## UNIVERSITY OF CALGARY

## Analytic Perturbation Theory for Matrix Functions

## By

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## Abstract

In this thesis we investigate perturbation problems for an analytic matrix function in two-parameter form. By considering one parameter as perturbation parameter and the other one as.the eigenvalue parameter, the perturbation issue is framed in the study of a matrix function depending analytically on two complex parameters. Two methods are used. One is the Newton's diagram method, and the other is the generating eigenvector method. Applications are made to classical eigenvalue problems and to gyroscopic systems.

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# List of Symbols 

| $\diamond$ |  |
| :---: | :---: |
| := |  |
| $\equiv$ |  |
| $\mathbb{C}$ |  |
| $\operatorname{det} A$ |  |
| $\operatorname{ker} A$ |  |
| $A^{*}$ |  |
| $\sigma(M)$ |  |
| $r(M)$ |  |
| $\\|M\\|$ |  |
| $\operatorname{In}(M)$ |  |
| $\pi(M)$ |  |
| $\nu(M)$ |  |
| $\delta(M)$ | 1 |
| $H\left(i_{1}, \ldots, i_{j}\right)$ |  |
| $\Delta_{j}$ |  |
| $\mathcal{K}$ |  |
| $\mathcal{K}^{\prime}$ |  |
| $[M]_{\mathcal{K}, \mathcal{K}^{\prime}}$ |  |
| $H=\left[L_{01}\right]_{\mathcal{K}, \mathcal{K}^{\prime}}$ |  |
| $L \begin{aligned} & L\left[\begin{array}{lll}i_{1} & i_{2} & . \\ k_{1} & k_{2} & .\end{array}\right. \\ & p(\mu)\end{aligned}$ | $\left.\begin{array}{cc}\ldots & i_{p} \\ \ldots & k_{p}\end{array}\right]$ |

the end of examples and proofs definition identically equal the field of the complex members the determinant of a matrix $A$ the right null space (or kernel) of a matrix $A$ the conjugate transpose of a matrix $A$ the spectrum of a square matrix $M$ the spectral radius of $M$ the spectral matrix norm of $M$ the inertia of an hermitian matrix $M, \mathrm{pl10}$ the number of positive eigenvalues of $M$, p110 the number of negative eigenvalues of $M, \mathrm{p} 110$ the number of zero eigenvalues of $M, \mathrm{p} 110$ a minor of $H, \mathrm{p} 41$ minors of $H, \mathrm{p} 43$ the kernel of matrix $L\left(\lambda_{0}, 0\right)$ the kernel of matrix $L\left(\lambda_{0}, 0\right)^{*}$ a $g \times g$ matrix defined in p49 the perturbation matrix, p51
a minor of $L$ of order $p, \mathrm{p} 70$
a pencil defined in p78

## Chapter 1

## Introduction

Consider an $n \times n$ matrix function $L(\lambda, \alpha)$ depending analytically on two complex parameters $\lambda$ and $\alpha$. The parameter $\lambda$ can be viewed as an eigenvalue parameter dependent on the free "physical" parameter $\alpha$ that causes perturbation of $L\left(\lambda, \alpha_{0}\right)$. Sometimes the two parameters $\lambda$ and $\alpha$ can also be treated symmetrically. Let $\operatorname{det} L(\lambda, \alpha)=f(\lambda, \alpha)$. Then $f(\lambda, \alpha)$ is a scalar analytic function in $\lambda$ and $\alpha$, and we are interested in the zeros for $f(\lambda, \alpha)=0$, they are so-called $\lambda$-eigenvalues and $\alpha$ eigenvalues of $L(\lambda, \alpha)$. Specifically, we can let $\alpha=\alpha_{0}$, find the $\lambda$-roots for $f\left(\lambda, \alpha_{0}\right)=$ 0 , assuming that $f\left(\lambda, \alpha_{0}\right) \not \equiv 0$. If $\lambda=\lambda_{0}$ is one of these roots, the main aim of this thesis is to study the solution function $\lambda(\alpha)$ in the neighborhood of $\lambda_{0}$ for $\alpha$ in the neighborhood of $\alpha_{0}$ determined by $f(\lambda(\alpha), \alpha) \equiv 0$, since $\lambda(\alpha)$ represents the eigenvalue behavior under the perturbation of $\alpha$ close to $\alpha_{0}$. $\lambda(\alpha)$ such that $\operatorname{det} L(\lambda(\alpha), \alpha) \equiv 0$ is called an eigenvalue function.

A function of one complex variable is said to be analytic on a domain if it has a derivative at each point of the domain. By Abel's theorem (P. 126 of [D]), a function $f(z)$ which is an absolutely convergent power series, $f(z)=\sum_{n=0}^{\infty} f_{n}\left(z-z_{0}\right)^{n}$ in a neighborhood of $z_{0}$ is analytic in the neighborhood of $z_{0}$. This criterion of analyticity is often used in the whole thesis. Conversely, by Taylor's theorem (P. 117 of [D]), if a function $f(z)$ is analytic in a neighborhood of $z_{0}$ (sometimes we call it analytic at $z_{0}$ ), then $f(z)$ has the Taylor series representation in the neighborhood of $z_{0}$. For a function of two complex variables being analytic means that the function is analytic
in each variable separately. It is known that a function which is analytic at a point in $\mathbb{C}^{2}$ is equivalent to that it has a uniformly convergent expansion as a two-variable power series in a neighborhood of the point (see [T]).

Historically, important contributions have been made to the classical perturbation theory by Rellich $[R]$, Kato $[K]$, Baumgartel $[B]$ and several others. Special attention has been paid to the hermitian case, namely the case of $L(\lambda(\alpha))^{*}=L(\bar{\lambda}, \bar{\alpha}) \cdot($ see $[\mathrm{R}]$, [GLRI], [HL]).

Some examples are:

Example 1.1 $L(\lambda, \alpha)=A+\alpha H-\lambda I$, where $A^{*}=A, H$ is a matrix which need not be hermitian. This example reflects the original meaning of "perturbation". H is the perturbation factor to $A$.

Example 1.2 $L(\lambda, \alpha)=A(\alpha)-\lambda I$, where $A(\alpha)^{*}=A(\bar{\alpha})$. This example comes from $[R]$.

Example 1.3 Vibrating systems frequently have'transfer functions of the form $L(\lambda, \alpha)=$ $\lambda^{2} A(\alpha)+\lambda B(\alpha)+C(\alpha)$. The dependence of $L(\lambda, \alpha)$ on $\lambda$ is simply quadratic. And the requirements of the dependence on $\alpha$ are generally hermitian, i.e. $A(\alpha)^{*}=A(\bar{\alpha})$, $B(\alpha)^{*}=B(\bar{\alpha})$ and $C(\alpha)^{*}=C(\bar{\alpha})$, and often definite or semidefinite conditions on $A(\alpha), B(\alpha)$ and $C(\alpha)$. (This example comes from [L3].)

Example 1.4 To study gyroscopic system $L(\lambda)=\lambda^{2} I+\lambda B+C$, where $B^{*}=B$, $B$ is indefinite and invertible, $C^{*}=C>0$. Suppose $B$ and $C$ have the form $B=$ $\left[\begin{array}{cc}B_{1} & 0 \\ 0 & -B_{2}\end{array}\right] C=\left[\begin{array}{cc}C_{11} & C_{12} \\ C_{12} & C_{22}\end{array}\right]$, where $B_{1}>0, B_{2}>0$.

Using perturbation theory for

$$
L(\lambda, \alpha)=\left[\begin{array}{ll}
\lambda^{2} I+\lambda B_{1}+C_{11} & \alpha C_{12} \\
\alpha C_{12}^{*} & \lambda^{2} I-\lambda B_{2}+C_{22}
\end{array}\right]
$$

or $L(\lambda, \alpha)=\lambda^{2} I+\lambda \alpha B+C$, some results and some simplified proofs are attained. (This example comes from [BLM] and [HKLP].)

Example 1.5 Or the entries of $L(\lambda, \alpha)$ can be any analytic functions in $\lambda$ and $\alpha$.

$$
L(\lambda, \alpha)=\left[\begin{array}{ll}
e^{\lambda}(1-\alpha) & \sin \alpha \\
\cos \lambda & \lambda^{2}
\end{array}\right]
$$

The focus here is on the general functions without the hermitian conditions. It is well-known that if $\lambda_{0}$ is an eigenvalue of $L\left(\lambda, \alpha_{0}\right)$ with algebraic multiplicity $m$, then there exist eigenvalue functions $\lambda_{j}(\alpha), j=1,2, \ldots, m$, for $\lambda$ near $\alpha_{0}$ such that ! $\lambda_{j}\left(\alpha_{0}\right)=\lambda_{0}$. Assume that $\alpha_{0}=0$. Then the $\lambda_{j}(\alpha)$ can be represented as branches of Puiseux series in $\alpha^{k / q_{v}}$ with natural numbers $q_{v}$. In other words, in general, the eigenvalue function $\lambda_{j}(\alpha)$ can be expanded in series of fractional powers of $\alpha$. In many applications it is important to know when there are no fractions in these expansions, i.e., when $\lambda$ depends analytically on $\alpha$. If the partial multiplicities of the eigenvalue $\lambda_{0}$ of $L(\lambda, 0)$ are denoted by $m_{1}, m_{2}, \ldots, m_{g}$, the eigenvalue $\lambda_{0}$ is said to have the complete regular splitting property if for each $m_{i}$ from $\lambda_{0}$ there emerge $m_{i}$ eigenvalues $\lambda_{i j}(\alpha)$ with Puiseux expansion

$$
\lambda_{i j}(\alpha)=\lambda_{0}+\lambda_{1, i j} \alpha^{1 / m_{i}}+o\left(|\alpha|^{1 / m_{i}}\right)
$$

for $\alpha \rightarrow 0, j=1,2, \ldots, m_{i}$, (hence the $q_{v}$ in the Puiseux expansions coincide with the partial multiplicities) and all the $\lambda_{1, i j} \mathrm{~S}$ are not equal to zero. So what are the splitting properties of the eigenvalues under perturbation? Do eigenvalue and eigenvector functions have analytic dependence on perturbation parameter? This thesis will discuss topics of this kind.

Currently, there are two methods in studying the analytic perturbation problems for matrix functions. One can be called the "Newton diagram method". It was significantly developed by Langer and Najman. The other can be called the "generating eigenvector method". It was introduced in the paper [HL] of Hryniv and Lancaster. Here Chapter 3 and Chapter 4 are developed on the two methods respectively. Chapter 2 is the preparation for Chapter 3. Chapter 5 and Chapter 6 are the applications in classical eigenvalue problems and gyroscopic systems. It• shows how a two-parameter matrix function arises from real life problems. In Chapter 7, the conclusion comments, there is comparison and connection between the two methods. The structure of this dissertation is illustrated by Figure 1.1. Each chapter contains the author's contributions. In Chapter 2, although there are innovations in the treatment from Section 2.1 to Section 2.7, essentially they are some reorganizations of material from hundreds of years ago. Section 2.8 is the author's work. The basic ideas of Chapter 3 come from the papers of Langer, Najman and Veselic. The notations which are consistent with the works of Lancaster are used. And since the original papers of Langer, Najman and Veselic are intricate and hard to follow, it is the author's intention here that, with the preparation of Chapter 2 and with the example-driven exposition, the method can be understood more easily. Sections 3.6 and 3.7 are due to the author. Chapter 4 is based on the paper [LMZ2] of three


Figure 1.1: Contents structure
authors. Example 4.1, Example 4.2, Theorem 4.11 and Corollary 4.13 are the author's main contributions, but the author also contributed to the proof of Lemma 4.1, Lemma 4.4, Theorem 4.6 and Theorem 4.8. Chapter 5 is basically the author's work. In Chapter 6, Section 6.2 is quoted from [LMZ1], where Example 6.1 and Example 6.2 are the author's contributions. Sections 6.3 and 6.4 are the author's works.

## Chapter 2

## Newton's Diagram

### 2.1 Introduction

Let $f(\lambda, \alpha)$ be a function of two complex variables $\lambda$ and $\alpha$. Of course, if $L(\lambda, \alpha)$ is a matrix valued function, then the determinant of $L(\lambda, \alpha), \operatorname{det} L(\lambda, \alpha)$ is such a scalar valued function. Let $f(\lambda, \alpha)$ be a polynomial in $\lambda$ of the form

$$
\begin{equation*}
f(\lambda, \alpha)=a_{0}(\alpha)+a_{1}(\alpha) \lambda+\ldots+a_{p}(\alpha) \lambda^{p} \tag{2.1}
\end{equation*}
$$

where each coefficient $a_{i}(\alpha)$ is itself a polynomial in $\alpha$. Then a function $\lambda(\alpha)$ defined for values $\alpha$ in a neighborhood of $\alpha=\alpha_{0}$ by the equation $f(\lambda, \alpha) \equiv 0$ is called an algebraic function. Algebraic functions have Puiseux series expansions which are absolutely convergent in a deleted neighborhood of $\alpha_{0}$. The Existence Theorem I in Section 2.2 will show that when $a_{i}(\alpha)$ are analytic at $\alpha_{0}$ (then $f(\lambda, \alpha)$ is called pseudo-polynomial in $\lambda$ in [VT]), the solutions for $f(\lambda, \alpha) \equiv 0$ are functions in Puiseux series form. The Existence Theorem II in Section 2.6 will show that when $f(\lambda, \alpha)$ are analytic in both $\lambda$ and $\alpha$, some solutions of $f(\lambda, \alpha) \equiv 0$ have Puiseux series expansions too. Newton's diagram method is the main tool used to investigate solution functions of $f(\lambda, \alpha) \equiv 0$ by determining each term of their expansion series. Newton's diagram is also known as Newton's polygon or Newton's parallelogram. Further development of Newton's method and its role in modern mathematics are treated in many books and articles (for example, $[\mathrm{Bl}],[\mathrm{VT}]$ ). Although this historic
method has been widely studied, it is useful to develop some details for the later reference. Because of the fundamental role it played in the investigation of analytic perturbations of matrix valued functions, the whole of Chapter 2 is devoted to this method as a preliminary to the later chapters. The Existence Theorem I and Newton's diagram I assume that $f(\lambda, \alpha)$ is a pseudo-polynomial in $\lambda$, while the Existence Theorem II and Newton's diagram II are for the general analytic function $f(\lambda, \alpha)$.

The traditional Newton's diagram only gives the procedure to calculate the leading terms of all the expansions of the solution functions of $f(\lambda, \alpha) \equiv 0$ and it says that the following terms can be calculated and so on and so forth. Theoretically that is true, but in practice we try to figure out what are the explicit determining equations for the coefficients of the second terms. Section 2.8 includes an attempt in this direction.

### 2.2 Existence Theorem I

Let $f(\lambda, \alpha)$ be a polynomial in $\lambda$ of the form (2.1), where $a_{0}(\alpha), \ldots, a_{p}(\alpha)$ are analytic at $\alpha_{0}$, and suppose $f\left(\lambda, \alpha_{0}\right) \not \equiv 0$. The number $p$ is called the $\lambda$-order of $f(\lambda, \alpha)$. If $q \leq p$ is the biggest integer such that $a_{q}\left(\alpha_{0}\right) \neq 0$, then $q$ is called the $\lambda$-order of $f(\lambda, \alpha)$ at $\alpha_{0}$.

Theorem 2.1 (Existence Theorem I) Let $p, q$ be the $\lambda$-order and $\lambda$-order at $\alpha_{0}$ of $f(\lambda, \alpha)$ in (2.1) respectively. Then there exist $p$ functions $\lambda(\alpha)$ of the form

$$
\begin{equation*}
\lambda(\alpha)=\lambda_{1}\left(\alpha-\alpha_{0}\right)^{\epsilon_{1}}+\lambda_{2}\left(\alpha-\alpha_{0}\right)^{\epsilon_{2}}+\ldots \tag{2.2}
\end{equation*}
$$

such that $f(\lambda(\alpha), \alpha) \equiv 0$ in a neighborhood $\mathcal{N}$ of $\alpha_{0}$, where $\epsilon_{1}<\epsilon_{2}<\ldots$ is a increasing sequence of rational numbers and $\lambda_{1}, \lambda_{2}, \ldots$ are all different from zero
(unless $\lambda(\alpha) \equiv 0$ ). If $q<p$, then of these $p$ functions, there are $p-q$ with negative power terms. That is, $\epsilon_{1}<0$ for these $p-q$ functions.

The proof is given by $[\mathrm{Bl}]$ and is long and technical. In $[\mathrm{BL}]$ the proof is for polynomials $f(\lambda, \alpha)$, but is applicable to pseudo-polynomials without any change. Here, we summarize a few of the underlying concepts and results that will be needed in the sequel.

First of all, the resultant of two polynomials

$$
\begin{aligned}
& h(\lambda)=h_{0} \lambda^{m}+h_{1} \lambda^{m-1}+\ldots+h_{m}, \quad m>0, \quad h_{0} \neq 0 \\
& g(\lambda)=g_{0} \lambda^{n}+g_{1} \lambda^{n-1}+\ldots+g_{n}, \quad n>0, \quad g_{0} \neq 0
\end{aligned}
$$

is the scalar

$$
R=\operatorname{det}\left[\begin{array}{c}
\lambda^{n-1} h \\
\cdots \\
\lambda h \\
h \\
\lambda^{m-1} g \\
\cdots \\
\lambda g \\
g
\end{array}\right]
$$

which stands for the determinant whose $n+m$ rows are the coefficients of the powers of $\lambda$ in the polynomials indicated. If $m=0$ or $n=0$, define $R=1$.

The discriminant $D$ of a polynomial $h(\lambda)$ is defined by the resultant of the derivative $h^{\prime}(\lambda)$ and the polynomial $m h-\lambda h^{\prime}$ where m is the degree of $h(\lambda)$. Therefore $D=0$ is a necessary and sufficient condition that $h(\lambda)$ has a multiple root. Hence,
a necessary and sufficient condition that $f(\lambda, \alpha)$ has no repeated irreducible factors involving $\lambda$ is $D(\alpha) \neq 0$, where $D(\alpha)$ is the discriminant of $f(\lambda, \alpha)$ as a polynomial in $\lambda$.

Now $\alpha=\alpha_{0}$ is called an ordinary point if $a_{p}\left(\alpha_{0}\right) \neq 0$ or $D\left(\alpha_{0}\right) \neq 0$. Otherwise it is called a singular point. By the implicit function theorem, we know that near an ordinary point $\alpha=\alpha_{0}$ the p values of the function $\lambda(\alpha)$ are defined by p convergent power series,

$$
\begin{equation*}
\lambda_{i}(\alpha)=\lambda_{i 0}+\lambda_{i 1}\left(\alpha-\alpha_{0}\right)+\lambda_{i 2}\left(\alpha-\alpha_{0}\right)^{2}+\ldots \tag{2.3}
\end{equation*}
$$

where $i=1, \ldots, p$ and the numbers $\lambda_{i 0}$ are the p distinct roots of $f\left(\lambda, \alpha_{0}\right)=0$. It can also be proved that the radii of convergence for the p series in (2.3) are at least equal to the distance from $\alpha_{0}$ to the nearest singular point.

If $a_{p}\left(\alpha_{0}\right)=0$, then some negative power terms will appear in $p-q$ functions of the form (2.2). Suppose $a_{p}(\alpha)=\left(\alpha-\alpha_{0}\right)^{s} b(\alpha)$ with $b\left(\alpha_{0}\right) \neq 0$ and $\lambda(\alpha)$ is an function such that $f(\lambda(\alpha), \alpha)=0$. It can be proved that in a properly chosen neighborhood of $\alpha=\alpha_{0},\left(\alpha-\alpha_{0}\right)^{s} \lambda(\alpha)$ is bounded. A corollary of this result is that in the expansion (2.2), the number of terms with negative exponents must be finite.

If $a_{p}\left(\alpha_{0}\right) \neq 0$, then none of the functions of the form (2.2) have negative power terms. It seems possible that the singularity $D\left(\alpha_{0}\right)=0$ will give rise to fractional exponents of the form (2.2). But this is not always the case. Let us examine some examples.

Example 2.1 Let $f(\lambda, \alpha)=(\lambda-\alpha)^{2}$. Then $D(0)=0$. In fact, $D(\alpha) \equiv 0$. There are two identical roots $\lambda=\alpha$ for $f(\lambda, \alpha)=0$.


Figure 2.1: Region $R$ is replaced by $r$ sheets of $t$-plane
Example 2.2 Let $f(\lambda, \alpha)=(\lambda-\alpha)(\lambda+\alpha)$. Then $D(0)=0$ and 0 is an isolated singular point. There are two distinct roots $\lambda=\alpha$ and $\lambda=-\alpha$ for $f(\lambda, \alpha)=0$.

Example 2.3 Let $f(\lambda, \alpha)=\lambda^{2}-\alpha$. Then $D(0)=0$ and 0 is again an isolated singular point. This time the solution functions are $\lambda=\alpha^{1 / 2}$ and $\lambda=-\alpha^{1 / 2}$.

The fractional exponents in the expansion of a solution function are due to the loss of the one to one correspondence between the $\alpha$ plane and the $\lambda$ plane. Let $\alpha_{0}$ be an isolated singular point. Let $R$ be a closed disk centered at $\alpha=\alpha_{0}$ with boundary circle $C$ containing no singular points except $\alpha=\alpha_{0}$. When the one to one correspondence is lost, there exists a finite integer $r>1$ such that we can use the transformation $\alpha-\alpha_{0}=t^{r}$, and the transformation replaces the region $R$ by a "pile" of $r$ such regions (Figure 2.1), each one corresponding to a single-valued function $\lambda^{[k]}(\alpha), k=1, \ldots, r$. They are called $r$ branches of a set (or a group) of solution functions. These $r$ branches can be described by power-series expansions in $\left(\alpha-\alpha_{0}\right)^{1 / r}$, known as Puiseux series:

$$
\begin{equation*}
\lambda^{[k]}(\alpha)=\lambda_{1}\left(\alpha-\alpha_{0}\right)^{\frac{\mu_{1}}{r}}+\ldots+\lambda_{i-1}\left(\alpha-\alpha_{0}\right)^{\frac{\mu_{i-1}}{r}}+\lambda_{i} \theta_{r, k}\left(\alpha-\alpha_{0}\right)^{\frac{\mu_{i}}{r}}+\ldots \tag{2.4}
\end{equation*}
$$

where $k=1, \ldots, r, \mu_{1}<\mu_{2}<\ldots$ are integers, and $\theta_{r, k}$ is one of the $r$-th roots of unity. Suppose the $r$ functions $\lambda^{[k]}(\alpha)$ share the same $i-1$ coefficients $\lambda_{1}, \ldots, \lambda_{i-1}$, while the $i$-th coefficients of the $r$ functions constitute the $r r$-th roots of $\lambda_{i}$. We may say that the $i$-th coefficient is the first coefficient to split. The splitting can occur at the first position, i.e., $\cdot i=1$. When $i>1$, however, $\mu_{1}, \ldots, \mu_{i-1}$ must be integer multiples of $r$. Note that in some monographs, the $r$ branches of the same set together may be called one algebraic function.

If $\alpha \in R$ and $\alpha \neq \alpha_{0}$, then $\alpha$ is an ordinary point, and there are $p$ distinct values of $\lambda$ corresponding to this $\alpha$ such that $f(\lambda, \alpha)=0$. Therefore the sum of the numbers $r$ for the different sets is $p$, the $\lambda$-order of $f(\lambda, \alpha)$. It is possible that a set has only one branch, i.e., $r=1$. In this case, $\lambda(\alpha)$ is an analytic single-valued function on $R$ except possibly when $\alpha=\alpha_{0}$.

To illustrate various situations, let us look at some more examples.

Example $2.4 f(\lambda, \alpha)=\left(\lambda^{3}-\alpha^{2}\right)\left(\lambda^{2}-\alpha\right) . \quad f(\lambda, \alpha)=0$ determines two sets of solution functions. One set has 3 branches given by $\lambda^{[k]}=\theta_{k} \alpha^{3 / 2}, \theta_{k}=e^{\frac{2 k \pi}{3} i}$, $k=1,2,3$. Another set has 2 branches given by $\lambda=\alpha^{1 / 2}$ and $\lambda=-\alpha^{1 / 2}$. The sum of all different branches is 5 , which is the $\lambda$-order of $f(\lambda, \alpha)$.

Example $2.5 f(\lambda, \alpha)=(\lambda-\alpha)^{2}-\alpha^{3}$. The two solution functions determined by $f(\lambda, \alpha)=0$ are $\lambda^{[1]}(\alpha)=\alpha+\alpha^{3 / 2}$ and $\lambda^{[2]}(\alpha)=\alpha-\alpha^{3 / 2}$. They belong to the same group. The splitting of coefficients happens at the second term.

Example $2.6 f(\lambda, \alpha)=\lambda^{3}-\alpha^{2}-\alpha^{3}$. Using the Newton diagram method introduced in the next section, we will see that the three solution functions given by $f(\lambda, \alpha)=0$ are infinite Puiseux series:

$$
\lambda^{[k]}(\alpha)=\theta_{k} \alpha^{2 / 3}+\frac{1}{3} \theta_{k} \alpha^{5 / 3}+\ldots
$$

where $\theta_{k}=e^{\frac{2 k \pi}{3} i}, k=1,2,3$.

### 2.3 Newton's diagram I

Suppose $f(\lambda, \alpha)$ is a pseudo-polynomial in $\lambda$ of the form (2.1). The purpose of Newton's diagram is to successively compute all the expansions for $\lambda(\alpha)$ in (2.2). The first phase will give all the possibilities for $\epsilon_{1}$, then calculate the coefficients for those values of $\epsilon_{1}$. The second phase will find all the values of $\epsilon_{2}$ and their coefficients $\lambda_{2}$, and so on.

Suppose the coefficients in (2.1) are written as

$$
\begin{gather*}
a_{0}(\alpha)=b_{0}\left(\alpha-\alpha_{0}\right)^{C_{0}}+\ldots \\
\vdots  \tag{2.5}\\
a_{p}(\alpha)=b_{p}\left(\alpha-\alpha_{0}\right)^{C_{p}}+\ldots
\end{gather*}
$$

where $b_{i} \neq 0$ (unless $a_{i}(\alpha) \equiv 0$ ), $C_{i} \geq 0$ are integers, $i=1, \ldots, p$. Then substituting $(2.5)$ and (2.2) in $f(\lambda(\alpha), \alpha) \equiv 0$ gives

$$
\begin{align*}
f(\lambda(\alpha), \alpha)= & \left(b_{0}\left(\alpha-\alpha_{0}\right)^{C_{0}}+\ldots\right)  \tag{2.6}\\
& +\left(b_{1}\left(\alpha-\alpha_{0}\right)^{C_{1}}+\ldots\right)\left(\lambda_{1}\left(\alpha-\alpha_{0}\right)^{\epsilon_{1}}+\ldots\right) \\
& +\ldots \\
& +\left(b_{p-1}\left(\alpha-\alpha_{0}\right)^{C_{p-1}}+\ldots\right)\left(\lambda_{1}^{p-1}\left(\alpha-\alpha_{0}\right)^{(p-1) \epsilon_{1}}+\ldots\right) \\
& +\left(b_{p}\left(\alpha-\alpha_{0}\right)^{C_{p}}+\ldots\right)\left(\lambda_{1}^{p}\left(\alpha-\alpha_{0}\right)^{p \epsilon_{1}}+\ldots\right) \equiv 0
\end{align*}
$$

Therefore the coefficients of all powers of $\left(\alpha-\alpha_{0}\right)$ in $f(\lambda(\alpha), \alpha)$ must vanish, in particular, the coefficient of the smallest power. The only possibilities for the smallest power are: $C_{0}, C_{1}+\epsilon_{1}, C_{2}+2 \epsilon_{1}, \ldots$, and $C_{p}+p \epsilon_{1}$. If $a_{k}(\alpha) \equiv 0$, there is not a candidate like $c_{k}+k \epsilon_{1}$.

For the sum of the coefficients of the smallest power of $\left(\alpha-\alpha_{0}\right)$ to cancel mutually, it is necessary that at least two of them are present. Hence $\epsilon_{1}$ must satisfy:
(1) $C_{k}+k \epsilon_{1} \geq C_{i}+i \epsilon_{1}, k=1, \ldots, p$, for some $i$,
(2) There is $j \neq i$ with $C_{i}+i \epsilon_{1}=C_{j}+j \epsilon_{1}$.

To find the values for $\epsilon_{1}$ that satisfy these two conditions we can use a diagram. Plot the points $A_{0}=\left(0, C_{0}\right), A_{1}=\left(1, C_{1}\right), \ldots, A_{p}=\left(p, C_{p}\right)$ on the Cartesian plane (Figure 2.2 is an example of such a diagram). Geometrically, Condition (2) gives $\epsilon_{1}=$ $\tan \tau$, where $\tau$ is the angle formed by $A_{i} A_{j}$ with respect to the negative horizontal axis. So $\tan \tau=-\operatorname{slope}\left(A_{i} A_{j}\right)=\frac{c_{j}-c_{i}}{i-j}$. Condition (1) means that all the other points should lie above the line $A_{i} A_{j}$, since the value $C_{k}+k \tan \tau$ is exactly the $y$-intercept of the straight line through $A_{k}$ defined by the angle $\tau$. (There is also a technical requirement that if the term $a_{k}(\alpha) \lambda^{k}$ in $f(\lambda, \alpha)$ is missing, i.e., $a_{k}(\alpha) \equiv 0$, then $A_{k}$ cannot be viewed as ( $k, 0$ ), and must be disregarded.) Connecting all $A_{i} A_{j}$ for which conditions (1) and (2) are satisfied, we obtain Newton's diagram. By this construction, we know that Newton's diagrams are convex.

Newton's diagram consists of three sections- the descending section, the horizontal section and the ascending section. The descending section yields small solutions, i.e., solutions which go to zero as $\alpha \rightarrow \alpha_{0}$. The horizontal section yields the solutions $\lambda=\lambda(\alpha)$ for which $\lambda\left(\alpha_{0}\right)=\lambda_{1}$, i.e., the leading terms of these series are
nonzero constants. The ascending section yields the solutions $\lambda=\lambda(\alpha)$ for which $\lim _{\alpha \rightarrow \alpha_{0}} \lambda(\alpha)=\infty$, i.e., their expansions include negative powers. By Theorem 2.1, the ascending section starts from the point $(q, 0)$, where $q$ is the $\lambda$-order of $f(\lambda, \alpha)$ at $\alpha_{0}$. As in Figure 2.2, $A_{0}$ to $A_{3}$ is the descending section, $A_{3}$ to $A_{4}$ is the horizontal section, and $A_{4}$ to $A_{5}$ is the ascending section.

The slope of each segment of Newton's diagram gives a possible value of $\epsilon_{1}$. To find the values of $\lambda_{1}$ for this $\epsilon_{1}$, we collect powers of $\alpha-\alpha_{0}$ from the left hand side of Equation (2.6), and equate the coefficient of the lowest power of $\alpha-\alpha_{0}$ to zero. This equation involving $\lambda_{1}$ is called the determining equation (D.E.). (In [VT], the determining equation is called the defining equation.)

