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INTRINSIC VOLUMES OF FINITE BALL-PACKINGS

BY

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Abstract

Let K be a convex, compact set in E^d, and B^d be the unit ball centered at the origin. We define the *ith* intrinsic volume $V_i(K)$ *via* the formula of Steiner; that is, for $\lambda > 0$,

$$V(K + \lambda B^d) = \sum_{i=0}^{d} \kappa_{d-i} \lambda^{d-i} \cdot V_i(K).$$

Note that $V_i(K)$ is proportional to the mean i-dimentional content of the projections of K onto the i-dimentional linear subspaces.

A finite set $\{x_1 + B^d, \ldots, x_n + B^d\}$ of unit balls is a packing if the interiors of any two balls are disjoint. The dissertation investigates the minimum properties of the convex hull of the balls with respect to the *ith* intrinsic volume for $i = 1, \ldots, d$, and the shape of $C_n = \text{conv}\{x_1, \ldots, x_n\}$ when $V_i(C_n + B^d)$ is minimal. In this case C_n is called a minimal body.

The shapes of minimal bodies depend on how large i and n are compared to d. If $i = 1, \ldots, d-1$ and n is large compared to d then a minimal body is basically ball. Assume that $n \leq d+1$. For small i the minimal body is probably a regular simplex, and if i is close to d then the minimal arrangement is probably the sausage-like one; that is, C_n is a segment. Finally we consider the case of the volume (if i=d). According to L. Fejes Tóth's celebrated Sausage Conjecture, for $d \geq 5$, the sausage arrangement is optimal for any n. We prove that a minimal body can not be far from being a segment.

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Chapter 1

Introduction

We present all the notions and theorems which we will require. In every section we quote the references containing the proofs.

We assume that the reader is familiar with the concepts of elementary set theory. Concerning the real number system R, we use the arithmetic properties of the real numbers, and the existence of functions like $\sin t$, $\cos t$, the logarithmic and the exponential functions. The references will be quoted at the point where they are applicable.

1.1 Linear algebra

In this section we consider some properties of R^d for $d \geq 1$. The proofs can be found in [13].

The letters p, q, r, s and t, together with the Greek letters α, β, ε, λ, μ and ρ denote real numbers, and d, i, j, k, m and n denote non-negative integers. The elements of R^d are denoted by u, v, w, x, y and z, and the zero-vector is denoted by 0.

Let $\lambda \in R$ and $x = (x^1, \ldots, x^d)$ and $y = (y^1, \ldots, y^d)$ be points of R^d. Define addition and scalar multiplication on R^d as

$$x + y = (x^1 + y^1, \ldots, x^d + y^d) \qquad \text{and} \qquad \lambda \cdot x = (\lambda\, x^1, \ldots, \lambda\, x^d).$$

We extend the definition of addition and scalar multiplication to subsets of R^d. For

$A, B \subset R^d$ and $\lambda \in R$,

$$A + B = \{a + b \mid a \in A \text{ and } b \in B\} \quad \text{and} \quad \lambda A = \{\lambda a \mid a \in A\}.$$

A subset $L \subset R^d$ is a *linear subspace* of R^d if $x + y \in L$ and $\lambda \cdot x \in L$ for any $x, y \in L$ and $\lambda \in R$..

Let $x_1, \ldots, x_n \in R^d$. A *linear combination* of $x_1, \ldots, x_n$ is the expression $\lambda_1 x_1 + \ldots + \lambda_n x_n$ for some $\lambda_1, \ldots, \lambda_n \in R$. The vectors $x_1, \ldots, x_n$ are *independent* if

$$\lambda_1 \cdot x_1 + \ldots + \lambda_n \cdot x_n = 0$$

implies that $\lambda_1 = \ldots = \lambda_n = 0$. Otherwise, the vectors are *dependent.* Let L be a subspace of R^d, $L \neq \{0\}$. The maximal cardinality n of an independent subset of L is called the *dimension* of L, in notation, $n = \dim L$. An independent set $\{x_1, \ldots, x_n\}$ is a *basis* of L; that is, any $y \in L$ can be written in a unique way as a linear combination of $x_1, \ldots, x_n$. Note that L can be identified with R^n, and hence $\dim L \leq d$, with equality if and only if $L = R^d$. We let $\dim\{0\} = 0$.

A map $T : R^d \to R^d$ is a *linear map* if for any $u, v \in R^d$ and $\lambda, \mu \in R$,

$$T(\lambda u + \mu v) = \lambda T(u) + \mu T(v).$$

If there is a map $S : R^d \to R^d$ such that $S \circ T = T \circ S = id_{R^d}$ then T is an *invertible* map, and $S = T^{-1}$ is the inverse of T. Here $S \circ T$ denotes the composition of S and T, and id_X is the identity map of X.

The standard basis of R^d is $\{e_1, \ldots, e_d\}$, where for $i = 1, \ldots, d$, the *ith* coordinate of e_i is 1, and all the other coordinates are 0. We always use this basis for R^d. To every linear map $T : R^d \to R^d$, we assign the matrix

$$[u_1, \ldots, u_d] = \begin{bmatrix} u_1^1 & \ldots & u_d^1 \\ \vdots & & \vdots \\ u_1^d & \ldots & u_d^d \end{bmatrix},$$

where $T(e_i) = u_i = (u_i^1, \ldots, u_i^d)$. The matrix and the map are called *integral* if all the entries of the matrix are integers.

Let Π_d be the group of permutations on the set $\{1, \ldots, d\}$, and sgn$t$ be the function satisfying sgn$t = 1$ if $t > 0$ and sgn$t = -1$ if $t < 0$. For $\pi \in \Pi_d$, define

$$\varepsilon(\pi) = \prod_{1 \le i < j \le d} \mathrm{sgn}(\pi(j) - \pi(i)).$$

The determinant of the matrix $[u_1, \ldots, u_d]$ is

$$\det[u_1, \ldots, u_d] = \sum_{\pi \in \Pi_d} \varepsilon(\pi) u_1^{\pi(1)} \cdot \ldots \cdot u_d^{\pi(d)}.$$

The determinant of a linear map is the determinant of the corresponding matrix. Then $\det(S \circ T) = \det T \cdot \det S$, and T is invertible if and only if $\det T \neq 0$. Assume that T is invertible. Then T is an *orientation-preserving* map if $\det T > 0$, and an *orientation-reversing* map otherwise.

Since we are interested in metrical properties, we put some additional structure on R^d. The quadratic form

$$< x, y >= x^1 \cdot y^1 + \ldots + x^d \cdot y^d \qquad \text{for} \quad x, y \in R^d,$$

is linear in both variables, and for any $x \in R^d$, $< x, x > \geq 0$, with equality if and only if $x = 0$. This form is called *scalar product.* With the help of this we define the norm, or length, of a vector x as

$$\|x\| = \sqrt{< x, x >}.$$

The norm satisfies the usual axioms; that is, for any $x, y \in R^d$ and $\lambda \in R$,

i) $\|\lambda \cdot x\| = |\lambda| \cdot \|x\|$,

ii) $\|x\| \geq 0$, with equality if and only if $x = 0$,

iii) $\|x + y\| \leq \|x\| + \|y\|$.

The Cauchy-Schwarz inequality states that

$$| < x, y > | \leq \|x\| \cdot \|y\|$$

for any $x, y \in R^d$. Hence for non-zero x and y, we may define $\alpha = \text{ang}(x, 0, y)$ by the properties $0 \leq \alpha \leq \pi$ and

$$\cos \alpha = \frac{< x, y >}{\|x\| \cdot \|y\|}.$$

In addition, for distinct $x, y, z \in R^d$, let $\text{ang}(x, y, z) = \text{ang}(x - y, 0, z - y)$.

We note that, by means of Hadamard's inequality,

$$\det[u_1, \ldots, u_d] \leq \|u_1\| \cdot \ldots \cdot \|u_d\|$$

for any $u_1, \ldots, u_d \in R^d$.

Two subspaces V and W of R^d are *orthogonal* (or *perpendicular*) if $< v, w >= 0$ for any $v \in V$ and $w \in W$. This yields that $\dim V + \dim W \leq d$, with equality if and only if $V + W = R^d$. In the case of equality, any $u \in R^d$ can be uniquely written in the form $u = v + w$ for $v \in V$ and $w \in W$.

Finally, a linear map $T : R^d \to R^d$ is *orthogonal* if $< T(x), T(y) >=< x, y >$ for any $x, y \in R^d$. Note that then $\det T = \pm 1$, and that the composition of orthogonal maps is also orthogonal.

As an example, let T be an orientation-preserving orthogonal transformation of

R^2. Then the matrix of T is

$$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

for some $-\pi \leq \alpha \leq \pi$, and T is called the *rotation with angle α around 0.* We remark that $\mathrm{ang}(x, 0, T(x)) = |\alpha|$ for any $x \neq 0$.

1.2 Metric spaces

We rely mainly on [20]. The book [6] is easier to read and contains most of the material except the notion of sequential compactness.

Let X be a non-empty set. A function $d : X \times X \to R$ is a *metric* on X if it satisfies

i) $d(x,y) \geq 0$, with equality if and only if $x = y$,

ii) $d(x,y) = d(y,x)$,

iii) $d(x,z) \leq d(x,y) + d(y,z)$.

The last property iii) is known as the *triangle inequality.* Observe that a non-empty subset of metric space is also a metric space with the inherited metric. For $x \in X$ and non-empty $A \subset X$, let

$$d(x,A) = \inf\{d(x,y) \,|\, y \in A\}.$$

Define the open ball $B(x,r)$, with center $x \in X$ and radius $r > 0$, as

$$B(x,r) = \{y \in X \,|\, d(x,y) < r\}.$$

Observe that $x \in B(x,r)$. A non-empty subset $\Sigma \subset X$ is *discrete* if for each $x \in \Sigma$, there is an $r > 0$ such that $B(x,r)$ contains no point of Σ besides x. A subset

$A \subset X$ is called *open* if for any $x \in A$ there exists an $r > 0$ such that $B(x,r) \subset A$. Note that union and finite intersection of open sets is open. A is *closed* provided its complement $X \backslash A$ is open.

Let $A \subset X$ and $x \in X$. Then A is a *neighbourhood* of x if there exists an $r > 0$ with $B(x,r) \subset A$. The *interior* of A, denoted by $\operatorname{int} A$, is the union of all the open sets C with $C \subset A$. The *boundary* of A, denoted by $\operatorname{bd}A$, is the set of all points $x \in X$ such that every neighbourhood of x contains some point of A and some point of $X \backslash A$. Note that A is open if and only if $A = \operatorname{int}A$ and A is closed if and only if $A = \operatorname{int}A \cup \operatorname{bd}A$.

Let $\{x_n\}_{n=0}^{\infty}$ be a sequence of points of X. The sequence *converges* to an $x \in X$ ($\{x_n\} \to x$ or $lim_{n\to\infty} x_n = x$) if for any $r > 0$ there is a $N > 0$ such that $x_n \in B(x,r)$ for $n > N$. The sequence $\{x_n\}_{n=0}^{\infty}$ is *convergent* if there is an $x \in X$ with $\{x_n\} \to x$.

Let (X,d) and (Y,d') be metric spaces. A function $f : X \to Y$ is *continuous* if $\{x_n\} \to x$ implies that $\{f(x_n)\} \to f(x)$ in Y for $x, x_n \in X$. Note that in this case, $f^{-1}(B)$ is open for any open $B \subset Y$. The function f is an *isometry* if f is a bijection and $d'(f(x), f(y)) = d(x,y)$ for any $x, y \in X$.

As an example, consider R^d for $d \geq 1$. Then the function $d(x,y) = |x - y|$ for $x, y \in R^d$, is a metric. Let $GL(d)$ be the set of invertible $d \times d$ matrices. Since $GL(d)$ can be considered as a subset of R^{d^2}, it is a metric space with the inherited metric. Then the determinant, as a function $GL(d) \to R$, is continuous.

Now we consider some possible properties of a metric space X. It is called *bounded* if there is an $M > 0$ such that $d(x,y) < M$ for any $x, y \in X$. X is *compact* if any open cover of X has a finite subcover. Note that a compact metric space is bounded, and a closed subset of a compact metric space is also compact. Another family of metric spaces is the *sequentially compact* spaces, which have the property that any infinite sequence of elements of X has a convergent subsequence. Actually, these two different notions are equivalent; that is, X is compact if and only if it is sequentially compact. Let $f : X \to R$ be continuous and X be compact. Then there is an $x \in X$ such that

$$f(x) = \inf\{f(y) \mid y \in X\}.$$

A subset $C \subset R^d$ is compact if and only if C is closed and bounded. The statement that closed and bounded subsets of R^d are sequentially compact is known as the Bolzano-Weierstrass theorem.

1.3 The Euclidean space

There are many books which give introduction to the Euclidean space. The classical survey is [4]. More recent ones are [1], [17] and [21].

Let $d \geq 1$, $0 \leq n \leq d$, $x \in R^d$ and L be a linear subspace of R^d with $\dim L = n$. Then $A = x + L$ is called an n-dimensional *affine subspace*. The points of R^d are the 0-dimensional affine subspaces. The 0 vector, as a point of R^d, is frequently called the *origin*. The one-dimensional affine subspaces are called *lines*, the two-dimensional affine subspaces are called *planes*, and the $d-1$ dimensional ones are the *hyperplanes* of R^d.

Remark: From a geometrical point of view, it makes no difference whether an affine subspace contains the origin. This fact is emphasised in [1]. The reason to start with linear subspaces is that this way we can use the powerful tools of algebra and analysis.

Observe that a non-empty intersection of affine subspaces is an affine subspace. For a non-empty $C \subset R^d$ we define aff$C$, the *affine hull* of C, as the intersection of all the affine subspaces of R^d containing C, and set dim$C$ =dim aff$C$. We also say that C *spans* aff$C$.

Let B_1 and B_2 be affine subspaces with $B_1 \cap B_2 \neq \emptyset$ where $\emptyset$ denotes the emptyset. Then

$$\dim \text{aff}(B_1 \cup B_2) + \dim(B_1 \cap B_2) = \dim B_1 + dim B_2.$$

A set $C \subset R^d$ is *convex* if it is non-empty and $\lambda x + (1-\lambda)y \in C$ for any $x, y \in C$ and $0 \leq \lambda \leq 1$. Since a non-empty intersection of convex sets is also convex, we define the *convex hull* of a non-empty $B \subset R^d$ as the intersection of all the convex sets containing B. The convex hull of B is denoted by conv$B$.

Let $d \geq 2$, $m \geq 3$ and $|\Sigma| = m$. The points of the set Σ are in *general position* if for any $1 \leq n \leq d-1$, each n-dimensional affine subspace contains at most $n+1$ elements of Σ. If $m \geq d+1$ then the points of Σ are in general position if each hyperplane contains at most $d+1$ elements of Σ.

If R^d is considered as an affine space with the scalar product then we call it a *d-dimensional Euclidean space*, and denote it by E^d. By E^0 we mean a one-point set. Let A be an affine subspace of E^d. Then A can be endowed with the structure of an n-dimensional Euclidean space, so that the metric on A is induced by the metric on E^d, and the affine subspaces of A are the affine subspaces of E^d which are contained in A.

Let C be a non-empty subset of E^d. Then the interior and the boundary of C in aff$C$ is denoted, respectively, by relint$C$ and relbd$C$.

Recall that e_i is the *ith* standard basis vector of E^d. The line aff$\{0, e_i\}$ is called the *ith coordinate axis.* Let $1 \leq n \leq d$. If we work in E^d then we refer to the affine hull of the first n coordinate axes as E^n.

Let L_i be a linear subspace of E^d with positive dimension, $x_i \in E^d$ and $A_i = x_i + L_i$ for $i = 0, 1$. We say that A_0 and A_1 are *parallel* if $L_0 \subset L_1$ or $L_1 \subset L_0$. The affine subspaces A_0 and A_1 are *orthogonal* to each other if L_0 and L_1 are orthogonal. A non-zero vector v is said to be *orthogonal* (or *parallel*) to an affine subspace A with positive dimension if the line $\text{aff}\{0, v\}$ is orthogonal (or parallel) to A. As a convention, the 0 vector is parallel and orthogonal to each affine subspace.

Let A be an affine subspace of E^d with $n = \dim A$ and $x \in E^d$. There exists a unique affine subspace B which is orthogonal to A, contains x and has dimension $d - n$. Note that A and B have exactly one common point. In addition, if $x \notin A$ then there is a unique line l which passes through x, intersects A and is orthogonal to A. The point $l \cap A$ is called the orthogonal projection of x onto A.

The hyperplanes of E^d have an important role in the geometry of E^d. Let H be a hyperplane and u be a non-zero vector perpendicular to H. Then there is an $\alpha \in R$ such that

$$H = \{x \in E^d | < x, u >= \alpha\}.$$

The sets

$$H^+ = \{x \in E^d | < x, u > \geq \alpha\} \quad \text{and} \quad H^- = \{x \in E^d | < x, u > \leq \alpha\}$$

are called the *halfspaces* of E^d determined by H. Observe that H^+ and H^- are convex, $E^d = H^+ \cup H^-$ and $\text{bd}H^+ = \text{bd}H^- = H$. Let B and C be non-empty subsets of E^d. We say that the hyperplane H *separates* B and C if B is contained in one of the halfspaces determined by H while C is contained in the other. If in addition, $B \cap H = C \cap H = \emptyset$ then H *strictly separates* B and C.

Let u and v be distinct points and $w = \frac{1}{2}(u+v)$. The hyperplane passing through w and perpendicular to $u - v$ is called the *hyperplane perpendicularly bisecting the segment* $\text{conv}\{u, v\}$. Note that

$$H = \{x \in E^d | \, d(x,u) = d(x,v)\}.$$

Let H^+ be the halfspace determined by H and containing u. Then $x \in H^+$ if and only if $d(x,u) \leq d(x,v)$.

Finally we turn to the transformations of E^d. A map $\varphi(x) = T(x) + y$ is called an *affine map* if T is an invertible linear map and $y \in E^d$. The affine transformations are the bijections of E^d which map any affine subspace onto an affine subspace of the same dimension, while preserving incidence. The composition of affine maps is again an affine map. The isometries of E^d are the affine transformations $\varphi(x) = T(x) + y$ where T is an orthogonal map. An isometry $\varphi(x) = T(x) + y$ is called *orientation-preserving* if T is *orientation-preserving*; that is, $\det T = 1$.

Note that if the points of both of the sets $\{x_0, \ldots, x_d\}$ and $\{y_0, \ldots, y_d\}$ are in general position then there is a unique affine map $\varphi(x)$ such that $\varphi(x_i) = y_i$ for $i = 0, \ldots, d$. If in addition $d(x_i, x_j) = d(y_i, y_j)$ for $0 \leq i < j \leq d$ then $\varphi(x)$ is an isometry. If for non-empty $C, D \subset E^d$ there is an isometry $\varphi(x)$ so that $\varphi(C) = D$ then C and D are *congruent.*

Let us list some isometries of E^d which occur later in the text. Let $y \in E^d$. The map $\varphi_0(x) = x + y$ is called *translation by* y. The *reflection through* y is defined by $\varphi_1(x) = -x + 2y$. Note that $y = \frac{1}{2}(x + \varphi_1(x))$ for any $x \in E^d$.

Let H be a hyperplane and for $x \in X$, let $\pi_H(x)$ be the projection onto H. We define $\varphi_2(x)$ as the unique point satisfying $\pi_H(x) = \frac{1}{2}(x + \varphi_2(x))$. The map $\varphi_2(x)$ is an isometry, and it is called *reflection through H*.

For $d \geq 2$, let g be a $(d-2)$-dimensional affine subspace and $\pi_g(x)$ be the projection onto g. An orientation-preserving isometry $\varphi_3(x)$, which fixes g pointwise, is called *rotation around g*. Let Π be a plane orthogonal to g and $\Pi \cap g = \{y\}$. Recall that in Section 1.1 we have defined the rotation of E^2 around the origin. Note that $\varphi_3(x)$ maps Π onto Π, and $\varphi_3(x)$ restricted to Π is a rotation around y. It follows that for $x \notin g$, the $\mathrm{ang}(x, \pi_g(x), \varphi_3(x))$ is independent of x, and it is called *the angle of the rotation.*

1.4 Compact, convex sets

We start with a simple theorem which connects the notions of convexity and dimension.

THEOREM 1.4.1 (Radon) *Let $d \geq 1$ and $m \geq d+2$, and $x_1, \ldots, x_m$ are points in E^d. Then the points can be partitioned into two non-empty sets, say $x_1, \ldots, x_n$ and $x_{n+1}, \ldots, x_m$, $2 \leq n \leq m-1$, such that*

$$conv\{x_1, \ldots, x_n\} \cap conv\{x_{n+1}, \ldots, x_m\} \neq \emptyset.$$

Let $d \geq 1$ and $\mathcal{K}^d$ be the family of compact, convex sets of E^d. A $K \in \mathcal{K}^d$ is called a *convex body* if $\dim K = d$, or equivalently, if $\mathrm{int} K \neq \emptyset$.

The isometries of E^d induce an equivalence relation on $\mathcal{K}^d$; namely, two elements of $\mathcal{K}^d$ are equivalent if they are congruent. Let $C_0, C_1 \in \mathcal{K}^d$. We write $C_0 \equiv C_1$ if C_0 and C_1 are congruent.

Let $K \in \mathcal{K}^d$. A hyperplane H is a *supporting hyperplane* of K if $H \cap K \neq \emptyset$ and one of the halfspaces determined by H contains K. For any $x \in \operatorname{bd} K$, there is a supporting hyperplane containing x.

Now we collect some properties of the linear combinations of compact, convex sets. For $\lambda_0, \lambda_1 \in R$, we have $\lambda_0 C_0 + \lambda_1 C_1 \in \mathcal{K}^d$. If $C_i = \operatorname{conv}\Sigma_i$ for some non-empty $\Sigma_i \in E^d$, $i = 0, 1$, then $\lambda_0 C_0 + \lambda_1 C_1 = \operatorname{conv}(\lambda_0 \Sigma_0 + \lambda_1 \Sigma_1)$. If C_0 is strictly contained in C_1 then $C_0 + K$ is also strictly contained in $C_1 + K$.

The first statement of the following lemma actually characterizes the elements of $\mathcal{K}^d$ among the compact sets of E^d.

LEMMA 1.4.2 *For any $K \in \mathcal{K}^d$ and $x \notin K$ there exists a unique $y \in K$ such that $d(x, y) = \min\{d(x, z) | z \in K\}$. If $1 \leq \dim K \leq d - 1$ and $y \in \operatorname{relint} K$ then $x - y$ is perpendicular to $\operatorname{aff} K$.*

For $C, K \in \mathcal{K}^d$, define

$$\delta(C, K) = \inf\{\, \rho \,|\, C \subset K + \rho B^d \text{ and } K \subset C + \rho B^d\}.$$

This function is a metric on $\mathcal{K}^d$, and it is called the *Hausdorff distance*. Endowed with this, $\mathcal{K}^d$ becomes a metric space.

A non-empty family $\mathcal{F} \subset \mathcal{K}^d$ is bounded with respect to the Hausdorff metric if and only if there is a $\rho > 0$ so that $K \subset \rho B^d$ for any $K \in \mathcal{F}$.

THEOREM 1.4.3 (Blaschke) *Each bounded, infinite sequence of compact convex sets has a convergent subsequence.*

In other words, the compact subspaces of $\mathcal{K}^d$ are the closed, bounded subsets.

An $x \in K$ is an *extreme point* of $K \in \mathcal{K}^d$ if $x = \lambda y + (1-\lambda)z$ for $y, z \in K$ and $0 < \lambda < 1$, implies that $x = y = z$. Denote the set of extreme points of K by ext$K$. Then $K = \text{conv ext}K$, and ext$K$ is the smallest set B with the property that $K = \text{conv}B$.

A polytope P is the convex hull of finitely many points, and hence $P \in \mathcal{K}^d$. A $F \subset \text{bd}P$ is called a *face* of P if there is a supporting hyperplane H of P with $H \cap P = F$. It is frequently convenient to consider P itself as a d-dimensional face. The faces of a polytope are polytopes themselves. The 0-dimensional faces of a polytope are called *vertices*, the one dimensional faces are called *edges*, the $(d-1)$-dimensional faces are called *facets.* If $d = 2$ then the edges are also called *sides*, and if $d = 3$ then the facets are simply called *faces.*

The vertices of P constitute ext$P$. Let $dimP \geq 1$ and $0 \leq m \leq n \leq dimP - 1$. Then each m-dimensional face of P is contained in some n-dimensional face of P, and hence if P is a polytope then bd$P$ is the union of the facets of P. In addition, if F is a face of P, and F' is a face of F then F' is a face of P.

Let $d \geq 1$, $C \in \mathcal{K}^d$ with dim$C = d - 1$ and x be a point outside of aff$C$. The set conv$(\{x\} \cup C)$ is called a *cone* with *base* C and $d(x, \text{aff}C)$ is the *height* of the cone. If $x_0, \ldots, x_d$ are points in general position then $S = \text{conv}\{x_0, \ldots, x_d\}$ is called *a d-simplex.* Observe that every d-simplex is a cone, and each facet of S is a $(d-1)$-dimensional simplex which is a base of S. If all the edges have equal length then S is a *regular d-simplex.* As a convention, we consider the points as 0-dimensional regular simplices.

A polytope P is called a *paralleletope* if there are independent vectors $x_1, \ldots, x_d$ so that P is congruent to the set

$$\{\lambda_1 x_1 + \ldots + \lambda_d x_d \,|\, 0 \leq \lambda_i \leq 1 \text{ for } i = 1 \ldots, d\}.$$

If all the x_i's have the same length and $< x_i, x_j >= 0$ for $i \neq j$ then P is a *cube.*

1.5 Simplices, ellipsoids and radii

Let $d \geq 1$, $y \in E^d$ and $r > 0$, and define

$$B^d = \{x \in E^d | \, \|x\| \leq 1\}.$$

We call $y + r\,B^d$ the *d-dimensional ball with center y and radius r*, and $y + B^d$ is a *unit ball.* The boundary of B^d is the *unit sphere* S^{d-1}. Note that S^{d-1} is compact and

$$S^{d-1} = \{x \in E^d \,|\, \|x\| = 1\}.$$

Let $K \in \mathcal{K}^d$ and $1 \leq m \leq d$. Then the *m-dimensional inner radius of K*, $r_m(K)$, is the radius of the largest m-dimensional ball contained in K. The *m-dimensional outer radius of K*, $R_m(K)$, is the minimum of $R_m \geq 0$ such that there exists an $(m-1)$-dimensional affine subspace g with $K \subset g + R_m B^d$. We remark that $dim K < m$, $r_m(K) = 0$ and $R_m(K) = 0$ are equivalent statements.

We define the *inradius* of K as $r(K) = r_d(K)$, the *circumradius* of K as $R(K) = R_0(K)$. There is a unique ball which contains K and has radius $R(K)$. This ball is called the *circumscribed ball* of K. In addition, $D(K) = 2r_1(K)$ is the *diameter* of K, which is the maximal distance $d(x, y)$ for $x, y \in K$.

Let u be a unit vector. The *width of K in the direction of u* is defined as

$$\Delta(K,u) = \max\{< x,u > | x \in K\} - \min\{< x,u > | x \in K\},$$

and the *width* of K is defined as

$$\Delta(K) = \min\{\Delta(K,u) | u \in S^{d-1}\}.$$

Note that $\Delta(K) = 2R_d(K)$. If $\dim K = m$ then the *relative width of K* is $\Delta_m(K) = 2R_m(K)$ which is the width of K in aff$K$.

The functions $D(K)$, $R(K)$ and $r(K)$ are continuous with respect to the Hausdorff distance (but for example, $r_m(K)$ is not continuous for $1 < m < d$).

Observe that if K is a ball of any dimension then $r_m(K) = R_m(K)$ for $1 \leq m \leq d$. According to [23],

THEOREM 1.5.1 *Let $1 \leq m \leq d$ and $K \in \mathcal{K}^d$. Then*

$$r_m(K) \leq R_m(K) \leq (m+1)r_m(K).$$

The upper bound is not precise. If $m = 0$ or $m = d$ then best estimates are known. Concerning the circumradius and the diameter, this is Jung's theorem:

THEOREM 1.5.2 *Let $d \geq 1$ and $K \in \mathcal{K}^d$. Then*

$$2R(K) \geq D(K) \geq \sqrt{\frac{2(d+1)}{d}}R(K).$$

In the case $m = d$, we have

THEOREM 1.5.3 *Let $d \geq 1$ and $K \in \mathcal{K}^d$. Then*

$$2r(K) \leq \Delta(K) \leq \begin{cases} \frac{2(d+1)}{\sqrt{d+2}}\, r(K) & \text{if } d \text{ is even} \\ 2\sqrt{d}\, r(K) & \text{if } d \text{ is odd} \end{cases}$$

Remark: If K is a regular d-simplex then equality holds in both theorems.

The affine images of B^d are called *ellipsoids.* Let M be an d-dimensional ellipsoid. There exists an isometry φ of E^d so that the equation of $\varphi(M)$ is

$$\frac{(x^1)^2}{a_1^2} + \cdots + \frac{(x^d)^2}{a_d^2} \leq 1$$

for some positive $a_1, \ldots, a_d$. The numbers $a_1, \ldots, a_d$ are called the *axes* of M.

Actually, the shape of a convex body can not be too far from the shape of some ellipsoid ([24]):

THEOREM 1.5.4 (John) *Let K be a convex body in E^d. Then there exists a point x and an ellipsoid M such that*

$$x + M \subset K \subset x + d\,M,$$

and if in addition K is centrally symmetric with center y then there exists an ellipsoid M with

$$y + M \subset K \subset y + \sqrt{d}\,M.$$

The rest of the section is concerned with simplices.

LEMMA 1.5.5 *For $d \geq 1$, let $S = conv\{x_0, \ldots, x_d\}$ be a d-simplex and $y \in S$ different from the vertices. Then*

i) $d(y, x_0) < \max_{i=1,\ldots,d}\{d(x_0, x_i)\}$, *and*

ii) *there is a $0 \leq i \leq d$ such that $d(y, x_i) \leq R(S)$.*

If $d = 2$ then the d-simplices are called *triangles.* By a side of a triangle we frequently mean not only the actual side but also its length. Let $C = \mathrm{conv}\{x_0, x_1, x_2\}$

be a triangle and $\{i,j,k\} = \{0,1,2\}$. Then $a_i = |x_j - x_k|$ is a side of C, and $\alpha_i = ang(x_j, x_i, x_k)$ is called an *angle of* C. The sides and angles of C obey the Law of cosines:

$$a_0^2 = a_1^2 + a_2^2 - 2a_1a_2\cos\alpha_0.$$

The next lemma follows from the Law of cosines.

LEMMA 1.5.6 *Let C_i be a triangle with sides a_i, b_i and c_i, and let γ_i be the angle of C_i opposite to c_i, $i = 0,1$. If $a_0 = a_1$, $b_0 = b_1$ and $\gamma_0 < \gamma_1$ then $c_0 < c_1$.*

Let $d \geq 0$. The regular d-simplex with edge length 2 is denoted by T^d. If $T^d = \text{conv}\{x_0, \ldots, x_d\}$ then the point $\frac{1}{d+1}(x_0 + \ldots + x_d)$is called the *center* of T^d. This point is the center of the circumscribed ball and the inscribed ball of T^d. The radius of the circumscribed ball is $R(T^d) = \sqrt{\frac{2d}{d+1}}$, and the radius of the inscribed ball is $r(T^d) = \sqrt{\frac{2}{d(d+1)}}$. The height h_d of T^d is the distance of any vertex of T^d from the opposite facet. Note that for $d \geq 1$,

$$h_d = \sqrt{4 - R(T^{d-1})^2} = \sqrt{\frac{2(d+1)}{d}}.$$

1.6 Some notions of calculus

An $r \in R$ is called *algebraic* if there exists a polynomial $p(t)$ with integer coefficients such that $p(r) = 0$. The sum and product of algebraic numbers is also algebraic. If $r \in R$ is not algebraic then it is *transcendental.* The sum or product of an algebraic and a transcendental number is transcendental.

Let n and i be non-negative integers, and define i *factorial* as $i! = 1 \cdot \ldots \cdot i$ if

$i > 0$ and $0! = 1$. In addition, let

$$\binom{n}{i} = \frac{n \cdot \ldots \cdot (n-i+1)}{i!}.$$

Note that $\binom{n}{i}$ is the number of i element subsets of a set of cardinality n.

We frequently approximate given real numbers. All the calculated values are cited up to four or five decimal digits, and they are not rounded. If a value is used later, we use a more accurate approximation with at least ten digits and not the cited one.

The *Landau symbols* make it easier to read asymptotic formulae. Let $d \geq 1$ and $f(x)$ and $g(x)$ be functions from a subset of E^d into R. Then we write $f(x) = O(g(x))$ if there exists a $c > 0$ such that $\|f(x)\| \leq c\,\|g(x)\|$. Observe that $f(x) = O(1)$ means that $f(x)$ is bounded. Let $d = 1$. We write $f(t) = o(g(t))$ if $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 0$, and $f(t) \sim g(t)$ if $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1$.

We assume that the reader is familiar with the notions of limit and derivative of a function $f : R \to R$ (see [28]). We quote only the results which we explicitly need.

Fix an open interval (a, b), where we allow a to be $-\infty$ and b to be ∞. Let $f, g : (a, b) \to R$ be differentiable functions, and $r, s \in (a, b)$ with $r < s$. The *mean value theorem* states that there is a t for which $r < t < s$ such that

$$f(s) - f(r) = f'(t)(s - r).$$

Suppose $g(t) \neq 0$ in a neighbourhood of b and

$$\lim_{t\to b^-} \frac{f'(t)}{g'(t)} = \alpha$$

for some $\alpha \in R$. In addition, assume that as $t \to b^-$ either $g(t) \to \infty$, or $f(t) \to 0$ and $g(t) \to 0$. By means of *L'Hospital's rule*, we conclude that

$$\lim_{t \to b^-} \frac{f(t)}{g(t)} = \alpha.$$

Note that a similar statement holds if $t \to a^+$.

Let $f : (a, b) \to R$ be a function. Then f is *convex* if for any $r, s \in (a, b)$ with $r < s$ and $0 < \lambda < 1$,

$$f(\lambda r + (1 - \lambda)s) \leq \lambda f(r) + (1 - \lambda)f(s).$$

If strict inequality holds for any suitable choice of r, s and λ then f is *strictly convex.* The function $f(t)$ is *concave* if $-f(t)$ is convex.

Let f be continuous. Then f is convex if and only if

$$f(\tfrac{1}{2}(r + s)) \leq \tfrac{1}{2}(f(r) + f(s))$$

for any $r, s \in (a, b)$. Assume that f is twice differentiable. Then f is convex if and only if $f''(t) \geq 0$ for any $t \in (a, b)$, and if $f''(t) > 0$ for any $t \in (a, b)$ then f is strictly convex.

So let f be a convex function on (a, b), and $r, s \in (a, b)$ with $r < s$. Note that for $r \leq t \leq s$,

$$f(t) \leq \frac{f(s) - f(r)}{s - r}(t - r).$$

Let $t \in (r, s)$. It follows that $f(t) \leq \max\{f(r), f(s)\}$, and if f is strictly convex then $f(t) < \max\{f(r), f(s)\}$.

If f is concave then the dual statements hold.

1.7 Integration, volume and the Gamma function

Let $d \geq 1$, C be a convex subset of E^d and $f : C \to R$ be a continuous function. We denote the Riemann integral of f on C (see [27]) by

$$\int_C f(x)\,dx,$$

provided the integral exists. If $d = 1$ and C is the interval (a, b), $-\infty \leq a < b \leq \infty$, then we use the notation

$$\int_C f(t)\,dt = \int_a^b f(t)\,dt.$$

Assume that $f(t) = F'(t)$ for some $F : (a, b) \to R$. The Fundamental Theorem of Calculus states that

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

We return to the case $d \geq 1$. Let T be an invertible linear map, $y \in E^d$ and $\varphi(x) = T(x) + y$. Then the Jacobian determinant of $\varphi(x)$ equals $\det T$ and the standard change of variables formula may be written as

$$\int_{\varphi(C)} f \circ \varphi^{-1}(x)\,dx = |det\,T| \cdot \int_C f(x)\,dx. \tag{1.1}$$

Let $C \in \mathcal{K}^d$. The *volume* of C is defined as

$$V(C) = \int_C 1\,dx.$$

If $d = 2$ then the volume is called *area*. It follows by 1.1 that the volume is isometry invariant. Let T be an invertible linear map and $\lambda > 0$. Then $V(T(C)) = |det\,T| \cdot V(C)$, and hence $V(\lambda C) = \lambda^d V(C)$. We denote $V(B^d)$ by κ_d.

For $1 \leq n \leq d$, denote the convex hull of the last n coordinate axes by $\tilde{E}^n$.

THEOREM 1.7.1 (Fubini) *If f is an integrable function on E^d and $1 \leq m < d$ then*

$$\int_{E^d} f(x)dx = \int_{E^m} \int_{\tilde{E}^{d-m}} f(u,v)\, dv\, du,$$

where $x=(u,v)$ with $u \in E^m$ and $v \in \tilde{E}^{d-m}$.

With the help of the Fundamental Theorem of Calculus it is not hard to prove

THEOREM 1.7.2 *Let $d \geq 1$ and assume that the integrable function $f : \rho B^d \to R$, $0 < \rho \leq \infty$, satisfies $f(x) = g(\|x\|)$ for some $g : R \to R$. Then*

$$\int_{\rho B^d} f(x)dx = d\,\kappa_d \int_0^\rho g(r) r^{d-1} dr. \tag{1.2}$$

In the rest of the section we work with the *Gamma function*, which is defined for $0 < t < \infty$ as

$$\Gamma(t) = \int_0^\infty s^{t-1} e^{-s} ds.$$

The integral is well-defined for these t. All we need to know about $\Gamma(t)$ can be found in [28]. First we list some basic properties:

i) $\Gamma(t+1) = t\,\Gamma(t)$ for $t > 0$,

ii) $\Gamma(n+1) = n!$ for n=0,1,2,...,

iii) $\ln \Gamma(t)$ is convex for $t > 0$.

