. The factored given data are normalized by the white noise b, and the optimal filters are $W(\lambda)$ and $G(\lambda)$. Notice the simple graphical relationship between the given data and the optimal filters $W(\lambda)$ and $G(\lambda)$.

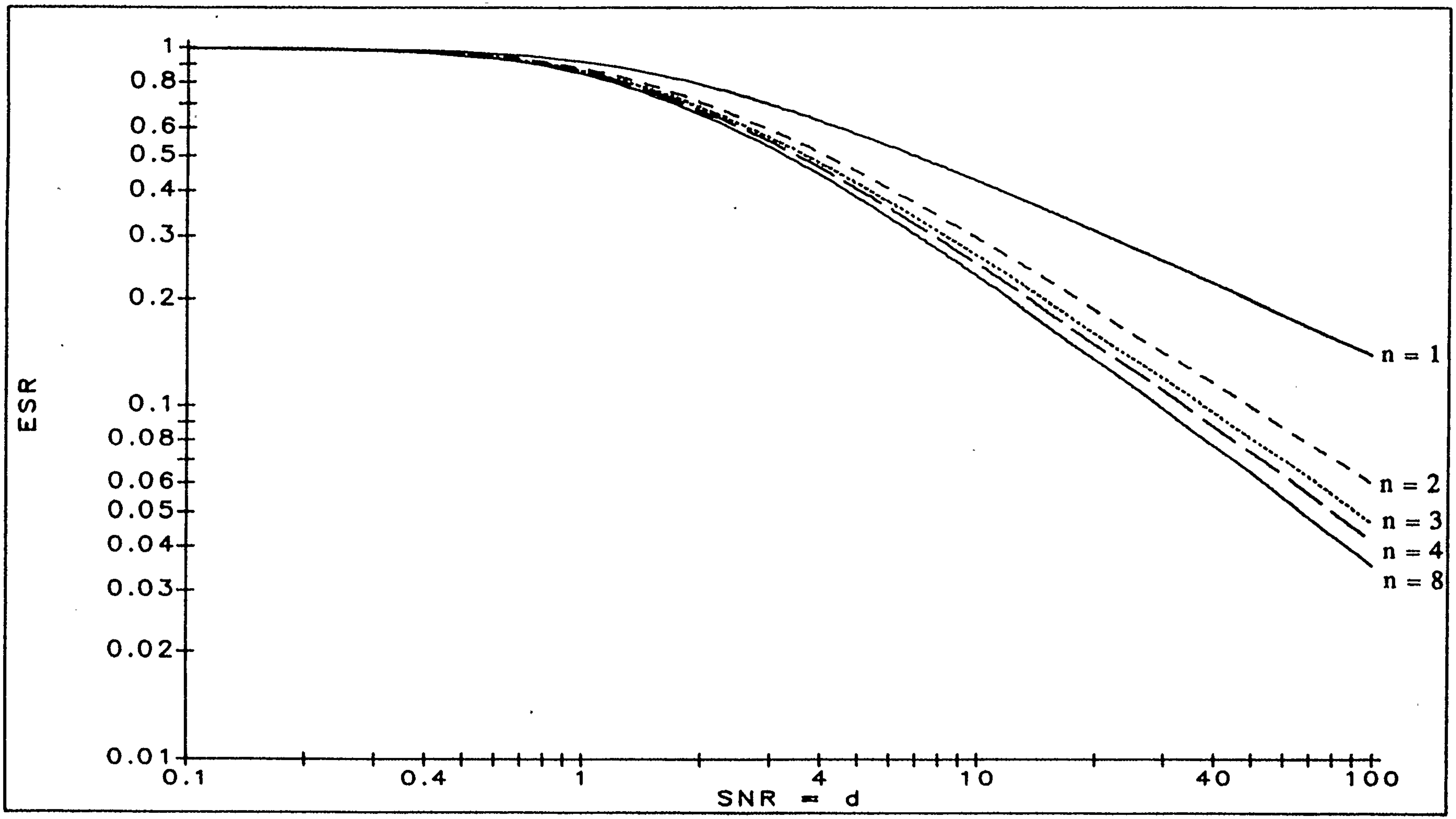


Fig. 2.3 The optimal filters performance. ESR versus SNR for n = 1, 2, 3, 4, and 8.

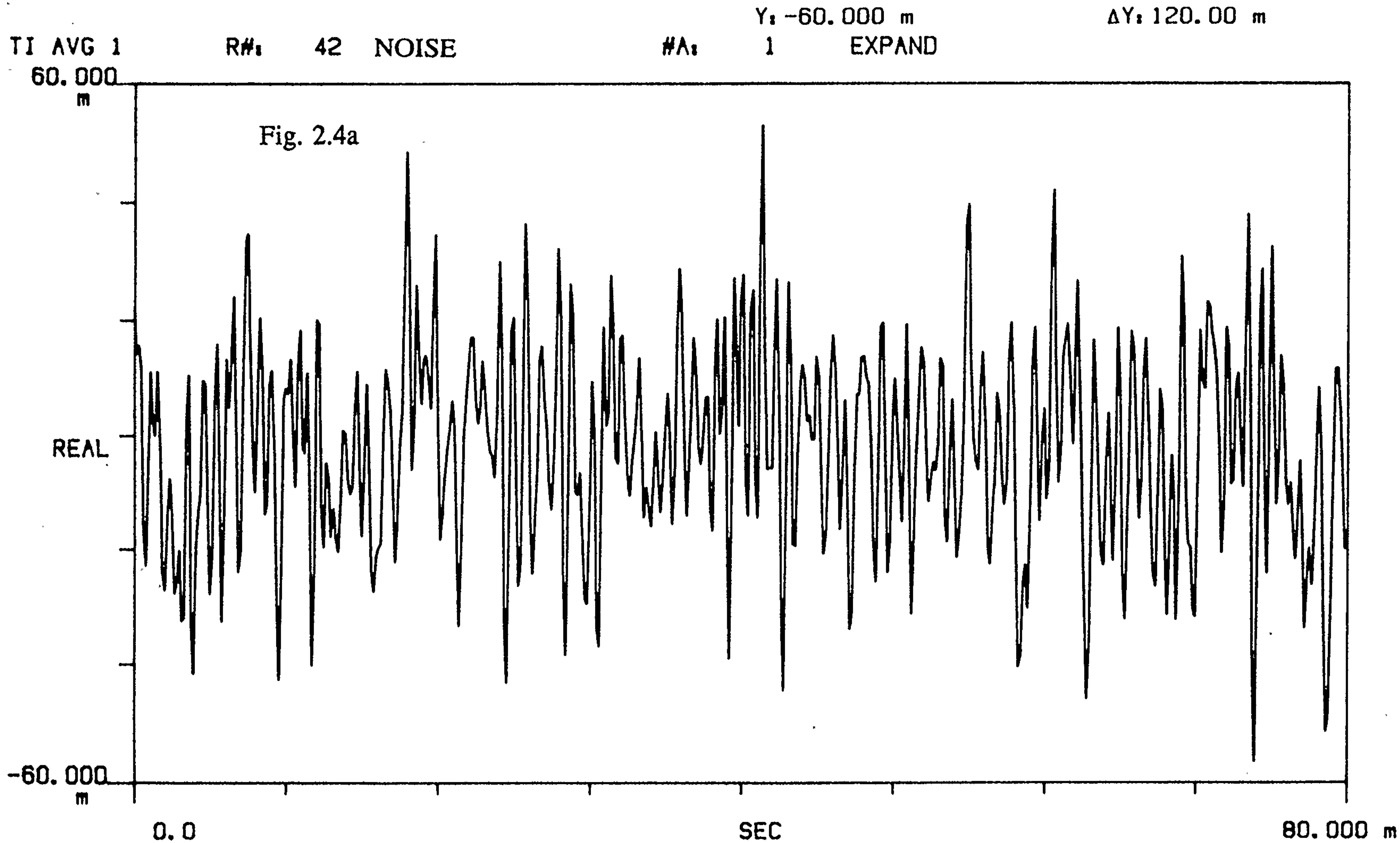
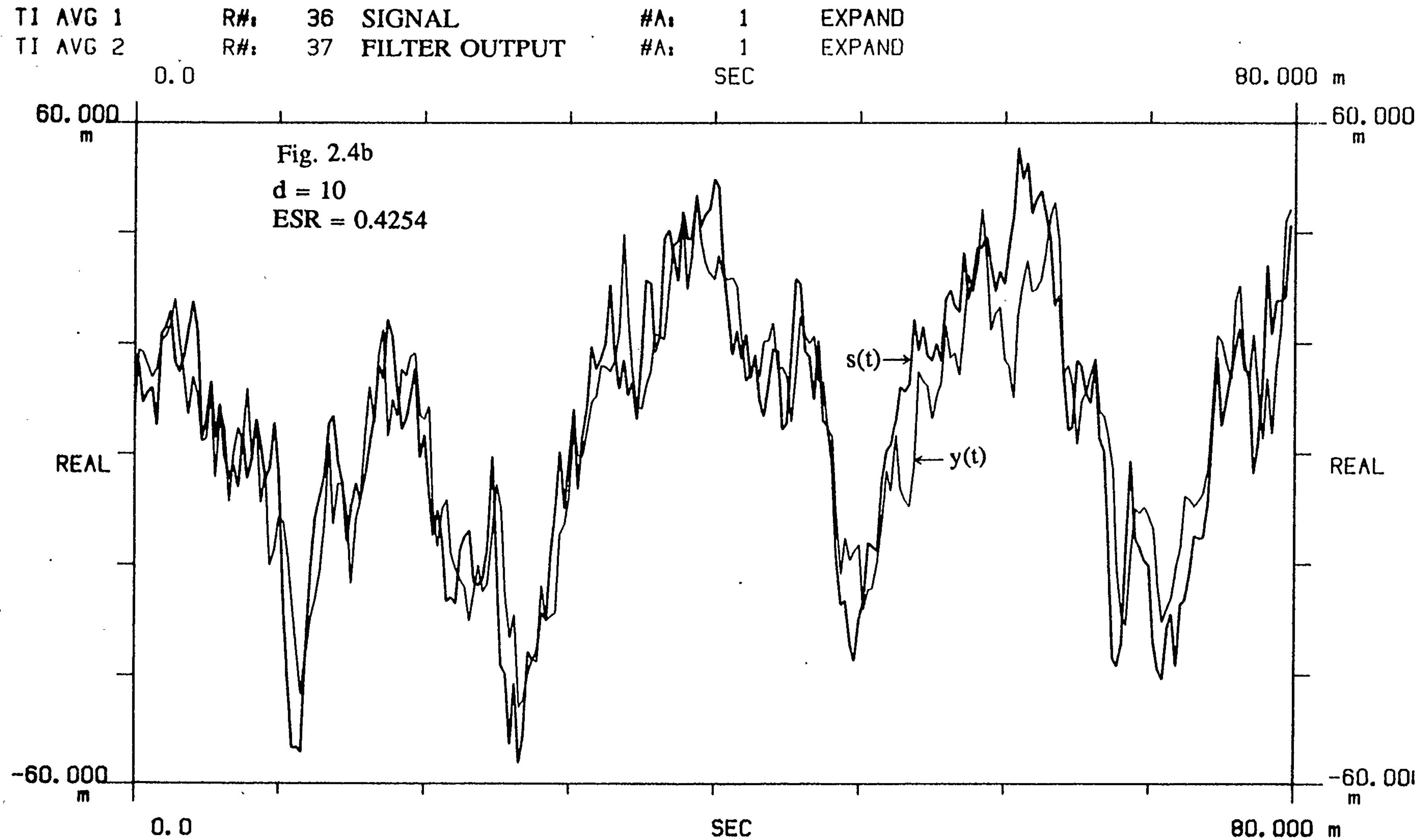


Fig. 2.4 A simulation of an optimal filter for a first order Butterworth signal n = 1, with a SNR = d = 10, and an ESR = σ_e/σ_s = 0.4254. (a) The noise n(t). (b) The input signal s(t), and the filter output y(t).


TI AVG 1 R#: 36 SIGNAL #A: 1 EXPAND
TI AVG 2 R#: 37 FILTER OUTPUT #A: 1 EXPAND
0.0
SEC
80.000 m
60.000 m
REAL
-60.000 m
60.000 m
REAL
-60.00 m
0.0
SEC
80.000 m
Fig. 2.4b
d = 10
ESR = 0.4254
s(t)
y(t)

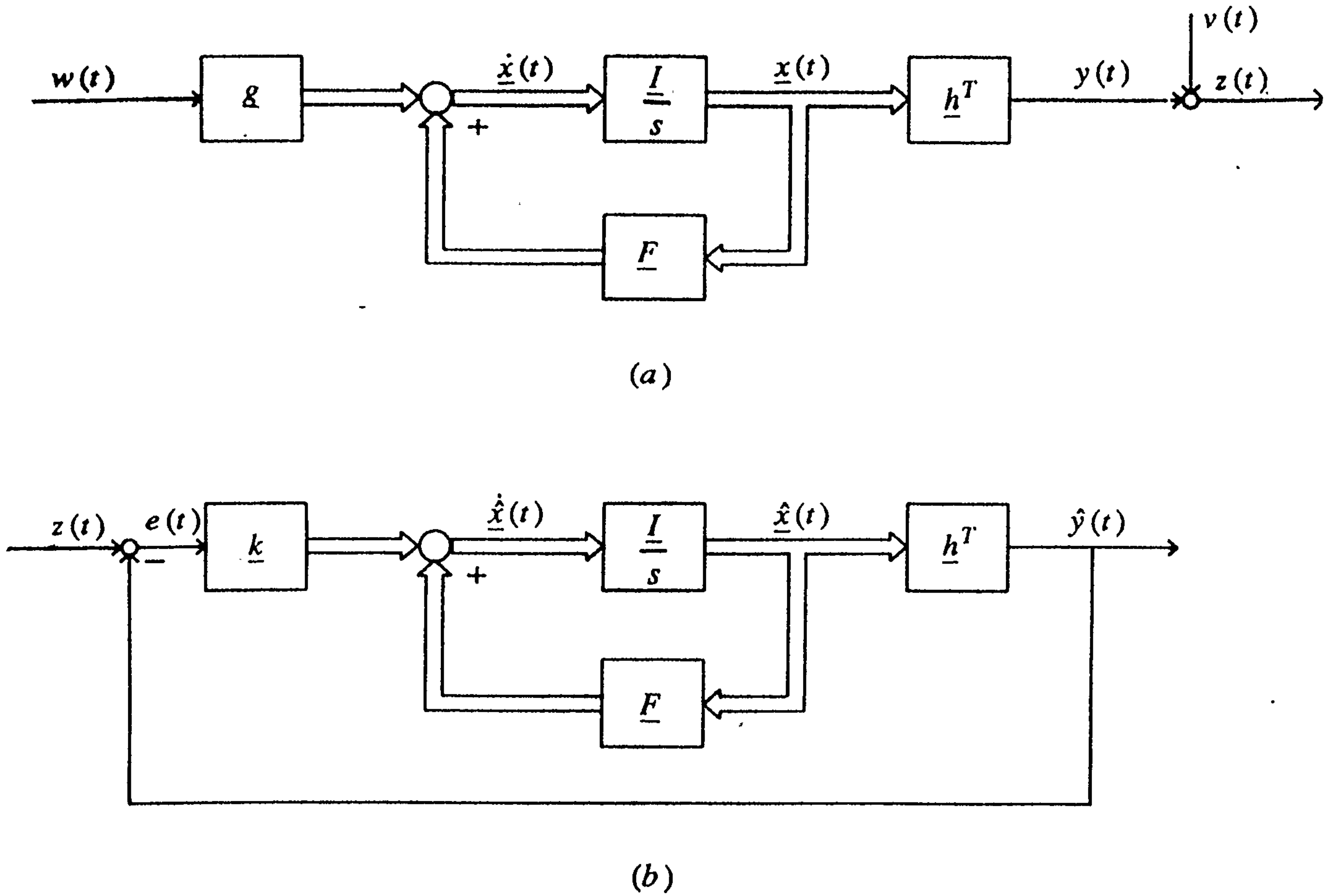
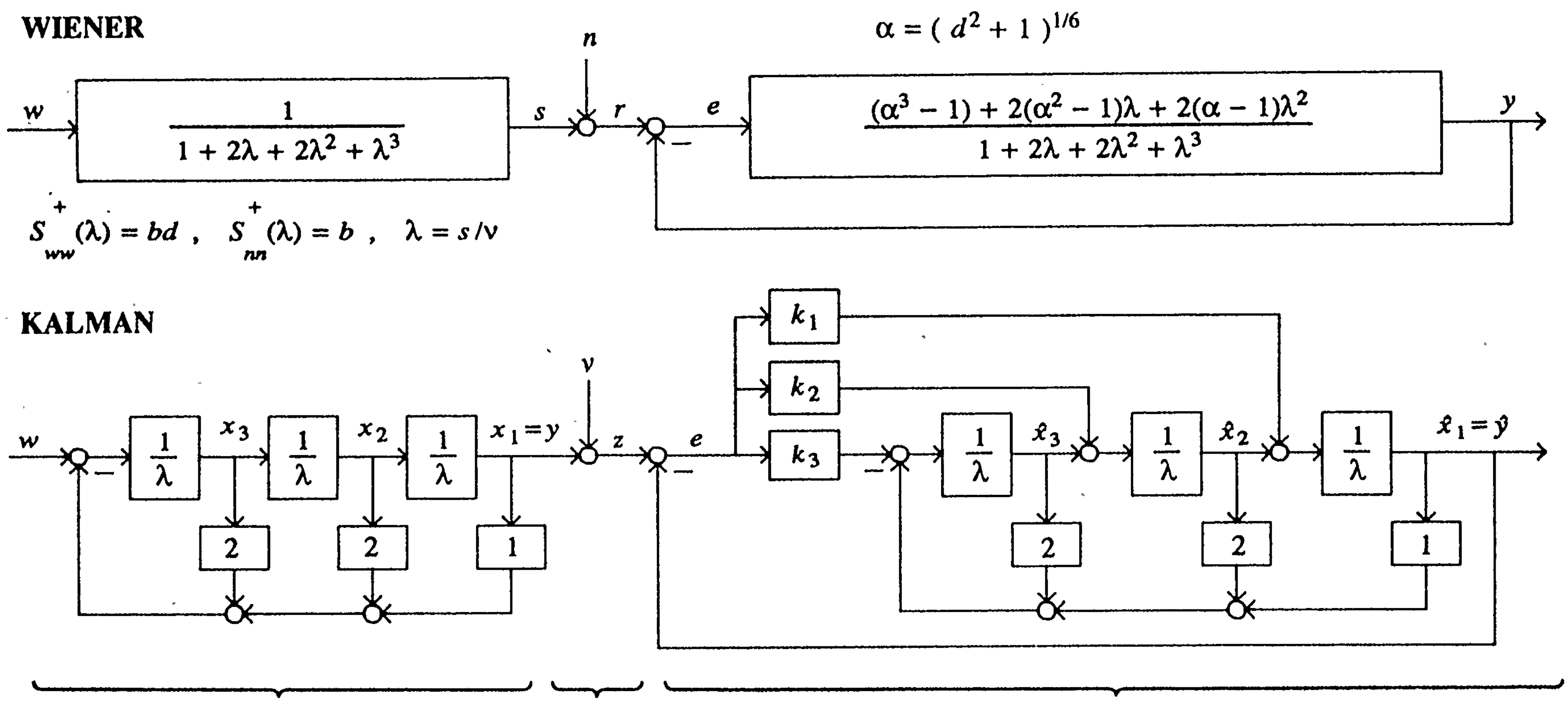


Fig. 2.5 The Kalman filter problem. (a) The message and observation model. (b) The optimal filter.



message model

observation model

optimal filter

$$\dot{\underline{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -2 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w$$

$$y = [\,1 \;\; 0 \;\; 0\,]\, \underline{x}\,, \quad \lambda = s/\nu$$

prior statistics

$$E[w^2] = b^2 d^2 \delta(t - \tau)$$

$$E[\nu^2] = b^2 \delta(t - \tau)$$

$$\alpha = (d^2 + 1)^{1/6}$$

$$\underline{k}^T = [\, 2(\alpha - 1),\;\; 2(\alpha - 1)^2,\;\; (\alpha - 1)(\alpha^2 - 3\alpha + 1)\,]$$

Fig. 2.6 Comparison of Wiener and Kalman filters, for third order Butterworth signals (n=3), in terms of normalized frequency λ ($\lambda = s/\nu$).