To find the higher order terms in the expansions of $\lambda(\alpha)$, we substitute (2.2) with known $\lambda_{1}$ and $\epsilon_{1}$, and unknown $\lambda_{2}$ and $\epsilon_{2}$ into (2.1). Using the same approach we can obtain the second lowest terms in the expansions. Continuing the process, we obtain an expansion for each of the $p$ solutions of $f(\lambda, \alpha)=0$.

Example 2.7 Let $f(\lambda, \alpha)=\alpha^{2}+2 \alpha^{3}-\left(\alpha+\alpha^{2}\right) \lambda+\alpha \lambda^{2}+\lambda^{3}$. Without loss of generality, we can always suppose $\alpha_{0}=0$, since the transformation $\alpha^{\prime}=\alpha-\alpha_{0}$ moves $\alpha_{0}$ to the origin.

With $A_{0}=(0,2), A_{1}=(1,1), A_{2}=(2,1), A_{3}=(3,0)$, the diagram takes the form of Figure 2.3. 'in:

So we see two possibilities for $\epsilon_{1}$ from the diagram: $\epsilon_{1}=1$ or $\epsilon_{1}=\frac{1}{2}$.
If $\epsilon_{1}=1, \lambda(\alpha)$ has the form

$$
\begin{equation*}
\lambda(\alpha)=\lambda_{1} \alpha+\ldots \tag{2.7}
\end{equation*}
$$



Figure 2.2: One example of Newton's diagram I


Figure 2.3: Newton's diagram for Example 2.7

To find $\lambda_{1}$ for $\epsilon_{1}=1$, we substitute (2.7) in $f(\lambda, \alpha)$ :

$$
\begin{equation*}
f(\lambda, \alpha)=\alpha^{2}-\lambda_{1} \alpha^{2}+2 \alpha^{3}-\lambda_{1} \alpha^{3}+\lambda^{2} \alpha^{3}+\lambda_{1} \alpha^{3}+\ldots . \tag{2.8}
\end{equation*}
$$

Making the coefficient of $\alpha^{2}$ in (2.8) equal to zero, we obtain the determining equation (D.E.) of $\lambda_{1}: 1-\lambda_{1}=0$. So $\lambda_{1}=1$.

$$
\text { If } \epsilon_{1}=\frac{1}{2} \text {, then } \lambda(\alpha)=\lambda_{1} \alpha^{1 / 2}+\ldots .
$$

The D.E. of $\lambda_{1}$ is: $-\lambda_{1}+\lambda_{1}^{3}=0$ and the nonzero solutions of the D.E. are: $\lambda_{1}=1$ and $\lambda_{1}=-1$.

Thus we obtain three different solution functions starting with the terms

$$
\begin{aligned}
\lambda^{[1]}(\alpha) & =\alpha+\ldots \\
\lambda^{[2]}(\alpha) & =\alpha^{1 / 2}+\ldots \\
\lambda^{[3]}(\alpha) & =-\alpha^{1 / 2}+\ldots .
\end{aligned}
$$

Example 2.8 Let $f(\lambda, \alpha)=\alpha+(1-2 \alpha) \lambda+\alpha \lambda^{2}, \alpha_{0}=0$. Since the coefficient of $\lambda^{2}$ will vanish when $\alpha=0$, we expect some negative powers to appear in one expansion for $\lambda(\alpha)$.

The diagram for finding $\epsilon_{1}$ is as Figure 2.4, which includes a descending section and an ascending section. Each section consists of one segment, and each gives one value of $\epsilon_{1}: \epsilon_{1}=1$ and $\epsilon_{1}=-1$.

The D.E. for $\epsilon_{1}=1$ is $1+\lambda_{1}=0$. The D.E. for $\epsilon_{1}=-1$ is $\lambda_{1}+\lambda_{1}^{2}=0$. So the coefficient $\lambda_{\mathbf{1}}=-1$ for both cases.

Continuing the procedure we have two solutions with the first few terms:

$$
\begin{aligned}
& \lambda^{[1]}=-\alpha+2 \alpha^{2}+3 \alpha^{3}+\ldots \\
& \lambda^{[2]}=-\alpha^{-1}+2+3 \alpha+\ldots
\end{aligned}
$$



Figure 2.4: Newton's diagram for Example 2.8

Theorem 2.2 Newton's diagram method gives all expansions of the $p$ functions for $f(\lambda, \alpha)$ of Theorem 2.1.

Proof: The procedure of Newton's diagram method looks for the expansions using only the necessary conditions. So if they exist, all of the expansions can be found by this method.

### 2.4 Special configurations of Newton's diagram I

We shall now consider five special cases of the solution functions. They can be viewed as some more examples which are helpful for getting familiar with Newton's diagram.

As in (2.1), suppose $f(\lambda, \alpha)=a_{0}(\alpha)+\dot{a}_{1}(\alpha) \lambda+\ldots+a_{p}(\alpha) \lambda^{p}$. In addition, let $\alpha_{0}=0, f(\lambda, 0) \not \equiv 0$ and $\lambda=0$ be a root of $f(\lambda, 0)$. Now $a_{0}(\alpha), a_{1}(\alpha), \ldots, a_{p}(\alpha)$ have


Figure 2.5: Instances of $f(0,0)=0$
the form:

$$
\begin{aligned}
a_{0}(\alpha) & =b_{0} \alpha^{C_{0}}+\ldots \\
a_{1}(\alpha) & =b_{1} \alpha^{C_{1}}+\ldots \\
\vdots & \\
a_{p}(\alpha) & =b_{p} \alpha^{C_{p}}+\ldots
\end{aligned}
$$

where $b_{i} \neq 0$ (unless $a_{i}(\alpha) \equiv 0$ ), $\dot{C}_{i}$ are non-negative integers, and $i=1, \ldots, p$. Especially, $C_{0}>0$, because 0 is a root of $f(\lambda, 0)=0$.

Case I, $p=2$.
First observe that the origin $A_{0}=(0,0)$ can never be in Newton's diagram. So the instances in Figure 2.5 can never happen.

Since $f(\lambda, 0) \not \equiv 0$, we do not need to consider the instances in Figure 2.6.
Then there are left 9 possibilities (a) to (i) shown in Figure 2.7. We proceed to examine each of these 9 cases.
$\mathrm{I},(\mathrm{a}), f(\lambda, \alpha)=\left(b_{2}+\ldots\right) \lambda^{2}, \lambda=0$ constitutes a double root of $f(\lambda, \alpha)=0$.


Figure 2.6: Cases of $f(\lambda, 0) \equiv 0$
$\mathrm{I},(\mathrm{b}), f(\lambda, \alpha)=\left(b_{1}+\ldots\right) \lambda+\left(b_{2}+\ldots\right) \lambda^{2}$. Newton's diagram consists of only the horizontal section. This yields the solution function with constant leading term

$$
\lambda(\alpha)=\lambda_{1}+\ldots
$$

The D.E. of $\lambda_{1}$ is $b_{1} \lambda_{1}+b_{2} \lambda_{1}^{2}=0$, so $\lambda_{1}=-\frac{b_{1}}{b_{2}}$.
Hence $f(\lambda, \alpha)=0$ has a solution of the form

$$
\lambda(\alpha)=-\frac{b_{1}}{b_{2}}+\ldots
$$

and also the solution $\lambda=0$.
$\mathrm{I},(\mathrm{c}), f(\lambda, \alpha)=\left(b_{1} \alpha^{C_{1}}+\ldots\right) \lambda+\left(b_{2}+\ldots\right) \lambda^{2}$. The only segment in Newton's diagram yields the exponent $C_{1}$. One solution $\lambda(\alpha)$ has the form

$$
\lambda(\alpha)=\lambda_{1} \alpha^{C_{1}}+\ldots
$$

The D.E. of $\lambda_{1}$ is $b_{1} \lambda_{1}+b_{2} \lambda_{1}^{2}=0$, so $\lambda_{1}=-\frac{b_{1}}{b_{2}}$.
Hence $f(\lambda, \alpha)=0$ has a solution of the form

$$
\lambda(\alpha)=-\frac{b_{1}}{b_{2}} \alpha^{C_{1}}+\ldots
$$


(d)

(g)


(e)

(f)

(h)

(i)
.Figure 2.7: Newton's diagrams of Case I
and also the solution $\lambda=0$.
$\mathrm{I},(\mathrm{d}), f(\lambda, \alpha)=\left(b_{1}+\ldots\right) \lambda+\left(b_{2} \alpha^{C_{2}}+\ldots\right) \lambda^{2}$. The two solution functions in this instance are:

$$
\lambda(\alpha)=-\frac{b_{1}}{b_{2}} \alpha^{-C_{2}}+\ldots
$$

and $\lambda=0$.
$\mathrm{I},(\mathrm{e}), f(\lambda, \alpha)=\left(b_{0} \alpha^{C_{0}}+\ldots\right)+\left(b_{2}+\ldots\right) \lambda^{2}$. From Newton's diagram, $\epsilon_{1}=\frac{C_{0}}{2}$.
The D.E. of $\lambda_{1}$ is $b_{0}+b_{2} \lambda_{1}^{2}=0$, so $\lambda_{1}= \pm \sqrt{-b_{0} / b_{2}}$, and the two branching functions are:

$$
\lambda(\alpha)= \pm \sqrt{-b_{0} / b_{1}} \alpha^{C_{0} / 2}+\ldots
$$

$\mathrm{I},(\mathrm{f}), f(\lambda, \alpha)=\left(b_{0} \alpha^{C_{0}}+\ldots\right)+\left(b_{1}+\ldots\right) \lambda+\left(b_{2}+\ldots\right) \lambda^{2}$. From Newton's diagram, $\epsilon_{1}=C_{0}$ or $\epsilon_{1}=0$.

For $\epsilon_{1}=C_{0}$, the D.E. is $b_{0}+b_{1} \lambda_{1}=0$, and $\lambda_{1}=-b_{0} / b_{1}$.
For $\epsilon_{1}=0$, the D.E. is $b_{1} \lambda_{1}+b_{2} \lambda_{1}^{2}=0$, and $\lambda_{1}=-b_{1} / b_{2}$.
The two solution functions are:

$$
\therefore 1 \mid
$$

$$
\lambda(\alpha)=-\left(b_{0} / b_{1}\right) \alpha^{C_{0}}+\ldots \quad \text { and } \quad \lambda(\alpha)=-b_{1} / b_{2}+\ldots
$$

$\mathrm{I},(\mathrm{g}), f(\lambda, \alpha)=\left(b_{0} \alpha^{C_{0}}+\ldots\right)+\left(b_{1}+\ldots\right) \lambda+\left(b_{2} \alpha^{C_{2}}+\ldots\right) \lambda^{2}$. From Newton's diagram, $\epsilon_{1}=C_{0}$ or $\epsilon_{1}=-C_{2}$.

The two solution functions are:

$$
\lambda(\alpha)=-\left(b_{0} / b_{1}\right) \alpha^{C_{0}}+\ldots \quad \text { and } \quad \lambda(\alpha)=-\left(b_{1} / b_{2}\right) \alpha^{-C_{2}}+\ldots
$$

$\mathrm{I},(\mathrm{h}), f(\lambda, \alpha)=\left(b_{0} \alpha^{C_{0}}+\ldots\right)+\left(b_{1} \alpha^{C_{1}}+\ldots\right) \lambda+\left(b_{2}+\ldots\right) \lambda^{2}$. From Newton's diagram, $\epsilon_{1}=C_{0}-C_{1}$ or $\epsilon_{1}=C_{1}$.

The two solution functions are:

$$
\lambda(\alpha)=-\left(b_{0} / b_{1}\right) \alpha^{C_{0}-C_{1}}+\ldots \quad \text { and } \quad \lambda(\alpha)=-\left(b_{1} / b_{2}\right) \alpha^{C_{1}}+\ldots
$$

$\mathrm{I},(\mathrm{i}), f(\lambda, \alpha)=\left(b_{0} \alpha^{C_{0}}+\ldots\right)+\left(b_{1} \alpha^{C_{0} / 2}+\ldots\right) \lambda+\left(b_{2}+\ldots\right) \lambda^{2}$. From Newton's diagram, $\epsilon_{1}=\frac{C_{0}}{2}$. The D.E. is

$$
b_{0}+b_{1} \lambda_{1}+b_{2} \lambda_{1}^{2}=0
$$

Suppose the two complex roots of the D.E. are $\lambda_{11}$ and $\lambda_{12}$. Then the two solution functions are:

$$
\lambda(\alpha)=\lambda_{11} \alpha^{C_{0} / 2}+\ldots \quad \text { and } \quad \lambda(\alpha)=\lambda_{12} \alpha^{C_{0} / 2}+\ldots
$$

Note that in this case, $C_{0}$ must be an even integer. If $\lambda_{11}$ and $\lambda_{12}$ are two different roots, then the solution functions are independent, i.e., they belong to different groups, and they are analytic (by Theorem 2.4 in the next section). If $\lambda_{11}=\lambda_{12}$ is a double root, then there is still a chance that the two solution functions are two branches of one group, but the coefficient splitting does not happen at the leading term.

Case II, $f(\lambda, \alpha)=\left(b_{0} \alpha+\ldots\right)+\ldots+\left(b_{p}+\ldots\right) \lambda^{p}, C_{0}=1, a_{p}(0) \neq 0$. $\therefore$

In this case $p=q$. Since $a_{p}(0) \neq 0$, Newton's diagram has no horizontal and ascending.sections. Also, it is apparent from Figure 2.8 that $\epsilon_{1}=\frac{1}{p}$.

The D.E. of $\lambda_{1}$ is $b_{0}+b_{p} \lambda_{1}^{p}=0$, and the $p$ roots are $\lambda_{1, k}=\theta_{p, k}\left(-b_{0} / b_{p}\right)^{1 / p}$, where $\theta_{p, k}=e^{\frac{2 k \pi}{p} i}$, and $\left(-b_{0} / b_{p}\right)^{1 / p}$ is the principle value of the $p$-th root of complex number $-b_{0} / b_{p}, k=1, \ldots, p$.


Figure 2.8: Newton's diagram of Case II

The $p$ branching solutions are $\lambda(\alpha)=\lambda_{1, k} \alpha^{1 / p}+\ldots$, where $k=1, \ldots, p$.

Case III, $C_{0}=2, a_{p}(0) \neq 0, C_{j}=1$, and $j<\frac{p}{2}$, and only the points $(0,2),(j, 1)$ and ( $p, 0$ ) are plotted in Newton's diagram.

We thus have

$$
f(\lambda, \alpha)=\left(b_{0} \alpha^{2}+\ldots\right)+\ldots+\left(b_{j} \alpha+\ldots\right) \lambda^{j}+\ldots+\left(b_{p}+\ldots\right) \lambda^{p}
$$

Newton's diagram takes the form of Figure 2.9.
Hence $\epsilon_{1}=\frac{1}{j}$ or $\epsilon_{1}=\frac{1}{p-j}$.
For $\epsilon_{1}=\frac{1}{j}$, the D.E. is $b_{0}+b_{j} \lambda_{1}^{j}=0$.
Denote the $j$ roots by

$$
\lambda_{11, k}=\theta_{j, k}\left(-b_{0} / b_{j}\right)^{1 / j}, \quad k=1, \ldots, j .
$$

For $\epsilon_{1}=\frac{1}{p-j}$, the D.E. is $b_{j} \lambda_{1}^{j}+b_{p} \lambda_{1}^{p}=0$.
Denote the $p-j$ nonzero roots by

$$
\lambda_{12, k}=\theta_{p-j, k}\left(-b_{j} / b_{p}\right)^{1 /(p-j)}, \quad k=1, \ldots, p-j .
$$



Figure 2.9: Newton's diagram of Case III

Then the branching functions of two groups are given by

$$
\begin{aligned}
& \lambda_{1}^{[k]}(\alpha)=\lambda_{11, k} \alpha^{1 / j}+\ldots, \quad k=1, \ldots, j \\
& \lambda_{2}^{[k]}(\alpha)=\lambda_{12, k} \alpha^{1 /(p-j)}+\ldots, \quad k=1, \ldots, p-j .
\end{aligned}
$$

Case IV, $C_{0}=2, a_{p}(0) \neq 0$, and only the points $(0,2)$ and $(p, 0)$ are plotted in Newton's diagram.

We thus have $f(\lambda, \alpha)=\left(b_{0} \alpha^{2}+\ldots\right)+\ldots+\left(b_{p}+\ldots\right) \lambda^{p}$, and Newton's diagram takes the form of Figure 2.10 .

So, $\epsilon_{1}=2 / p$ and the D.E. is $b_{0}+\dot{b}_{p} \lambda_{1}^{p}=0$.
If $p$ is odd, then the $p$ branches constitute one group:

$$
\lambda^{[k]}(\alpha)=\theta_{p, k}\left(-b_{0} / b_{p}\right)^{1 / p} \alpha^{2 / p}+\ldots, \quad k=1, \ldots, p .
$$

If $p$ is even, then the $p$ solutions are divided into two groups, each of which consists of $p / 2$ branches:

$$
\begin{aligned}
& \lambda_{1}^{[k]}(\alpha)=\theta_{p / 2, k}\left(\sqrt{-b_{0} / b_{p}}\right)^{2 / p} \alpha^{2 / p}+\ldots, \quad k=1, \ldots, p / 2 \\
& \lambda_{2}^{[k]}(\alpha):=\theta_{p / 2, k}\left(-\sqrt{-b_{0} / b_{p}}\right)^{2 / p} \alpha^{2 / p}+\ldots, \quad k=1, \ldots, p / 2 .
\end{aligned}
$$



Figure 2.10: Newton's diagram of Case IV


Figure 2.11: Newton's diagram of Case V

Case V, $C_{0}=2, a_{p}(0) \neq 0, p$ is even, $p=2 j$, and $C_{j}=1$, only the points $(0,2)$, $(j, 1)$ and $(p, 0)$ are plotted in Newton's diagram.

We thus have

$$
f(\lambda, \alpha)=\left(b_{0} \alpha^{2}+\ldots\right)+\ldots+\left(b_{j} \alpha+\ldots\right) \lambda^{j}+\ldots+\left(b_{p}+\ldots\right) \lambda^{p},
$$

and Newton's diagram takes the form of Figure 2.11.
So, $\epsilon_{1}=1 / j$ and the D.E. is $b_{0}+b_{j} \lambda_{1}^{j}+b_{p} \lambda_{1}^{2 j}=0$.


Figure 2.12: An example of Newton's diagram with three D.E.s

Suppose $\lambda_{1}^{j}=\eta_{1}$ and $\lambda_{1}^{j}=\eta_{2}$ are two complex roots of the D.E..
If $\eta_{1} \neq \eta_{2}$, then there are two sets of the solution functions, each of which has $j$ branches:

$$
\begin{array}{ll}
\lambda_{1}^{[k]}(\alpha)=\theta_{j, k}\left(\eta_{1}\right)^{1 / j} \alpha^{1 / j}+\ldots, & k=1, \ldots, j \\
\lambda_{2}^{[k]}(\alpha)=\theta_{j, k}\left(\eta_{2}\right)^{1 / j} \alpha^{1 / j}+\ldots, & k=1, \ldots, j
\end{array}
$$

If $\eta_{1}=\eta_{2}$, then additional investigation is required.

### 2.5 Determining Equations

We shall elucidate a number of properties of the determining equations.
Let us begin by considering an example of Newton's diagram with three D.E.s. In Figure 2.12, we observe that three segments determine three possible exponents of the leading terms in the expansions of solution functions. Let us call them $\epsilon_{1}^{\prime}, \epsilon_{1}$
and $\epsilon_{1}^{\prime \prime}$. They are respectively the absolute values of the slopes of the corresponding segments.

It is clear that

$$
\begin{equation*}
\epsilon_{1}^{\prime}>\epsilon_{1}>\epsilon_{1}^{\prime \prime} \tag{2.9}
\end{equation*}
$$

Recall that one property of Newton's diagrams is that the graphs are convex. So in the descending section, the exponent given by one segment is less than the exponents given by the higher segments and bigger than the exponents given by the lower segments.

Suppose the segment with slope $-\epsilon_{1}$ has $\left(i, C_{i}\right)$ and $\left(j, C_{j}\right)$ for end points, and passes through the point $\left(k, C_{k}\right)$.

So

$$
\epsilon_{1}=\frac{C_{i}-C_{j}}{j-i}=\frac{C_{i}-C_{k}}{k-i}=\frac{C_{k}-C_{j}}{j-k} .
$$

Or

$$
C_{i}+\epsilon_{1} i=C_{k}+\epsilon_{1} k=C_{j}+\epsilon_{1} j .
$$

We now compute the D.E. for $\epsilon_{1}$.
If we substitute $\lambda(\alpha)=\lambda_{1} \alpha^{\epsilon_{1}}+\ldots$ into the equation $f(\lambda, \alpha)=0$, then the smallest power of $\alpha$ in $f(\lambda ; \alpha)$ must be $C_{i}+\dot{\epsilon}_{1} i=C_{k}+\epsilon_{1} k=C_{j}+\epsilon_{1} j$. This is because, except for the terms which have the corresponding points on Newton's diagram, the other terms in $f(\lambda, \alpha)$ are higher power terms. Suppose $\left(h, C_{h}\right)$ is a point on the segment from $\left(0, C_{0}\right)$ to $\left(i, C_{i}\right)$, and $\left(l, C_{l}\right)$ is a point on the segment from $\left(j, C_{j}\right)$ to $(q, 0)$.

So (2.9) implies that

$$
\frac{C_{h}-C_{i}}{i-h}>\epsilon_{1}>\frac{C_{j}-C_{l}}{l-j} .
$$

Or

$$
\begin{align*}
C_{i}+\epsilon_{1} i & <C_{h}+\epsilon_{1} h \\
C_{j}+\epsilon_{1} j & <C_{l}+\epsilon_{1} l \tag{2.10}
\end{align*}
$$

The geometric meaning of the values $C_{h}+\epsilon_{1} h, C_{i}+\epsilon_{1} i, C_{j}+\epsilon_{1} j$ and $C_{l}+\epsilon_{1} l$ (compared with Figure 2.3) can explain the inequalities in (2.10) too.

Hence the D.E. for $\epsilon_{1}$ is

$$
\begin{equation*}
b_{i} \lambda_{1}^{i}+b_{k} \lambda_{1}^{k}+b_{j} \lambda_{1}^{j}=0 \tag{2.11}
\end{equation*}
$$

There are $i$ zero roots of (2.11) which can be explained by the fact that among all the solutions, $i$ of them start with terms higher than $\alpha^{\epsilon_{1}}$.

There are $j-i$ nonzero roots of (2.11). If these roots are simple, then they give the first coefficients of the expansions of $j-i$ different solutions $\lambda(\alpha)$. If some roots are multiple, we continue the investigation for the second coefficients.

In this example, each determining equation has as many nonzero roots as the length of the projection of the particular segment of the diagram onto the X-axis. Together, the number of the roots of all determining equations is equal to the $\lambda$-order of $f(\lambda, \alpha)$ (counting multiplicities). By Theorem 2.1 (Existence Theorem I), we know that the total number of solution functions determined by $f(\lambda, \alpha)=0$ is equal to the $\lambda$-order of $f(\lambda, \alpha)$. And Theorem 2.2 shows that each of leading coefficients in the expansions of these solution functions is one root of a determining equation. Since the number of the nonzero roots of all the D.E.s is equal to the number of the solution functions (counting multiplicities), so there is a one to one correspondence between the nonzero roots of determining equations and the first coefficients in the
expansions of the solution functions. We can write this result as (see Section 15 of [BL]):

Theorem 2.3 The set of nonzero roots of the determining equations determines the first coefficients in the expansions of all nonidentically zero solution functions of $f(\lambda, \alpha) \equiv 0$.

Another property of the determining equations is included here in a theorem without proof (see sections 2.4 and 2.5 of [VT]).

Theorem 2.4 Suppose $\lambda_{1}$ is a simple root of the determining equation for $\epsilon_{1}=\frac{s}{r}$, $s$ and $r$ are integers and $s$ is prime to $r$. Then all the exponents in the expansion for $\lambda(\alpha)$ starting from $\bar{\lambda}_{1} \alpha^{s / r}$ are integer multiples of $1 / r$. In other words, $r$ is the common denominator of the exponents in the expansion.

The next example shows that the condition that $\lambda_{1}$ is a simple root is necessary in Theorem 2.4.

Example $2.9 f(\lambda, \alpha)=4 \alpha^{2}-\alpha^{3}+4 \alpha \lambda+\lambda^{2}=(\lambda+2 \alpha)^{2}-\alpha^{3}$. The D.E. for $\epsilon_{1}=1$ is $4+4 \lambda_{1}+\lambda_{1}^{2}=0$. So $\lambda_{1}=-2$ is a double root. It turns out that $\epsilon_{2}=\frac{3}{2}$, where the denominator 2 is greater than 1.

### 2.6 Existence Theorem II

More generally, consider a function $f(\lambda, \alpha)$ which is analytic in $\lambda$ and $\alpha, f\left(\lambda, \alpha_{0}\right) \not \equiv 0$. Let $\lambda=\lambda_{0}$ be a zero of $f\left(\lambda, \alpha_{0}\right)$ of multiplicity $m$.

Lemma 2.5 In a neighborhood of $\lambda_{0}$, the total number (counting multiplicities) of $\lambda$-zeros of $f(\lambda, \alpha)$ for $\alpha$ in a neighborhood of $\alpha_{0}$ is $m$.

Proof: Since $\lambda_{0}$ is an isolated zero of $f\left(\lambda, \alpha_{0}\right)$, there exists a deleted neighborhood $\mathcal{U}$ of $\lambda_{0}$ such that for $\lambda \in \mathcal{U}, f\left(\lambda, \alpha_{0}\right) \neq 0$, and there exists a circle $\Gamma$ in $\mathcal{U}$ around $\lambda_{0}$ such that $\min \left|f\left(\lambda, \alpha_{0}\right)\right|=\epsilon>0$. Now the continuity of function $f(\lambda, \alpha)$ on $\alpha$ means that for $\epsilon$, there exists a neighborhood $\mathcal{V}$ of $\alpha_{0}$ such that if $\alpha \in \mathcal{V},\left|f(\lambda, \alpha)-f\left(\lambda, \alpha_{0}\right)\right|<\epsilon$. Consider $f(\lambda, \alpha)$ and $f\left(\lambda, \alpha_{0}\right)$ as functions of the single variable $\lambda$. By Rouche's Theorem, we can conclude that the multiplicity $m$ of $\lambda_{0}$, which is the only zero of $f\left(\lambda, \alpha_{0}\right)$ inside $\Gamma$, is equal to the total number of $\lambda$-zeros of $f(\lambda, \alpha)$ inside $\Gamma$ for $\alpha \in \mathcal{V}$. $\diamond$

Theorem 2.6 (Existence Theorem II) Let $f(\lambda, \alpha)$ be a function depending analytically on complex variables $\lambda$ and $\alpha$. Suppose $f\left(\lambda, \alpha_{0}\right) \not \equiv 0$ and $\lambda_{0}$ is a zero of $f\left(\lambda, \alpha_{0}\right)$ of multiplicity $m$. Then for $\lambda$ in a neighborhood of $\lambda_{0}$, and $\alpha$ in a neighborhood of $\alpha_{0}$, there exist $m$ functions of the form

$$
\begin{equation*}
\lambda(\alpha)=\lambda_{0}+\lambda_{1}\left(\alpha-\alpha_{0}\right)^{\epsilon_{1}}+\lambda_{2}\left(\alpha-\alpha_{0}\right)^{\epsilon_{2}}+\ldots \tag{2.12}
\end{equation*}
$$

such that $f(\lambda(\alpha), \alpha) \equiv 0$, where $0<\epsilon_{1}<\epsilon_{2}<\ldots$ can be rational numbers, and $\lambda_{1}, \lambda_{2}, \ldots$ are all different from zero.

Proof: For $\lambda$ in a neighborhood of $\lambda_{0}$ and $\alpha$ in a neighborhood of $\alpha_{0}$ the solution of $f(\lambda(\alpha), \alpha)=0$ are continuous functions. By Lemma 2.5 the number of these functions $\lambda(\alpha)$ is $m$. For each of the $\lambda(\alpha), \alpha_{0}$ is not a pole and $\lambda\left(\alpha_{0}\right)=\lambda_{0}$. Thus by Theorem 2.2 of $[\mathrm{VT}]$, each of the functions has the form (2.12).

The underlying analysis and the connection between the Existence theorems I and II are given by the Weierstrass preparation theorem (Theorem 3.10 of [M], for example). The theorem says that in a neighborhood of $\lambda_{0}$ and a neighborhood of $\alpha_{0}$.

$$
\begin{equation*}
f(\lambda, \alpha)=f_{1}(\lambda, \alpha) g(\lambda, \alpha) \tag{2.13}
\end{equation*}
$$

where $g(\lambda, \alpha)$ is analytic at $\left(\lambda_{0}, \alpha_{0}\right), g\left(\lambda_{0}, \alpha_{0}\right) \neq 0$, and $f_{1}(\lambda, \alpha)$ is a polynomial in $\lambda$, with $\lambda$-order $m$, i.e.,

$$
f_{1}(\lambda, \alpha)=a_{0}(\alpha)+a_{1}(\alpha)\left(\lambda-\lambda_{0}\right)+\ldots+a_{m-1}(\alpha)\left(\lambda-\lambda_{0}\right)^{m-1}+\left(\lambda-\lambda_{0}\right)^{m},
$$

where $a_{i}\left(\alpha_{0}\right)=0, i=0,1, \ldots, m-1$.
By Theorem 2.1, $f_{1}(\lambda, \alpha) \equiv 0$ determines $m$ functions of the form (2.12). Since the solutions of equations $f(\lambda, \alpha) \equiv 0$ and $f_{1}(\lambda, \alpha) \equiv 0$ are identical, these $m$ functions are the solutions of $f_{1}(\lambda, \alpha) \equiv 0$.

Corollary 2.7 Let $f(\lambda, \alpha)$ be a function depending analytically on complex variables $\lambda$ and $\alpha$. Suppose $f\left(\lambda, \alpha_{0}\right) \not \equiv 0$ and $\lambda_{0}$ is a zero of $f\left(\lambda, \alpha_{0}\right)$ of multiplicity m. Let $g(\lambda)$ be a function which is analytic at $\lambda_{0}$ and $g\left(\lambda_{0}\right) \neq 0$. Then $f(\lambda, \alpha)=0$ and $g(\lambda) f(\lambda, \alpha)=0$ have the same $m$ solutions $\lambda(\alpha)$ of the form (2.12) in a neighborhood of $\alpha_{0}$.

Proof: Since $\lambda_{0}$ is a zero of $g(\lambda) f\left(\lambda, \alpha_{0}\right)$ of multiplicity $m$, so by Theorem 2.6, there are $m$ and only $m$ solutions $\lambda(\alpha)$ such that $g(\lambda) f(\lambda, \alpha)=0$. However, the $m$ solutions $\lambda(\alpha)$ of $f(\lambda, \alpha)=0$ are solutions of $g(\lambda) f(\lambda, \alpha)=0$ too. Hence the solutions for $f(\lambda, \alpha)=0$ and $g(\lambda) f(\lambda, \alpha)=0$ coincide.