Note that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Let t and s be positive. Then iii) yields that

$$\left(\Gamma\left(\frac{t+s}{2}\right)\right)^2 \leq \Gamma(t) \cdot \Gamma(s). \tag{1.3}$$

The function $\Gamma(t)$ can be used to evaluate the integral

$$\int_0^1 \tau^{t-1}(1-\tau)^{s-1}\, d\tau = \frac{\Gamma(t)\,\Gamma(s)}{\Gamma(t+s)}. \tag{1.4}$$

If t tends to infinity then Stirling's formula states that

$$\begin{aligned} \ln\Gamma(t+1) &= t\ln t+\frac{1}{2}\ln t-t+\ln\sqrt{2\pi}+o(1) \\ &= t\ln t+\frac{1}{2}\ln t-t+O(1). \end{aligned} \tag{1.5}$$

1.8 The intrinsic volumes

Let $d \geq 1$ and $K \in \mathcal{K}^d$. According to Steiner's theorem, there are scalars $V_i(K)$, $i = 0, \dots, d$, such that

$$V(K+\lambda B^d)=\sum_{i=0}^{d}\kappa_{d-i}\lambda^{d-i}\cdot V_i(K)$$

for any $\lambda > 0$. The scalar $V_i(K)$ is called the *ith intrinsic volume.* They were introduced in this form in [22]. The existence of these scalars and their main properties are discussed in all the cited works about convexity (the approach in [1] is probably the simplest).

Note that $V_0(K) = 1$ and if $\dim K \leq i$ then $V_i(K)$ is the i-dimensional volume of K. We define $2\,V_{d-1}(K)$ to be the *hypersurface area* of K, denoted by $S(K)$ (we shall justify this definition later). If $d = 2$ then the hypersurface area is called the *perimeter.*

Let $1 \leq i \leq d$ and $C, K \in \mathcal{K}^d$. The function $V_i(K)$ is non-negative, continuous and monotonic; that is, $C \subset K$ yields that $V_i(C) \leq V_i(K)$. If $\dim C \geq i$, $C \subset K$ and $C \neq K$ then $V_i(C) < V_i(K)$. The reason to choose the intrinsic volumes and not another normalization (for example the *Quermassintegrals* in [17]), is that the intrinsic volumes are dimension-invariant. Assume that $K \subset E^m$ for $i \leq m \leq d$ and denote by $V_i'(K)$ the *ith* intrinsic volume of K in E^m. Then $V_i'(K) = V_i(K)$.

The first intrinsic volume is linear; namely, for $\lambda, \mu > 0$,

$$V_1(\lambda K + \mu C) = \lambda V_1(K) + \mu V_1(C).$$

We give an equivalent definition for the intrinsic volumes of polytopes with non-empty interior. Let $d \geq n \geq 1$, C be a closed, convex set in E^d with $\dim C = n$, and $x \in C$. Then C is called an *unbounded cone with apex* x if for any $\lambda > 0$ and $y \in C \backslash \{x\}$, we have $x + \lambda(y - x) \in C$. We define

$$\alpha(C) = \frac{V_n(C \cap (x + B^d))}{\kappa_n}.$$

Note that $\alpha(E^d) = 1$.

By Lemma 1.4.2, for any $x \notin \mathrm{int}P$ there is a unique point $\phi(x) \in \mathrm{bd}P$ which is the closest to x among the points of P. Hence for $z \in \mathrm{bd}P$, we define $C(z)$ to be the set of the points $x \notin \mathrm{int}P$ with $\phi(x) = z$. Observe that $C(z)$ is an unbounded cone with apex z. Let F be an at most $(d-1)$-dimensional face of P and $z \in \mathrm{relint}F$. Then $\alpha(C(z))$ is independent of the choice of z, and it is called the *external angle* $\gamma(F, P)$. We define $\gamma(P, P) = 1$.

Let $i = 0, \dots, d$, P be a polytope with non-empty interior and $\mathcal{F}_i$ be the set of i-dimensional faces of P. Then

$$V_i(P) = \sum_{F \in \mathcal{F}_i} V_i(F) \cdot \gamma(F, P).$$

We note that

$$S(P) = 2\, V_{d-1}(P) = \sum_{F \in \mathcal{F}_{d-1}} V_{d-1}(F),$$

as one would expect.

If $C \cup K \in \mathcal{K}^d$ then $C \cap K \neq \emptyset$ and

$$V_i(C \cup K) = V_i(C) + V_i(K) - V_i(C \cap K).$$

Functions satisfying this property are called *valuations*. The definition of the intrinsic volumes yields

LEMMA 1.8.1 *Let $d \geq 1$ and $K \in \mathcal{K}^d$. Then*

$$V_i(K + B^d) = \frac{1}{\kappa_{d-i}} \sum_{j=0}^{i} \binom{d-j}{d-i} \kappa_{d-j} V_j(K)$$

for $i = 1, \ldots, d$.

Let $K \in \mathcal{K}^d$, H be a hyperplane and K' be the image of K by the reflection through H. In addition, denote the set of lines perpendicular to H by $\mathcal{L}(H)$. For $\lambda > 0$, we define

$$S_{H,\lambda}(K) = \cup\{(1-\lambda) \cdot (l \cap K) + \lambda \cdot (l \cap K') | l \in \mathcal{L}(H) \text{ and } l \cap K \neq \emptyset\}.$$

Then $S_{H,\lambda}(K)$ is a compact and convex set. If $\lambda = \frac{1}{2}$ then $S_{H,\lambda}(K)$ is the *Steiner symmetrization* of K with respect to H.

This symmetrization process has many useful properties. If u is one of the unit vectors perpendicular to H then

$$\Delta(S_{H,\lambda}(K), u) \leq \Delta(K, u).$$

The next property can be found in [17] and [21].

THEOREM 1.8.2

i) *Let $d \geq 1$, $K \in \mathcal{K}^d$, H be a hyperplane and $K' = S_{H,\lambda}(K)$ for $0 < \lambda < 1$. Then*

$$V_i(K') \leq V_i(K)$$

for any $i = 1, \ldots, d$.

ii) *Assume $dimK = d$. If $i = d$ then we have equality above. If $i < d$ then equality holds if and only if K is symmetric with respect to a hyperplane parallel to H.*

COROLLARY 1.8.3 (Alexandrov-Fenchel) *Let* $d \geq 1$, $\rho > 0$ *and* $K \in \mathcal{K}^d$. *Then for* $i = 1, \ldots, d$,

i) *if* $V(K) = V(\rho B^d)$ *then* $V_i(K) \geq V_i(\rho B^d)$,

ii) $\dfrac{V(K)^{i/d}}{V_i(K)} \leq \dfrac{V(B^d)^{i/d}}{V_i(B^d)}$ *for* $dim K \geq i$.

If $dim K = d$ *then equality holds if and only if* K *is a ball.*

We refer to this corollary as the *Alexandrov-Fenchel inequality* (this is a very special case of the original Alexandrov-Fenchel inequality).

1.9 Intrinsic volumes of some specific convex bodies

We have denoted $V(B^d)$ by κ_d. It is well known (see [10], p.130 for an elegant proof) that

$$\kappa_d = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}.$$

With the help of $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(1) = 1$ and $\Gamma(t+1) = t\,\Gamma(t)$, one can actually calculate κ_d for any $d \geq 0$. In Chapter 4 we shall also need the recursive formula

$$\frac{\kappa_{d+1}}{\kappa_d} = \frac{d}{d+1} \cdot \frac{\kappa_{d-1}}{\kappa_{d-2}}. \tag{1.6}$$

In [3], for $d \geq 1$ the authors establish the estimate

$$\sqrt{\frac{2\pi}{d+1}} < \frac{\kappa_d}{\kappa_{d-1}} < \sqrt{\frac{2\pi}{d}}. \tag{1.7}$$

It is easy to prove by the definition of the intrinsic volumes that for $i = 0, \ldots, d$,

$$V_i(B^d) = \binom{d}{i} \frac{\kappa_d}{\kappa_{d-i}}.$$

Recall that ellipsoids are affine images of B^d. This fact allows one to calculate (for $i = d$) or estimate (for $1 \leq i < d$) the *ith* intrinsic volume of an ellipsoid.

LEMMA 1.9.1 *Let $d \geq 2$, $i = 1, \ldots, d-1$ and M be a d-dimensional ellipsoid with half-axes $a_1 \geq \ldots \geq a_d > 0$. Then*

i) $V(M) = a_1 \cdot \ldots \cdot a_d \cdot \kappa_d$,

ii) $a_{d-i+1} \cdot \ldots \cdot a_d \cdot V_i(B^d) \ \leq \ V_i(M) \ \leq \ a_1 \cdot \ldots \cdot a_i \cdot V_i(B^d)$.

Let $d \geq 2$ and K be a cone with base C and height h. Fubini's theorem yields that

$$V(K) = \frac{1}{d} \cdot h \cdot V_{d-1}(C).$$

Now we collect some information about simplices. Let C be a triangle with sides a, b and c, and α be the angle opposite to a. Then

$$A(C) = \tfrac{1}{2}\, b\, c \sin \alpha.$$

For $d \geq 2$, we are interested only in the value of $V_i(T^d)$, $i = 1, \ldots, d$. As the height of T^d is $h_d = \sqrt{\frac{2(d+1)}{d}}$, one can establish by induction that

$$V(T^d) = \frac{\sqrt{i+1}}{i!} \cdot 2^{i/2}.$$

Note that $\int_0^\infty e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}$. Hence we can define the function $h : (-\frac{\pi}{2}, \frac{\pi}{2}) \to R$ with the equation

$$\int_0^{h(s)} e^{-t^2}\, dt = s.$$

In [18], H. Hadwiger was able to give a formula for $V_i(T^d)$, $i = 1, \ldots, d$; namely,

$$V_i(T^d) = \binom{d+1}{i+1} \frac{i+1}{i!}\, 2^{i/2} \cdot \Phi(d, i), \tag{1.8}$$

where $\Phi(d, s)$ is defined for $s > 0$ as

$$\Phi(d, s) = \int_0^1 e^{-s[h(\sqrt{\pi}(1/2-t))]^2}\, t^{d-s}\, dt. \tag{1.9}$$

We note that the function $f(t) = e^{-[h(\sqrt{\pi}(1/2-t))]^2}$, $0 \leq t \leq 1$, is concave and $f(t) = f(1-t)$. In addition, $f(0) = f(1) = 0$ and $f(\frac{1}{2}) = 1$.

Let $\zeta = 1/(1+2\sqrt{\pi}e) = 0.09401$ and $1 \leq s \leq \zeta\, d$. According to [18], we have the upper bound

$$\Phi(d,s) < \left(\frac{1}{2\sqrt{\pi}}\,\frac{d-s}{s}\right)^{-s}\left(\ln\frac{1}{2\sqrt{\pi}}\,\frac{d-s}{s}\right)^{s/2}. \tag{1.10}$$

With the help of these formulae, we can give bounds for $V_i(T^d)$.

Finally we note that if

$$P = \{\lambda_1 u_1 + \ldots + \lambda_d u_d \,|\, 0 \leq \lambda_i \leq 1 \text{ for } i = 1\ldots, d\}$$

for some independent $u_1, \ldots, u_d$ then $V(P) = |\det[u_1, \ldots, u_d]|$.

1.10 Lattices

Let $d \geq 1$. A discrete subset Λ of E^d is called *lattice* if $\dim\Lambda = d$ and $u \pm v \in \Lambda$ for any $u, v \in \Lambda$. The letter Λ will always denote a lattice. In particular, Z^d is the *integer lattice*. Considering references, [16] is a handbook on this topic, and [7] is a more accessible introduction which browses through the most important facts.

The independent vectors $u_1, \ldots, u_d \in \Lambda$ form a *basis* of Λ if each $w \in \Lambda$ can be written as a linear combination of $u_1, \ldots, u_d$ with integer coefficients. The matrix $[u_1, \ldots, u_d]$ is called a *basis matrix* of Λ. Every lattice has some basis and let $u_1, \ldots, u_d$ be a basis of Λ. For a linear transformation T, the vectors $T(u_1), \ldots, T(u_d)$ form a basis of Λ if and only if T is integral and $\det T = \pm 1$. In particular, the determinant of any two basis matrices of Λ may differ only in the sign, and we define

$$\det\Lambda = |\det[u_1, \ldots, u_d]|.$$

The parallelotope

$$P = \{\lambda_1 u_1 + \ldots + \lambda_d u_d \,|\, 0 \leq \lambda_i \leq 1 \;\text{ for }\; i = 1 \ldots, d\}$$

is called a *fundamental parallelotope* of Λ. Observe that $V(P) = \det\Lambda$. The next lemma follows *via* Hadamard's inequality.

LEMMA 1.10.1 *If $v_1, \ldots, v_d$ are independent vectors of the lattice Λ in E^d then*

$$det\,\Lambda \leq \|v_1\| \cdot \ldots \cdot \|v_d\|.$$

To each lattice Λ in E^d assign a basis matrix $A(\Lambda)$ of Λ. Let $\{\Lambda_n\}_{n=0}^{\infty}$ be a sequence of lattices. We say that $\{\Lambda_n\}_{n=0}^{\infty}$ converges to the lattice Λ, if there is a basis matrix B_n of each Λ_n so that $\{B_n\} \to A(\Lambda)$ in $GL(d)$, which in turn yields that $\{\det\Lambda_n\} \to \det\Lambda$. If there exists such a Λ then the sequence $\{\Lambda_n\}_{n=0}^{\infty}$ is convergent.

The lattice Λ is (ε, D)-bounded for some $\varepsilon, D > 0$, if $\det\Lambda \leq D$ and $|u| \geq \varepsilon$ for any non-zero $u \in \Lambda$. The sequence $\{\Lambda_n\}_{n=0}^{\infty}$ is *bounded* if there exist $\varepsilon, D > 0$ so that each Λ_n is (ε, D)-bounded.

THEOREM 1.10.2 (Mahler) *If $\{\Lambda_\alpha\}$ is a bounded infinite family of lattices then the family contains a convergent sequence of pairwise different lattices.*

A lattice Λ is called *packing lattice* if $|u| \geq 2$ for any non-zero $u \in \Lambda$. The following theorem is due to H. Minkowski.

THEOREM 1.10.3 (Minkowski) *There exists a packing lattice Λ in E^d, $d \geq 1$, with $det\,\Lambda \leq 2^d \cdot \kappa_d$.*

Actually, the Minkowski theorem is somewhat stronger, but this form is more convenient for us. Finally, we quote a simple result which belongs also to the realm of integral geometry.

LEMMA 1.10.4 *Let $d \geq 1$, $K \in \mathcal{K}^d$ and Λ be a lattice. Then there exists a point x such that*

$$|K \cap (x + \Lambda)| \geq \frac{V(K)}{det\, \Lambda}.$$

For a lattice Λ and $x \in E^d$, the set $\Gamma = x + \Lambda$ is called a *grid.*

1.11 Packings

Let $d \geq 1$. A set $\{K_n\}$ of convex bodies of E^d is a *packing* if $\mathrm{int}K_i \cap \mathrm{int}K_j = \emptyset$ for $i \neq j$. The packing is called a *tiling* if $\cup\{K_n\} = E^d$. Besides the works quoted in the previous section, we refer to [5] and [26]. The book [10] is a beautiful introduction into the two- and three-dimensional case but we concentrate rather on high-dimensional spaces.

A discrete set Σ is a *packing set* if $\Sigma + B^d$ is a packing. In other words, Σ is the set of the centers in a packing of unit balls. Observe that every packing lattice Λ is a packing set, and we call the set $x + \Lambda$ a *packing grid.*

Let Σ be a packing set and for $\lambda > 0$, define

$$\Sigma^\lambda = \{x \in \Sigma | x + B^d \subset W_\lambda\},$$

where W_λ is the cube $[-\lambda, \lambda]^d$. Note that $V(W_\lambda) = 2^d\lambda^d$. The *upper density* $\delta_+(\Sigma)$ and the *lower density* $\delta_-(\Sigma)$ of the ball packing $\Sigma + B^d$ are defined as

$$\delta_+(\Sigma) = \limsup_{\lambda\to\infty} \frac{|\Sigma^\lambda| \cdot \kappa_d}{2^d\lambda^d} \quad \text{and} \quad \delta_-(\Sigma) = \liminf_{\lambda\to\infty} \frac{|\Sigma^\lambda| \cdot \kappa_d}{2^d\lambda^d}.$$

If $\delta_+(\Sigma) = \delta_-(\Sigma)$ then the common value is the *density of the packing* (in notation $\delta(\Sigma)$).

A packing set Σ is called a *periodic packing set* if there is a finite set $\{x_1, \ldots, x_n\}$ and a $\lambda > 0$ so that $\Sigma = \{x_1, \ldots, x_n\} + 2\lambda Z^d$. Then W_λ is a *fundamental cube* of the periodic packing and we may assume that $x_i \in W_\lambda$.

LEMMA 1.11.1 *Let Σ_0 be a finite subset of W_λ with the property that $\Sigma = \Sigma_0 + 2\lambda Z^d$ is a periodic packing set. Then*

$$\delta(\Sigma) = \frac{|\Sigma_0| \cdot \kappa_d}{2^d \lambda^d}.$$

Note that for any packing set Σ and $\varepsilon > 0$, there is a periodic packing set Σ' such that $\delta_+(\Sigma) < \delta(\Sigma') + \varepsilon$. The *packing density* is defined as

$$\delta_d = \sup\{\delta_+(\Sigma) \,|\, \Sigma \text{ is a packing set}\}.$$

It follows that

$$\delta_d = \sup\{\delta(\Sigma) \,|\, \Sigma \text{ is a periodic packing set}\}. \tag{1.11}$$

If Λ is a packing lattice then $\delta(\Lambda) = \frac{\kappa_d}{\det \Lambda}$. By Mahler's selection theorem, there is a packing lattice Λ_0 such that

$$\delta(\Lambda_0) = \max\{\delta(\Lambda) | \Lambda \text{ is a packing lattice}\}.$$

We call $\bar{\delta}_d = \delta(\Lambda_0)$ the *density of the densest lattice*. Readily, we have $\bar{\delta}_d \leq \delta_d$.

If $d = 2$ then $\delta_2 = \bar{\delta}_2 = \pi/\sqrt{12} = 0.90689$. This density is attained by the hexagonal lattice packing. This lattice has basis vectors u and v so that $\operatorname{conv}\{u, v, 0\}$ is congruent to T^2.

For $d \geq 3$, let D_d be the sublattice of Z^d defined by

$$D_d = \{(x^1, \ldots, x^d) \in Z^d |\, x^1 + \ldots + x^d \text{ is even}\}.$$

If $d \in \{3, 4\}$ then the lattice $\sqrt{2}\, D_d$ determines the densest lattice packing of unit balls in E^d ($\sqrt{2}\, D_3$ is known as the *face-centered cubic lattice*). It follows that

$$\delta_3 \geq \bar{\delta}_3 = \delta(\sqrt{2}\, D_3) = 0.74048 \quad \text{and} \quad \delta_4 \geq \bar{\delta}_4 = \delta(\sqrt{2}\, D_4) = 0.61685.$$

If d is large then

$$2^{-d} \leq \bar{\delta}_d \leq \delta_d \leq 2^{-0.599d + o(d)}.$$

Minkowski's theorem yields the lower bound, and the upper bound was proved in [19].

1.12 Summary of the results

Let $\mathcal{G}_n^d$ be the set of all $C_n \in \mathcal{K}^d$ in E^d such that there is a packing of n unit balls for which the center of the balls contained in C_n. In the dissertation we also consider the subfamily $\mathcal{H}_n^d$ of all $C_n \in \mathcal{G}_n^d$ for which the corresponding packing can be chosen to be a grid packing. Here we do not quote the results about $\mathcal{H}_n^d$ since most of them are analogous to the results about $\mathcal{G}_n^d$.

We denote by $\mathcal{P}_{i,n}^d$ the minimal body in $\mathcal{G}_n^d$ with respect to the *ith* intrinsic volume, $i = 1, \ldots, d$; that is,

$$V_i(\mathcal{P}_{i,n}^d + B^d) = \min\{V_i(C_n + B^d) |\, C_n \in \mathcal{G}_n^d\}.$$

We call a packing of $n \geq 2$ unit balls a sausage arrangement if the convex hull of the balls is a segment with length $2(n-1)$.

The main results of Chapter 2 are Theorem 2.5.1 and its corollary. The Theorem states that if $i = 1, \ldots, d-1$ and n approach infinity then

$$V_i(\mathcal{P}^d_{i,n} + B^d) \sim V_i(B^d) \cdot \delta_d{}^{-i/d} \cdot n^{i/d}.$$

This yields (see Corollary 2.5.2) that for $i \le d-1$,

$$\lim_{n\to\infty} \frac{r(\mathcal{P}^d_{i,n})}{R(\mathcal{P}^d_{i,n})} = 1.$$

Some of the lemmas leading to the proof of the Theorem may have independent interest. For example, Lemma 2.1.1 shows that not only cubes can be used in the definition of the density of a packing.

Chapter 3 and Chapter 4 deal with packings of $n \le d+1$ of balls. With respect to the first intrinsic volume, the minimal bodies seem to be regular simplices. According to Theorem 3.3.7, $\dim\mathcal{P}^d_{1,n} \ge 2$ for $n = 3, \ldots, d+1$; moreover, $\dim\mathcal{P}^d_{1,n} \ge \frac{1}{2}\ln n$ if n is large. On the other hand, if $\dim\mathcal{P}^d_{1,n} \ge n-2$ then $\mathcal{P}^d_{1,n} \equiv T^{n-1}$. Our probably most interesting tool is Lemma 3.3.2 which says that 'closing' a simplex decreases the first intrinsic volume.

In Section 4.1, we prove (Theorem 4.1.1) that $\mathcal{P}^3_{2,4} \equiv T^3$. The considerations in Section 4.2 indicate that for small n, $\mathcal{P}^d_{i,n}$ is probably either a segment or a regular simplex. We compare these two arrangements for $n = d+1$ in Section 4.3. According to Theorem 4.3.1, the regular simplex arrangment of $d+1$ unit balls is more optimal than the sausage arrangement for $i < d^{0.5+o(1)}$, and the sausage arrangement is more optimal for $i > d^{0.5+o(1)}$.

Probably, the questions raised in Chapters 2-4 have not been studied so far. As a contrast, the Sausage Conjecture of L. Fejes Tóth, which is the topic of Chapter 5, has received much attention in the past 10 years. It states that $\mathcal{P}^d_{d,n}$ is a segment for $d \geq 5$ and for any n; or in other words, the sausage arrangment is the densest arrangment of n unit balls for $d \geq 5$.

Let $d \geq 5$ and C_n be the convex hull of n unit balls which form a packing. It has been proved previously that if either $\dim C_n < \varphi(d)$ for a certain $\varphi(d)$ or C_n is almost a ball (see p.143) then the packing has lower density than the sausage arrangement. Note that $\varphi(d) \sim \frac{7}{12} d$. We narrowed the gap between the two type of results in Theorem 5.3.2 and in Theorem 5.3.4. Namely, the sausage arrangment is more optimal than the packing corresponding to C_n if either $r_{\varphi(d)}(C_n) > O(\ln d/d)$ or $R_{\varphi(d)}(C_n) > O(\ln d)$.

Chapter 2

The ith intrinsic volume, $1 \le i \le d-1$

In this chapter, we assume $d \geq 2$ and $1 \leq i \leq d-1$ (several results also hold for $i = d$, and this is mentioned where needed).

This chapter focuses on packings of large number of balls and serves as a prelude for the third and the fourth chapters. Let C_n be the convex hull of the centers of n unit balls in a packing, n large. We show that if the ith intrinsic volume $V_i(C_n + B^d)$ is minimal then C_n is ball-like, $i = 1, \ldots, d-1$.

We recall from Chapter 1 that a packing set Σ is the set of centers of unit balls in some packing. If in addition, Σ is a lattice or a grid then it is called a packing lattice or a packing grid, respectively. The theme of Chapters 2, 3 and 4 originates from the following problem :

Let Σ be a packing set. Find non-negative numbers λ_i, $i = 0, \ldots, d$, such that for any $K \in \mathcal{K}^d$ and Σ,

$$\sum_{i=0}^{d} \lambda_i V_i(K) \geq |\Sigma \cap K|.$$

The required property of the numbers λ_i can be stated in a slightly different way: Let $n \geq 1$ and $\mathcal{G}_n^d$ be the set of all $C_n \in \mathcal{K}^d$ such that there is a Σ with $|\Sigma \cap C_n| \geq n$. Then

$$\inf_{C_n \in \mathcal{G}_n^d} \sum_{i=0}^{d} \lambda_i V_i(C_n) \geq n.$$

Frequently one puts some restrictions on Σ; for example, it is a packing lattice or $\Sigma = 2Z^d$. This problem has been investigated since the late sixties mostly by H.

Hadwiger, J. Wills, P. Gritzmann, and others. In spite of the effort that has been put into these investigations, no satisfactory coefficients are known for any reasonable Σ. That inspired us to work with the minimum properties of the individual intrinsic volumes, and the second version of the original problem. Notice that if C_n is the segment of length $2(n-1)$ then $C_n \in \mathcal{G}_n^d$ and $V_i(C_n) = 0$, $i = 2, \ldots, d$. Hence we minimize $V_i(C_n + B^d)$ instead of $V_i(C_n)$, for $C_n \in \mathcal{G}_n^d$ and given $1 \leq i \leq d$.

2.1 Packing sets contained in a compact, convex set

Fix a $K \in \mathcal{K}^d$. If a packing set Σ is contained in K then $\Sigma + B^d \subset K + B^d$ and this yields that

$$|\Sigma| \leq \frac{V(K + B^d)}{\kappa_d}.$$

Thus we define $\nu(K)$ as the maximum cardinality of the packing sets contained in K. In addition, $\bar{\nu}(K)$ is the maximum value of n such that there exists a packing grid Γ satisfying $|\Gamma \cap K| = n$. Readily, $\bar{\nu}(K) \leq \nu(K)$.

As an example, consider the ball rB^d. Minkowski's theorem states that there is a packing lattice Λ with $\det\Lambda \leq 2^d \kappa_d$. Applying Lemma 1.10.4 for Λ yields that

$$\bar{\nu}(rB^d) \geq \frac{r^d \kappa_d}{2^d \kappa_d} = \frac{r^d}{2^d}. \tag{2.1}$$

The following lemma is a generalization of the definition of the packing density. Assume that the inradius of $K \in \mathcal{K}^d$ is large and consider a packing of maximal number of unit balls in $K + B^d$. The lemma yields that independently of the shape of K, the density of this packing is close to δ_d. Remember that in the definition of the density of a packing, we used only cubes.

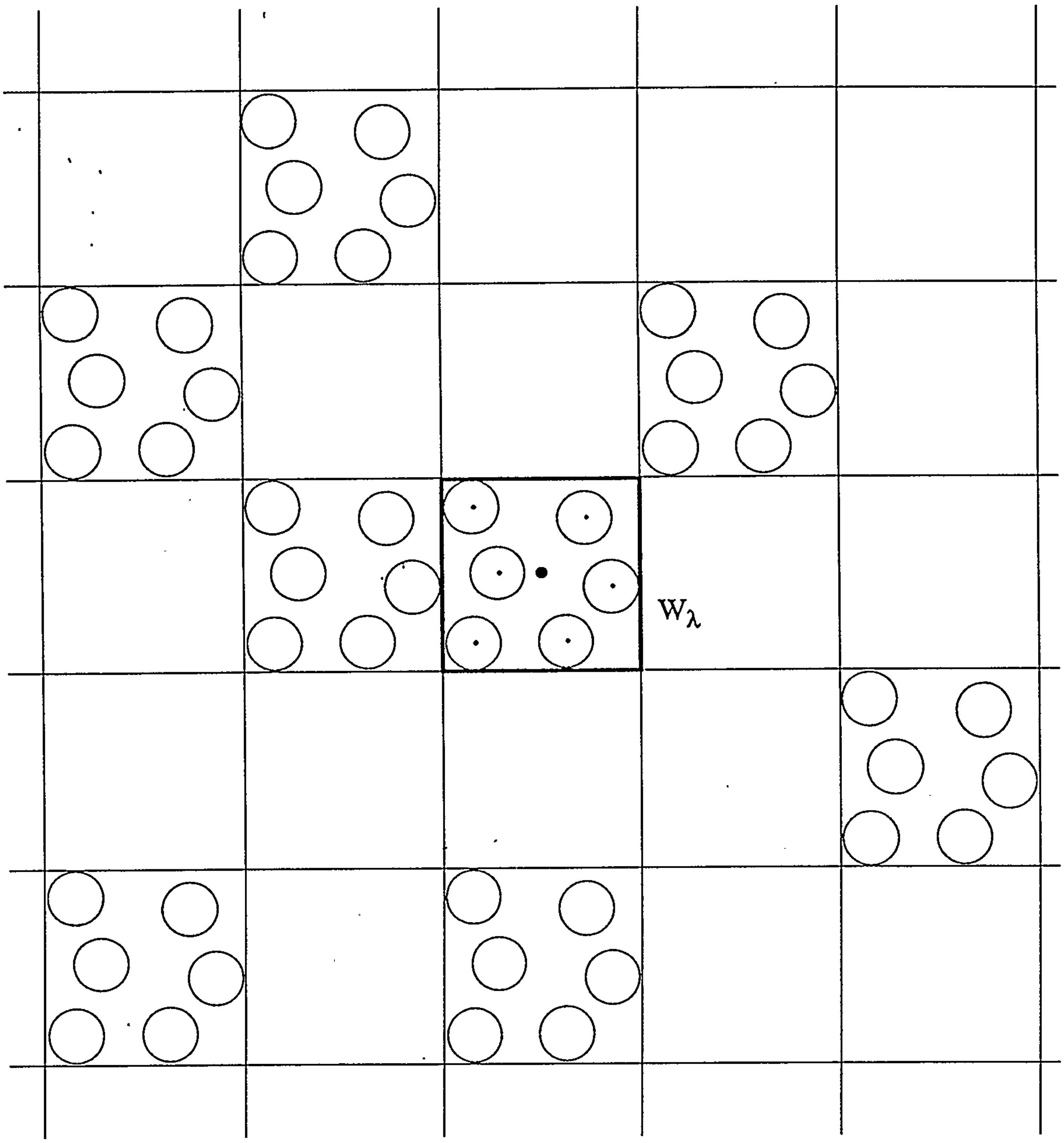

Figure 2.1

A periodic packing by a tiling

LEMMA 2.1.1 *Let $\{K_m\}$ be a sequence of convex bodies with $r(K_m) \to \infty$. Then*

i) $$\lim_{m\to\infty} \frac{\nu(K_m)\cdot\kappa_d}{V(K_m+B^d)} = \delta_d,$$

ii) $$\lim_{m\to\infty} \frac{\bar{\nu}(K_m)\cdot\kappa_d}{V(K_m+B^d)} = \bar{\delta}_d.$$

Remark: The second statement was proved, for example, in [16].

Proof: We prove i) and observe that ii) can be proved similarly by replacing δ_d with $\bar{\delta}_d$, and any packing set with a suitable subset of a packing grid. Recall from Chapter 1 that W_λ is the cube $[-\lambda, \lambda]^d$ and has volume $2^d \cdot \lambda^d$.

In the proof, our main tool is to consider the tiling of the space with copies of W_λ, λ large, and also periodic packings determined by the tiling (see Figure 2.1). Let $K \in \mathcal{K}^d$ such that $r(K)$ is large even compared to λ, and pack some unit balls into $K + B^d$ with density δ. There is a copy of W_λ, say exactly W_λ, with the property that the subpacking of the balls whose center is in W_λ has density almost δ in $W_{\lambda+1}$ (see Figure 2.2). This finite packing yields a periodic packing having $W_{\lambda+1}$ as a fundamental cube, and consequently has density close to δ.

We will frequently make use of the following (see Figure 2.3).

Let $\Sigma \subset 2\lambda Z^d$, $\lambda > 0$, and

$$C_0 \subset \Sigma + W_\lambda \subset C_1$$

for $C_0, C_1 \in \mathcal{K}^d$. Then $V(C_0) \le |\Sigma| \cdot V(W_\lambda) \le V(C_1)$ and hence

$$\frac{V(C_0)}{2^d\lambda^d} \le |\Sigma| \le \frac{V(C_1)}{2^d\lambda^d}. \tag{2.2}$$

Let $\theta > \delta_d$. We prove that there is a $\varrho > 0$ such that if $r(K) > \varrho$ for a $K \in \mathcal{K}^d$ then

$$\frac{\nu(K)\cdot\kappa_d}{V(K+B^d)} < \theta, \tag{2.3}$$

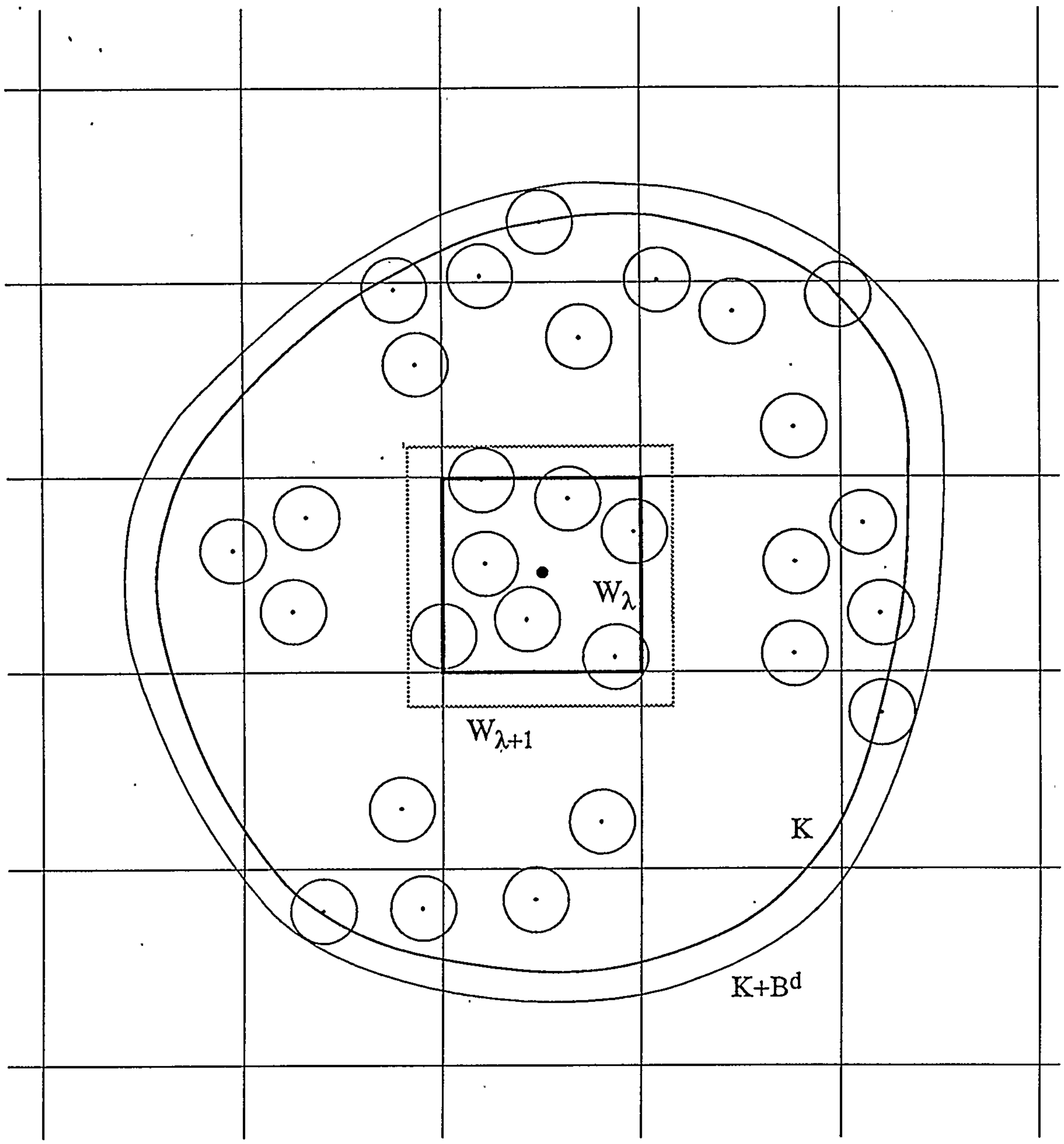

Figure 2.2

Subpacking with high density

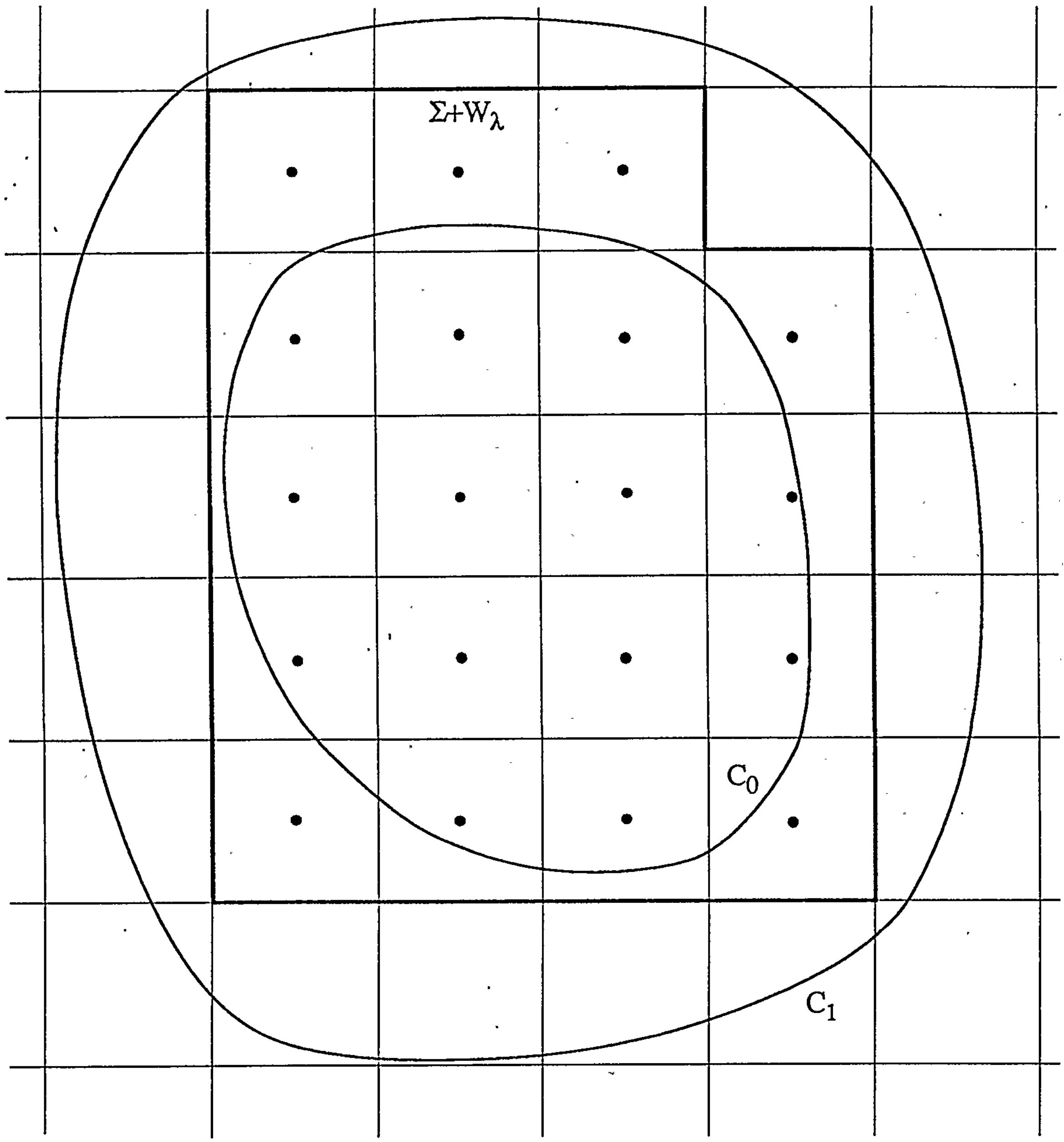

Figure 2.3

Tiles which are contained in C_1 and cover C_0

which in turn yields by the arbitrariness of $\theta > \delta_d$ that

$$\limsup_{m\to\infty} \frac{\nu(K_m)\cdot\kappa_d}{V(K_m+B^d)} \leq \delta_d. \tag{2.4}$$

Choose θ_1 and θ_2 so that

$$\theta > \theta_1 > \theta_2 > \delta_d,$$

positive ε and λ with the property that

$$\frac{\lambda^d}{(\lambda+1)^d} > \frac{\theta_2}{\theta_1} \qquad \text{and} \qquad \frac{1}{(1+\varepsilon)^d} > \frac{\theta_1}{\theta},$$

and finally a ϱ satisfying

$$D(W_\lambda) < \varepsilon\cdot\varrho.$$

Let $K \in \mathcal{K}^d$ and assume that $r(K) > \varrho$. Then

$$D(W_\lambda) < \varepsilon\cdot r(K), \tag{2.5}$$

and hence K is 'much larger' than W_λ and has the property that there is a packing of unit balls in $K+B^d$ with high density.