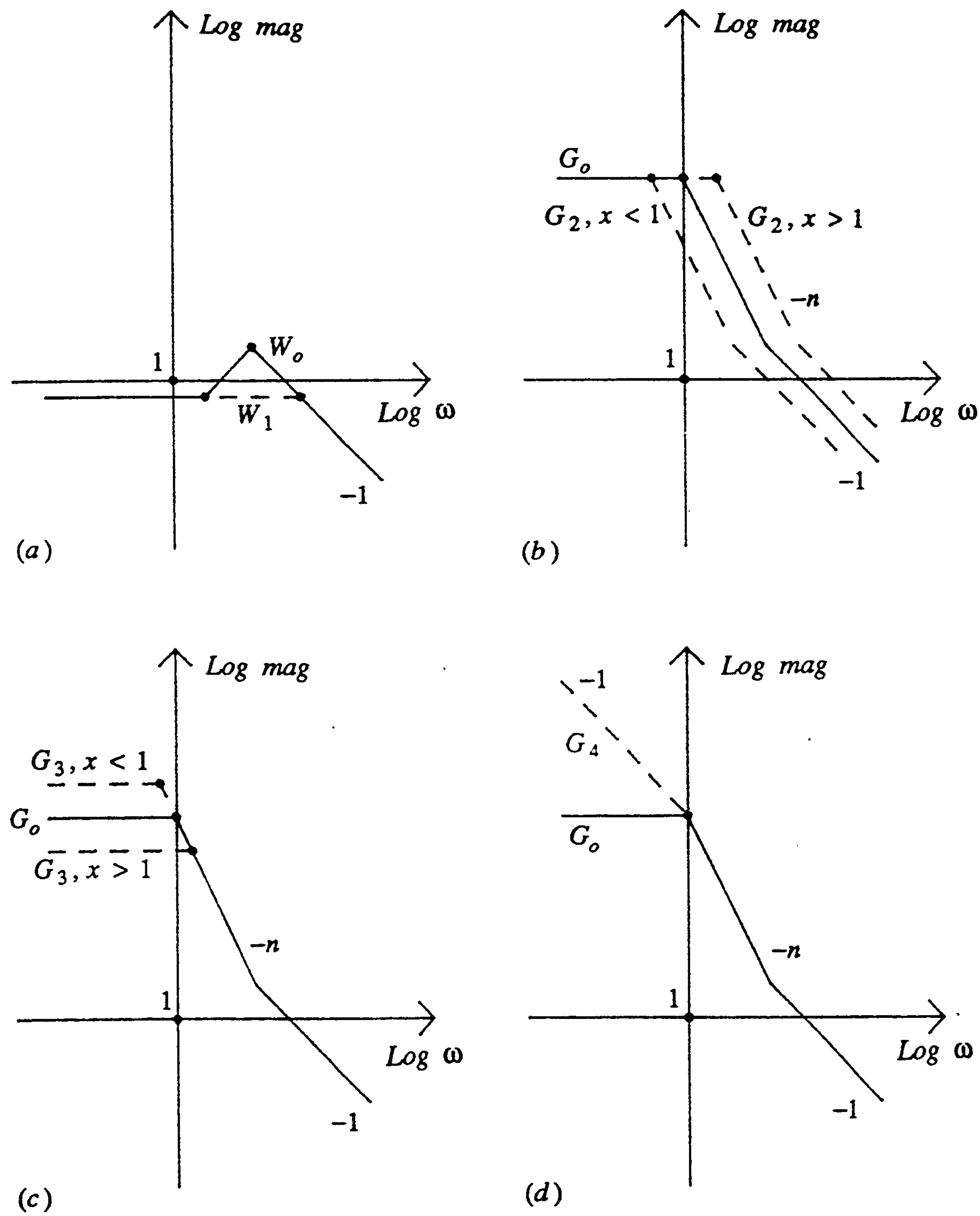


Fig. 2.7 Bode plots for suboptimal filters. (a) Suboptimal filter W_1. (b) Suboptimal open-loop filter G_2. (c) Suboptimal open-loop filter G_3. (d) Suboptimal open-loop filter G_4.

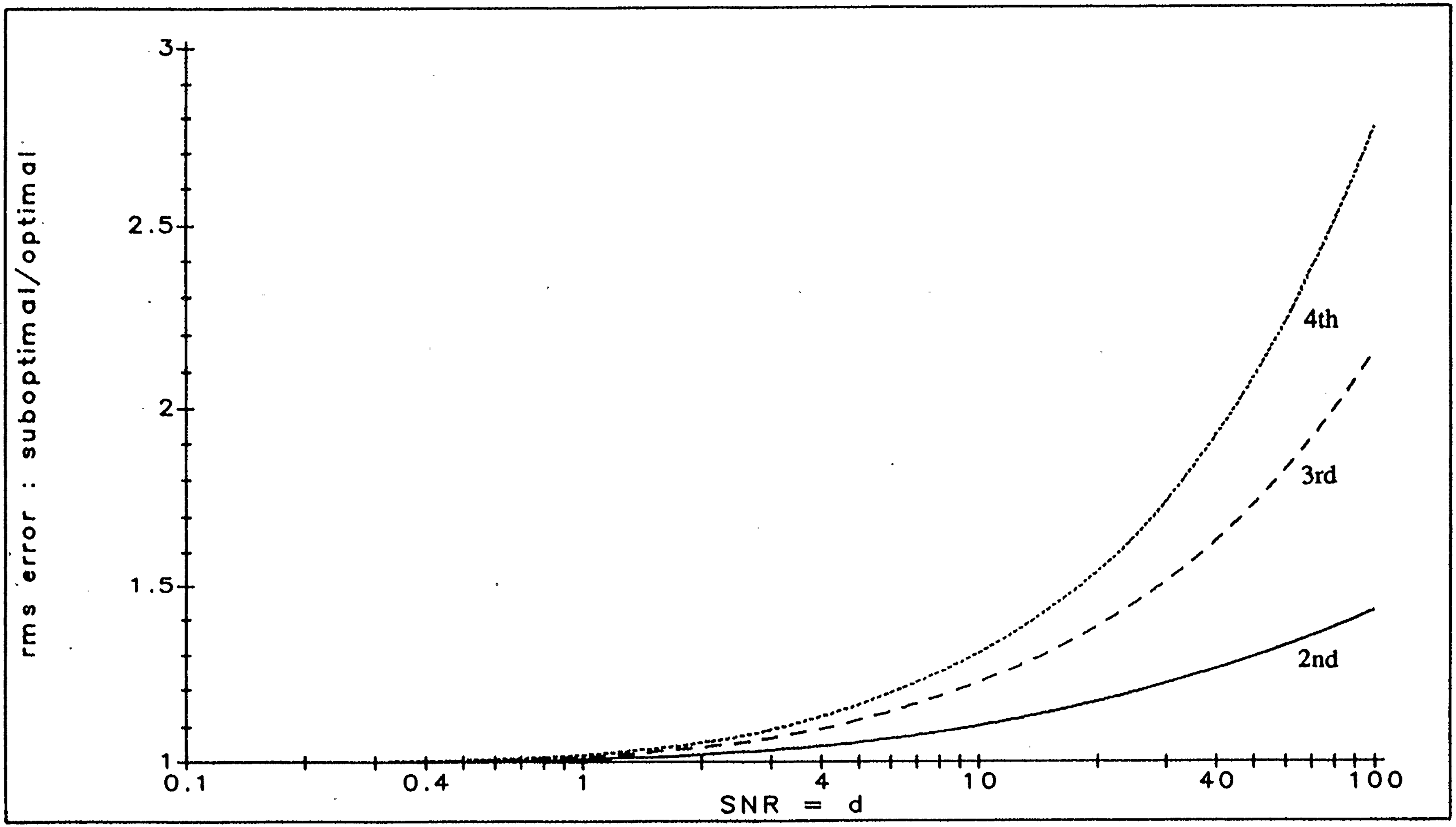


Fig. 2.8 Increase in rms error for the suboptimal filter $W_1(\lambda)$.

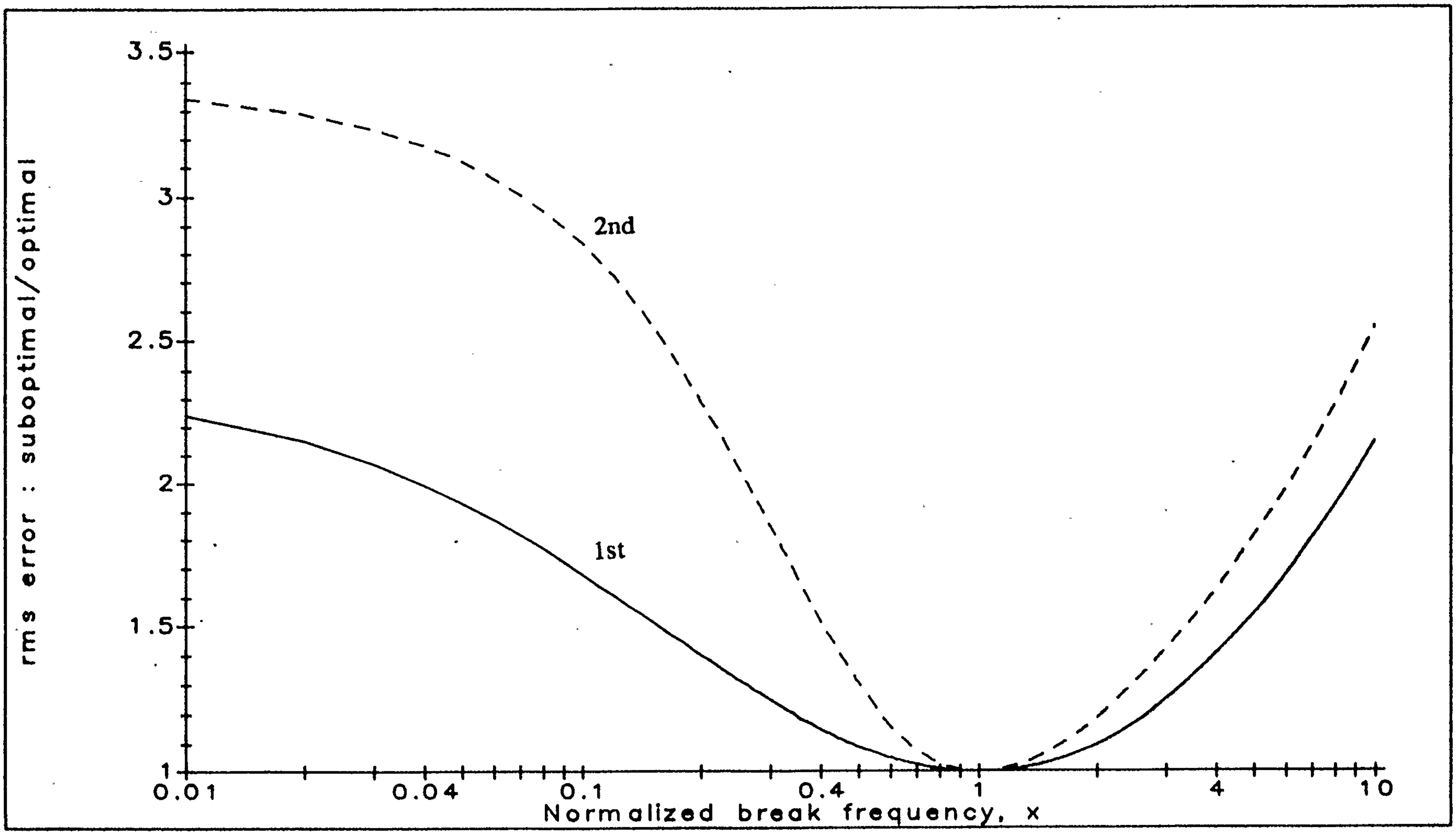


Fig. 2.9 Increase in rms error for the suboptimal filter $G_2(\lambda)$.

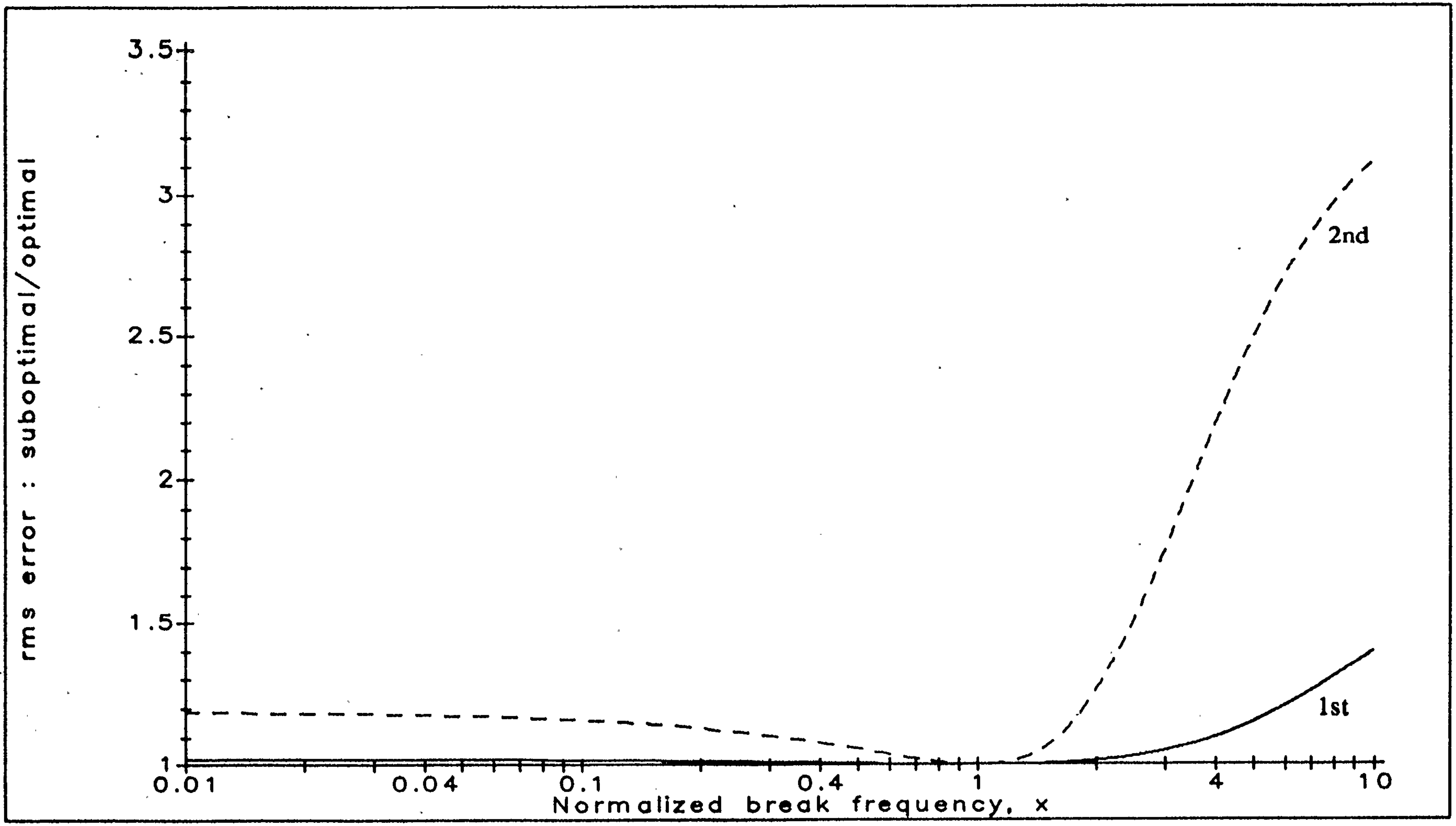


Fig. 2.10 Increase in rms error for the suboptimal filter $G_3(\lambda)$.

CHAPTER 3

A NEW APPROACH TO SERIES COMPENSATION OF CONTROL SYSTEMS

3.1. Introduction

A first undergraduate course in control systems has been a basic part of the curriculum of almost all electrical engineering departments for the last four decades. An excellent traditional text is Melsa and Schultz [15], a recent text with the popular accompanying software is Sinha [16], and an example of a recent paper which attempts to improve these techniques is [17]. These courses assume single-input single-output fixed plants. To simplify the discussion, we also assume that the fixed plant is minimum-phase. The *most interesting* part of these courses is compensation, or how to make complex dynamic systems using negative feedback perform satisfactorily. There are two approaches: series compensation; and state variable compensation. In this chapter, we consider series compensation. The extension of these results to state variable compensation is routine, after we have chosen a desired system, see [15] Chap. 9.

The traditional approach to series compensation is to present several methods such as: root locus, Bode plots, Nyquist charts, Nichols charts, etc. These traditional approaches are *trial−and−error* procedures, rather lengthy, and somewhat arbitrary. There is a problem with the traditional approach. There is a limit to the performance which can be obtained by any real physical plant, which is ignored by the traditional approach. This limit involves both the third order pole in the fixed plant, and the saturation nonlinearity ($\pm L$) at its input. This is the problem we consider.

Optimization theory, Wiener or Kalman, for compensation of control systems, involves the minimization of system error, subject to a constraint on control effort. Minimization of system error involves the maximization of the speed of response and the maximization of the quality of response. The constraint on the control effort is required by the saturation nonlinearity at the input to the fixed plant. However, in neither Wiener or Kalman theory is any guidance presented on how to select the constraint. We present a specific solution to this problem.

Fig. 3.1 presents the series compensation problem considered in this thesis. It presents the basic block diagram where the fixed plant $P(s)$ has at least a pole-zero-excess (PZE) of 3, and the saturation nonlinearity ($\pm L$) which is at the input to the fixed plant. The series compensation is $KF(s)$. Fig. 3.1 also presents the definitions of the open-loop system in a unity-feedback configuration $G(s)$, and of the closed-loop system $W(s)$,

$$G(s) = KF(s)P(s) \,, \quad W(s) = \frac{G(s)}{1 + G(s)} \,. \tag{3.1}$$

An outline of the chapter is as follows. In the remainder of this introduction we introduce two basic concepts. First, we present a new approach to the use of Bode plots. Second, we present a new approach to what constitutes a realistic fixed plant. In Sec. 3.2, we present new significant results for the problem of system specifications. The most important of these results are as follows. First and most important is that, the quality of response, which is given by the percent overshoot, is inversely proportional to a new concept, the required horizontal symmetrical (α, $1/\alpha$) Band of Minus One Slope (BMOS) around magnitude = 1. Second, the speed of response, which is given by the time delay (t_d), is inversely proportional to the crossover frequency (ω_c). In Sec. 3.3 we use the preceding results, to present a simple new procedure for series compensation which is considerably more than an order of magnitude faster than traditional methods. In Sec. 3.4 we extend the preceding results to consider the real problem of series compensation, and one the literature does not consider, and that is the saturation nonlinearity at the input to the fixed plant.

3.1.1. Bode Plots

In this thesis, Bode plot denotes a straight line Bode magnitude plot. The efficient use of Bode plots is very important in this chapter. Horizontal and vertical scales are assumed to equal, so that slope retains its conventional meaning, and slope is also proportional to the PZE at that frequency. We use "engineering paper" with a scale of 1 in. = a factor of 10.

Another very important concept, which is shown in Fig. 3.2, is the algebraic Bode plot relationship (ABPR) between any two points (a, A) and (b, B) which are connected by a slope of $-k$,

$$A\, a^k = B\, b^k \, . \tag{3.2}$$

Note that on a Bode plot we indicate the actual value, but plot it as its logarithm.

This relationship is important for two reasons. First, consider numerical problems, such as Examples 1 through 3 in the Sec. 3.3. We can use a simple graphical design procedure, which yields a great deal of insight into the problem, but which has a rather low precision. We can then repeat these calculations using ABPR (3.2) using a pocket calculator, and obtain any degree of precision we require, but with little insight. Having solved the problem two very different ways, we have an efficient self-checking procedure. Second, consider algebraic problems, such as Examples 4 through 6 in Sec. 3.4. As long as we have an inequality ordering of the roots, we can obtain an algebraic solution using a Bode plot and the ABPR (3.2). This is very useful.

3.1.2. Fixed Plants

Introductory control textbooks assume linear systems. Many of their examples assume second-order fixed plants. This is not a realistic problem formulation because, given a linear second-order fixed plant and series compensation, one can achieve any set of specifications that one can conceive of, and this is not true of real fixed plants.

A realistic fixed plant is defined as one with a pole-zero-excess of at least 3, and a saturation nonlinearity ($\pm L$) at its input. Note that zeros, while important, are somewhat rare in fixed plants. Fourth and higher order poles are ignored, on the basis that the third-order pole, and the saturation nonlinearity almost always limit the performance which is achievable, and this is the problem which we shall consider.

For a numerical example we use our undergraduate control systems laboratory unit, a Feedback Limited Model 150, as identified by an HP 5423A Digital Signal Analyzer,

$$P(s) = \frac{k}{s\,(1 + s\,/\,a)(1 + s\,/\,b)} = \frac{1.25}{s\,(1 + s\,/\,4.00)(1 + s\,/\,160.)}\,. \tag{3.3}$$

The actual numbers vary somewhat with the actual operating conditions, and a simplified typical set is used here.

The most important results in this chapter are: (1) Figs. 3.6, 3.7, and 3.8 which present new results for the specification problem; (2) Examples 1, 2, and 3 which greatly simplify traditional series compensation, and (3) Example 6 which is the recommended approach to series compensation with a saturation nonlinearity for a system with a random input.

3.2. Advances in Specifications

The specification problem is to relate a description of how we want the system to behave, which is given by time-domain specifications on the system output $y(t)$, to a description of how we want to design the system, which is given by frequency-domain specifications on the open-loop transfer function $G(s)$. We present some significant advances.

Basic specifications are as follows. Time domain specifications on system output $y(t)$ to a unit step are: (1) a quality of response specification which is given by the percent overshoot (PO); and (2) a speed of response specification which is given by the time delay (t_d). These are illustrated in Fig. 3.3a. Frequency domain specifications on the open-loop system $G(s)$ are: (1) crossover frequency (ω_c) which is given by $|G(j\omega_c)| = 1$; and (2) phase margin (PM) which is given by $PM = 180^o - \text{Arg. } G(j\omega_c)$ which is easy to calculate. These are illustrated in Fig. 3.3b.

The traditional approach to relate specifications on $y(t)$ and $G(s)$ is to consider a unit-numerator second-order system because it is analytically tractable, see Fig. 3.4a,

$$W(s) = \frac{1}{1 + 2\zeta s / \omega_n + (s / \omega_n)^2}, \quad G(s) = \frac{\omega_n / 2\zeta}{s(1 + s / 2\zeta\omega_n)} . \tag{3.4}$$

The classic presentation of this approach is given in [18]. However, Fig. 3.4a is not what a well compensated system usually looks like, it usually looks like Fig. 3.4b, e.g. see Melsa and Schultz [15] Chap. 10. We consider the more realistic problem below.