### 2.7 Newton's diagram II

Although the polynomial $f_{1}(\lambda, \alpha)$ in (2.13) is uniquely determined by the Weierstrass' preparation theorem, it is not needed in explicit form. However, the $m$ solutions $\lambda(\alpha)$ can be found from Newton's diagram directly constructed by $f(\lambda, \alpha)$.

Let the Taylor expansion of $f(\lambda, \alpha)$ at $\left(\lambda_{0}, \alpha_{0}\right)$ be

$$
\begin{align*}
f(\lambda, \alpha)= & f\left(\lambda_{0}, \alpha_{0}\right)+\frac{\partial f}{\partial \lambda}\left(\lambda_{0}, \alpha_{0}\right)\left(\lambda-\lambda_{0}\right)+\frac{1}{2!} \frac{\partial^{2} f}{\partial \lambda^{2}}\left(\lambda_{0}, \alpha_{0}\right)\left(\lambda-\lambda_{0}\right)^{2}+\ldots \\
& +\frac{\partial f}{\partial \alpha}\left(\lambda_{0}, \alpha_{0}\right)\left(\alpha-\alpha_{0}\right)+\frac{1}{2!} \frac{\partial^{2} f}{\partial \lambda \partial \alpha}\left(\lambda_{0}, \alpha_{0}\right)\left(\lambda-\lambda_{0}\right)\left(\alpha-\alpha_{0}\right)+\ldots \\
& +\frac{1}{2!} \frac{\partial^{2} f}{\partial \alpha^{2}}\left(\lambda_{0}, \alpha_{0}\right)\left(\alpha-\alpha_{0}\right)^{2}+\ldots \\
& +\ldots \\
= & \sum_{j=0}^{\infty} \frac{1}{j!} \frac{\partial^{j} f}{\partial \alpha^{j}}\left(\lambda_{0}, \alpha_{0}\right)\left(\alpha-\alpha_{0}\right)^{j} \\
& +\left(\sum_{j=0}^{\infty} \frac{1}{j!} \frac{\partial^{j+1} f}{\partial \alpha^{j} \partial \lambda}\left(\lambda_{0}, \alpha_{0}\right)\left(\alpha-\alpha_{0}\right)^{j}\right)\left(\lambda-\lambda_{0}\right) \\
& +\left(\sum_{j=0}^{\infty} \frac{1}{2!} \frac{1}{j!} \frac{\partial^{j+2} f}{\partial \alpha^{j} \partial \lambda^{2}}\left(\lambda_{0}, \alpha_{0}\right)\left(\alpha-\alpha_{0}\right)^{j}\right)\left(\lambda-\lambda_{0}\right)^{2} \\
& +\ldots \tag{2.14}
\end{align*}
$$

Now let us draw a picture. If $\frac{\partial^{k+h} f}{\partial \alpha^{k} \partial \lambda^{h}}\left(\lambda_{0}, \alpha_{0}\right) \neq 0$, we can plot a point $(h, k)$ on the Cartesian plane (Figure 2.13 is an example of such a diagram). Note that since $f\left(\lambda_{0}, \alpha_{0}\right)=0$, if the point $(0, i)$ is on the graph; then $i>0$. Also, if $\lambda_{0}$ is a zero of $f\left(\lambda, \alpha_{0}\right)$ of multiplicity $m$, then ( $m, 0$ ) is on the graph. As in $\S 2.3$, we are only concerned with the point $(h, k)$ such that $k \leq k^{\prime}$ for all the points $\left(h, k^{\prime}\right)$ on the diagram with the same $h$. With $h \leq m$, these points constitute the extremal part of Newton's diagram. In other words, they are the lowest points on each vertical line before and including the line $h=m$ (the points with circles in Figure 2.13).

By the same reasoning used in $\S 2.3$, among these extremal points only the pair which have other extremal points lying above the line passing through the pair can determine an exponent in the expansion. The collection of such pairs is called the essential part of Newton's diagram (the points with crosses in Figure 2.13).


Figure 2.13: One example of Newton's diagram II

An easier way to find the essential points is to rotate the vertical axis counterclockwise around the extremal point $(0, n)$ until it meets the first extremal point. Then rotate about the lowest extremal point on it until it meets a new extremal point, and so on. All the extremal points touched by these rotating lines, including the points $(0, n)$ and $(m, 0)$, constitute the essential part (as illustrated in Figure 2.13). The sum of the terms in $f(\lambda, \alpha)$ corresponding to the extremal part is called the extremal part of the function $f(\lambda, \alpha)$. Similarly, we have the definition for the essential part of the function $f(\lambda, \alpha)$.

Remark 2.7.1 Without loss of generality, it is assumed throughout that $\alpha_{0}=0$. Note as well that the transformation $\lambda^{\prime}=\lambda-\lambda_{0}$ shifts $\lambda_{0}$ to the origin of the $\lambda$ plane, so we can also assume $\lambda_{0}=0$. But to avoid the ambiguity that zero is referred to $\lambda_{0}$ or $\alpha_{0}$, we remain the notation $\lambda_{0}$.

Remark 2.7.2 The terms in $\left(\lambda-\lambda_{0}\right)$ with higher order than $m$ in (2.14) play no part in Newton's diagram method II. In a deleted neighborhood of $\alpha_{0}$, there might be some functions with $\alpha_{0}$ as a pole. However, Newton's diagram II does not take them into consideration. Instead, Newton's diagram II only has a descending section which determines $m$ solutions $\lambda(\alpha)$ such that lim $m_{\alpha \rightarrow \alpha_{0}} \lambda(\alpha)=\lambda_{0}$. We call them small solutions (as in [N],[LN1],[LN2] and [LN3], where the descending section is called the falling part).

Theorem 2.8. Let $f_{1}(\lambda, \alpha)$ and $f_{2}(\lambda, \alpha)$ be two functions which are analytic at $\left(\lambda_{0}, 0\right)$. If the essential parts of $f_{1}(\lambda, \alpha)$ and $f_{2}(\lambda, \alpha)$ coincide, then $f_{1}(\lambda, \alpha)=0$ and $f_{2}(\lambda, \alpha)=$ 0 have the same leading terms in their small solutions.

Proof: If the essential parts of $f_{1}(\lambda, \alpha)$ and $f_{2}(\lambda, \alpha)$ coincide, then the essential parts' of 'Newton's diagrams for the two functions are the same. This gives that the exponents of the leading terms of their small solutions $\lambda(\alpha)$ are the same. Then since the terms corresponding to the essential parts of the diagrams in $f_{1}(\lambda, \alpha)$ and $f_{2}(\lambda, \alpha)$ coincide, the determining equations for two functions coincide, and they give the same solutions for the coefficients of the leading terms.

Suppose the $m$ small solutions of function $f(\lambda, \alpha)$ have the form

$$
\lambda=\lambda_{0}+\lambda_{1} \alpha^{\epsilon_{1}}+\ldots
$$

In fact, the essential part of $f(\lambda, \alpha)$ also determines the determining equation for $\lambda_{1}$, which has $m$ solutions.

## Example 2.10 Let

$$
f(\lambda, \alpha)=\lambda^{2}+\lambda^{3}+\alpha e^{\lambda}, \quad \lambda_{0}=0
$$

By Corollary 2.7, since $\frac{1}{1+\lambda}$ is analytic at 0, instead of treating $f(\lambda, \alpha)$ directly, it is the same to look for solutions for

$$
\begin{aligned}
\frac{1}{1+\lambda} f(\lambda, \alpha) & =\lambda^{2}+\alpha e^{\lambda} \frac{1}{1+\lambda} \\
& =\lambda^{2}+\alpha\left(1+\lambda+\frac{\lambda^{2}}{2!}+\ldots\right)\left(1 \dot{-} \lambda+\lambda^{2}+\ldots\right)
\end{aligned}
$$

The essential part of this function is $\lambda^{2}+\alpha$. Suppose $\lambda=\lambda_{1} \alpha^{\epsilon_{1}}+\ldots$. Hence $\epsilon_{1}=\frac{1}{2}$. the D.E. for $\lambda_{1}$ is $\lambda_{1}^{2}+1$, so $\lambda_{1}= \pm i$. $f(\lambda, \alpha)=0$ has two small solutions:

$$
\lambda(\alpha)=i \alpha^{1 / 2}+\ldots \quad \text { and } \quad \lambda(\alpha)=-i \alpha^{1 / 2}+\ldots
$$

### 2.8 Conditions for analyticity

In this section, we shall give two sufficient conditions for the solution function $\lambda(\alpha)$ of $f(\lambda, \alpha) \equiv 0$ to be analytic at the origin.

Theorem 2.9 Let $f(\lambda, \alpha)$ be analytic at $\left(\lambda_{0}, 0\right)$ and $\lambda_{0}$ be a zero of $f(\lambda, 0)$ of multiplicity $m$. Suppose the Taylor expansion of $f(\lambda, \alpha)$ at $\left(\lambda_{0}, 0\right)$ is:

$$
\begin{equation*}
f(\lambda, \alpha)=\sum_{i=1, j=0}^{\infty} b_{i j}\left(\lambda-\lambda_{0}\right)^{i} \alpha^{j}+\sum_{i=m}^{\infty} b_{i 0}\left(\lambda-\lambda_{0}\right)^{i} \tag{2.15}
\end{equation*}
$$

If the following conditions hold:

$$
\begin{aligned}
& 1 b_{01}=b_{02}=\ldots=b_{0, m-2}=b_{0, m-1}=0, b_{0 m} \neq 0 \\
& \quad b_{11}=b_{12}=\ldots=b_{1, m-2}=0 \\
& \quad \vdots \\
& b_{m-2,1}=0 \\
& \text { 2 } \lambda_{11} \text { is a simple root of the D.E. of } \lambda_{1},
\end{aligned}
$$

then there is at least one solution function determined by $f(\lambda, \alpha)=0$,

$$
\lambda(\alpha)=\lambda_{0}+\lambda_{11} \alpha+\ldots
$$

which is analytic in a neighborhood of $\alpha=0$.
Proof: By Newton's diagram, Condition 1 guarantees that $\epsilon_{1}=1$. By Theorem 2.4, Condition 2 implies that $\lambda(\alpha)=\lambda_{0}+\lambda_{11} \alpha \ldots$ is analytic in a neighborhood of $\alpha=0$. $\diamond$

Now if the condition 2 is relaxed so that $\lambda_{11}$ is a multiple root of the D.E. of $\lambda_{1}$ with multiplicity $s>1$, then there are $s$ solutions of $f(\lambda, \alpha)=0$ of the form

$$
\begin{equation*}
\lambda(\alpha)=\lambda_{0}+\lambda_{11} \alpha+\lambda_{2} \alpha^{\epsilon_{2}}+\ldots \tag{2.16}
\end{equation*}
$$

To investigate what is Newton's diagram for $\epsilon_{2}$ and what is the D.E. for $\lambda_{2}$, we use the transformation

$$
\bar{\lambda}=\lambda-\lambda_{0}-\lambda_{11} \alpha
$$

Equivalently,

$$
\begin{equation*}
\lambda-\lambda_{0}=\bar{\lambda}+\lambda_{11} \alpha \tag{2.17}
\end{equation*}
$$

Substitute (2.17) in (2.15), we have

$$
f(\lambda, \alpha)=\sum_{i=1, j=0}^{\infty} b_{i j}\left(\bar{\lambda}+\lambda_{11} \alpha\right)^{i} \alpha^{j}+\sum_{i=m}^{\infty} b_{i 0}\left(\bar{\lambda}+\lambda_{11} \alpha\right)^{i}=f(\bar{\lambda}, \alpha)
$$

If $f(\lambda, \alpha)$ also satisfies condition 1 , then

$$
\begin{align*}
f(\bar{\lambda}, \alpha)= & \left(b_{0 m} \alpha^{m}+b_{0, m+1} \alpha^{m+1}+\ldots\right) \\
& +\left(b_{1, m-1} \alpha^{m-1}+b_{1 m} \alpha^{m}+\ldots\right)\left(\bar{\lambda}+\lambda_{11} \alpha\right) \\
& +\left(b_{2, m-2} \alpha^{m-2}+b_{2, m-1} \alpha^{m-1}+\ldots\right)\left(\bar{\lambda}^{2}+2 \lambda_{11} \alpha \bar{\lambda}+\lambda_{11}^{2} \alpha^{2}\right) \\
& \ldots \\
& +\left(b_{m-1,1} \alpha+b_{m-1,2} \alpha^{2}+\ldots\right)\left(\bar{\lambda}^{m-1}+(m-1) \lambda_{11} \alpha \bar{\lambda}^{m-2}+\ldots+\lambda_{11}^{m-1} \alpha^{m-1}\right) \\
& +\left(b_{m 0} \alpha+b_{m 1} \alpha+\ldots\right)\left(\bar{\lambda}^{m}+m \lambda_{11} \alpha \bar{\lambda}^{m-1}+\ldots+\lambda_{11}^{m} \alpha^{m}\right) \\
& \ldots \\
= & \left(\overline{b_{0 m} \alpha^{m}}+\overline{b_{0, m+1}} \alpha^{m+1}+\ldots\right) \\
& +\left(\overline{b_{1, m-1}} \alpha^{m-1}+\overline{b_{1, m}} \alpha^{m}+\ldots\right) \bar{\lambda} \\
& +\left(\overline{b_{2, m-2}} \alpha^{m-2}+\overline{b_{2, m-1}} \alpha^{m-1}+\ldots\right) \bar{\lambda}^{2} \\
& \ldots  \tag{2.18}\\
& +\left(\overline{b_{s, m-s}} \alpha^{m-s}+\overline{b_{s, m-s+1}} \alpha^{m-s+1}+\ldots\right) \bar{\lambda}^{s} \\
& \ldots
\end{align*}
$$

Suppose the D.E. of $\lambda_{1}$ is

$$
\begin{aligned}
f_{0}\left(\lambda_{1}\right) & =b_{0 m}+b_{1, m-1} \lambda_{1}+b_{2, m-2} \lambda_{1}^{2}+\ldots+b_{m 0} \lambda_{1}^{m} \\
& =\left(\lambda_{1}-\lambda_{11}\right)^{s} g\left(\lambda_{1}\right)=0
\end{aligned}
$$

where $g\left(\lambda_{11}\right) \neq 0$. Then, in (2.18),

$$
\begin{aligned}
& \overline{b_{0 m}}=b_{0 m}+b_{1, m-1} \lambda_{11}+b_{2, m-2} \lambda_{11}^{2}+\ldots+\lambda_{m 0} \lambda_{11}^{m}=f_{0}\left(\lambda_{11}\right)=0 \\
& \overline{b_{1, m-1}}=b_{1, m-1}+2 b_{2, m-2} \lambda_{11}+\ldots+m b_{m 0} \lambda_{11}^{m-1}=f_{0}^{\prime}\left(\lambda_{11}\right)=0 \\
& \overline{b_{2, m-2}}=f_{0}^{\prime \prime}\left(\lambda_{11}\right)=0
\end{aligned}
$$


$s$

Figure 2.14: Newton's diagram for $\epsilon_{2}$
$\vdots$
$\overline{b_{s, m-s}}=f_{0}^{(s)}\left(\lambda_{11}\right) \neq 0$.
Define Condition 3 as:

$$
\begin{gathered}
\overline{b_{0, m+1}}=\overline{b_{0, m+2}}=\ldots=\overline{b_{0, m+s-2}}=\overline{b_{0, m+s-1}}, \overline{b_{0, m+s}} \neq 0, \\
\cdot \overline{b_{1, m}}=\overline{b_{1, m+1}}=\ldots=\overline{b_{1, m+s-3}}=0, \\
\vdots \\
\overline{b_{s-2, m-s+3}}=0
\end{gathered}
$$

If condition 3 is satisfied, then Newton's diagram for $\epsilon_{2}$ in (2.16)(Figure 2.14) shows that $\epsilon_{2}=2$. And the D.E. for $\lambda_{2}$ may be written as

$$
\overline{b_{0, m+s}}+\overline{b_{1, m+s-2}} \lambda_{2}+\ldots+\overline{b_{s-1, m-s+2}} \lambda_{2}^{s-1}+\overline{b_{s, m-s}} \lambda_{2}^{s}=0
$$

with the possibility that some of $\overline{b_{i, m+s-2 i}}, i=1, \ldots, s-1$ vanish.
Additionally define Condition 4 as: $\lambda_{21}$ is a simple root of the D.E. of $\lambda_{2}$.
Now Theorem 2.9 can be refined to give the following theorem.

Theorem 2.10 Suppose $\lambda_{11}$ is a multiple root of the D.E. of $\lambda_{1}$ with multiplicity $s>1$. If conditions 1, 3 and 4 hold, then there is at least one solution function determined by $f(\lambda, \alpha)=0$,

$$
\lambda(\alpha)=\lambda_{0}+\lambda_{11} \alpha+\lambda_{21} \alpha^{2}+\ldots
$$

which is analytic in a neighborhood of $\alpha=0$.

## Chapter 3

# Perturbation Theory for Analytic Matrix Functions: The Newton diagram method 

### 3.1 Introduction

Let $L(\lambda, \alpha)$ be a matrix function with values in the $n \times n$ complex matrices depending analytically on complex variables $\lambda$ and $\alpha$. The value $\lambda(\alpha)$ such that $\operatorname{det} L(\lambda(\alpha), \alpha)=$ 0 is called an eigenvalue function of matrix function $L(\lambda, \alpha)$. Therefore it is natural to use Newton's diagram to investigate the eigenvalue functions of analytic matrix functions. It previously appeared in the book of [B]. A significant development of this method was given by Langer, Najman and Veselic in their "remarks" ([N], [LN1], [LN2], [LN3], [LNV]). The main part of this chapter is only rewriting the basic ideas in [LN1] and [LNV], using the notations which are unified with the works of Lancaster([L3], [LMZ2]). The applications of the general theory to the semisimple case (section 3.6) and matrix polynomial (section 3.7) are new.

### 3.2 Leading terms of the eigenvalue expansions for a special simple form

In this section we examine the leading terms of the eigenvalues for a special $g \times g$ matrix function

$$
L(\lambda, \alpha)=D_{1}(\lambda)+\alpha H
$$

where $D_{1}(\lambda)=\operatorname{diag}\left(\left(\lambda-\lambda_{0}\right)^{m_{1}},\left(\lambda-\lambda_{0}\right)^{m_{2}}, \ldots,\left(\lambda-\lambda_{0}\right)^{m_{g}}\right), 0<m_{1} \leq m_{2} \leq \ldots \leq m_{g}$ and $H$ is a constant matrix $H=\left(h_{i j}\right)$.

Without loss of generality, we need to discuss this special form only. This will be seen in the sequel.

We need to introduce some notations which basically are inherited from [LN1]. We divide $m_{1}, \ldots, m_{g}$ into $k$ groups of mutually equal $m_{i}$, the $j$ th group containing $n_{j}$ elements:

$$
m_{1}=\ldots=m_{n_{1}}<m_{n_{1}+1}=\ldots=m_{n_{1}+n_{2}}<\ldots<m_{n_{1}+\ldots+n_{k-1}+1}=\ldots=m_{g}
$$

and define $\tilde{n}_{j}=n_{1}+\ldots+n_{j}, j=1, \ldots, k$. So obviously $\tilde{n}_{k}=g$, and $0<m_{\tilde{n}_{1}}<m_{\tilde{n}_{2}}<$ $\ldots<m_{\tilde{n}_{k}}$ are $k$ different values of $m_{i}$. And define $k$-values $e_{j}=n_{1} m_{\tilde{n}_{1}}+\ldots+n_{j} m_{\tilde{n}_{j}}$, $j=1, \ldots, g$. So $e_{g}=m$, where $m:=m_{1}+\ldots+m_{g}$.

Also, if $1 \leq i_{1}<i_{2}<\ldots<i_{j} \leq g$, we denote by $H\left(i_{1}, \ldots, i_{j}\right)$ the determinant of the matrix obtained from $H$ by deleting the rows and columns with indices $i_{1}, \ldots, i_{j}$. So $H\left(i_{1}, \ldots, i_{j}\right)$ is a minor of $H$ of order $g-j$.

As preparation, we need a lemma.

Lemma 3.1, For a $g \times g$ diagonal matrix $D=\operatorname{diag}\left(d_{1}, d_{2}, \ldots, d_{g}\right)$ and $a g \times g$ matrix $H=\left(h_{i j}\right)$,

$$
\begin{align*}
\operatorname{det}(D+H)= & \operatorname{det} H+\sum_{i=1}^{g} d_{i} H(i)+\sum_{1 \leq i_{1}<i_{2} \leq g} d_{i_{1}} d_{i_{2}} H\left(i_{1}, i_{2}\right) \\
& +\ldots+\sum_{1 \leq i_{1}<\ldots<i_{g-1} \leq g} d_{i_{1}} \ldots d_{i_{g-1}} H\left(i_{1}, \ldots, i_{g-1}\right) \\
& +d_{1} \ldots d_{g} \tag{3.1}
\end{align*}
$$

Proof: It can be proved by induction, only the description is tedious.

Another point of view is to regard $\operatorname{det}(D+H)$ as a polynomial of variables $d_{1}, \ldots, d_{g}$, by determining the coefficients of each term of the polynomial, (3.1) can be proved. For example, what is the coefficient of $d_{i_{1}} d_{i_{2}}$ ? By the definition of the determinant, obviously it is $H\left(i_{1}, i_{2}\right)$. Giving all the possible terms of the polynomial $\operatorname{det}(D+H),(3.1)$ is proved.

Before we come to the general result, it may be convenient to specialize more to an example.

## Example 3.1 Suppose

$$
D_{1}(\lambda)=\operatorname{diag}\left(\left(\lambda-\lambda_{0}\right),\left(\lambda-\lambda_{0}\right),\left(\lambda-\lambda_{0}\right)^{2},\left(\lambda-\lambda_{0}\right)^{2},\left(\lambda-\lambda_{0}\right)^{2},\left(\lambda-\lambda_{0}\right)^{3}\right)
$$

So $m_{1}=m_{2}=1, m_{3}=m_{4}=m_{5}=2, m_{6}=3$. We have three groups of $m_{i}$, so $k=3$. And $g=6, m=\sum_{i=1}^{6} m_{i}=11$. Also, $e_{1}=m_{1}+m_{2}=2, e_{2}=\sum_{i=1}^{5} m_{i}=8$, and $e_{3}=m=11$.

By Lemma 3.1,

$$
\begin{aligned}
\operatorname{det}\left(D_{1}(\lambda)+\alpha H\right)= & \alpha^{6} \operatorname{det} H+\alpha^{5}\left(\lambda-\lambda_{0}\right)(H(1)+H(2)) \\
& +\alpha^{5}\left(\lambda-\lambda_{0}\right)^{2}(H(3)+H(4)+H(5))+\alpha^{5}\left(\lambda-\lambda_{0}\right)^{3} H(6) \\
& +\alpha^{4}\left(\lambda-\lambda_{0}\right)^{2} H(1,2)+\alpha^{4}(\ldots) \\
& +\alpha^{3}\left(\lambda-\lambda_{0}\right)^{4}(H(1,2,3)+H(1,2,4)+H(1,2,5))+\alpha^{3}(\ldots) \\
& +\alpha^{2}\left(\lambda-\lambda_{0}\right)^{6}(H(1,2,3,4)+H(1,2,3,5)+H(1,2,4,5))+\alpha^{2}(\ldots) \\
& +\alpha\left(\lambda-\lambda_{0}\right)^{8}\left(H(1,2,3,4,5)+\alpha(\ldots)+\left(\lambda-\lambda_{0}\right)^{11} .\right.
\end{aligned}
$$

In the equation above, the brackets with the dots denote the terms with the exponents of $\left(\lambda-\lambda_{0}\right)$ higher than the terms ahead. So they play no part in Newton's


Figure 3.1: Newton's diagram for Example 3.1
diagram. If the coefficients of the terms in the essential part of $\operatorname{det}\left(D_{1}(\lambda)+\alpha H\right) d o$ not vanish, namely, if $\operatorname{det} H \neq 0, H(1)+H(2) \neq 0, H(1,2) \neq 0, \sum_{i=3}^{5} H(1,2, i) \neq 0$, $\sum_{3 \leq i_{1}<i_{2} \leq 5} H\left(1,2, i_{1}, i_{2}\right) \neq 0$, and $H(1,2,3,4,5) \neq 0$, then Newton's diagram of $\operatorname{det}(D(\lambda)+\alpha H)$ takes the form of Figure 3.1.

Generally, consider $D_{1}(\lambda)=\operatorname{diag}\left(\left(\lambda-\lambda_{0}\right)^{m_{1}},\left(\lambda-\lambda_{0}\right)^{m_{2}}, \ldots,\left(\lambda-\lambda_{0}\right)^{m_{g}}\right)$. Since $m_{i}$ are ordered from small to big, on Newton's diagram of $\operatorname{det}\left(D_{1}(\lambda)+\alpha H\right)$, the turning points (including the end points) correspond to the terms of $\alpha^{g}, \alpha^{g-\tilde{n}_{1}}\left(\lambda-\lambda_{0}\right)^{e_{1}}$, $\alpha^{g-\tilde{n}_{2}}\left(\lambda-\lambda_{0}\right)^{e_{2}}, \ldots,\left(\lambda-\lambda_{0}\right)^{m}$. And the coefficients of these terms are:

$$
\operatorname{det} H, H\left(1, \ldots, \tilde{n}_{1}\right), H\left(1, \ldots, \tilde{n}_{2}\right), \ldots, 1
$$

Define $\Delta_{0}=\operatorname{det} H, \Delta_{j}=H\left(1, \ldots, \tilde{n}_{j}\right), j=1, \ldots, k-1$. So $\Delta_{j}$ is the coefficient of the term $\alpha^{g-\tilde{n}_{j}}\left(\lambda-\lambda_{0}\right)^{e_{j}}$. Note that the coefficient of $\left(\lambda-\lambda_{0}\right)^{m}$ is always 1 , so if all
of $\Delta_{j}$ are not equal to zero, i.e.,

$$
\begin{equation*}
\Delta_{0} \Delta_{1} \ldots \Delta_{k-1} \neq 0 \tag{3.2}
\end{equation*}
$$

then on Newton's diagram of $\operatorname{det}\left(D_{1}(\lambda)+\alpha H\right)$ there are $k$, the number of the groups of $m_{i}$, segments. Each segment has the slope of $-\frac{1}{m_{\bar{n}_{j}}}$. Especially, if $\Delta_{j-1} \Delta_{j} \neq 0$, there is one segment with slope of $-\frac{1}{m_{\bar{n}_{j}}}$ on the diagram, that means there are $n_{j} m_{\tilde{n}_{j}}$ eigenvalues starting from:

$$
\begin{equation*}
\lambda^{(\theta)}(\alpha)=\lambda_{0}+\lambda_{1 \theta} \alpha^{1 / m_{\tilde{n}_{j}}}+\ldots, \quad \theta=1, \ldots, n_{j} m_{\tilde{n}_{j}} \tag{3.3}
\end{equation*}
$$

And $\lambda_{1 \theta}, \theta=1, \ldots, n_{j} m_{\tilde{n}_{j}}$ are $n_{j} m_{\tilde{n}_{j}}$ roots of the D.E.:

$$
\begin{align*}
\Delta_{j-1} & +\lambda_{1} \sum_{\tilde{n}_{j=1}<i<\tilde{n}_{j}} H\left(1, \ldots, \tilde{n}_{j-1}, i\right)+\lambda_{1}^{m_{\tilde{n}_{j}}} \sum_{\tilde{n}_{j-1}<i_{1}<i_{2}<\tilde{n}_{j}} H\left(1, \ldots, \tilde{n}_{j-1}, i_{1}, i_{2}\right) \\
& +\ldots+\lambda_{1}^{n_{j} m_{\tilde{n}_{j}}} \Delta_{j}=0 \tag{3.4}
\end{align*}
$$

### 3.3 Local Smith normal form

The first part of this section is quoted from [L3]. Let $A(\lambda)$ be an $n \times n$ analytic matrix-valued function of a complex variable $\lambda$. Suppose $\operatorname{det} A(\lambda)$ does not vanish identically. Then a value $\lambda_{0}$ such that $\operatorname{det} A\left(\lambda_{0}\right)=0$ is defined as an eigenvalue of matrix function $A(\lambda)$. There are two important multiplicities associated with $\lambda_{0}$. First, the algebraic multiplicity, denoted by $m$, is the multiplicity of $\lambda_{0}$ as a zero of the characteristic function $\operatorname{det} A(\lambda)$. Then the geometric multiplicity of the eigenvalue $\lambda_{0}$ is the dimension of the null space, or kernel $\mathcal{K}$ of the matrix $A\left(\lambda_{0}\right)$ and is denoted by $g=\operatorname{dim} \mathcal{K}$. When $m=g=1, \lambda_{0}$ is said to be a simple eigenvalue of $A(\lambda)$. In general, $m \geq g$ and, to see that inequality might hold, just consider $n=1$
and $A(\lambda)=\left(\lambda-\lambda_{0}\right)^{p}$, for integer $p>1$. Here $\bar{m}=p$ and $g=1$. When $m=g$, the eigenvalue $\lambda_{0}$ of $A(\lambda)$ is said to be semisimple. Obviously, a simple eigenvalue is also semisimple.

The main features of the local Smith normal form are quoted in the next theorem without proof. (See chapter 1 of [BGR])

Theorem 3.2 Let $A(\lambda)$ be a matrix function analytic at an eigenvalue $\lambda_{0}$ and $\operatorname{det} A(\lambda) \not \equiv$ 0 . Then there are matrix functions $E(\lambda), F(\lambda)$ which are analytic and nonsingular in a neighborhood of $\lambda=\lambda_{0}$, and such that

$$
F(\lambda) A(\lambda) E(\lambda)=\left[\begin{array}{llllll}
\left(\lambda-\lambda_{0}\right)^{m_{1}} & & & &  \tag{3.5}\\
& \ddots & & \\
& & \left(\lambda-\lambda_{0}\right)^{m_{g}} & & \\
& & & 1 & \\
& & & & \ddots & \\
& & & & & 1
\end{array}\right]=D(\lambda)
$$

where $0<m_{1} \leq m_{2} \leq \ldots \leq m_{g}$ are integers uniquely associated to $\lambda_{0}$ and $A(\lambda)$.

In Theorem 3.2 $D(\lambda)$ is called the local Smith normal form of $A(\lambda)$ at $\lambda_{0}$. The positive integers $m_{1}, \ldots, m_{g}$ are known as the partial multiplicities of $\lambda_{0}$ as an eigenvalue of $A(\lambda)$. Taking determinants it is seen immediately that $\sum_{j=1}^{g} m_{j}=m$. When $\lambda_{0}$ is semisimple, $m_{1}=\ldots=m_{g}=1$.