We prove that K satisfies 2.3. Assume to the contrary that

$$\frac{\nu(K)\cdot\kappa_d}{V(K+B^d)} \geq \theta. \tag{2.6}$$

We construct a periodic packing of unit balls having density greater than δ_d, which contradicts the definition of the packing density. Translate K so that

$$r(K)\,B^d \subset K.$$

Let us consider the cubes in the tiling determined by W_λ which intersects K (see Figure 2.4). The union of those cubes covers K. Denote the set of midpoints of

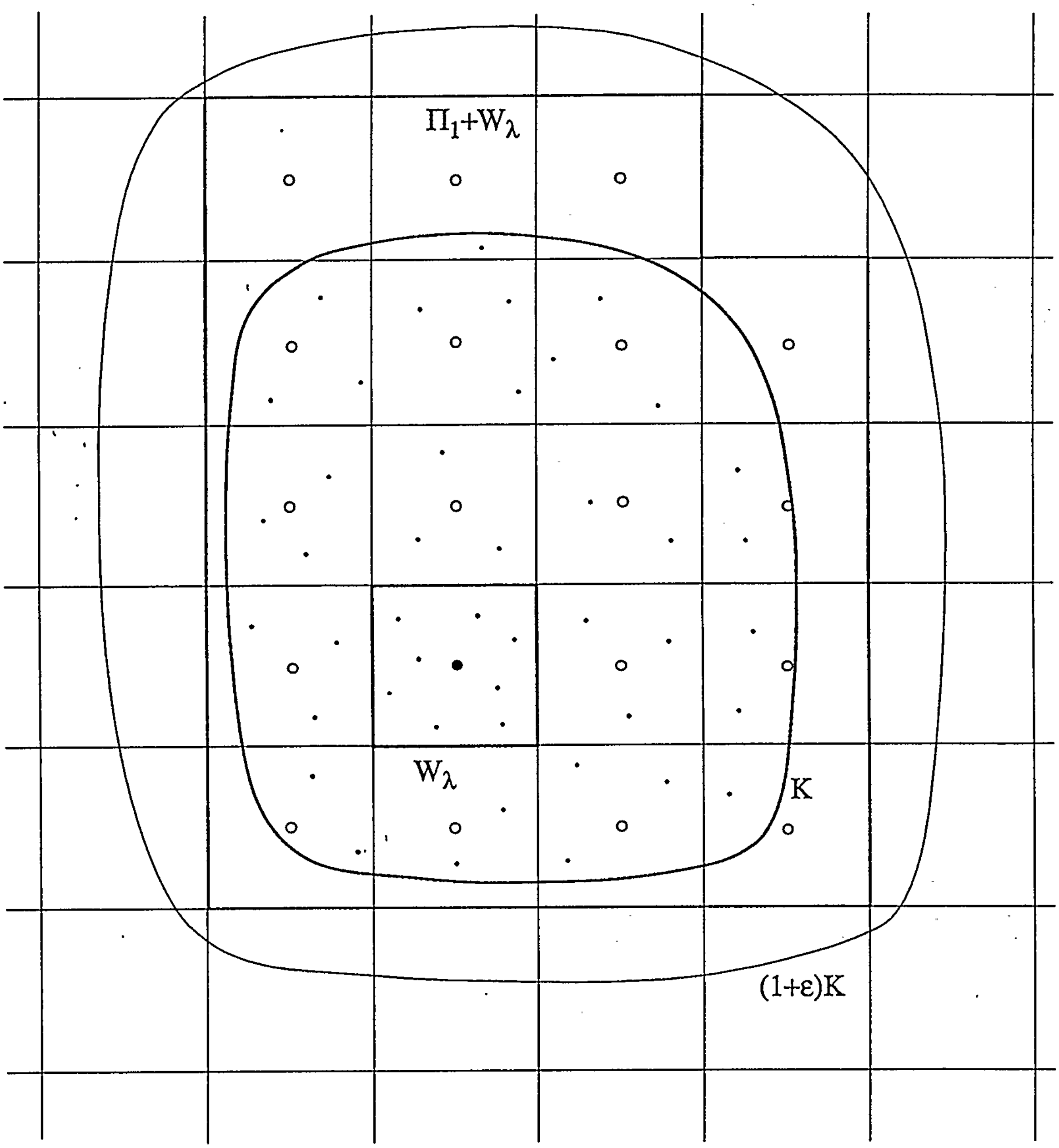

Figure 2.4

The tiles intersecting K

these tiles by Π_1 and let $y \in (x+W_\lambda) \cap K$ for some $x \in \Pi_1$. It follows by 2.5 (see Figure 2.5) that

$$\begin{aligned} x + W_\lambda &\subset y + D(W_\lambda)B^d \subset K + \varepsilon \cdot r(K)B^d \\ &\subset K + \varepsilon \cdot K \subset (1+\varepsilon)(K+B^d). \end{aligned}$$

We have shown that if $(x+W_\lambda) \cap K \neq \emptyset$ then $x + W_\lambda \subset (1+\varepsilon)(K+B^d)$. Hence, $\Pi_1 + W_\lambda = \cup_{x \in \Pi_1}\{x + W_\lambda\} \subset (1+\varepsilon)(K+B^d)$, and by 2.2, the number of tiles intersecting K is

$$|\Pi_1| \leq \frac{(1+\varepsilon)^d V(K+B^d)}{2^d \lambda^d}. \tag{2.7}$$

By definition, there exists a packing set Σ_1 contained in K with $|\Sigma_1| = \nu(K)$. Those $\nu(K)$ points are contained in the $|\Pi_1|$ cubes of $\Pi_1 + W_\lambda$ (see Figure 2.4) and hence, one of those cubes contains at least $\frac{V(K)}{|\Pi_1|}$ points of Σ_1. We may assume that W_λ is that particular cube, and denote by Σ_2 the subset of Σ_1 contained in W_λ. By 2.7,

$$|\Sigma_2| \geq \frac{\nu(K)}{|\Pi_1|} \geq \frac{\nu(K) \cdot 2^d \lambda^d}{(1+\varepsilon)^d V(K+B^d)}. \tag{2.8}$$

The definition of Σ_2 yields that

$$\Sigma_2 + B^d \subset W_\lambda + B^d \subset W_{\lambda+1}.$$

Thus, with the help of the tiling of copies of $W_{\lambda+1}$, we form the periodic packing set

$$\Sigma_3 = \{x + \Sigma_2 | x \in 2(\lambda+1)Z^d\}.$$

By 2.8 and Lemma 1.11.1, the density of Σ_3 is

$$\begin{aligned} \delta(\Sigma_3) &= \frac{|\Sigma_2| \cdot \kappa_d}{2^d(\lambda+1)^d} \\ &\geq \frac{\nu(K) \cdot 2^d \lambda^d}{(1+\varepsilon)^d V(K+B^d)} \cdot \frac{\kappa_d}{2^d(\lambda+1)^d} \\ &> \theta \cdot \frac{\theta_1}{\theta} \cdot \frac{\theta_2}{\theta_1} = \theta_2 > \delta_d. \end{aligned}$$

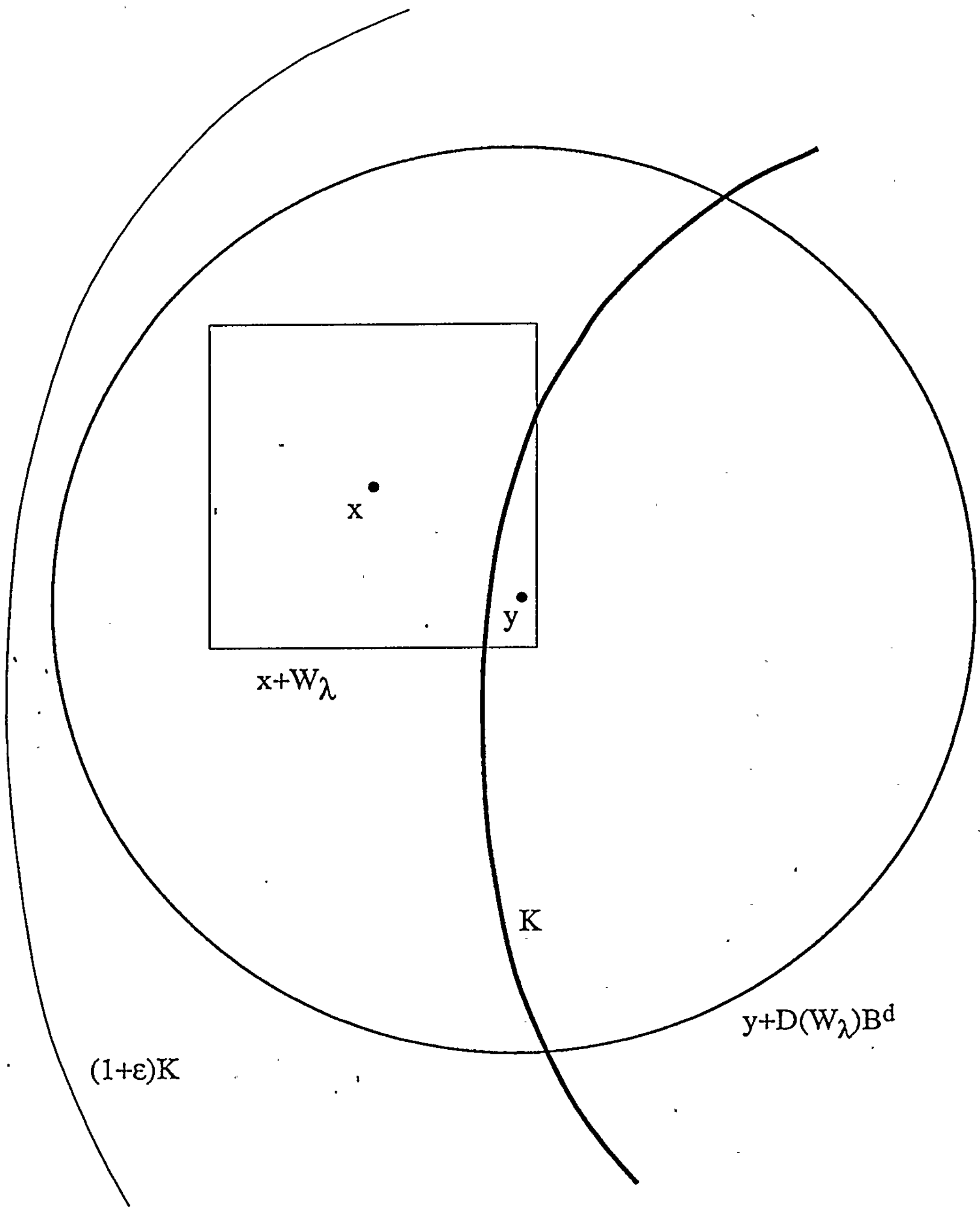

Figure 2.5

Any tile which intersects K is in (1+ε)K

This contradiction proves 2.4.

On the other hand, we prove that

$$\liminf_{m\to\infty} \frac{\nu(K_m)\cdot\kappa_d}{V(K_m+B^d)} \geq \sigma \tag{2.9}$$

for any fixed $0<\sigma<\delta_d$. Combining with 2.4, this yields the lemma. In the course of the proof of 2.9, we redefine the quantities K, ε and λ. We construct a packing set of high cardinality contained in K_m, m large, using a periodic packing of high density.

Choose a σ_1 satisfying $\sigma<\sigma_1<\delta_d$ and a $0<\varepsilon<1$ with

$$(1-\varepsilon)^{d+1} > \frac{\sigma}{\sigma_1}.$$

By 1.11, there is a periodic packing with density greater then σ_1. Let W_λ, $\lambda>0$, be a fundamental cube of it and denote by Σ_4 the centers of the balls contained in W_λ. According to Lemma 1.11.1,

$$\frac{|\Sigma_4|\cdot\kappa_d}{2^d\lambda^d} > \sigma_1. \tag{2.10}$$

Let M be an index such that if $m>M$ then

$$D(W_\lambda) < \varepsilon\cdot r(K_m) \tag{2.11}$$

and

$$\frac{V(K_m)}{V(K_m+B^d)} > 1-\varepsilon. \tag{2.12}$$

The conditions can be fulfilled because $r(K_m)\to\infty$ and $\frac{r(K_m)}{r(K_m+B^d)}$ is less than 1 and tends to 1. We use the condition on $D(W_\lambda)$ to ensure that if a cube, congruent to W_λ, intersects $(1-\varepsilon)K$ then that cube is contained in K. Let $K=K_m$ for some $m>M$ and $r=r(K)$. We translate K so that

$$r(K)\,B^d\subset K.$$

Suppose that a cube in the form of $x + W_\lambda$ intersects $(1-\varepsilon)K$ and y is a point of the intersection. Then by 2.11 (see Figure 2.5),

$$\begin{aligned} x + W_\lambda &\subset y + D(W_\lambda)B^d \subset (1-\varepsilon)K + \varepsilon \cdot r\, B^d \\ &\subset (1-\varepsilon)K + \varepsilon\, K = K. \end{aligned} \tag{2.13}$$

In the tiling of W_λ, keeping Σ_4 inside, consider those tiles which intersect $(1-\varepsilon)K$ (the situation is similar to that in Figure 2.4, only one replaces K by $(1-\varepsilon)K$ and $(1+\varepsilon)K$ by K). They cover $(1-\varepsilon)K$ and hence by 2.2, there is at least $\frac{V((1-\varepsilon)K)}{2^d\lambda^d}$ of them. On the other hand, they determine the packing set

$$\Sigma_5 = \{x + \Sigma_4 | x \in 2\lambda Z^d \ \text{ and } \ (x + W_\lambda) \cap (1-\varepsilon)K \neq \emptyset\}$$

which is contained in K by 2.13. It follows that

$$\nu(K) \ \geq\ |\Sigma_5| \ \geq\ |\Sigma_4| \cdot \frac{(1-\varepsilon)^d V(K)}{2^d \lambda^d}.$$

Combining this inequality with 2.10 and 2.12 yields that

$$\begin{aligned} \frac{\nu(K)\cdot\kappa_d}{V(K+B^d)} &\geq \frac{|\Sigma_4|(1-\varepsilon)^d V(K)}{2^d\lambda^d} \cdot \frac{\kappa_d}{V(K+B^d)} \\ &\geq \sigma_1(1-\varepsilon)^d \frac{V(K)}{V(K+B^d)} \\ &\geq \sigma_1(1-\varepsilon)^d(1-\varepsilon). \end{aligned}$$

Thus by the choice of ε,

$$\frac{\nu(K)\cdot\kappa_d}{V(K+B^d)} \geq \sigma,$$

which proves 2.9 and the lemma.

□

In Chapter 5, we use the inequality 2.3 in the following form:

COROLLARY 2.1.2 *For any $\alpha < \kappa_d/\delta_d$ there is a $\varrho(\alpha)$ so that if $r(K) > \varrho(\alpha)$ then*

$$V(K+B^d) > \alpha \cdot \nu(K).$$

2.2 General and grid packings

We have seen in Section 2.1 that $\bar{\nu}(K) \leq \nu(K)$ for any $K \in \mathcal{K}^d$. We give a $K \in \mathcal{K}^d$, at least for odd $d \geq 3$, with the property that $\dim K = d$ and $\bar{\nu}(K) < \nu(K)$. Up to the point where we need the restriction that d is odd, the only restriction on d is that $d \geq 2$.

Let T be a copy of T^d such that the origin is a vertex of T, and denote the other vertices by $u_1, \ldots, u_d$. It follows that $|u_i| = 2$ and $< u_i, u_j >= 2 \cdot 2 \cdot \cos(\pi/3) = 2$ for $1 \leq i < j \leq d$. Let $v = \frac{2}{d}\sum_{i=1}^{d} u_i$ and define $U^d = \mathrm{conv}\{0, u_1, \ldots, u_d, v\}$. As $\frac{1}{d}\sum_{i=1}^{d} u_i$ is the center of $\mathrm{conv}\{u_1, \ldots, u_d\} \equiv T^{d-1}$, U^d is congruent to the union of two copies of T^d whose intersection is a common face.

By symmetry, $d(0,v)$ is twice the height of T^d, and hence $d(0,v) = 2\sqrt{\frac{2(d+1)}{d}} > 2$. Consequently the vertices of U^d form a packing set, and $\nu(U^d) \geq d+2$.

If x is a point of T, different from all its vertices, then $d(x,0)$ and $d(x,u_i)$ are less then 2 by Lemma 1.5.5, and hence $d(x,y) < 2$ for any $y \in T$. This observation yields that the only way to have a packing set of cardinality at least 3 in U^d is if all the points are vertices.

It follows that $\nu(U^d) = d+2$ and if $\bar{\nu}(U^d) = d+2$ then there is a packing lattice Λ with $u_1, \ldots, u_d, v \in \Lambda$. Assume now that d is odd. Then $(d-1)/2$ is an integer, and

$$w = \sum_{i=1}^{d} u_i - \frac{d-1}{2} v = \sum_{i=1}^{d} u_i - \frac{d-1}{d} \sum_{i=1}^{d} u_i = \frac{1}{d} \sum_{i=1}^{d} u_i$$

is a point of Λ, different from the origin. As $|w|$ is just the height of T^d, we have $|w| = \sqrt{\frac{2(d+1)}{d}} < \sqrt{\frac{2\cdot 4}{3}} < 2$. This contradicts the assumption that Λ is a packing lattice, and hence $\bar{\nu}(U^d) < d+2$.

Actually $\bar{\nu}(U^d) = \bar{\nu}(T^d) = d+1$. We prove it by showing that the lattice Λ_0 with base $u_1, \ldots, u_d$ is a packing lattice. Let $m_1, \ldots, m_d$ be integers, not all of them 0. Then for $x = \sum_{i=1}^{d} m_i u_i \in \Lambda_0$,

$$|x|^2 = \sum_{i=1}^{d} m_i^2 u_i^2 + 2 \sum_{1 \leq i < j \leq d} m_i m_j < u_i, u_j >= 4(\sum_{i=1}^{d} m_i^2 + \sum_{1 \leq i < j \leq d} m_i m_j).$$

Since $|x|^2$ is a non-zero integer, divisible by 4, we have $|x| \geq 2$. Then $|y-z| = d(y-z, 0) = d(y,z)$ yields that Λ_0 is a packing lattice.

How much smaller can $\bar{\nu}(K)$ be than $\nu(K)$? We claim that there is a positive constant $c(d)$, depending only on d, such that $c(d) \cdot \nu(K) \leq \bar{\nu}(K)$. The constant we give is probably much worse than the best possible one but we are interested only in the existence of $c(d)$.

Let $K \in \mathcal{K}^d$ and $U_i(K)$ be the *ith* inner Quermassintegral of K; that is,

$$U_i(K) = \max\{V_i(K \cap E) | \text{E is a } i\text{-dimensional affine subspace}\}.$$

We may write 'max' because of Blaschke's selection theorem. $U_i(K)$ is readily monotonic in K, and if $\lambda > 0$ then $U_i(\lambda K) = \lambda^i U_i(K)$. Also, by the monotonicity of the intrinsic volumes, $U_i(K) \leq V_i(K)$.

We want to have some upper bounds for the ratio of $V_i(K)$ over $U_i(K)$. Here we consider the case where K is an ellipsoid.

LEMMA 2.2.1 *Let $1 \leq i \leq m$ and M be a m-dimensional ellipsoid. Then*

$$V_i(M) \leq \frac{\kappa_m}{\kappa_i\, \kappa_{m-i}} \binom{m}{i} U_i(M).$$

Proof: Let M be the ellipsiod in E^m with the equation

$$\frac{(x^1)^2}{a_1^2} + \dots + \frac{(x^m)^2}{a_m^2} \leq 1, \quad 0 < a_m \leq \ldots \leq a_1.$$

The i-dimensional coordinate subspace $x^{i+1} = \ldots = x^m = 0$ intersects M in the i-dimensional ellipsoid N with axes $a_1, \ldots, a_i$, and hence, by Lemma 1.9.1,

$$U_i(M) \geq V_i(N) = \kappa_i \cdot a_1 \cdot \ldots \cdot a_i. \tag{2.14}$$

On the other hand, Lemma 1.9.1 yields that

$$V_i(M) \leq V_i(B^m) \cdot a_1 \cdot \ldots \cdot a_i = \frac{\kappa_m}{\kappa_{m-i}} \binom{m}{i} a_1 \cdot \ldots \cdot a_i.$$

Combining this with 2.14 yields the lemma.

□

THEOREM 2.2.2 *Let $d \geq 2$. There is a positive constant $c(d)$, depending only on d, such that $c(d) \cdot \nu(K) \leq \bar{\nu}(K)$ for any $K \in \mathcal{K}^d$.*

Proof: Note that $\bar{\nu}(K) \geq 1$ for any $K \in \mathcal{K}^d$. Hence let $n = \nu(K) \geq 2$, $m = \dim K \leq d$ and assume that K is contained in E^m. By the definition of $\nu(K)$, we have

$$\sum_{i=0}^{m} \kappa_{m-i} V_i(K) = V_m(K + B^m) \geq n \cdot \kappa_m.$$

Then $V_0(K) = 1$ yields $\sum_{i=1}^{m} \kappa_{m-i} V_i(K) \geq (n-1) \cdot \kappa_m$, and hence there is a $1 \leq i \leq m$ with the property that

$$\kappa_{m-i} V_i(K) \geq \frac{1}{m} \cdot (n-1) \kappa_m.$$

Finally, $n - 1 \geq \frac{1}{2} n$ yields that

$$V_i(K) \geq \frac{\kappa_m}{\kappa_{m-i}} \cdot \frac{n-1}{m} \geq \frac{\kappa_m}{2m\kappa_{m-i}} \cdot n. \tag{2.15}$$

By Theorem 1.5.4, there is a m-dimensional ellipsoid M such that after a suitable translation,

$$M \subset K \subset m \cdot M.$$

It follows that

$$\begin{aligned} V_i(K) &\leq V_i(m \cdot M) = m^i V_i(M) \\ &\leq \frac{m^i \kappa_m}{\kappa_i \kappa_{m-i}} \binom{m}{i} U_i(M) \leq \frac{m^i \kappa_m}{\kappa_i \kappa_{m-i}} \binom{m}{i} U_i(K). \end{aligned} \tag{2.16}$$

Let

$$\bar{c}(m,i) = \frac{\kappa_i \kappa_{m-i}}{m^i \kappa_m} \binom{m}{i}^{-1} \frac{\kappa_m}{2m\kappa_{m-i}} = \frac{\kappa_i}{2\, m^{i+1}} \binom{m}{i}^{-1}.$$

Combining 2.15 and 2.16 shows that there is a i-dimensional section C of K with

$$\frac{m^i \kappa_m}{\kappa_i \kappa_{m-i}} \binom{m}{i} V_i(C) \geq \frac{\kappa_m}{2m\kappa_{m-i}} \cdot n,$$

and hence

$$V_i(C) \geq \bar{c}(m,i) \cdot n.$$

Translate K again, now in order to have the origin contained in C. According to Minkowski's theorem, there is a i-dimensional packing lattice Λ in aff$C$ with $\det\Lambda \leq 2^i \kappa_i$. Hence, by Theorem 1.10.4,

$$\bar{\nu}(C) \geq \frac{V_i(C)}{2^i \kappa_i} \geq \frac{\bar{c}(m,i)}{2^i \kappa_i} \cdot n.$$

At the moment our constant depends on m and i. Let $c(d) = \min_{1\leq i\leq m\leq d}\{\frac{\bar{c}(m,i)}{2^i\kappa_i}\}$. Then $c(d)$ is positive and as $\bar{\nu}(K) \geq \bar{\nu}(C)$, it follows that $\bar{\nu}(K) \geq \dot{c}(d) \cdot n$.

□

Our estimates are very rough. For example, we essentially used the inequality

$$\sum_{i=0}^{m} \frac{\kappa_{m-i}}{\kappa_m} V_i(K) \geq \nu(K)$$

for $K \in K^m$. A conjecture (see [14]), which may not be far from the truth, suggests that

$$\sum_{i=0}^{m} \frac{\sigma_i}{\kappa_i} V_i(K) \geq \nu(K)$$

for any $K \in K^m$. Here σ_i is Rogers' constant (see [26]), and $\sigma_i \sim 2^{-i/2}$ for large i.

2.3 Radii and the intrinsic volumes

Let $K \in \mathcal{K}^d$. In this section, we investigate how the relation between $V(K)$ and $V_i(K)$, $i = 1, \ldots, d-1$, affects the shape of K.

Assume $\dim K = d$. Let u be a unit vector such that the width $\Delta(K)$ of K is in the direction of u. Consider a hyperplane H orthogonal to u and denote by K^* the Steiner-symmetrization of K with respect to H (see Figure 2.6). There exists a right cylinder containing K^* and having a base which is a translation of $C = K^* \cap H$ and height at most $\Delta(K)$. It follows that

$$V(K^*) \leq V_{d-1}(C) \cdot \Delta(K). \tag{2.17}$$

Let $1 \leq i \leq d-1$. The Alexandrov-Fenchel inequality can be formulated as

$$V_{d-1}(C) \leq c_1 \cdot V_i(C)^{\frac{d-1}{i}},$$

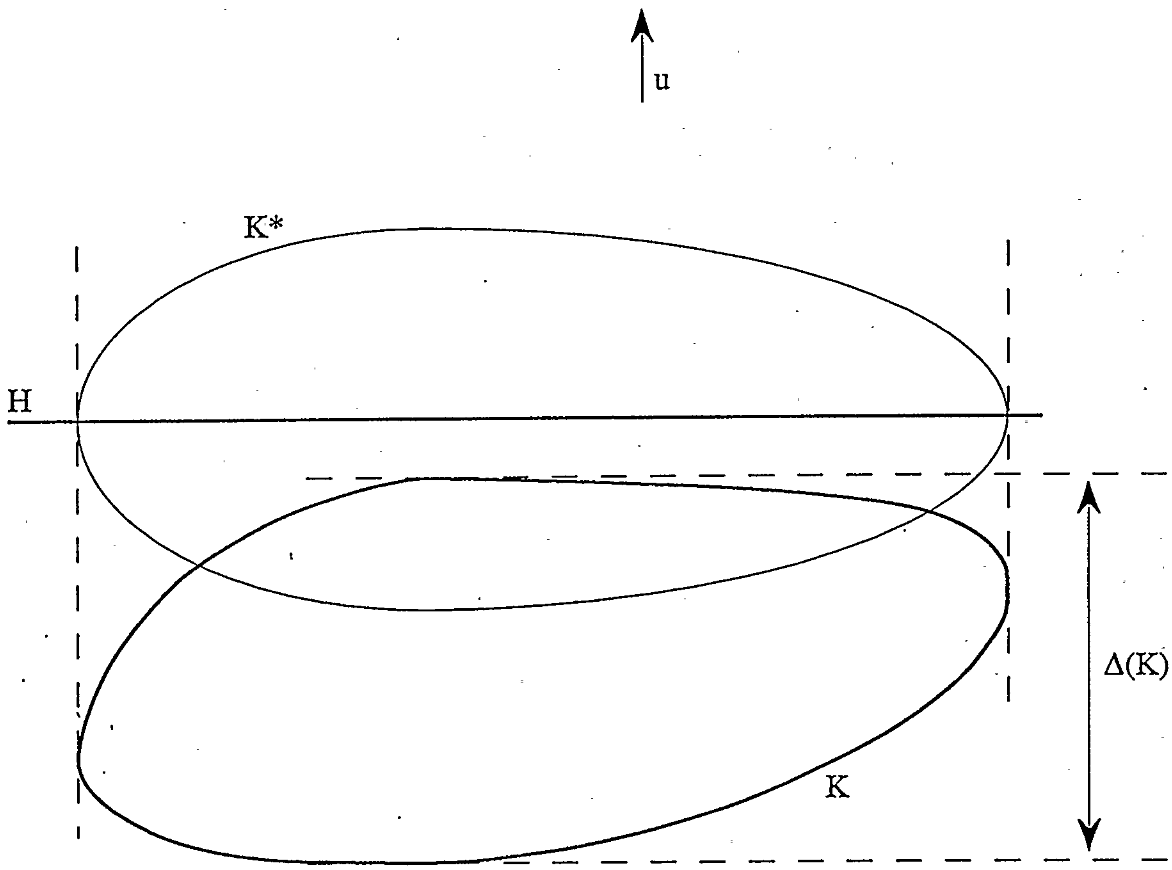

Figure 2.6

Steiner symmetrization

where c_1 is a constant depending only on d and i. Substituting this inequality into 2.17 yields that

$$\begin{aligned} V(K^*) &\leq c_1 \cdot V_i(C)^{\frac{d-1}{i}} \Delta(K) \\ &\leq c_1 \cdot V_i(K^*)^{\frac{d-1}{i}} \Delta(K). \end{aligned}$$

According to Theorem 1.8.2, we have $V(K^*) = V(K)$ and $V_i(K^*) \leq V_i(K)$. It follows that

$$V(K) \leq c_1 \cdot V_i(K)^{\frac{d-1}{i}} \Delta(K). \tag{2.18}$$

By Theorem 1.5.3, there exists a constant c_2 with the property that

$$\Delta(K) \leq c_2 \cdot r(K).$$

Combining this inequality with 2.18 yields

LEMMA 2.3.1 *Let $d \geq 2$ and $1 \leq i \leq d-1$. Then there exists a constant c such that*

$$V(K) \leq c \cdot V_i(K)^{\frac{d-1}{i}} r(K)$$

for any $K \in \mathcal{K}^d$.

Notice that if $\dim K \leq d-1$ then $V(K) = 0$ and any positive constant c satisfies the Lemma.

Let $K \in \mathcal{K}^d$ and $1 \leq i \leq d-1$. The Alexandrov-Fenchel inequality yields an upper bound for $V(K)$ if $V_i(K)$ is known. The previous lemma refines it by taking also the inradius into the picture. Now we consider another extension of the Alexandrov-Fenchel inequality, giving an upper bound for $\frac{V(K)^{i/d}}{V_i(K)}$ in terms of the ratio of the inradius and the circumradius of K instead of the constant $\frac{V(B^d)^{i/d}}{V_i(B^d)}$. This result is an improvement if the inradius is much smaller than the circumradius.

LEMMA 2.3.2 *Let $1 \leq i \leq d-1$. Then there exists a positive constant $\bar{c}$ such that*

$$\frac{r(K)}{R(K)} \geq \bar{c} \cdot \left(\frac{V(K)^{i/d}}{V_i(K)} \right)^d = \bar{c} \cdot \frac{V(K)^i}{V_i(K)^d}$$

for any $K \in \mathcal{K}^d$ with dim$K \geq i$.

The inequality is readily true if dim$K < d$, and hence assume dim$K = d$. In addition, we assume that $r(K)\, B^d \subset K$ and also $R(K) = 1$ (since taking λK instead of K with a positive λ does not change the inequality). By Theorem 1.5.2, there are $x, y \in K$ (see Figure 2.7) with $|y| \leq |x|$ and

$$d(x,y) \geq \sqrt{\frac{2(d+1)}{d}}\, R(K) > \sqrt{2}.$$

Hence $|x| \geq \sqrt{2}/2$ because of the triangle inequality.

Consider now some $(i-1)$-dimensional subspace through the origin, perpendicular to x. The intersection of the subspace and $r(K)B^d$ determines a i-dimensional cone C having height $|x| \geq \sqrt{2}/2$ and base congruent to $r(K)\, B^{i-1}$ (see Figure 2.7). That cone yields that

$$V_i(K) \geq V_i(C) \geq \frac{1}{i} \frac{\sqrt{2}}{2} \kappa_{i-1} \cdot r(K)^{i-1} = c_3 \cdot r(K)^{i-1} \tag{2.19}$$

for the obvious c_3. According to Lemma 2.3.1,

$$V(K) \leq c \cdot V_i(K)^{\frac{d-1}{i}} \cdot r(K)$$

for a suitable c. Taking the *ith* power of this inequality and using 2.19 yield that

$$\begin{aligned} V(K)^i &\leq c^i \cdot V_i(K)^{d-1} r(K)^i = c^i \cdot V_i(K)^{d-1} r(K) \cdot r(K)^{i-1} \\ &\leq c^i \cdot V_i(K)^{d-1} r(K) \cdot \frac{1}{c_3} V_i(K) = \frac{c^i}{c_3} \cdot V_i(K)^d r(K). \end{aligned}$$

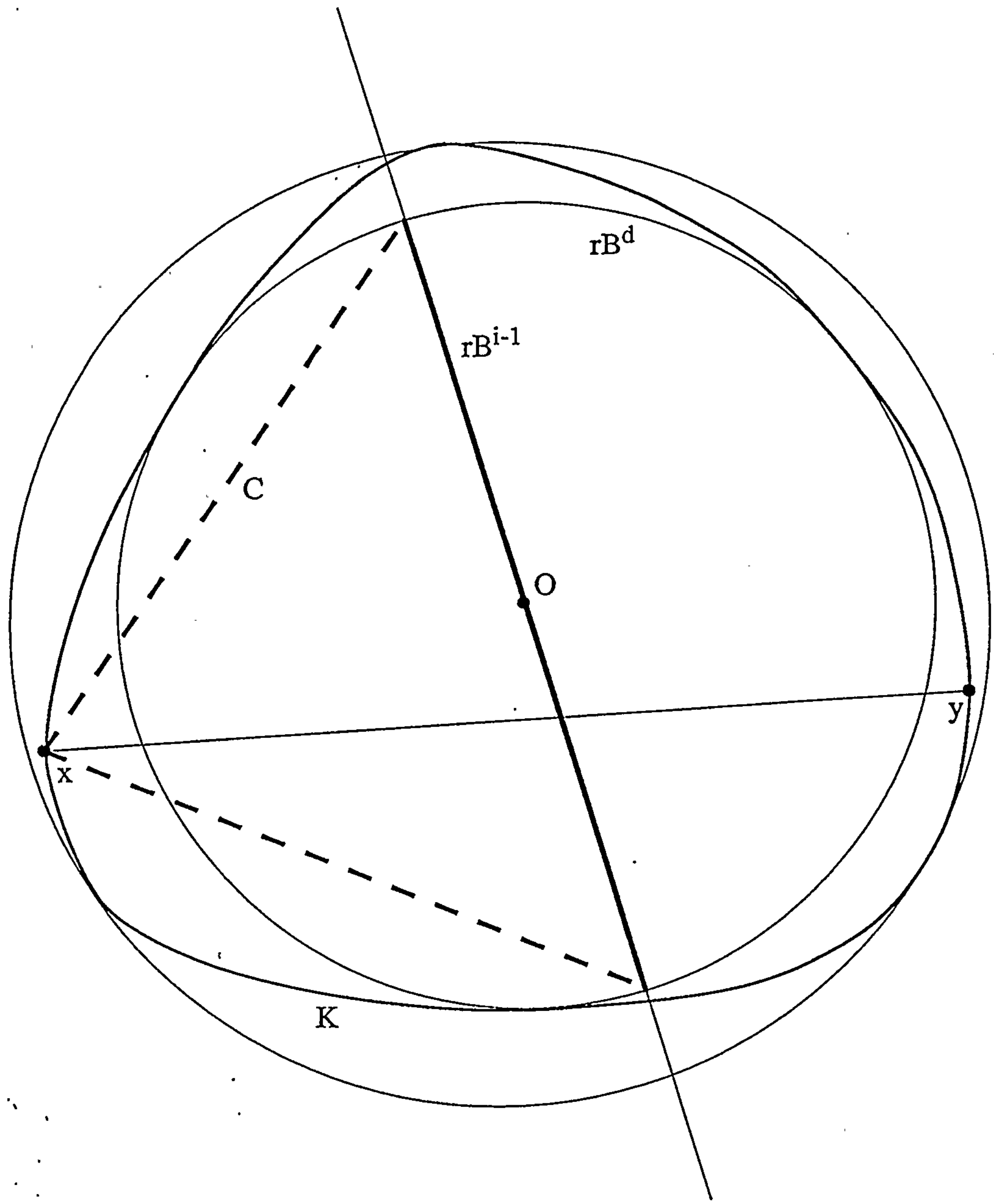

Figure 2.7

An i-dimensional cone contained in K

Since $R(K) = 1$, the lemma follows.