3.2.1. A Realistic Compensated System

Consider the problem of maximizing the phase margin for the -2, -1, -2 configuration given in Fig. 3.5a, where

$$G(s) = \frac{b\,\omega_c\,(1 + s / b)}{s^2(1 + s / c)}, \quad b < c , \tag{3.5}$$

$$PM = \tan^{-1}(\omega_c / b) - \tan^{-1}(\omega_c / c) \text{ ; and}$$

$$\frac{d}{d\omega_c} PM = 0 \text{ , yields } \omega_c = \sqrt{b\,c} \text{ ,}$$

the geometric mean, or on a log scale the mid-point. Notation is simplified by introducing the parameter alpha (α),

$$c / \omega_c = \omega_c / b = (c / b)^{1/2} \equiv \alpha , \tag{3.6}$$

$$PM = \tan^{-1}(\alpha) - \tan^{-1}(1 / \alpha) .$$

Recall that horizontal and vertical scales are equal. Therefore, PM is maximized by providing a horizontal symmetrical (α, $1 / \alpha$) Band of Minus One Slope (BMOS) about mag. = 1, as shown in Fig. 3.5b. This is an important concept.

Reconsider Fig. 3.4b, a -1, -2, -1, -2, -3 configuration assuming that $b = 1/\alpha$ and $c = \alpha$ to maximize PM. The poles at a and d have a relatively small influence on the above, but are required for a realistic problem. In real problems the poles at a and d are not necessarily symmetric with respect to ω_c; however we shall assume that they are, $a = \omega_c / \beta$ and $d = \beta\omega_c$, in order to retain a simple relation for the PM (3.6). An actual PM calculation should correct for this assumption.

To further simplify notation, we use the normalized frequency, $\lambda = s/\omega_c$. Therefore, as the system which we will use to relate specifications on $y(t)$ to specifications on $G(\lambda)$ we have

$$G_0(\lambda) = \frac{\beta(1+\alpha\lambda)}{\alpha\lambda(1+\beta\lambda)(1+\lambda/\alpha)(1+\lambda/\beta)}\,, \quad \lambda = s/\omega_c\,, \tag{3.7}$$

see Fig. 3.6; and

$$PM = 90^o - \tan^{-1}(\beta) + \tan^{-1}(\alpha) - \tan^{-1}(1/\alpha) - \tan^{-1}(1/\beta)\,,$$

$$\tan^{-1}(\beta) + \tan^{-1}(1/\beta) = 90^o\,,$$

$$PM = \tan^{-1}(\alpha) - \tan^{-1}(1/\alpha)\,, \tag{3.8}$$

$$\tan^{-1}(\alpha) + \tan^{-1}(1/\alpha) = 90^o\,,$$

$$\alpha = \tan((PM + 90^o)/2)\,.$$

To obtain a computer solution we must somehow fix β. For a "typical" case we assume that $\beta = \alpha^2$, which results in $G_2(\lambda)$, see Fig. 3.6,

$$G_2(\lambda) = \frac{\alpha(1+\alpha\lambda)}{\lambda(1+\alpha^2\lambda)(1+\lambda/\alpha)(1+\lambda/\alpha^2)} \,. \tag{3.9}$$

In addition, we consider the two extreme cases which are: (1) $\beta = \infty$, which results in $G_1(\lambda)$,

$$G_1(\lambda) = \frac{(1+\alpha\lambda)}{\alpha\lambda^2(1+\lambda/\alpha)} \,; \tag{3.10}$$

and (2) $\beta = \alpha$, which results in $G_3(\lambda)$,

$$G_3(\lambda) = \frac{1}{\lambda(1+\lambda/\alpha)^2} \,. \tag{3.11}$$

Again see Fig. 3.6. In all three cases, we still have

$$\alpha = \tan((PM + 90^o)/2) \,.$$

$G_2(\lambda)$ is a much more realistic system for series compensation than the traditional second-order system (3.4). The problem now is to relate the specifications on $y(t)$ to specifications on $G_2(\lambda)$.

The solution to this still computationally intensive problem, has been obtained using MACSYMA [1]. Fig. 3.7 presents phase margin plus percent overshoot, $PM + PO$, as a function alpha, for $2 \le \alpha \le 5$, for the typical case $\beta = \alpha^2$ and for the two extreme cases, $\beta = \infty$ and $\beta = \alpha$. This shows that

$$PM + PO \approx 74^o \tag{3.12}$$

is a good approximation. Fig. 3.8 presents time delay, t_d, as function of α for $2 \le \alpha \le 5$ for the same three cases. This shows that

$$t_d \approx 0.9 / \omega_c \tag{3.13}$$

is a useful approximation.

To summarize, a realistic compensated system for traditional series compensation is

$$G_o(\lambda) = \frac{\beta(1 + \alpha\lambda)}{\lambda(1 + \beta\lambda)(1 + \lambda/\alpha)(1 + \lambda/\beta)}, \quad \beta > \alpha,$$

which has one zero and four poles; and we have computed simple design formulas to relate given specifications on $y(t)$, to design specifications on $G(s)$. These are as follows. (1) Given a desired percent overshoot, a quality of response specification, we can determine the required phase margin by

$$PM \approx 74^o - PO .$$

The required PM is obtained by providing a required horizontal symmetrical $(\alpha, 1/\alpha)$ band of -1 slope (BMOS) about mag. = 1. The parameter α is given by

$$\alpha = \tan((PM + 90^o)/2) .$$

Briefly, α is inversely proportional to PO. (2) Given a desired time delay (t_d), a speed of response specification, we can determine the required crossover frequency (ω_c) by

$$\omega_c \approx 0.9 / t_d .$$

Briefly, ω_c is inversely proportional to t_d. Sec. 3.3 presents three examples which illustrate the use of these formulas.

3.2.2. Control Effort

Another important problem is control effort, or how hard we drive the fixed plant. For deterministic signals, the default assumption is a unit step, and we are concerned with the maximum value of the input to the fixed plant or max. $u(t)$. An easily obtained estimate of max. $u(t)$ is given by

$$u(0+) \leq \text{max. } u(t) \tag{3.14}$$

where the equality is often valid. The value for $u(0+)$ can be obtained by the initial value theorem

$$u(0+) = \lim_{s \to \infty} s \frac{KF(s)}{1 + KF(s)P(s)} \frac{1}{s} = KF(\infty) \, , \tag{3.15}$$

because $P(s)$ has a pole-zero-excess of 3. Note that this is a specification on how much the high-frequency asymptote (HFA) at $\omega = 1$ may be increased.

3.2.3. The Best Compensated System

We now reconsider the entire series compensation problem, and inquire "what is the best we can do?". Our basic problem is a trade off between speed of response and control effort. We know the follows. The quality of the response is given by alpha α, that is by the required horizontal symmetrical (α, $1/\alpha$) band of -1 slope, at mag. = 1 (BMOS), which we assume is fixed. The speed of response t_d is inversely proportional to crossover frequency ω_c, so ω_c should be maximized. The control effort is approximated by the high-frequency asymptote (HFA) at $\omega = 1$ or the allowable increase in gain at high frequencies. If we assume that HFA is fixed by the saturation nonlinearity, and consider $G_1(\lambda)$, $G_2(\lambda)$, and $G_3(\lambda)$, it is obvious that ω_c is maximized by $G_3(\lambda)$. Therefore, the best compensated system is

$$G_3(\lambda) = \frac{1}{\lambda(1+\lambda/\alpha)^2}\ , \quad \lambda = s/c\ , \tag{3.16}$$

where from previous MACSYMA calculations we have

$$PM \approx 66^o - PO\ , \tag{3.17}$$

$$\alpha = \tan((PM + 90^o)/2)\ , \text{ and}$$

$$\omega_c \approx 1.1/t_d\ . \tag{3.18}$$

This will be used in Sec. 3.4.

3.3. Simplification of Traditional Series Compensation

Given: the fixed plant $P(s)$ in Bode plot form (3.3), and the following specifications.

(1) Steady-state error.

Add enough gain (K) to satisfy the traditional steady-state velocity error specification (e_{ssv}), or velocity-error constant (K_v),

$$K\,k = K_v = 1/e_{ssv}\ .$$

(2) Quality of response, or percent overshoot.

Modify the open-loop system $G(s)$ to obtain the required horizontal symmetrical (α, $1/\alpha$) band of -1 slope (BMOS) about mag. = 1. Given a required percent overshoot, we can approximately determine a required phase margin, which translates into a required horizontal symmetrical (α, $1/\alpha$) band of -1 slope

(BMOS) about mag. = 1;

$$PM \approx 74^o - PO \ ,$$

$$\alpha = \tan((PM + 90^o)/2) \ .$$

(3) A third specification, which is one of the following.

(a) Lag compensation.

Drop the highest frequency existing section of -1 slope into the required horizontal symmetrical (α, $1/\alpha$) band of -1 slope (BMOS) about mag. = 1. See Example 1.

(b) Lead compensation.

Bend an existing section of -2 slope in the required horizontal symmetrical (α, $1/\alpha$) band of -1 slope (BMOS) about mag. = 1 into a -1 slope. See Example 2.

(c) Speed of response, or time delay t_d.

Crossover frequency ω_c is approximately inversely proportional to time delay.

$$\omega_c \approx 0.90/t_d \ .$$

Start with ω_c, and using the required horizontal symmetrical (α, $1/\alpha$) band of -1 slope (BMOS) about mag. = 1, work backward to obtain a lag-lead network. The required series compensation is the difference between what we have $P(s)$, and what we want $G_d(s) = KF(s)P(s)$. See Example 3.

3.3.1. Example 1, Lag Network

Given: the fixed plant,

$$P(s) = \frac{1.25}{s\,(1 + s/4.00)(1 + s/160.)} \;;$$

and the specifications:

(1) steady state velocity error (e_{ssv}) = 2.5% ,

(2) $PO \approx 12\%$, (3) lag network.

Solution.

(1) $K_v = 1/e_{ssv} = 40 = LFA = 1.25K,\; K = 32.0,$ plot $KP(s)$.

(2) $PM \approx 74^o - PO = 62^o,$

$\alpha = \tan((PM + 90^o)/2) \approx 4.00,$

= horizontal symmetrical (α, $1/\alpha$) band of -1 slope (BMOS) about mag. = 1.

(3) For lag network, *drop* the highest frequency existing section of -1 slope into the required band.

(4) Draw the Bode plot, Fig. 3.9. Check by using ABPR (3.2) and pocket calculator. Note, all steps are shown.

$\alpha = 4.00\;,\quad \omega_c = 4/\alpha = 1.00\;,\quad a = 4/\alpha^2 = 0.250\;,$

$\alpha\,(a)^2 = B\,(b)^2$ and $B\,(b)^1 = 40\,(1)^1\;,$

$\alpha a^2 = 40.0\,b\;,\quad b = 0.00625\;,$

$32.0\,(b)^1 = A\,(a)^1\;,\quad A = 0.800 = u(0+)\;.$

(5) The lag compensation is

$$KF(s) = 32.0\frac{(1 + s/0.250)}{(1 + s/0.00625)}$$

$$t_d \approx 0.90/\omega_c = 0.900 \text{ s.}$$

$$u(0+) = KF(\infty) = 0.800 = \text{estimate of control effort} .$$

Note that this is about a 10 minute procedure to this point.

(6) As a check, simulate on a personal computer using one of the many control software packages which are available (such as MatrixX/PC). The step response $y(t)$ and control effort $u(t)$ are shown in Fig. 3.10. See Table 3.1.

3.3.2. Example 2, Lead Network

Given: the fixed plant,

$$P(s) = \frac{1.25}{s(1 + s/4.00)(1 + s/160.)} ;$$

and the specifications:

(1) $e_{ssv} = 2.5\%$, (2) $PO \approx 12\%$, (3) lead network.

Solution.

(1) $K_v = 1/e_{ssv} = 40 = LFA = 1.25K$, $K = 32.0$, plot $KP(s)$.

(2) $PM \approx 74^o - PO = 62^o$,

$\alpha = \tan((PM + 90^o)/2) \approx 4.00$,

= horizontal symmetrical $(\alpha, 1/\alpha)$ band of -1 slope (BMOS) about mag. = 1.

(3) For lead network, *bend* a existing section of -2 slope into the required band.

(4) Draw the Bode plot, Fig. 3.11. Check by using ABPR (3.2) and pocket calculator.

$\alpha = 4.00$, $40\,(1)^1 = A\ (4)^1$ and $A\ (4)^2 = \alpha\,(a)^2$

$40\,(4) = \alpha\,(a)^2$, $a = 6.32$,

$\omega_c = a\,\alpha = 25.3$, $b = a\,\alpha^2 = 101.$,

$32\,(a)^{-1} = u\,(0+)\,(b)^{-1}$, $u\,(0+) = 512.$

(5) The lead compensation is

$$KF(s) = 32.0\,\frac{(1 + s\,/\,6.32)}{(1 + s\,/\,101.)}$$

$t_d \approx 0.90\,/\,\omega_c = 0.0356$ s.

$u\,(0+) = KF\,(\infty) = 512. =$ estimate of control effort .

(6) As a check, simulate. This not included, see Example 1, and Table 3.1.

3.3.3. Example 3, Lag-lead Network

Given: the fixed plant,

$$P(s) = \frac{1.25}{s\ (1 + s/4.00)(1 + s/160.)}\ ;$$

and the specifications:

(1) $e_{ssv} = 2.5\%$, (2) $PO \approx 12\%$, (3) $t_d = 0.180$ s.

Solution.

(1) $K_v = 1\,/\,e_{ssv} = 40 = LFA = 1.25\,K$, $K = 32.0$, plot $KP\,(s)$.

(2) $PM \approx 74^o - PO = 62^o$,

$\alpha = \tan((PM + 90^o)/2) \approx 4.00$,

= horizontal symmetrical (α, $1/\alpha$) band of -1 slope (BMOS) about mag. = 1.

(3) The speed of response or time delay specification requires a lag-lead network.

$$\omega_c = 0.90/t_d = 5.00 .$$

Start with ω_c, and provide the required band of -1 slope.

(4) Draw the Bode plot, Fig. 3.12. Check by using ABPR (3.2) and pocket calculator.

$\alpha = 4.00$, $a = \omega_c \alpha = 20.0$, $b = \omega_c / \alpha = 1.25$,

$\alpha (b)^2 = C (c)^2$ and $C (c)^1 = K_v (1)^1$,

$\alpha b^2 = 40 c$, $c = 0.156$.

(5) The lag-lead compensation is

$$KF(s) = 32.0 \frac{(1 + s/1.25)(1 + s/4.00)}{(1 + s/0.156)(1 + s/20.0)}$$

$$t_d \approx 0.90/\omega_c = 0.180 \text{ s.}$$

$u(0+) = KF(\infty) = 20.0$ = estimate of control effort .

(6) As a check, simulate. See Table 3.1.

3.4. Design for a Saturation Nonlinearity

3.4.1. Introduction

The real problem in series compensation is the trade off between speed of response, and control effort or how hard we drive the fixed plant. This is rarely treated. However, consideration of Wiener optimal controls, or of Kalman optimal or linear quadratic (LQ) regulators, clearly demonstrates the fundamental nature of this trade off.

In practice the fixed plant always contains one or more variables, which are subject to a saturation nonlinearity ($\pm L$), and which limit system performance. We assume that there is only one variable which has a saturation nonlinearity and that it is the input to plant, to simplify the presentation.

In Example 4 we consider the default specification which is a step input, and the specification that we should maximize the speed of response, subject to the constraint that the system operates in its linear region. This is not a particularly good specification, only a frequently encountered one. There is much to argue for dual-mode control, particularly in this age of microprocessors. The main purpose of Example 4 is to provide a logical transition from the traditional approach to series compensation to the best approach which considers the saturation nonlinearity ($\pm L$). The main concepts here are as follows. (1) When trying to constrain control effort, we are dealing with the allowable increase in gain at high frequencies or increase in the high-frequency asymptote (HFA). This is very different from the traditional steady-state error specification which is a specification on the low-frequency asymptote (LFA). (2) Given the above, and that we are trying to maximize speed of response, and that we have a choice between $G_1(\lambda)$, $G_2(\lambda)$, $G_3(\lambda)$ from Sec. 3.2, the obvious best choice is $G_3(\lambda)$. We note that an optimal system for the given problem may perform a little better, and usually have a very similar structure, with damping ratio $\zeta \approx 0.5$ rather than $\zeta = 1.0$.

Sec. 3.4.3 presents a very brief review of random signals, and of the calculation of mean-square values. It presents Booton's method [19] for handling a saturation nonlinearity with a random input, and an approximation to it. It also presents Streets' [20] simple approximate method for calculation of mean-square values.

In Example 5 we consider a random input, the default being $S_r^+(s) = e/s$, and we calculate the exact value for the mean-square value of the input to the fixed plant σ_u^2, and set it equal to Booton's value to obtain the best compensation. This procedure requires some careful normalization to achieve reasonable simple results.

In Example 6 we reconsider Example 5 in very general terms which include an arbitrary: fixed plant, factored PSD of the random input, and desired system. This is accomplished by the calculation of an approximate value for the mean-square value of the input to the fixed plant. Again we set this equal to Booton's value. This is the best solution to series compensation for a saturation nonlinearity with a random input.

3.4.2. Example 4, Step Input, Linear Operation Constraint

Given: the fixed plant,

$$P(s) = \frac{k}{s(1+s/a)(1+s/b)} ;$$

with nominal values: $k = 1.25$, $a = 4.00$, $b = 160.$; and the following specifications.

(1) A saturation nonlinearity $\pm L$; nominal: $L = 13$.

(2) Maximize the system performance, subject to the constraint that the system operates in a linear mode for a step input $r(t) = e\,u(t)$; nominal: $e = 1$.

(3) Speed of response; minimize time delay t_d.

(4) Quality of response; percent overshoot $\approx$ 12%.

Solution.

(1) $u(0+) = KF(\infty)$ is an easily obtained estimate of max. $u(t)$, which we wish to limit, $u(0+) \leq$ max. $u(t) = \mathrm{L}$. As a first try let $K = L = 13$. $K\,k = 16.3$, HFA $= K\,k\,a\,b = 1.04\times10^4$. Plot $KP(s)$ in Fig. 3.13.

(2) To maximize speed of response (minimize t_d) with a saturation nonlinearity, the desired system from Sec. 3.2.3 is $G_3(\lambda) = G_d(\lambda)$,

$$G_d(\lambda) = \frac{1}{\lambda(1+\lambda/\alpha)^2}\,, \quad \lambda = s/c\ .$$

$$PM \approx 66^o - PO\ , \quad t_d \approx 1.1/\omega_c\ .$$

(3) $PM \approx 54^o$, $\alpha = \tan((PM + 90^o))/2 \approx \sqrt{10} = 3.16$.

(4) Find c in $\lambda = s/c$ such that $G_d(\lambda)$ agrees with $KP(s)$ at high frequencies, HFA $= c^3\alpha^2 = K\,k\,a\,b$, $c = 10.1$, $\alpha c = 32.0$. Plot $G_d(\lambda)$ in Fig. 3.13.

(5) The required series compensation is

$$KF(s) = \frac{G_d(\lambda)}{P(s)} = (c/k)\,\frac{(1+s/a)(1+s/b)}{(1+s/\alpha c)^2}$$

$$= 8.11\,\frac{(1+s/4.00)(1+s/160.)}{(1+s/32.0)^2}\,, \qquad \text{nominal.}$$

(6) However, simulation gives max. $u(t)$ = 26.3 instead of 13.0. We have encountered a problem where $u(0+) <$ max. $u(t)$. We need to reduce

MF = max. $KF(s)$, see Fig. 3.13,

$$(c/k)(a)^{-1} = (MF)(\alpha c)^{-1}, \quad MF = c^2\alpha/ak = 64.5 .$$

(7) Second try.

$$MF = (13.0/26.3)\,64.5 = 31.9 , \quad c = (MFak/\alpha)^{1/2} = 7.10 , \quad \alpha c = 22.5 .$$

Simulation gives max. $u(t)$ = 13.4. See Table 3.2.

The required series compensation is

$$KF(s) = 5.68 \frac{(1 + s/4.00)(1 + s/160.)}{(1 + s/22.5)^2}$$

$$t_d \approx 1.1/\omega_c = 0.155 \text{ s.}$$

3.4.3. Random Signals and a Saturation Constraint

A Gaussian random signal [4] is mathematically described by its power spectral density (PSD), $S_x(s)$. A PSD can be easily measured by modern signal processing computer systems such as a HP 5423A or a B&K 2032. An important concept is the factored or superscript plus PSD, $\overset{+}{S}_x(s)$, which consists of the left-half s-plane poles and zeros, and the square-root of the constant, of the PSD. The input-output relation for a minimum-phase system $H(s)$ with a random signal input $x(t)$ and an output $y(t)$ is

$$\overset{+}{S}_y(s) = H(s)\overset{+}{S}_x(s) . \tag{3.19}$$

Note the similarity to the input-output relation for deterministic signals

$Y(s) = H(s)X(s)$.