Equation (3.5) can be used to generate a basis for $\mathcal{K}$, the kernel of the matrix $A\left(\lambda_{0}\right)$. The vectors in $\mathcal{K}$ are called eigenvectors of $A(\lambda)$ corresponding to eigenvalue $\lambda_{0}$. Let $u_{j}$ be the $j$ th unit coordinate vector and, for $j=1,2, \ldots, g$, denote $e_{j}=$ $E\left(\lambda_{0}\right) u_{j}$, i.e., the $j$ th column of $E\left(\lambda_{0}\right)$. Then $\left\{e_{1}, \ldots, e_{g}\right\}$ form the required. basis.

This is because $\operatorname{det} F\left(\lambda_{0}\right) \neq 0$, so $A\left(\lambda_{0}\right) E\left(\lambda_{0}\right)=F\left(\lambda_{0}\right)^{-1} D\left(\lambda_{0}\right)$, thus $A\left(\lambda_{0}\right) e_{j}=0$, $e_{j} \in \mathcal{K}, j=1, \ldots, g$. On the other hand $\operatorname{det} E\left(\lambda_{0}\right) \neq 0$, so $e_{1}, \ldots, e_{g}$ are linearly independent. Also $g=\operatorname{dim} \mathcal{K}$. Hence, $E\left(\lambda_{0}\right)$ can be used to construct a basis of the eigenvectors corresponding to $\lambda_{0}$. Similarly, after taking conjugate transposes in (3.5), it is found that vectors $f_{j}:=F\left(\lambda_{0}\right)^{*} u_{j}, j=1,2, \ldots, g$, i.e., the first $g$ rows of the conjugate of $F\left(\lambda_{0}\right)$, form a basis of eigenvectors for $\mathcal{K}^{\prime}:=\operatorname{ker}\left(A\left(\lambda_{0}\right)^{*}\right)$.

There is a practical method to calculate the local Smith normal form $D(\lambda)$ and the associated $E(\lambda)$ and $F(\lambda)$, namely, by means of equivalence transformations. This is a natural extension of the Gauss reduction process for a constant matrix. First we define the elementary row and column operations on a matrix function which is analytic at $\lambda_{0}$. They are:

1. Multiply any row (column) by a function $a(\lambda)$ which is analytic at $\lambda_{0}$ and $a\left(\lambda_{0}\right) \neq 0$.
2. Interchange any two rows (columns)
3. Add to any row(column) any other row.(column) multiplied by a function $b(\lambda)$ which is analytic at $\lambda_{0}$.

Then a finite sequence of elementary operations is called an equivalence transformation. It is easy to verify that performing an elementary row operation is equivalent to premultiplication of an appropriate matrix and that postmultiplication by an appropriate matrix produces an elementary column operation.

Now we use an example to illustrate the process of obtaining the local Smith normal form by means of equivalence transformation.

Example 3.2 Let

$$
A(\lambda)=\left[\begin{array}{cc}
2 \lambda^{2} & 2 \lambda+1 \\
2 \lambda-1 & 2 \lambda^{2}
\end{array}\right]
$$

Since $\operatorname{det} A(\lambda)=\left(2 \lambda^{2}-1\right)^{2}=4(\lambda+1 / \sqrt{2})^{2}(\lambda-1 / \sqrt{2})^{2}, \lambda_{0}=1 / \sqrt{2}$ and $\lambda_{0}=-1 / \sqrt{2}$ are two eigenvalues of $A(\lambda)$. We are going to show how to obtain the local Smith normal form at $\lambda_{0}=1 / \sqrt{2}$ below. An operation written above the arrow presents an elementary row operation, while an operation written below the arrow presents an elementary column operation.

$$
\begin{aligned}
& A(\alpha) \xrightarrow{\frac{1}{\lambda} \times \mathbf{R 1}}\left[\begin{array}{cc}
2 \lambda & 2+\frac{1}{\lambda} \\
2 \lambda-1 & 2 \lambda^{2}
\end{array}\right] \xrightarrow{-1 \times \mathbf{R} 2+\mathbf{R 1}}\left[\begin{array}{cc}
1 & 2+\frac{1}{\lambda}-2 \lambda^{2} \\
2 \lambda-1 & 2 \lambda^{2}
\end{array}\right] \\
& \xrightarrow{(1-2 \lambda) \times \mathbf{R} 1+\mathbf{R} 2}\left[\begin{array}{cc}
1 & 2+\frac{1}{\lambda}-2 \lambda^{2} \\
0 & 2 \lambda^{2}+(1-2 \lambda)\left(2+\frac{1}{\lambda}-2 \lambda^{2}\right)
\end{array}\right] \\
& \xrightarrow[-\left(2+1 / \lambda-2 \lambda^{2}\right) \times \mathbf{C} 1+\mathbf{C} \mathbf{2}]{\longrightarrow}\left[\begin{array}{cc}
1 & 0 \\
0 & 2 \lambda^{2}+(1-2 \lambda)\left(2+\frac{1}{\lambda}-2 \lambda^{2}\right)
\end{array}\right] \\
& \xrightarrow[{\frac{\lambda}{2(\sqrt{2} \lambda+1)^{2}} \times \mathbf{C}} 2]{ }\left[\begin{array}{cc}
1 & 0 \\
0 & \left(\lambda-\frac{1}{\sqrt{2}}\right)^{2}
\end{array}\right] \xrightarrow[\mathbf{C} 1 \leftrightarrows \mathbf{C} 2]{ }\left[\begin{array}{cc}
0 & \left(\lambda-\frac{1}{\sqrt{2}}\right)^{2} \\
1 & 0
\end{array}\right] \\
& \xrightarrow[\mathbf{C} 1 \leftrightarrows \mathbf{C} 2]{ }\left[\begin{array}{cc}
\left(\lambda-\frac{1}{\sqrt{2}}\right)^{2} & 0 \\
0 & 1
\end{array}\right]=D(\lambda)
\end{aligned}
$$

$E(\lambda)$ and $F(\lambda)$ are the products of the sequence of matrices corresponding to the
row and column operations respectively.

$$
\begin{aligned}
F(\lambda) & =\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]\left[\begin{array}{cc}
1 & 0 \\
1-2 \lambda & 1
\end{array}\right]\left[\begin{array}{cc}
1 & -1 \\
0 & 1
\end{array}\right]\left[\begin{array}{cc}
1 / \lambda & 0 \\
0 & 1
\end{array}\right] \\
& =\left[\begin{array}{cc}
\frac{1-2 \lambda}{\lambda} & 2 \lambda \\
\frac{1}{\lambda} & -1
\end{array}\right] \\
E(\lambda) & =\left[\begin{array}{ll}
1 & -\left(2+\frac{1}{\lambda}-2 \lambda^{2}\right) \\
0 & 1
\end{array}\right]\left[\begin{array}{ll}
1 & 0 \\
0 & \frac{\lambda}{2(\sqrt{2} \lambda+1)^{2}}
\end{array}\right]\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right] \\
& =\left[\begin{array}{ll}
\frac{-2 \lambda-1+2 \lambda^{3}}{2(\sqrt{2} \lambda+1)^{2}} & 1 \\
\frac{\lambda}{2(\sqrt{2} \lambda+1)^{2}} & 0
\end{array}\right]
\end{aligned}
$$

And the first column of $E\left(\frac{1}{\sqrt{2}}\right),\left[\begin{array}{c}\frac{1}{8 \sqrt{2}}(-1-\sqrt{2}) \\ \frac{1}{8 \sqrt{2}}\end{array}\right]$, is an eigenvector of $A(\lambda)$ at $\lambda_{0}=\frac{1}{\sqrt{2}}$. The first row of $F\left(\frac{1}{\sqrt{2}}\right),\left[\begin{array}{c}\sqrt{2}-2 \\ \sqrt{2}\end{array}\right]$, is an eigenvector of $A\left(\frac{1}{\sqrt{2}}\right)^{*}$.

### 3.4 The essential parts

Let the analytic matrix function $L(\lambda, \alpha)$ have the Taylor expansion at $\left(\lambda_{0}, 0\right)$ :

$$
\begin{align*}
L(\lambda, \alpha)= & L\left(\lambda_{0}, 0\right)+\frac{\partial L}{\partial \lambda}\left(\lambda_{0}, 0\right)\left(\lambda-\lambda_{0}\right)+\frac{1}{2!} \frac{\partial^{2} L}{\partial^{2} \lambda}\left(\lambda_{0}, 0\right)\left(\lambda-\lambda_{0}\right)^{2}+\ldots \\
\cdots & +\frac{\partial L}{\partial \alpha}\left(\lambda_{0}, 0\right) \alpha+\frac{1}{2!} \frac{\partial^{2} L}{\partial \lambda \partial \alpha}\left(\lambda_{0}, 0\right)\left(\lambda-\lambda_{0}\right) \alpha+\ldots \\
& +\frac{1}{2!} \frac{\partial^{2} L}{\partial \alpha^{2}}\left(\lambda_{0}, 0\right) \alpha^{2}+\ldots \\
& +\ldots  \tag{3.6}\\
= & \sum_{i, j=0}^{\infty}\left(\lambda-\lambda_{0}\right)^{i} \alpha^{j} L_{i j} \tag{3.7}
\end{align*}
$$

where $L_{i j}=\frac{I}{(i+j)!} \frac{\partial^{(i+j)} L}{\partial \lambda^{i} \partial \alpha^{j}}\left(\lambda_{0}, 0\right)$. Note that in (3.6) the first line is the expansion for $L(\lambda, 0)$ and the first column is the expansion for $L\left(\lambda_{0}, \alpha\right)$. Let $\lambda_{0}$ be an eigenvalue of $L(\lambda, 0)$, i.e., $\operatorname{det}\left(L\left(\lambda_{0}, 0\right)\right)=0$. We write $\mathcal{K}=\operatorname{ker} L_{0}$ and $\mathcal{K}^{\prime}=\operatorname{ker}\left(L_{0}^{*}\right)$, where $L_{0}:=L_{00}$. Suppose the geometric multiplicity of $\lambda_{0}$ as the eigenvalue of $L(\lambda, 0)$ is $g$, i.e., $\operatorname{dim} \mathcal{K}=g$. Of course $\operatorname{rank} L_{0}=\operatorname{rank} L_{0}^{*}$, so $\operatorname{dim} \mathcal{K}^{\prime}=g$.

Another useful notation is: for an $n \times n$ matrix $M$, if we choose bases $\left\{e_{k}\right\}_{1}^{g}$ for $\mathcal{K}$ and $\left\{f_{j}\right\}_{1}^{g}$ for $\mathcal{K}^{\prime}$, then we can define a matrix

$$
\begin{equation*}
[M]_{\mathcal{K}, \mathcal{K}^{\prime}}:=\left[f_{1}, \ldots, f_{g}\right]^{*} M\left[e_{1}, \ldots, e_{g}\right] \tag{3.8}
\end{equation*}
$$

where $\left[e_{1}, \ldots, e_{g}\right]$ and $\left[f_{1}, \ldots, f_{g}\right]$ are $n \times g$ matrices constructed by column vectors $e_{1}, \ldots, e_{g}$ and $f_{1}, \ldots, f_{g}$ respectively. Obviously, this matrix depends on the choice of basis vectors, if we choose a different pair of bases $B_{2}=\left\{e_{1}^{\prime}, \ldots, e_{g}^{\prime}\right\} \&\left\{f_{1}^{\prime}, \ldots, f_{g}^{\prime}\right\}$ in $\mathcal{K}$ and $\mathcal{K}^{\prime}$ respectively and write $[M]_{\mathcal{K}, \mathcal{K}^{\prime}}^{B_{2}}=\left[f_{1}^{\prime}, \ldots ; f_{g}^{\prime}\right]^{*} M\left[e_{1}^{\prime}, \ldots, e_{g}^{\prime}\right]$, then there are nonsingular $g \times g$ matrices $U$ and $V$ such that $[M]_{\mathcal{K}, \mathcal{K}^{\prime}}=U[M]_{\mathcal{K}, \mathcal{K}^{\prime}}^{B_{2}} V$. But this difference is not important in the sequel and is suppressed in the notation.

As defined in section 2.7, the essential part of function $\operatorname{det} L(\lambda, \alpha)$ is needed to determine the leading terms of the eigenvalue functions of $L(\lambda, \alpha)$. In this section, it turns out that under some condition the essential part of $\operatorname{det} L(\lambda, \alpha)$ is the same as the essential part of $\operatorname{det}\left(D_{1}(\lambda)+\alpha H\right)$, the special simple form in Section 3.2.

Suppose $L(\lambda, 0)$ has local Smith normal form

$$
D(\lambda)=\left[\begin{array}{cccc}
\left(\lambda-\lambda_{0}\right)^{m_{1}} & & &  \tag{3.9}\\
& \ddots & & \\
& & \left(\lambda-\lambda_{0}\right)^{m_{g}} & \\
& & & \\
& & & I_{n-g}
\end{array}\right]=\left[\begin{array}{cc}
D_{1}(\lambda) & 0 \\
0 & I_{n-g}
\end{array}\right]
$$

By Theorem 3.2, there are $n \times n$ matrix functions $E(\lambda), F(\lambda)$ which are analytic and invertible near $\lambda_{0}$ such that

$$
F(\lambda) L(\lambda, 0) E(\lambda)=D(\lambda) .
$$

Also, the first $g$ columns of $E\left(\lambda_{0}\right)$ and $F\left(\lambda_{0}\right)^{*}$ form bases for $\mathcal{K}$ and $\mathcal{K}^{\prime}$, respectively.
Proposition 3.3 The solution functions $\lambda(\alpha)$ for $\operatorname{det} L(\lambda, \alpha)=0$ and

$$
\operatorname{det}\left(D(\lambda)+F(\lambda)\left(\sum_{i=0, j=1}^{\infty}\left(\lambda-\lambda_{0}\right)^{i} \alpha^{j} L_{i j}\right) E(\lambda)\right)=0
$$

are the same.

Proof: Since $E(\lambda)$ and $F(\lambda)$ are analytic at $\lambda_{0}$ and $\operatorname{det} E\left(\lambda_{0}\right) \neq 0, \operatorname{det} F\left(\lambda_{0}\right) \neq 0$, by Corollary 2.7, the result follows.

Now we write the Taylor expansion again at $\left(\lambda_{0}, 0\right)$ for

$$
\bar{L}(\lambda, \alpha):=D(\lambda)+F(\lambda)\left(\sum_{i=0, j=1}^{\infty}\left(\lambda-\lambda_{0}\right)^{i} \alpha^{j} L_{i j}\right) E(\lambda) .
$$

And partition and denote it as:

$$
\begin{align*}
\bar{L}(\lambda, \alpha) & =D(\lambda)+\sum_{i=0, j=1}^{\infty}\left(\lambda-\lambda_{0}\right)^{i} \alpha^{j} \bar{L}_{i j} \\
& =\left[\begin{array}{cc}
D_{1}(\lambda) & 0 \\
0 & I_{n-g}
\end{array}\right]+\alpha\left[\begin{array}{ll}
G_{11}(\lambda, \alpha) & G_{12}(\lambda, \alpha) \\
G_{21}(\lambda, \alpha) & G_{22}(\lambda, \alpha)
\end{array}\right] . \tag{3.10}
\end{align*}
$$

Note that $\bar{L}_{01}=F\left(\lambda_{0}\right) L_{01} E\left(\lambda_{0}\right)$ and the leading term of $G_{11}(\lambda, \alpha)$ is $\left[L_{01}\right]_{\mathcal{K}, \mathcal{K}^{\prime}}$.
Define $H=\left[L_{01}\right]_{\mathcal{K}, \mathcal{K}^{\prime}}$. It plays an important role in our discussion, as in [LN3], it is called the perturbation matrix. Divide the partial multiplicities of eigenvalue $\lambda_{0}$ of $L(\lambda, 0)$ into $k$ groups of mutually equal ones. Then we can make the definitions of $\Delta_{j}, j=0, \ldots, k-1$, for $H$ as in Section 3.2.

Proposition 3.4 If $\Delta_{0} \Delta_{1} \ldots \Delta_{k-1} \neq 0$ (condition (3.2)), then the essential parts of functions $\operatorname{det} \bar{L}(\lambda, \alpha)$ and $\operatorname{det}\left(D_{1}(\lambda)+\alpha H\right)$ are the same.

Proof: Since

$$
\begin{align*}
\operatorname{det} \bar{L}(\lambda, \alpha) & =\operatorname{det}\left(\left[\begin{array}{cc}
D_{1}(\lambda) & 0 \\
0 & I_{n-g}
\end{array}\right]+\alpha\left[\begin{array}{ll}
G_{11} & G_{12} \\
G_{21} & G_{22}
\end{array}\right]\right) \\
& =\operatorname{det}\left(\left[\begin{array}{cc}
D_{1}(\lambda)+\alpha G_{11} & 0 \\
\alpha G_{21} & I_{n-g}
\end{array}\right]+\left[\begin{array}{cc}
D_{1}(\lambda)+\alpha G_{11} & \alpha G_{12} \\
\alpha G_{21} & \alpha G_{22}
\end{array}\right]\right) \\
& =\operatorname{det}\left(D_{1}(\lambda)+\alpha G_{11}\right)+o\left(\operatorname{det}\left(D_{1}(\lambda)+\alpha G_{11}\right)\right) \\
& =\operatorname{det}\left(D_{1}(\lambda)+\alpha H\right)+o\left(\operatorname{det}\left(D_{1}(\lambda)+\alpha H\right)\right) \tag{3.11}
\end{align*}
$$

as $\alpha \rightarrow 0$.
If condition $\Delta_{0} \Delta_{1} \ldots \Delta_{k-1} \neq 0$ is satisfied, then the Newton's diagram of $\operatorname{det} \bar{L}(\lambda, \alpha)$ is constructed by $k$ segments with the end points $(0, g),\left(n_{1} m_{\tilde{n}_{1}}, g-\tilde{n}_{1}\right), \ldots,\left(n_{1} m_{\tilde{n}_{1}}+\right.$ $\left.\ldots+n_{j} m_{\tilde{n}_{j}}, g-\tilde{n}_{j}\right), \ldots,(m, 0)$. And $o\left(\operatorname{det}\left(D_{1}(\lambda)+\alpha H\right)\right.$ is the part standing above these segments on Newton's diagram.

The condition(3.2): $\Delta_{0} \Delta_{1} \ldots \Delta_{k-1} \neq 0$ is critical for us to reduce the general question to the form of $D_{1}(\lambda)+\alpha H$. The necessity is illustrated by the next example.

## Example 3.3 Consider

$$
L(\lambda, \alpha)=\left[\begin{array}{ccc}
\left(\lambda-\lambda_{0}\right) & & \alpha \\
& \left(\lambda-\lambda_{0}\right)^{2} & \alpha \\
\alpha & \alpha & 1+\alpha
\end{array}\right] .
$$

In this extreme case $H=0$, condition (3.2) is not satisfied. If $H \neq 0$ and condition (3.2) is satisfied, for $D_{1}(\dot{\lambda})=\left[\begin{array}{cc}\left(\lambda-\lambda_{0}\right) & \\ & \left(\lambda-\lambda_{0}\right)^{2}\end{array}\right]$, we expect that Newton's


Figure 3.2: Newton's diagram for Example 3.3
diagram is constructed by $(0,2),(1,1)$ and $(3,0)$ (shown in Figure 3:2 by a thin line). But it turns out that the real'Newton's diagram of $\operatorname{det} L(\lambda, \alpha)$ consists of points $(1,2)$ and $(3,0)$ (shown in Figure 3.2 by a thick line).

### 3.5 Splitting properties

Let $\lambda_{0}$ be an eigenvalue of the unperturbed matrix function $L(\lambda, 0)$ with partial multiplicities $m_{1}, m_{2}, \ldots, m_{g}$. Existence Theorem II in Chapter 2 guarantees that there are numbers $\epsilon>0$ and $\delta>0$ such that, for $|\alpha|<\epsilon$, the spectrum of $L(\lambda, \alpha)$ in $\left|\lambda-\lambda_{0}\right|<\delta$ consists of $m=\sum_{i=1}^{g} m_{i}$ eigenvalues $\lambda(\alpha)$ which can be represented by branches of several Puiseux series

$$
\begin{equation*}
\lambda_{\nu}(\alpha)=\lambda_{0}+\sum_{k=1}^{\infty} C_{\nu k} \alpha^{k / q_{\nu}} \tag{3.12}
\end{equation*}
$$

$\nu=1,2, \ldots, r, q_{\nu}$ are positive integers. The function $\lambda_{\nu}(\alpha)$ is an algebraic $q_{\nu}$-valued function on a cut neighborhood of $\alpha=0$ (say $|\alpha|<\epsilon,-\pi<\arg \alpha \neq \pi)$. It determines
$q_{\nu}$ values of $\lambda_{\nu}^{[j]}(\alpha)$ and $\alpha^{1 / q_{\nu}}$, namely

$$
|\alpha|^{1 / q_{\nu}} \exp \left(i(\arg \alpha+2 \pi j) / q_{\nu}\right), \quad j=0,1, \ldots, q_{\nu}-1
$$

To be consistent with section 2.2, here $r$ is the number of the groups of branches, for each group of $\nu$, there are $q_{\nu}$ branches of algebraic functions, and $q_{\nu}$ is also the common denominator for the exponents in (3.12) (compare with equation (2.4)). In general, the only connection between the numbers $\left\{m_{i}\right\}_{i=1}^{g}$ and $\left\{q_{\nu}\right\}_{\nu=1}^{r}$ is the equality

$$
\begin{equation*}
\sum_{i=1}^{g} m_{i}=\sum_{\nu=1}^{r} q_{\nu} \tag{3.13}
\end{equation*}
$$

As defined by Langer, Najman and Veselić (see [LNV]), an eigenvalue $\lambda_{0}$ of $L(\dot{\lambda}, 0)$ with partial multiplicities $m_{1}, m_{2}, \ldots, m_{g}$ has the regular splitting property (or the RS property), if, for each $m_{i}$ there emerge from $\lambda_{0}$ (in the complex plane) $m_{i}$ eigenvalues $\lambda_{i}^{[j]}(\alpha)$ with Puiseux expansions for which

$$
\begin{equation*}
\lambda_{i}^{[j]}(\alpha)=\lambda_{0}+\lambda_{1, i j} \alpha^{1 / m_{i}}+o\left(|\alpha|^{1 / m_{i}}\right) \tag{3.14}
\end{equation*}
$$

holds as $\alpha \longrightarrow 0, i=1,2, \ldots, g, j=1,2, \ldots, m_{i}$, and $\lambda_{1, i j} \neq 0$ whenever $m_{i}>1$. If, in addition, $\lambda_{1, i j} \neq 0$ for all $i, j$ in (3.14), then there is a complete regular splitting at $\lambda_{0}$, we say that $\lambda_{0}$ has the CRS property.

Returning to the Puiseux series (3.12), if we have the RS property and the CRS property, in addition to the equality in (3.13), we can also conclude that $r=g$ and $m_{i}=q_{i}$, for $i=1, \ldots, r$. The only difference between the RS and the CRS properties is that for $m_{i}=1$, the RS property admits the possibility that the coefficient of the leading term, $\lambda_{1, i j}$ in (3.14), is equal to zero.

As an example, we think of a scalar function $f(\lambda, \alpha)$. Since $n=g=1$, the only partial multiplicity is the multiplicity $m$ of $\lambda_{0}$ as a zero of $f(\lambda, 0)$. Write the normal form of $f(\lambda, \alpha)$ as (see (3.10)):

$$
\bar{f}(\lambda, \alpha)=\left(\lambda-\lambda_{0}\right)^{m}+\bar{f}_{01} \alpha+\ldots
$$

If $m>1$, then the necessary and sufficient condition that $\lambda_{0}$ has the RS or CRS property is $\bar{f}_{01} \neq 0$. If $m=1$, then $f(\lambda, \alpha)$ always has the $R S$ property at $\lambda_{0}$, but for the CRS property, it is still necessary that $\bar{f}_{01} \neq 0$. The configuration of Newton's diagram in this case is the same as case II, Section 2.4 , where $p$ is the multiplicity of $\lambda_{0}=0$.

Now we write the primary result of this chapter as a theorem.

Theorem 3.5 For an analytic matrix function $L(\lambda, \alpha)$, suppose $L(\lambda, 0)$ has the local Smith normal form (3.9) at eigenvalue $\lambda_{0}$. Let the perturbation matrix $H=\left[L_{01}\right]_{\mathcal{K}, \mathcal{K}}$. Then a sufficient condition that $\lambda_{0}$ has the CRS property is

$$
\Delta_{0} \Delta_{1} \ldots \Delta_{k-1} \neq 0
$$

If the partial multiplicities of the first group is equal to 1 :

$$
\begin{equation*}
m_{1}=\ldots=m_{n_{1}}=1 \tag{3.15}
\end{equation*}
$$

then a sufficient condition that $\lambda_{0}$ has the $R S$ property can be reduced to:

$$
\Delta_{1} \Delta_{2} \ldots \Delta_{k-1} \neq 0
$$

Proof: First of all, by Proposition 3.4, condition $\Delta_{0} \Delta_{1} \ldots \Delta_{k-1} \neq 0$ is a sufficient condition that we can reduce the problem to the normal form $\operatorname{det}\left(D_{1}(\lambda)+\alpha H\right)=0$.

Then as in the arguments in Section 3.2, we can see that it is also a sufficient condition for the CRS property. When condition (3.15) is satisfied, although $\Delta_{0}=\operatorname{det} H=0$, the slope of the first segment in Newton's diagram has absolute value not less than one. We can say that $\lambda_{0}$ has the RS property.

Notice that there is an important refinement in Theorem 3.5 to [LNV]. The condition (3.2) is a sufficient condition for the CRS property, not only for the RS property.

### 3.6 The semisimple case

If $\lambda_{0}$ is a semisimple eigenvalue of $L(\lambda, 0)$, then $m_{1}=\ldots=m_{g}=1$. So the group number $k=1$, the condition (3.2) is reduced to

$$
\Delta_{0}=\operatorname{det} H \neq 0
$$

And Theorem 3.5 gives the following corollary.

Corollary 3.6 A semisimple eigenvalue $\lambda_{0}$ of $L(\lambda, 0)$ has the RS property. If also the perturbation matrix at $\lambda_{0}, H=\left[L_{01}\right]_{\mathcal{K}, \mathcal{K}^{\prime}}$, is nonsingular, then $\lambda_{0}$ has the $C R S$ property.

Corollary 3.6 is illustrated by Figure 3.3. If $\operatorname{det} H \neq 0$, Newton's diagram is constructed by only one segment with end points $(0, g)$ and $(g, 0)$ (shown in Figure 3.3 by thin line). If $\operatorname{det} H=0$, it is impossible that the real Newton's diagram lies lower than the thin line (the thick line in Figure 3.3 is one example). Since the segment $A C$ is steeper than $A B$ and segment $C D$ is steeper than $A C$, the exponents


Figure 3.3: Newton's diagram for the semisimple case
determined by AC and CD are greater than one. So in this case we still can say that $\lambda_{0}$ has the RS property.

When $\operatorname{det} H \neq 0$, the essential part of Newton's diagram of $\operatorname{det} L(\lambda, \alpha)=0$ includes points $(0, g)$ and $(g, 0)$ definitely, and possibly points of $(1, g-1),(2, g-$ $2), \ldots,(g-1,1)$, depending on the coefficients of terms

$$
\left(\lambda-\lambda_{0}\right) \alpha^{g-1},\left(\lambda-\lambda_{0}\right)^{2} \alpha^{g-2}, \ldots,\left(\lambda-\lambda_{0}\right)^{g-1} \alpha
$$

namely the values of

$$
\begin{equation*}
\sum_{1 \leq i \leq g} H(i), \sum_{1 \leq i_{1} \leq i_{2} \leq g} H\left(i_{1}, i_{2}\right), \ldots, \sum_{1 \leq i_{1} \ldots \leq i_{g-1} \leq g} H\left(i_{1}, \ldots, i_{g-1}\right) \tag{3.16}
\end{equation*}
$$

being zero or not. As in equation (3.4), for the eigenvalues

$$
\begin{equation*}
\lambda^{[j]}(\alpha)=\lambda_{0}+\lambda_{1 j} \alpha+\ldots, \tag{3.17}
\end{equation*}
$$

$\lambda_{1 j}, j=1, \ldots, g$, are roots of the D.E.:

$$
\begin{aligned}
\Delta_{0} & +\lambda_{1} \sum_{1 \leq i \leq g} H(i)+\lambda_{1}^{2} \sum_{1 \leq i_{1} \leq i_{2} \leq g} H\left(i_{1}, i_{2}\right)+\ldots \\
& +\lambda_{1}^{g-1} \sum_{1 \leq i_{1}<\ldots<i_{g-1} \leq g} H\left(i_{1}, \ldots, i_{g-1}\right)+\lambda_{1}^{g} \\
& =\operatorname{det}\left(\lambda_{1} I+H\right)=0
\end{aligned}
$$

When $\operatorname{det} H=0$, since $\lambda_{0}$ has the RS property, we can assume the eigenvalue functions have the form

$$
\begin{equation*}
\lambda(\alpha)=\lambda_{0}+\mu \alpha+o(|\alpha|) \tag{3.18}
\end{equation*}
$$

Now substitute (3.18) into (3.10), examine the block structure of it as in (3.11), and it follows that $\mu$ satisfies the equation

$$
\begin{equation*}
\operatorname{det}(\mu I+H)=0 \tag{3.19}
\end{equation*}
$$

If the values of (3.16) are not all zero, suppose the highest essential point of Newton's diagram is $(j, g-j)$ (see illustration Figure 3.4). Then (3.19) is the D.E. of $\mu$, and it has $g-j$ nonzero roots and $j$ zero roots. If all the values of (3.16) are zero, the line given by $(0, g)$ and $(g, 0)$ is totally missing on Newton's diagram (see Figure 3.3). So strictly speaking, we can not call (3.19) the D.E. of $\mu$, but it is still admissible to say that values of $\mu$ in (3.18) are given by equation (3.19), because in that case (3.19) only has zero roots.

The following theorem is now established and generalizes a result of [LNI] and [LN2]-after reduction to the semisimple case. There it is required that $\operatorname{det} H \neq 0$ and, here this assumption is not made.

Theorem 3.7 Let $L(\lambda, \alpha)$ be an analytic matrix function of $\lambda$ and $\alpha$ with a semisimple eigenvalue $\lambda_{0}$ at $\alpha=0$ of multiplicity $g$. Then there are exactly $g$ eigenvalues

. Figure 3.4: Newton's diagram for $\operatorname{det} H=0$
$\lambda^{[j]}(\alpha), j=1,2, \ldots, g$ of $L(\lambda, \alpha)$ for which $\lambda^{[j]}(\alpha) . \rightarrow \lambda_{0}$ as $\alpha \rightarrow 0$. These eigenvalues have Puiseux expansions for which (3.17) holds and there is a one-to-one correspondence between the coefficients $\lambda_{1 j}$ and the roots of equation (3.19).