$\square$

With the help of the previous lemma, we strengthen the equality case in the Alexandrov-Fenchel inequality.

LEMMA 2.3.3 *A sequence $\{K_m\}$ of convex bodies has the property*

$$\frac{V(K_m)^{i/d}}{V_i(K_m)} \to \frac{V(B^d)^{i/d}}{V_i(B^d)} \quad \textit{if and only if} \quad \frac{r(K_m)}{R(K_m)} \to 1.$$

Proof: Let $C_m = \frac{1}{R(K_m)} K_m$ and assume that it contains the origin. Then $r(C_m) \leq R(C_m) = 1$ and we need to show that

$$\frac{V(C_m)^{i/d}}{V_i(C_m)} \to \frac{V(B^d)^{i/d}}{V_i(B^d)} \quad \text{if and only if} \quad r(C_m) \to 1. \tag{2.20}$$

First we establish the necessary part. Let $\rho = \liminf_{m\to\infty} r(C_m)$. By Blaschke's selection theorem and the continuity of the inradius, we may assume that $C_m \to C$ for some $C \in \mathcal{K}^d$ with $r(C) = \rho$. We note that the circumradius is also continuous and consequently $R(C) = 1$.

2.20 and Lemma 2.3.2 yield that

$$\begin{aligned} r(C) &= \lim_{m\to\infty} r(C_m) \geq \lim_{m\to\infty} \bar{c} \cdot \left(\frac{V(C_m)^{i/d}}{V_i(C_m)}\right)^d \\ &= \bar{c} \cdot \left(\frac{V(B^d)^{i/d}}{V_i(B^d)}\right)^d. \end{aligned}$$

Hence $r(C) > 0$, which in turn ensures that $V_i(C) > 0$. Thus by 2.20; the continuity of the intrinsic volumes and $C_m \to C$ yield that

$$\frac{V(C)^{i/d}}{V_i(C)} = \lim_{m\to\infty} \frac{V(C_m)^{i/d}}{V_i(C_m)} = \frac{V(B^d)^{i/d}}{V_i(B^d)}.$$

Now by the equality case in the Alexandrov-Fenchel inequality, C is a ball. Hence $\rho = 1$ and $r(C_m)$ tends to 1.

In order to prove the sufficient part, first observe that for any convex body K, denoting $r(K)$ by r and $R(K)$ by R,

$$V(rB^d)^{i/d} \leq V(K)^{i/d} \leq V(RB^d)^{i/d} \quad \text{and} \quad V_i(rB^d) \leq V_i(K) \leq V_i(RB^d).$$

These inequalities, with $C_m = K$, yield that

$$\frac{r(C_m)^i \cdot V(B^d)^{i/d}}{R(C_m)^i \cdot V_i(B^d)} \leq \frac{V(C_m)^{i/d}}{V_i(C_m)} \leq \frac{R(C_m)^i \cdot V(B^d)^{i/d}}{r(C_m)^i \cdot V_i(B^d)}.$$

Since $\frac{r(C_m)}{R(C_m)} = r(C_m)$ tends to 1, $\frac{V(C_m)^{i/d}}{V_i(C_m)}$ is forced to tend to $\frac{V(B^d)^{i/d}}{V_i(B^d)}$.

□

2.4 Minimal bodies

Let $n \geq 1$ and $\mathcal{G}_n^d$ be the family of all $K \in \mathcal{K}^d$ with the property that K contains the centers of n balls, which then form a packing in $K + B^d$. In other words,

$$\mathcal{G}_n^d = \{K \in \mathcal{K}^d | \nu(K) \geq n\}.$$

Considering n-ball packings which are part of a grid packing, define

$$\mathcal{H}_n^d = \{K \in \mathcal{K}^d | \bar{\nu}(K) \geq n\}.$$

Notice that $\mathcal{H}_n^d \subset \mathcal{G}_n^d$. The considerations of this section also apply for the case $i = d$. Hence, let $1 \leq i \leq d$ and $\vartheta_{i,n}^d$ be the infinum of the *ith* intrinsic volume of the convex hull of n-ball packings, and denote by $\bar{\vartheta}^d_{i,n}$ the corresponding infimum for grid packings. In other words,

$$\vartheta^d_{i,n} = \inf\{V_i(C_n + B^d) | C_n \in \mathcal{G}^d_n\}$$

and

$$\bar{\vartheta}^d_{i,n} = \inf\{V_i(C_n + B^d) | C_n \in \mathcal{H}^d_n\}.$$

Readily $\vartheta^d_{i,n} \leq \bar{\vartheta}^d_{i,n}$.

Actually, one may write 'min' instead of 'inf' in the definitions. We argue this for the case of gridpackings, and indicate the argument for general packings.

If $n = 1$ then C_n may be a point and $\bar{\vartheta}^d_{i,n} = V_i(B^d)$. Hence assume $n \geq 2$. Consider a sequence $\{K_j\}$ of elements in $\mathcal{H}^d_n$ satisfying

$$V_i(K_j + B^d) \to \bar{\vartheta}^d_{i,n}.$$

We may assume that the origin is contained in each of the K_j's and that there is a packing-lattice Λ_j with $|\Lambda_j \cap K_j| \geq n$ for every j.

Our goal, using Blaschke's and Mahler's selection theorems, is to find $K \in \mathcal{H}^d_n$ with the property that $\bar{\vartheta}^d_{i,n} = V_i(K)$. In order to use the theorems, both $\{K_j\}$ and $\{\Lambda_j\}$ have to be bounded. If $D(K_j)$ approaches infinity then $V(K_j + B^d)$ also tends to infinity, and, by the Alexandrov-Fenchel inequality, $V_i(K_j + B^d)$ becomes large and does not approach $\bar{\vartheta}^d_{i,n}$. Thus the sequence $\{D(K_j)\}$ is bounded, and we may apply Blaschke's selection theorem.

Turning to $\{\Lambda_j\}$, we do some preparations. Since the intrinsic volumes are monotonic, we assume that

$$K_j = \mathrm{conv}(\Lambda_j \cap K_j),$$

and, taking a suitable subsequence if necessary, that $\dim K_j = m$ for some $1 \leq m \leq d$. Finally, we rotate the K_j's so that they are all contained in some Euclidean subspace

E^m. We note that

$$\Lambda'_j = \Lambda_j \cap E^m$$

is a packing lattice in E^m.

If $m < d$ then those vectors of Λ_j which are not in E^m may be arbitrarily long. Hence $\{\Lambda_j\}$ is not necessarily bounded. On the other hand, for an index j, let $v_1, \dots, v_m \in \Lambda'_j \cap K_j$ be independent. Then by Lemma 1.10.1,

$$\det\Lambda'_j \leq |v_1| \cdot \ldots \cdot |v_m| \leq D(K_j)^m.$$

Since Λ'_j is a packing lattice and $\det\Lambda'_j$ is bounded by the boundedness of $D(K_j)$, also the sequence $\{\Lambda'_j\}$ is bounded (in E^m). Combining Blaschke's and the Mahler's selection theorems yields that, taking a suitable subsequence if necessary, $K_j \to K$ and $\Lambda'_j \to \Lambda'$ (see Figure 2.8), where $K \in \mathcal{K}^m$ and Λ' is a m-dimensional packing lattice. We also have $|K \cap \Lambda'| \geq n$. Notice that Λ' can be extended to a d-dimensional packing lattice Λ with $\Lambda \cap E^m = \Lambda'$. Let $\mathcal{Q}^d_{i,n} = K$. Then $\mathcal{Q}^d_{i,n} \in \mathcal{H}^d_n$, and by the continuity of the intrinsic volumes,

$$V_i(\mathcal{Q}^d_{i,n} + B^d) = \bar{\vartheta}^d_{i,n}.$$

If the sequence $\{K_j\}$ is chosen from $\mathcal{G}^d_n$ with

$$V_i(K_j + B^d) \to \vartheta^d_{i,n}$$

then again $D(K_j)$ is bounded. Combining Blaschke's and the Bolzano-Weierstrass theorems yields a subsequence converging to a $\mathcal{P}^d_{i,n} \in \mathcal{G}^d_n$; that is, an element of $\mathcal{G}^d_n$ satisfying

$$V_i(\mathcal{P}^d_{i,n} + B^d) = \vartheta^d_{i,n}.$$

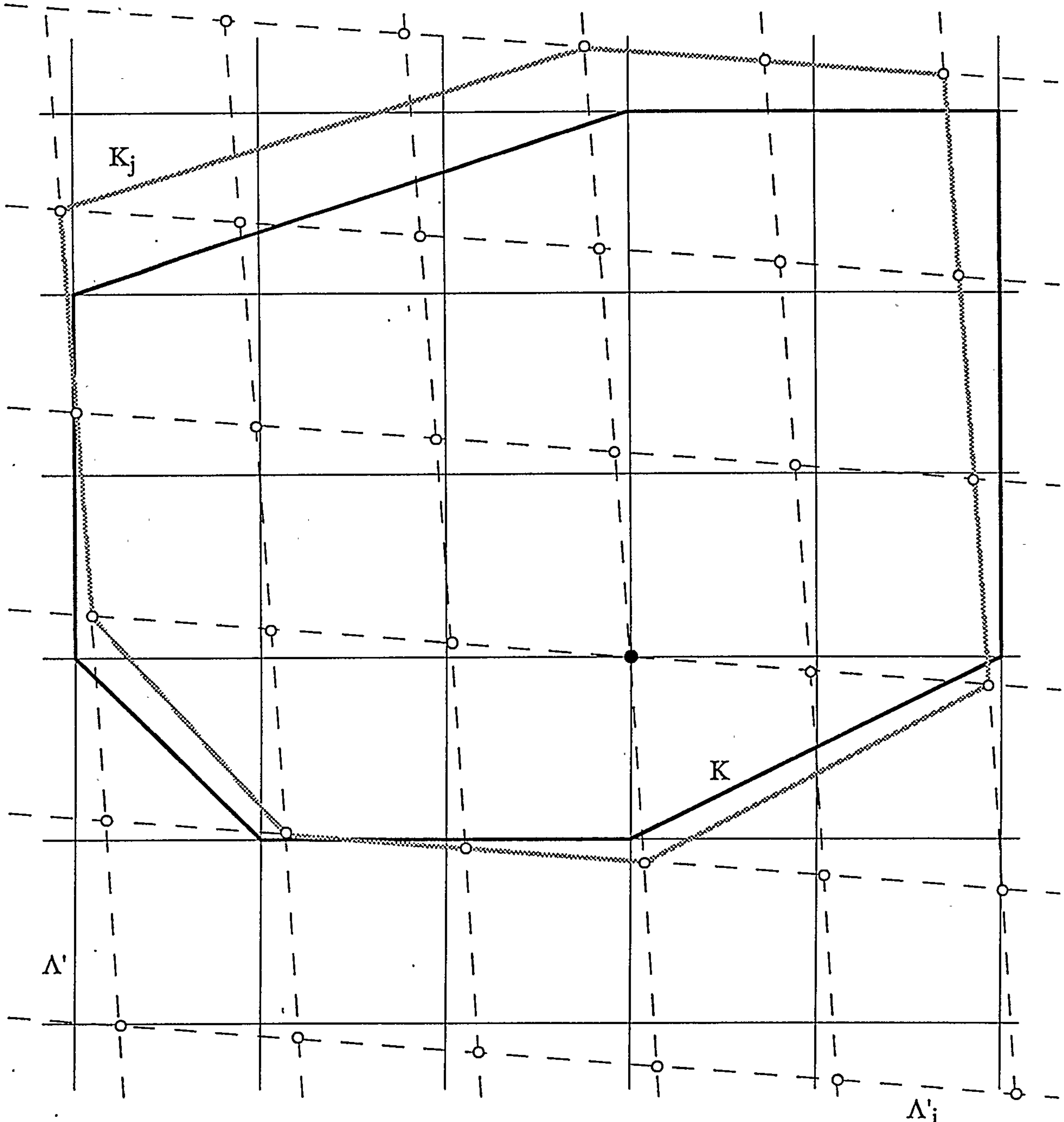

Figure 2.8

Convergent sequence of packing lattices

Notice that there may be more minimal bodies in $\mathcal{G}_n^d$ or in $\mathcal{H}_n^d$ for a given i but we choose only one of them.

We now collect some basic properties of $\mathcal{P}_{i,n}^d$ and $\mathcal{Q}_{i,n}^d$. Let $n \geq 1$ and note that there exists a packing set $\Sigma \subset \mathcal{P}_{i,n}^d$ with

$$|\Sigma| = \nu(\mathcal{P}_{i,n}^d) \geq n.$$

Then $\mathrm{conv}\Sigma \in \mathcal{G}_n^d$ and $\mathrm{conv}\Sigma + B^d \subset \mathcal{P}_{i,n}^d + B^d$. Since the intrinsic volumes are strictly monotonic on the space of convex bodies, the minimality property of $\mathcal{P}_{i,n}^d$ yields that $\mathcal{P}_{i,n}^d + B^d = \mathrm{conv}\Sigma + B^d$ and consequently that

$$\mathcal{P}_{i,n}^d = \mathrm{conv}\Sigma.$$

If $|\Sigma|$ is greater than n then remove an extremal point of $\mathcal{P}_{i,n}^d$, and denote the resulting set by Σ' (see Figure 2.9). Then for $C = \mathrm{conv}\Sigma'$, $\nu(C) \geq |\Sigma'| \geq n$ and C is strictly contained in $\mathcal{P}_{i,n}^d$. This implies that $V_i(C + B^d) < V_i(\mathcal{P}_{i,n}^d + B^d)$, which contradicts the minimality of $V_i(\mathcal{P}_{i,n}^d + B^d)$. Therefore,

$$n = |\Sigma| = \nu(\mathcal{P}_{i,n}^d).$$

Assume that $\Sigma = \{x_0, \dots, x_{n-1}\}$ and x_0 is an extreme point of $\mathcal{P}_{i,n}^d$. If $d(x_0, x_i) > 2$ for any $i = 1, \dots, n-1$ then move x_0 into $\mathrm{relint}\,\mathcal{P}_{i,n}^d$ to a position x_0' so that still $d(x_0', x_i) > 2$. The convex hull C' of the points $x_0', x_1, \dots, x_{n-1}$ is strictly contained in $\mathcal{P}_{i,n}^d$ because x_0, being an extreme point of $\mathcal{P}_{i,n}^d$, is not in C'. This yields the inequality $V_i(C' + B^d) < V_i(\mathcal{P}_{i,n}^d + B^d)$ which can not hold by the definition of $\mathcal{P}_{i,n}^d$. Therefore, for every $x \in \mathrm{ext}\,\mathcal{P}_{i,n}^d$ there is a $y \in \Sigma$ with $d(x, y) = 2$.

Similar considerations can be applied to $\mathcal{Q}_{i,n}^d$. They yield a packing grid Γ such

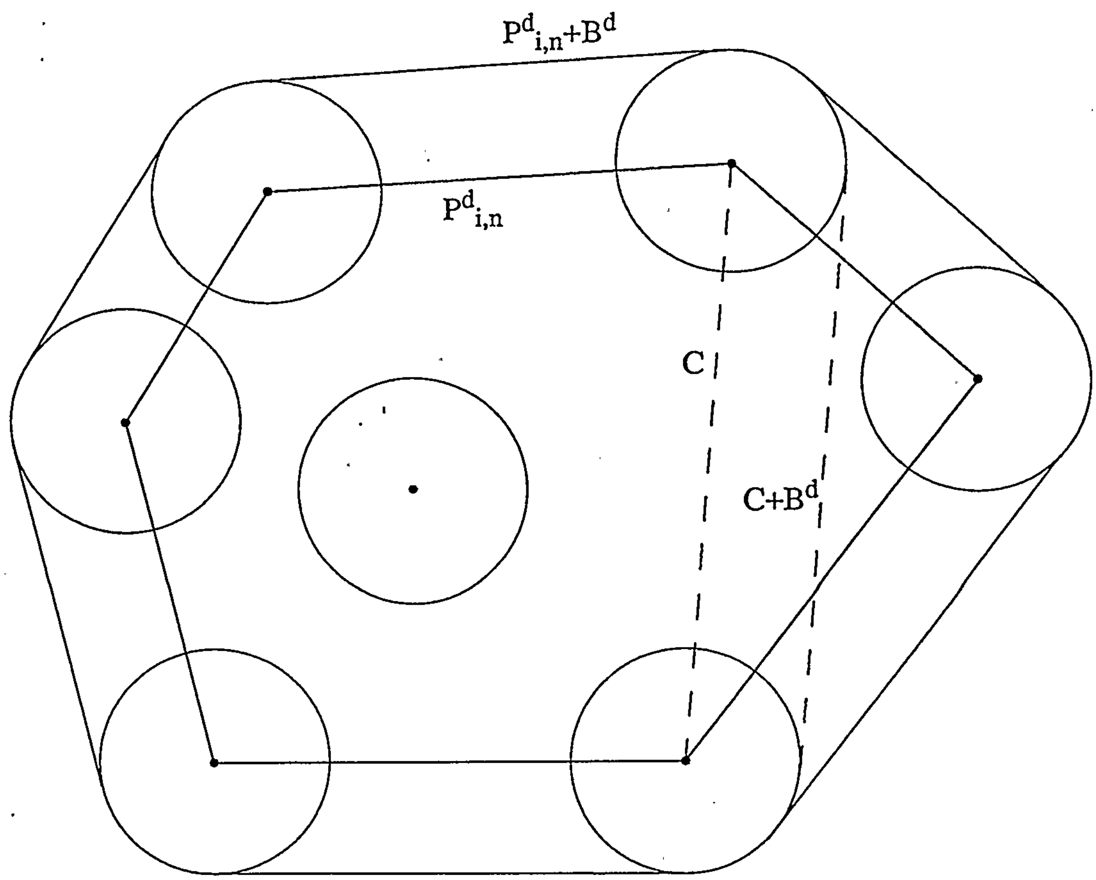

Figure 2.9

A minimal body and the corresponding packing that $\Sigma_0 = \Gamma \cap \mathcal{Q}_{i,n}^d$ satisfies $\nu(\mathcal{Q}_{i,n}^d) = |\Sigma_0| = n$ and $\mathcal{Q}_{i,n}^d = \mathrm{conv}\Sigma_0$ (the assertion about the extreme points of $\mathcal{P}_{i,n}^d$ may not have an equivalent for $\mathcal{Q}_{i,n}^d$).

Next, we give some bounds for $\vartheta_{i,n}^d$ and $\bar{\vartheta}_{i,n}^d$. As a reference, we mention that

$$V_i(B^d) = \binom{d}{i} \frac{\kappa_d}{\kappa_{d-i}}.$$

THEOREM 2.4.1 *Let* $1 \leq i \leq d$ *and* $n \geq 1$. *Then*

$$V_i(B^d) \cdot n^{i/d} \leq \vartheta_{i,n}^d \leq \bar{\vartheta}_{i,n}^d \leq 3^i \cdot V_i(B^d) \cdot n^{i/d}.$$

Proof: Any $C_n \in \mathcal{G}_n^d$ has the property that

$$V(C_n + B^d) \geq n \cdot V(B^d) = V(n^{1/d} B^d).$$

Hence the Alexandrov-Fenchel inequality yields that

$$V_i(C_n + B^d) \geq V_i(n^{1/d} B^d) = V_i(B^d) \cdot n^{i/d}.$$

Turning to the upper bound, define $r = 2\, n^{\frac{1}{d}} \geq 2$. According to 2.1,

$$\bar{\nu}(rB^d) \geq \frac{r^d}{2^d} = n,$$

which yields that $rB^d \in \mathcal{H}_n^d$. It follows, by the definition of r, that

$$\begin{aligned} \bar{\vartheta}_{i,n}^d &\leq V_i(rB^d + B^d) = V_i(B^d) \cdot (r+1)^i \\ &= V_i(B^d) \cdot (r+1)^i \cdot \frac{2^i n^{i/d}}{r^i}. \end{aligned}$$

In the final expression, $r \geq 2$ yields that

$$(r+1)^i \cdot \frac{2^i}{r^i} = (2 + \frac{2}{r})^i \leq 3^i, \tag{2.21}$$

and the upper bound is established as well.

□

One can easily improve the bounds for large n. For example, since $r = 2n^{1/d}$ tends to infinity, 2.21 also yields that

$$\bar{\vartheta}^d_{i,n} \leq (2 + \varepsilon(n))^i \cdot V_i(B^d) \cdot n^{i/d}$$

where $\varepsilon(n) \to 0$. But in the next section, at least in the cases $i = 1, \ldots, d-1$, we give even the exact asymptotic behavior of $\vartheta^d_{i,n}$ and $\bar{\vartheta}^d_{i,n}$.

2.5 Packings of large numbers of balls

In this section we exclude again the case $i = d$. We mention later why the considerations do not apply for this case.

Let $1 \leq i \leq d-1$ and $r(n)B^d$ be the smallest ball contained in $\mathcal{G}^d_n$. We prove that for $r = r(n)$, the density of the n-ball packings contained in $rB^d + B^d$ tends to δ_d. In other words,

$$\frac{n \cdot V(B^d)}{V(rB^d + B^d)} \sim \delta_d.$$

Taking the $(i/d)th$ power yields, after some rearrangement, that

$$\frac{V(rB^d + B^d)^{i/d}}{V(B^d)^{i/d}} \sim n^{i/d} \cdot {\delta_d}^{-i/d}. \tag{2.22}$$

Observe that for $\rho > 0$,

$$\frac{V(\rho B^d)^{i/d}}{V(B^d)^{i/d}} = \rho^i = \frac{V_i(\rho B^d)}{V_i(B^d)}.$$

Substituting this into 2.22, with $\rho = r + 1$, results in

$$V_i(rB^d + B^d) \sim n^{i/d}\, V_i(B^d)\, {\delta_d}^{-i/d}.$$

We prove, with the help of the Alexandrov-Fenchel inequality, that $r(n)B^d$ is a good approximation for $\mathcal{P}^d_{i,n}$ (or for $\mathcal{Q}^d_{i,n}$); that is,

THEOREM 2.5.1 *Let $d \geq 2$, $1 \leq i \leq d-1$ and n approach infinity. Then*

i) $\vartheta^d_{i,n} \sim V_i(B^d) \cdot \delta_d{}^{-i/d} \cdot n^{i/d}$,

ii) $\bar{\vartheta}^d_{i,n} \sim V_i(B^d) \cdot \bar{\delta}_d{}^{-i/d} \cdot n^{i/d}$.

Proof: We present the proof only for i). It can be done similarly for ii), with the natural change of notions. The definition of $r(n)$ is equivalent to

$$r(n) = \min\{r | \nu(rB^d) \geq n\}.$$

Readily $r(n)$ tends to infinity, and we may assume that n is large enough to ensure $r = r(n) > 3$. Observe that $\nu(rB^d)$ may be greater than n. Yet, as we prove it soon, $\nu(r(n)B^d) \sim n$.

There exists a packing set Σ_n, contained in $r(n)B^d$, with the property that $|\Sigma_n| = \nu(rB^d) \geq n$. Denote by Σ'_n the set of those points in Σ_n which are not in $(r-1)B^d$ (see Figure 2.10). Then

$$\Sigma'_n + B^d \subset (r+1)B^d \backslash (r-2)B^d, \qquad \text{and}$$

$$|\Sigma'_n| \leq \frac{V((r+1)B^d) - V((r-2)B^d)}{\kappa_d} = (r+1)^d - (r-2)^d.$$

Since $|\Sigma_n \cap (r-1)B^d| < n$ by the definition of $r(n)$, we have

$$\nu(rB^d) < n + (r+1)^d - (r-2)^d,$$

which in turn yields

$$\nu(rB^d) - (r+1)^d + (r-2)^d < n \leq \nu(rB^d). \tag{2.23}$$

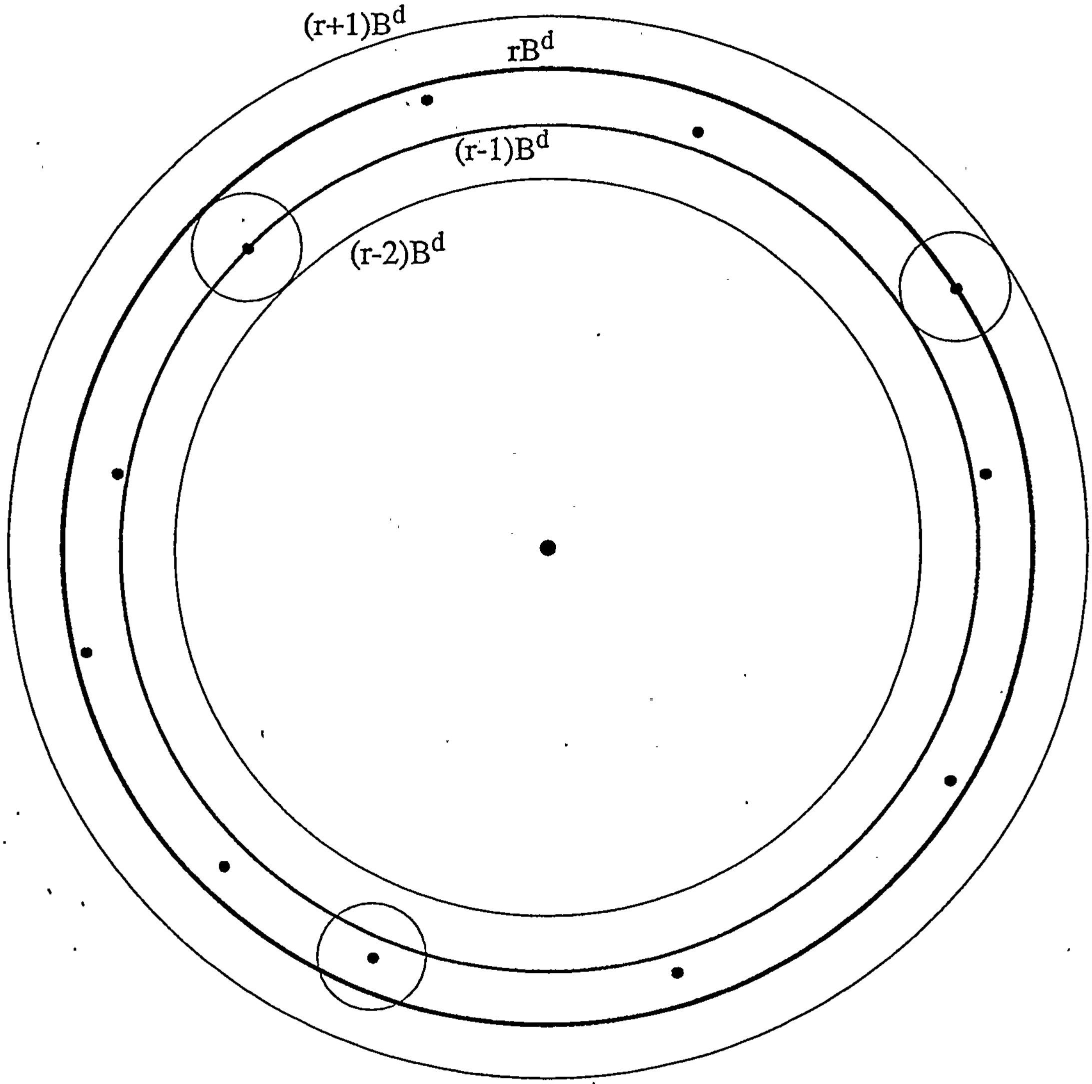

Figure 2.10

The points of Σ'_n in $r(n)B^d$

Observe that

$$\lim_{r\to\infty} \frac{(r+1)^d - (r-2)^d}{(r+1)^d} = 0. \tag{2.24}$$

It is time to recall Lemma 2.1.1. It follows that

$$\lim_{n\to\infty} \frac{\nu(rB^d)\cdot \kappa_d}{V(rB^d + B^d)} = \delta_d, \tag{2.25}$$

which is equivalent to

$$\lim_{n\to\infty} \frac{\nu(rB^d)}{(r+1)^d} = \delta_d.$$

Combining this fact with 2.23 and 2.24 yields that

$$\lim_{n\to\infty} \frac{n}{(r+1)^d} = \delta_d.$$

Thus, by 2.24, we have proved that $\nu(rB^d) \sim n$. It also follows that we may replace $\nu(rB^d)$ by n in 2.25; that is,

$$\lim_{n\to\infty} \frac{n\cdot V(B^d)}{V(rB^d + B^d)} = \delta_d. \tag{2.26}$$

We note that 2.25 refers to the density of n-ball packings in $r(n)B^d + B^d$.

Now let us consider $\mathcal{P}^d_{i,n}$ and denote $\mathcal{P}^d_{i,n} + B^d$ by D_n. Observe that $V_i(D_n) \leq 3^i \cdot V_i(B^d)\cdot n^{i/d}$ by Theorem 2.4.1 and $V(D_n) \geq n\kappa_d$. Substitute these inequalities into

$$V(D_n) \leq c\cdot V_i(D_n)^{\frac{d-1}{i}}\, r(D_n)$$

of Lemma 2.3.1. This results in

$$n\cdot \kappa_d \leq c_0 \cdot n^{\frac{d-1}{d}} r(D_n),$$

with $c_0 = c\cdot 3^{d-1}\, V_i(B^d)^{\frac{d-1}{i}}$. Thus,

$$r(D_n) \geq \frac{\kappa_d}{c_0}\cdot n^{1/d}.$$

Since $r(D_n) = r(\mathcal{P}_{i,n}^d) + 1$, the previous inequality yields that $\lim_{n\to\infty} r(\mathcal{P}_{i,n}^d) = \infty$. This enables us to use Lemma 2.1.1 with $K_n = \mathcal{P}_{i,n}^d$. It follows, by $\nu(\mathcal{P}_{i,n}^d) = n$, that

$$\lim_{n\to\infty} \frac{n \cdot V(B^d)}{V(D_n)} = \delta_d. \tag{2.27}$$

Here we pause for a moment, in order to indicate why our proof breaks down in the case $i = d$. The reason is that for $C_n \in \mathcal{G}_n^d$, the condition $V(C_n + B^d) \leq c_1 \cdot n$, where c_1 is a constant independent of n, does not force $r(C_n)$ to tend to infinity. For example, if C_n is the segment S_n with length $2(n-1)$, then $r(n) = 0$ and (see Figure 2.11)

$$V(C_n + B^d) \leq \kappa_{d-1} \cdot 2 \cdot n.$$

(Actually, according to the Sausage Conjecture which is discussed in Chapter 5, possibly $\mathcal{P}_{d,n}^d \equiv S_n$ for $d \geq 5$ and any $n \geq 1$.) Hence we can not use Lemma 2.1.1 which is the tool to relate to the packing density.

After these remarks, let us continue our argument. We combine the information about D_n and $rB^d + B^d$.

Observe that $\frac{V(D_n)}{V(rB^d+B^d)}$ is $\frac{nV(B^d)}{V(rB^d+B^d)}$ divided by $\frac{nV(B^d)}{V(D_n)}$. Thus 2.26 and 2.27 yield

$$\lim_{n\to\infty} \frac{V(D_n)}{V(rB^d + B^d)} = 1. \tag{2.28}$$

Take the $(i/d)th$ power in 2.28. By definition, $V_i(D_n) \leq V_i(rB^d + B^d)$, and hence

$$\liminf_{n\to\infty} \frac{V(D_n)^{i/d}}{V(rB^d + B^d)^{i/d}} \cdot \frac{V_i(rB^d + B^d)}{V_i(D_n)} \geq 1, \tag{2.29}$$

which in turn, simplifying by $(r+1)^i$, can be written in the form

$$\liminf_{n\to\infty} \frac{V(D_n)^{i/d}}{V_i(D_n)} \geq \frac{V(B^d)^{i/d}}{V_i(B^d)}.$$

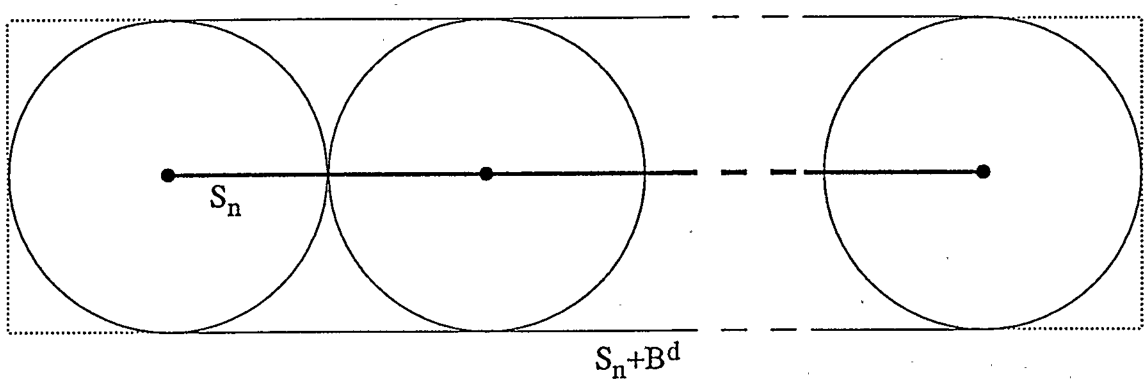

Figure 2.11

The cylinder containing the sausage S_n+B^d

On the other hand, according to the Alexandrov-Fenchel inequality,

$$\frac{V(D_n)^{i/d}}{V_i(D_n)} \leq \frac{V(B^d)^{i/d}}{V_i(B^d)}.$$

It follows that

$$\lim_{n\to\infty} \frac{V(D_n)^{i/d}}{V_i(D_n)} = \frac{V(B^d)^{i/d}}{V_i(B^d)}. \tag{2.30}$$

We mention that the analogous statement

$$\lim_{n\to\infty} \frac{V(Q^d_{i,n} + B^d)^{i/d}}{V_i(Q^d_{i,n} + B^d)} = \frac{V(B^d)^{i/d}}{V_i(B^d)} \tag{2.31}$$

can be proved similarly.

We now work backwards from 2.30 and obtain, instead of 2.29, that

$$\lim_{n\to\infty} \frac{V(D_n)^{i/d}}{V(rB^d + B^d)^{i/d}} \cdot \frac{V_i(rB^d + B^d)}{V_i(D_n)} = 1.$$

Hence, by 2.28,

$$\lim_{n\to\infty} \frac{V_i(rB^d + B^d)}{V_i(D_n)} = 1.$$

As we have already noted in the beginning of this section, 2.26 can be written as

$$\frac{n \cdot V(B^d)}{V(rB^d + B^d)} \sim \delta_d,$$

which in turn yields that

$$V_i(rB^d + B^d) \sim V_i(B^d)\, \delta_d{}^{-i/d}\, n^{i/d}.$$

Since $V_i(rB^d + B^d)$ and $V_i(D_n)$ have the same asymptotic behavior, the theorem follows.

□

With regard to Theorem 2.5.1, we note that $V_i(B^d) = \binom{d}{i}\frac{\kappa_d}{\kappa_{d-i}}$ and

$$2^{0.599i(1+\varepsilon(d))} \leq \delta_d^{-i/d} \leq \bar{\delta}_d^{-i/d} \leq 2^i,$$

where $\varepsilon(d) \to 0$.

COROLLARY 2.5.2 *Let $d \geq 2$ and $1 \leq i \leq d-1$. Then*

$$\lim_{n\to\infty} \frac{r(\mathcal{P}^d_{i,n})}{R(\mathcal{P}^d_{i,n})} = \lim_{n\to\infty} \frac{r(\mathcal{Q}^d_{i,n})}{R(\mathcal{Q}^d_{i,n})} = 1.$$

Proof: It follows by 2.30 and Lemma 2.3.3 that $\lim_{n\to\infty} \frac{r(D_n)}{R(D_n)} = 1$ for $D_n = \mathcal{P}^d_{i,n} + B^d$. Since $R(D_n) = R(\mathcal{P}^d_{i,n}) + 1$ and $r(D_n) = r(\mathcal{P}^d_{i,n}) + 1$, we have

$$\lim_{n\to\infty} \frac{r(\mathcal{P}^d_{i,n}) + 1}{R(\mathcal{P}^d_{i,n}) + 1} = 1.$$

Notice that $n\,\kappa_d \leq V(D_n) \leq R(D_n)^d\, V(B^d)$. Hence $R(D_n)$, and consequently also $R(\mathcal{P}^d_{i,n}) = R(D_n) - 1$, tends to infinity, which in turn yields

$$\lim_{n\to\infty} \frac{r(\mathcal{P}^d_{i,n})}{R(\mathcal{P}^d_{i,n})} = 1.$$

One uses a similar argument for $\mathcal{Q}^d_{i,n}$, only in this case it is based on 2.31.