The mean-square value for a random signal $\sigma_x^2 = E\{x^2\}$ is best calculated by use of standard tables. A abbreviated table is given on p. 126 of [4], and the complete table is given in Newton, et. al. [7].

$$\sigma_x^2 = E\{x^2\} = I_n(\overset{+}{S}_x(s)) \equiv \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \overset{+}{S}_x(s)\overset{+}{S}_x(-s)\,ds\ , \tag{3.20a}$$

where

$$\overset{+}{S}_x(s) = \frac{c(s)}{d(s)} = \frac{c_0 + c_1 s + \cdots + c_{n-1}s^{n-1}}{d_0 + d_1 s + \cdots + d_n s^n}\ . \tag{3.20b}$$

Using normalized frequency $\lambda = s/\nu$, a useful simplification occurs,

$$\sigma_x^2 = \nu I_n(\overset{+}{S}_x(\lambda))\ . \tag{3.21}$$

The effect of a saturation nonlinearity on a control system with a random input, Fig. 3.14, which is a very important design consideration has been analyzed by Booton [19]. He derived an equivalent linear gain,

$$K_l = erf\,(L/\sqrt{2}\sigma_u)\ , \tag{3.22}$$

where the error function is defined by

$$erf\,(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

This is difficult to handle. Consideration of Taylor series shows that an approximation for K_l is

$$
\begin{aligned}
K_{la} &\approx 1 \quad , \quad \sigma_u \leq \sqrt{2/\pi}\,L \\
&\approx \sqrt{2/\pi}\,L\,/\,\sigma_u \quad , \quad \sigma_u \geq \sqrt{2/\pi}\,L
\end{aligned}
\tag{3.23}
$$

Fig. 3.15 presents a log-log plot of K_l and K_{la}, which demonstrates that K_{la} is a good approximation to K_l. Note that for $\sigma_u > \sqrt{2/\pi}\,L$, the rms value of the saturation nonlinearity output is constant,

$$\sigma_{ul} = K_{la}\,\sigma_u = \sqrt{2/\pi}\,L \; . \tag{3.24}$$

Considering the above, we present the following best design procedure for control systems with a random input, and a saturation nonlinearity. (1) Assume the system is linear. (2) Calculate the rms value of the input to the fixed plant, σ_u. (3) Set it equal to Booton's value, $\sigma_{ul} = \sqrt{2/\pi}\,L$. This will be illustrated in Examples 5 and 6.

A very useful graphical approximation for calculating mean-square input has been presented in [20]. Briefly summarized the procedure is as follows. (1) Draw a Bode plot of the factored PSD, $S_x^+(s)$. (2) Determine a line with a -1/2 slope which is tangent to or a least upper bound (LUB) to $S_x^+(s)$. (3) In an appropriate band of -1/2 slope directly below the LUB, use the exact expression for $S_x^+(s)$, below that simplify the expression for $S_x^+(s)$. Obviously, the wider the band of -1/2 slope, the better the results. This procedure is presented in detail in [20], and it will be illustrated in Example 6.

3.4.4. Example 5, Random Input, Exact Rms Value Constraint

Note, three key steps are required to make the answer for the exact evaluation of the mean-square value σ_u^2 reasonably simple.

Given: the fixed plant (as before) is in the following generalized form. Key step 1 is to define $P(s)$ in terms of the geometric mean of its poles, where for two real poles $w = \sqrt{ab}$, $2zw = a + b$,

$$P(s) = \frac{kw^2}{s(s^2 + 2zws + w^2)};$$

with nominal values: $k = 1.25$, $w = 25.3$, $z = 3.24$; and the following specifications.

(1) A saturation nonlinearity $\pm L$, nominal: $L = 13$.

(2) The factored power spectral density of the random input, nominal: $e = 1$.

$$S_r^+(s) = \frac{e}{s}.$$

(3) Determine a good approximation for best series compensation to minimize the mean-square error of the system for the preceding specifications, using Booton's value for a saturation nonlinearity, and an exact evaluation of the mean-square value of the input to the fixed plant.

Solution.

(1) The basic approach is as follows.

Calculate the exact value of mean-square value of the input to the linear fixed plant σ_u^2, and set it equal Booton's value for a saturation nonlinearity, $\sigma_u^2 = (2/\pi)L^2$, to determine the series compensation.

(2) To maximize speed of response (minimize t_d) with a saturation nonlinearity, the desired system from Sec. 3.2.3 is $G_3(\lambda) = G_d(\lambda)$,

$$G_d(\lambda) = \frac{1}{\lambda(1+\lambda/\alpha)^2}, \quad \lambda = s/c .$$

(3) Key step 2 is to define c in terms of w defined above, $c = vw$, where v is the variable to be determined.

$$W_d(s) = \frac{\alpha^2 v^3 w^3}{s^3 + 2\alpha v w s^2 + \alpha^2 v^2 w^2 s + \alpha^2 v^3 w^3} .$$

Key step 3 is to introduce a new normalized frequency $\lambda = s/w$. Thus we have,

$$P(\lambda) = \frac{k/w}{\lambda(\lambda^2 + 2z\lambda + 1)} ,$$

$$W_d(\lambda) = \frac{\alpha^2 v^3}{\lambda^3 + 2\alpha v \lambda^2 + \alpha^2 v^2 \lambda + \alpha^2 v^3} ,$$

$$\overset{+}{S}_r(\lambda) = \frac{e/w}{\lambda} .$$

(4) The factored power spectral density of the input to the linear fixed plant is

$$\overset{+}{S}_u(\lambda) = \frac{W_d(\lambda)}{P(\lambda)} \overset{+}{S}_r(\lambda)$$

$$= (\alpha^2 v^3 e/k) \frac{1 + 2z\lambda + \lambda^2}{\alpha^2 v^3 + \alpha^2 v^2 \lambda + 2\alpha v \lambda^2 + \lambda^3} .$$

(5) The exact mean-square value of

$$\overset{+}{\underset{x}{S}}(\lambda) = \frac{k\,(c_0 + c_1\lambda + c_2\lambda^2)}{(d_0 + d_1\lambda + d_2\lambda^2 + d_3\lambda^3)}\,, \quad \lambda = s\,/\,v$$

is given by standard tables as

$$E\{x^2\} = \sigma_x{}^2 = v\,I_3(\overset{+}{\underset{x}{S}}(\lambda))$$

$$= vk^2\,\frac{[c_2{}^2 d_o d_1 + (c_1{}^2 - 2c_0 c_2) d_0 d_3 + c_0{}^2 d_2 d_3]}{2d_0 d_3(-d_0 d_3 + d_1 d_2)}\,.$$

For this problem we have

$$\sigma_u{}^2 = \frac{e^2\alpha w}{k^2 2(2\alpha - 1)}\,[\,\alpha^3 v^5 + 2(2z^2 - 1)\alpha v^3 + 2v\,]\,.$$

(6) Setting this equal to Booton's value, $\sigma_u{}^2 = (2\,/\,\pi)L^2$, yields,

$$\alpha^3 v^5 + 2(2z^2 - 1)\alpha v^3 + 2v \;-\; (2kL\,/\,e)^2(2\alpha - 1)\,/\,(\pi\alpha w) = 0\,. \tag{3.25}$$

Solving this 5^{th} order polynomial for nominal values yields $v = 0.540$, and $c = vw = 13.7$.

(7) The required series compensation is

$$KF(s) = \frac{G_d(\lambda)}{P(s)} = (c\,/k)\,\frac{1 + 2zs\,/\,w + (s\,/\,w)^2}{(1 + s\,/\,c\,\alpha)^2}$$

$$= 17.0\,\frac{(1 + s\,/\,4.00)(1 + s\,/\,160.)}{(1 + s\,/\,43.0)^2}\,, \qquad \text{nominal}\,.$$

(8) Extensive computer simulations for Example 5 and 6 are presented in Figs. 3.16 and 3.17. Fig. 3.16 presents the rms value of the system error as a function of normalized frequency, v. Fig. 3.17 presents the rms value of the saturation

nonlinearity limited input to the fixed plant, for the same conditions. The computer algebra software MACSYMA is used to obtain two theoretical results: Booton's exact value (3.22); and Booton's approx. value (3.23). The minimum for Booton's exact theoretical solution is $v = 0.794$ or $c = vw = 20.1$. Note, we are operating on a broad minimum as is usual in optimal solutions. The approximate solution for Example 5 is $v = 0.540$ or $c = 13.7$. The more approximate solution for Example 6 is $v = 0.727$ or $c = 18.4$. Both of these are shown to be excellent approximations, which are on the conservative side. In addition simulation studies have been conducted on an actual system, our undergraduate servo lab. This has been done for three values: (a) linear or $v = 0.282$; (b) partially nonlinear, the design value of Example 5 or $v = 0.540$; and (c) totally nonlinear, or $v = 1.12$. The experimental verification of the theory is very good for 5 of the 6 points. The discrepancy of the experimental data for system error for $v = 1.12$ is probably due to an inadequacy in our simulation of $KF(s)$. This problem is being investigated. However, Booton's theoretical results and our experimental results from a real physical system agree quite well.

3.4.5. Example 6, Random Input, Approximate Rms Value Constraint

Given: the fixed plant,

$$P(s) = \frac{k}{s(1 + s/a)(1 + s/b)} ;$$

with nominal values: $k = 1.25$, $a = 4.00$, $b = 160.$. Saturation nonlinearity ($\pm L$), nominal: L = 13.0 . Factored power spectral density of the random input, nominal: $e = 1.00$, $f = 2.00$.

$$\overset{+}{S}_r(s) = \frac{e/f}{1+s/f}, \quad f < a .$$

Find: Series compensation using Booton's value, and approximate evaluation of the mean-square value, $\sigma_u{}^2$.

Solution.

From Sec. 3.2, the desired system which maximizes the speed of response (minimizes t_d) while constraining control effort is given by

$$G_d(\lambda) = \frac{1}{\lambda(1+\lambda/\alpha)^2}, \quad \lambda = s/c \text{ , or}$$

$$W_d(\lambda) = \frac{\alpha^2}{\lambda^3 + 2\alpha\lambda^2 + \alpha^2\lambda + \alpha^2}, \qquad \text{nominal: } \alpha = \sqrt{10} .$$

We factor the 3rd order polynomial, and let $w_n = 1.478\,c$, $\zeta = 0.5778$, $g = 3.095$. Thus,

$$W_d(s) = \frac{1}{[1 + 2\zeta s/w_n + (s/w_n)^2](1 + s/gw_n)},$$

where w_n is to be determined. Note the high-frequency asymptote

$$\lim_{s\to\infty} W_d(s) = \frac{10c^3}{s^3} = \lim_{\lambda\to\infty} G_d(\lambda) .$$

A reasonable engineering assumption is that the desired system's crossover frequency w_n occurs where the fixed plant has a slope of -2, $a < w_n < b$. The input to the fixed plant is

$$\overset{+}{S}_{u}(s) = \frac{W_d(s)}{P(s)} \overset{+}{S}_{r}(s)$$

$$= (e/fk) \frac{s(1+s/a)(1+s/b)}{(1+s/f)[1+2\zeta s/w_n + (s/w_n)^2](1+s/gw_n)}\,; \qquad (3.26)$$

or in terms of nominal values,

$$\overset{+}{S}_{u}(s) = 0.400 \frac{s(1+s/4.00)(1+s/160.)}{(1+s/2.00)[1+2\zeta s/w_n + (s/w_n)^2](1+s/gw_n)}\,. \qquad (3.27)$$

Fig. 3.18 is an algebraic Bode plot of (3.26). For tutorial purposes, this is drawn to scale for the nominal values (3.27), and the solution of Example 5. We do not wish to evaluate an I_4 by hand, so we seek a simpler good approximation to $\overset{+}{S}_{u}(s)$. We draw a line with -1/2 slope which is tangent to $\overset{+}{S}_{u}(s)$. In the band, immediately below this line we use the exact expression, but below this band we approximate freely to simplify the expression,

$$\overset{+}{S}_{ua}(s) = (e/fk) \frac{s(s/a)(1)}{(s/f)[1+2\zeta s/w_n + (s/w_n)^2](1)}\,.$$

See Fig. 3.18, and note that we have an algebraic problem formulation which is valid for any set of parameters which satisfy the inequality ordering of roots, $f < a < w_n < b/g$. Simplifying, and defining a new normalized frequency, $\lambda = s/w_n$, we obtain,

$$\overset{+}{S}_{ua}(\lambda) = (ew_n/ak) \frac{(0+\lambda)}{(1+2\zeta\lambda+\lambda^2)}\,.$$

The mean-square value of

$$\overset{+}{\underset{x}{S}}(\lambda) = \frac{k\,(c_0 + c_1\lambda)}{d_0 + d_1\lambda + d_2\lambda^2}\,, \quad \lambda = s\,/\,v$$

is given by standard tables as

$$E\{x^2\} = \sigma_x{}^2 = v\,I_2(\overset{+}{\underset{x}{S}}(\lambda)) = vk^2\,\frac{c_0^2 d_2 + c_1^2 d_0}{2d_0 d_1 d_2}\,.$$

For this problem, we have

$$\sigma_u{}^2 = w_n{}^3(e\,/ak\,)^2/\,4\zeta\,.$$

Setting this equal to Booton's value $(2\,/\,\pi)L^2$, we obtain

$$w_n = \left[\,\frac{8\zeta}{\pi}\,(\,\frac{akL}{e}\,)^2\,\right]^{1/3}, \tag{3.28}$$

an approximate algebraic solution. Substituting nominal values for this example, we obtain $w_n = 18.4$. Comparison with the computer simulations presented in Fig. 3.16 and 3.17 for Example 5 shows that Example 6 is a good approximation to Example 5. In addition (3.28) has the advantage that it clearly shows what the important trade-offs are. Note that this simple graphical but algebraic procedure will work for any given: fixed plant, saturation nonlinearity, factored PSD of a random input, and desired system. This makes it a very useful design tool. This is the recommended design procedure.

3.5. Conclusions

We have considered series compensation for a realistic fixed plant, which is defined as a fixed plant with poles-zero-excess of at least 3, and which has saturation nonlinearity ($\pm L$) which is usually at its input. Extensive use had been made of a computer algebra software program called MACSYMA.

First, we have presented a major simplification in design techniques for the traditional approach to series compensation. Series compensation is the most interesting part of a traditional first undergraduate course in control. We have presented a specific, very fast, easily understood Bode plot technique for traditional series compensation, which makes the presentation of this subject in current textbooks obsolete.

Second, we have presented a solution for the real problem of compensation, which is the trade off between speed of response and control effort. Our approach is based on: Booton's linearized gain for a saturation nonlinearity with a random input; and Streets' technique for approximate evaluation of mean-square values. We have presented a specific, very fast, easily understood Bode plot technique for the solution of this problem. Computer simulations verify our approach. Because optimal control problems typically have a broad minimum, our approximations which tend to be conservative appear to be very good.

We have given a specific solution for a frequently encountered real problem in control systems. Traditional courses in control systems do not solve this problem. Optimal control theory can provide a little better performance, if the designer only knew how to pick the control effort constraint. The one we use is Booton's value. The choice is simple, either: (1) we don't know; or (2) we will use a perhaps less than perfect solution.

TABLE 3.1

This table illustrates three points. (1) That the approximations used in the design for PO, t_d and $u(0+)$ are useful engineering approximations. (2) That the estimate of control effort $u(0+)$ is inversely proportional to the square of the speed of response for the cases considered. (3) That $u(0+) = \text{max.}\ u(t)$ is often a reasonable estimate of max. $u(t)$.

Approximation	PO	t_d	$u(0+)$	$t_d^2 u(0+)$
Lag	12	0.900	0.800	0.648
Lag-lead	12	0.180	20.0	0.648
Lead	12	0.0356	512.	0.649
Simulation	PO	t_d	$u(0+)$	max. $u(t)$
Lag	16.9	0.792	0.800	0.808
Lag-lead	15.3	0.162	20.0	20.0
Lead	8.33	0.0370	511.	511.

TABLE 3.2

Example 5, the solution for a saturation constraint for a random input using Booton's value, is very different than the solutions given in Examples 1 through 4. However, Example 5 is the best solution.

		PO	t_d	max. $u(t)$ step	σ_u random
	Specifications	12.0	min.	13.0	10.4
Ex. 1	Lag	16.9	0.792	0.808	0.717
Ex. 2	Lag-lead	15.3	0.162	20.0	3.62
Ex. 3	Lead	8.33	0.0370	511.	38.6
Ex. 4	Saturation, step	10.4	0.158	13.4	4.08
Ex. 5	Saturation, random	10.3	0.0827	48.0	10.4

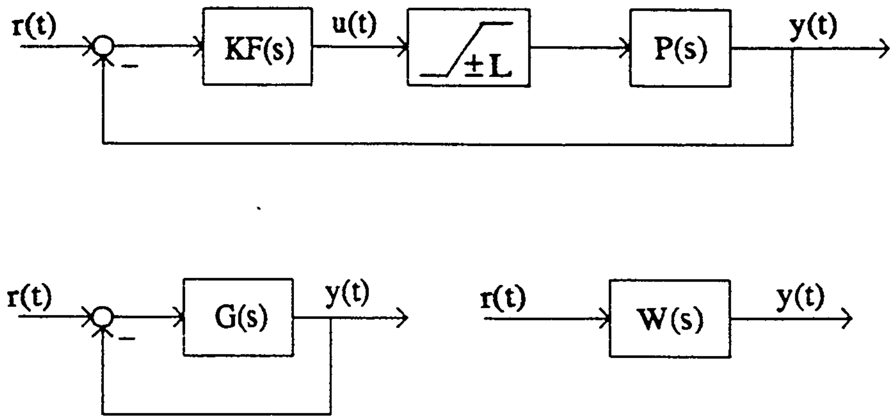


Fig. 3.1 The realistic series compensation problem considered. P(s) has PZE ≥ 3, and saturation constraint nonlinearity (±L) at its input.

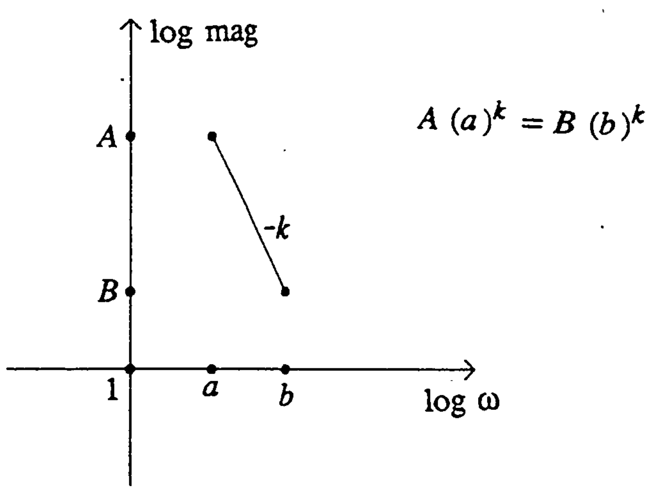
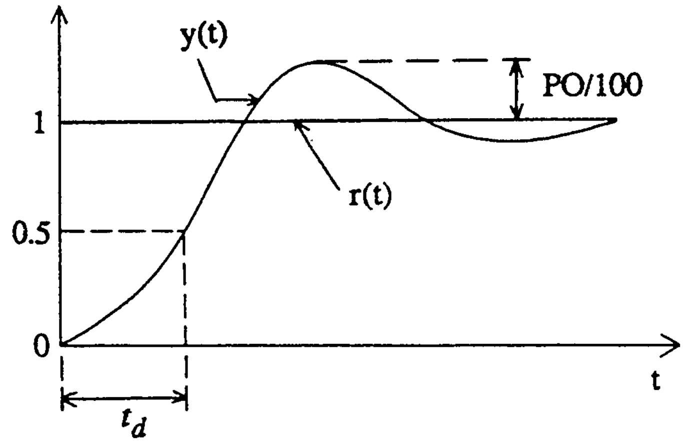
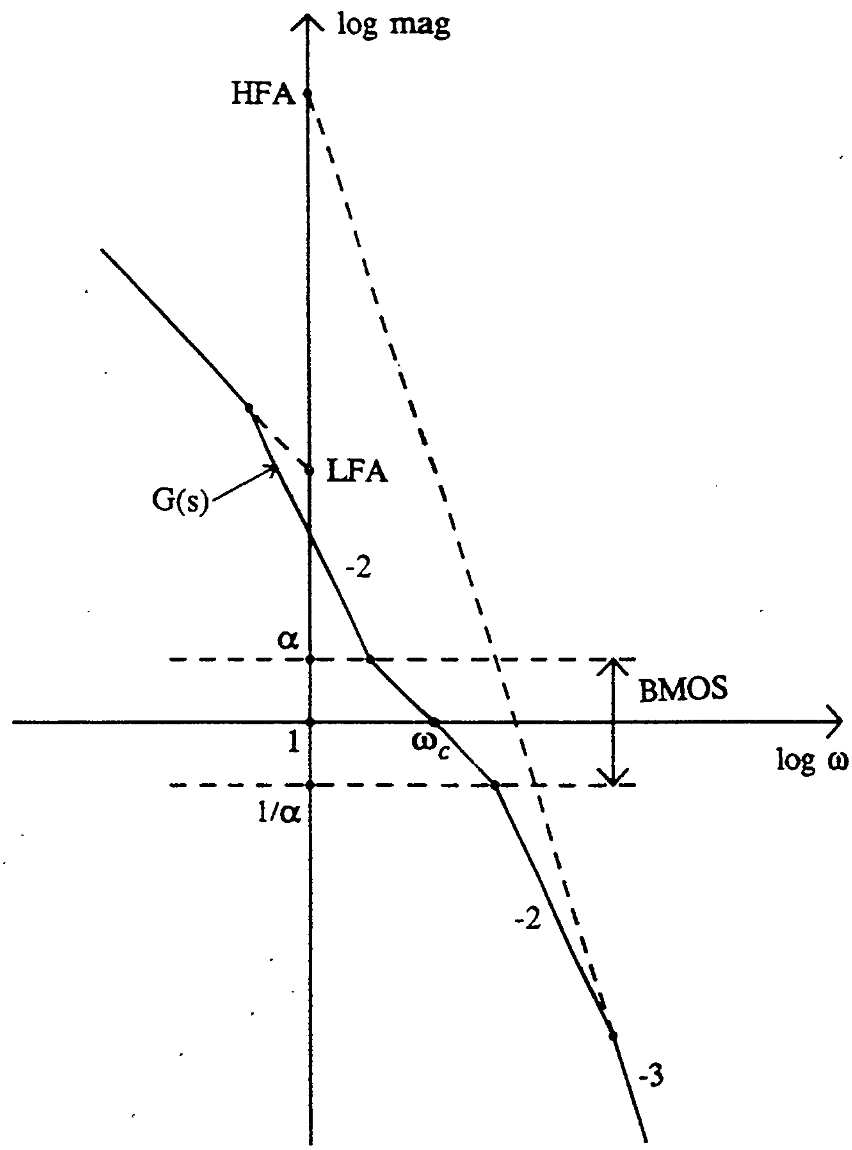


Fig. 3.2 The algebraic Bode plot relationship (ABPR) between any two points (a, A) and (b, B) which are connected by a slope of -k.