## Example 3.4

$$
L(\lambda, \alpha)=\left[\begin{array}{ccc}
-1+\lambda-2 \alpha & \alpha & \lambda^{3} \\
\alpha & -\lambda+\lambda^{2} & \alpha+\alpha^{2} \\
0 & \lambda \alpha & \lambda^{2}
\end{array}\right]
$$

There is an eigenvalue $\lambda_{0}=1$ at $\alpha=0$. By the technique of equivalence transformation, it is found that, with

$$
\begin{gathered}
E(\lambda)=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & \lambda^{-1} & 0 \\
0 & 0 & \lambda^{-2}
\end{array}\right], \quad F(\lambda)=\left[\begin{array}{ccc}
1 & 0 & -\lambda \\
0 & 1 & 0 \\
0 & 0 & 1
\end{array}\right], \\
F(\lambda) L(\lambda, 0) E(\lambda)=\left[\begin{array}{ccc}
\lambda-1 & 0 & 0 \\
0 & \lambda-1 & 0 \\
0 & 0 & 1
\end{array}\right]
\end{gathered}
$$

the Smith normal form. Bases for $\mathcal{K}$ and $\mathcal{K}^{\prime}$ can then be formed fram $E(1)$ and $F(1)$ :

$$
\mathcal{K}=\operatorname{span}\left\{\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]\right\}, \quad \mathcal{K}^{\prime}=\operatorname{span}\left\{\left[\begin{array}{c}
1 \\
0 \\
-1
\end{array}\right],\left[\begin{array}{l}
0 \\
1 \\
0
\end{array}\right]\right\} .
$$

Then

$$
H=\left[L_{01}\right]_{\mathcal{K}, \mathcal{K}^{\prime}}=\left[\begin{array}{cc}
-2 & 0 \\
1 & 0
\end{array}\right]
$$

and

$$
\operatorname{det}(\mu I+H)=\left[\begin{array}{cc}
\mu-2 & 0 \\
1 & \mu
\end{array}\right]
$$

To find the determining equation directly from the definition, write

$$
L(\lambda, \alpha)=L_{0}+(\lambda-1) L_{10}+\alpha L_{01}+\ldots
$$

and substitute $\lambda(\alpha)=1+\mu \alpha+o(\alpha)$ to obtain

$$
L(\lambda, \alpha)=\left[\begin{array}{lll}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{array}\right]+\left[\begin{array}{ccc}
(\mu-2) & 1 & 3 \mu \\
1 & \mu & 1 \\
0 & 1 & 2 \mu
\end{array}\right] \alpha+(\text { terms of } o(\alpha))
$$

Then it is found that

$$
\operatorname{det} L(\lambda, \alpha)=\mu(\mu-2) \alpha^{2}+o\left(\alpha^{2}\right)
$$

Thus, the determining equation is $\mu(\mu-2)=0$, and agrees with the equation of (3.19)

Now using the results of Section 2.8, we can analyze the analyticity of the semisimple eigenvalues.

The condition 1 in Theorem 2.9 looks complicated, but it is naturally satisfied for a semisimple eigenvalue with $\operatorname{det} H \neq 0$. So when condition 2 is also satisfied, we have the following theorem.

Theorem 3.8 Let $\lambda_{0}$ be a semisimple eigenvalue of $L(\lambda, 0)$ with $\operatorname{det} H \neq 0$. Suppose also that $\lambda_{11}$ is a simple root of $\operatorname{det}\left(\lambda_{1} I+H\right)=0$. Then there exists a simple eigenvalue function

$$
\lambda(\alpha)=\lambda_{0}+\lambda_{11} \alpha+o(|\alpha|)
$$

which is analytic in a neighborhood of $\alpha=0$.

### 3.7 Eigenvalues for matrix polynomials

Consider matrix polynomials in $\lambda$ :

$$
\begin{equation*}
L(\lambda, \alpha)=A_{0}(\alpha)+A_{1}(\alpha) \lambda+\ldots+A_{p}(\alpha) \lambda^{p} \tag{3.20}
\end{equation*}
$$

where $A_{0}(\alpha), \ldots, A_{p}(\alpha)$ are analytic matrix functions in $\alpha$.
Denote

$$
\begin{equation*}
L(\lambda, 0)=A_{0}+A_{1} \lambda+\ldots+A_{p} \lambda^{p}, \tag{3.21}
\end{equation*}
$$

where $A_{i}=A_{i}(0)$ are $n \times n$ matrices with complex entries, $i=1, \ldots, p$.
In the previous sections of this chapter, we are using Newton's diagram II to discuss the eigenvalues $\lambda(\alpha)$ in a neighborhood of $\lambda_{0}$ which is one eigenvalue of $L(\lambda, 0)$. Now by Newton's diagram I, for a matrix polynomial $L(\lambda, \alpha)$, we can discuss all the eigenvalues $\lambda(\alpha)$ starting from all the finite eigenvalues $\lambda_{i}$ of $L(\lambda, 0)$, not only these small solutions, but also the eigenvalues $\lambda(\alpha)$ with negative exponents of $\alpha$ in the Puiseux expansion (2.2) defined in a deleted neighborhood of $\alpha=0$.

From 'Theorem 2.1, we obtain the following conclusion:
If $\operatorname{det} A_{p} \neq 0$, there are $n p$ eigenvalues of $L(\lambda, \alpha)$ with the form

$$
\begin{equation*}
\lambda^{[i j]}(\alpha)=\lambda_{i}+o(1), \quad j=1, \ldots, m_{i}, \tag{3.22}
\end{equation*}
$$

as $\alpha \rightarrow 0$, where $\lambda_{i}$ covers all the eigenvalues of $L(\lambda, 0), m_{i}$ is the algebraic multiplicity of $\lambda_{i}$. We have $\sum m_{i}=n p$.

If $\operatorname{det} A_{p}=0$, and $q=\operatorname{deg}(\operatorname{det} L(\lambda, 0))$, then there are $q$ eigenvalues of $L(\lambda, \alpha)$ of the form (3.22) defined in a neighborhood of $\alpha=0$, there are $n p-q$ eigenvalues with the form .

$$
\lambda^{[k]}(\alpha)=\lambda_{k 1} \alpha^{\epsilon_{k 1}}+o\left(\alpha^{\left(\epsilon_{k 1}\right)}, \quad \epsilon_{k 1}<0, k=1, \ldots, n p-q .\right.
$$

## Example 3.5

$$
\begin{aligned}
& L(\lambda, \alpha)=\left[\begin{array}{cc}
\alpha & \lambda \\
2 \alpha-1 & 1+\lambda^{2}
\end{array}\right] \\
&=\left[\begin{array}{cc}
\alpha & 0 \\
2 \alpha-1 & 1
\end{array}\right]+\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right] \lambda+\left[\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right] \lambda^{2} \\
& \operatorname{det} L(\lambda, \alpha)=\alpha+(1-2 \alpha) \lambda+\alpha \lambda^{2} \\
& L(\lambda, 0)=\left[\begin{array}{cc}
0 & \lambda \\
-1 & 1+\lambda^{2}
\end{array}\right] \quad \operatorname{det} L(\lambda, 0)=\lambda
\end{aligned}
$$

$\lambda=0$ is the only eigenvalue of $L(\lambda, 0)$, and it has multiplicity 1. By Newton's diagram $I$, there are two eigenvalues of $L(\lambda, \alpha)$ defined in a deleted neighborhood of $\alpha=0$. They are

$$
\begin{gathered}
\lambda^{[1]}(\alpha)=-\alpha+o(|\alpha|), \\
\lambda^{[2]}(\alpha)=-\alpha^{-1}+o\left(|\alpha|^{-1}\right) .
\end{gathered}
$$

## Chapter 4

## Perturbation Theory for Analytic Matrix Functions: The generating eigenvector method

### 4.1 Introduction

In the preceding chapter, the only concern is the eigenvalue functions of an analytic matrix function. Now we are going to take the corresponding eigenvector functions into consideration. This method is called the generating eigenvector method. The "generating eigenvector" method was first introduced and systematically studied by [HL]. Many important results of this method are given in [HL], and are quoted here. In this chapter, the basic idea, the related concepts and results of the generating eigenvector method are introduced, collected and deduced. The application to the semisimple case as developed in [LMZ2] is included.

Part of Section 4.2 is quoted from [Z]. Theorem 4.6 is a cornerstone that the later analysis lays on, however it is not independent of Chapter 3. In the three authors' paper [LMZ2], one of the author's main contributions is using Newton's diagram to give the proof of this theorem. For completeness, Section 4.6 is quoted from [LMZ2].

### 4.2 An equivalent definition for partial multiplicities

In Section 3.3, the partial multiplicities of an eigenvalue $\lambda_{0}$ of an analytic matrix function $A(\lambda)$ are defined using the local Smith normal form. Now we define them
using "null chains", and prove that the two definitions are equivalent.

Definition 4.1 Let the matrix function $A(\lambda)$ be analytic in a neighborhood $U$ of $\lambda_{0}$, and have Taylor expansion in $U$ :

$$
A(\lambda)=\sum_{j=0}^{\infty}\left(\lambda-\lambda_{0}\right)^{j} A_{j}
$$

where $A_{j} \in \mathbb{C}^{n \times n}, A_{0} \neq 0$. A vector-valued function $\phi(\lambda)$, such that $\phi(\lambda)$ is analytic at $\lambda_{0} ; \phi\left(\lambda_{0}\right) \neq 0$ and $\left.A(\lambda) \phi(\lambda)\right|_{\lambda=\lambda_{0}}=0$, is called a null function of $A(\lambda)$. The order of $\lambda_{0}$ as a zero of $A(\lambda) \phi(\lambda)$ is called the order of the null function $\phi(\lambda)$.

Suppose the order of a null function $\phi(\lambda)$ is $k$. Develop the null function in powers of $\left(\lambda-\lambda_{0}\right)$ :

$$
\phi(\lambda)=\sum_{j=0}^{\infty}\left(\lambda-\lambda_{0}\right)^{j} \phi_{j}
$$

where $\phi_{j} \in \mathbb{C}^{n}$. It follows from the definition of null functions that $\phi_{0}=\phi\left(\lambda_{0}\right) \neq 0$. The vector $\phi_{0}$ is called an eigenvector and $\phi_{0}, \phi_{1}, \ldots, \phi_{k-1}$ is called a null chain of $A(\lambda)$ of length $k$ corresponding to $\lambda_{0}$. (See Section 3.1 and 3.2 of $[B G R]$. The same definitions are given in [GS], where a null function is called a root function and a null chain is called a chain of eigenvector and associated vectors. In [HL], a null function is called a generating function and a null chain is called a Jordan chain.)

It will be convenient to introduce the block Toeplitz matrices,

$$
A^{(k)}=\left[\begin{array}{ccccc}
A_{0} & 0 & 0 & \ldots & 0  \tag{4.1}\\
A_{1} & A_{0} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{k} & A_{k-1} & A_{k-2} & \ldots & A_{0}
\end{array}\right]
$$

a square matrix of size $n(k+1)$, and also the $n(k+1)$ column vector

$$
\phi^{(k)}=\left[\begin{array}{c}
\phi_{0} \\
\phi_{1} \\
\vdots \\
\phi_{k}
\end{array}\right]
$$

If $\phi_{0}, \phi_{1}, \ldots, \phi_{k}$ is a null chain of length $k+1$ of $A(\lambda)$ at $\lambda_{0}$, then

$$
\left\{\begin{array}{l}
A_{0} \phi_{0}=0  \tag{4.2}\\
A_{1} \phi_{0}+A_{0} \phi_{1}=0 \\
\ldots \\
A_{k} \phi_{0}+\ldots+A_{0} \phi_{k}=0
\end{array}\right.
$$

or, $A^{(k)} \phi^{(k)}=0$ with $\phi_{0} \neq \dot{0}$.
To see this, write

$$
A(\lambda) \phi(\lambda)=\left(\sum_{j=0}^{\infty}\left(\lambda-\lambda_{0}\right)^{j} A_{j}\right)\left(\sum_{j=0}^{\infty}\left(\lambda-\lambda_{0}\right)^{j} \phi_{j}\right)
$$

and equate to zero the coefficients of $\left(\lambda-\lambda_{0}\right)^{j}$ for $j=0, \ldots, k$. Observe that $A_{0}$ is necessarily singular.

Conversely, if $\phi_{0}, \phi_{1}, \ldots, \phi_{k}$ is a solution of the system (4.2) with $\phi_{0} \neq 0$, then it is a null chain of $A(\lambda)$ of length $k+1$ corresponding to $\lambda_{0}$, since using null function

$$
\phi(\lambda)=\sum_{j=0}^{k}\left(\lambda-\lambda_{0}\right)^{j} \phi_{j}+\left(\lambda-\lambda_{0}\right)^{k+1} \psi(\lambda)
$$

for some $\psi(\lambda)$ which is analytic in a neighborhood of $\lambda_{0}$, it is easy to see that the order of $\phi(\lambda)$, or the order of $\lambda_{0}$ as a zero of $A(\lambda) \phi(\lambda)$, is $k+1$.

Now we say that a null chain starting with $\phi_{0}$ has maximal length $k$, when

$$
A^{(k-1)} \phi^{(k-1)}=0
$$

has solution $\phi_{0}, \phi_{1}, \ldots, \phi_{k-1}$, but the system

$$
A^{(k)} \phi^{(k)}=0
$$

has no solution $\phi_{k}$.
A canonical system of null chains corresponding to $\lambda_{0}$,

$$
\phi_{k 0}, \phi_{k 1}, \ldots, \phi_{k, m_{k-1}}, \quad k=1, \ldots, g
$$

obeys the following rules:

1. the vectors $\phi_{10}, \ldots, \phi_{g 0}$ form a basis of $\operatorname{ker} A\left(\lambda_{0}\right)$.
2. each chain $\phi_{k 0}, \phi_{k 1}, \ldots, \phi_{k, m_{k}-1}$ for $k=1, \ldots, g$ has the maximal length $m_{k}$.

A canonical system is not defined uniquely, however; it can be shown that the numbers $m_{1}, \ldots, m_{g}$ do not depend on the choice of null chains. So sometimes the partial multiplicities are defined as:

Definition 4.2 In a canonical system of null chains of an eigenvalue $\lambda_{0}$, the lengths of the null chains $m_{1}, \ldots, m_{g}$ are called partial multiplicities of eigenvalue $\lambda_{0}$.

For $D(\lambda)=\operatorname{diag}\left(\left(\lambda-\lambda_{0}\right)^{m_{1}}, \ldots,\left(\lambda-\lambda_{0}\right)^{m_{g}}, 1, \ldots, 1\right)$. Let $u_{j}$ be the $j$ th unit coordinate vector, it is easy to check that the sequence of vectors $u_{j}, 0, \ldots, 0$ with the length $m_{j}$ form a null chain of $D(\lambda)$ and it can not be extended to length $m_{j}+1$. So

$$
u_{j}, \underbrace{0, \ldots, 0}_{m_{j}-1} \quad j=1, \ldots, g
$$

is a canonical system of $D(\lambda)$.

Suppose $F(\lambda) A(\lambda) E(\lambda)=D(\lambda)$, where $E(\lambda)$ and $F(\lambda)$ are analytic and nonsingular at $\lambda_{0}$. By Definition 4.1, it is easy to see that if $\phi(\lambda)$ is a null function for $D(\lambda)$ of order $k$ at $\lambda_{0}$ then $E(\lambda) \phi(\lambda)$ is a null function for $A(\lambda)$ of order $k$ at $\lambda_{0}$, while if $\phi(\lambda)$ is a null function for $A(\lambda)$ of order $k$ at $\lambda_{0}$ then $E(\lambda)^{-1} \phi(\lambda)$ is a null function for $D(\lambda)$ of order $k$ at $\lambda_{0}$. As shown above, a null function of a matrix function corresponds to a null chain in a canonical system of the matrix function. We actually proved that the partial multiplicities defined in Definition 4.2 are the partial multiplicities in the local Smith normal form.

Using the new definition for partial multiplicities, an eigenvalue $\lambda_{0}$ is semisimple if and only if, for every eigenvector $x$ associated to $\lambda_{0}$, the singular equation

$$
A_{0} y=-A_{1} x
$$

where $A_{0}=A\left(\lambda_{0}\right), A_{1}=A^{\prime}\left(\lambda_{0}\right)$, has no solution $y$.

### 4.3 Generating eigenvectors

Now we take an eigenvector $x(\alpha)$ associated with eigenvalue $\lambda(\alpha)$ such that $L(\lambda(\alpha), \alpha) x(\alpha)=$ 0 into consideration. Assume that the Puiseux series in a deleted cut neighborhood $\mathcal{N}$ of $\alpha=0$ for $\lambda(\alpha)$ and $x(\alpha)$ are respectively:

$$
\begin{gather*}
\lambda(\alpha)=\lambda_{0}+\sum_{j=1}^{\infty} c_{j} \alpha^{j / q}  \tag{4.3}\\
x(\alpha)=\sum_{j=0}^{\infty} \xi_{j} \alpha^{j / l} \tag{4.4}
\end{gather*}
$$

where $q$ and $l$ are integers but do not necessarily coincide.

Let the Taylor expansion of $L(\lambda(\alpha), \alpha)$ at $\lambda_{0}$ be:

$$
\begin{equation*}
L(\lambda(\alpha), \alpha)=\sum_{j=0}^{\infty}\left(\lambda(\alpha)-\lambda_{0}\right)^{j} A_{j}(\alpha) \tag{4.5}
\end{equation*}
$$

Substitute (4.3) into (4.5), where the procedure of double summation means that

$$
L(\lambda(\alpha), \alpha)=\lim _{m \rightarrow \infty} \sum_{j=0}^{m}\left(\lim _{n \rightarrow \infty}\left(\sum_{i=1}^{n} c_{i} \alpha^{i / q}\right)^{j} A_{j}(\alpha)\right)
$$

(the absolute convergences of (4.3) and (4.5) at a point $\alpha$ can guarantee that the double limit exists at the point $\alpha$, see [C]) and denote it as:

$$
\begin{equation*}
L(\lambda(\alpha), \alpha)=\sum_{j=0}^{\infty} \bar{A}_{j} \alpha^{j / q} \tag{4.6}
\end{equation*}
$$

Now substitute (4.6) and (4.4) in $L(\lambda(\alpha), \alpha) x(\alpha)=0$, set the coefficients of all the terms of different powers of $\alpha$ to zero.

We have a linear system:

$$
\left\{\begin{array}{l}
\bar{A}_{0} \xi_{0}=0  \tag{4.7}\\
\cdots
\end{array}\right.
$$

So $x(\alpha)$ of (4.4) is an eigenvector associated with eigenvalue (4.3) if and only if $\xi_{j}$ satisfy (4.7). The second and the following equations in (4.7) depend on whether $q$ and $l$ have common divisors. For example, if $q=2, l=3$ in (4.6) and (4.4) respectively, then the first few equations in (4.7) are:

$$
\left\{\begin{array}{l}
\bar{A}_{0} \xi_{0}=0 \\
\bar{A}_{0} \xi_{1}=0 \\
\bar{A}_{1} \xi_{0}=0 \\
\bar{A}_{0} \xi_{2}=0 \\
\bar{A}_{0} \xi_{3}+\bar{A}_{2} \xi_{0}=0 \\
\vdots
\end{array}\right.
$$

But if $q=2, l=4$ in (4.6) and (4.4) respectively, then the first few equations in (4.7) are:

$$
\left\{\begin{array}{l}
\bar{A}_{0} \xi_{0}=0 \\
\bar{A}_{0} \xi_{1}=0 \\
\bar{A}_{0} \xi_{2}+\bar{A}_{1} \xi_{0}=0 \\
\vdots
\end{array}\right.
$$

Obviously if $x(\alpha)$ in (4.4) is an eigenvector associated with eigenvalue (4.3) then $\left(\alpha^{1 / q}\right)^{ \pm k} x(\alpha)$ and $\left(\alpha^{1 / l}\right)^{ \pm k} x(\alpha)$, where $k$ is an integer, are also eigenvectors associated with $\lambda(\alpha)$.

The following Lemma shows that for an analytic eigenvalue function there exists a corresponding analytic eigenvector function. This and more general results are known (see Theorem 18.2.1. of [GLR3], for example), but a new proof is provided here.

Lemma 4.1 If $L(\lambda, \alpha)$ is an analytic matrix function in a neighborhood of $\left(\lambda_{0}, 0\right)$ and has an eigenvalue function $\lambda(\alpha)$ which is analytic at $\alpha=0$, then there is an eigenvector function $x(\alpha)$ which is analytic at $\alpha=0$.

Proof: Suppose that $L(\lambda, \alpha)$ has an eigenvalue $\lambda(\alpha)=\sum_{j=0}^{\infty} c_{j} \alpha^{j}$ in a neighborhood of $\alpha=0$, and that $L(\lambda(\alpha), \alpha)$ has rank $r$ in a deleted neighborhood $\mathcal{N}$ of $\alpha=0$. Without loss of generality, assume that the minor

$$
L\left[\begin{array}{llll}
1 & 2 & \ldots & r \\
1 & 2 & \ldots & r
\end{array}\right] \neq 0, \quad \text { in } \mathcal{N}
$$

where

$$
L\left[\begin{array}{cccc}
i_{1} & i_{2} & \ldots & i_{p} \\
k_{1} & k_{2} & \ldots & k_{p}
\end{array}\right]:=\operatorname{det}\left(\left[\begin{array}{cccc}
l_{i_{1} k_{1}} & l_{i_{1} k_{2}} & \ldots & l_{i_{1} k_{p}} \\
l_{i_{2} k_{1}} & l_{i_{2} k_{2}} & \ldots & l_{i_{2} k_{p}} \\
\ldots & \ldots & \ldots & \ldots \\
l_{i_{p} k_{1}} & l_{i_{p} k_{2}} & \ldots & l_{i_{p} k_{p}}
\end{array}\right]\right)
$$

for any $1 \leq p \leq n$ (see [G]).
Consider the minor of order $r+1$ :

$$
L\left[\begin{array}{cccc}
1 & 2 & \ldots & r+1 \\
1 & 2 & \ldots & r+1
\end{array}\right]=\sum_{j=1}^{r+1} l_{r+1, j}(\alpha) x_{j}(\alpha)=0
$$

where $x_{1}(\alpha), \ldots, x_{r+1}(\alpha)$ are cofactors of the $r+1$ row in the determinant on the left.

Observe also that

$$
x_{r+1}(\alpha)=L\left[\begin{array}{cccc}
1 & 2 & \ldots & r \\
1 & 2 & \ldots & r
\end{array}\right] \neq 0
$$

and complete the construction of a nonzero vector $x(\alpha)=\left[\begin{array}{c}x_{1}(\alpha) \\ \vdots \\ x_{n}(\alpha)\end{array}\right]$ by setting
$x_{r+2}=\ldots=\dot{x}_{n}=0$.
Then, for $r=1,2, \ldots, n$, consider the $j$-th entry of the column vector $L(\lambda, \alpha) x(\alpha)$. It has the form

$$
\begin{aligned}
\sum_{k=1}^{n} l_{j k}(\alpha) x_{k}(\alpha) & =\sum_{k=1}^{r+1} l_{j k}(\alpha) x_{k}(\alpha) \\
& =L\left[\begin{array}{ccccc}
1 & 2 & \ldots & r & j \\
1 & 2 & \ldots & r & r+1
\end{array}\right]
\end{aligned}
$$

However, the last expression is zero because, for $j \leq r$ two rows of the minor agree and, for $j \geq r+1$, we have a minor of order $r+1>r$. Thus there is a nonzero vector $x(\alpha)$, analytic in $\mathcal{N}$, such that $L(\lambda(\alpha), \alpha) x(\alpha)=0$ in $\mathcal{N}$. If this vector has a zero of order $k$ at $\alpha=0$, then the vector function $x(\alpha) \alpha^{-k}$ has all the required properties. $\diamond$

The following lemma concerning Puiseux series for eigenvectors will also be useful.

## Lemma 4.2 For every eigenvalue

$$
\begin{equation*}
\lambda(\alpha)=\lambda_{0}+\sum_{j=1}^{\infty} c_{j} \alpha^{j / q} \tag{4.3}
\end{equation*}
$$

of $L(\lambda, \alpha)$ defined on a cut neighborhood of $\alpha=0$ there exists an associated eigenvector on this same neighborhood of the form

$$
\begin{equation*}
x(\alpha)=\sum_{k=0}^{\infty} \xi_{k} \alpha^{k / q}, \quad \xi_{0} \neq 0 \tag{4.8}
\end{equation*}
$$

In these two expansions, $\alpha^{1 / q}$ denotes the same branch of the corresponding $q$-valued function and $\alpha^{k / q}=\left(\alpha^{1 / q}\right)^{k}$.

Proof: This follows immediately from Proof of Lemma 4.1 on replacing $\alpha$ by $\alpha^{1 / q}$. $\diamond$
. With the preparation of Lemma 4.2, we have the definition:

Definition 4.3 Let $\lambda(\alpha), x(\alpha)$ be an eigenvalue-eigenvector pair of the form (4.3), (4.8), respectively. Then $x(0)=\xi_{0} \neq 0$ is called a generating eigenvector of $L(\lambda, \alpha)$ (at the point $\left(\lambda_{0}, 0\right)$ and associated with $\lambda(\alpha)$ ).

The next lemma is quoted from [HL] (Lemma 3.2). It throws some light on the concept of generating eigenvector.

Lemma 4.3 Let $\lambda_{1}(\alpha), \ldots, \lambda_{q}(\alpha)$ be eigenvalues of $L(\lambda, \alpha)$ which tend to $\lambda_{0}$ as $\alpha \rightarrow$ 0 and which constitute all the different branches of the same Puiseux series. Then there are corresponding eigenvectors of the form (4.8), say $x_{1}(\alpha), \ldots, x_{q}(\alpha)$ for which $x_{1}(0)=\ldots=x_{q}(0)$

Recall that in Section 3.5, for an eigenvalue $\lambda_{0}$ of $L(\lambda, 0)$ with algebraic multiplicity $m$, there exist $m$ eigenvalue functions in a neighborhood of $\alpha$ which can be represented by branches of several Puiseux series

$$
\begin{equation*}
\lambda_{\nu}(\alpha)=\lambda_{0}+\sum_{k=1}^{\infty} c_{\nu k} \alpha^{k / q_{\nu}} \tag{3.12}
\end{equation*}
$$

$\nu=1,2, \ldots, r$. Let us call the positive integers $q_{1}, \ldots, q_{r}$ splitting multiplicities. We know that

$$
\sum_{\nu=1}^{r} q_{\nu}=m .
$$

According to Lemma 4.2, there exist eigenvectors which have Puiseux series with the corresponding splitting multiplicities $q_{\nu}$. It follows from Lemma 4.3 that each group of branches $\lambda_{\nu}(\alpha)$ has at least one generating eigenvector.

The next example shows that not all eigenvectors.in $\mathcal{K}$ can be extended as a generating eigenvector. It is pointed out in [HL] (Proposition 3.5) that, for a semisimple eigenvalue, if the kernel $\mathcal{K}$ can be spanned by generating eigenvectors then all the eigenvalue functions are analytic.

## Example 4.1

$$
L(\lambda, \alpha)=\left[\begin{array}{cc}
\lambda & -\alpha  \tag{4.9}\\
-\alpha^{2} & \lambda
\end{array}\right]
$$

When $\alpha=0, \lambda_{0}=0$. is the only eigenvalue. 'As $\alpha \rightarrow 0$, there are two eigenvalue functions which are two branches of the same group:

$$
\begin{equation*}
\lambda(\alpha)= \pm \alpha^{3 / 2} \tag{4.10}
\end{equation*}
$$

The only splitting multiplicity is 2. All nonzero vectors in $\mathbb{C}^{2}$ are eigenvectors at $\lambda_{0}$. However substituting (4.10) into (4.9) shows that

$$
L(\lambda, \alpha)=\left[\begin{array}{cc} 
\pm \alpha^{3 / 2} & -\alpha \\
-\alpha^{2} & \pm \alpha^{3 / 2}
\end{array}\right]=\left[\begin{array}{cc}
0 & -1 \\
0 & 0
\end{array}\right] \alpha \pm\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right] \alpha^{3 / 2}+\left[\begin{array}{cc}
0 & 0 \\
-1 & 0
\end{array}\right] \alpha^{2} .
$$

And only

$$
x(\alpha)=C\left[\begin{array}{c}
1 \\
\pm \alpha^{1 / 2}
\end{array}\right]=\left[\begin{array}{l}
C \\
0
\end{array}\right]+\left[\begin{array}{c}
0 \\
\pm C
\end{array}\right] \alpha^{1 / 2}
$$

where $C$ is a nonzero constant, can be the eigenvector associated with $\lambda(\alpha)$. So orily the vector

$$
\left[\begin{array}{l}
1 \\
0
\end{array}\right]
$$

and its nonzero scalar multiples are generating eigenvectors.

## $4.4 \alpha$-semisimple

Although, as mentioned in Chapter 1, physically the parameter $\lambda$ can be regarded as spectral parameter while $\alpha$ is regarded as the perturbation-parameter, mathematically there is no difference between the positions of the two parameters. The Taylor expansion of $L(\lambda, \alpha)$ in (3.6) shows strongly the symmetry of the two parameters. Therefore some results in regard to $\lambda$ parameter can be applied to $\alpha$ parameter (Theorem 4.8 of [HL] is an example).

Here we symmetrically define the semisimple property with respect to the $\alpha$ parameter. Thus, the eigenvalue $\lambda_{0}$ at $\alpha=0$ is said to be $\alpha$-semisimple if for all nonzero $x \in \mathcal{K}$, the singular equation $L_{0} y=-L_{01} x$ has no solution $y$.

Our analysis uses the following useful notation.
As in Chapter 3, we denote $\mathcal{K}=\operatorname{ker} L_{0}$ and $\mathcal{K}^{\prime}=\operatorname{ker}\left(L_{0}^{*}\right)$. Choose a basis $\left\{f_{j}\right\}_{1}^{g}$ for $\mathcal{K}^{\prime}$, define a linear transformation $R$ from $\mathbb{C}^{n}$ to $\mathcal{K}^{\prime}$ by

$$
\begin{equation*}
R=\sum_{j=1}^{g}\left(., f_{j}\right) f_{j} \tag{4.11}
\end{equation*}
$$

Note that if $\left\{f_{j}\right\}_{1}^{g}$ is an orthonormal basis, then $R$ is the orthogonal projector onto $\mathcal{K}^{\prime}$. With respect to the unit coordinate basis $\left\{u_{j}\right\}_{1}^{g}$ of $\mathbb{C}^{n}$ and the basis $\left\{f_{j}\right\}_{1}^{g}$ of $\mathcal{K}^{\prime}$, the linear transformation $R$ has the matrix representation of the $g \times n$ matrix $\left[f_{1}, \ldots, f_{g}\right]^{*}$.

Recall the notation of (3.8) for an $n \times n$ matrix $M$ :

$$
\begin{equation*}
[M]_{\mathcal{K}, \mathcal{K}^{\prime}}:=\left[e_{1}, \ldots, e_{g}\right]^{*} M\left[f_{1}, \ldots, f_{g}\right] \tag{3.8}
\end{equation*}
$$

where $\left\{e_{j}\right\}_{1}^{g}$ and $\left\{f_{j}\right\}_{1}^{g}$ are chosen basis for $\mathcal{K}$ and $\mathcal{K}^{\prime}$ respectively. We denote the $g \times g$ matrix $[M]_{\mathcal{K}, \mathcal{K}^{\prime}}$ by $\hat{M}$ as well.:

The following two lemmas are Lemma 3 and Lemma 4 in [LMZ2].