$\square$

2.6 Some remarks about the case of few balls

The considerations of this chapter show that if n is large then in order to optimize the *ith* intrinsic volume of the convex hull of an n-ball packing, the balls should be packed as clusters, $i = 1, \ldots, d-1$. What is the shape of $\mathcal{P}^d_{i,n}$ when n is small, say,

$n = d+1$? Does $\mathcal{P}^d_{i,d+1}$ equal the regular simplex T^d with edge length 2, or maybe the segment S_{d+1}, $i = 1, \dots, d-1$? The upcoming two chapters are mostly concerned with these questions.

Chapter 3

The first intrinsic volume

Recall that the first intrinsic volume is linear. This property helps us to gain some more specific information about the the minimal properties of finite packings with respect to the first intrinsic volume.

3.1 General properties

In Section 2.4 we defined $\mathcal{G}_n^d$ as the family of all $C_n \in \mathcal{K}^d$ which contains the centers of some n-ball packing. The minimum of the *ith* intrinsic volume of the convex hull of the balls is denoted by $\vartheta_{i,n}^d$ for $i = 1, \ldots, d$. If the packing is part of a gridpacking then $C_n \in \mathcal{H}_n^d$, and the corresponding minimum for $\mathcal{H}_n^d$ is denoted by $\bar{\vartheta}_{i,n}^d$. We chose some minimal bodies $\mathcal{P}_{i,n}^d$ and $\mathcal{Q}_{i,n}^d$, which have the properties that $V_i(\mathcal{P}_{i,n}^d + B^d) = \vartheta_{i,n}^d$ and $V_i(\mathcal{Q}_{i,n}^d + B^d) = \bar{\vartheta}_{i,n}^d$.

By the linearity of the first intrinsic volume, $V_1(K + B^d) = V_1(K) + V_1(B^d)$ for $K \in \mathcal{K}^d$, and it follows that

$$V_1(\mathcal{P}_{1,n}^d) = \min\{V_1(C_n)|C_n \in \mathcal{G}_n^d\} \qquad \text{and} \qquad V_1(\mathcal{Q}_{1,n}^d) = \min\{V_1(C_n)|C_n \in \mathcal{H}_n^d\}.$$

We also saw in Section 2.4 that $\mathcal{P}_{1,n}^d = \operatorname{conv}\{x_1, \ldots, x_n\}$ for some packing set $\{x_1, \ldots, x_n\}$, and hence

$$V_1(\mathcal{P}_{1,n}^d) = \min\{V_1(\operatorname{conv}\{x_1, \ldots, x_n\})|\{x_1, \ldots, x_n\} \text{ is a packing set}\}.$$

Let $n = 3, \ldots, d+1$. Observe that $\dim C_n \leq n-1$ for $C_n = \operatorname{conv}\{x_1, \ldots, x_n\}$, and so we may assume that $\mathcal{P}^d_{1,n} = \mathcal{P}^{n-1}_{1,n}$. As a consequence, it is sufficient to examine $\mathcal{P}^d_{1,d+1}$.

Recall from Chapter 2 that $\mathcal{P}^d_{1,n}$ is very much ball-like if n is large compared to d. The shape of $\mathcal{P}^d_{i,d+1}$ seems to depend on i. In Section 4.3 we will prove that the sausage arrangement becomes more optimal against the regular simplex arrangement as i increases; that is,

$$\begin{aligned} V_i(T^d + B^d) &< V_i(S_{d+1} + B^d) \qquad \text{if } i < B(d), \\ V_i(T^d + B^d) &> V_i(S_{d+1} + B^d) \qquad \text{if } i > B(d), \end{aligned}$$

where $S_{d+1} \in \mathcal{G}^d_{d+1}$ is the seqment of length $2d$ and $\log_d B(d) \sim \log_d \sqrt{d}$. We believe that $\mathcal{P}^d_{1,d+1} \equiv T^d$, and the main goal of this chapter is to prove some results supporting the conjecture.

But first we have a look at $V_1(\mathcal{P}^d_{1,n})$ for general n. Theorem 2.5.1 and $\vartheta^d_{1,n} = V_1(\mathcal{P}^d_{1,n}) + V_1(B^d)$ yield that, as n tends to infinity,

$$V_1(\mathcal{P}^d_{1,n}) \sim \vartheta^d_{1,n} \sim \delta_d{}^{-1/d} \cdot V_1(B^d) \cdot n^{-1/d},$$

and similarly

$$V_1(\mathcal{Q}^d_{1,n}) \sim \bar{\vartheta}^d_{1,n} \sim \bar{\delta}_d{}^{-1/d} \cdot V_1(B^d) \cdot n^{-1/d}.$$

If $n = 1$ then $\mathcal{P}^d_{1,n}$ and $\mathcal{Q}^d_{1,n}$ are just a point.If $n \geq 2$ then $\mathcal{P}^d_{1,n}$ contains a segment of length 2, and readily

$$2 \leq V_1(\mathcal{P}^d_{1,n}) \leq V_1(\mathcal{Q}^d_{1,n}).$$

LEMMA 3.1.1 *Let $d \geq 2$ and $n \geq 2$. Then*

$$V_1(\mathcal{P}^d_{1,n}) \leq V_1(\mathcal{Q}^d_{1,n}) \leq 2 \cdot V_1(B^d) \cdot n^{-1/d}.$$

Proof: Let $r = 2n^{-1/d}$. Then 2.1 yields $\bar{\nu}(r B^d) \geq \frac{r^d}{2^d} = n$, and consequently that $rB^d \in \mathcal{H}_n^d$. It follows that

$$V_1(\mathcal{Q}_{1,n}^d) \leq V_1(r B^d) = 2 \cdot n^{-1/d} \cdot V_1(B^d).$$

□

The upper bound is probably close to being accurate when n is large. We note that some prominent geometers (see [7]) believe that the order of $\log_2 \bar{\delta}_d$ is $-d$. Assume that this is true, and let d be large. Then $\bar{\delta}_d^{\,-1/d}$ is approximately 2, and hence $V_1(\mathcal{Q}_{1,n}^d)$ is approximately $2 \cdot V_1(B^d) \cdot n^{-1/d}$.

In the rest of the chapter we consider $\mathcal{P}_{1,n}^d$ when n is small; namely, $3 \leq n \leq d+1$. Let $C_n \in \mathcal{G}_n^d$. Our impression about the behavior of the first intrinsic volume, roughly speaking, is that the smaller the diameter of C_n the less the first intrinsic volume of C_n. Hence we prove in Section 3.3 that $V_1(C_n) > V_1(T^{n-1})$ if $\dim C_n$ is small compared to n; namely, either $\dim C_n = 1$ or n is large and $\dim C_n \leq \frac{1}{2} \ln n$. We note that there is some C_n such that $\dim C_n$ is approximately $\ln n$ and $V_1(C_n)$ is close to $V_1(T^{n-1})$. With these results we turn to the case $\dim C_n \geq n-2$. We prove that if $V_1(C_n)$ is a local minimum on $\mathcal{G}_n^d$ then $C_n \equiv T^{n-1}$. Finally, we establish that $\mathcal{P}_{1,n}^d \equiv T^{n-1}$ for $n = 3, 4, 5$ and $d \geq n-1$.

First we prove that, as d tends to infinity,

$$V_1(T^d) = (2\sqrt{2\pi} + o(1))\sqrt{\ln d}. \tag{3.1}$$

As $V_1(\mathcal{P}_{1,n}^d) \leq V_1(T^{n-1})$, it follows that, for $d \geq n-1$,

$$V_1(\mathcal{P}_{1,n}^d) \leq (2\sqrt{2\pi} + \varepsilon(n))\sqrt{\ln n},$$

where $\varepsilon(n)$ tends to zero when n tends to infinity.

3.2 The value of $V_1(T^d)$

By the Landau symbol $o(1)$, we denote functions which tends to 0 as d tends to infinity.

Recall from Chapter 1 the function $h(s)$, $-\frac{\pi}{2} < s < \frac{\pi}{2}$ is defined by the property that $\int_0^{h(s)} e^{-t^2}\,dt = s$. Let $g(t) = h(\sqrt{\pi}(1/2 - t))$, $0 < t < 1$, which then satisfies

$$\int_0^{g(t)} e^{-s^2}\,ds = \int_0^{h(\sqrt{\pi}(\frac{1}{2}-t))} e^{-s^2}\,ds = \sqrt{\pi}(\frac{1}{2} - t) = \frac{\sqrt{\pi}}{2} - \sqrt{\pi}t.$$

Note that $g(t) = -g(1-t)$ by $h(s) = -h(-s)$. By means of 1.8, we have

$$V_1(T^d) = \binom{d+1}{2} \cdot 2 \cdot \sqrt{2} \cdot \Phi(d,1) = \sqrt{2}d(d+1) \cdot \Phi(d,1),$$

where $\Phi(d,1) = \int_0^1 e^{-g^2(t)} t^{d-1}\,dt$. We prove that

$$\Phi(d,1) = (2\sqrt{\pi} + o(1))\frac{\sqrt{\ln d}}{d^2}, \tag{3.2}$$

which in turn yields the required formula

$$V_1(T^d) = \sqrt{2}d(d+1) \cdot (2\sqrt{\pi} + o(1))\frac{\sqrt{\ln d}}{d^2} = (2\sqrt{2\pi} + o(1))\sqrt{\ln d}.$$

Before focusing on 3.2, we derive a simple upper bound for $V_1(T^d)$, which we also need in the following section. Observe that

$$\Phi(d,1) = \int_0^1 e^{-g^2(t)} t^{d-1}\,dt < \int_0^1 t^{d-1}\,dt = \frac{1}{d},$$

which in turn yields that

$$V_1(T^d) < \sqrt{2}d(d+1) \cdot \frac{1}{d} = \sqrt{2}(d+1). \tag{3.3}$$

We start with some basic formulae. Let c_1 and c_2 be real constants, $c_1 > 0$ and s tend to infinity. We note that by L'Hospital's Rule:

i) $\lim_{t\to 0^+} t\ln t = 0$

ii) $\lim_{s\to\infty} \frac{\ln s}{s^{c_1}} = 0$

iii) $\frac{1}{(s+c_2)(s+c_3)} = \frac{1}{s^2} + o\left(\frac{1}{s^2}\right)$

iv) $\left(1 - \frac{\ln s}{s}\right)^s \sim \frac{1}{s}$

v) $\sqrt{\ln s + c_2 \ln\ln s} = (1+o(1))\sqrt{\ln s}$

vi) $\left(1 - \frac{1}{s\ln s}\right)^s \sim 1$

Let $0 < t < 1$. Since $\Phi(d,1) = \int_0^1 e^{-g^2(t)} t^{d-1}\,dt$, first we give an approximation for $e^{-g^2(t)}$ if t is close to 0. Since $g(t) = -g(1-t)$, this yields an approximation for $e^{-g^2(t)}$ also if t is close to 1.

3.2.1 Approximating $e^{-g^2(t)}$

Let $0 < t < (3\,e\sqrt{\pi})^{-1}$. First we define two functions $\alpha(t)$ and $\beta(t)$ which satisfy $\beta(t) < g(t) < \alpha(t)$. Since the function $\frac{1}{2s}e^{-s^2}$ is strictly decreasing, there is a unique $s > 1$ with the property that $\frac{1}{2s}e^{-s^2} = \sqrt{\pi}\,t$ for $0 < t < (3\,e\sqrt{\pi})^{-1}$. Hence we define $\alpha(t) > 1$ with

$$\frac{e^{-\alpha^2(t)}}{2\alpha(t)} = \sqrt{\pi}\,t, \qquad 0 < t < \frac{1}{3\,e\sqrt{\pi}}.$$

The identity

$$\frac{d}{ds}\left(\frac{-e^{-s^2}}{2s}\right) = e^{-s^2} + \frac{e^{-s^2}}{2s^2} > e^{-s^2}$$

yields that

$$\int_\alpha^\infty e^{-s^2}\,ds < \int_\alpha^\infty \frac{d}{ds}\left(\frac{-e^{-s^2}}{2s}\right)ds = \frac{e^{-\alpha^2}}{2\alpha}.$$

It follows that

$$\int_0^\alpha e^{-s^2}\,ds = \int_0^\infty e^{-s^2}\,ds - \int_\alpha^\infty e^{-s^2}\,ds = \frac{\sqrt{\pi}}{2} - \int_\alpha^\infty e^{-s^2}\,ds > \frac{\sqrt{\pi}}{2} - \frac{e^{-\alpha^2}}{2\alpha} \geq \frac{\sqrt{\pi}}{2} - \sqrt{\pi}t.$$

Recall that $g(t)$ is defined by

$$\int_0^{g(t)} e^{-s^2}\, ds = \frac{\sqrt{\pi}}{2} - \sqrt{\pi} t.$$

We conclude for $g(t)$ that

$$\int_0^{g(t)} e^{-s^2}\, ds = \frac{\sqrt{\pi}}{2} - \sqrt{\pi} t < \int_0^{\alpha} e^{-s^2}\, ds,$$

and hence $g(t) < \alpha(t)$.

If $s > 1$ then the function $\frac{s}{2s^2+1} e^{-s^2} = \frac{1}{2s+\frac{1}{s}} e^{-s^2}$ is also strictly increasing. By $0 < t < (3\,e\sqrt{\pi})^{-1}$, there is a unique $s > 1$ with $\frac{s}{2s^2+1} e^{-s^2} = \sqrt{\pi}\, t$. Hence we define $\beta(t) > 1$ as

$$\frac{\beta(t)}{2\beta^2(t)+1} e^{-\beta^2(t)} = \sqrt{\pi}\, t, \qquad 0 < t < \frac{1}{3\,e\sqrt{\pi}}.$$

We note that $\beta(t)$ tends to infinity as $t \to 0^+$. If $s > \beta$ then

$$\frac{2\,s^2+1}{s^2} = 2 + \frac{1}{s^2} < 2 + \frac{1}{\beta^2} = \frac{2\beta^2+1}{\beta^2}.$$

It follows that

$$\frac{d}{ds}\left(\frac{-\beta^2}{2\beta^2+1}\,\frac{e^{-s^2}}{s}\right) = \frac{\beta^2}{2\beta^2+1}\left(2\,e^{-s^2} + \frac{e^{-s^2}}{s^2}\right) = \frac{\beta^2}{2\beta^2+1}\cdot\frac{2\,s^2+1}{s^2}\,e^{-s^2} < e^{-s^2},$$

and we deduce that

$$\int_\beta^\infty e^{-s^2}\, ds > \int_\beta^\infty \frac{d}{ds}\left(\frac{-\beta^2}{2\beta^2+1}\,\frac{e^{-s^2}}{s}\right) ds = \frac{\beta^2}{2\beta^2+1}\,\frac{e^{-\beta^2}}{\beta} = \frac{\beta}{2\beta^2+1} e^{-\beta^2}.$$

This inequality yields that

$$\begin{aligned}\int_0^\beta e^{-s^2}\, ds &= \int_0^\infty e^{-s^2}\, ds - \int_\beta^\infty e^{-s^2}\, ds = \frac{\sqrt{\pi}}{2} - \int_\beta^\infty e^{-s^2}\, ds \\ &< \frac{\sqrt{\pi}}{2} - \frac{\beta}{2\beta^2+1} e^{-\beta^2} \le \frac{\sqrt{\pi}}{2} - \sqrt{\pi} t.\end{aligned}$$

Consequently,

$$\int_0^{g(t)} e^{-s^2}\,ds = \frac{\sqrt{\pi}}{2} - \sqrt{\pi}t > \int_0^{\beta} e^{-s^2}\,ds,$$

and hence $g(t) > \beta(t)$. In summary,

$$\beta(t) < g(t) < \alpha(t) \qquad \text{for} \qquad 0 < t < \frac{1}{3\,e\sqrt{\pi}}.$$

The defining equalities of $\alpha(t)$ and $\beta(t)$ can be written in the form

$$\frac{e^{-\alpha^2(t)}}{2\sqrt{\pi}\,t\,\alpha(t)} = 1 \qquad \text{and} \qquad \frac{e^{-\beta^2(t)}}{2\sqrt{\pi}\,t\,\beta(t)} = 1 + \frac{1}{2\beta^2(t)} \tag{3.4}$$

for $0 < t < (3\,e\sqrt{\pi})^{-1}$. Since $\frac{1}{2\,s}e^{-s^2}$ is a strictly decreasing function and $\beta(t) < g(t) < \alpha(t)$, 3.4 yields that

$$1 < \frac{e^{-g^2(t)}}{2\sqrt{\pi}\,t\,g(t)} < 1 + \frac{1}{2\beta^2(t)},$$

and hence by $\lim_{t\to 0^+} \beta(t) = \infty$,

$$\lim_{t\to 0^+} \frac{e^{-g^2(t)}}{2\sqrt{\pi}\,t\,g(t)} = 1. \tag{3.5}$$

The formula 3.5 is equivalent to

$$\lim_{t\to 0^+} (-g^2(t) - \ln(2\sqrt{\pi}) - \ln t - \ln g(t)) = 0.$$

Observe that $-\ln t$ tends to infinity as $t \to 0^+$, and so $g(t)$ also tends to infinity as $t \to 0^+$. It follows that

$$\lim_{t\to 0^+} \left(-1 - \frac{\ln(2\sqrt{\pi})}{g^2(t)} + \frac{-\ln t}{g^2(t)} - \frac{\ln g(t)}{g^2(t)}\right) = 0,$$

which in turn yields that

$$\lim_{t\to 0^+} \left(\frac{-\ln t}{g^2(t)} - 1\right) = 0.$$

Hence $\lim_{t\to 0+} \frac{\sqrt{-\ln t}}{g(t)} = 1$, and combining this with 3.5 results in

$$\lim_{t\to 0+} \frac{e^{-g^2(t)}}{2\sqrt{\pi}\cdot t\sqrt{-\ln t}} = 1.$$

Now let $1-(3\,e\sqrt{\pi})^{-1} < t < 1$. Since $g(t) = -g(1-t)$, we deduce that

$$\lim_{t\to 1^-} \frac{e^{-g^2(t)}}{2\sqrt{\pi}\cdot(1-t)\sqrt{-\ln(1-t)}} = 1,$$

which in turn yields that

$$e^{-g^2(t)} = (2\sqrt{\pi}+\varepsilon(t))(1-t)\sqrt{-\ln(1-t)}, \tag{3.6}$$

where $\lim_{t\to 1^-}\varepsilon(t) = 0$. Actually, we may assume that $\varepsilon(t)$ is defined for any $0 < t < 1$.

3.2.2 Some observations on $\Phi(d,1)$

Recall that we wish to prove

$$\Phi(d,1) = \int_0^1 e^{-g^2(t)}t^{d-1}\,dt = (2\sqrt{\pi}+o(1))\frac{\sqrt{\ln d}}{d^2}.$$

The dominant part of the integral is between $1-\frac{\ln d}{d}$ and 1 since $e^{-g^2(t)} \le 1$ and by iv),

$$\int_0^{1-\frac{\ln d}{d}} e^{-g^2(t)}t^{d-1}\,dt \le \int_0^{1-\frac{\ln d}{d}} t^{d-1}\,dt = \frac{1}{d}\left(1-\frac{\ln d}{d}\right)^d = O\left(\frac{1}{d^2}\right);$$

that is, $\int_0^{1-\frac{\ln d}{d}} e^{-g^2(t)}t^{d-1}\,dt$ is small. The formula 3.6 yields that

$$\begin{aligned}
\Phi(d,1) &= \int_{1-\frac{\ln d}{d}}^1 e^{-g^2(t)}t^{d-1}\,dt + \int_0^{1-\frac{\ln d}{d}} e^{-g^2(t)}t^{d-1}\,dt \\
&= \int_{1-\frac{\ln d}{d}}^1 (2\sqrt{\pi}+\varepsilon(t))(1-t)\sqrt{-\ln(1-t)}\,t^{d-1}\,dt + O\left(\frac{1}{d^2}\right) \\
&= 2\sqrt{\pi}\,A(d) + \int_{1-\frac{\ln d}{d}}^1 \varepsilon(t)(1-t)\sqrt{-\ln(1-t)}\,t^{d-1}\,dt + O\left(\frac{1}{d^2}\right),
\end{aligned}$$

where

$$A(d) = \int_{1-\frac{\ln d}{d}}^{1} (1-t)\sqrt{-\ln(1-t)}\, t^{d-1}\, dt.$$

Defining $\mu(d) = \sup_{1-\frac{\ln d}{d} \le t < 1} |\varepsilon(t)|$, we have

$$\left| \int_{1-\frac{\ln d}{d}}^{1} \varepsilon(t)(1-t)\sqrt{-\ln(1-t)}\, t^{d-1}\, dt \right| \le \mu(d) \int_{1-\frac{\ln d}{d}}^{1} (1-t)\sqrt{-\ln(1-t)}\, t^{d-1}\, dt.$$

Since $\lim_{t\to 1^-} \varepsilon(t) = 0$ yields that $\lim_{d\to\infty} \mu(d) = 0$, we deduce that

$$\Phi(d,1) = (2\sqrt{\pi} + o(1))A(d) + O\left(\frac{1}{d^2}\right). \tag{3.7}$$

Before estimating $A(d)$, we evaluate a frequently used integral. Let $0 \le a < b \le 1$ and $m > 0$. Then

$$\begin{aligned}
\int_{1-b}^{1-a} (1-t)t^m\, dt &= \int_{1-b}^{1-a} t^m\, dt - \int_{1-b}^{1-a} t^{m+1}\, dt \\
&= \frac{(1-a)^{m+1}}{m+1} - \frac{(1-b)^{m+1}}{m+1} - \frac{(1-a)^{m+2}}{m+2} + \frac{(1-b)^{m+2}}{m+2} \\
&= (1-a)^{m+1}\left(\frac{1}{m+1} - \frac{1-a}{m+2}\right) \\
&\quad -(1-b)^{m+1}\left(\frac{1}{m+1} - \frac{1-b}{m+2}\right).
\end{aligned}$$

We use this in the form

$$\begin{aligned}
\int_{1-b}^{1-a} (1-t)t^m\, dt &= (1-a)^{m+1}\frac{1+a(m+1)}{(m+1)(m+2)} \\
&\quad -(1-b)^{m+1}\frac{1+b(m+1)}{(m+1)(m+2)}.
\end{aligned} \tag{3.8}$$

First we give a lower bound for $A(d)$. If $1 - \frac{\ln d}{d} \le t < 1$ then

$$\sqrt{-\ln(1-t)} \ge \sqrt{-\ln\frac{\ln d}{d}} = \sqrt{\ln d - \ln\ln d} = (1+o(1))\sqrt{\ln d}.$$

It follows by 3.8 that

$$\begin{aligned}A(d) &\geq (1+o(1))\sqrt{\ln d}\int_{1-\frac{\ln d}{d}}^{1}(1-t)t^{d-1}\,dt\\ &= (1+o(1))\sqrt{\ln d}\left(\frac{1}{d(d+1)}-\left(1-\frac{\ln d}{d}\right)^{d}\frac{1+\ln d}{d(d+1)}\right).\end{aligned}$$

Observe that by iii),

$$\frac{1}{d(d+1)}=\frac{1}{d^2}+o\left(\frac{1}{d^2}\right),$$

and by iv) and ii),

$$\left(1-\frac{\ln d}{d}\right)^{d}\frac{1+\ln d}{d(d+1)}=O\left(\frac{1}{d}\right)\cdot O\left(\frac{\ln d}{d^2}\right)=O\left(\frac{\ln d}{d^3}\right)=o\left(\frac{1}{d^2}\right).$$

Thus

$$A(d)\geq(1+o(1))\sqrt{\ln d}\left(\frac{1}{d^2}+o\left(\frac{1}{d^2}\right)\right)=(1+o(1))\frac{\sqrt{\ln d}}{d^2}. \tag{3.9}$$

Finally we determine an upper bound for $A(d)$ which is of the same order as the lower bound.

3.2.3 Upper bound for $A(d)$

We write $A(d)$ as $A(d)=A_1(d)+A_2(d)$ where

$$A_1(d)=\int_{1-\frac{1}{d\ln d}}^{1}(1-t)\sqrt{-\ln(1-t)}\,t^{d-1}\,dt$$

and

$$A_2(d)=\int_{1-\frac{\ln d}{d}}^{1-\frac{1}{d\ln d}}(1-t)\sqrt{-\ln(1-t)}\,t^{d-1}\,dt.$$

First we estimate $A_1(d)$. If $f(t)=(1-t)\sqrt{-\ln(1-t)}$ and $1-\frac{1}{d\ln d}\leq t<1$ then

$$\begin{aligned}f'(t) &= -\sqrt{-\ln(1-t)}+(1-t)\cdot\frac{1}{2}\cdot\frac{1}{\sqrt{-\ln(1-t)}}\cdot\frac{1}{(1-t)}\\ &= -\sqrt{-\ln(1-t)}\left(1+\frac{1}{2\ln(1-t)}\right)<0.\end{aligned}$$

Hence $f(t)$ is maximal for $t = 1 - \frac{1}{d \ln d}$, which in turn yields that

$$(1-t)\sqrt{-\ln(1-t)} \leq \frac{1}{d\ln d}\sqrt{-\ln\frac{1}{d\ln d}} = \frac{1}{d\ln d}\sqrt{\ln d + \ln\ln d} < \frac{1}{d}$$

for $1 - \frac{1}{d\ln d} \leq t < 1$. It follows that

$$\begin{aligned} A_1(d) &= \int_{1-\frac{1}{d\ln d}}^{1} (1-t)\sqrt{-\ln(1-t)}\, t^{d-1}\, dt < \frac{1}{d}\int_{1-\frac{1}{d\ln d}}^{1} t^{d-1}\, dt \\ &= \frac{1}{d}\cdot\frac{1}{d}\left(1-\left(1-\frac{\ln d}{d}\right)^{d}\right) < \frac{1}{d^2}. \end{aligned} \tag{3.10}$$

Turning to $A_2(d)$, first observe that as $t \to 1^-$, the function $\sqrt{-\ln(1-t)}$ is increasing, and hence for $1 - \frac{\ln d}{d} \leq t \leq 1 - \frac{1}{d\ln d}$,

$$\sqrt{-\ln(1-t)} \leq \sqrt{-\ln\frac{1}{d\ln d}} = \sqrt{\ln d + \ln\ln d}.$$

It follows that

$$\begin{aligned} A_2(d) &= \int_{1-\frac{\ln d}{d}}^{1-\frac{1}{d\ln d}} (1-t)\sqrt{-\ln(1-t)}\, t^{d-1}\, dt \\ &\leq \sqrt{\ln d + \ln\ln d}\int_{1-\frac{\ln d}{d}}^{1-\frac{1}{d\ln d}} (1-t)\, t^{d-1}\, dt \\ &= \sqrt{\ln d + \ln\ln d}\cdot A_3(d), \end{aligned}$$

where

$$A_3(d) = \int_{1-\frac{\ln d}{d}}^{1-\frac{1}{d\ln d}} (1-t)\, t^{d-1}\, dt.$$

By 3.8 and the asymptotic formulae iv) and vi),

$$\begin{aligned} A_3(d) &= \left(1-\frac{1}{d\ln d}\right)^{d}\frac{1+\frac{1}{\ln d}}{d(d+1)} - \left(1-\frac{\ln d}{d}\right)^{d}\frac{1+\ln d}{d(d+1)} \\ &= (1+o(1))\left(1+\frac{1}{\ln d}\right)\cdot\frac{1}{d(d+1)} - O\left(\frac{1}{d}\right)\cdot\frac{1+\ln d}{d(d+1)}. \end{aligned}$$

Note that

$$\frac{1}{d(d+1)} = \frac{1}{d^2} + o\left(\frac{1}{d^2}\right) \qquad \text{and} \qquad \frac{1+\ln d}{d(d+1)} = O\left(\frac{\ln d}{d^2}\right),$$

and hence by ii),

$$\begin{aligned} A_3(d) &= \frac{1}{d^2} + o\left(\frac{1}{d^2}\right) + O\left(\frac{1}{d}\right) \cdot O\left(\frac{\ln d}{d^2}\right) = \frac{1}{d^2} + o\left(\frac{1}{d^2}\right) + O\left(\frac{\ln d}{d^3}\right) \\ &= \frac{1}{d^2} + o\left(\frac{1}{d^2}\right) + o\left(\frac{1}{d^2}\right) = (1+o(1))\frac{1}{d^2}. \end{aligned}$$

As $\sqrt{\ln d + \ln\ln d} = (1+o(1))\sqrt{\ln d}$ by v), for $A_2(d)$ we have the estimate

$$A_2(d) \leq (1+o(1))\frac{\sqrt{\ln d}}{d^2}.$$

Recall that according to 3.10, $A_1(d) < \frac{1}{d^2}$. Thus

$$A(d) = A_1(d) + A_2(d) \leq (1+o(1))\frac{\sqrt{\ln d}}{d^2},$$

which in turn yields by 3.9, that

$$A(d) = (1+o(1))\frac{\sqrt{\ln d}}{d^2}.$$

Finally, by 3.7,

$$\Phi(d,1) = (2\sqrt{\pi} + o(1))A(d) + O\left(\frac{1}{d^2}\right) = (2\sqrt{\pi} + o(1))\frac{\sqrt{\ln d}}{d^2},$$

and hence 3.2 yields that

$$V_1(T^d) = \sqrt{2}d(d+1) \cdot (2\sqrt{\pi} + o(1))\frac{\sqrt{\ln d}}{d^2} = (2\sqrt{2\pi} + o(1))\sqrt{\ln d}.$$

□

3.3 Minimal simplices

First we prove that $\dim\mathcal{P}^d_{i,n}$ can not be very small compared to n. We have seen in Section 3.1 that in order to gain information about $\mathcal{P}^d_{1,n}$, $n = 3, \ldots, d+1$, it is sufficent to consider $\mathcal{P}^d_{1,d+1}$.

LEMMA 3.3.1 *Let $d \geq 2$. Then $\dim\mathcal{P}^d_{1,d+1} \geq 2$, and if d is large then $\dim\mathcal{P}^d_{1,d+1} > \frac{1}{2}\ln(d+1)$.*

Proof: Let $C \in \mathcal{G}^d_{d+1}$. We prove that if $\dim C = 1$ or if d is large and $\dim C \leq \frac{1}{2}\ln(d+1)$ then $V_1(C) > V_1(T^d)$. Since $V_1(T^d) \geq V_1(\mathcal{P}^d_{1,d+1})$ by definition, it follows that $C \not\equiv \mathcal{P}^d_{1,d+1}$.

Let $\dim C = 1$, and so $V_1(C) \geq 2d$. Note that

$$V_1(T^2) = 3 < V_1(C)$$

for $d = 2$. If $d \geq 3$ then $\sqrt{2}(d+1) < 2d$. As $V_1(T^d) < \sqrt{2}(d+1)$ by 3.3, we have

$$V_1(T^d) < \sqrt{2}(d+1) < 2d \leq V_1(C).$$

In summary,

$$V_1(T^d) < V_1(C) \tag{3.11}$$

if $d \geq 2$ and $\dim C = 1$, and hence $C \not\equiv \mathcal{P}^d_{1,d+1}$.

Now let $2 \leq m \leq \frac{1}{2}\ln(d+1)$ and $C \in \mathcal{G}^m_{d+1}$. Then $V_m(C + B^m) \geq V_m(B^m) \cdot (d+1)$ because one may pack $d+1$ m-dimensional unit balls into $C + B^m$, and the Alexandrov-Fenchel inequality in E^m yields that $V_1(C+B^m) \geq V_1(B^m) \cdot (d+1)^{1/m}$. We deduce from the linearity of the first intrinsic volume that

$$V_1(C) = V_1(C+B^m) - V_1(B^m) > V_1(B^m)((d+1)^{1/m} - 1).$$

By 1.7, we have the estimate

$$V_1(B^m) = m \cdot \frac{\kappa_m}{\kappa_{m-1}} > m \cdot \sqrt{\frac{2\pi}{m+1}} = \sqrt{2\pi} \cdot \sqrt{\frac{m}{m+1}} \cdot \sqrt{m}.$$

Note that $\sqrt{\frac{m}{m+1}} \geq \sqrt{\frac{2}{3}}$ because $m \geq 2$. Thus

$$V_1(C) > \frac{2\sqrt{\pi}}{\sqrt{3}} \cdot \sqrt{m} \cdot ((d+1)^{1/m} - 1).$$

The function $t^{1/2}((d+1)^{1/t} - 1)$ decreases on the interval $(2, \frac{1}{2}\ln(d+1))$, since its derivative is

$$\begin{aligned}
\frac{d}{dt} t^{1/2}((d+1)^{1/t} - 1) &= \tfrac{1}{2} t^{-1/2}((d+1)^{1/t} - 1) + t^{1/2} \left(-\frac{1}{t^2}\right) \ln(d+1) \cdot (d+1)^{1/t} \\
&< \tfrac{1}{2} t^{-1/2}(d+1)^{1/t} - t^{-3/2} \ln(d+1) \cdot (d+1)^{1/t} \\
&= \tfrac{1}{2} t^{-1/2}(d+1)^{1/t} \left(1 - \frac{2\ln(d+1)}{t}\right) \leq 0
\end{aligned}$$

by $\frac{\ln(d+1)}{t} \geq 2$. It follows that

$$\begin{aligned}
V_1(C) &> \frac{2\sqrt{\pi}}{\sqrt{3}} \sqrt{\tfrac{1}{2} \ln(d+1)}\, ((d+1)^{\frac{2}{\ln(d+1)}} - 1) > \frac{\sqrt{2\pi}}{\sqrt{3}} (e^{\ln(d+1) \cdot \frac{2}{\ln(d+1)}} - 1)\sqrt{\ln d} \\
&= \frac{e^2 - 1}{\sqrt{3}} \sqrt{2\pi}\sqrt{\ln d} > 3\sqrt{2\pi} \cdot \sqrt{\ln d}.
\end{aligned}$$

On the other hand, $V_1(T^d) \sim 2\sqrt{2\pi}\sqrt{\ln d}$ by 3.1. Observe that $V_1(T^d) < V_1(C)$ if d is large, and hence $C \not\equiv \mathcal{P}^d_{1,d+1}$.

□

Remark: Actually, using bounds of the type $V_1(\mathcal{P}^m_{1,d+1}) \geq V_1(B^m) \cdot (d+1)^{1/m}$, one can not gain much more information about the dimension of $\mathcal{P}^d_{1,d+1}$. Let d be large and $m = [2\ln d]$; that is, the integer part of $2\ln d$. Note that

$$\lim_{d\to\infty} (d+1)^{1/m} = \lim_{d\to\infty} d^{\frac{1}{[2\ln d]}} = \sqrt{e}.$$

It follows by 1.7, that as d tends to infinity,

$$V_1(B^m) = m \cdot \frac{\kappa_m}{\kappa_{m-1}} \sim m \cdot \sqrt{\frac{2\pi}{m}} = \sqrt{2\pi\, m} \sim 2\sqrt{\pi}\sqrt{\ln d}.$$

Let $r = 2(d+1)^{1/m}$. According to 2.1, $\bar{\nu}(rB^m) \geq \frac{r^m}{2^m} = d+1$, and hence

$$V_1(\mathcal{P}^m_{1,d+1}) \leq V_1(rB^m) = 2(d+1)^{1/m} V_1(rB^m) \sim 2\sqrt{e} \cdot 2\sqrt{\pi}\sqrt{\ln d}.$$

As $\frac{2\sqrt{e}}{\sqrt{2}} = 2.33164$ and $V_1(T^d) \sim \sqrt{2} \cdot 2\sqrt{\pi}\sqrt{\ln d}$, it follows that $V_1(\mathcal{P}^m_{1,d+1}) < 3\ V_1(T^d)$ if d is large enough, or in other words, $V_1(\mathcal{P}^m_{1,d+1})$ is close to $V_1(T^d)$.

Now we turn to the case when $\dim C \geq n-2$. As we mentioned earlier, probably $\mathcal{P}^d_{1,n} \equiv T^{n-1}$, $n = 3, \ldots, d+1$. With the help of some lemmas, we prove that $V_1(T^{n-1})$ is a local minimum, and if $\dim \mathcal{P}^d_{1,n} \geq n-2$ then $\mathcal{P}^d_{1,n} \equiv T^{n-1}$ for $d \geq n-1$. We conclude the section showing that $\mathcal{P}^d_{1,n} \equiv T^{n-1}$ for $n = 3, 4, 5$ and $d \geq n-1$.

Let $d \geq 2$, $K = \text{conv}\{x_0, \ldots, x_d\}$ have dimension at least $d-1$ and denote $\text{aff}\{x_2, \ldots, x_d\}$ by g. We say that g is an *axis* of K if either $\dim K = d$ or $\dim K = d-1$, $\dim g = d-2$ and g strictly separates x_0 and x_1 in $\text{aff}\, K$. Assume that g is an axis of K, and let $H = \text{aff}\{x_1, \ldots, x_d\}$. Observe that H is a hyperplane in E^d as $x_1 \notin g$, and denote by H^+ the halfspace of E^d determined by H and containing x_0. By rotating x_1 towards x_0 around g we mean a rotation of x_1 around g into $\mathit{int} H^+$.

Note that this rotation brings x_1 closer to x_0. Let Π be the plane through x_1 perpendicular to g, and y be the orthogonal projection of x_0 onto Π (see Figure 3.1). Then $d(x_0, x_1)^2 = d(x_0, y)^2 + d(x_1, y)^2$ and x_1 stays in Π throughout the rotation. As the rotation decreases $d(x_1, y)$ in Π, it also decreases $d(x_0, x_1)$.

The following lemma has a key role in the future considerations.