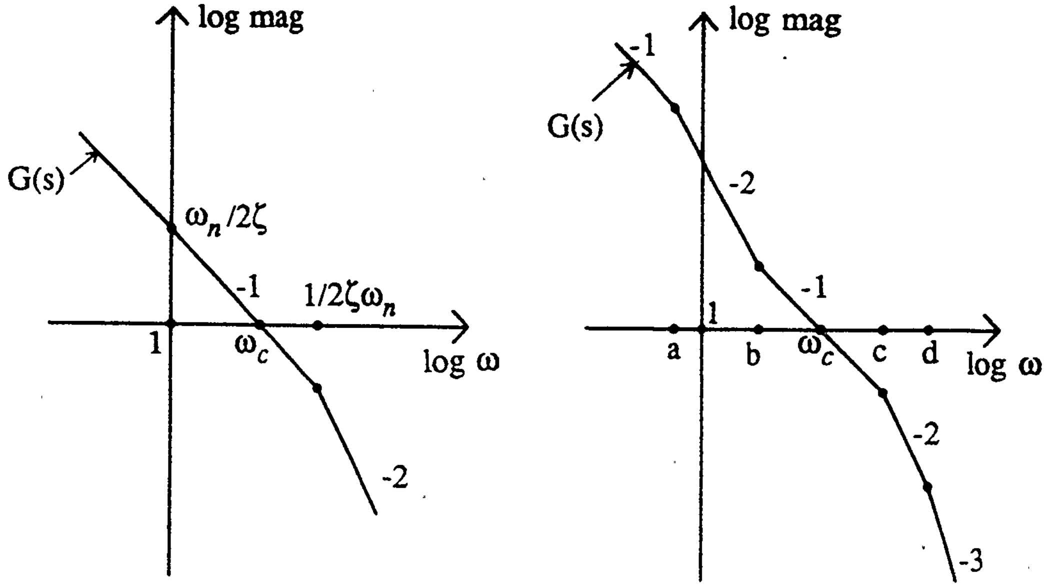


(a)



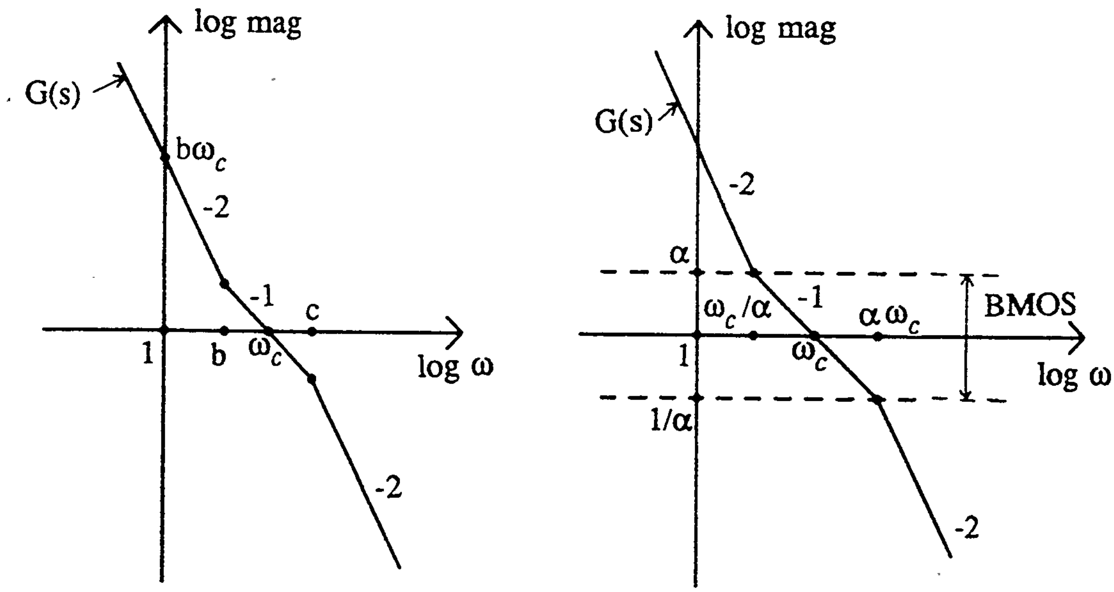
(b)

Fig. 3.3 (a) Basic specifications (specs). Time domain specs. PO = percent overshoot, a quality of response spec. t_d = time delay, a speed of response spec. (b) Frequency domain specs. LFA = low frequency asymptote; ω_c = crossover frequency; BMOS = the horizontal symmetrical (α, $1/\alpha$) band of a -1 slope at mag. = 1; and HFA = high frequency asymptote.



(a) (b)

Fig. 3.4 Systems used to relate time domain specs on y(t) to frequency domain specs on G(s). (a) Traditional. (b) This Chapter.



(a) (b)

Fig. 3.5 The maximize phase margin problem. (a) The problem. (b) The solution.

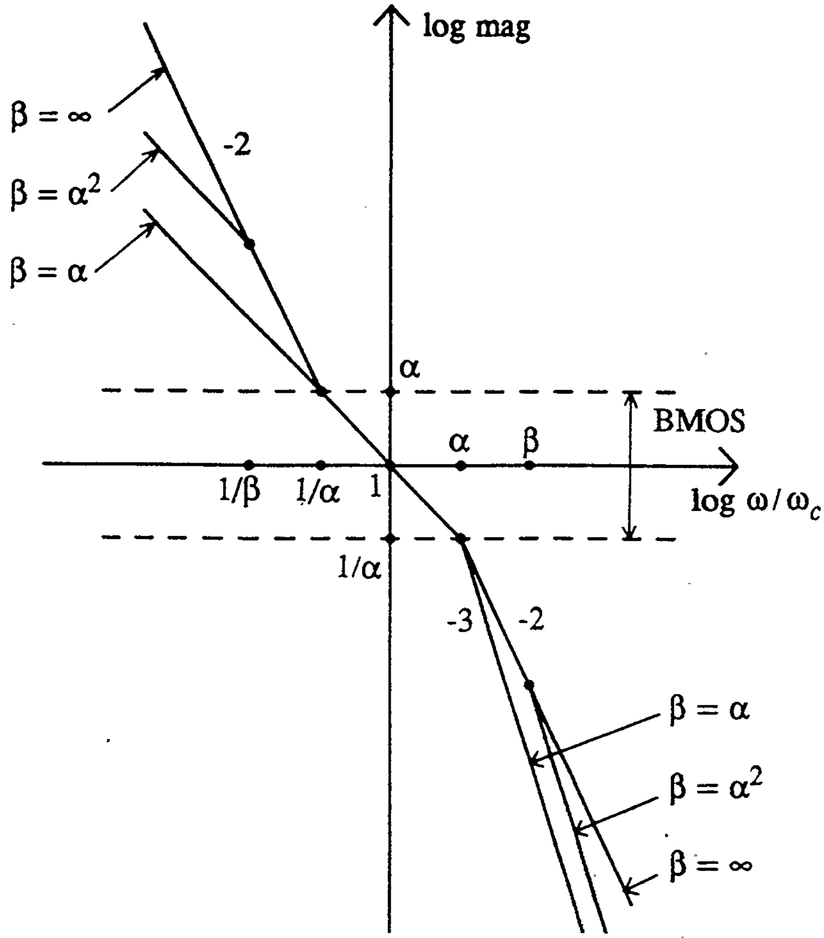


Fig. 3.6 The compensated systems which are used to relate time domain specs on y(t), to frequency domain specs on G(s).

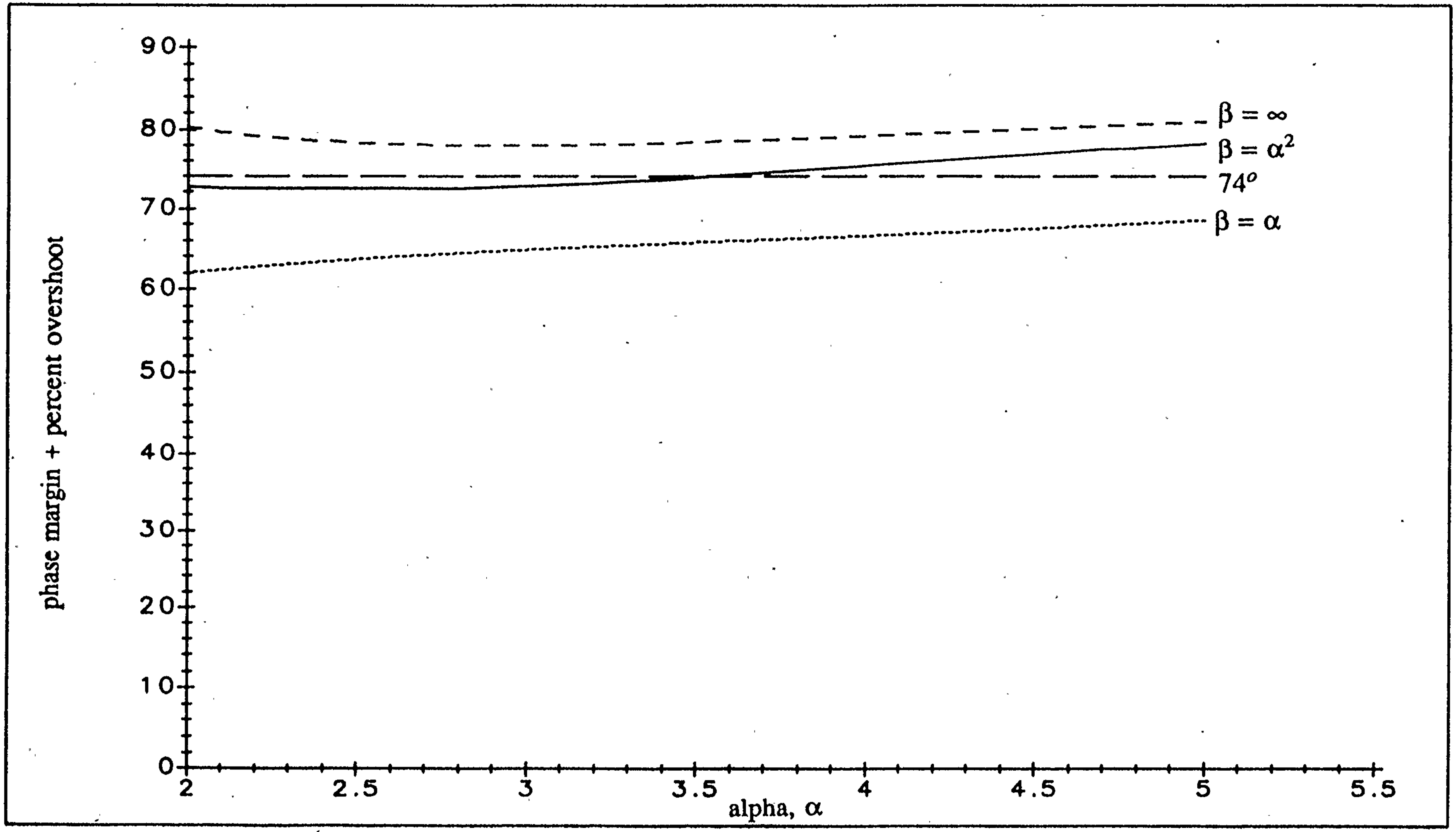


Fig. 3.7 MACSYMA results for phase margin plus percent overshoot as a function of alpha for $2 \leq \alpha \leq 5$. This shows that PO + PM $\approx 74^o$.

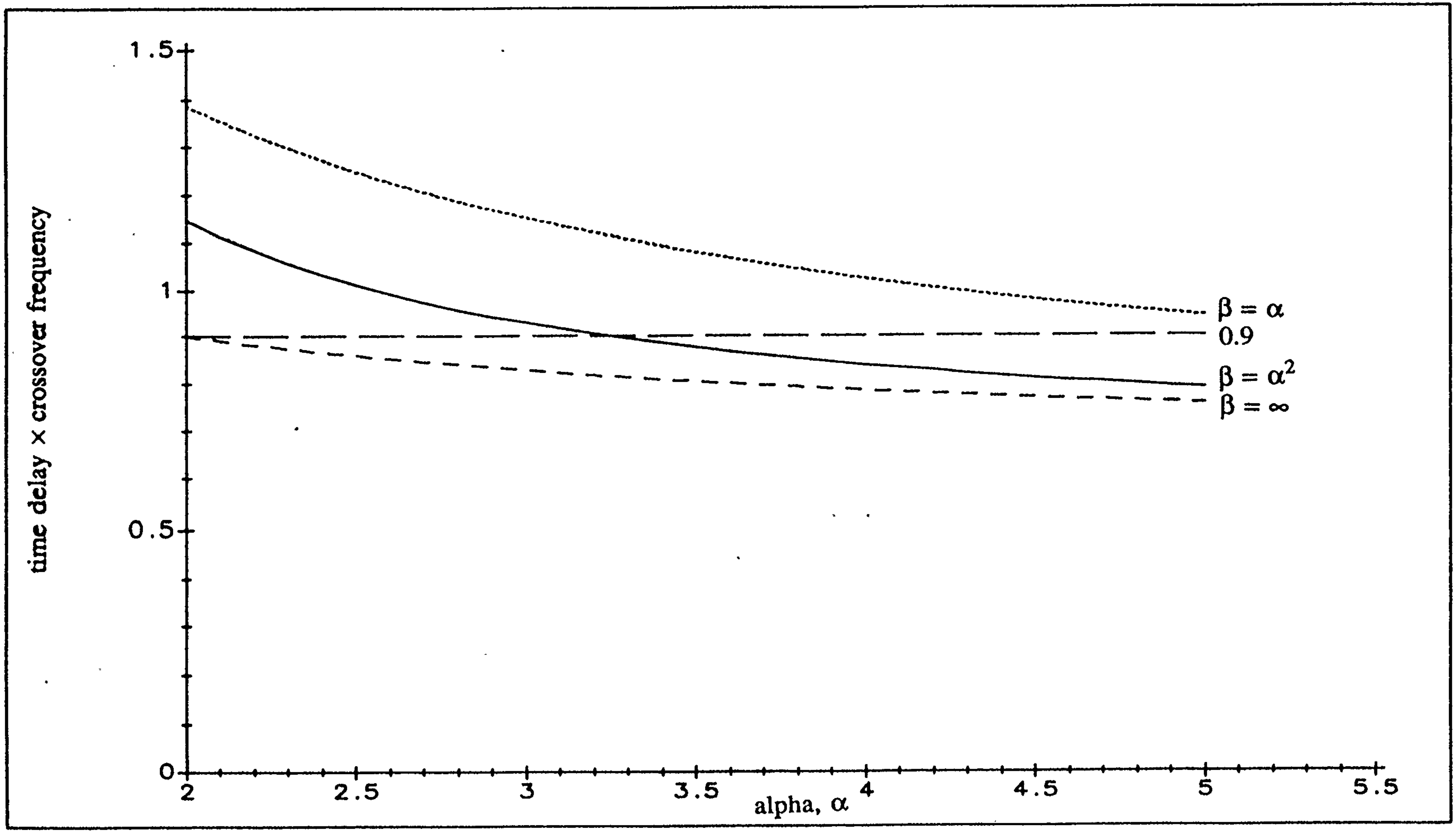


Fig. 3.8 MACSYMA results for time delay as a function of alpha for $2 \leq \alpha \leq 5$. This shows that $t_d \approx 0.9 / \omega_c$.

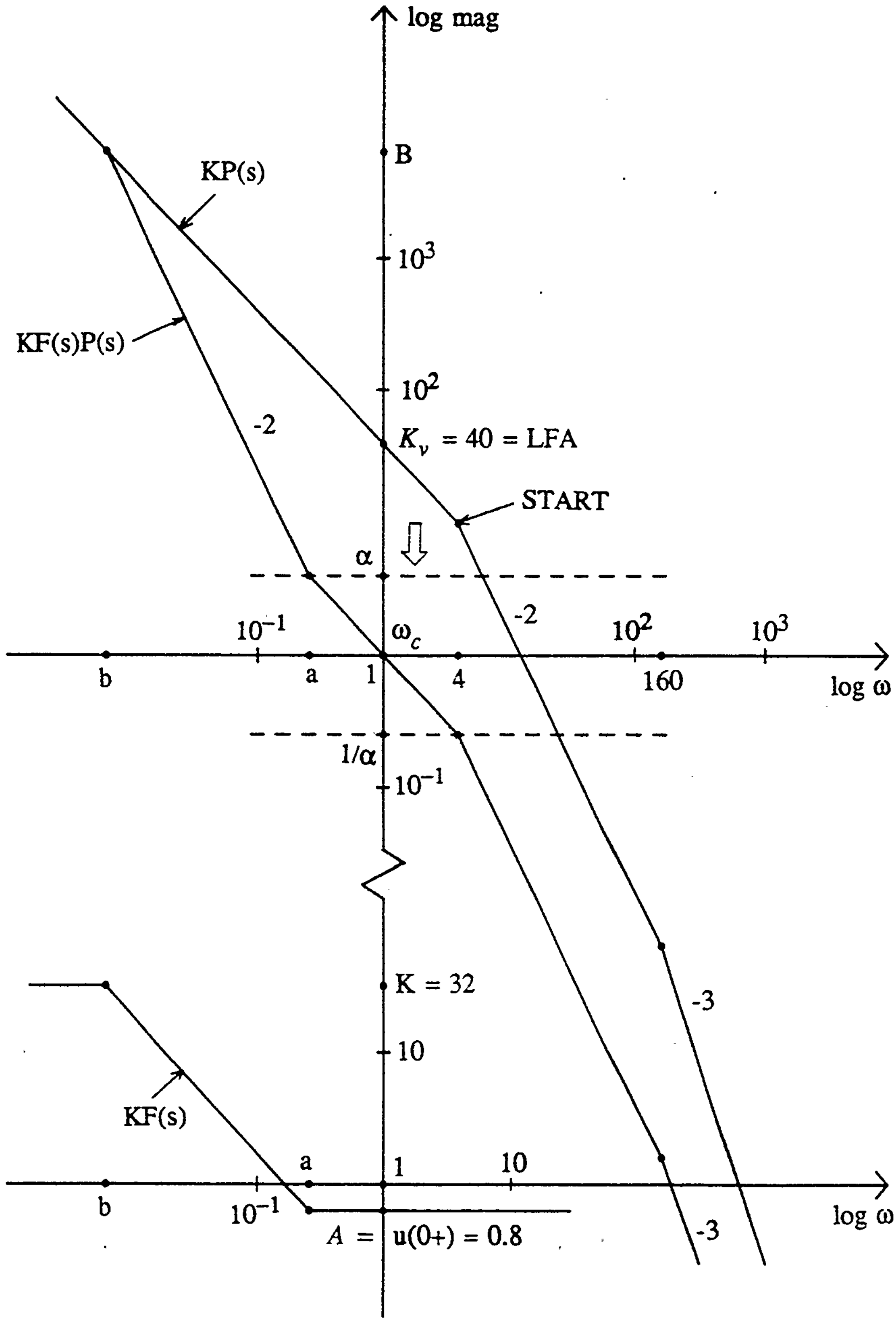


Fig. 3.9 Example 1, design procedure for a lag network.