Lemma 4.4 (a) An eigenvalue $\lambda_{0}$ at $\alpha=0$ is semisimple if and only if $\left[L_{10}\right]_{\mathcal{K}, \mathcal{K}^{\prime}}$ is nonsingular.
(b) An eigenvalue $\lambda_{0}$ at $\alpha=0$ is $\alpha$-semisimple if and only if $\left[L_{01}\right]_{\mathcal{K}, \mathcal{K}^{\prime}}$ is nonsingular.

Proof (a) Let $x \in \mathcal{K}, x \neq 0$, and consider the equation

$$
\begin{equation*}
L_{0} y=-L_{10} x \tag{4.12}
\end{equation*}
$$

for $y$. There exists a solution for (4.12) if and only if $L_{10} x$ is in $\operatorname{Im} L_{0}$, but $\mathcal{K}^{\prime}:=$ $\operatorname{ker}\left(L_{0}^{*}\right)=\left(\operatorname{Im} L_{0}\right)^{\perp}$, so there exists a solution if and only if $L_{10} x$ is orthogonal to $\mathcal{K}^{\prime}$. This is equivalent to

$$
\begin{equation*}
R L_{10} x=0 \tag{4.13}
\end{equation*}
$$

Since $x \in \mathcal{K}$, for the basis $\left\{e_{j}\right\}_{1}^{g}$ of $\mathcal{K}$, there exists a vector $\phi \in \mathbb{C}^{g}, \phi \neq 0$ such that

$$
x=\left[e_{1}, \ldots, e_{g}\right] \phi
$$

So (4.13) is equivalent to

$$
\begin{equation*}
\hat{L}_{10} \phi=0 \tag{4.14}
\end{equation*}
$$

There exists a solution for (4.12) if and only if the equation (4.14) has a nonzero solution $\phi$, hence if and only if $\left[L_{10}\right]_{\mathcal{K}, \mathcal{K}^{\prime}}=\hat{L}_{10}$ is singular.

The proof for $(b)$ is similar.
As we are particularly interested in the case in which $\lambda_{0}$ is a semisimple eigenvalue, note that from (3.14) regular splitting corresponds to the existence of asymptotic relations

$$
\begin{equation*}
\lambda^{[[]}(\alpha)=\lambda_{0}+\lambda_{1 j} \alpha+o(|\alpha|) . \tag{4.15}
\end{equation*}
$$

as $\alpha \rightarrow 0$ for $j=1,2, \ldots, g$ and complete regular splitting means that $\lambda_{1 j} \neq 0$ for each $j$.

We now extend Lemma 3.10 of [HL] as follows: .
Lemma 4.5 A semisimple eigenvalue has the regular splitting property and, if the eigenvalue is also $\alpha$-semisimple, then it has the CRS property.

Proof Let $\lambda_{0}$ be semisimple with multiplicity $g$. Consider an eigenvalue function of the form (4.3) where $1 \leq q \leq g$. By Lemma 4.2 there is an associated eigen-
vector function of the form (4.8). Denote the first nonzero coefficient in (4.3) by $c_{k^{\prime}}$ and assume that $k^{\prime}<q$. Now compare coefficients of $\alpha^{k / q}$ for $k=\dot{0}, 1, \ldots$ in $L(\lambda(\alpha), \alpha) x(\alpha) \equiv 0$, i.e. in
$\left\{L_{0}+\left(c_{k^{\prime}} \alpha^{k^{\prime} / q}+c_{k^{\prime}+1} \alpha^{\left(k^{\prime}+1\right) / q}+\ldots\right) L_{10}+\alpha L_{01}+\cdots\right\}\left\{\xi_{0}+\xi_{1} \alpha^{1 / q}+\cdots+\xi_{q} \alpha+\ldots\right\}=0$.

It is found that $L_{0} \xi_{j}=0$ for $j=0,1, \ldots, k^{\prime}-1$ and

$$
c_{k^{\prime}} L_{10} \xi_{0}+L_{0} \xi_{k^{\prime}}=0
$$

It follows that $\xi_{0}, \xi_{k^{\prime}} / c_{k^{\prime}}$ form a null chain for $\lambda_{0}$ and the assumption that $\lambda_{0}$ is semisimple is contradicted. Consequently, $k^{\prime} \geq q$ and there must be a regular splitting.

Suppose now that the splitting is not complete. Then there is an integer $r>0$ and an eigenvalue

$$
\lambda(\alpha)-\lambda_{0}=c_{q+r} \alpha^{(q+r) / q}+\cdots
$$

with $c_{q+r} \neq 0$, and

$$
\left\{L_{0}+\left(c_{q+r} \alpha^{(q+r) / q}+\cdots\right) L_{10}+\alpha L_{01}+\cdots\right\}\left\{\xi_{0}+\xi_{1} \alpha^{1 / q}+\cdots\right\}=0
$$

The coefficients of $\alpha^{0}$ and $\alpha^{1}$ yield

$$
L_{0} \xi_{0}=0 \quad \text { and } \quad L_{0} \xi_{q}+L_{01} \xi_{0}=0
$$

This contradicts the definition of an $\alpha$-semisimple eigenvalue and concludes the proof. $\diamond$

Thus, for any eigenvalue function $\lambda(\alpha)$ (emanating from a semisimple $\lambda_{0}$ ),

$$
\lambda(\alpha)=\lambda_{0}+\lambda_{1} \alpha+o(|\alpha|), \quad \text { as } \alpha \rightarrow 0
$$

where $\lambda_{1}=c_{q}$ of equation (4.3), and $\lambda(\alpha)$ is said to be real differentiable at $\alpha=0$. Then the equation $L(\lambda(\alpha), \alpha) x(\alpha)=0$ implies

$$
\left\{L_{0}+\alpha\left(\lambda_{1} L_{10}+L_{01}\right)+\cdots\right\}\left\{\xi_{0}+\xi_{1} \alpha^{1 / q}+\cdots\right\}=0, \quad(1 \leq q \leq g),
$$

whence

$$
L_{0} \xi_{q}+\left(\lambda_{1} L_{10}+L_{01}\right) \xi_{0}=0
$$

Using a basis $\left\{e_{1}, e_{2}, \ldots, e_{g}\right\}$ for $\mathcal{K}$, write $E=\left[e_{1} \cdots e_{g}\right]$ and $\xi_{0}=E \phi, 0 \neq \phi \in \mathbb{C}^{g}$. Now introduce a basis $\left\{f_{1}, \ldots, f_{g}\right\}$ for $\mathcal{K}^{\prime}$ so that $L_{0}^{*} f_{j}=0$ and

$$
\left(\left(\lambda_{1} L_{10}+L_{01}\right) E \phi, f_{j}\right)=0, \quad . j=1,2, \ldots, g
$$

i.e.

$$
\left(\lambda_{1}\left[L_{10}\right]_{\mathcal{K}, \mathcal{K}^{\prime}}+\left[L_{01}\right]_{\mathcal{K}, \mathcal{K}^{\prime}}\right) \phi=0, \quad \phi \neq 0,
$$

and $\lambda_{1}$ is an eigenvalue of the pencil

$$
\begin{equation*}
\mathcal{P}(\mu):=\mu\left[L_{i 0}\right]_{\mathcal{K}, \mathcal{K}^{\prime}}+\left[L_{01}\right]_{\mathcal{K}, \mathcal{K}^{\prime}} . \tag{4.16}
\end{equation*}
$$

Lemma 4.5 is the same as corollary 3.6. But the new proof is from the view point of eigenvectors and it is informative. Comparing with Section 3.6, notice that with the basis constructed by $E(\lambda)$ and $F(\lambda)$ in the local Smith normal form (3.5), [ $\left.L_{10}\right]_{\mathcal{K}, \mathcal{K}^{\prime}}=I_{g}$ and $\left[L_{01}\right]_{\mathcal{K}, \mathcal{K}^{\prime}}=H$, so (3.19) is equivalent to $\operatorname{det} P(\mu)=0$. Not only $\lambda_{1 j}$ for each $j$ in (4.15) is an eigenvalue of $P(\mu)$, by Newton's diagram, theorem 3.7 proved that each eigenvalue of $P(\mu)$ is one of the coefficients $\lambda_{1 j}$ of (4.15).

The argument above shows:
Theorem 4.6 Let $L(\lambda, \alpha)$ be an analytic matrix function of $\lambda$ and $\alpha$ with a semisimple eigenvalue $\lambda_{0}$ at $\alpha=0$ of multiplicity $g$. Then there are exactly $g$ eigenvalues
$\lambda^{[j]}(\alpha), \quad j=1,2, \ldots, g$ of $L(\lambda, \alpha)$ for which $\lambda^{[j]}(\alpha) \rightarrow \lambda_{0}$ as $\alpha \rightarrow 0$. These eigenvalues have Puiseux expansions for which (4.15) holds and there is a one-to-one correspondence between the coefficients $\lambda_{1 j}$ and the eigenvalues of the pencil $\mathcal{P}(\mu)$.

For every eigenvalue $\lambda^{[j]}(\alpha)$ of $L(\lambda, \alpha)$ there is a corresponding eigenvector $x_{j}(\alpha)$ which also has a Puiseux expansion about $\alpha=0$ and, if $x_{j}(0)=\left[e_{1}, \ldots, e_{g}\right] \phi_{j}$, then the vector $\phi_{j} \in \mathbb{C}^{g}$ is an eigenvector of the pencil $\mathcal{P}(\mu)$ corresponding to $\lambda_{1 j}$.

### 4.5 Analytic eigenvalues

The main interest of this section is the formulation of conditions guaranteeing the existence of analytic eigenvalue functions and corresponding eigenvector functions. Lemma 4.1 shows that the second property follows from the first. The main result of this section depends on techniques and results developed in [HL]. Some preparation is needed and are contained in the following definition and lemmas.
'. Consider the adjoint matrix function $L_{*}$ defined by

$$
L_{*}(\lambda, \alpha)=(L(\bar{\lambda}, \bar{\alpha}))^{*} .
$$

Using $L_{*}(\lambda, \alpha)$ instead of $L(\lambda, \alpha)^{*}$ requires reformulation of some results from [HL]. Thus, the next lemma is equivalent to Lemmas 3.1 of [HL] and the proof is the same. Note, in particular, that $\lambda(\alpha)$ is an eigenvalue of $\dot{L}(\lambda, \alpha)$ if and only if $\overline{\lambda(\bar{\alpha})}$ is an eigenvalue of $L_{*}(\lambda, \alpha)$.

Lemma 4.7 Let $\lambda_{1}(\alpha)$ and $\lambda_{2}(\alpha)$ be different eigenvalue functions of the form (4.3) on the same cut neighborhood of $\alpha=0$. Let $x_{1}(\alpha)$ be an eigenvector of the form (4.8) associated with the eigenvalue $\lambda_{1}(\alpha)$ of $L(\lambda, \alpha)$, and let $y_{2}(\alpha)$ be an eigenvector of the
form (4.8) associated with the eigenvalue $\overline{\lambda_{2}(\bar{\alpha})}$ of $L_{*}(\lambda, \alpha)$. Then $\left(L_{10} x_{1}(0), y_{2}(0)\right)=$ 0.

Theorem 3.6 of [HL] can be refined in the following way. (The proof is practically unaltered.)

Theorem 4.8 Let $\lambda_{1}(\alpha)$ be an eigenvalue function of $L(\lambda, \alpha)$ of the form (4.3). Assume that, for every generating eigenvector $x$ corresponding to $\lambda_{1}(\alpha)$, there exists a generating eigenvector $y$ of $L_{*}(\lambda, \alpha)$ associated with $\overline{\lambda_{1}(\bar{\alpha})}$ such that

$$
\left(L_{10} x, y\right) \neq 0
$$

Then $\lambda_{1}(\alpha)$ depends on $\alpha$ analytically, and there is a corresponding eigenvector $x(\alpha)$ which is analytic in $\alpha$ for $\alpha$ sufficiently close to zero.

Proof Assume that $\lambda_{1}(\alpha)$ is non-analytic. Then by assumption of the form (4.3) $\lambda_{1}(\alpha)$ is a branch of a Puiseux series at $\lambda_{0}$. Let $\lambda_{2}(\alpha)$ be a different branch of the same algebraic function. By Lemma 4.3, there are corresponding continuous eigenvectors $x_{1}(\alpha)$ and $x_{2}(\alpha)$ of the form (4.8) such that $x_{1}(0)=x_{2}(0):=x_{0}$.

Now let $\lambda(\alpha)$ and $y(\alpha)$ be any eigenvalue-eigenvector pair of $L_{*}(\lambda, \alpha)$. Then $\overline{\lambda(\bar{\alpha})}$ is an eigenvalue of $L(\lambda, \alpha)$ and, since $\overline{\lambda(\bar{\alpha})}$ cannot coincide identically with both $\lambda_{1}(\alpha)$ and $\lambda_{2}(\alpha)$, it follows from Lemma 4.7 that $\left(L_{10} x_{0}, y(0)\right)=0$, and this contradicts our hypothesis. Hence $\lambda_{1}(\alpha)$ depends on $\alpha$-analytically. Also, it follows from Lemma 4.1 that, in some neighborhood of $\alpha=0$, there is an associated analytic eigenvector function $x(\alpha)$.

In the following statement a simple eigenvalue of $\mathcal{P}(\mu)$ is just a simple zero of $\operatorname{det} \mathcal{P}(\mu)$.

Lemma 4.9 If $\lambda_{0}$ is a semisimple eigenvalue of $L(\lambda, \alpha)$ and $\lambda_{1}$ is a simple eigenvalue of $\mathcal{P}(\mu)$ then, for each generating eigenvector $x_{0}$ of $L(\lambda, \alpha)$ corresponding to the eigenvalue $\lambda(\alpha)=\lambda_{0}+\lambda_{1} \alpha+o(|\alpha|)$, there exists a generating eigenvector $y_{0}$ of $L_{*}(\lambda, \alpha)$ at $\left(\bar{\lambda}_{0}, 0\right)$ such that

$$
\left(L_{10} x_{0}, y_{0}\right) \neq 0 .
$$

Proof By the definition of generating eigenvector, there is an eigenvector $x(\alpha)$ of the form (4.8) corresponding to $\lambda(\alpha)$ for which $x(0)=x_{0}$. Consider the eigenvalue $\overline{\lambda(\bar{\alpha})}$ of $L_{*}(\lambda, \alpha)$, and let $y(\alpha)$ be a corresponding eigenvector of the form (4.8). Set $y_{0}=y(0) \neq 0$.

Now it follows from Theorem 4.6 that $\lambda(\alpha)=\lambda_{0}+\lambda_{1} \alpha+o(|\alpha|)$ as $\alpha \rightarrow 0$, and

$$
\begin{equation*}
\mathcal{P}\left(\lambda_{1}\right) u=0 \tag{4.17}
\end{equation*}
$$

where vector $u=\left[u_{j}\right]_{j=1}^{g}\left(\in \mathbb{C}^{g}\right)$ is defined via the decomposition of $x_{0}$ with respect to the basis $\left\{e_{j}\right\}$ for $\mathcal{K}$, i.e. $x_{0}=\sum_{j=1}^{g} u_{j} e_{j}$.

Now consider $L_{*}(\lambda, \alpha), \overline{\lambda(\bar{\alpha})}$, and $y_{0}$. Decompose $y_{0}$ with respect to the basis $\left\{f_{k}\right\}_{k=1}^{g,} ; \cdot y_{0}=\sum_{k=1}^{g} v_{k} f_{k}$, and $v=\left[v_{k}\right] \in \mathbb{C}^{g}$. Clearly $\overline{\lambda(\bar{\alpha})}=\overline{\lambda_{0}}+\overline{\lambda_{1}} \alpha+o(|\alpha|)$ as $\alpha \rightarrow 0$.
-Applying the arguments above to the matrix function $L_{*}(\lambda, \alpha)$ leads to the definition of the pencil

$$
\begin{equation*}
\mathcal{P}_{*}(\mu)=\mu\left[L_{10}^{*}\right]_{\mathcal{K}^{\prime}, \mathcal{K}}+\left[L_{01}^{*}\right]_{\mathcal{K}^{\prime}, \mathcal{K}}, \tag{4.18}
\end{equation*}
$$

and, as in the proof of Lemma 4.5

$$
\begin{equation*}
\mathcal{P}_{*}\left(\bar{\lambda}_{1}\right) v=0 . \tag{4.19}
\end{equation*}
$$

However, it follows from definition (3.8) that $\left[M^{*}\right]_{\mathcal{K}^{\prime}, \mathcal{K}}=\left([M]_{\mathcal{K}, \mathcal{K}^{\prime}}\right)^{*}$, and hence

$$
\begin{equation*}
\mathcal{P}_{*}(\bar{\mu})=(\mathcal{P}(\mu))^{*} . \tag{4.20}
\end{equation*}
$$

Since $\lambda_{1}$ is a simple eigenvalue of $\mathcal{P}(\mu)$, there are no Jordan chains at $\lambda_{1}$. Hence, with $u$ from (4.17) the equation

$$
\begin{equation*}
\mathcal{P}\left(\lambda_{1}\right) w=-\left[L_{10}\right]_{\mathcal{K}, \mathcal{K}^{\prime}} u \tag{4.21}
\end{equation*}
$$

has no solution $w . \operatorname{By}(4.19), \operatorname{dim}\left(\operatorname{Ker} \mathcal{P}_{*}\left(\overline{\lambda_{1}}\right)\right)=\operatorname{dim}\left(\operatorname{Ker} \mathcal{P}\left(\lambda_{1}\right)\right)=1$. Therefore, $\operatorname{Ker} \mathcal{P}_{*}\left(\overline{\lambda_{1}}\right)=\operatorname{span}\{v\}$ and (using the well-known criterion for the solvability of inhomogeneous equations) the fact that (4.21) has no solution implies

$$
\begin{equation*}
\left(\left[L_{10}\right]_{\mathcal{K}, \mathcal{K}^{\prime}} u, v\right) \neq 0 \tag{4.22}
\end{equation*}
$$

Finally, using (3.8)

$$
\begin{aligned}
\left(\left[L_{10}\right]_{\mathcal{K}, \mathcal{K}^{\prime}} u, v\right) & =\sum_{j=1}^{g} \sum_{k=1}^{g}\left(L_{10} e_{k}, f_{j}\right) u_{k} \bar{v}_{j} \\
& =\left(L_{10} \sum_{k=1}^{g} u_{k} e_{k}, \sum_{j=1}^{g} v_{j} f_{j}\right) \\
& =\left(L_{10} x_{0}, y_{0}\right)
\end{aligned}
$$

Thus, the lemma follows from (4.22).
Lemma 4.9 now admits a direct application of Theorem 4.8 to obtain the main result:

Theorem 4.10 Let $\lambda_{0}$ be a semisimple eigenvalue of $L(\lambda, 0)$. Suppose also that $\lambda_{1}$ is a simple eigenvalue of $\mathcal{P}(\mu)$ with corresponding eigenvector $u$. Then for some $\varepsilon>0$ there exists a simple eigenvalue function $\lambda(\alpha)$ of $L(\lambda, \alpha)$ which is analytic in $|\alpha|<\varepsilon$ and satisfies $\lambda(0)=\lambda_{0}, \lambda^{\prime}(0)=\lambda_{1}$. A corresponding eigenvector $x(\alpha)$ can be chosen analytic in $|\alpha|<\varepsilon$ and such that $x(0)=\xi$, where $\xi=\sum_{i=1}^{g} u_{i} e_{i}$ and $\left[u_{i}\right]_{1}^{g}=u$.

Proof By Theorem 4.6 there exists an eigenvalue $\lambda(\alpha)$ with the representation

$$
\lambda(\alpha)=\lambda_{0}+\lambda_{1} \alpha+o(|\alpha|) \quad \text { as } \quad \alpha \rightarrow 0
$$

It follows from Lemma 4.9 that, for every generating eigenvector $x_{0}$ of $L(\lambda, \alpha)$ at $\left(\lambda_{0}, 0\right)$ corresponding to $\lambda(\alpha)$ there is a generating eigenvector $y_{0}$ of $L_{*}(\lambda, \alpha)$ at $\left(\overline{\lambda_{0}}, 0\right)$ such that $\left(L_{10} x_{0}, y_{0}\right) \neq 0$. By Theorem 4.8, this implies that the eigenvalue $\lambda(\alpha)$ is analytic in a neighborhood of $\alpha=0$, and the corresponding eigenvector $x(\alpha)$ can be chosen analytic there. The statement connecting the vectors $x(0)$ and $u$ follows from Theorem 4.6.

It remains to prove that, for $\alpha \neq 0, \cdot \lambda(\alpha)$ is a simple eigenvalue of $L(\lambda, \alpha)$. Suppose that $\hat{\lambda}(\alpha)$ is another eigenvalue of $L(\lambda, \alpha)$ defined on a neighborhood of $\alpha=0$ with an asymptotic representation

$$
\hat{\lambda}(\alpha)=\lambda_{0}+\hat{\lambda}_{1} \alpha+o(|\alpha|) .
$$

It follows from Theorem 4.6 that $\hat{\lambda}_{1}=\lambda_{1}$. Since

$$
\lambda(\alpha)-\hat{\lambda}(\alpha)=\left(\lambda_{1}-\hat{\lambda}_{1}\right) \alpha+o(|\alpha|)
$$

it follows that $\lambda(\alpha)=\hat{\lambda}(\alpha)$ when $0<|\alpha|<\varepsilon$.

- Note that the conclusions of this theorem (with, additionally, $\lambda_{1} \neq 0$ ) were deduced in [LN1] under the additional assumption that $\lambda_{0}$ is also $\alpha$-semisimple. For the classical eigenvalue problem $L(\lambda, \alpha)=\lambda I-A(\alpha)$, the theorem is well -known (see, for example; p. 269 of [B]).

The next example shows that the assumption that $\lambda_{1}$ is a simple eigenvalue for $\mathcal{P}(\mu)$ cannot be relaxed to admit a multiple semisimple eigenvalue.

Example 4.2 Conisider the classical eigenvalue problem:

$$
L(\lambda, \alpha)=\left[\begin{array}{cc}
\lambda-\alpha & \alpha^{2} \\
\alpha^{3} & \lambda-\alpha
\end{array}\right],
$$

with a double semisimple eigenvalue $\lambda_{0}=0$ at $\alpha=0$.
It is easily seen that $\mathcal{P}(\mu)=\mu I_{2}-I_{2}$ so that $\mathcal{P}(\mu)$ has a semisimple eigenvalue $\mu=1$. However, the eigenvalue functions are $\lambda_{1}(\alpha)=\alpha+\alpha^{5 / 2}, \lambda_{2}(\alpha)=\alpha-\alpha^{5 / 2}$, and are not analytic at $\alpha=0$.

However, if $\lambda_{1}$ is only a semisimple eigenvalue of $P(\mu)$, we still have the following result:

Theorem 4.11 Let $L(\lambda, \alpha)$ be an analytic matrix function of $\lambda$ and $\alpha$ with a semisimple eigenvalue $\lambda_{0}$ at $\alpha=0$. Let $\lambda_{1}$ be a semisimple eigenvalue of the pencil $P(\mu)$ of multiplicity $s$. Then there exist seigenvalues $\lambda^{[j]}(\alpha)$ which can be represented as:

$$
\begin{equation*}
\ldots \quad \lambda^{[j]}(\alpha)=\lambda_{0}+\lambda_{1} \alpha+\lambda_{2 j} \alpha^{2}+o\left(|\alpha|^{2}\right), \tag{4.23}
\end{equation*}
$$

as $\alpha \rightarrow 0$ for $j=1, \ldots, s$.
Proof: Since $\lambda_{0}$ is a semisimple eigenvalue of $L(\lambda, \alpha)$ at $\alpha=0$, and $\lambda_{1}$ is a semisimple eigenvalue of the pencil $P(\mu)$ of multiplicity $s$, by Theorem 4.6 , we know that there are $s$ eigenvalue functions $\lambda(\alpha)$ with Puiseux expansions of the form

$$
\begin{equation*}
\lambda(\alpha)=\lambda_{0}+\lambda_{1} \alpha+c \alpha^{k / q}+\ldots \tag{4.24}
\end{equation*}
$$

where $c$ is the first nonzero coefficient after the term $\lambda_{1} \alpha$, so $k>q$. Suppose $[L(\lambda(\alpha), \alpha)]_{\mathcal{K}, \mathcal{K}^{\prime}}$ has an associated eigenvector function of the form:

$$
x(\alpha)=\phi_{0}+\phi_{1} \alpha^{1 / q}+\phi_{2} \alpha^{2 / q}+\ldots
$$

with $\phi_{0} \neq 0$.
Assume $k<2 q$, then in

$$
\begin{gathered}
{[L(\lambda(\alpha), \alpha)]_{\mathcal{K}, \mathcal{K}^{\prime}} x(\alpha)} \\
\left.=\left(\lambda \hat{L}_{10}+\hat{L}_{01}\right) \alpha+c \hat{L}_{10} \alpha^{k / q}+\ldots\right)\left(\phi_{0}+\phi_{1} \alpha^{1 / q}+\phi_{2} \alpha^{2 / q}+\ldots\right)=0,
\end{gathered}
$$

the coefficients of $\alpha$ and $\alpha^{k / q}$ yield

$$
\left(\lambda_{1} \hat{L}_{10}+\hat{L}_{01}\right) \phi_{0}=0 \quad \text { and } \quad c \hat{L}_{10} \phi_{0}+\left(\lambda_{1} \hat{L}_{10}+\hat{L}_{01}\right) \phi_{k-q}=0 .
$$

It follows that $\phi_{0}, \phi_{k-q} / c$ form a null chain for $\lambda_{1}$, so the hypothesis that $\lambda_{1}$ is semisimple is contradicted. Consequently $k \geq 2 q$, so the form (4.23) is permitted with the possibility that $\lambda_{2 j}=0$.

Theorem 4.10 immediately implies:
Corollary 4.12 Let $\lambda_{0}$ be a semisimple eigenvalue of $L(\lambda, 0)$. Suppose also that the eigenvalues $\left\{\mu_{j}\right\}_{j=1}^{g}$ of the pencil $\mathcal{P}(\mu)$ are distinct and let $\left\{u_{j}\right\}_{1}^{g}$ be a corresponding set of eigenvectors. Then there are numbers $\epsilon>0$ and $\delta .>0$ such that, for all $\alpha$ satisfying $0<|\alpha|<\epsilon$ the spectrum of $L(\lambda, \alpha)$ in $\left|\lambda-\lambda_{0}\right|<\delta$ consists of. $g$ distinct eigenvalues $\lambda_{1}(\alpha), \ldots, \lambda_{g}(\alpha)$ which are analytic in $|\alpha|<\epsilon$ and, for $j=1,2, \ldots, g$,

$$
\lambda_{j}(0)=\lambda_{0}, \quad \lambda_{j}^{\prime}(0)=\mu_{j} .
$$

Corresponding eigenvectors $x_{j}(\alpha)$ of $L(\lambda, \alpha)$ can be chosen analytic in $|\alpha|<\epsilon$ and such that, for $j=1,2, \ldots, g$,

$$
x_{j}(0)=\xi_{j},
$$

where $\xi_{j}=\sum_{i=1}^{g} u_{j i} e_{i}$ and $\left[u_{j i}\right]_{i=1}^{g}=u_{j}$.

A convenient way to determine those eigenvectors in $\mathcal{K}$ which are generating is not immediately obvious and would be useful in applications. Note, in particular, that linear combinations of generating eigenvectors are not necessarily generating. The last theorem provides a way to find the generating eigenvectors associated with a semisimple eigenvalue.

Corollary 4.13 Let $\lambda_{0}$ be a semisimple eigenvalue of $L(\lambda, 0)$. Suppose also that all eigenvalues of $\mathcal{P}(\mu)$ are distinct. Then $\xi_{0}=\sum_{i=1}^{g} u_{i} e_{i}$ is a generating eigenvector at $\lambda_{0}$ if and only if $u=\left[u_{i}\right]$ is an eigenvector of $\mathcal{P}(\mu)$.

Proof Corollary 4.12 shows that each eigenvector of $\mathcal{P}(\mu)$ can be extended analytically into a neighborhood of $\alpha=0$ as an eigenvector of $L(\lambda, \alpha)$ and, therefore, each of these $g$ linearly independent eigenvectors is generating.

Conversely, given a generating eigenvector $\xi_{0}{ }^{\circ}$ of $L(\lambda, \alpha)$ at $\lambda_{0}$, there is an eigenvector function $x(\alpha)$ with a Puiseux expansion (4.8) and, as in Theorem 4.6, it follows that the representation of $\xi_{0}$ with respect to basis $\left\{e_{i}\right\}_{1}^{g}$ is an eigenvector of $\mathcal{P}(\mu) . \diamond$

Example 4.3 The following simple example is instructive. Consider the self-adjoint function

$$
L(\lambda, \alpha)=\left[\begin{array}{cc}
\lambda & \alpha \\
\alpha & -\lambda
\end{array}\right]
$$

There is a semisimple eigenvalue $\lambda_{0}=0$ at $\alpha=0$. Eigenvalue functions emanating from $\lambda_{0}$ have the form $\pm i \alpha$ and are analytic but not real.

$$
\mathcal{P}(\mu)=\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] \mu+\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right]
$$

with eigenvalues $\pm i$.

All nonzero vectors in $\mathbb{C}^{2}$ are eigenvectors at $\lambda_{0}$, but (by Corollary 4.13) generating eigenvectors are confined to the nonzero scalar multiples of

$$
\left[\begin{array}{l}
1 \\
i
\end{array}\right], \quad\left[\begin{array}{c}
1 \\
-i
\end{array}\right] .
$$

### 4.6 Taylor coefficients

Consider the Taylor decomposition of $L(\lambda, \alpha)$, valid in some neighborhood of $\left(\lambda_{0}, 0\right)$ where $\lambda_{0}$ is a semisimple eigenvalue of $L(\lambda, 0)$ :

$$
\begin{equation*}
L(\lambda, \alpha)=\sum_{i, j=0}^{\infty}\left(\lambda-\lambda_{0}\right)^{i} \alpha^{j} L_{i j} \tag{3.7}
\end{equation*}
$$

and $L_{00}=L_{0}$. In'the sequel, we will ignore the difference between the $n \times 1$ eigenvector in $\mathcal{K}$ and its $g \times 1$ representation with respect to the basis $\left\{e_{i}\right\}_{1}^{g}$ of $\mathcal{K}$. It is clear from the context that a vector is in $\mathbb{C}^{n}$ or $\mathbb{C}^{g}$.