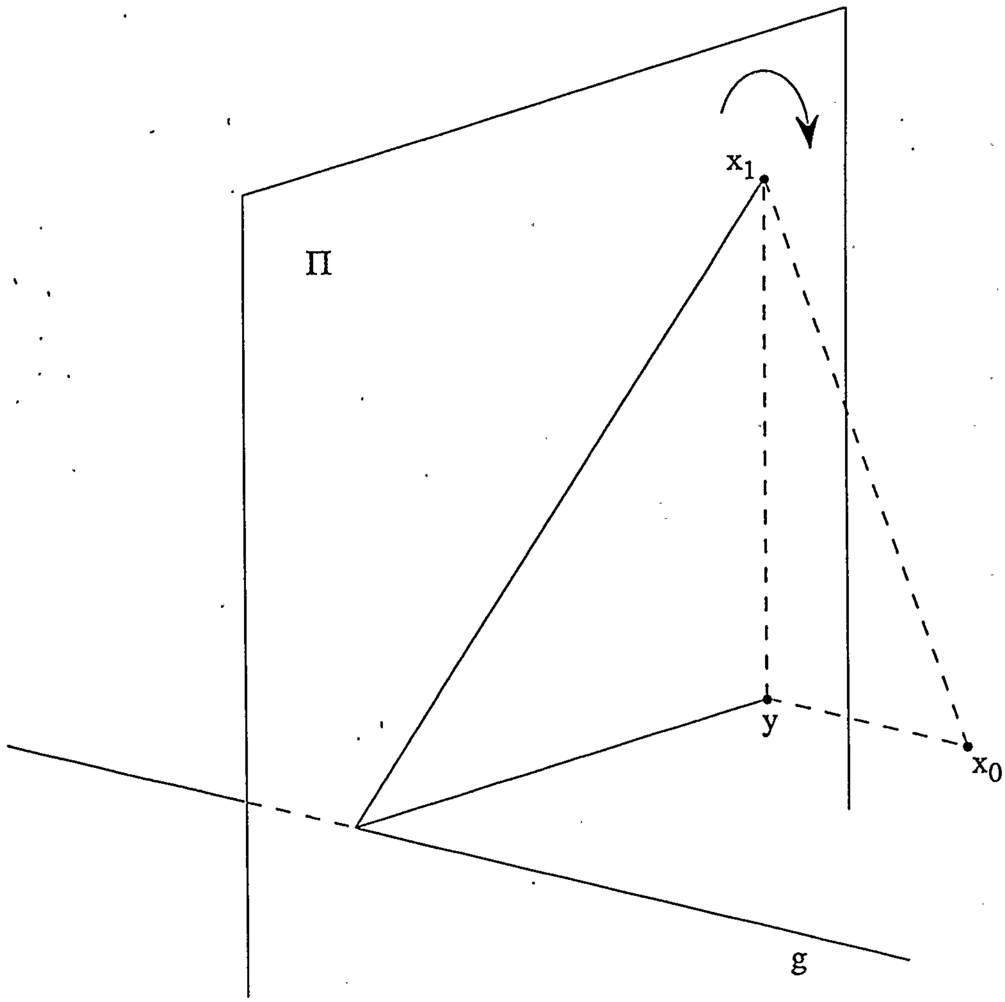

Figure 3.1

Rotating around an affine (d-2)-space

LEMMA 3.3.2 (Bending) *Let $d \geq 2$, $K = conv\{x_0, \dots, x_d\}$ with $dimK \geq d - 1$, and $g = aff\{x_2, \dots, x_d\}$ be an axis of K. Then rotating x_1 towards x_0 strictly decreases $V_1(K)$.*

Proof: Denote by y_1 the new position of x_1, by H the hyperplane perpendicularly bisecting the segment $\text{conv}\{x_1, y_1\}$, and let H^+ be the halfspace containing y_1 (see Figure 3.2). Observe that $g \subset H$, and that $x_0 \in intH^+$ by $d(y_1, x_0) < d(x_1, x_0)$.

For any $x \in E^d$, let $\varphi(x)$ be the image of x by the reflection through H and let $y_0 = \varphi(x_0)$. The sets

$$K' = \text{conv}\{y_0, y_1, x_2, \dots, x_d\}, \qquad M = \text{conv}\{x_0, y_1, x_2, \dots, x_d\}$$
$$\text{and} \qquad M' = \text{conv}\{y_0, x_1, x_2, \dots, x_d\}$$

satisfy $K' = \varphi(K)$ and $M' = \varphi(M)$, and the lemma states that $V_1(M) < V_1(K)$.

By the linearity of the intrinsic volumes, $V_1(M) = V_1(M_0)$ and $V_1(K) = V_1(K_0)$ for $M_0 = \frac{1}{2}(M + M')$ and $K_0 = \frac{1}{2}(K + K')$. We prove that M_0 is strictly contained in K_0, which in turn yields that $V_1(M) < V_1(K)$.

The points $u_0 = \frac{1}{2}(x_0 + y_1)$, $v_0 = \frac{1}{2}(y_0 + x_1)$, $u_1 = \frac{1}{2}(x_0 + x_1)$ and $v_1 = \frac{1}{2}(y_0 + y_1)$ satisfy $v_i = \varphi(u_i)$, $i = 0, 1$. These points occur in the sets

$$\sigma_K = \tfrac{1}{2}(\{x_0, x_1, x_2, \dots, x_d\} + \{y_0, y_1, x_2, \dots, x_d\})$$
$$\text{and} \qquad \sigma_M = \tfrac{1}{2}(\{x_0, y_1, x_2, \dots, x_d\} + \{y_0, x_1, x_2, \dots, x_d\}).$$

We note that $K_0 = \text{conv}\,\sigma_K$ and $M_0 = \text{conv}\,\sigma_M$, and that $\sigma_K \backslash \sigma_M = \{u_0, v_0\}$ and $\sigma_M \backslash \sigma_K = \{u_1, v_1\}$.

As $y_i = \varphi(x_i)$, we may consider the trapezoid $\text{conv}\{x_0, y_0, x_1, y_1\}$, and the line l which is parallel to $\text{aff}\{x_i, y_i\}$ in the plane $\text{aff}\{x_0, y_0, x_1, y_1\}$, $i = 0, 1$, and have the

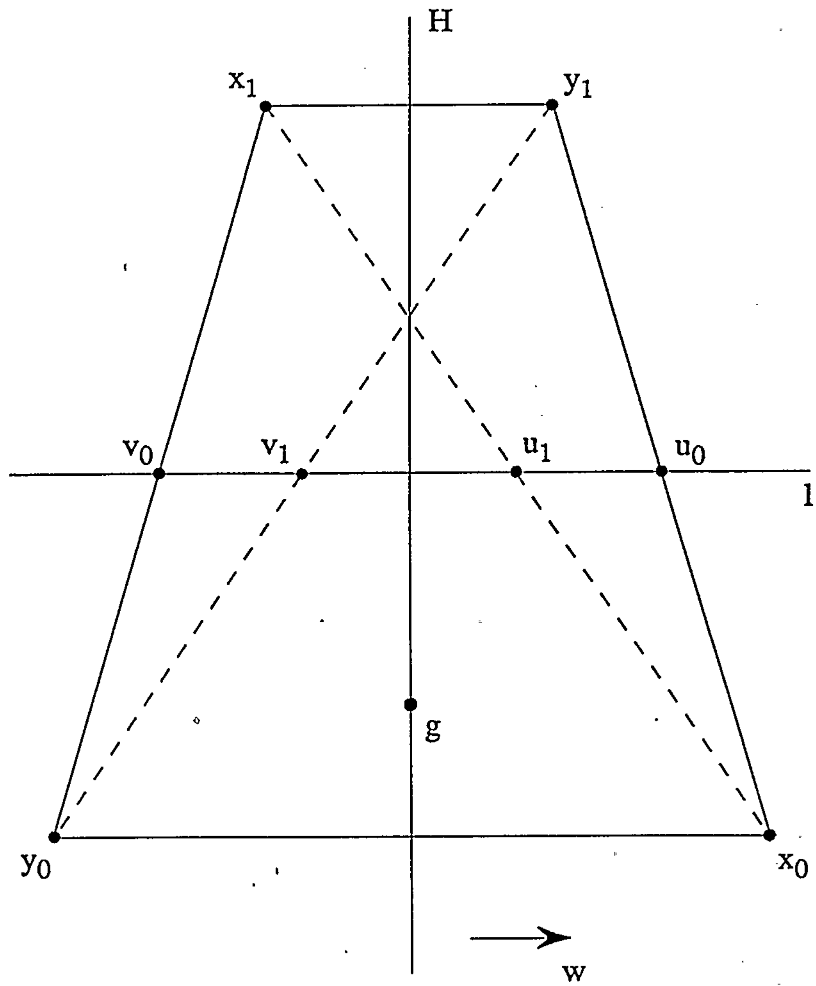

Figure 3.2

The Bending of a simplex

same distance from aff$\{x_0, y_0\}$ and aff$\{x_1, y_1\}$. Then l passes through u_i and v_i, $i = 0, 1$, and the points u_0 and v_0 lie on the boundary of the trapezoid. We deduce that $u_1, v_1 \in \text{conv}\{u_0, v_0\}$, and since u_1 and v_1 are the only points in $\sigma_M \backslash \sigma_K$, we have $M_0 \subset K_0$.

In order to establish the strict inclusion, assume that H contains the origin and let w be the unit normal vector to H pointing into H^+. Define μ as

$$\mu = \max\{< w, x_0 >, < w, y_1 >\} = \max\{< w, z > | z \in M\}.$$

Any $z_0 \in M_0$ can be written in the form $z_0 = \frac{1}{2}(z + z')$ for some $z \in M$ and $z' \in M'$. Thus $< w, z' > \leq 0$ and $< w, z > \leq \mu$ yield that

$$< w, z_0 > = \tfrac{1}{2} < w, z > + \tfrac{1}{2} < w, z' > \leq \tfrac{1}{2}\mu.$$

On the other hand, as $< w, x_0 >$ and $< w, y_1 >$ are positive and one of them is μ,

$$< w, v_0 > = \tfrac{1}{2} < w, x_0 > + \tfrac{1}{2} < w, y_1 > \; > \tfrac{1}{2}\mu,$$

which in turn yields that $u_0 \in K_0$ but $u_0 \notin M_0$. Therefore M_0 is strictly contained in K_0, and so $V_1(K) < V_1(M)$.

□

As the first application of the Bending Lemma, we prove that

THEOREM 3.3.3 *Let $d \geq 2$ and $n = 3, \ldots, d+1$. Then $V_1(T^{n-1})$ is a local minimum on $\mathcal{G}_n^d$.*

Proof: Let $K = \text{conv}\{x_1, \ldots, x_n\}$. If $d(x_i, x_j)$ tends to 2 for any $1 \leq i < j \leq n$ then K tends T^{n-1}. Thus we may set a $\varepsilon > 0$ with the property that if $2 \leq d(x_i, x_j) \leq 2 + \varepsilon$ for any $1 \leq i < j \leq n$ then $\dim K = n - 1$.

Let $\mathcal{U}$ be the family of all $C \in \mathcal{G}_n^d$ with $D(C) \leq 2 + \varepsilon$. Then $\mathcal{U}$ is a closed neighbourhood of T^{n-1} in $\mathcal{G}_n^d$, and there exists a $C_0 \in \mathcal{U}$ with

$$V_1(C_0) = \min\{V_1(C) | C \in \mathcal{U}\}$$

by Blaschke's selection theorem. There is a packing set $\{x_1, \ldots, x_n\} \subset C_0$. The points satisfy $2 \leq d(x_i, x_j) \leq 2 + \varepsilon$ for $1 \leq i < j \leq n$, and hence $\dim K = n - 1$ for $K = \text{conv}\{x_1, \ldots, x_n\}$. The minimality of $V_1(C_0)$ and $K \subset C_0$ yield that $K = C_0$, and hence we may assume that $K \subset E^{n-1}$.

Assume that, say, $d(x_1, x_2) > 2$. In E^{n-1}, rotate x_1 around $\text{aff}\{x_3, \ldots, x_n\}$ towards x_2 so that still $d(x_1', x_2) \geq 2$ for the new position x_1', and let $K' = \{x_1', x_2, \ldots, x_n\}$. Then $V_1(K') < V_1(K)$ by the Bending Lemma. Since $K' \in \mathcal{U}$, we have contradicted the minimality of $V_1(C_0)$, and hence $d(x_i, x_j) = 2$ for $1 \leq i < j \leq n$ and $C_0 = T^{n-1}$.

□

Let $n = 3, \ldots, d+1$. In order to prove that $V_1(T^{n-1})$ is the unique local minimum of $V_1(C)$, $C \in \mathcal{G}_n^d$, with the property that $\dim C \geq n - 2$, we need the following two lemmas. Recall from Section 1.5 that

$$h_d = \sqrt{4 - R(T^{d-1})^2} = \sqrt{\frac{2(d+1)}{d}} > \sqrt{2},$$

where h_d is the height of T^d, and that $R(T^d) = \sqrt{\frac{2d}{d+1}} < \sqrt{2}$.

LEMMA 3.3.4 *Let $d \geq n \geq 1$, $K = conv\{x_1, \ldots, x_n\}$ be congruent to T^{n-1}, $x_0 \in E^d$ and $d(x_0, x_i) \geq 2$ for $i = 1, \ldots, n$. Then $d(x_0, y) > \sqrt{2}$ for any $y \in K$.*

Proof: We proceed by induction on n. If $n = 1$ then $K = \{x_1\}$ and $d(x_0, x_1) \geq 2 > \sqrt{2}$.

Let $n \geq 2$ and assume that for any smaller value of n the statement of the lemma holds. Let y be the closest point of K to x_0. It is sufficient to prove that $d(x_0, y) > \sqrt{2}$.

If $y \in relbd K$ then we may assume that $y \in \operatorname{conv}\{x_1, \ldots, x_m\}$, $1 \leq m \leq n$. As $\operatorname{conv}\{x_1, \ldots, x_m\}$ is congruent to T^{m-1}, the induction hypothesis yields that $d(x_0, y) > \sqrt{2}$.

If $y \in \operatorname{relint} K$ then Lemma 1.5.5 yields that there is a vertex of K, say x_1, with $d(y, x_1) < R(T^{n-1})$. By Lemma 1.4.2, the line $\operatorname{aff}\{y, x_0\}$ is perpendicular to $\operatorname{aff} K$, and hence

$$d(y, x_0) = \sqrt{d(x_1, x_0)^2 - d(x_1, y)^2} \geq \sqrt{4 - R(T^{n-1})^2} = h_n > \sqrt{2}.$$

□

Actually the same proof shows that the conditions of Lemma 3.3.4 yield $d(y, x_0) \geq h_n = \sqrt{\frac{2(n+1)}{n}}$.

LEMMA 3.3.5 *Let $d \geq 1$ and $\{x_0, \ldots, x_{d+1}\}$ be a packing set which spans E^d. Then the set has two points, say, x_0 and x_1, such that $d(x_0, x_1) > 2$ and $H = \mathit{aff}\{x_2, \ldots, x_{d+1}\}$ is a hyperplane which strictly separates x_0 and x_1.*

Proof: We proceed by induction on d. If $d = 1$ then we may assume that x_2 separates x_0 and x_1, and readily $d(x_0, x_1) > 2$.

Assume that $d \geq 2$ and the statement of the lemma holds if the dimension is less than d. If the points are not in general position then $d+1$ of them, say $x_0, \ldots, x_d$, are contained in a hyperplane H_0. The points $x_0, \ldots, x_d$ span H_0 because $x_0, \ldots, x_{d+1}$ span E^d. By the induction hypothesis, we may assume that $d(x_0, x_1) > 2$, and that $g = \mathrm{aff}\{x_2, \ldots, x_d\}$ has dimension $d-2$ and strictly separates x_0 and x_1 in H_0. Since $x_{d+1} \notin H_0$, $H = \mathrm{aff}\{x_2, \ldots, x_{d+1}\}$ is a hyperplane separating x_0 and x_1.

Hence assume that the points $x_0, \ldots, x_{d+1}$ are in general position. By Radon's theorem, we can reorder the points so that there is a $y \in K_1 \cap K_2$ for $K_1 = \mathrm{conv}\{x_0, \ldots, x_m\}$ and $K_2 = \mathrm{conv}\{x_{m+1}, \ldots, x_{d+1}\}$, $0 \leq m \leq d$. As $x_0, \ldots, x_{d+1}$ are in general position,

$$\dim K_1 + \dim K_2 = m + (d - m) = d,$$

and hence $\mathrm{aff} K_1 \cap \mathrm{aff} K_2 = \{y\}$. It also follows that y is in the relative interior of both K_1 and K_2.

If $K_2 \equiv T^{d-m}$ then by Lemma 1.5.5, we may assume that $d(y, x_{d+1}) \leq R(T^{d-m}) < \sqrt{2}$. Assume in addition, that K_1 is congruent to T^m. Then Lemma 3.3.4 yields, as $y \in K_1$, that $d(x_{d+1}, y) > \sqrt{2}$. This contradiction proves that one of K_1 and K_2, say K_1, is not a regular simplex with edge length 2, and hence we may assume that $m \geq 1$ and $d(x_0, x_1) > 2$.

Let g be $\{y\}$ if $m = 1$ and $g = \{y, x_2, \ldots, x_m\}$ if $m \geq 2$. In $\mathrm{aff} K_1$, g strictly separates x_0 and x_1 because $y \in \mathrm{relint}\, K_1$. Since $\mathrm{aff} K_1 \cap \mathrm{aff} K_2 = \{y\}$, the hyperplane $H = \mathrm{aff}\{x_2, \ldots, x_{d+1}\}$ also separates x_0 and x_1.

□

Now we prove the statement about the partial uniqueness of $V_1(T^{n-1})$ as a local minimum on $\mathcal{G}_n^d$.

LEMMA 3.3.6 *Let $d \geq 2$, $n = 3, \ldots, d+1$ and $K \in \mathcal{G}_n^d$ with $\dim K \geq n-2$. If $V_1(K)$ is a local minimum on $\mathcal{G}_n^d$ then $K \equiv T^{n-1}$.*

Proof: Note that $V_1(T^{n-1})$ is a local minimum according to Theorem 3.3.3.

Let $V_1(K)$ be a local minimum on $\mathcal{G}_n^d$ and $\dim K \geq n-2$. There is a packing set $\{x_0, \ldots, x_{n-1}\} \subset K$, and denote $\mathrm{conv}\{x_0, \ldots, x_{n-1}\}$ by K_1. For $0 \leq \lambda \leq 1$, define

$$K_\lambda = (1-\lambda) \cdot K + \lambda \cdot K_1.$$

Then $K = K_0$ and $K_1 \subset K_\lambda \subset K$. Observe that $K_\lambda \in \mathcal{G}_n^d$ as $K_1 \subset K_\lambda$, and K_λ tends to K as $\lambda \to 0^+$.

If $K_1 \neq K$ then also $K_\lambda \neq K$ for $0 < \lambda < 1$, which in turn yields that $V_1(K_\lambda) < V_1(K)$ for $0 < \lambda < 1$ by the strict monotonicity of the first intrinsic volume. This contradicts the local minimality of $V_1(K)$, and hence $K = \mathrm{conv}\{x_0, \ldots, x_{n-1}\}$.

Assume that $K \subset E^{n-1}$ and let $g = \mathrm{aff}\{x_2, \ldots, x_{n-1}\}$. By Lemma 3.3.5, if $\dim K = n-2$ then we may assume that $d(x_0, x_1) > 2$ and g is an axis of K in E^{n-1}. If $\dim K = n-1$ and $K \not\equiv T^{n-1}$ then again g is an axis of K in E^{n-1}, and we may assume that $d(x_0, x_1) > 2$. By means of the Bending Lemma, in both cases K can be deformed a little, by a rotation through g in E^{n-1}, into a $K' \in \mathcal{G}_n^{n-1} \subset \mathcal{G}_n^d$ with $V_1(K') < V_1(K)$. Therefore, if $K \in \mathcal{G}_n^d$, $\dim K \geq n-2$ and $V_1(K)$ is a local minimum on $\mathcal{G}_n^d$ then $K \equiv T^{n-1}$.

□

In summary, as $\mathcal{P}_{1,n}^d = \mathcal{P}_{1,n}^{n-1}$ for $n = 3, \ldots, d+1$, Lemma 3.3.1 and Lemma 3.3.6 yield

THEOREM 3.3.7 *Let $d \geq 2$ and $n = 3, \ldots, d+1$. Then $dim\mathcal{P}^d_{1,n} \geq 2$; moreover, $dim\mathcal{P}^d_{1,n} \geq \frac{1}{2}\ln n$ if n is large. If $dim\mathcal{P}^d_{1,n} \geq n-2$ then $\mathcal{P}^d_{1,n} \equiv T^{n-1}$.*

Theorem 3.3.7 yields that if $d \geq n-1$ then $\mathcal{P}^d_{1,n} \equiv T^{n-1}$ for $n = 3, 4$. Denote by $\mathcal{E}$ be the set of edges of T^3, and by $\gamma(e, T^3)$ the external angle at e for $e \in \mathcal{E}$. Let e_0 be an edge of T^3. Simple calculations show that $\gamma(e_0, T^3) = (2\pi)^{-1}arccos(-\frac{1}{3})$, and hence

$$V_1(\mathcal{P}^3_{1,4}) = V_1(T^3) = \sum_{e\in\mathcal{E}} V_1(e)\cdot\gamma(e, T^3) = 6\cdot 2\cdot\gamma(e, T^3) = 3.64904.$$

Turning to the case $n = 5$, recall that U^3 is the union of two copies of T^3 glued together by a common face. It follows that

$$V_1(U^3) = V_1(T^3) + V_1(T^3) - V_1(T^2) = 4.29808.$$

We remark that with the help of some laborious considerations, one can prove that the unique choice for $\mathcal{P}^3_{1,5}$ is U^3. If $\dim\mathcal{P}^4_{1,5} \leq 3$ then $V_1(\mathcal{P}^4_{1,5}) = V_1(\mathcal{P}^3_{1,5})$, and so Theorem 3.3.7 yields that $\mathcal{P}^4_{1,5} \equiv T^4$. But we choose a much shorter path to prove $\mathcal{P}^4_{1,5} \equiv T^4$, and hence $\mathcal{P}^d_{1,5} \equiv T^4$, $d \geq 4$.

LEMMA 3.3.8 *Let $\{x_0, x_1, x_2, x_3\}$ be a packing set in E^2 such that $x_0 \in K$ for $K = conv\{x_1, x_2, x_3\}$ (see Figure 3.3). Then $V_1(K) > V_1(U^3)$.*

Proof: We frequently make use of the value $V_1(U^3) = 4.29808$. We may assume that $\dim K = 2$ since if $\dim K = 1$ then $V_1(K) \geq 6 > V_1(U^3)$.

Let $\alpha_i = \text{ang}(x_j, x_0, x_k)$ where $\{i, j, k\} = \{1, 2, 3\}$, and $\beta = \pi - \alpha_1$. Assume that $\alpha_1 \leq \alpha_2 \leq \alpha_3$. Since $x_0 \in K$, we have $\alpha_3 \leq \pi$ and $\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$, and

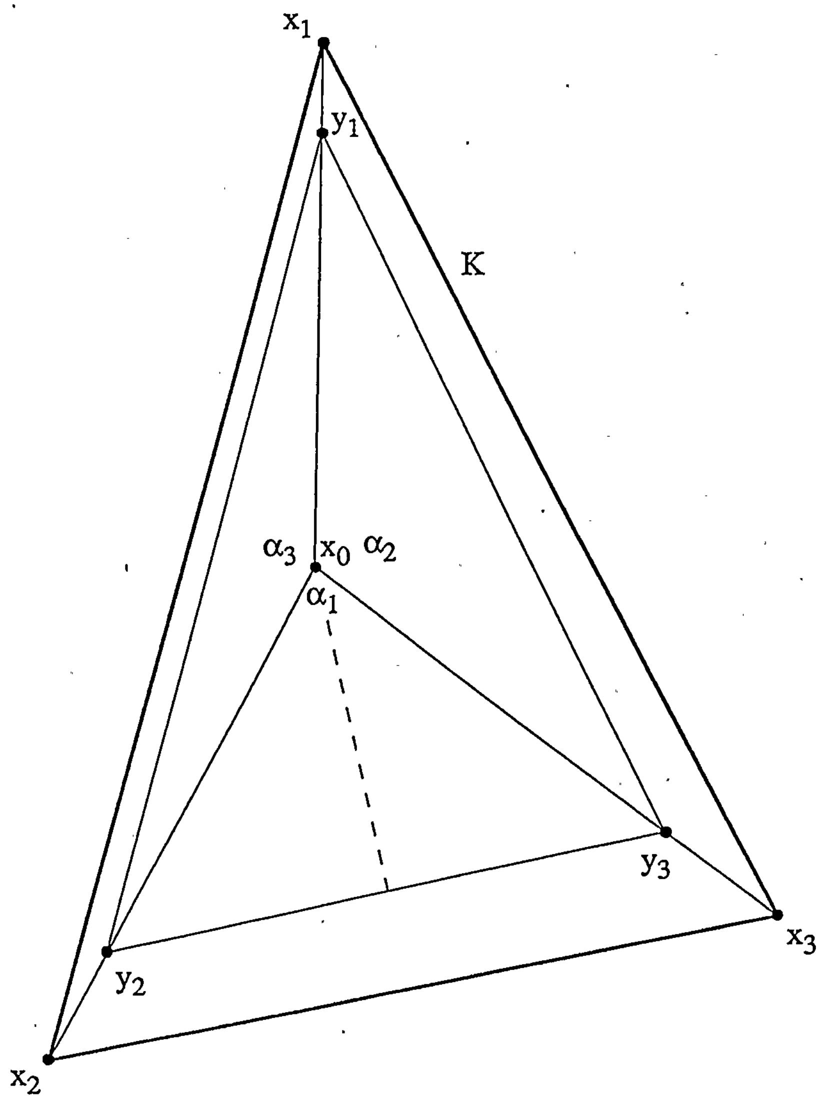

Figure 3.3

The perimeter of $\text{conv}\{x_0,x_1,x_2,x_3\}$

hence $\alpha_1 \leq \frac{2\pi}{3}$ and $\alpha_2 \geq \frac{\pi}{2}$. Note that $\pi + \beta = 2\pi - \alpha_1 = \alpha_2 + \alpha_3$. In addition, $\alpha_1 + \alpha_2 \geq \pi \geq \alpha_3$ yields that $\pi - \beta = \alpha_1 \geq \alpha_3 - \alpha_2$, and hence

$$\begin{aligned} \sin\frac{\alpha_3}{2} + \sin\frac{\alpha_2}{2} &= 2\sin\frac{\alpha_3+\alpha_2}{2}\cos\frac{\alpha_3-\alpha_2}{2} \geq 2\sin\frac{\pi+\beta}{2}\cos\frac{\pi-\beta}{2} \\ &= \sin\frac{\pi}{2} + \sin\frac{\beta}{2} = 1 + \sin\frac{\beta}{2}. \end{aligned} \tag{3.12}$$

Let y_i be the point of $\mathrm{conv}\{x_0, x_i\}$ with $d(x_0, x_i) = 2$, $i = 1, 2, 3$. Observe that if $\alpha_i \geq \frac{\pi}{2}$ then $\mathrm{conv}\{x_j, x_k\}$ is the diameter of $\mathrm{conv}\{x_j, x_0, x_k\}$, and so $d(x_j, x_k) \geq d(y_j, y_k) = 4\sin\frac{\alpha_i}{2}$.

Since $\frac{\pi}{2} \leq \alpha_2 \leq \alpha_3$, it follows by 3.12 that

$$d(x_1, x_2) + d(x_1, x_3) \geq 4\sin\frac{\alpha_3}{2} + 4\sin\frac{\alpha_2}{2} \geq 4 + 4\sin\frac{\beta}{2}.$$

If $\alpha_1 \geq \frac{\pi}{2}$ then $d(x_2, x_3) \geq 4\sin\frac{\alpha_1}{2} \geq 4\sin\frac{\pi}{4} = 2\sqrt{2}$. Since $\beta = \pi - \alpha_1 \geq \frac{\pi}{3}$, we have $4\sin\frac{\beta}{2} \geq 4\sin\frac{\pi}{6} = 2$, and hence

$$V_1(K) = \tfrac{1}{2}(d(x_1, x_2) + d(x_1, x_3) + d(x_2, x_3)) \geq \tfrac{1}{2}(4 + 2 + 2\sqrt{2}) = 4.41421 > V_1(U^3).$$

If $\alpha_1 \leq \frac{\pi}{2}$ then $\beta \geq \frac{\pi}{2}$ and $4\sin\frac{\beta}{2} \geq 4\sin\frac{\pi}{4} = 2\sqrt{2}$. We deduce by $d(x_2, x_3) \geq 2$, that

$$V_1(K) = \tfrac{1}{2}(d(x_1, x_2) + d(x_1, x_3) + d(x_2, x_3)) \geq \tfrac{1}{2}(4 + 2\sqrt{2} + 2) > V_1(U^3).$$

□

THEOREM 3.3.9 *Let $n = 3, 4, 5$ and $d \geq n - 1$. Then the unique choice for $\mathcal{P}^d_{1,n}$ is T^{n-1}.*

Proof: The cases $n = 3, 4$ readily follow from Theorem 3.3.7. Let $n = 5$ and $\{x_0, \ldots, x_4\}$ be a packing set such that $\dim K = 2$ for $K = \text{conv}\{x_0, \ldots, x_4\}$. We claim that $V_1(K) > V_1(U^3)$. If K is a pentagon then

$$V_1(K) = \tfrac{1}{2} P(K) \geq 5 > V_1(U^3).$$

If K is the quadrilateral $\text{conv}\{x_1, x_2, x_3, x_4\}$ then the diagonal $\text{conv}\{x_1, x_3\}$ divides K into two triangles, one of which contains x_0. It follows that if K is a quadrilateral or a triangle then we may assume that $x_0 \in \text{conv}\{x_1, x_2, x_3\}$, and hence $V_1(K) > V_1(U^3)$ by Lemma 3.3.8.

As $V_1(\mathcal{P}^d_{1,5}) \leq V_1(U^3)$, we conclude that $\dim\mathcal{P}^d_{1,5} \neq 2$. By Theorem 3.3.7, $\mathcal{P}^d_{1,5}$ is congruent to T^4.

$\square$

Since $T^{n-1} \in \mathcal{H}^d_n$ for $d \geq n-1$, it follows that

COROLLARY 3.3.10 *Let $n = 3, 4, 5$ and $d \geq n - 1$. Then the unique choice for $Q^d_{1,n}$ is T^{n-1}.*

Chapter 4

Packings of small numbers of balls

After considering the minimal properties of finite packings with respect to the first intrinsic volume, we consider the minimal properties of finite packings with respect to the ith intrinsic volume, $i = 1, \ldots, d$. In Sections 4.2 and 4.3 we concentrate on the relation between i and the shape of a minimal body.

4.1 The surface of four-ball packings in E^3

We prove that if for a four-ball packing in E^3, the surface of the convex hull of the balls is minimal then the centers are vertices of a regular tetrahedron. In other words,

THEOREM 4.1.1 *Let $K \in \mathcal{G}_4^3$ with $V_2(K + B^3) = \vartheta_{2,4}^3$. Then $K \equiv T^3$, and hence $\mathcal{P}_{2,4}^3 \equiv T^3$.*

In Section 2.2 we have seen that $T^3 \in \mathcal{H}_4^3$. It yields

COROLLARY 4.1.2 *If $V_2(K + B^3) = \bar{\vartheta}_{2,4}^3$ for a $K \in \mathcal{H}_4^3$ then $K \equiv T^3$. In particular, $Q_{2,4}^3 \equiv T^3$.*

If K is a minimal body in $\mathcal{G}_4^3$; that is, $V_2(K + B^3) = \vartheta_{2,4}^3$, then $K = \text{conv}\{x_0, x_1, x_2, x_3\}$ with $d(x_i, x_j) \geq 2$ for $i \neq j$ (see Section 2.4, p.60).

Let $K \in K^3$. Since $\kappa_1 = 2$, $\kappa_2 = \pi$ and $V_0(K) = 1$, it follows from Lemma 1.8.1 that

$$\begin{aligned} V_2(K+B^3) &= \frac{1}{\kappa_1}(3\kappa_3 V_0(K) + 2\kappa_2 V_1(K) + \kappa_1 V_2(K)) \\ &= \tfrac{1}{2}(3\,\kappa_3 + 2\pi V_1(K) + 2\,V_2(K)) \\ &= \frac{3}{2}\kappa_3 + \pi\, V_1(K) + V_2(K) = \frac{3}{2}\kappa_3 + \phi(K), \end{aligned}$$

where $\phi(K) = \pi V_1(K) + V_2(K)$. The theorem is equivalent to the following statement:

(*) *Let $K = conv\{x_0, \ldots, x_3\}$ with $d(x_i, x_j) \geq 2$ for $i \neq j$. Then $\phi(K) \geq \phi(T^3)$, with equality if and only if $K \equiv T^3$.*

We prove (*) in Subsection 4.1.1 for the case $\dim K \leq 2$, and in Subsection 4.1.2 if $\dim K = 3$. But before that let us have some useful definitions and observations.

Let $\{i, j, k, l\} = \{0, 1, 2, 3\}$ and $M_l = \mathrm{conv}\{x_i, x_j, x_k\}$.

In the course of the proof we frequently estimate the area or a side (edge) of a triangle. Let a, b and c be the edges of the triangle M, and α be the angle opposite to a. In addition, let $\pi/2 \leq \theta < \pi$. If $b \geq b_0$, $c \geq c_0$ and $\alpha \geq \theta$ then $\cos\alpha$ is non-positive, and by the Law of cosines,

$$a^2 = b^2 + c^2 - 2\,bc\cos\alpha \geq b_0^2 + c_0^2 - 2\,b_0 c_0 \cos\theta. \qquad \text{(L)}$$

If $\pi - \theta \leq \alpha \leq \theta$ then the area of M is

$$A(M) = \tfrac{1}{2}bc\sin\alpha \geq \tfrac{1}{2}b_0 c_0 \sin\theta. \qquad (\text{ A}(\theta)\text{ })$$

We note that if α is the largest angle of M and $\frac{2\pi}{3} \leq \theta < \pi$ then $\alpha \geq \frac{\pi}{3} \geq \pi - \theta$.

Let us recall some properties of the first and the second intrinsic volumes of a polytope Q. Both $V_1(Q)$ and $V_2(Q)$ are monotonic. If Q is planar then $V_1(Q) = \frac{1}{2}P(Q)$, where $P(Q)$ is the perimeter, and $V_2(Q) = A(Q)$. If Q is 3-dimensional then $V_2(Q) = \frac{1}{2}S(Q)$, where $S(Q)$ is the surface of Q. Let F be a face of Q. We denote the area of F by $V_2(F)$, if we want to use that $V_2(F) < V_2(Q)$, and by $A(F)$, if we consider F as part of the surface of Q.

Lastly, we note that $\phi(T^3) = 14.92790$. We calculated it with the help of $V_2(T^3) = \frac{1}{2}S(T^3) = 2A(T^2)$ and the value of $V_1(T^3)$ from Chapter 3.

4.1.1 dim$K$ is at most 2

We prove that in this case $\phi(K) > \phi(T^3)$. We may write $\phi(K)$ as $\phi(K) = \pi V_1(K) + A(K)$, and we usually evaluate $V_1(K)$ as $V_1(K) = \frac{1}{2}P(K)$.

If K is a segment then $V_1(K) \geq 6$ and $A(K) = 0$, and hence

$$\phi(K) = \pi V_1(K) + A(K) = \pi V_1(K) \geq 18.84955 > \phi(T^3).$$

Assume that K is planar, and say $K = \text{conv}\{x_1, x_2, x_3\}$ (see Figure 4.1). Denote $\text{ang}(x_i, x_0, x_j)$ by α_{ij}, $1 \leq i < j \leq 3$, and assume that $\alpha_{12} \leq \alpha_{13} \leq \alpha_{23}$. Since $x_0 \in K$, we have $\alpha_{12} + \alpha_{13} \geq \pi$ and $\alpha_{12} + \alpha_{13} + \alpha_{23} = 2\pi$, and consequently also $\alpha_{12} \leq \frac{2\pi}{3}$.