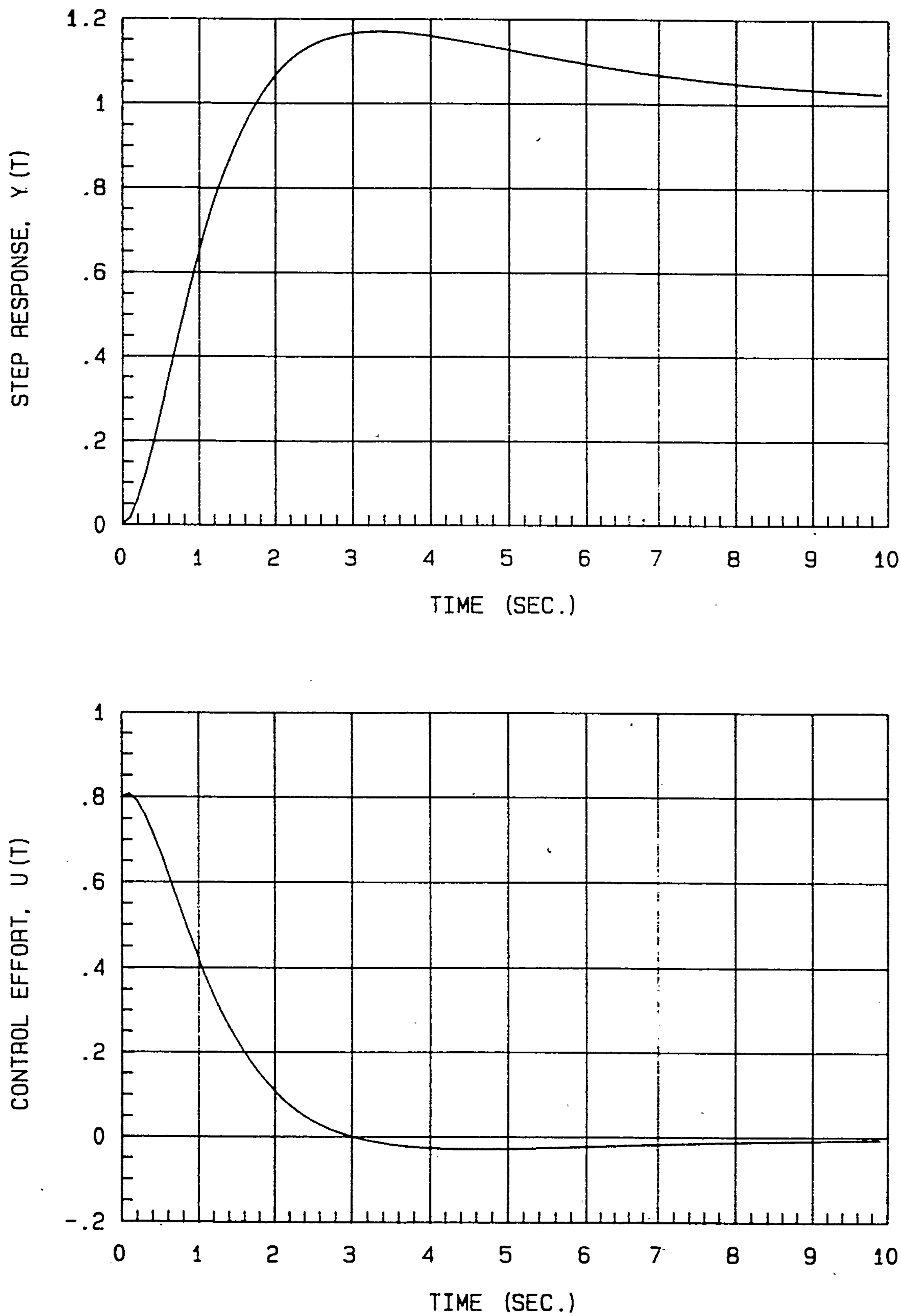


Fig. 3.10. Example 1, computer simulation of step response and control effort.

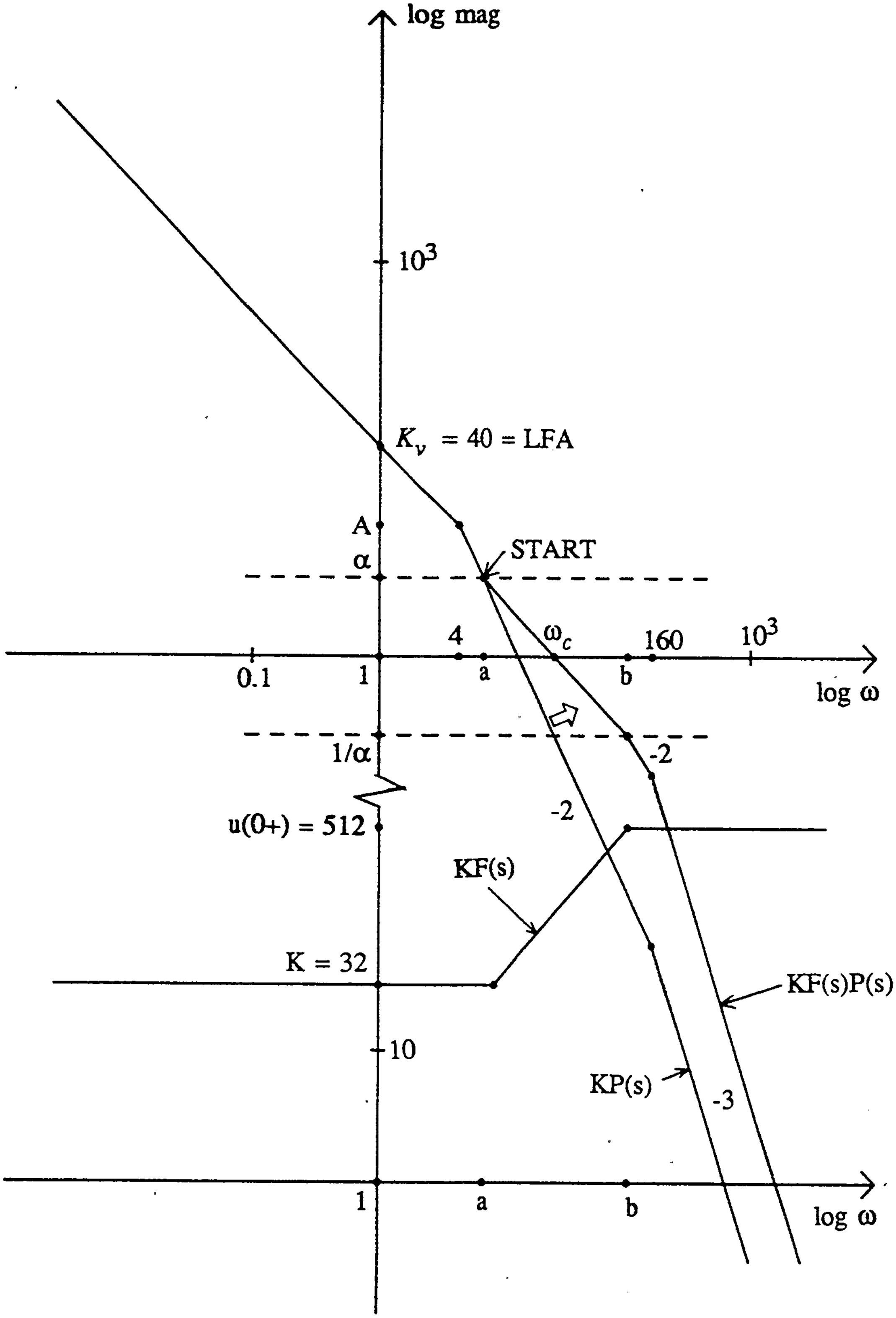


Fig. 3.11 Example 2, design procedure for a lead network.

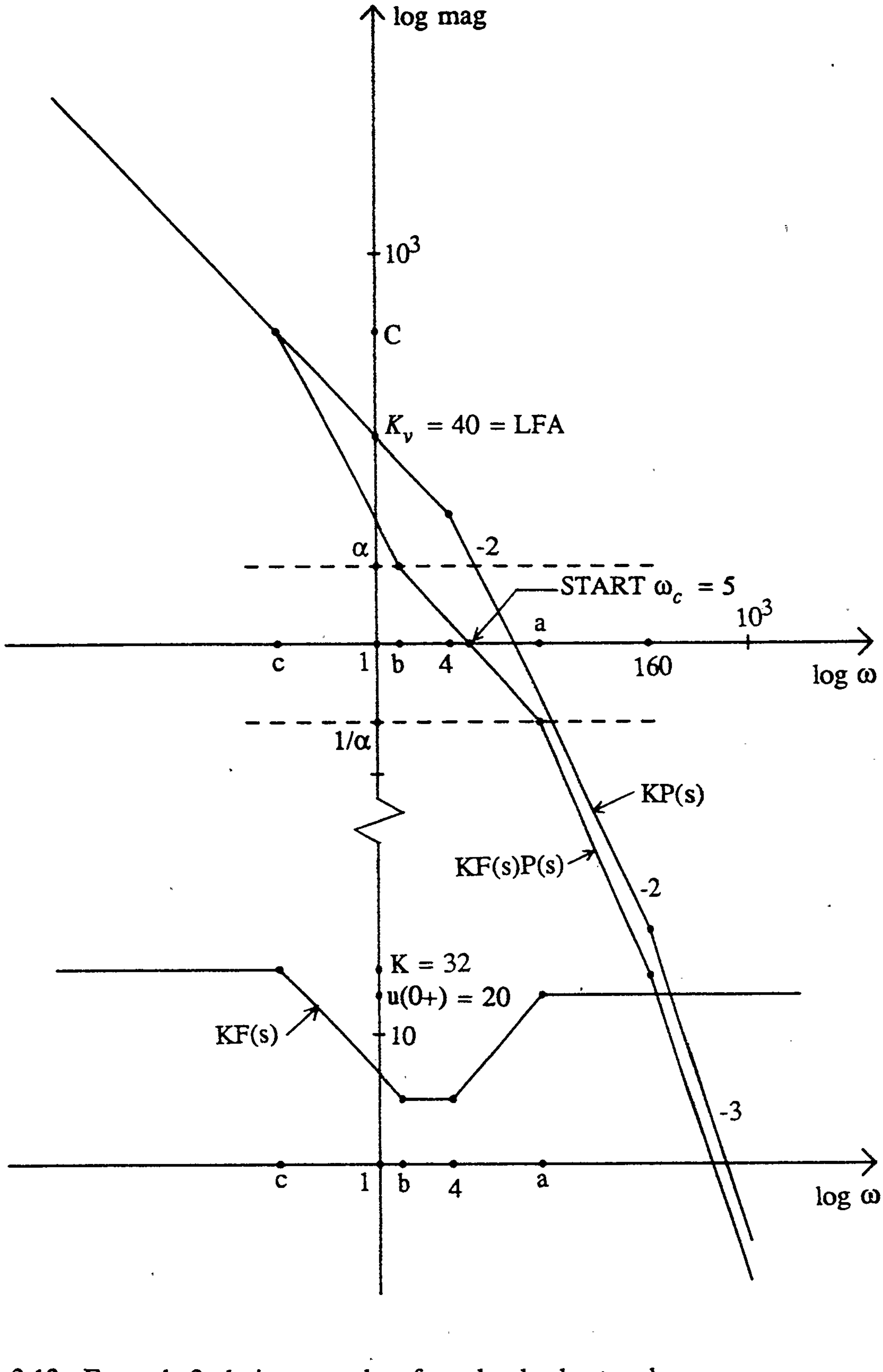


Fig. 3.12 Example 3, design procedure for a lag-lead network.

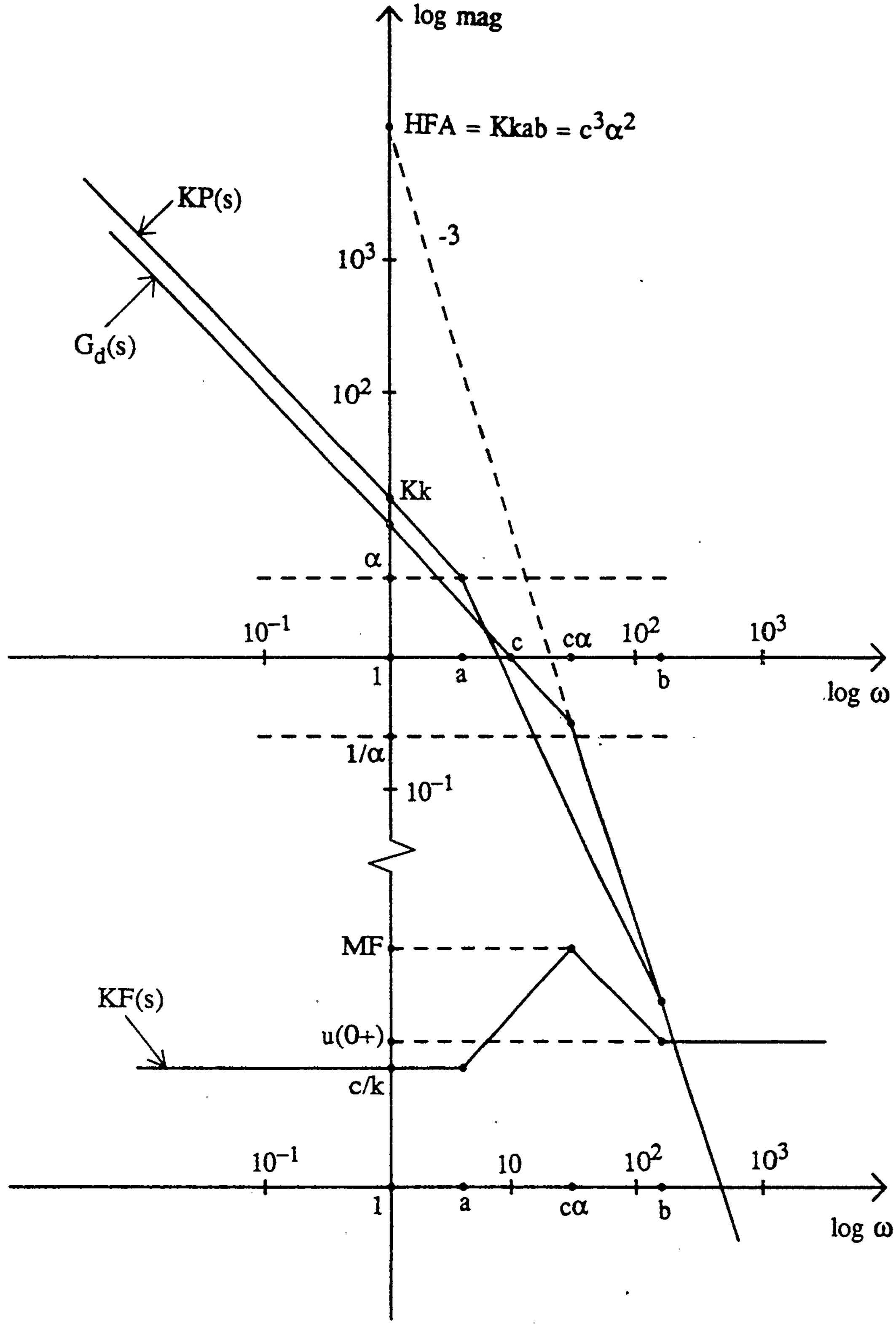


Fig. 3.13 Example 4, design procedure for a saturation constraint for a unit step.

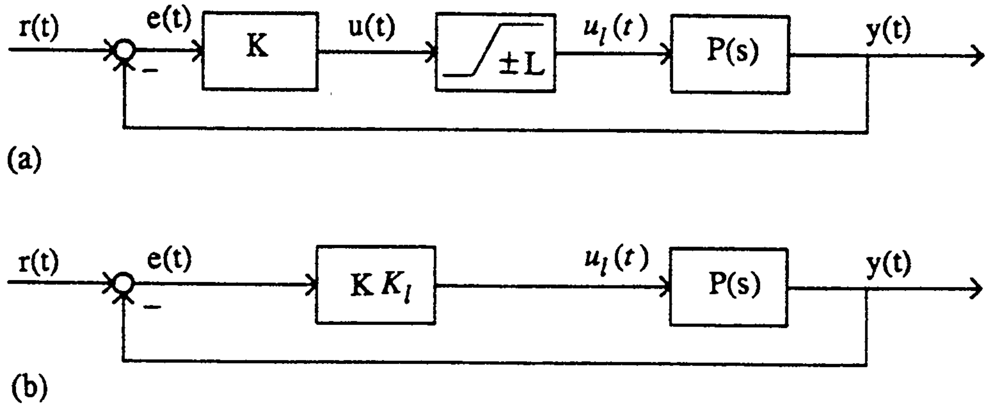


Fig. 3.14 Booton's analysis for a control system with a random input. (a) Actual nonlinear system. (b) Linearized model.

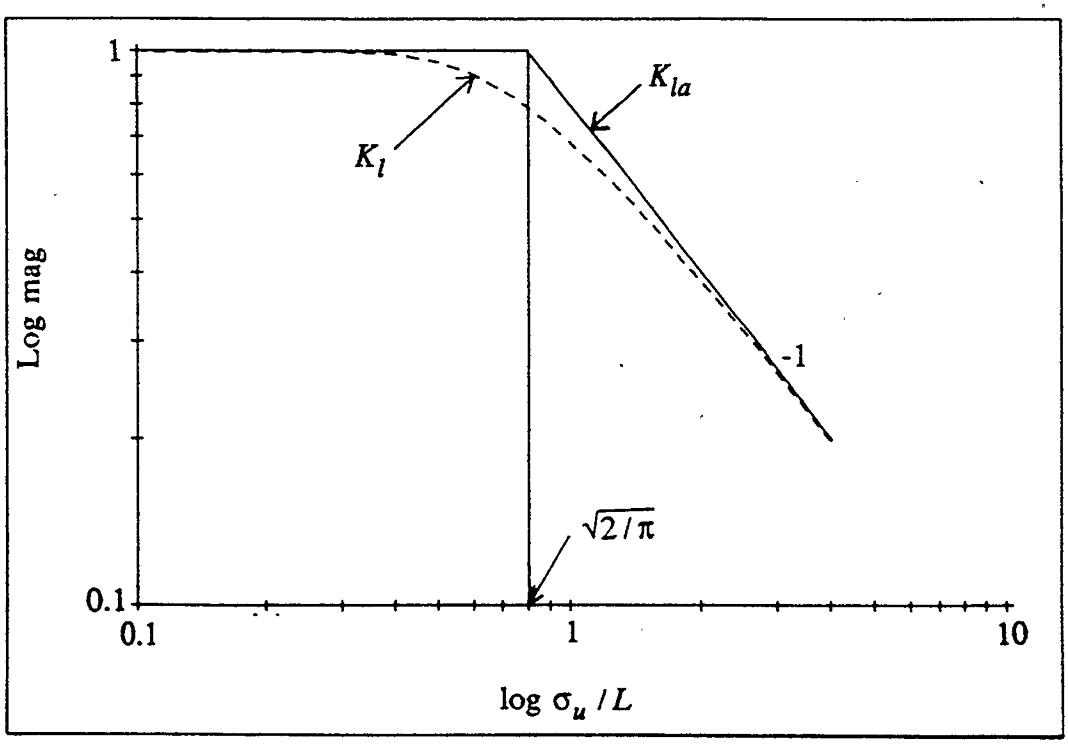


Fig. 3.15 Booton's equivalent gain K_l, and an approximation to it K_{la}.

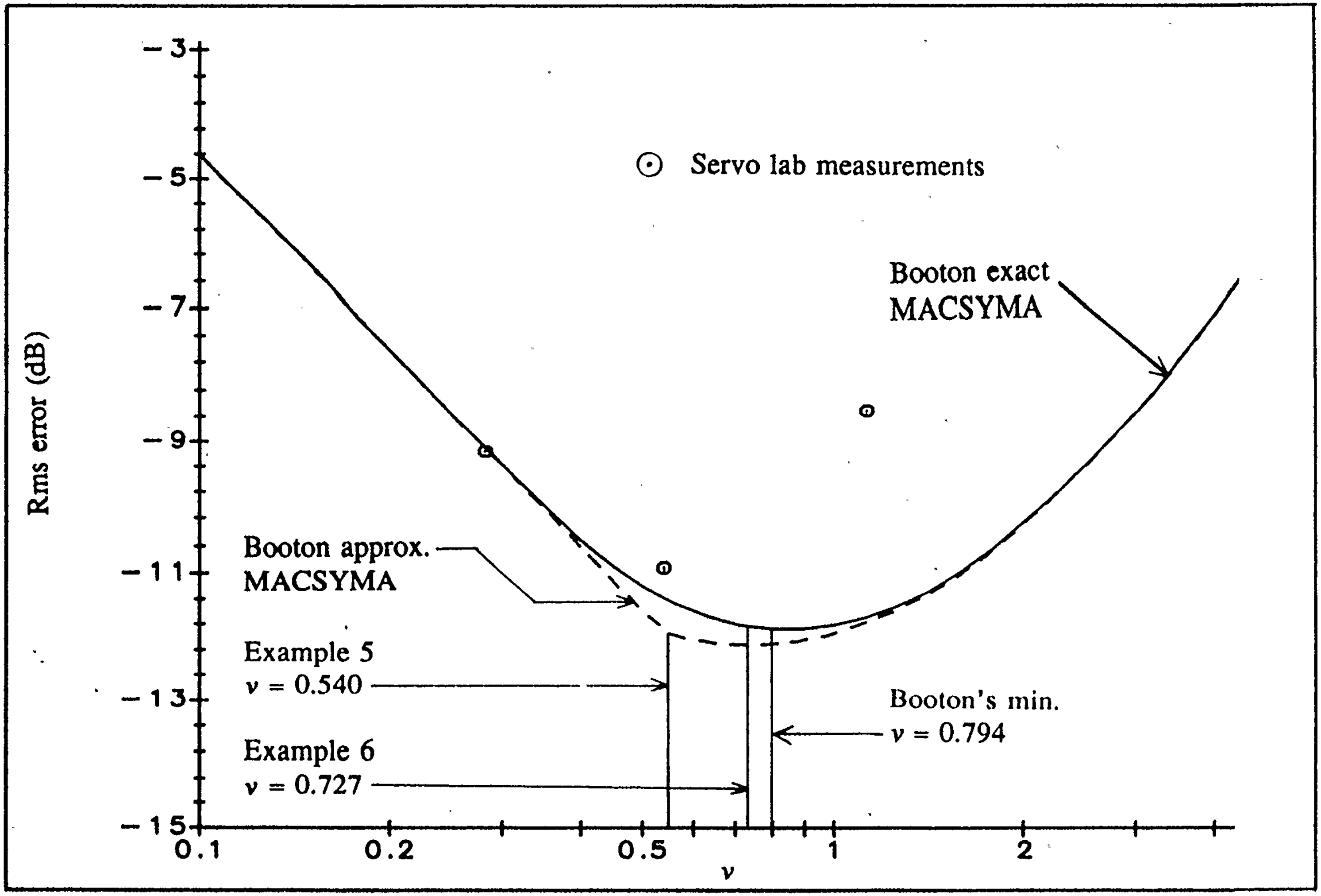


Fig. 3.16 Simulation results for Examples 5 and 6. Rms error versus the normalized frequency ν where $c = \nu w$.