Let the pencil $\mathcal{P}(\mu)$ have a simple eigenvalue $b_{1}$ with associated eigenvector $\xi_{0}$. By Theorem 4.10 there is an analytic eigenvalue function of $L(\lambda, \alpha)$;

$$
\begin{equation*}
\lambda(\alpha)=\lambda_{0}+\sum_{k=1}^{\infty} b_{k} \alpha^{k} \tag{4.25}
\end{equation*}
$$

and a corresponding analytic eigenvector function

$$
\begin{equation*}
x(\alpha)=\sum_{k=0}^{\infty} \xi_{k} \alpha^{k} \tag{4.26}
\end{equation*}
$$

both valid in a neighborhood of $\alpha=0$, and $b_{1}, \xi_{0}$ are the eigenvalue and eigenvector of $\mathcal{P}(\mu)$ introduced above. The series of (4.25) and (4.26) can be substituted in the identity

$$
L(\lambda(\alpha), \alpha) x(\alpha)=0
$$

to obtain

$$
\begin{equation*}
\sum_{i, j=0}^{\infty}\left(\sum_{k=1}^{\infty} b_{k} \alpha^{k}\right)^{i} \alpha^{j} L_{i j}\left(\sum_{m=0}^{\infty} \xi_{m} \alpha^{m}\right)=0 \tag{4.27}
\end{equation*}
$$

The constant term on the left is $L_{0} \xi_{0}=0$. Equating coefficients of $\alpha^{j}$ to zero for $j=1,2, \ldots$, an infinite system of equations is obtained for the numbers $b_{2}, b_{3}, \ldots$ and vectors $\xi_{1}, \xi_{2}, \ldots$ Thus,

$$
\begin{align*}
& L_{0} \xi_{1}+\left(b_{1} L_{10}+L_{01}\right) \xi_{0}=0  \tag{4.28}\\
& L_{0} \xi_{2}+\left(b_{1} L_{10}+L_{01}\right) \xi_{1}+\left(b_{2} L_{10}+b_{1}^{2} L_{20}+b_{1} L_{11}+L_{02}\right) \xi_{0}=0  \tag{4.29}\\
& L_{0} \xi_{3}+\left(b_{1} L_{10}+L_{01}\right) \xi_{2}+\left(b_{2} L_{10}+b_{1}^{2} L_{20}+b_{1} L_{11}+L_{02}\right) \xi_{1}+ \\
&\left(b_{3} L_{10}+2 b_{1} b_{2} L_{20}+b_{2} L_{11}+b_{1}^{3} L_{30}+b_{1}^{2} L_{21}+b_{1} L_{12}+L_{03}\right) \xi_{0}=0  \tag{4.30}\\
& \vdots \\
& L_{0} \xi_{n}+\left(b_{1} L_{10}+L_{01}\right) \xi_{n-1}+\cdots+  \tag{4.31}\\
&\left(b_{n-1} L_{10}+\cdots+L_{0, n-1}\right) \xi_{1}+\left(b_{n} L_{10}+\cdots+L_{o n}\right) \xi_{0}=0
\end{align*}
$$

and so on.

Theorem 4.14 Let $\lambda_{0}$ be a semisimple eigenvalue of $L(\lambda, 0)$, and let $b_{1}$ be a simple eigenvalue of $\mathcal{P}(\mu)$ with associated eigenvector $\xi_{0}$. Then the infinite system (4.28), (4.29), $\cdots$ has a solution $\left\{b_{j}\right\}_{2}^{\infty},\left\{\xi_{j}\right\}_{1}^{\infty}$ such that the series (4.25) and (4.26) converge in a neighborhood of $\alpha=0$ and represent there an eigenvalue-eigenvector pair of $L(\lambda, \alpha)$.

The numbers $\left\{b_{j}\right\}_{2}^{\infty}$ are uniquely determined by this system. The solution $\left\{b_{j}\right\}_{2}^{\infty},\left\{\xi_{j}\right\}_{1}^{\infty}$ which gives an analytic eigenvalue-eigenvector pair $\lambda(\alpha), x(\alpha)$ can be found by successive computation of the unknowns $\xi_{1}, b_{2}, \xi_{2}, b_{3}, \ldots$.

For the proof, the readers are referred to [LMZ2]. One remark on this theorem is that although essentially the proof of Theorem 6.4 depends on Newton's diagram, it provides an independent algorithm for the eigenvalue series (4.25), and even for the eigenvector series (4.26).

## Chapter 5

# Applications to classical eigenvalue problems 

### 5.1 Introduction

We consider an analytic matrix function

$$
\begin{equation*}
A(\alpha)=A_{0}+\alpha A_{1}+\alpha^{2} A_{2}+\ldots \tag{5.1}
\end{equation*}
$$

with $A_{k} \in \mathbb{C}^{n \times n}, k=0,1, \ldots$ and $\alpha$ is contained in a small enough neighborhood of the origin. As we call $A_{0}$ the unperturbed matrix, $A(\alpha)$ are viewed as the perturbed matrices or the perturbation of matrix $A_{0}$. By letting

$$
\begin{equation*}
L(\lambda, \alpha)=A(\alpha)-\lambda I \tag{5.2}
\end{equation*}
$$

the classical eigenvalue problems for the perturbation of a matrix become a special case of our two-parameter perturbation problems.

In $[\mathrm{AH}]$, several algorithms are given to compute the eigenvectors for matrix function $A(\alpha)$. Here a criterion for the index of annihilation (as defined in [KPK]) which is useful in one algorithm in [AH] will be given. It extends the applicability of the author's work in [Z].

### 5.2 Some concepts

An analytic matrix function is called regular if $\operatorname{det} A(\alpha) \not \equiv 0 ;$ otherwise if $\operatorname{det} A(\alpha) \equiv 0$ then $A(\alpha)$ is called singular. All the scalar-valued meromorphic functions form a
field and meromorphic vector functions form a linear space over the field. We would expect a singular matrix function to have dependent rows and columns.

## Example 5.1

$$
\begin{gathered}
A(\alpha)=\left[\begin{array}{cc}
\sin \alpha & 1+\cos \alpha \\
1-\cos \alpha & \sin \alpha
\end{array}\right] \\
\operatorname{det} A(\alpha) \equiv 0
\end{gathered}
$$

Since

$$
\sin \alpha\left[\begin{array}{c}
\sin \alpha \\
1-\cos \alpha
\end{array}\right]-(1-\cos \alpha)\left[\begin{array}{c}
1-\cos \alpha \\
\sin \alpha
\end{array}\right]=0
$$

we see the columns of $A(\alpha)$ are linear dependent.

As usual, the rank of a matrix function is defined by the size of its largest minor that is not identically zero. An $n \times n$ matrix function is regular if and only if it is full rank, i.e., its rank is $n$. For an $A(\alpha)$ in (5.1) we assume $\alpha=0$ is an eigenvalue of $A(\alpha)$, so we assume $\operatorname{det} A(0)=\operatorname{det} A_{0}=0$. Denote $r=\operatorname{rank} A_{0}<n$. Then using the local Smith normal form at $\alpha=0$ defined in Section 3.3, we have matrix functions $E(\alpha), F(\alpha)$ which are analytic and non-singular in a neighborhood of $\alpha=0$, and
such that
where $0=m_{1}=\ldots=m_{r}<m_{r+1} \leq \ldots \leq m_{\bar{r}}$. This $\bar{r}$ coincides with the rank of $A(\alpha)$.

Also $\bar{r}=\max _{\alpha \in \Omega} \operatorname{rank} A(\alpha)$, where $\Omega$ is a neighborhood of $\alpha=0$ and rank $A(\alpha)$ is the rank of a matrix $A(\alpha)$ for $\alpha \in \Omega$. Actually, there exists a deleted neighborhood $\Omega$ of $\alpha=0$, in which the ranks of all the matrices $A(\alpha), \alpha \in \Omega$ are the same, and equal to $\bar{r}$. The set of all eigenvalues of $A(\alpha)$, that is $\sigma(A(\alpha))=\{\alpha \in \mathbb{C}: \operatorname{det} A(\alpha)=0\}$, is a set of the zeros of a scalar analytic functions. By Liouville's theorem or the maximum principle in complex analysis, it is a discrete set of points in $\mathbb{C}$. Except for these points, the ranks of matrices $A(\alpha)$ are all $\bar{r}$. Now we have two kinds of rank. One is the rank of a matrix, the other is the rank of a matrix function over the field of meromorphic scalar-valued functions. In the following, $\operatorname{rank} A(\alpha)$ is referred to the rank of matrix function $A$.

Define $m=\operatorname{dim}\left(\operatorname{ker} A_{0}\right)=n-r$ and $\bar{m}=n-\bar{r}$. Hence $m$ is also the geometric multiplicity of eigenvalue zero of matrix $A_{0}$. By our assumption that $\operatorname{det} A_{0}=0$, we
have $m>0$. We denote by $v_{i}, i=1, \ldots, m, m$ linear independent eigenvectors of $A_{0}$ corresponding to the eigenvalue zero and form the matrix $V:=\left[v_{1}, \ldots, v_{m}\right]$. This $n \times m$ matrix $V$ satisfies

$$
A_{0} V=0 .
$$

Theorem 18.2.1 of [GLR3] guarantees that there exist $\bar{m}$ analytic vector functions $\bar{v}_{i}(\alpha), i=1, \ldots, \bar{m}$ which constitute a basis for the null space of $A(\alpha)$ for $\alpha$ in a neighborhood of $\alpha=0$. Similarly, construct the $n \times \bar{m}$ matrix function $\bar{V}(\alpha):=$ $\left[\bar{v}_{1}(\alpha), \ldots, \bar{v}_{\bar{m}}(\alpha)\right]$. Therefore, $\bar{V}(\alpha)$ satisfies

$$
\begin{equation*}
A(\alpha) \bar{V}(\alpha)=0 \tag{5.3}
\end{equation*}
$$

and can be expressed as a power series in some neighborhood of zero

$$
\begin{equation*}
\bar{V}(\alpha)=V_{0}+\alpha V_{1}+\alpha^{2} V_{2}+\ldots \tag{5.4}
\end{equation*}
$$

Define the block Toeplitz matrix $A^{(k)}$ by the coefficient matrices $A_{0}, \ldots, A_{k}$ as in (4.1):

$$
A^{(k)}=\left[\begin{array}{ccccc}
A_{0} & 0 & 0 & \ldots & 0  \tag{5.5}\\
A_{1} & A_{0} & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{k} & A_{k-1} & A_{k-2} & \ldots & A_{0}
\end{array}\right], k=0,1,2, \ldots
$$

For the eigenvalue functions

$$
\begin{equation*}
\bar{v}_{i}(\alpha)=v_{i 0}+\alpha v_{i 1}+\alpha^{2} v_{i 2}+\ldots \tag{5.6}
\end{equation*}
$$

define the $n(k+1)$ column vectors

$$
v_{i}^{(k)}=\left[\begin{array}{c}
v_{i 0}  \tag{5.7}\\
v_{i 1} \\
\vdots \\
v_{i k}
\end{array}\right]
$$

Then the equation

$$
A(\alpha) \bar{v}_{i}(\alpha)=0
$$

is equivalent to the systems

$$
\begin{equation*}
A^{(k)} v_{i}^{(k)}=0 \tag{5.8}
\end{equation*}
$$

for $k=0,1,2, \ldots$.
Using the concept of null chains defined in Section 4.2, the coefficients $v_{i 0}, v_{i 1}, \ldots$ of an eigenvector function $\bar{v}_{i}(\alpha)$ in (5.6) with $v_{i 0} \neq 0$ form a null chain of $A(\alpha)$. Note that for these chains, even though they have only finitely many nonzero members, as null chains we can add infinitely many zero vectors at their tails. So we have $\bar{m}$. infinite null chains of $A(\lambda)$.

Using the concept of generating eigenvectors defined in Section 4.3 (Definition 4.2), the $\bar{m}$ linearly independent vectors $v_{10}, \ldots, v_{\bar{m} 0}$ are generating eigenvectors.

Generally, we have $m \geq \bar{m}$ i.e., $r \leq \bar{r}$. It is very possible that $m>\bar{m}$, because under the perturbation of $\alpha$, some zero eigenvalues of $A_{0}$ may vary as functions of $\alpha$. So there may be a jump of the dimension of the null space of $A(\alpha)$ or a reduction in the rank of $A(\alpha)$ from $\alpha=0$ to its neighborhood. The following examples illustrate the concepts introduced above.

## Example 5.2

$$
A(\alpha)=\left[\begin{array}{cc}
\alpha & \alpha^{2} \\
\alpha^{2} & \alpha^{3}
\end{array}\right], A_{0}=0
$$

$\bar{r}=\operatorname{rank} A(\alpha)=1, \bar{m}=1$.
$r=\operatorname{rank} A_{0}=0, m=2$.
$\operatorname{ker} A_{0}=\operatorname{span}\left\{\left[\begin{array}{l}1 \\ 0\end{array}\right],\left[\begin{array}{l}0 \\ 1\end{array}\right]\right\}, \operatorname{ker} A(\alpha)=\operatorname{span}\left\{\left[\begin{array}{c}-\alpha \\ 1\end{array}\right]\right\}$.
Let

$$
v_{10}=\left[\begin{array}{l}
1 \\
0
\end{array}\right], v_{20}=\left[\begin{array}{l}
0 \\
1
\end{array}\right], v_{21}=\left[\begin{array}{c}
-1 \\
0
\end{array}\right] .
$$

Only $v_{20}$ is a generating eigenvector. The following two null chains constitute a canonical system of null chains (as defined in Section 4.2) corresponding to $\alpha=0$ :

$$
\begin{aligned}
& v_{10} ; \\
& \ddot{v}_{20}, v_{21}, 0,0, \ldots .
\end{aligned}
$$

The first chain has length 1 while the second one is infinitely long.

Example 5.3

$$
\begin{aligned}
& A(\alpha)=\left[\begin{array}{ll}
1 & \alpha \\
0 & 0
\end{array}\right]=\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]+\alpha^{2}\left[\begin{array}{ll}
0 & 1 \\
0 & 0
\end{array}\right] \\
& A_{0}=\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right] .
\end{aligned}
$$

$\bar{r}=\operatorname{rank} A(\alpha)=1, \bar{m}=1$.
$r=\operatorname{rank} A_{0}=1, m=1$.

$$
\begin{array}{r}
\operatorname{ker} A_{0}=\operatorname{span}\left\{\left[\begin{array}{l}
0 \\
1
\end{array}\right]\right\}, \operatorname{ker} A(\alpha)=\operatorname{span}\left\{\left[\begin{array}{c}
-\alpha \\
1
\end{array}\right]\right\} . \text { Let } \\
v_{10}=\left[\begin{array}{l}
0 \\
1
\end{array}\right], v_{11}=\left[\begin{array}{c}
-1 \\
0
\end{array}\right]
\end{array}
$$

$v_{10}$ is a generating eigenvector.
A canonical system of null chains of $A(\alpha)$ corresponding to $\alpha=0$ is:

$$
v_{10}, v_{11}, 0,0, \ldots .
$$

It is an infinitely long chain.
Example 5.4

$$
\begin{aligned}
& A(\alpha)=\left[\begin{array}{ll}
1 & 0 \\
0 & \alpha^{2}
\end{array}\right]=\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]+\alpha\left[\begin{array}{ll}
0 & 0 \\
0 & 1
\end{array}\right] \\
& A_{0}=\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]
\end{aligned}
$$

$\bar{r}=\operatorname{rank} A(\alpha)=2, \bar{m}=0$.
$r=\operatorname{rank} A_{0}=1, m=1$.
$\operatorname{ker} A_{0}=\operatorname{span}\left\{\left[\begin{array}{l}0 \\ 1\end{array}\right]\right\}, \operatorname{ker} A(\alpha)=0 . \operatorname{Let}$
$v_{10}=\left[\begin{array}{l}0 \\ 1\end{array}\right]$,
$v_{10}$ is a generating eigenvector.
A canonical system of null chains of $A(\alpha)$ corresponding to $\alpha=0$ is: .

It is a null chain of length 2.

## Example 5.5

$$
\begin{aligned}
A(\alpha)=\left[\begin{array}{lll}
\alpha & 1 & 0 \\
0 & 1 & \alpha \\
0 & 0 & 0
\end{array}\right] & =\left[\begin{array}{lll}
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{array}\right]+\alpha\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{array}\right] \\
A_{0} & =\left[\begin{array}{lll}
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{array}\right] .
\end{aligned}
$$

$\bar{r}=\operatorname{rank} A(\alpha)=2, \bar{m}=1$.
$r=\operatorname{rank} A_{0}=1, m=2$.
$\operatorname{ker} A_{0}=\operatorname{span}\left\{\left[\begin{array}{l}1 \\ 0 \\ 0\end{array}\right],\left[\begin{array}{l}0 \\ 0 \\ 1\end{array}\right]\right\}, \operatorname{ker} A(\alpha)=\operatorname{span}\left\{\left[\begin{array}{c}1 \\ -\alpha \\ 1\end{array}\right] \cdot\right\} \cdot$ Let

$$
v_{1}=\left[\begin{array}{l}
1 \\
0 \\
0
\end{array}\right], v_{2}=\left[\begin{array}{l}
0 \\
0 \\
1
\end{array}\right], v_{10}=\left[\begin{array}{l}
1 \\
0 \\
1
\end{array}\right], v_{11}=\left[\begin{array}{c}
0 \\
-1 \\
0
\end{array}\right]
$$

$v_{10}$ belongs to ker $A_{0}$ and is a generating eigenvector.
The null chains
$v_{1} ;$
$v_{10}, v_{11}, 0,0, \ldots$.
constitute a canonical system of $A(\alpha)$ corresponding to $\alpha=0$, while the null chains
$v_{2} ;$
$v_{10}, v_{11}, 0,0, \ldots$.
are a canonical system too.

Now we give some other definitions. Let $M_{t}$ be subspaces of $\operatorname{ker} A_{0}, t \geq 0$ are integers:

$$
\begin{array}{ll}
M_{t}=\left\{v \mid v \in \operatorname{ker} A_{0}\right. & \text { such that } v \text { can be extended to } \\
& \text { a null chain of } A(\alpha) \text { with length } t+1\} .
\end{array}
$$

So $M_{0}=\operatorname{ker} A_{0}$ and the dimensions of the subspaces are non-increasing with $t$. Define
$\tau:=$ the smallest $t$ such that the minimum of $\operatorname{dim} M_{t}$ is attained.

Hence we have

$$
M_{0} \supseteq M_{1} \supseteq M_{2} \supseteq \ldots \supseteq M_{\tau}=M_{\tau+1}=\ldots .
$$

For example, suppose we have a canonical system of five chains:

```
v10
v20
v30
v40
v
```

In this case

$$
\begin{aligned}
& M_{0}=\operatorname{span}\left\{v_{10}, v_{20}, v_{30}, v_{40}, v_{50}\right\} \\
& M_{1}=\operatorname{span}\left\{v_{10}, v_{20}, v_{30}, v_{40}, v_{50}\right\} \\
& M_{2}=\operatorname{span}\left\{v_{20}, v_{30}, v_{40}, v_{50}\right\} \\
& M_{3}=\operatorname{span}\left\{v_{40}, v_{50}\right\} \\
& M_{4}=\operatorname{span}\left\{v_{50}\right\} \\
& M_{5}=\operatorname{span}\left\{v_{50}\right\}
\end{aligned}
$$

and $\tau=4$.
As pointed out before, a null chain with length $t+1$ is a solution of (5.8), and a solution of (5.8) is an eigenvector function in the form (5.6) for $A(\alpha)$ associated to eigenvalue zero. With this viewpoint, we can see that $\operatorname{dim} M_{\tau}=\bar{m}$ and $\tau$ is the longest length of the finitely long null chains. If $A(\alpha)$ has no finitely long null chain, as in Example 5.3, then $\tau=0$. In Examples 5.2 and 5.5, $\tau=1$. In Example 5.4, $\tau=2$. Connected to the Jordan canonical form for a matrix polynomials, in [KPK], $\tau$ is called the index of annihilation. Here we continue to use this name for $\tau$ by calling it the index of annihilation of $A(\alpha)$ at $\alpha=0$.

For an $n \times n$ analytic matrix function $A(\alpha)$, if $A(\alpha)$ is regular, then it is invertible in the sense that there exists an $n \times n$ matrix function $A^{-1}(\alpha)$ such that $A(\alpha) A^{-1}(\alpha)=A^{-1}(\alpha) A(\alpha)=I$. Normally $A^{-1}(\alpha)$ is a meromorphic matrix function, and it has a Laurent expansion in a neighborhood of $\alpha=0$ :

$$
A^{-1}(\alpha)=\alpha^{-s} B_{-s}+\alpha^{-s+1} B_{-s+1}+\ldots
$$

with $B_{-s} \neq 0$. Hence $\alpha=0$ is a pole of $A^{-1}(\alpha)$ of order $s$. By the result of Theorem 7.1 of [GS], we know $s$ is equal to $\tau$, the index of annihilation of $A(\alpha)$ at $\alpha=0$.

In [AH], assuming $\bar{m}>0$, the authors provide some recursive algorithms for the coefficients in the Laurent expansion of $\bar{V}(\alpha)$ in (5.4). In one of these algorithms (Theorem 4 of [AH]), the index of annihilation $\tau$ needs to be pre-determined. In [Z], for regular matrix functions, i.e., in the case of $\bar{m}=0$, the author provides a rank criterion for $\tau$. Now in the next section, we are going to extend the criterion to the case of $\bar{m}>0$. As in [Z], the criterion simply involves the calculation of the ranks of a sequence of matrices. And it can be done in a computationally efficient and reliable way, for example by using singular value decompositions (see [GV]).

The results in [AH] with the assumption $\bar{m}=0$ can easily be applied to the general perturbed eigenvalue problem

$$
\begin{equation*}
A(\alpha) x(\alpha)=\lambda(\alpha) x(\alpha) \tag{5.9}
\end{equation*}
$$

First we use the method of Newton's diagram to determine the coefficients and the index $p$ in the Puiseux series of an eigenvalue function:

$$
\lambda(\alpha)=\lambda_{0}+\lambda_{1} \alpha^{1 / p}+\lambda_{2} \alpha^{2 / p}+\ldots
$$

Next, introduce a new variable $\eta:=\alpha^{1 / p}$ and a matrix function in $\eta$ :

$$
B(\eta)=A\left(\eta^{p}\right)-\lambda(\eta) I
$$

Note that $B(\eta)$ is analytic in variable $\eta$ now and $\operatorname{det} B(\eta) \equiv 0$. Consequently, system (5.9) is transformed to

$$
B(\eta) x(\eta)=0,
$$

which can be effectively solved by the algorithms in [AH].

### 5.3 Computing the index of annihilation

Theorem 5.1 Let an $n \times n$ analytic matrix function $A$ have Taylor expansion about $\alpha=0$ :

$$
A(\alpha)=A_{0}+\alpha A_{1}+\alpha^{2} A_{2}+\ldots
$$

where $A_{k} \in \mathbb{C}^{n \times n}, k=0,1, \ldots$. If rank $A_{0}=\operatorname{rank} A$, where $\operatorname{rank} A$ is the rank of matrix function $A$, then the index $\tau$ of annihilation of $A(\alpha)$ at $\alpha=0$ is zero. If $\operatorname{rank} A_{0}<\operatorname{rank} A$, then the index $\tau$ of annihilation of $A(\alpha)$ at $\alpha=0$ is the first $k$ such that $\operatorname{rank} A^{(k)}=\operatorname{rank} A^{(k-1)}+\operatorname{rank} A$, where $A^{(k-1)}, A^{(k)}$ are defined in (5.5).

Proof: By the definition of the index $\tau$ of annihilation of $A(\alpha)$ at $\alpha=0$, it is the longest length of the finitely long null chains. If $\operatorname{rank} A_{0}=\operatorname{rank} A$, then there is no finitely long null chain for $A(\alpha)$ at $\alpha=0$, each eigenvector in $\operatorname{ker} A_{0}$ is a generating eigenvector, so $\tau=0$. If $\operatorname{rank} A_{0}<\operatorname{rank} A$, define $\bar{m}=n-\operatorname{rank} A$, then there are $\bar{m}$ linear independent eigenvectors in $\operatorname{ker} A_{0}$ which can be extended to infinitely long null chains, i.e., $\bar{m}$ eigenvectors are generating eigenvectors. Consider the block structure of $A^{(k)}$ as

$$
A^{(k)}=\left[\begin{array}{cc}
A_{0} & 0 \\
* & A^{(k-1)}
\end{array}\right]
$$

it is clear that $\operatorname{dim}\left(\operatorname{ker} A^{(k-1)}\right) \leq \operatorname{dim}\left(\operatorname{ker} A^{(k)}\right)$.
Suppose $\operatorname{dim}\left(\operatorname{ker} A^{(k-1)}\right)<\operatorname{dim}\left(\operatorname{ker} A^{(k)}\right)$. Then there must be some $n(k+1)$ column vector $v_{i}^{(k)}$ in form (5.7) such that $v_{i 0} \neq 0$ satisfies

$$
A^{(k)} v_{i}^{(k)}=0
$$

This means that a chain of length $k$ can be extended to length $k+1$. But we know that $\bar{m}$ chains which can be extended from length $k$ to length $k+1$. Hence we can
conclude that $\tau$ is the first $k$ such that

$$
\begin{equation*}
\operatorname{dim}\left(\operatorname{ker} A^{(k)}\right)=\operatorname{dim}\left(\operatorname{ker} A^{(k-1)}\right)+\bar{m} \tag{5.10}
\end{equation*}
$$

However

$$
\operatorname{dim}\left(k e r A^{(k)}\right)=n(k+1)-\operatorname{rank}\left(A^{(k)}\right)
$$

and

$$
\operatorname{dim}\left(\operatorname{ker} A^{(k-1)}\right)=n k-\operatorname{rank}\left(A^{(k-1)}\right)
$$

So that Equation (5.10) is equivalent to:

$$
\operatorname{rank} A^{(k)}=\operatorname{rank} A^{(k-1)}+\operatorname{rank} A
$$

To check the criterion, we can examine Example 5.5. There

$$
A_{0}=\left[\begin{array}{lll}
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{array}\right], A_{1}=\left[\begin{array}{lll}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{array}\right], \operatorname{rank} A=2
$$

$\operatorname{rank} A^{(0)}=1, \operatorname{rank} A^{(1)}=3$, so 1 is the number that first makes the equation rank $A^{(1)}=\operatorname{rank} A^{(1-1)}+2$. So $\tau=1$.

## Chapter 6

## Applications in gyroscopic systems

### 6.1 Introduction

The so-called "gyroscopic systems" model an important class of problems in the linear theory of vibrations and are characterized by transfer functions of the form

$$
\begin{equation*}
L(\lambda)=\lambda^{2} I+\lambda B+C \tag{6.1}
\end{equation*}
$$

where $B$ and $C$ are $n \times n$ hermitian matrices with
(a) $C>0$, i.e. $C$ is positive definite, and
(b) $B$ invertible and indefinite.

More specifically, the case in which $B=i G$ and $G$ is real and skew-symmetric is important and was the main topic of the papers [BL] and [HKLP]. The more general form of the coefficient $B$ was examined in [BLM]. For the purpose here, we merely consider system (6.1) as a finite matrix function, but there are generalizations to operator functions of the form (6.1) acting on an infinite-dimensional Hilbert space $H$ (as in [BLM]).

Section 6.2 gives a wider class of stable gyroscopic systems (quoted from [LMZ1]). Section 6.3 introduces another parameter $h$ to the gyroscopic system (6.1), and shows a connection to Chapter 3 and Chapter 4, as $h$ is regarded as the "perturbation parameter".

### 6.2 More general stable gyroscopic systems

We say a gyroscopic system (6.1) is strongly stable if and only if all eigenvalues of the quadratic matrix polynomial (6.1) are real and of definite type (this idea is in [GLR2], but see also [LMM2] and [L2], for example). Since (6.1) is transformed from the original transfer function of gyroscopic differential equaiton by the substitution $\lambda=i \mu$, where $\mu$ is the eigenvalue of the original system, so when $\lambda$ is real, $\mu$ stays on the imaginary axis and the solutions of the original gyroscopic differential equation are bounded. This is the normal meaning of "stable". The "strongly" means that under small hermitian perturbations (in norm) of the coefficients $B$ and $C$, the system remains stable. The term "strongly stable" is the same as "stably bounded", "stably r-diagonable" in [GLR2] (see p. 234 and p.257): An eigenvalue $\lambda_{0}$ is of definite type means that the kernel (or nullspace) of $L\left(\lambda_{0}\right)$ is a definite subspace with respect to $L^{\prime}\left(\lambda_{0}\right)$, i.e. we have

$$
\left(L^{\prime}\left(\lambda_{0}\right) x, x\right)>0, \quad \text { or } \quad\left(L^{\prime}\left(\lambda_{0}\right) x, x\right)<0
$$

for every nonzero $x \in \operatorname{Ker} L\left(\lambda_{0}\right)$. Obviously, when (a) holds and $B=0$ the system is unstable, and our concern is with what conditions on the "size" of $B$ (relative to $C$ and $I$ ) ensure strong stability.

A sufficient condition for strong stability is established and developed in [BLM], [L2], and elsewhere. It takes the form:
(c) $|B|>k I+k^{-1} C$ for some $k>0$,
where $|B|$ is the positive definite square root of $B^{2}$. When conditions (a), (b) and (c) hold the system (6.1) is said to be gyroscopically stabilized and is therefore known as
a GS system. (Note that such systems are also quasihyperbolic in the sense of [L1] and [LMM1].) It is our purpose here to formulate a more general class of gyroscopic systems by relaxing condition (c).

An intermediate class of problems will be examined before making our final generalization. Let $\sigma(M)$ denote the spectrum of a square matrix $M$ and let $r(M):=$ $\max _{\lambda \in \sigma(M)}|\lambda|$ be the spectral radius of $M$.

Lemma 6.1 Condition (c) is equivalent to

$$
\begin{equation*}
r\left(|B|^{-1}\left(k I+k^{-1} C\right)\right)<1 \quad \text { for some } \quad k>0 \tag{6.2}
\end{equation*}
$$

Proof: (c) is equivalent to

$$
\left(\left(k I^{1}+k^{-1} C\right) f, f\right)<\left(|B|^{1 / 2} f,|B|^{1 / 2} f\right), \quad(f \neq 0)
$$

or,

$$
|B|^{-1 / 2}\left(k I+k_{-.}^{-1} C\right)|B|^{-1 / 2}<I
$$

But then the last inequality is equivalent to

$$
r\left(|B|^{-1 / 2}\left(k I+k^{-1} C\right)|B|^{-1 / 2}\right)<1
$$

which, in turn, is equivalent to (6.2).

Since the spectral radius of matrix $|B|^{-1}\left(\omega I+\omega^{-1} C\right)$ is a continuous function of $\omega$, condition (6.2) can also be written in the form

$$
\left(\mathrm{c}^{\prime}\right) \quad . \quad \min _{\omega>0} r\left(|B|^{-1}\left(\omega I+\omega^{-1} C\right)\right)<1
$$

The idea of using this condition in place of condition (c) arises on consideration of the methods used in [HKLP]. This leads one to investigate the condition

$$
\text { (d) } \quad \min _{\omega>0} r\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right)<1,
$$

which will be seen to be weaker than (c) (or ( $c^{\prime}$ )). Then (d) will itself be replaced by a weaker condition (e) and, finally, it will be established that systems satisfying conditions (a), (b) and (e) (or, of course, (a), (b) and (d)) are stable in the above strong sense.