Assume that in addition $\alpha_{12} \geq \frac{5\pi}{18}$. This condition forces $A(M_3)$ to be large. As for $\frac{13\pi}{18} = \pi - \frac{5\pi}{18}$ we have $\pi - \frac{13\pi}{18} \leq \alpha_{12} \leq \frac{2\pi}{3} < \frac{13\pi}{18}$, (A($\frac{13\pi}{18}$)) yields that

$$A(M_3) \geq \tfrac{1}{2} \cdot 2 \cdot 2 \cdot \sin \frac{13\pi}{18} = 2 \sin \frac{13\pi}{18}.$$

By Lemma 3.3.8, $\pi V_1(K) > \pi V_1(U^3) = 13.50282$. It follows that

$$\begin{aligned} \phi(K) &= \pi V_1(K) + A(K) > \pi V_1(U^3) + A(M_3) \\ &\geq \pi V_1(U^3) + 2 \sin \frac{13\pi}{18} = 15.03490 > \phi(T^3). \end{aligned}$$

If $\alpha_{12} < \frac{5\pi}{18}$ then the lengths of the edges of K force $V_1(K)$ to be large. By $\alpha_{12} + \alpha_{13} \geq \pi$, we have $\alpha_{13} \geq \pi - \alpha_{12} > \frac{13\pi}{18}$. This implies that the edge $\text{conv}\{x_1, x_3\}$

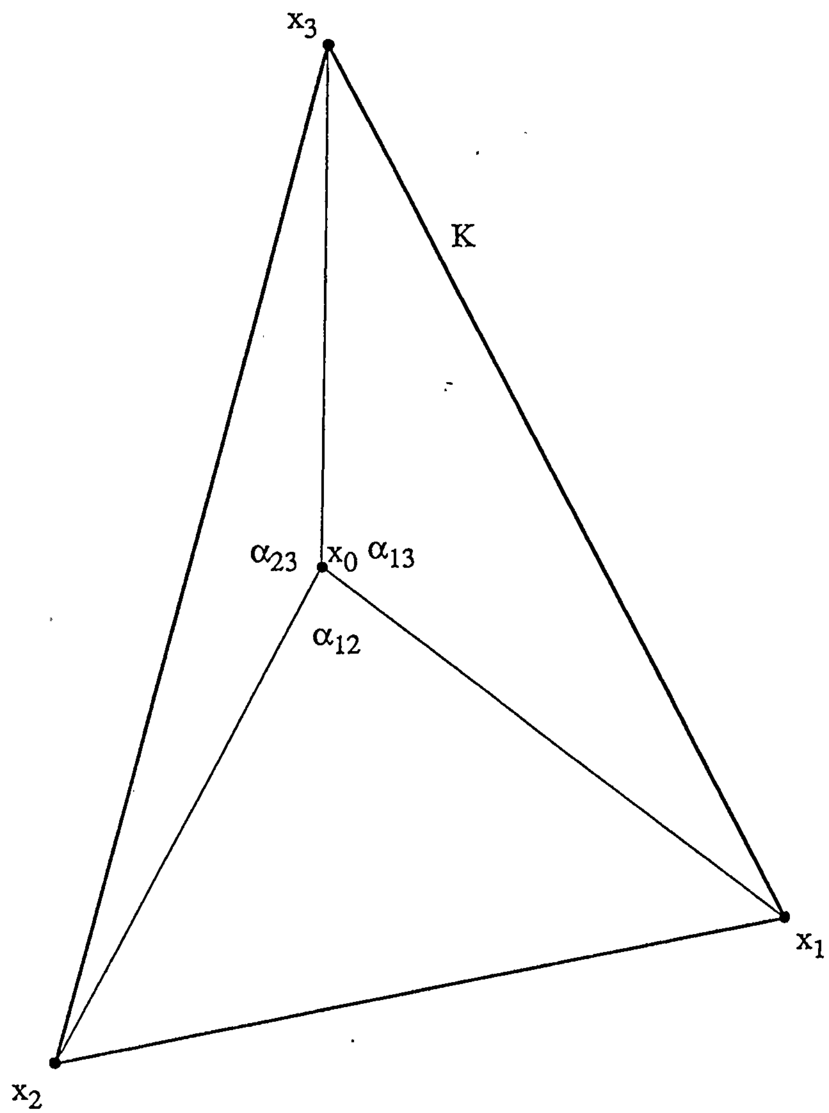

Figure 4.1

K=conv$\{x_0,x_1,x_2,x_3\}$ is a triangle is long because for M_2, (L) yields that

$$d(x_1, x_3) \geq \sqrt{2^2 + 2^2 - 2 \cdot 2 \cdot 2 \cdot \cos \frac{13\pi}{18}} = 3.62523.$$

Since $\alpha_{13} + \alpha_{23} = 2\pi - \alpha_{12} > \frac{31\pi}{18}$ and $\alpha_{23} \geq \alpha_{13}$, we have $\alpha_{23} \geq \frac{1}{2}(2\pi - \alpha_{12}) > \frac{31\pi}{36}$. Now for M_1, (L) yields that

$$d(x_2, x_3) \geq \sqrt{2^2 + 2^2 - 2 \cdot 2 \cdot 2 \cdot \cos \frac{31\pi}{36}} = 3.90518.$$

With these lower bounds and $d(x_1, x_2) \geq 2$,

$$\begin{aligned} \phi(K) &= \pi V_1(K) + A(K) > \pi V_1(K) = \pi \tfrac{1}{2} P(K) \\ &= \pi \tfrac{1}{2}(d(x_1, x_2) + d(x_1, x_3) + d(x_2, x_3)) \geq 14.97034 > \phi(T^3). \end{aligned}$$

Henceforth, assume that K is the quadrilateral $\operatorname{conv}\{x_0, x_1, x_2, x_3\}$ (see Figure 4.2).

Note that if the length of at least one of the edges of K is at least $2\sqrt{2 + \sqrt{2}}$ then $V_1(K)$ itself is large enough and

$$\begin{aligned} \phi(K) &= \pi V_1(K) + A(K) > \pi V_1(K) = \pi \tfrac{1}{2} P(K) \\ &\geq \pi \tfrac{1}{2}(2 + 2 + 2 + 2\sqrt{2 + \sqrt{2}}) = 15.22968 > \phi(T^3). \end{aligned}$$

The value $2\sqrt{2 + \sqrt{2}}$ comes from the fact that a triangle, with edgelengths at least two, and an angle of at least $\frac{3\pi}{4}$, has an edge of length at least $2\sqrt{2 + \sqrt{2}}$ by (L), since

$$2\sqrt{2 + \sqrt{2}} = \sqrt{2^2 + 2^2 - 2 \cdot 2 \cdot 2 \cdot \cos \frac{3\pi}{4}}.$$

Assume that K has two neighbouring angles which are at least $\frac{3\pi}{4}$, say, $\operatorname{ang}(x_3, x_0, x_1)$ and $\operatorname{ang}(x_0, x_1, x_2)$. This implies that $V_1(K)$ is large. Observe that

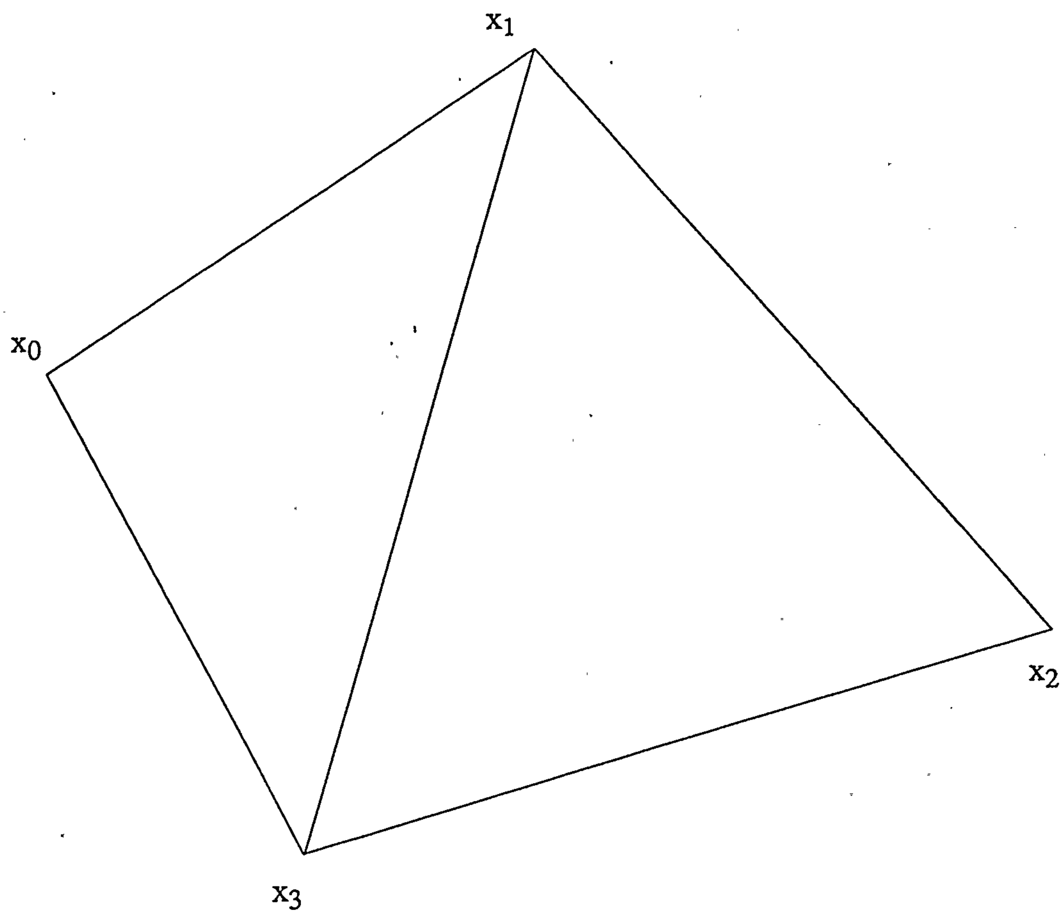

Figure 4.2

$K=\mathrm{conv}\{x_0,x_1,x_2,x_3\}$ is a quadrilateral

$d(x_1, x_3) \geq 2\sqrt{2+\sqrt{2}}$. In M_2, $\text{ang}(x_0, x_1, x_3) < \frac{\pi}{4}$ since $\text{ang}(x_3, x_0, x_1) \geq \frac{3\pi}{4}$, and hence

$$\text{ang}(x_3, x_1, x_2) = \text{ang}(x_0, x_1, x_2) - \text{ang}(x_0, x_1, x_3) > \frac{3\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}.$$

The edge $\text{conv}\{x_2, x_3\}$ is the longest one of M_0 because the opposite angle is obtuse, and hence $d(x_2, x_3) > d(x_1, x_3) > 2\sqrt{2+\sqrt{2}}$. As $\text{conv}\{x_2, x_3\}$ is an edge of K, it follows that $\phi(K) > \phi(T^3)$ by the previous observation.

The only case which is open is if K is a quadrilateral such that for any two neighbouring angles of K, one of them is at most $\frac{3\pi}{4}$. Then there is a pair of opposite angles of K, say, at x_0 and x_2 such that each of them is at most $\frac{3\pi}{4}$.

If $\text{ang}(x_0, x_1, x_3)$ is at least $\frac{3\pi}{4}$ then $d(x_0, x_3) \geq 2\sqrt{2+\sqrt{2}}$. Thus K has an edge of length at least $2\sqrt{2+\sqrt{2}}$, and $\phi(K) > \phi(T^3)$.

Assume that each of the angles of both M_0 and M_2 is at most $\frac{3\pi}{4}$. This forces $A(K)$ to be large by (A($\frac{3\pi}{4}$)). Observe that the area of both M_0 and M_2 is at least $\frac{1}{2} \cdot 2 \cdot 2 \cdot \sin\frac{3\pi}{4} = \sqrt{2}$. As $V_1(K) = \frac{1}{2}P(K) \geq 4$, we have

$$\phi(K) = \pi V_1(K) + A(K) \geq \pi \cdot 4 + 2\sqrt{2} = 15.39479 > \phi(T^3).$$

Therefore $\phi(K) > \phi(T^3)$ for any choice of K with $\dim K \leq 2$. It follows that if $\phi(K) = \phi(\mathcal{P}^3_{2,4})$ (or equivalently, K is a minimal body) then $\dim K = 3$.

4.1.2 K=conv$\{x_0, x_1, x_2, x_3\}$ is a non-degenerate 3-simplex

The verification of (*) requires some preliminary results concerning edgelengths. We assume for the rest of the section that $\text{conv}\{x_0, x_1\}$ is the longest edge of K. We list some possible properties of K:

i) For any x_i, there is an x_j with $d(x_i, x_j) = 2$.

ii) If $\mathrm{ang}(x_i, x_k, x_j)$ and $\mathrm{ang}(x_i, x_l, x_j)$ are at most $\pi/2$ then $d(x_i, x_j) = 2$ (see Figure 4.3).

iii) If $d(x_2, x_3) > 2$ then $d(x_0, x_i) = d(x_1, x_i)$, $i = 2, 3$ (see Figure 4.4).

iv) If $d(x_i, x_k) = d(x_j, x_k) = 2$ and $d(x_i, x_j) \leq 2\sqrt{2}$ then $d(x_i, x_j) = 2$.

Note that iv) follows from ii). We assume ii), $d(x_i, x_k) = d(x_j, x_k) = 2$ and $d(x_i, x_j) \leq 2\sqrt{2}$. Then $d(x_i, x_j)^2 \leq 8 = d(x_i, x_k)^2 + d(x_j, x_k)^2$ and (L) imply that $\mathrm{ang}(x_i, x_k, x_j) \leq \pi/2$. Since $d(x_i, x_j)^2 \leq 8 = d(x_i, x_l)^2 + d(x_j, x_l)^2$, we also have that $\mathrm{ang}(x_i, x_l, x_j) \leq \pi/2$. Consequently, $d(x_i, x_j) = 2$ by ii).

We need another property of the triangles. Let a, b and c be edges of the triangle M, and α be the angle opposite to a. Let $\alpha \leq \frac{\pi}{2}$. Since $A(M) = \frac{1}{2}bc\sin\alpha$, the area of M decreases if α decreases.

LEMMA 4.1.3 *Assume $\phi(K) = \phi(\mathcal{P}^3_{2,4})$; that is, K is a minimal body. Then the properties i), ii), and iii) hold.*

Proof: i) was proved in Section 2.4, p.60. With respect to ii) and iii), we prove that if they do not hold then the points $x_0, \ldots, x_3$ can be moved a little so that they are still at least distance 2 apart but the first and the second intrinsic volumes of their convex hull decrease.This contradicts the minimality of $\phi(K)$.

In order to prove ii), suppose that both $\mathrm{ang}(x_i, x_k, x_j)$ and $\mathrm{ang}(x_i, x_l, x_j)$ are at most the right angle and $d(x_i, x_j) > 2$. Rotate x_i towards x_j around $\mathrm{aff}\{x_l, x_k\}$ so that $d(x_i, x_j)$ is still at least 2. Then $V_1(K)$ decreases by the Bending Lemma.

Since $d(x_i, x_j)$ decreases, both $\mathrm{ang}(x_i, x_k, x_j)$ and $\mathrm{ang}(x_i, x_l, x_j)$ decreases. These angles were at most $\pi/2$ at beginning of the process, which yields that the areas of both M_k and M_l have been decreased. Notice that the other two faces remained

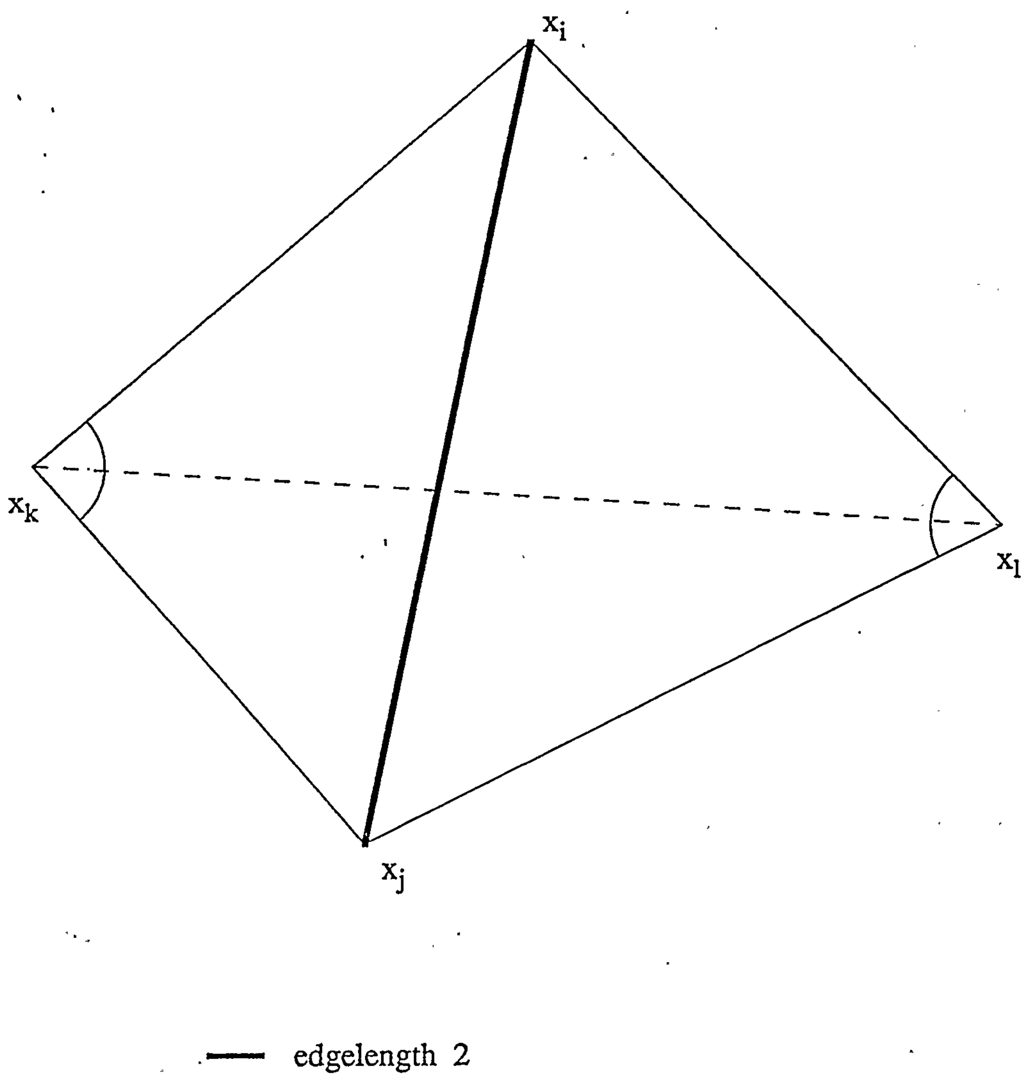

Figure 4.3

A simplex K with property ii)

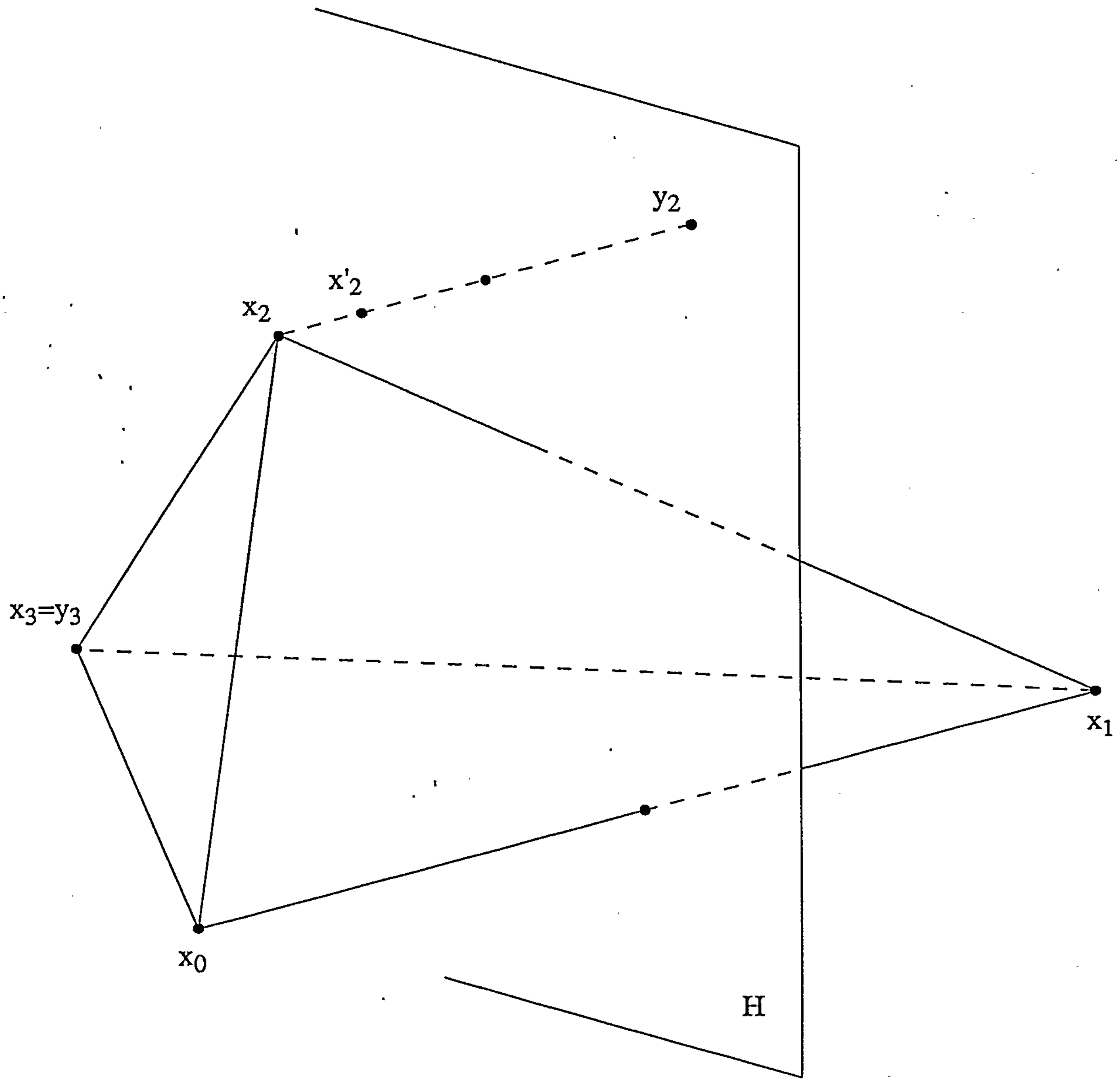

Figure 4.4

A simplex K with property iii)

unchanged. It follows that $V_2(K)$ (which is half of the surface), and consequently also $\phi(K)$, has been decreased.

For iii), we use the Steiner symmetrization. So assume that $d(x_2, x_3) > 2$ and $d(x_0, x_2) < d(x_1, x_2)$. Let H be the plane perpendicularly bisecting the segment $\mathrm{conv}\{x_0, x_1\}$, and y_i be the image of x_i by the reflection through H, $i = 2, 3$ (see Figure 4.4). Let $0 < \varepsilon < \frac{1}{2}$ with the property that $d(x_2', x_3') \geq 2$ for $x_i' = (1-\varepsilon)x_i + \varepsilon\, y_i$, $i = 2, 3$. Then x_i' is in the same open halfspace of H as x_i.

Recall that $\mathrm{conv}\{x_0, x_1\}$ is the longest edge of K. It follows that $\mathrm{ang}(x_0, x_2, x_2') > \pi/2$, and hence $d(x_0, x_2') > d(x_0, x_2) \geq 2$. Since x_2' is in the same open halfspace of H as x_2, $d(x_1, x_2') > d(x_0, x_2') > 2$. Similar cosiderations show that $d(x_i, x_3') \geq 2$, $i = 0, 1$ (equality occurs if $x_3 \in H$ and $d(x_i, x_3) = 2$). Hence for $K' = \mathrm{conv}\{x_0, x_1, x_2', x_3'\}$, all the edges of K' are at least 2.

The condition $d(x_0, x_2) < d(x_1, x_2)$ yields that K is not symmetric in H, and in any plane parallel to H. It follows that $V_i(S_{H,\varepsilon}(K)) < V_i(K)$ for $i = 1, 2$ by Theorem 1.8.2. All the points x_0, x_1, x_2', x_3' are contained in $S_{H,\varepsilon}(K)$ by definition. Thus $K' \subset S_{H,\varepsilon}(K)$, which in turn yields $V_i(K') < V_i(K)$, $i = 1, 2$, and hence $\phi(K') < \phi(K)$.

□

The graph of the edges with length 2

Let G be the graph on $\{x_0, x_1, x_2, x_3\}$ such that a pair $\{x_i, x_j\}$ is an edge if and only if $d(x_i, x_j) = 2$.

We assume that K is minimal, and hence it has the properties i) to iv). We show that if $d(x_0, x_1) > 2$ then G has at least three edges and $\phi(K) > \phi(T^3)$; that is, K is not minimal. Hence it follows that $d(x_0, x_1) = 2$ and $K \equiv T^3$.

By i), G has no isolated points. Hence G has at least three edges unless it has exactly two which have no common vertex. Since $d(x_0, x_1) > 2$, we may assume that $d(x_0, x_2) = d(x_1, x_3) = 2$. By iii), either $d(x_2, x_3) = 2$ or $d(x_1, x_2) = d(x_0, x_2)(= 2)$ and $d(x_0, x_3) = d(x_1, x_3)(= 2)$. Therefore G has at least three edges in any case.

G has exactly three edges

Observe that two of the edges of G definitely have a common vertex. Since G has no isolated points, it is either a path, say $d(x_0, x_2) = d(x_2, x_3) = d(x_3, x_1) = 2$ (see Figure 4.5 and Lemma 4.1.4) or a star, say $d(x_3, x_i) = 2$, $i = 0, 1, 2$ (see Figure 4.6 and Lemma 4.1.5).

LEMMA 4.1.4 *If $d(x_0, x_2) = d(x_2, x_3) = d(x_3, x_1) = 2$ and the other edges of K are greater than 2 then $\phi(K) > \phi(T^3)$.*

Proof: If $d(x_0, x_3) \leq 2\sqrt{2}$ then $d(x_0, x_2) = d(x_2, x_3) = 2$ yields $d(x_0, x_3) = 2$ by iv). Hence $d(x_0, x_3) > 2\sqrt{2}$, and similarly $d(x_1, x_2) > 2\sqrt{2}$. Let $\text{ang}(x_0, x_2, x_1) \geq \text{ang}(x_0, x_3, x_1)$. Then ii) and $d(x_0, x_1) > 2$ yield that $\text{ang}(x_0, x_2, x_1) > \pi/2$.

These observations and (L) yield for M_3 that

$$d(x_0, x_1) > \sqrt{d(x_0, x_2)^2 + d(x_1, x_2)^2} > \sqrt{2^2 + (2\sqrt{2})^2} = 2\sqrt{3},$$

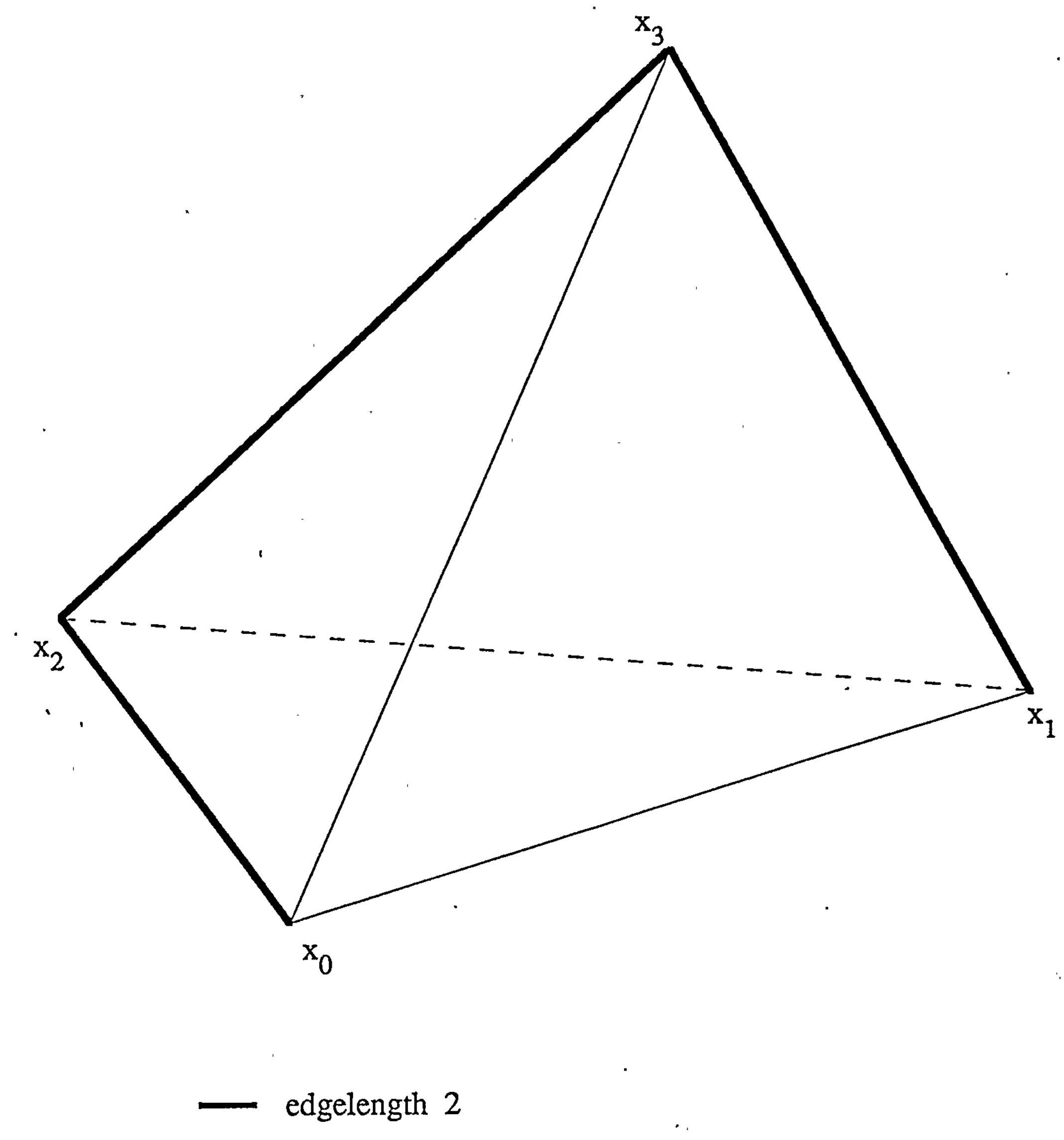

Figure 4.5

G is a path

and consequently

$$V_1(M_3) = \pi\,\tfrac{1}{2}(d(x_0,x_1) + d(x_1,x_2) + d(x_2,x_0)) > \pi\,\tfrac{1}{2}(2\sqrt{3} + 2\sqrt{2} + 2) = 13.02587.$$

If $\operatorname{ang}(x_0,x_2,x_1) \le \frac{2\pi}{3}$ then $A(M_3)$ is also large; namely, $(\mathrm{A}(\frac{2\pi}{3}))$ yields that

$$V_2(M_3) \ge \tfrac{1}{2}\cdot 2\cdot 2\sqrt{2}\cdot \sin\frac{2\pi}{3} = \sqrt{6} = 2.44948.$$

Combining this with the lower bound for $V_1(M_3)$ results in

$$\phi(K) > \phi(M_3) = \pi V_1(M_3) + V_2(M_3) > 15.47536 > \phi(T^3).$$

Hence assume that $\operatorname{ang}(x_0,x_2,x_1) > \frac{2\pi}{3}$, and we increase the lower bound for $V_1(M_3)$. As $d(x_0,x_2) = 2$ and $d(x_1,x_2) > 2\sqrt{2}$, (L) yields that

$$d(x_0,x_1) > \sqrt{2^2 + (2\sqrt{2})^2 - 2\cdot 2\cdot 2\sqrt{2}\cdot\cos\frac{2\pi}{3}} = 2\sqrt{3+\sqrt{2}},$$

which in turn yields that

$$V_1(M_3) = \pi\,\tfrac{1}{2}(d(x_0,x_1)+d(x_1,x_2)+d(x_2,x_0)) > \pi\,\tfrac{1}{2}(2\sqrt{3+\sqrt{2}}+2\sqrt{2}+2) = 14.18497.$$

If in addition, $\operatorname{ang}(x_1,x_3,x_2) \le \frac{3\pi}{4}$ then $V_2(M_0)$ is large. Observe that $\operatorname{ang}(x_1,x_3,x_2)$ is the greatest angle of M_0, and hence by $(\mathrm{A}(\frac{3\pi}{4}))$,

$$A(M_0) = V_2(M_0) \ge \tfrac{1}{2}\cdot 2\cdot 2\cdot \sin\frac{3\pi}{4} = \sqrt{2}.$$

Therefore we have the lower bound

$$\phi(K) = \pi V_1(K) + V_2(K) > \pi V_1(M_3) + V_2(M_0) > 15.59918 > \phi(T^3).$$

The only possibility left is if $\operatorname{ang}(x_1,x_3,x_2) > \frac{3\pi}{4}$, in which case $V_1(M_3)$ itself forces $\phi(K)$ to be large. Observe that by (L), $\operatorname{ang}(x_1,x_3,x_2) > \frac{3\pi}{4}$ implies for M_0 that

$$d(x_1,x_2) > \sqrt{2^2 + 2^2 - 2\cdot 2\cdot 2\cdot\cos\frac{3\pi}{4}} = 2\sqrt{2+\sqrt{2}}.$$

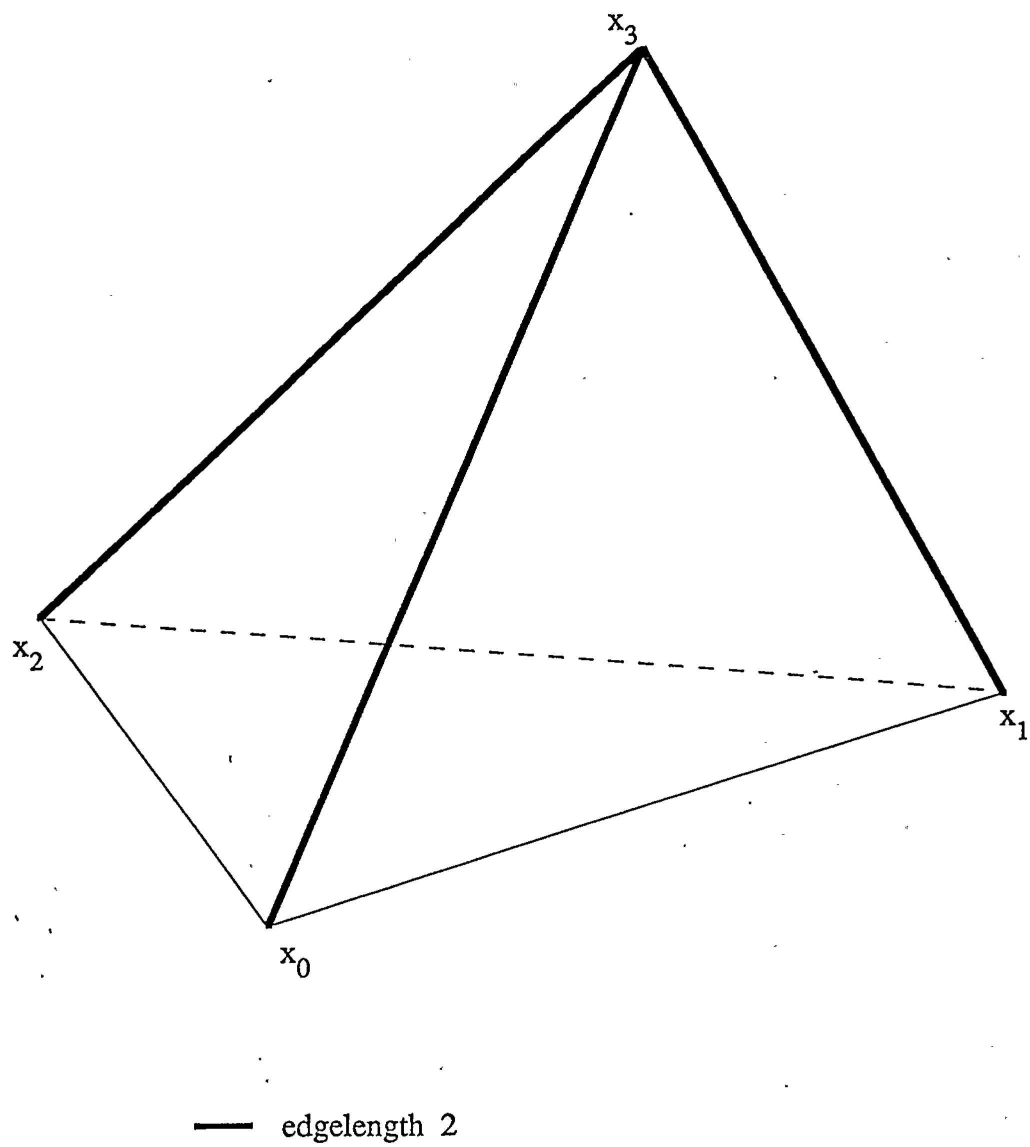

Figure 4.6

G is a star

Hence $\mathrm{ang}(x_0, x_2, x_1) > \pi/2$ and (L) yield for M_3 that

$$d(x_0, x_1) > \sqrt{2^2 + (2\sqrt{2+\sqrt{2}})^2} = 2\sqrt{3+\sqrt{2}},$$

which in turn yields that

$$\begin{aligned}\phi(K) &= \pi V_1(K) + V_2(K) > \pi V_1(M_3) = \pi \tfrac{1}{2} P(M_3) \\ &> \pi \tfrac{1}{2}(2 + 2\sqrt{2+\sqrt{2}} + 2\sqrt{3+\sqrt{2}}) = 15.54699 > \phi(T^3).\end{aligned}$$

□

LEMMA 4.1.5 *If $d(x_3, x_i) = 2$, $i = 0, 1, 2$, and all the other edges of K are greater than 2 then $\phi(K) > \phi(T^3)$.*

Proof: iv) yields that each edge of M_3 is greater than $2\sqrt{2}$. By the triangle inequality applied for M_i, $i = 0, 1, 2$, the edges of M_3 are at most 4.

Note that

$$d(x_0, x_1)^2 \leq 16 = (2\sqrt{2})^2 + (2\sqrt{2})^2 < d(x_0, x_2)^2 + d(x_1, x_2)^2,$$

and hence all the angles of M_3 are acute by (L). As a consequence,

$$\begin{aligned}\phi(K) &= \pi V_1(K) + V_2(K) > \pi V_1(M_3) + V_2(M_3) \\ &\geq \pi \tfrac{1}{2}(2\sqrt{2} + 2\sqrt{2} + 2\sqrt{2}) + \tfrac{1}{2} \cdot 2\sqrt{2} \cdot 2\sqrt{2} \cdot \sin\frac{\pi}{3} = 16.79275 > \phi(T^3).\end{aligned}$$

□

G has at least four edges

If G contains a cycle then, as $d(x_0, x_1) > 2$, we have

$$d(x_0, x_2) = d(x_2, x_1) = d(x_1, x_3) = d(x_3, x_0) = 2$$

(see Figure 4.7 and Lemma 4.1.6). Otherwise, observe that there is a vertex which is contained in 3 edges of G but G can not have 5 or 6 edges. Therefore G contains a star and an additional edge, say, $d(x_3, x_i) = 2$, $i = 0, 1, 2$, and $d(x_0, x_2) = 2$ (see Figure 4.9 and Lemma 4.1.8).

LEMMA 4.1.6 *If* $d(x_0, x_2) = d(x_2, x_1) = d(x_1, x_3) = d(x_3, x_0) = 2$ *and* $d(x_0, x_1) > 2$ *then* $\phi(K) > \phi(T^3)$.