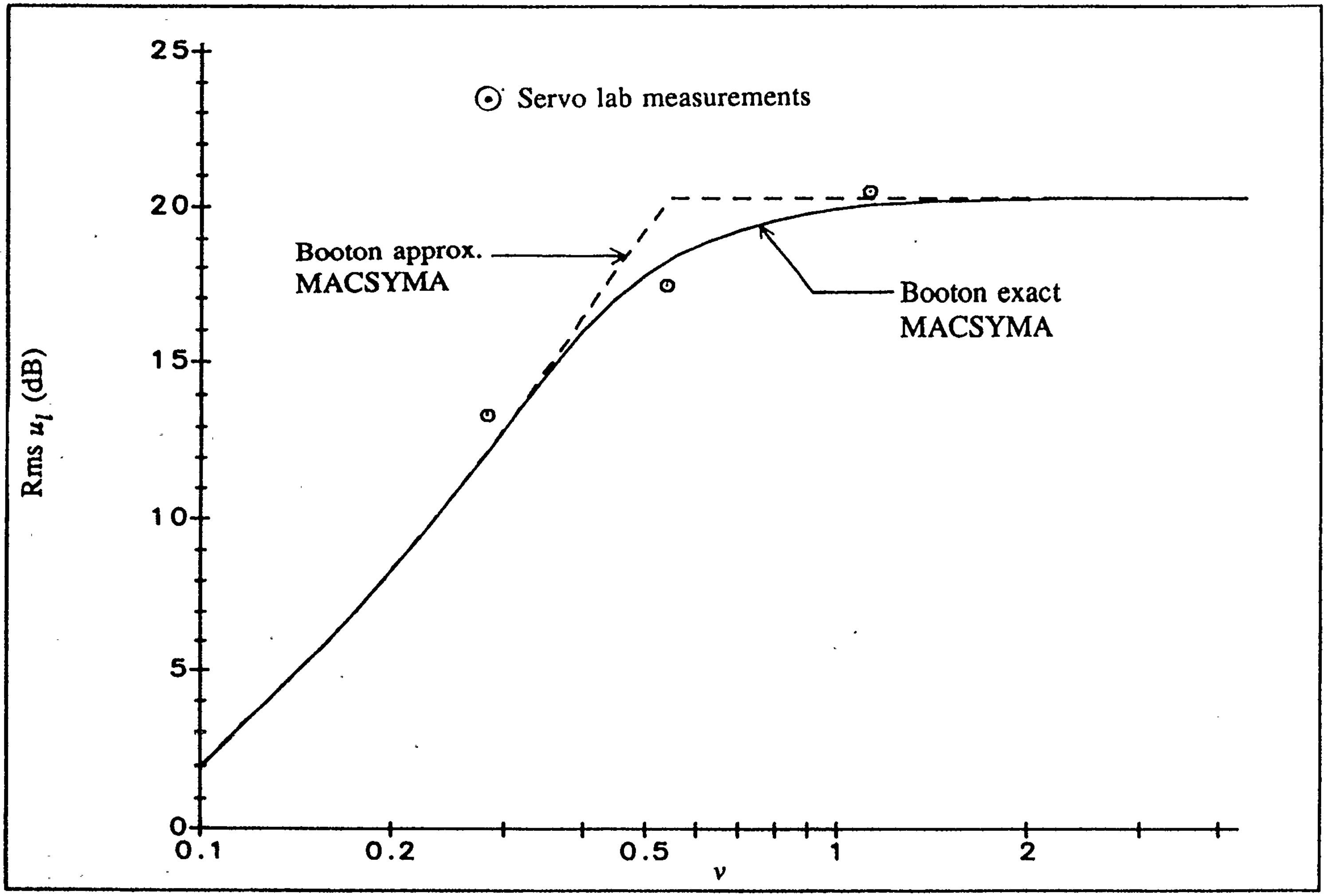


Fig. 3.17 Continuation of Fig. 3.16. Rms value of the limited input to the fixed plant σ_u versus normalized frequency ν.

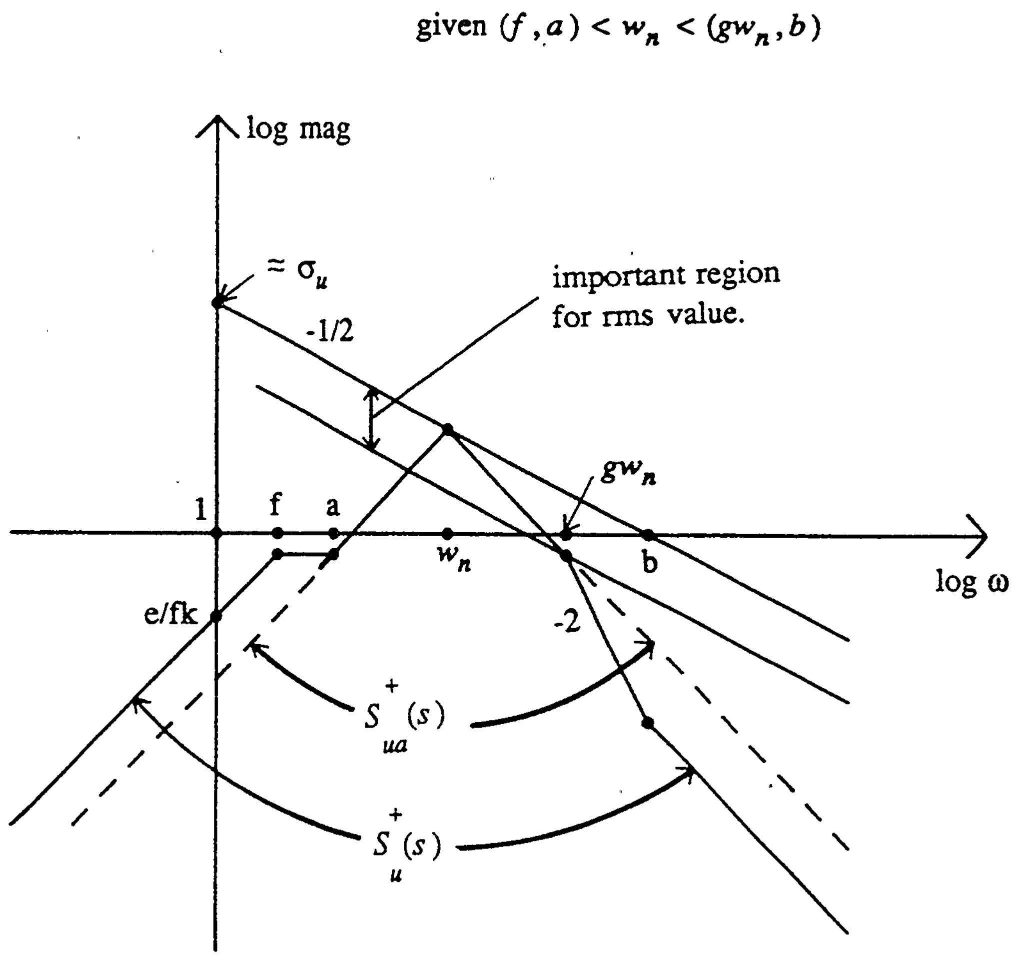


Fig. 3.18 Example 6, design procedure for the approximate evaluation of the mean-square value, σ_u^2.

CHAPTER 4

AN ALGEBRAIC SOLUTION FOR A LQ REGULATOR FOR A TYPICAL THIRD-ORDER MOTOR

4.1. Introduction

Algebraic solutions for linear-quadratic (LQ) or Kalman regulators are very rare. We consider the single-input single-output (SISO) LQ output regulator problem. By assuming one very specific but realistic fixed plant, we are able to obtain an algebraic solution for the LQ output regulator. This specific solution provides considerable insight into the general solution. The fixed plant is presented below.

A typical motor consists of an integrator and two real or complex poles

$$P(s) = \frac{kab}{s(s+a)(s+b)}, \text{ or}$$

$$= \frac{kc^2}{s[s^2 + 2\zeta cs + c^2]}, \quad \zeta \le 1 .$$

The transitional case ($a = b = c$, $\zeta = 1$) is

$$P(s) = \frac{kc^2}{s(s+c)^2} .$$

Notation is simplified by introducing normalized frequency, and ignoring k,

$$P(\lambda) = \frac{1}{\lambda(\lambda+1)^2}, \quad \lambda = s/c . \tag{4.1}$$

In state-variable form

$$\underline{\dot{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{b}\,u(t) \tag{4.2}$$

$$y(t) = \underline{c}^T \underline{x}(t) \ ,$$

this is,

$$\underline{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \underline{c}^T = [\ 1 \ \ 0 \ \ 0\ ] \ . \tag{4.3}$$

This is the specific realistic fixed plant which is assumed, and which enables us to obtain an algebraic solution.

A software program for "computer algebra" called MACSYMA [1] has been very beneficial as a computational aid in this thesis. A simplified design procedure for the LQ output regulator problem which has been presented by Schultz and Melsa [21] is also used.

The performance of the LQ output regulator (percent overshoot, time delay, and control effort) has been calculated numerically and is presented graphically. It is shown that the basic trade-off is that the control effort tends to increase as the cube of the improvement in the speed of response.

A new application for the LQ output regulator is presented, in which the LQ regulator is used as the desired system for traditional series compensation. We consider a control system with a stochastic input, and a saturation nonlinearity at the input to the fixed plant. Using Booton's [19] approximation for a saturation nonlinearity with a stochastic input, a simple approximate procedure is presented for designing the best compensation considering the saturation nonlinearity.

4.2. Review of LQ Regulator

Consider the SISO fixed plant (4.2), and the performance index to be minimized

$$\mathrm{PI} = \int_0^\infty [\underline{x}^T(t)\underline{Q}\,\underline{x}(t) + u(t)P\,u(t)]\,dt \tag{4.4}$$

where $\underline{Q}$ is symmetric non-negative definite and P is positive definite. The well known solution to this problem is as follows.

(1) Solve the algebraic Riccati equation [22] for symmetric matrices $\underline{R}$ which satisfy

$$\underline{A}^T\underline{R} + \underline{R}\,\underline{A} - \underline{R}\,\underline{b}\,P^{-1}\underline{b}^T\underline{R} + \underline{Q} = 0\,. \tag{4.5}$$

(2) Select the one matrix $\underline{R}$ which is positive definite.

(3) The optimal gain vector is

$$\underline{k} = \underline{R}\,\underline{b}\,P^{-1}\,, \tag{4.6}$$

and the optimal control is given by a linear constant feedback control law

$$\underline{u} = -\underline{k}^T\underline{x}(t)\,.$$

There are two problems with this solution. (1) The critically important problem of quadratic weight ($\underline{Q}$ and P) selection is invariably one of trial and error iteration [23]. (2) The solution of the algebraic Riccati equation is invariably numerical.

We consider the LQ output regulator problem where

$$\underline{Q} = \underline{c}\,\underline{c}^T\,,\ \text{and}\ P = 1/\rho^2 \tag{4.7}$$

where $1/\rho^2$ is a Lagrangian multiplier which determines the trade-off between control effort and speed of response. The performance index to be minimized is

$$PI = \int_0^{\infty} [y^2(t) + (1/\rho^2) u^2(t)]\, dt \;. \tag{4.8}$$

Schultz and Melsa [21] have presented a simple solution to this problem which is summarized as follows,

$$W(s) = \frac{y(s)}{r(s)} = \frac{\rho P(s)}{[\,1 + \rho^2 P(-s) P(s)\,]^+} \;, \tag{4.9}$$

where

$[\cdot]^+$ = the left half s-plane poles and zeros of $[\cdot]$ and the square root of the constant. This is known as spectrum factorization or the superscript plus operation.

4.3. Solution for LQ Regulator

Given the fixed plant (4.3), and letting $\rho = \alpha(\alpha^2 - 1)$, we have

$$\underline{Q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad P = \frac{1}{\alpha^2(\alpha^2 - 1)^2} \;. \tag{4.10}$$

The solution of (4.5) is given by

$$\underline{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{12} & r_{22} & r_{23} \\ r_{13} & r_{23} & r_{33} \end{bmatrix} , \tag{4.11}$$

where

$$r_{11} = (2\alpha^2 - 1) / [\alpha(\alpha^2 - 1)]$$

$$r_{12} = (2\alpha - 1) / [\alpha(\alpha^2 - 1)]$$

An analytical solution using Schultz and Melsa's solution (4.9), and the fixed plant (4.1) is

$$W(\lambda) = \frac{\rho P(\lambda)}{[\,1 + \rho^2 P(-\lambda)P(\lambda)\,]^+} ,$$

$$[\,1 + \rho^2 P(-\lambda)P(\lambda)\,]^+ = \frac{[\,-\lambda^6 + 2\lambda^4 - \lambda^2 + \rho^2\,]^+}{\lambda(\lambda+1)^2} .$$

The key step is to let $\rho = \alpha(\alpha^2 - 1) > 0$, where $\alpha \geq 1$.

$$[-\lambda^6 + 2\lambda^4 - \lambda^2 + \rho^2]^+ = [\,(-\lambda^2 + \alpha^2)\,(\lambda^4 + (\alpha^2 - 2)\lambda^2 + (\alpha^2 - 1)^2)\,]^+ \tag{4.14a}$$

$$= (\lambda + \alpha)[\,\lambda^2 + \alpha\lambda + (\alpha^2 - 1)\,] \tag{4.14b}$$

$$W(\lambda) = \frac{\alpha(\alpha^2 - 1)}{(\lambda + \alpha)\,[\lambda^2 + \alpha\lambda + (\alpha^2 - 1)]} . \tag{4.15}$$

Using Mason's rule, the closed-loop transfer function of the LQ regulator as shown in Fig. 4.1 is

$$W(\lambda) = \frac{K}{\lambda^3 + (2 + Kk_3)\lambda^2 + (1 + Kk_3 + Kk_2)\lambda + Kk_1} . \tag{4.16}$$

Equating coefficients of (4.15) to (4.16), we obtain (4.12b).

Understanding of the LQ regulator is aided by plotting its root locus, see Sec. 8.5 in [21]. The root locus relation, $1 + K\,N(s)/D(s) = 0$, can be written as

$$\mathrm{Im}\,\{D(s)N^*(s)\}\Big|_{\,s\,=\,\sigma + j\,\omega} = 0 \tag{4.17}$$

to obtain an analytical solution. For the fixed plant (4.1), the Bode plot of the LQ regulator is a straight line and a hyperbola,

$$r_{13} = 1/[\alpha(\alpha^2 - 1)]$$

$$r_{22} = (3\alpha - 1)/[\alpha(\alpha^2 - 1)(\alpha + 1)]$$

$$r_{23} = 2/[\alpha(\alpha^2 - 1)(\alpha + 1)]$$

$$r_{33} = 2/[\alpha^2(\alpha^2 - 1)(\alpha + 1)] \, .$$

The optimal gain vector for the regulator is

$$\underline{k}^T = \left[\alpha(\alpha^2 - 1), \quad \frac{2\alpha(\alpha^2 - 1)}{\alpha + 1}, \quad \frac{2(\alpha^2 - 1)}{\alpha + 1} \right] . \tag{4.12a}$$

For the control problem shown in Fig. 4.1 we have

$$K = \alpha(\alpha^2 - 1) \, , \quad \underline{k}^T = \left[1, \quad \frac{2}{\alpha + 1}, \quad \frac{2}{\alpha(\alpha + 1)} \right] . \tag{4.12b}$$

Using MACSYMA it is easy to verify: that (4.10) and (4.11) satisfy (4.5), that (4.11) is positive definite, and that (4.6) yields (4.12a).

The direct solution of (4.5) given (4.10) using "algsys" in MACSYMA fails. However, if we multiply $\underline{Q}$, P, and $\underline{R}$ by ρ, which does not change the problem, then MACSYMA yields (4.13), and (4.12a).

$$\rho \underline{R} = \begin{bmatrix} 2\alpha^2 - 1 & 2\alpha - 1 & 1 \\ 2\alpha - 1 & \dfrac{3\alpha - 1}{\alpha + 1} & \dfrac{2}{\alpha + 1} \\ 1 & \dfrac{2}{\alpha + 1} & \dfrac{2}{\alpha(\alpha + 1)} \end{bmatrix} . \tag{4.13}$$

Note that the denominator of (4.13) is much simplier than that of (4.11). Computer algebra occasionally requires a little help from the user.

$$\omega = 0\ ; \quad \frac{\sigma^2}{(c\,/\sqrt{3})^2} - \frac{\omega^2}{c^2} = 1\ . \tag{4.18}$$

We plot the root locus of $P(s)P(-s)$, for negative values of gain because the pole-zero-excess is odd, and we consider only the LHP loci, see Fig. 4.2. The root locus for the LQ regulator is easy to sketch for the general case where the motor has an integrator and two real or complex poles. However, a simple analytic expression exists only for the transitional case.

4.4. Performance of LQ Regulator

The desired closed-loop system $W_d(\lambda)$ (4.15) is given in term of $\alpha \geq 1$. This algebraic solution enables us to make some conclusions about reasonable values for α. A $W_d(\lambda)$ with three real poles is too slow, therefore $\alpha > 2/\sqrt{3} = 1.155$. A $W_d(\lambda)$ which has a bandwidth $BW = (\alpha^2 - 1)^{1/2} = 1$, which is the frequency where the phase of the fixed plant equals -180^o, is a high performance system. We will consider $\alpha = \sqrt{2} = 1.414$ as our center line design

$$W_d(\lambda) = \frac{\sqrt{2}}{(\lambda + \sqrt{2})\,[\lambda^2 + \sqrt{2}\lambda + 1\,]}\ . \tag{4.19}$$

For values $\alpha > \sqrt{2}$, the control effort problem must be examined carefully. The actual selection of α is most effectively accomplished by appropriate trade off curves which are given in Fig. 4.3 and discussed below.

To evaluate the system's performance for a deterministic input, we assume a unit step unit $r(s) = 1/s$; and we calculate: (1) the percent overshoot (PO), (2) the time delay (t_d) of time to achieve 50% of the final value, and (3) the control effort. For this problem shown in Fig. 4.1, the control effort is given by

$$\max.\ u(t) = u(0+) = K\ . \tag{4.20}$$

MACSYMA is used to calculate PO and t_d, by using the inverse Laplace transform and Newton's method for finding one real root of f(x) = 0. The results are presented in Fig. 4.3.

It is also useful to evaluate the systems performance for a stochastic input. To simplify the presentation, we assume that the stochastic input is white noise passed through an integrator, and that it has a factored power spectral density (PSD) of

$$S_{rr}^{+}(s) = \frac{e}{s}\ . \tag{4.21}$$

In this case the control effort is given by the rms value of the input to the fixed plant σ_u.

An introduction to the stochastic signals and the calculation of mean-square values has been given in Sec. 3.4.3.

We wish to calculate the mean-square value of the input to the fixed plant from

$$S_{uu}^{+}(\lambda) = \frac{W(\lambda)}{P(\lambda)} S_{rr}^{+}(\lambda)\ , \quad \lambda = s/c \tag{4.22}$$

$$= \alpha(\alpha^2 - 1)\, e/c\ \frac{1 + 2\lambda + \lambda^2}{\alpha(\alpha^2 - 1) + (2\alpha^2 - 1)\lambda + 2\alpha\lambda^2 + \lambda^3}\ .$$

Using standard tables, see Sec. 3.4.3, this is

$$\sigma_u^2 = \frac{e^2}{c}\ \frac{\alpha(\alpha^2 - 1)(2\alpha^4 - \alpha^2 + 1)}{2(3\alpha^2 - 1)}\ . \tag{4.23}$$

The rms value σ_u is shown in Fig. 4.3.

An asymptotic approximation for $\alpha >> 1$ for this problem is given by $P = 1/\lambda^3$, and $\rho = \alpha^3$. This solution is given by the dashed lines in Fig. 4.3.

Fig. 4.3 presents a practical engineering approach, trade-off curves for the important specifications, for the selection of the critical parameter α. To summarize, optimality guarantees stability and a good speed of response. The basic trade off is as follows. For a deterministic signal the control effort tends to increase as the cube of an improvement in the speed of response. For the more probable stochastic input the control effort tends to increase as the 2.5 power of the improvement in the speed of response. This is a drastic trade off.

4.5. An Application

Physical control systems frequently have a stochastic input r(t), and a saturation nonlinearity at the input to the fixed plant $P(s)$. We want to design a control system with the highest possible performance under these conditions. That is, we want to minimize the rms system error to a stochastic input considering the saturation nonlinearity.

In Sec. 4.4 a linear fixed plant was assumed and the rms value of the input to the fixed plant as function of alpha, $\sigma_u(\alpha)$ was calculated. However, the selection of the best value for σ_u or α would involve a trial and error procedure using a digital computer simulation of a continuous system with a stochastic input and a saturation nonlinearity. A simple specific design procedure for this problem would be valuable even if it was approximate.

LQ regulators are optimal for the deterministic signals and they treat the trade off between speed of response and control effort. A new application of LQ regulator is as the desired system for series or state-variable compensation where the system has a stochastic input and a saturation nonlinearity.