## Lemma 6.2

$$
\begin{equation*}
r\left(B^{-1}\left(\omega I+\omega^{-1} \dot{C}\right)\right) \leq r\left(|B|^{-1}\left(\omega I+\omega^{-1} C\right)\right) \tag{6.3}
\end{equation*}
$$

Proof: Since for $\omega>0, \omega I+\omega^{-1} C>0$, we can define $H=\left(\omega I+\omega^{-1} C\right)^{1 / 2}>0$. Also let $A=B^{-1}$. For these hermitian matrices $A, H$, it is well-known (P. 363 of [LT], the following norm is the spectral matrix norm) that

$$
r\left(A H^{2}\right)=r(H A H)=\|H A H\|=\sup _{\|f\|=1}|(H A H f, f)|
$$

Let $A_{+}, A_{-}$be the positive and negative parts of $A$, respectively, defined by

$$
A_{+}=\frac{1}{2}(|A|+A), \quad A_{-}=\frac{1}{2}(|A|-A)
$$

Then we have $A_{+} \geq 0, A_{-} \geq 0$, and $A=A_{+}-A_{-},|A|=A_{+}+A_{-}$. Thus

$$
\begin{aligned}
\sup _{\|f\|=1}|(H A H f, f)| & =\sup _{\|f\|=1}\left|\left(H A_{+} H f, f\right)-\left(H A_{-} H f, f\right)\right| \\
& \leq \sup _{\|f\|=1}\left(\left(H A_{+} H f, f\right)+\left(H A_{-} H f, f\right)\right) \\
& =\sup _{\|f\|=1}|(H|A| H f, f)|=\|H|A| H\|=r(H|A| H) \\
& =r\left(|A| H^{2}\right)=r\left(|B|^{-1}\left(\omega I+\omega^{-1} C\right)\right) .
\end{aligned}
$$

By Lemma 6.2, condition ( $c^{\prime}$ ) always implies (d). The next example shows that the reverse implication does not hold and, moreover, the left-hand-side of (6.3) can be arbitrarily small when the right-hand-side is arbitrarily large. Furthermore, this is the case even when $B=i G$ and $G$ is real, skew-symmetric.

## Example 6.1 Let

$$
B=i M^{1 / 4}\left[\begin{array}{llll}
0 & 0 & 0 & M \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
-M & 0 & 0 & 0
\end{array}\right], \quad C=\left[\begin{array}{llll}
M^{2} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right]
$$

where $M>1$ is a real number. Some computations show that

$$
\min _{\omega>0} r\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right)=2 M^{-1 / 4}
$$

Furthermore,

$$
r\left(|B|^{-1}\left(\omega I+\omega^{-1} C\right)\right)= \begin{cases}\omega+\omega^{-1} M^{2} & \text { for } 0<\omega \leq M^{1 / 2} \\ M\left(\omega+\omega^{-1}\right) & \text { for } \omega \geq M^{1 / 2}\end{cases}
$$

whence -

$$
\min _{\omega>0} r\left(|B|^{-1}\left(\omega I+\omega^{-1} C\right)\right)=M^{-3 / 4}(1+M)
$$

Finally, we see that

$$
\min _{\omega>0} r\left(|B|^{-1}\left(\omega I+\omega^{-1} C\right)\right) \rightarrow \infty, \quad \min _{\omega>0} r\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right) \rightarrow 0
$$

: as $M \rightarrow \infty$.

Now define functions $r_{+}$and $r_{-}$by

$$
\begin{aligned}
& r_{+}\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right)=\max \left(\lambda>0: \lambda \in \sigma\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right)\right) \\
& r_{-}\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right)=\max \left(\lambda>0:-\lambda \in \sigma\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right)\right)
\end{aligned}
$$

and then, since the spectrum of $B^{-1}\left(\omega I+\omega^{-1} C\right)$ is real,

$$
r\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right)=\max \left(r_{+}\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right), r_{-}\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right)\right)
$$

Writing

$$
\begin{equation*}
\rho(B, C):=\max \left(\min _{\omega>0} r_{+}\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right), \min _{\omega>0} r_{-}\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right)\right) \tag{6.4}
\end{equation*}
$$

we now have

$$
\begin{equation*}
\rho(B, \dot{C}) \leq \min _{\omega>0} r\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right) \tag{6.5}
\end{equation*}
$$

The next example shows that the left-hand-side of (6.5) can be arbitrarily small when the right-hand-side is arbitrarily large. However, there is equality here if $B$ has the special form $B=i G$ where $G$ is real and skew-symmetric (cf. [HKLP]).

## Example 6.2 Let

$$
B=\operatorname{diag}\left[M^{1 / 4},-M^{5 / 4}\right], \quad C=\operatorname{diag}\left[1, M^{2}\right]
$$

where $M>1$. Then some calculation shows that

$$
\min _{w>0} r\left(B^{-1}\left(w I+w^{-1} C\right)\right)=M^{1 / 4}+M^{-3 / 4}
$$

On the other hand,

$$
\min _{w>0} r_{+}\left(B^{-1}\left(w I+w^{-1} C\right)\right)=\min _{w>0} r_{-}\left(B^{-1}\left(w I+w^{-1} C\right)\right)=2 M^{-1 / 4}
$$

and it follows that $\rho(B, C)=2 M^{-1 / 4}$.

Thus, the more general case is that the condition (d) is replaced by

$$
\text { (e) } \quad \rho(B, C)<1
$$

where $\rho(B, C)$ is defined in (6.4), i.e. if conditions (a), (b), and (e) are satisfied then, for brevity, the system is said to be extended gyroscopically stabilized (EGS). It is proved in [LMZ1] that an EGS system is strongly stable and this result will be proved in Theorem 6.7.

### 6.3 Gyroscopic systems with two-parameters

Another perspective on the problem of strong stability of a gyroscopic system is developed in [HKLP] and we now investigate system (6.1) from this point of view. A complex parameter $h$ is introduced as follows:

$$
\begin{equation*}
L(\lambda, h)=\lambda^{2} I+\lambda h B+C \tag{6.6}
\end{equation*}
$$

and, instead of studying $L(\lambda, h)$ directly, consider the two parameter matrix function:

$$
\begin{equation*}
A(\omega, h):=\omega I+\omega^{-1} C+h B \tag{6.7}
\end{equation*}
$$

Note that if $C>0, \lambda=0$ can never be an eigenvalue of $L(\lambda, h)$ for any $h$, so the eigenvalue problems for $L(\lambda, h)$ and $A(\omega, h)$ are equivalent. We say that $A(\dot{\omega}, h)$ is strongly stable at a fixed $h$ when all $\omega$-eigenvalues of $A(\omega, h)$ are real and of definite type.

Now the question can be asked this way: since for $h=0, A(\omega, 0)$ is unstable and when $h$ is very big, the "size" of $h B$ can ensure the strong stability of $A(\omega, h)$ (see
[BLM] or [LMZ1]), what is the critical value of $h$ at which $A(\omega, h)$ turns from an unstable system to a stable system? We can define this value of $h$ as $h_{0}$ :

$$
\begin{equation*}
h_{0}=\inf \{h>0: A(\omega, h) \text { is strongly stable. }\} \tag{6.8}
\end{equation*}
$$

When $h=1$, the system $A(\omega, 1)$ is equivalent to the system (6.1), so the question examined by [LMZ1] can be translated as: what kind of condition on $B$ and $C$ for system $A(\omega, h)$ in (6.7) can ensure that $A(\omega, 1)$ is strongly stable? The sharpest condition of this kind obtained in [LMZ1] is condition (e).

The objectives of the following two sections are to confirm this result from the "two-parameter" viewpoint, and to generalize the theory of [HKLP] (in that work $B$ has the form $B=i G$, where $G$ is real and skew-symmetric) to the case of general indefinite $B$.

Now let us examine $\rho(B, C)$ in our "two-parameter" language.
It will be useful to introduce the inertia of an hermitian matrix $M$, defined as $\operatorname{In}(M)=(\pi, \nu, \delta)$, where $\pi(M), \nu(M), \delta(M)$ are, respectively, the numbers of positive, negative and zero eigenvalues of $M$ (counted with multiplicities).

Suppose the inertia of $B$ is $(p, n-p, 0)$. For the pencil $E+\lambda B$ with $E>0$, notice that all eigenvalues are nonzero and that for an eigenvector $x \neq 0$,

$$
(E+\lambda B) x=0 \Leftrightarrow\left(\lambda^{-1} I-E^{-i / 2}(-B) E^{-1 / 2}\right) E^{1 / 2} x=0
$$

so the pencil has $n-p$ positive eigenvalues. Similarly, if $E<0$, the pencil has $p$ positive eigenvalues.

Consider a fixed $\omega \in R^{+}, \omega I+\omega^{-1} C>0$, by the argument above, $A(\omega, h)$ has $n-p$ positive $h$-eigenvalues $h_{1}^{+}(\omega) \geq h_{2}^{+}(\omega) \geq \ldots h_{n-p}^{+}(\omega)>0$. At a fixed negative $\omega$,
$\omega I+\omega^{-1} C<0, A(\omega, h)$ has $p$ positive eigenvalues $h_{1}^{-}(\omega) \geq h_{2}^{-}(\omega) \geq \ldots h_{p}^{-}(\omega)>0$. These functions $h_{j}^{+}(\omega)$ and $h_{k}^{-}(\omega)$ are continuous functions of nonzero real variable $\omega$ and describe continuous curves in the $\omega, h$ coordinate plane. Let us examine these curves more carefully.

Let $c_{n}$ be the minimal eigenvalue of $C$ and $b_{1}$ be the maximal eigenvalue of $-B$. Let $x_{j}$ be an associated normalized eigenvector of $A(\omega, h)$ for $h_{j}^{+}(\omega)$ at a fixed $w>0$. Then

$$
\begin{equation*}
h_{j}^{+}(\omega)=\frac{x_{j}^{*}\left(\omega I+\omega^{-1} C\right) x_{j}}{x_{j}^{*}(-B) x_{j}} \geq \frac{\omega}{x_{j}^{*}(-B) x_{j}} \geq \frac{\omega}{b_{1}}>0 . \tag{6.9}
\end{equation*}
$$

Also

$$
\begin{equation*}
h_{j}^{+}(\omega) \geq \frac{\omega^{-1} x_{j}^{*} C x_{j}}{x_{j}^{*}(-B) x_{j}} \geq \frac{c_{n}}{\omega \dot{b_{1}}}>0 \tag{6.10}
\end{equation*}
$$

Hence, we can see that $h_{j}^{+}(\omega) \rightarrow \infty$ as $\omega \rightarrow \infty$ and also as $\omega \rightarrow 0$ from the right-hand side for each $j=1, \ldots, n-p$. Similarly, we have $h_{j}^{-}(w) \rightarrow \infty$ as $\omega \rightarrow-\infty$ and also as $\omega \rightarrow 0$ from the left-hand side for $j=1, \ldots, p$. Because $h_{1}^{+}$and $h_{1}^{-}$are continuous and bounded below, we can define

$$
\hat{h}^{+}=\min _{\omega>0} h_{1}^{+}(\omega), \quad \hat{h}^{-}=\min _{\omega<0} h_{1}^{-}(\omega), \quad \hat{h}=\max \left(\hat{h}^{+}, \hat{h}^{-}\right)
$$

The situation is sketched in Figure 6.1.
Using the symmetry of the eigenvalue $h(\omega)$ of " $A(\omega, h)$ with respect to the origin we have

$$
\begin{align*}
h_{1}^{+}(\omega)=r_{+}\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right), & \text { for } \omega>0  \tag{6.11}\\
h_{1}^{-}(-\omega)=r_{-}\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right), & \text { for } \omega>0 \tag{6.12}
\end{align*}
$$

So actually

$$
\begin{equation*}
\rho(B, C)=\hat{h} \tag{6.13}
\end{equation*}
$$



Figure 6.1: Eigenvalue curves $h_{j}^{+}(\omega), h_{k}^{-}(\omega)$

We have another important function

$$
\begin{equation*}
r\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right)=\max \left(r_{+}\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right), r_{-}\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right)\right) \tag{6.14}
\end{equation*}
$$

If we define $\bar{h}=\min _{\omega>0} r\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right)$, since

$$
\begin{aligned}
& r_{+}\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right) \leq r\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right) \\
& r_{-}\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right) \leq r\left(B^{-1}\left(\omega I+\omega^{-1} C\right)\right)
\end{aligned}
$$

so

$$
\hat{h}^{+} \leq \bar{h}, \quad \text { and } \quad \hat{h}^{-} \leq \bar{h}
$$

Thus

$$
\hat{h} \leq \bar{h}
$$

This is illustrated in Figure 6.2.
The'following lemma is useful for counting eigenvalue multiplicities.


Figure 6.2: The case: $\hat{h} \leq \bar{h}$

Lemma 6.3 For a matrix function $A(\omega, h)$ depending analytically on two complex parameters $\omega$ and $h$, the multiplicity of $\omega_{0}$ as an eigenvalue of $A\left(\omega, h_{0}\right)$ is equal to the total number of $\omega$-eigenvalues (counting multiplicities) in a neighborhood of $\omega_{0}$ for any fixed $h$ in a neighborhood of $h_{0}$.

Proof: Let $f(\omega)=\operatorname{det} A\left(\omega, h_{0}\right)$. The multiplicity of $\omega_{0}$ as an eigenvalue of $A\left(\omega, h_{0}\right)$ is the multiplicity of $\omega_{0}$ as zero of $f(\omega)$. Since $\omega_{0}$ is an isolated zero of $f(\omega)$, there exists a deleted neighborhood $U$ of $\omega_{0}$ such that for $\omega \in U, f(\omega) \neq 0$, and there exists a circle $\Gamma$ in $U$ around $\omega_{0}$ such that $\min _{\omega \in \Gamma}|f(\omega)|=\epsilon>0$. Now the continuity of function $\operatorname{det} A(\omega, h)$ in $h$ implies that for $\epsilon$, there exists a neighborhood $V$ of $h_{0}$ such that if $h \in V$, then $|f(\omega)-\operatorname{det} A(\omega, h)|<\epsilon$. Thus, by Rouché's Theorem (see [D]), we can conclude that the multiplicity of $\omega_{0}$, which is the only zero of $f(\omega)$ inside $\Gamma$, is equal to the total number of $\omega$-zeros of $\operatorname{det} A(\omega, h)$ inside $\Gamma$ for $h \in V$.

This lemma shows that in the two situations illustrated in Figures 6.3 and 6.4,


Figure 6.3: Multiple eigenvalue $\omega_{0}$, case 1
$\omega_{0}$ has multiplicity at least 2 .


Figure 6.4: Multiple eigenvalue $\omega_{0}$, case 2

### 6.4 The main results

Theorem 6.4 If $h>\hat{h}$, then all $\omega$-eigenvalues of $A(\omega, h)$ are real and semisimple.

Proof: By definition of $\hat{h}$ and the properties of $\ddot{h}_{j}^{+}(\omega), j=1, \ldots, n-p, h_{j}^{+}(\omega), j=$ $1, \ldots, p$, any horizontal line above, or coincident with $\hat{h}$ intersects each curve $h_{j}^{+}(\omega)$, $h_{j}^{-}(\omega)$ at least twice (see Figure 6.1). Using Lemma 6.3 to count multiplicities, "it follows that when $h \geq \hat{h}, A(\omega, h)$ has at least $2 n$ real $\omega$-eigenvalues. However, $A(\omega, h)$ can have at most $2 n$ complex $\omega$-eigenvalues. Hence, when $h \geq \hat{h}, \dot{A}(\omega, h)$ has exactly $2 n$ real $\omega$-eigenvalues.
$A(\omega, h)$ is a Hermitian matrix function and $\frac{\partial A}{\partial h}=B$. For any eigenvector $x$ of $A(\omega, h), x^{*}\left(\omega I+\omega^{-1} C+h B\right) x=0$, so if $\omega>0, h>0$, then $x^{*} B x<0$, and if $\omega<0$, $h>0$, then $x^{*} B x>0$. It follows from Corollary 4.3 of [HL] that if the eigenvalue $\omega$ is not semisimple then, after small perturbation in $h$ there are non-real $\omega$-eigenvalues. But this is not possible for $h>\hat{h}$, so all $\omega$-eigenvalues must be semisimple.

Theorem 6.5 If $\hat{h}=\hat{h}^{+}$(respectively $\hat{h}=\hat{h}^{-}$), then there is a unique $\omega_{\text {opt }}$ for which $\hat{h}=h_{1}^{+}\left(\omega_{\text {opt }}\right)\left(\right.$ respectively $\left.\hat{h}=h_{1}^{-}\left(\omega_{\text {opt }}\right)\right)$.

Proof: The proof uses the same idea as the proof of Theorem 6.4. By definition of $\hat{h}=\max \left(\hat{h}^{+}, \hat{h}^{-}\right)$, so $\hat{h}=\hat{h}^{+}$or $\hat{h}=\hat{h}^{-}$. Here the proof is only for the case of $\hat{h}=\hat{h}^{+}$, and the proof for the case of $\hat{h}=\hat{h}^{-}$(then $\left.\hat{h}=h_{1}^{-}\left(\omega_{\text {opt }}\right)\right)$ is similar. Suppose there are two different values $\omega_{1} \neq \omega_{2}$ such that $\hat{h}=h_{1}^{+}\left(\omega_{1}\right)=h_{1}^{+}\left(\omega_{2}\right)$. By the definition of $\hat{h}, h_{1}^{+}\left(\frac{1}{2}\left(\omega_{1}+\omega_{2}\right)\right) \geq \hat{h}$. If $h_{1}^{+}\left(\frac{1}{2}\left(\omega_{1}+\omega_{2}\right)\right)=\hat{h}$, then the horizontal line $h=\hat{h}$ has at least three intersections with the graph of $h_{1}^{+}$. If $h_{1}^{+}\left(\frac{1}{2}\left(\omega_{1}+\omega_{2}\right)\right)>\hat{h}$, then some horizontal line above $h=\hat{h}$ has at least four intersections with the graph of $h_{1}^{+}$, and it also intersects the other curves $h_{j}^{+}, j=2, \ldots, n-p$, and $h_{k}^{-}, k=1, \ldots, p$ at least twice. However, $A(\omega, h)$ can have at most $2 n \omega$-eigenvalues. This is a contradiction. Hence the $\omega$ at which $\hat{h}$ is attained is unique.

The proof of Theorem 6.5 corrects and simplifies an argument of [HKLP].
The following Theorem 6.7 is parallel to the main theorem in [LMZ1]. Before that, we need a useful lemma. It can be obtained from more general results of Section 12.4 of [GLR1]:

Lemma 6.6 Let $M(\omega)$ be a matrix polynomial, $a$ and $b$ be real numbers with $a<b$. If $M(a)$ and $M(b)$ are invertible and

$$
\nu(M(b))-\nu(M(a))=s>0
$$

then $M(\omega)$ has at least $s$ eigenvalues on $(a, b)$ (counting multiplicities) and, if there are exactly s eigenvalues on this interval, then they all have negative type.

Theorem 6.7 Let $\omega_{+}>0$ and $\omega_{-}<0$ be two $\omega$-values such that $\hat{h}^{+}$and $\hat{h}^{-}$are achieved, i.e. $\hat{h}^{+}=\hat{h}^{+}\left(\omega_{+}\right)$and $\hat{h}^{-}=\hat{h}^{-}\left(\omega_{-}\right)$. Let $\operatorname{In} B=(p, n-p, 0)$. Then for $h>\hat{h}$, all $\omega$-eigenvalues of $A(\omega, h)$ are real with definite type and there are:

## $p$ eigenvalues of negative type in $\left(-\infty, \omega_{-}\right)$,

$p$ eigenvalues of positive type in $\left(\omega_{-}, 0\right)$,
$n-p$ eigenvalues of negative type in $\left(0, \omega_{+}\right)$,
$n-p$ eigenvalues of positive type in $\left(\omega_{+},+\infty\right)$.

Proof: First consider a fixed $h>\hat{h}^{+}$.
By considering large positive $\omega$, we may write $A(+\infty, h)>0$. On the other hand, since $C>0$ there is small positive $\omega$, say $\omega=\epsilon>0$, such that $A(\epsilon, h)>0$.

Now let us check the inertia of matrix $A\left(\omega_{+}, h\right)$. First of all, since $h>\hat{h}^{+}$, matrix $A\left(\omega_{+}, h\right)$ is invertible.

If we can find a subspace $S$ such that $\operatorname{dim} S=n-p$ and $\left(A\left(\omega_{+}, h\right) f, f\right)<0$ for all nonzero $f \in S$, then $\nu\left(A\left(\omega_{+}, h\right)\right) \geq n-p$, and it follows from Lemma 6.3 that there must be at least $n-p \quad \omega$-eigenvalues of $A(\omega, h)$ (seen as a polynomial of $\omega$ ) in $\left(0, \omega_{+}\right)$, and also in $\left(\omega_{+}, \infty\right)$. Furthermore, if there are exactly $n-p$ eigenvalues in each interval, then they must all have definite type; negative type in $\left(0, \omega_{+}\right)$and positive type in $\left(\omega_{+}, \infty\right)$.

To find such a subspace $S$, we first define a new definite inner product

$$
[f, g]=\left(\left(\omega_{+} I+\omega_{+}^{-1} C\right) f, g\right)
$$

and it is easily verified that $B^{-1}\left(\omega_{+} I+\omega_{+}^{-1} C\right)$ is self-adjoint with respect to [., .]. So there exist orthogonal eigenvectors $x_{1}, \ldots, x_{n-p}$ corresponding to eigenvalues $h_{1}\left(\omega_{+}\right), \ldots, h_{n-p}\left(\omega_{+}\right):$

$$
\begin{equation*}
\left(\left(\omega_{+} I+\omega_{+}^{-1} C\right) x_{i}, x_{j}\right)=0 \quad i \neq j \tag{6.15}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\omega_{+} I+\omega_{+}^{-1} C+h_{i} B\right) x_{i}=0 \quad i=1, \ldots, n-p \tag{6.16}
\end{equation*}
$$

Furthermore

$$
\begin{equation*}
\hat{h}^{+}=h_{1} \geq h_{2} \geq \ldots \geq h_{n-p}>0 \tag{6.17}
\end{equation*}
$$

It follows from (6.15) and (6.17) and the positivity of $\omega_{+} I+\omega_{+}^{-1} C$ that

$$
\begin{equation*}
\left(B x_{i}, x_{i}\right)<0 \tag{6.18}
\end{equation*}
$$

If we define $\mathcal{S}=\operatorname{span}\left\{x_{i}\right\}_{i=1}^{n-p}$ and take $f=\sum_{i=1}^{n-p} a_{i} x_{i} \in \mathcal{S}$, then using (6.15):

$$
\begin{aligned}
\left(\left(\omega_{+} I+\omega_{+}^{-1} C\right) f, f\right) & =\left(\sum_{i} a_{i}\left(\omega_{+} I+\omega_{+}^{-1} C\right) x_{i}, \sum_{j} a_{j} x_{j}\right) \\
& =\sum\left|a_{i}\right|^{2}\left(\left(\omega_{+} I+\omega_{+}^{-1} C\right) x_{i}, x_{i}\right) \\
& =-\sum\left|a_{i}\right|^{2} h_{i}\left(B x_{i}, x_{i}\right)
\end{aligned}
$$

Since $\left(B x_{i}, x_{i}\right)<0$ and $h>\hat{h}^{+} \geq h_{i}$,

$$
\begin{aligned}
\left(\left(\omega_{+} I+\omega_{+}^{-1} C\right) f, f\right) & <-h \sum|a i|^{2}\left(B x_{i}, x_{i}\right) \\
& =-\left(h B\left(\sum a_{i} x_{i}\right), \sum a_{j} x_{j}\right) \\
& =-(h B f, f)
\end{aligned}
$$

In other words, $\left(A\left(\omega_{+}, h\right) f, f\right)=\left(\left(\omega_{+} I+\omega_{+}^{-1} C+h B\right) f, f\right)<0$. So the subspace $S$ has all the required properties.

Similarly, for a fixed $h>\hat{h}^{-}$, if we check the changes in the inertias for $A(-\infty, h)$, $A\left(\omega_{-}, h\right)$ and $A(-\epsilon, h)$, it can be proved that $A(\omega, h)$ has at least $p$ eigenvalues in both $\left(-\infty, \omega_{-}\right)$and $\left(\omega_{-}, 0\right)$ and if exactly $p$, then all of them are of negative (respectively, positive) type. When $h>\hat{h}$, by the definition of $\hat{h}, h>\hat{h}^{+}$and $h>\hat{h}^{-}$. As the total number of $\omega$-eigenvalues of $A(\omega, h)$ is $2 n$, it follows that there must be exactly $p$ or $n-p$ eigenvalues in each interval, as appropriate. In particular, all eigenvalues are real and of definite type.

Corollary $6.8 h_{0} \leq \hat{h}$.

Proof: This follows from the definition of $h_{0}$ (as defined in Equation 6.8) and Theorem 6.7.

Example 6.3 Let $A(\omega, h) \doteq \omega I+\dot{+}^{-1} C+h \dot{B}$ where $B=\operatorname{diag}[-1,-2,4]$ and $C=$ $\operatorname{diag}[1,4,16]$.

The positive eigenvalue functions of $A(\omega, h)$ are sketched in Figure 6.5. Some calculation shows that $h_{0}=2$ and $\hat{h}=3 / \sqrt{2} \approx 2.12$.

Theorem 6.9 If eigenvalue function $h(\omega)>0$ is differentiable at $\omega_{0}>0$, and $\left.\frac{d h}{d \omega}\right|_{\omega=\omega_{0}}>0\left(<0\right.$, respectively), then $\omega_{0}$ is an eigenvalue of positive(negative, respectively) type.


Figure 6.5: The positive eigenvalue functions of Example 6.3

Proof: Suppose $h_{0}=h\left(\omega_{0}\right)>0$. For any $x \in \operatorname{ker} A\left(\omega_{0}, h_{0}\right), h(\omega)$ is the function determined implicitly by the equation in two variables

$$
F(\omega, h)=(A(\omega, h) x, x)=0 .
$$

By the implicit function theorem,

$$
\begin{aligned}
\left.\frac{\dot{d} \dot{h}}{d \omega}\right|_{\omega=\omega_{0}} & =-\frac{\left(\frac{\partial A}{\partial \omega}\left(\omega_{0}, h_{0}\right) x, x\right)}{\left(\frac{\partial A}{\partial h}\left(\omega_{0}, h_{0}\right) x, x\right)}, \\
& =-\frac{\left(\left(I-\omega_{0}^{-2} C\right) x, x\right)}{(B x, x)} .
\end{aligned}
$$

Since

$$
A\left(\omega_{0}, h_{0}\right) x=.\left(\omega_{0}+\omega_{0}^{-1} C+h_{0} B\right) x=0
$$

so !,

$$
(B x, x)=-\frac{1}{h_{0}}\left(\left(\omega_{0}+\omega_{0}^{-1} C\right) x, x\right)<0 .
$$

Hence the sign of $\left(\frac{\partial A}{\partial \omega}\left(\omega_{0}, h_{0}\right) x, x\right)$ coincides with the sign of $\left.\frac{d h}{d \omega} \right\rvert\, \omega=\omega_{0}$.

Finally, some comparisons with the results of other papers are made. Notice that when $h=1$, the eigenvalue problem of $A(\omega, 1)$ is exactly the same as that of $L(\lambda)$ in Equation (6.1). By. our Theorem 6.7, the condition (e), $\rho(B, C)<1$ implies that $A(\omega, 1)$ is strongly stable, so this result is equivalent to a main result of [LMZ1]. Namely, the condition (e) implies that $L(\lambda)$ is strongly stable. In other words, the result of [LMZ1] shows that $\rho(h B, C)<1$ can guarantee that $A(\omega, h)$ is strongly stable. Actually condition $\rho(h B, C)<1$ is equivalent to $\hat{h}=\rho(B, C)<h$, as our condition in Theorem 6.7.

In the paper [BLM], it is shown that if $|h B|>k I+k^{-1} C$ for some $k$, then $A(\omega, h)$ is strongly stable. Our condition $\hat{h}<h$ is better than that condition in the sense that $\hat{h}<h$ implies $|h B|>k I+k^{-1} C$ for some $k$, but the converse implication is not true.

## Chapter 7

## Conclusion

In this thesis we have studied the analytic perturbation theory for matrix functions by two methods. One is the Newton diagram method, and the other is the generating eigenvector method. Since the eigenvalue function $\lambda(\alpha)$ of a matrix function $L(\lambda, \alpha)$ is a solution function determined by the equation $\operatorname{det} L(\lambda, \alpha)=0$, and Newton's diagram is the main tool used to investigate these solution functions, using the Newton's diagram method to study the eigenvalue function $\lambda(\alpha)$ is natural and basic. The generating eigenvector method adds information on corresponding eigenvectors $x(\alpha)$ associated to $\lambda(\alpha)$. The "spirit" of this method is to collect the terms with the same power in $\alpha$ in $L(\lambda(\alpha), \alpha) x(\alpha)$ and choose $x$ to make the coefficient vanish. With the "help" of the eigenvector $x(\alpha)$, Theorem 4.14 establishes a successive computation procedure for the coefficients $\left\{b_{j}\right\}_{2}^{\infty}$ :for the eigenvalue series (4.25). This algorithm is better than the algorithm of Newton's diagram, since, in general we can not give the explicit determining equation even for $b_{2}$. The algorithm is also more productive, since a by-product is the solution $\left\{\xi_{j}\right\}_{1}^{\infty}$ for the eigenvector series (4.26). However the algorithm is limited to the case that $\lambda_{0}$ is a semisimple eigenvalue of $L(\lambda, 0)$ and $b_{1}$ is a simple root of the determining equation $\operatorname{det} P(\mu)=0$. Also in the theoretical part of Theorem 4.14, the proof of the existence and uniqueness of $\left\{b_{j}\right\}_{2}^{\infty}$ is given by Newton's diagram.

Interesting open questions are how to generalize the results in Theorem 4.14 to the cases of when $b_{1}$ is a multiple semisimple root of $\operatorname{det} P(\mu)=0$, or further when
$\lambda_{0}$ is a general multiple eigenvalue. A deeper question underneath or connected to these open questions is: parallel to the result that $\operatorname{det} P(\mu)=0$ is the determining equation for $b_{1}$, what is the determining equation for $b_{2}$ ? Comparing equations (4.28) and (4.29), the intuition suggests that if $\mathcal{K}_{1}=\operatorname{ker} P\left(b_{1}\right)$ and $\mathcal{K}_{1}^{\prime}=\operatorname{ker}\left(P\left(b_{1}\right)^{*}\right)$, then the determining equation for $b_{2}$ could be
where the notation [ $]_{\mathcal{K}, \mathcal{K}^{\prime}}$ is defined in (3.8), and [ $]_{\mathcal{K}_{1}, \mathcal{K}_{1}^{\prime}}$ can be defined similarly. To prove this guess, we need the knowledge of the D.E. for the second coefficients in the Puiseux series expansions in the scalar valued function case. In Section 2.8, there is an attemp in this direction, but the question is not totally solved.

I look forward to further investigation of some of these problems.

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