A saturation nonlinearity is a difficult problem to analyze. Booton [19] has presented an approximation for a saturation nonlinearity with a stochastic input, which has been presented in Sec. 3.4.3. A simplified approximate design procedure using Booton's approximation has also been presented there, as is used below.

We set the mean-square value of the input to the linear fixed plant (4.23) equal to Booton's value (3.24)

$$\sigma_u^2 = \frac{e^2}{c}\,\frac{\alpha(\alpha^2-1)(2\alpha^4-\alpha^2+1)}{2(3\alpha^2-1)} = \frac{2}{\pi}L^2 . \tag{4.24}$$

For a simple example, we choose $e = 1$, $c = 1$ and $L = 5$. Therefore we have to factor the 7^{th} order polynomial

$$2\pi\alpha^7 - 3\pi\alpha^5 + 2\pi\alpha^3 - 300\alpha^2 - \pi\alpha + 100 = 0 \tag{4.25}$$

looking for one real root $\alpha > 1$. The result is $\alpha = 2.27$.

The optimal solution which considers the saturation nonlinearity is $\alpha = 2.27$. The desired open-loop system, calculated from (4.15), is

$$G_d(\lambda) = \frac{W_d(\lambda)}{1-W_d(\lambda)} = \frac{\alpha(\alpha^2-1)}{\lambda[\lambda^2+2\alpha\lambda+(2\alpha^2-1)]}\,, \quad \lambda = s/c\ . \tag{4.26}$$

The series compensation is

$$KF(\lambda) = \frac{G(\lambda)}{P(\lambda)} = \alpha(\alpha^2 - 1)\frac{(\lambda + 1)^2}{\lambda^2 + 2\alpha\lambda + (2\alpha^2 - 1)} . \tag{4.27}$$

A Bode plot of $KF(\lambda)$ is shown in Fig. 4.6, which uses the ABPR presented in Chap. 3. The state-variable compensation is given by Fig. 4.1 and (4.12b).

For the example considered in this section we present in Fig. 4.4 the rms value of the system error: (1) as calculated by MACSYMA using Booton's approximation (3.22); (2) as calculated by MACSYMA using our approximation (3.23); and (3) as calculated by a continuous system simulation software package called Enhanced Desire [24]. Fig. 4.4 demonstrates that: (1) Booton's approximation (3.22) is valid, and yields a broad minimum; and (2) that our design procedure yields excellent although slightly conservative results.

4.6. Conclusions

A very specific but realistic fixed plant has been assumed. An algebraic solution has been obtained for the structure of the LQ output regulator. Numerical analysis yields a complete set of trade off curves for the LQ output regulator's performance. This total solution for a very specific problem yields considerable insight into the general LQ output regulator problem.

A new application of LQ output regulators is presented, which is as the desired system for series or state-variable compensation where the system has a stochastic input and a saturation nonlinearity. A simple specific design procedure is presented which is based on Booton's approximation for a saturation nonlinearity. Simulation verifies this procedure is valid.

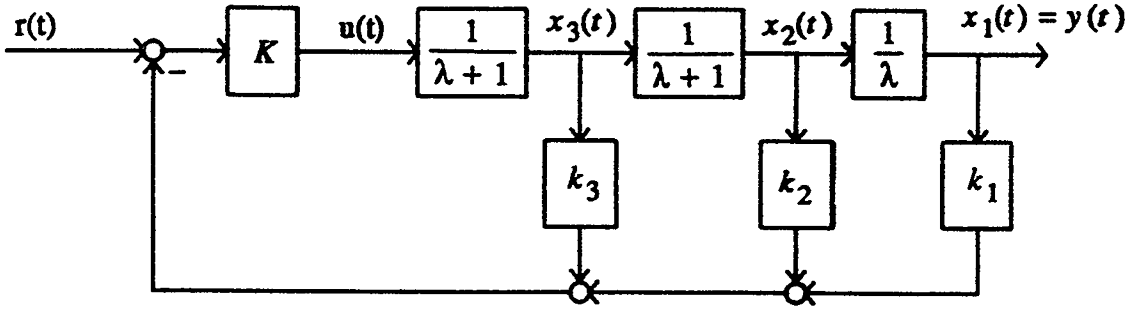


Fig. 4.1 The LQ regulator

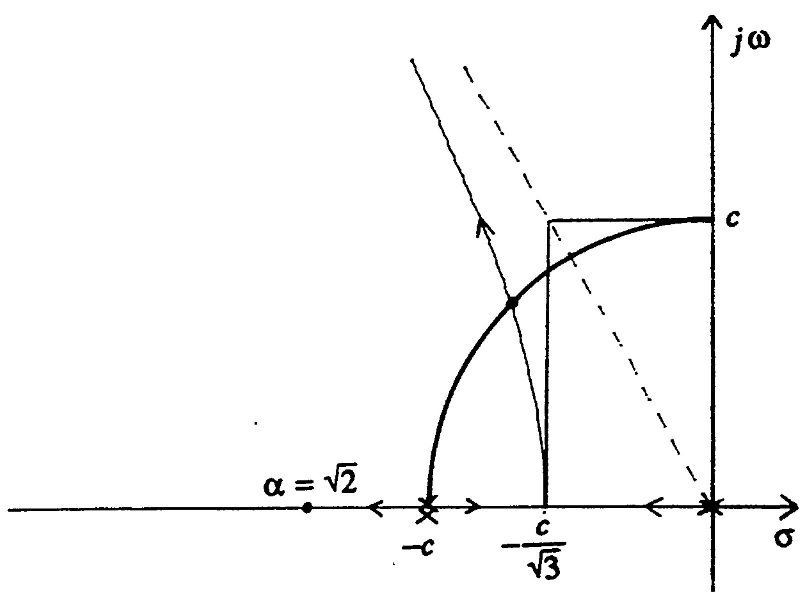


Fig. 4.2 Root locus of the LQ regulator

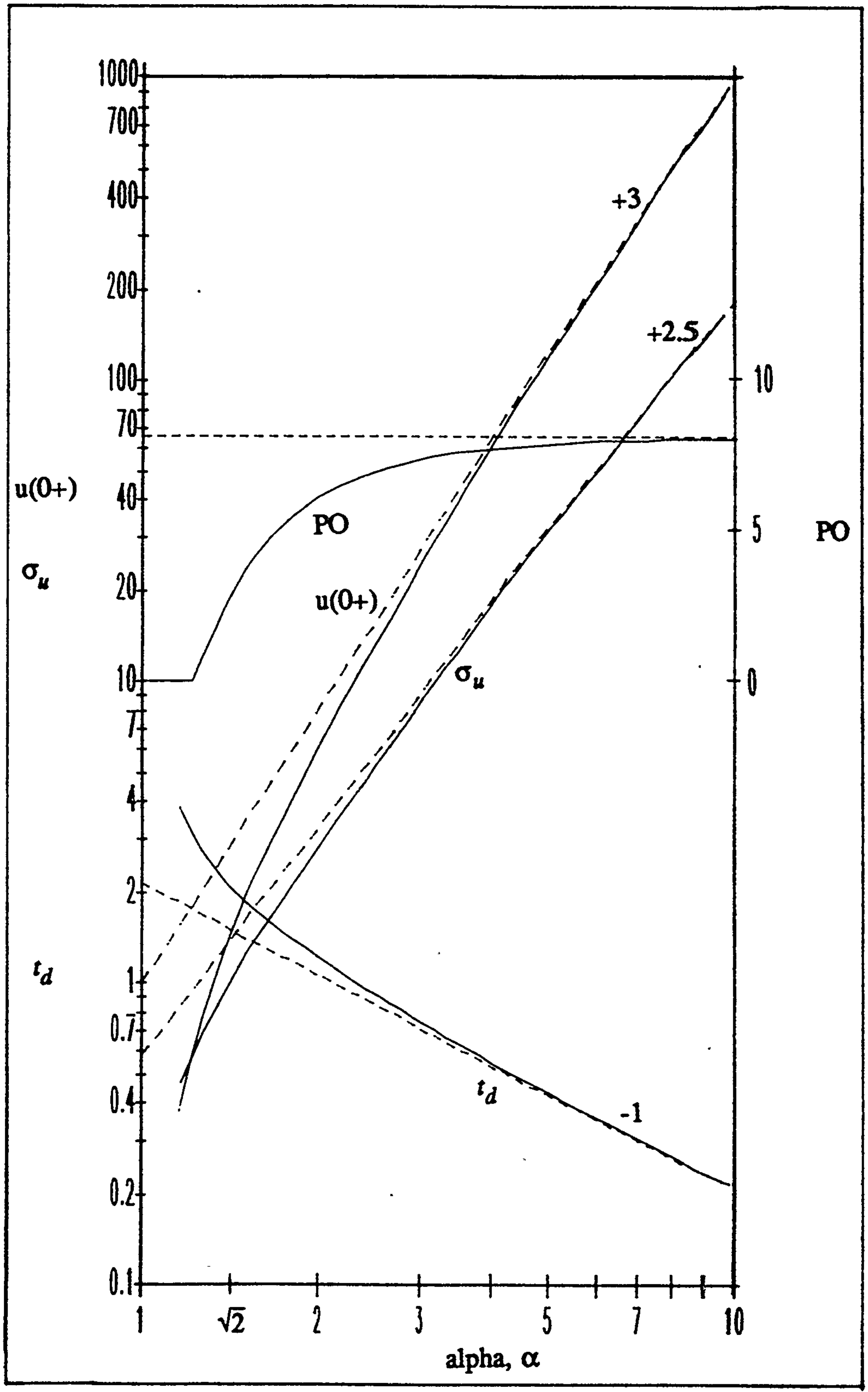


Fig. 4.3 Complete set of trade off curves for the LQ regulator.

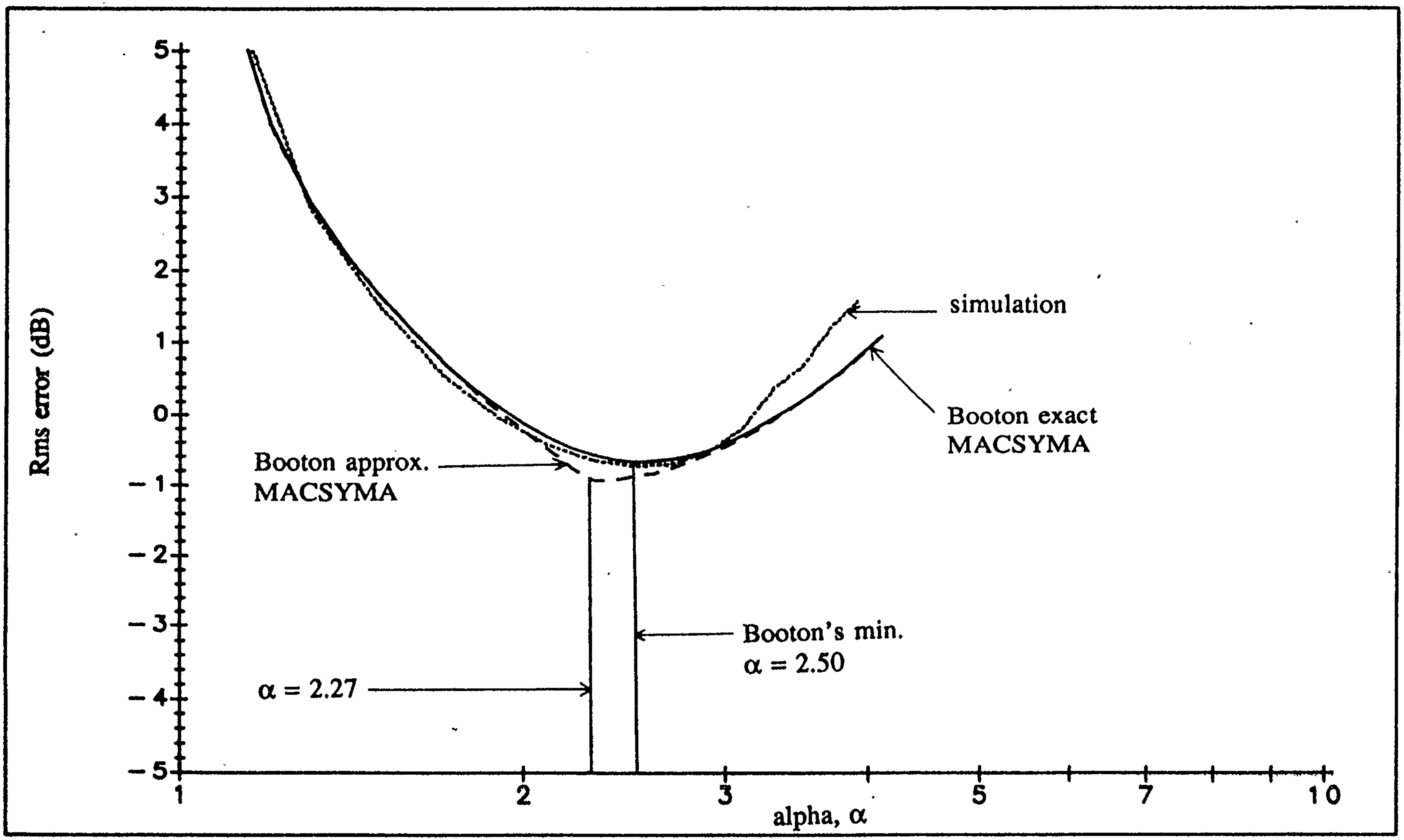


Fig. 4.4 Rms error of the LQ regulator with the saturation nonlinearity for a stochastic input.

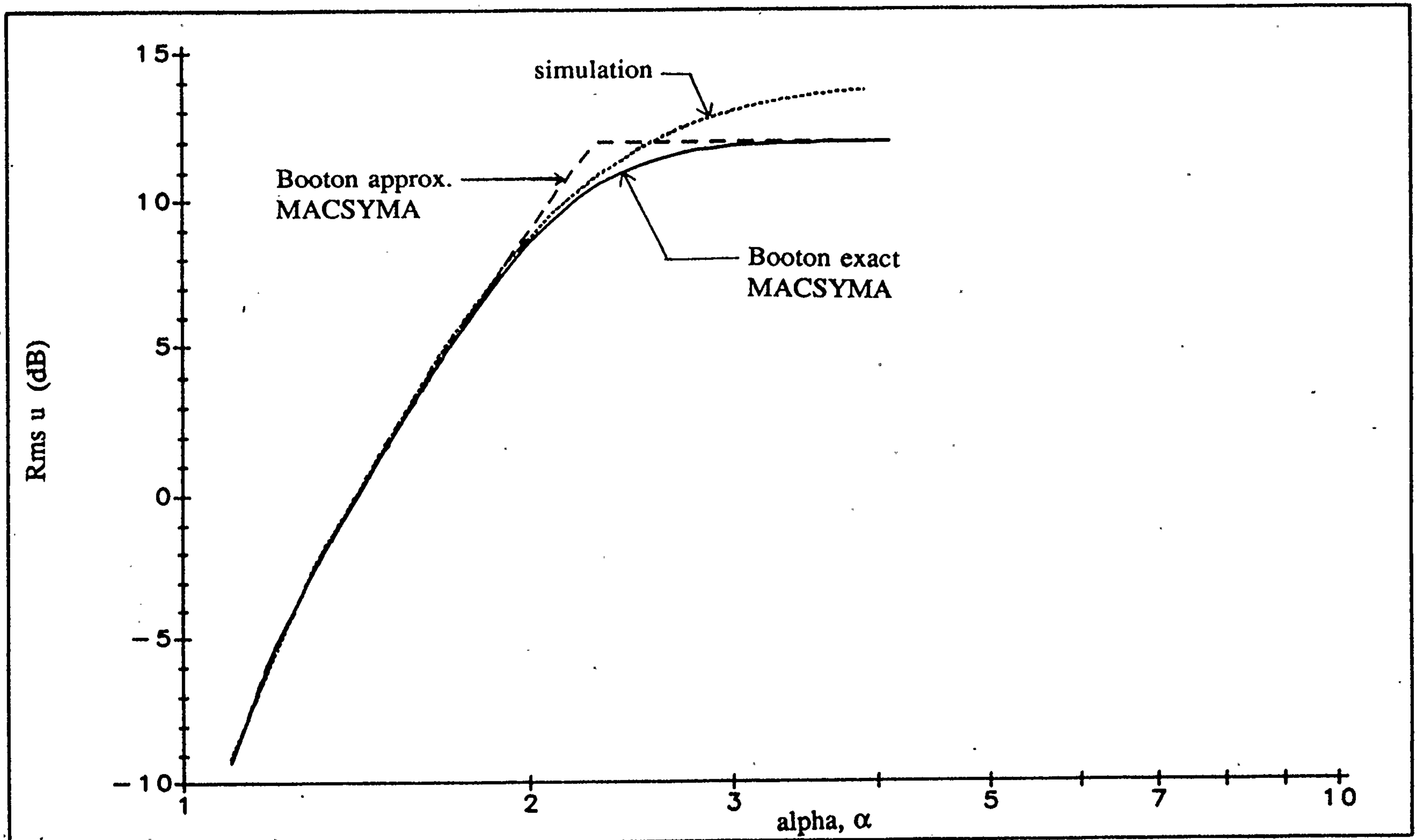


Fig. 4.5 Rms control effort of the LQ regulator with the saturation nonlinearity for a stochastic input.

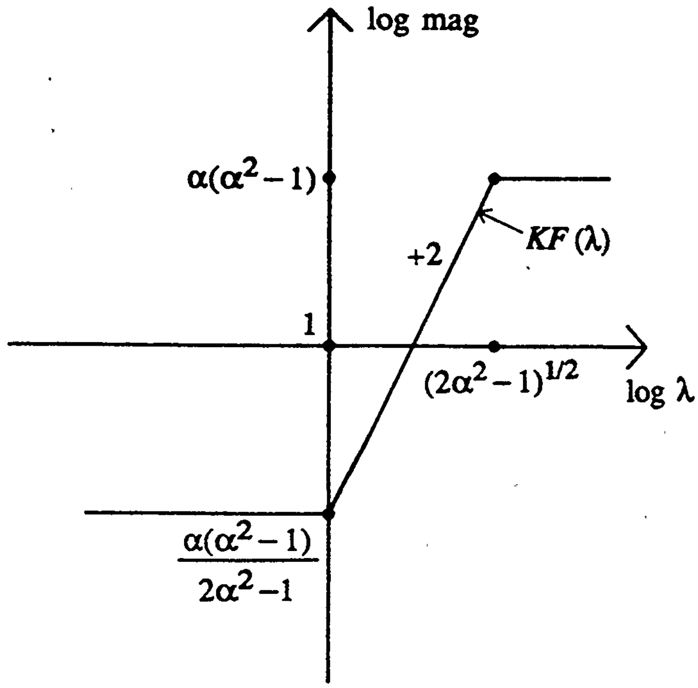


Fig. 4.6 Bode plot of the series compensation $KF(\lambda)$, considering the saturation nonlinearity and which uses the ABPR.

CHAPTER 5

CONCLUSIONS

5.1. Conclusions

We have presented a major simplification in design techniques for the traditional approach to series compensation. Series compensation is the most interesting part of a traditional first undergraduate course in control. We have presented a specific, very fast, easily understood Bode plot technique for traditional series compensation, which makes the presentation of this subject in current textbooks obsolete. We have presented a solution for the real problem of compensation, which is the trade off between speed of response and control effort. Our approach is based on: Booton's linearized gain for a saturation nonlinearity with a random input; and Streets' technique for approximate evaluation of mean-square values. We have presented a specific, very fast, easily understood Bode plot technique for the solution of this problem. Computer simulations verify our approach. Because optimal control problems typically have a broad minimum, our approximations which tend to be conservative appear to be very good.

The only major limitation of this work is the single-input single-output (SISO) assumption. However, a thorough understanding of the SISO problem is a significant asset when working with multiple-input multiple-output (MIMO) problems. The assumption that the fixed plant is minimum phase was made to simplify the presentation. The use of both Bode magnitude and phase plots handles RHP poles and zeros, and a pure time delay can be handled by a 2-pole 2-zero Padé approximation.

The assumption that the saturation nonlinearity is at the input to the fixed plant was made to simplify the presentation. If the saturation nonlinearity is on an internal variable of the fixed plant, redefining the fixed plant and compensation will solve this problem. The assumption that random processes we consider are stationary from minus infinity to infinity is inherent in our use of random data theory and of modern signal processing computer systems. This assumption is sometimes criticized as being unrealistic. However, one 10 minute portion of an undergraduate DSP lab. clearly demonstrates that the system identification of drastically changing system can be accomplished in two record lengths, and that this criticism is totally invalid.

5.2. Suggestions for Further Research

In extending the results of this research, it is a good idea to program our series compensation design technique on a popular personal computer using existed software simulation packages, e.g. an IBM PC/AT and MatrixX/PC. This program should first allow an user to define the fixed plant, the saturation nonlinearity ($\pm L$) at the input of the fixed plant, and the system input PSD. It then solves the series compensation problem for the given fixed plant considering the saturation nonlinearity. That is, it finds the parameters of the system when the rms value of input to the fixed plant equals Booton's value assuming that the system is linear. The algorithm is as follows. (1) Define the fixed plant. (2) Estimate the unknown parameter, c, of the desired compensated system (3.16) through iterations. A good first guess for c is (3.28). (3) The last step is to analyze the performance of the compensated system for the user specified input PSD, and for a deterministic step input. This program will definitely enable an user, who might not be very familiar with our series compensation design technique, to solve the real problem of compensation which is the trade off between speed of response and control effort very easily.

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