Proof: Rotate x_1 around $\text{aff}\{x_2, x_3\}$ towards x_0. Since the edgelengths of the triangles M_0 and M_1 are the same, the triangles are congruent, and there is a position x_1' for x_1 where $d(x_0, x_1') = 2$. Now rotate x_2 towards x_3 around $\text{aff}\{x_0, x_1'\}$ if $d(x_2, x_3) > 2$. Similarly as above, there is a position x_2' with $d(x_2', x_3) = 2$, and hence the resulted simplex is congruent to T^3. Since the first rotation certainly strictly decreased the first intrinsic volume by the Bending Lemma, $V_1(K) > V_1(T^3)$.

Let y be the midpoint of the edge $\text{conv}\{x_2, x_3\}$ (see Figure 4.7). Then the segment $\text{conv}\{y, x_1\}$ is the height of the triangle M_0. Since $d(x_2, x_3) \geq 2$, it follows that $d(y, x_1)$ is at most the height of T^2, and hence $d(y, x_1) \leq \sqrt{3}$. Similarly, $d(y, x_0) \leq \sqrt{3}$, and consequently

$$d(x_0, x_1) < d(y, x_0) + d(y, x_1) \leq 2\sqrt{3}.$$

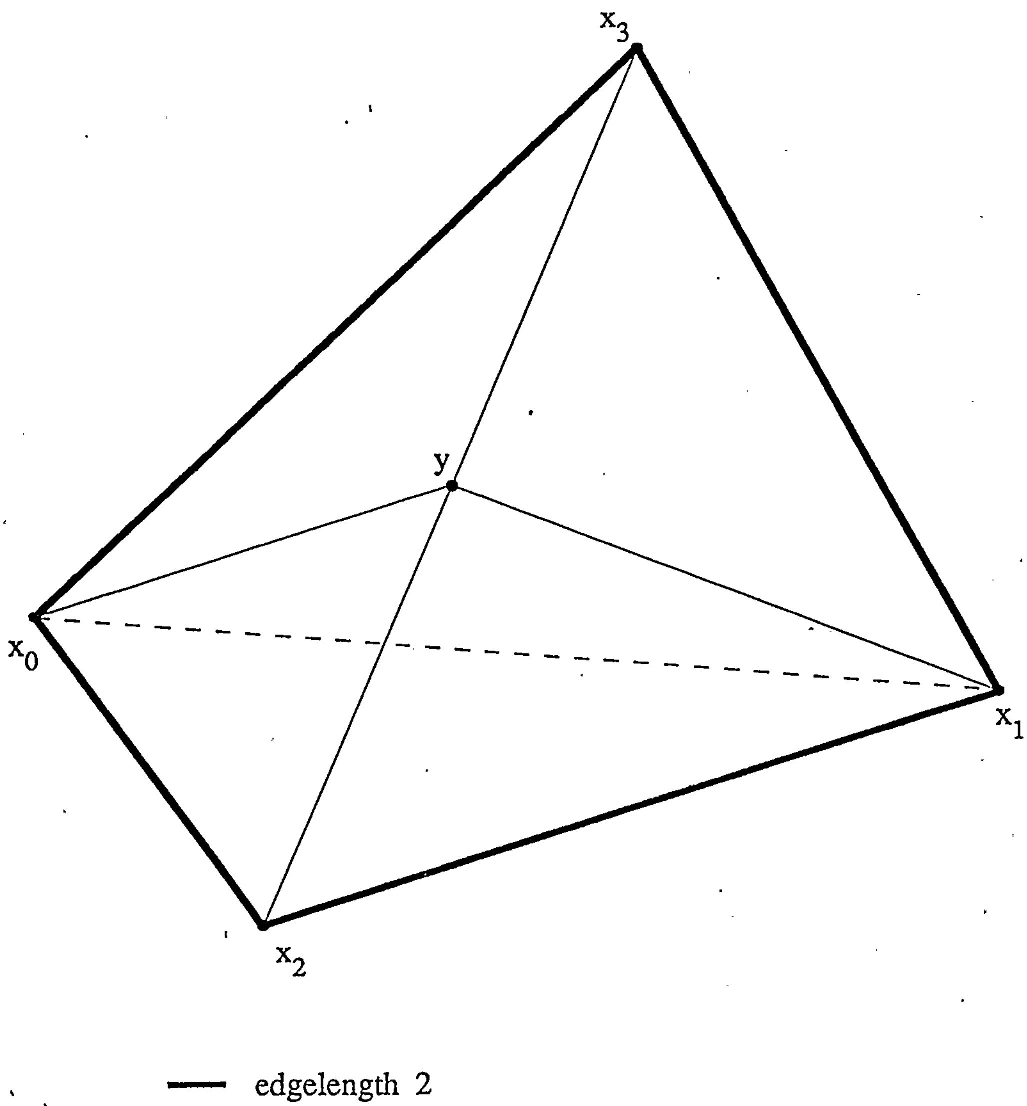

Figure 4.7

G is a cycle

Let α be the greatest angle of the isosceles triangle M_3. Since

$$\cos\alpha = \frac{2^2 + 2^2 - d(x_0, x_1)^2}{2 \cdot 2 \cdot 2} > \frac{8 - 12}{8} = -\frac{1}{2},$$

α is at most $\frac{2\pi}{3}$. Thus the area of M_3 is at least $A(T^2)$ by $(\mathrm{A}(\frac{2\pi}{3}))$.

Similar considerations show that that the area of each faces of K is at least $A(T^2)$, and consequently $V_2(K) \geq V_2(T^3)$. Combining this fact with $V_1(K) > V_1(T^3)$ yields that $\phi(K) = \pi V_1(K) + V_2(K) > \phi(T^3)$.

□

Before we consider the last possibility for G, we prove an auxiliary statement.

LEMMA 4.1.7 *Denote the triangle conv$\{y_0, y_1, y_2\}$ by M, and assume that $d(y_0, y_1)$ $= 2$ and $2\sqrt{2} \leq d(y_0, y_2) \leq d(y_1, y_2) \leq 4$ (see Figure 4.8). Then $A(M) \geq \sqrt{7}$.*

Proof: Denote by v the midpoint of conv$\{y_0, y_1\}$ and by w the projection of y_2 onto the line aff$\{y_0, y_1\}$. The point v separates y_1 and w (possibly $v = w$) because $d(y_0, y_2) \leq d(y_1, y_2)$.

Let $h = d(y_2, w)$ and $p = d(v, w)$, then $d(y_1, w) = 1 + p$ and $d(y_0, w) = |1 - p|$. It follows that

$$8 \leq d(y_0, y_2)^2 = (1-p)^2 + h^2$$

and

$$16 \geq d(y_1, y_2)^2 = (1+p)^2 + h^2,$$

which in turn implies that

$$4p + 8 \leq 4p + (1-p)^2 + h^2 = (1+p)^2 + h^2 \leq 16.$$

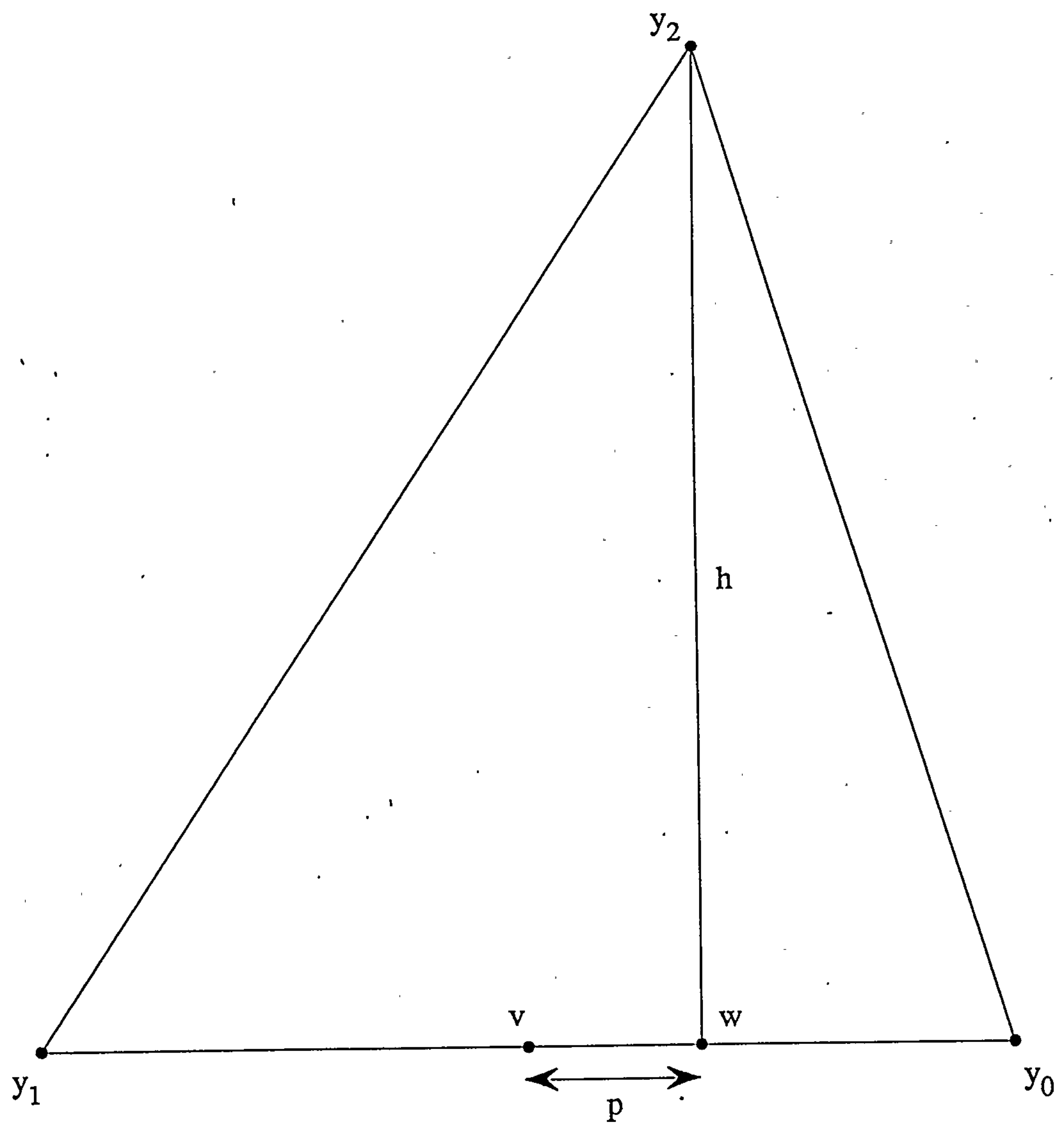

Figure 4.8

Determining the area of $M=\mathrm{conv}\{y_0,y_1,y_2\}$

Thus $p \le 2$, and consequently $h = \sqrt{8-(1-p)^2} \ge \sqrt{7}$. Since h is the height of M, the area of M is at least $\sqrt{7}$.

□

LEMMA 4.1.8 *If $d(x_3,x_0) = d(x_3,x_1) = d(x_3,x_2) = d(x_0,x_2) = 2$ and the other two edges of K are longer than 2 then $\phi(K) > \phi(T^3)$ (see Figure 4.9).*

Proof: Since $d(x_3,x_1) = d(x_3,x_2) = 2$ and $d(x_1,x_2) > 2$, iv) yields that $d(x_1,x_2) > 2\sqrt{2}$. Similarly, $d(x_1,x_0) > 2\sqrt{2}$, and hence $A(M_3) = V_2(M_3) \ge \sqrt{7}$ by Lemma 4.1.7. If both $d(x_0,x_1)$ and $d(x_1,x_2)$ are greater than $2\sqrt{3}$ then $V_1(M_3) > \frac{1}{2}(2+2\sqrt{3}+2\sqrt{3}) = 1+2\sqrt{3}$ and

$$\begin{aligned}\phi(K) &= \pi V_1(K) + V_2(K) > \pi V_1(M_3) + V_2(M_3)\\ &> \pi\cdot(1+2\sqrt{3}) + \sqrt{7} = 16.67014 > \phi(T^3).\end{aligned}$$

Assume that, say, $d(x_1,x_2) \le 2\sqrt{3}$, or equivalently, that $\mathrm{ang}(x_1,x_3,x_2) \le \frac{2\pi}{3}$. Observe that $A(M_0) \ge A(T^2) = \sqrt{3}$ by $(\mathrm{A}(\frac{2\pi}{3}))$. Since $A(M_1) = A(T^2) = \sqrt{3}$ and $d(x_1,x_i) > 2\sqrt{2}$, $i = 0,2$, it follows that

$$\begin{aligned}\phi(K) &= \pi V_1(K) + V_2(K) > \pi V_1(M_3) + \tfrac{1}{2}(A(M_0)+A(M_1)+A(M_3))\\ &> \pi\tfrac{1}{2}(2+2\sqrt{2}+2\sqrt{2}) + \tfrac{1}{2}(\sqrt{3}+\sqrt{3}+\sqrt{7}) = 15.08228 > \phi(T^3).\end{aligned}$$

□

Remark: So far we have seen that in E^3, T^3 is the unique minimal body which can be chosen either for $\mathcal{P}^3_{1,4}$ or for $\mathcal{P}^3_{2,4}$. Turning to the volume, note that

$$V(T^3+B^3) = \sum_{i=0}^{3} \kappa_{3-i} V_i(T^3) = 23.52360.$$

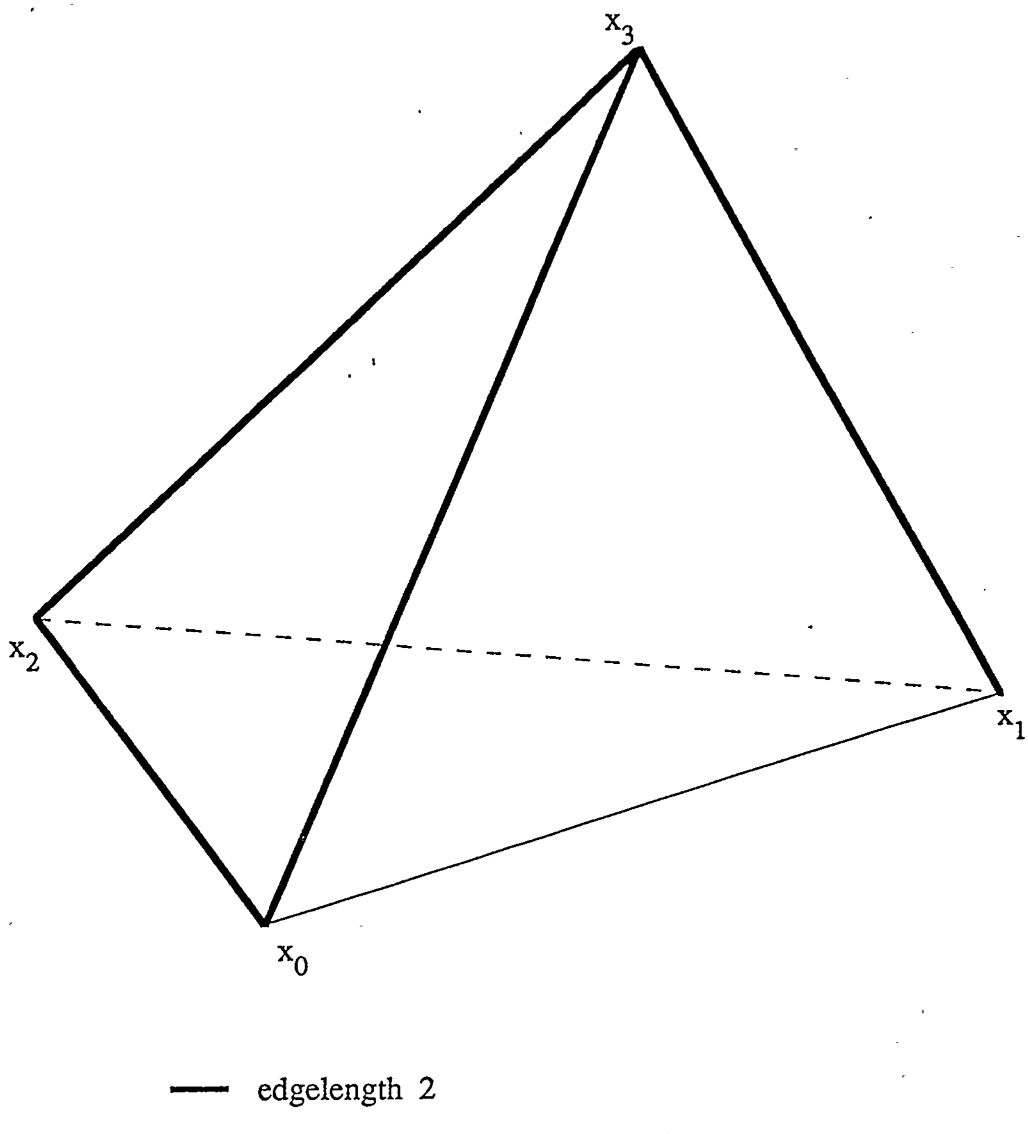

Figure 4.9

G is not a cycle and has four edges

On the other hand, the segment S_4 with length 6 is an element of $\mathcal{G}_4^3$ and

$$V(S_4 + B^3) = 6 \cdot \kappa_2 + \kappa_3 = 23.03834 < V(T^3 + B^3).$$

Therefore T^3 is not a minimal body with respect to $V(K + B^3)$, $K \in \mathcal{G}_4^3$.

4.2 Three-ball packings

Let $d \geq 2$ and $1 \leq i \leq d$. By the monotonicity of the intrinsic volumes, $\mathcal{P}_{i,1}^d$ is a point and $\mathcal{P}_{i,2}^d$ is a segment of length 2. Having three balls, it is not too complicated to determine $\mathcal{P}_{i,3}^d$, but the shape of it depends on i. We consider this case so as to illustrate the relation between i and the shape of a minimal body.

So let $M \in \mathcal{G}_3^d$ with the property that $V_i(M+B^d) = \vartheta_{i,3}^d$. By the considerations in Section 2.4, p.60, $M = \operatorname{conv}\{x_0, x_1, x_2\}$ with $d(x_i, x_j) \geq 2$ for $i \neq j$. Let $\operatorname{conv}\{x_1, x_2\}$ be one of the longest sides of M. If $d(x_1, x_2) = 2$ then all the sides have length 2. Otherwise, again by the considerations in Section 2.4, p.60, we have $d(x_0, x_1) = d(x_0, x_2) = 2$ (see Figure 4.10). Denote $\operatorname{ang}(x_0, x_1, x_2)$ by γ. Then $0 \leq \gamma \leq \pi/3$, $d(x_1, x_2) = 4\cos\gamma$ and $\operatorname{ang}(x_1, x_0, x_2) = \pi - 2\gamma$. It follows that

$$\begin{aligned} V_1(M) &= \tfrac{1}{2}(2 + 2 + 4\cos\gamma) = 2 + 2\cos\gamma, \\ \text{and} \qquad V_2(M) &= \tfrac{1}{2} \cdot 2 \cdot 2 \cdot \sin(\pi - 2\gamma) = 2\sin 2\gamma. \end{aligned}$$

According to Lemma 1.8.1,

$$\kappa_{d-i} V_i(M + B^d) = \sum_{j=0}^{i} \binom{d-j}{d-i} \kappa_{d-j} V_j(M).$$

Since $V_j(M) = 0$ for $j \geq 3$, combining the last three formulae yield that

$$\kappa_{d-i} V_i(M + B^d) = \binom{d}{d-i} \kappa_d + \binom{d-1}{d-i} \kappa_{d-1}(2 + 2\cos\gamma) + \binom{d-2}{d-i} \kappa_{d-2} \cdot 2\sin 2\gamma.$$

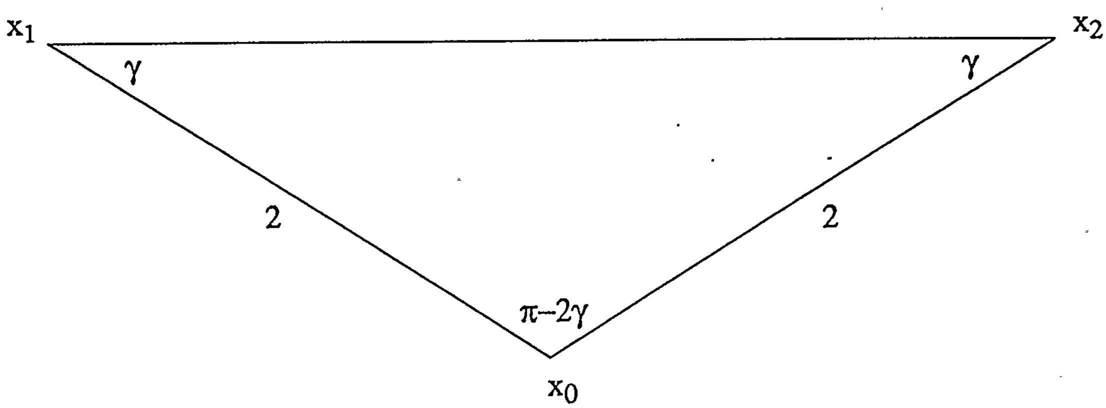

Figure 4.10

Convex hull of the centers in a three-ball packing

(Recall that if $i = 1$ then $\binom{d-2}{d-i} = 0$).

We conclude that M is a minimal body if and only if

$$\psi_i(t) = \binom{d-1}{d-i}\kappa_{d-1}\cos t + \binom{d-2}{d-i}\kappa_{d-2}\sin 2t$$

is minimal on $[0, \pi/3]$ for $t = \gamma$. Since

$$\psi_i''(t) = -\binom{d-1}{d-i}\kappa_{d-1}\cos t - 4\binom{d-2}{d-i}\kappa_{d-2}\sin 2t < 0,$$

the function $\psi_i(t)$ is strictly concave on $[0, \pi/3]$, and hence it takes its minimum value either for $t = 0$ or for $t = \pi/3$.

Note that

$$\binom{d-2}{d-i} = \frac{(d-2)\cdot\ldots\cdot(i-1)}{(d-i)!} = \frac{i-1}{d-1}\binom{d-1}{d-i}.$$

It follows that

$$\begin{aligned}\psi_i(0) - \psi_i(\tfrac{\pi}{3}) &= \frac{1}{2}\binom{d-1}{d-i}\kappa_{d-1} - \frac{\sqrt{3}}{2}\binom{d-2}{d-i}\kappa_{d-2} \\ &= \frac{1}{2}\binom{d-1}{d-i}\kappa_{d-2}\left(\frac{\kappa_{d-1}}{\kappa_{d-2}} - \frac{\sqrt{3}(i-1)}{d-1}\right).\end{aligned}$$

By 1.6, $\frac{\kappa_{m+1}}{\kappa_m} = \frac{m}{m+1}\cdot\frac{\kappa_{m-1}}{\kappa_{m-2}}$ for $m \geq 2$. As $\kappa_1/\kappa_0 = 2$ and $\kappa_2/\kappa_1 = \pi/2$, the quotient $\kappa_{m-1}/\kappa_{m-2}$ is rational if m is even and transcendental if m is odd. Since $\sqrt{3}$ is algebraic, it follows that $\psi_i(0) \neq \psi_i(\pi/3)$ for $i = 1, \ldots, d$. Therefore, $M = T^2$ (or equivalently, $\psi_i(\pi/3) < \psi_i(0)$) if and only if

$$i < \frac{d-1}{\sqrt{3}}\cdot\frac{\kappa_{d-1}}{\kappa_{d-2}} + 1,$$

and $M = S_3$ otherwise.

Let $C(d) = \frac{d-1}{\sqrt{3}} \cdot \frac{\kappa_{d-1}}{\kappa_{d-2}} + 1$. The inequalities $\sqrt{\frac{2\pi}{d}} < \frac{\kappa_{d-1}}{\kappa_{d-2}} < \sqrt{\frac{2\pi}{d-1}}$ of 1.7 yield that

$$\sqrt{\frac{2\pi}{3}} \cdot \frac{d-1}{\sqrt{d}} + 1 < C(d) < \sqrt{\frac{2\pi}{3}} \cdot \frac{d-1}{\sqrt{d-1}} + 1,$$

and so $C(d) \sim \sqrt{\frac{2\pi}{3}}\sqrt{d}$ as d tends to infinity.

We conclude that in case of three-ball packings, there is a unique minimal body with respect to the *ith* intrinsic volume, $i = 1, \ldots, d$. This is T^2 if $i < C(d)$ and S_3 if $i > C(d)$, where $C(d)$ has order of $\sqrt{d}$.

4.3 $(d+1)$-ball packings in E^d for large d

In this section we do not attempt to find any minimal bodies. We compare the sausage arrangment of $d+1$ unit balls with the arrangement determined by T^d. They are two possible candidates, for a given $1 \leq i \leq d$, for the minimal arrangement with respect to the *ith* intrinsic volume of the convex hull of the balls. The situation is similar to the one in the previous section; that is,

THEOREM 4.3.1 *Let $d \geq 2$ and $i = 1, \ldots, d$. There exists a function $B(d)$ with the property that*

$$V_i(T^d + B^d) < V_i(S_{d+1} + B^d) \quad \text{if} \quad i < B(d),$$

and

$$V_i(T^d + B^d) > V_i(S_{d+1} + B^d) \quad \text{if} \quad i > B(d).$$

For large d, $\ln B(d) \sim \ln \sqrt{d} = \frac{1}{2} \ln d$.

We establish the existence of $B(d)$ in Subsection 4.3.1, and determine its asymptotic behavior in Subsection 4.3.5. First we reformulate the problem. Let $h(s)$ be the function with the property that $\int_0^{h(s)} e^{-t^2}\,dt = s$. Recall Hadwiger's expression 1.8, which states that

$$V_j(T^d) = \binom{d+1}{j+1} \frac{j+1}{j!}\, 2^{j/2}\, \Phi(d,j)$$

for $1 \le j \le d$, where

$$\Phi(d,j) = \int_0^1 e^{-j[h(\sqrt{\pi}(1/2-t))]^2} t^{d-j}\,dt.$$

Let $1 \le j \le i \le d$. In the course of the proof we need the rather complicated expression

$$\Psi(d,i,j) = \frac{\sqrt{\pi}}{2} \cdot \frac{2^{j/2}}{\pi^{j/2}} \cdot \frac{d+1}{i} \cdot \frac{\Gamma(i+1)}{\Gamma(i-j+1)} \cdot \frac{1}{\Gamma(j+1)^2} \cdot \frac{\Gamma(\frac{d-1}{2}+1)}{\Gamma(\frac{d-j}{2}+1)}.$$

In addition, we define

$$A(d,i,j) = \Psi(d,i,j) \cdot \Phi(d,j) \quad \text{and} \quad f(d,i) = \sum_{j=1}^{i} A(d,i,j).$$

LEMMA 4.3.2 *Let $d \ge 2$ and $i = 1, \ldots, d$. Then*

$$V_i(T^d + B^d) < V_i(S_{d+1} + B^d) \quad \textit{if} \quad f(d,i) < 1,$$

$$\textit{and} \quad V_i(T^d + B^d) > V_i(S_{d+1} + B^d) \quad \textit{if} \quad f(d,i) > 1.$$

Proof: Lemma 1.8.1 and $V_0(T^d) = 1$ yield that

$$\begin{aligned}
\kappa_{d-i} V_i(T^d + B^d) &= \sum_{j=0}^{i} \binom{d-j}{d-i} \kappa_{d-j} V_j(T^d) \\
&= \binom{d}{d-i} \kappa_d + \sum_{j=1}^{i} \binom{d-j}{d-i} \kappa_{d-j} \binom{d+1}{j+1} \frac{j+1}{j!}\, 2^{j/2}\, \Phi(d,j).
\end{aligned}$$

On the other hand, $V_1(S_{d+1}) = 2d$ and $V_j(S_{d+1}) = 0$ for $j = 2, \ldots, d$ yield that

$$\kappa_{d-i}V_i(S_{d+1}+B^d) = \binom{d}{d-i}\kappa_d + \binom{d-1}{d-i}\kappa_{d-1}\cdot 2d.$$

Hence $V_i(T^d+B^d) < V_i(S_{d+1}+B^d)$ if and only if

$$\sum_{j=1}^{i}\binom{d-j}{d-i}\kappa_{d-j}\binom{d+1}{j+1}\frac{j+1}{j!}2^{j/2}\Phi(d,j) \;<\; \binom{d-1}{d-i}\kappa_{d-1}\,2d,$$

which is equivalent to $\sum_{j=1}^{i} A_0(d,i,j) < 1$ for $A_0(d,i,j) = \Psi_0(d,i,j)\cdot\Phi(d,j)$, where

$$\Psi_0(d,i,j) = \kappa_{d-j}\cdot\binom{d-j}{d-i}\cdot\binom{d+1}{j+1}\cdot\frac{j+1}{j!}\cdot 2^{j/2}\cdot\frac{1}{\kappa_{d-1}}\cdot\frac{1}{2d}\cdot\binom{d-1}{d-i}^{-1}.$$

We prove $\Psi_0(d,i,j) = \Psi(d,i,j)$ in two phases. First consider

$$\begin{aligned}
&\binom{d-j}{d-i}\cdot\binom{d+1}{j+1}\cdot\frac{j+1}{j!}\cdot\frac{1}{2d}\cdot\binom{d-1}{d-i}^{-1}\\
&\quad= \frac{(d-j)!}{(i-j)!(d-i)!}\cdot\frac{(d+1)!}{(d-j)!(j+1)!}\cdot\frac{j+1}{j!}\cdot\frac{1}{2d}\cdot\frac{(d-i)!(i-1)!}{(d-1)!}\\
&\quad= \frac{(d+1)!}{2d(d-1)!}\cdot\frac{(i-1)!}{(i-j)!}\cdot\frac{j+1}{(j+1)!j!} = \frac{d+1}{2i}\cdot\frac{\Gamma(i+1)}{\Gamma(i-j+1)}\cdot\frac{1}{\Gamma(j+1)^2}.
\end{aligned}$$

We used the fact that $\Gamma(n+1) = n!$ for any natural number n. The remaining terms in $\Psi_0(d,i,j)$ are

$$2^{j/2}\cdot\frac{\kappa_{d-j}}{\kappa_{d-1}} = 2^{j/2}\cdot\frac{\pi^{(d-j)/2}\cdot}{\Gamma(\frac{d-j}{2}+1)}\cdot\frac{\Gamma(\frac{d-1}{2}+1)}{\pi^{(d-1)/2}} = \frac{2^{j/2}\sqrt{\pi}}{\pi^{j/2}}\cdot\frac{\Gamma(\frac{d-1}{2}+1)}{\Gamma(\frac{d-j}{2}+1)},$$

and hence $\Psi_0(d,i,j) = \Psi(d,i,j)$.

□

4.3.1 The existence of $B(d)$

LEMMA 4.3.3 *Let $d \geq 2$. Then $f(d,i) = \sum_{j=1}^{i} A(d,i,j)$ is strictly increasing in i, $i = 1,\ldots,d$.*

Proof: Define $A(d,i,j) = 0$ if $i < j$. Hence we may write

$$\sum_{j=1}^{i} A(d,i,j) = \sum_{j=1}^{d} A(d,i,j).$$

If $i \geq j$ then (for fixed d and j) the value of $A(d,i,j) = \Psi(d,i,j) \cdot \Phi(d,j)$ depends on

$$\frac{\Gamma(i+1)}{i\Gamma(i-j+1)} = \frac{i!}{i \cdot (i-j+1)!} = \frac{(i-1)!}{(i-j+1)!}.$$

This is $(j-1)!$ for $i = j$ and $(i-1) \cdot \ldots \cdot (i-j+1)$ for $i > j$. Thus $A(d,i,j)$ is increasing for $i = 1, \ldots, d$. If $j = 2$ then $A(d,i,j) = 0$ for $i = 1$, and $\frac{(i-1)!}{(i-2)!} = i-1$ for $i \geq 2$. Observe that $A(d,i,2)$ is strictly increasing in i. It follows that in the sum $\sum_{j=1}^{d} A(d,i,j)$ each term inreases as i increases, and one of them strictly increases. Therefore $\sum_{j=1}^{i} A(d,i,j) = \sum_{j=1}^{d} A(d,i,j)$ is also strictly increasing in i.

□

Now we define a function $B(d)$ satisfying the requirements of Theorem 4.3.1. Let $f(d,0) = 1$, and consider

$$i_0 = \max\{i \,|\, i = 0, \ldots, d \quad \text{and} \quad f(d,i) \leq 1\}.$$

Set $B(d) = i_0$ if $f(d,i_0) = 1$, and $B(d) = i_0 + \frac{1}{2}$ if $f(d,i_0) < 1$. By Lemma 4.3.3 and Lemma 4.3.2, if $i < B(d)$ then $f(d,i) < 1$, and hence $V_i(T^d) < V_i(S_{d+1})$; and if $i > B(d)$ then $f(d,i) > 1$, and hence $V_i(T^d) > V_i(S_{d+1})$, $i = 1, \ldots, d$.

Note that $V_1(T^d + B^d) < V_1(S_{d+1} + B^d)$ by 3.11, and that $V(T^d + B^d) > V(S_{d+1} + B^d)$, as it was proved in [8], which yield $1 < B(d) < d$. In the rest of the section we improve these bounds for d large.

By definition, if $i \le i_0$ then

$$\sum_{j=1}^{i} \Psi(d,i,j) \cdot \Phi(d,j) = \sum_{j=1}^{i} A(d,i,j) = f(d,i) \le 1.$$

Our task is to determine, for what i is the inequality satisfied when d is large.

4.3.2 $\ln\Psi(d,i,j)$ for large d

First we make some general observations. Recall that $g(t) = O(1)$ means that the function $g(t)$ is bounded, and that 1.5 states that

$$\ln\Gamma(t+1) = t\ln t - t + \tfrac{1}{2}\ln t + O(1)$$

for $t \ge 1$. If $-\frac{1}{2} \le t \le \frac{1}{2}$ then $\ln\frac{1}{2} \le \ln(1+t) \le \ln\frac{3}{2}$, and since $\frac{d}{dt}\ \ln(1+t) = 1/(1+t)$, even $|\ln(1+t)| \le 2|t|$ by the mean value theorem.

Let $1 \le p \le \frac{1}{2}q$. By the observation on $\ln(1+t)$,

$$(q-p)\ln(1-\frac{p}{q}) = (q-p)O(\frac{p}{q}) = O(p). \tag{4.1}$$

We frequently meet the expression

$$\ln\frac{\Gamma(q+1)}{\Gamma(q-p+1)\Gamma(p+1)} = \ln\Gamma(q+1) - \ln\Gamma(q-p+1) - \ln\Gamma(p+1).$$

By 1.5, we may write

$$\begin{aligned}\ln\Gamma(q+1) - \ln\Gamma(p+1) &= q\ln q - q + \tfrac{1}{2}\ln q - p\ln p + p - \tfrac{1}{2}\ln p + O(1)\\ &= q\ln q - q + \tfrac{1}{2}\ln q - p\ln p + O(p).\end{aligned}$$

By $q - p = q(1-\frac{p}{q})$, the remaining term is

$$\begin{aligned}-\Gamma(q-p+1) &= -(q-p)\ln(q-p) + q - p - \tfrac{1}{2}\ln(q-p) + O(1)\\ &= (p-q)[\ln q + \ln(1-\frac{p}{q})] + q - p - \tfrac{1}{2}[\ln q + \ln(1-\frac{p}{q})] + O(1).\end{aligned}$$

Having this, 4.1 yields that

$$-\Gamma(q-p+1) = p\ln q - q\ln q + q - \tfrac{1}{2}\ln q + O(p).$$

Therefore, we may deduce that

$$\ln\frac{\Gamma(q+1)}{\Gamma(q-p+1)\Gamma(p+1)} = p\ln q - p\ln p + O(p). \tag{4.2}$$

Let $\alpha = \alpha(d,i) = \log_d i$ and $\beta = \beta(d,j) = \log_d j$. We assume that $\alpha > 1/3$ (and hence i tends to infinity as d does) and that $1 \le j \le \frac{1}{2}d$. It is convenient to write $\Psi(d,i,j)$ as $\Psi = \Psi_1 \cdot \Psi_2 \cdot \Psi_3$ for

$$\Psi_1 = \frac{\sqrt{\pi}}{2}\cdot\frac{2^{j/2}}{\pi^{j/2}}\cdot\frac{d+1}{i}, \qquad \Psi_2 = \frac{\Gamma(i+1)}{\Gamma(i-j+1)\Gamma(j+1)}\cdot\frac{1}{\Gamma(j+1)}$$

$$\text{and} \qquad \Psi_3 = \frac{\Gamma(\frac{d-1}{2}+1)}{\Gamma(\frac{d-j}{2}+1)}.$$

Let us start with Ψ_1. Using $d+1 = d(1+\frac{1}{d})$, $j \ge 1$ and $\ln i = \ln d^\alpha = \alpha \ln d$,

$$\ln\Psi_1 = \ln\frac{\sqrt{\pi}}{2} + \frac{j}{2}\ln\frac{2}{\pi} + \ln d + \ln(1+\frac{1}{d}) - \ln i = (1-\alpha)\ln d + O(j).$$

Note that $O(j) + O(1) = O(j)$ because $j \ge 1$.

Turning to Ψ_3, assume that $1 \le m \le \frac{1}{2}d$ and consider

$$\ln\Gamma(\frac{d-m}{2}+1) = \frac{d-m}{2}\ln\frac{d-m}{2} - \frac{d-m}{2} + \frac{1}{2}\ln\frac{d-m}{2} + O(1).$$

Since $m/d \le 1/2$, the formula 4.1 yields that

$$\begin{aligned}\frac{d-m}{2}\ln\frac{d-m}{2} &= \frac{d-m}{2}\left(\ln d + \ln\left(1-\frac{m}{d}\right) - \ln 2\right)\\ &= \tfrac{1}{2}d\ln d - \tfrac{1}{2}m\ln d - \frac{\ln 2}{2}d + O(m).\end{aligned}$$

We also have

$$\begin{aligned}
-\frac{d-m}{2}+\frac{1}{2}\ln\frac{d-m}{2} &= -\frac{d}{2}+\frac{m}{2}+\frac{1}{2}\left(\ln d+\ln\left(1-\frac{m}{d}\right)-\ln 2\right)+O(1)\\
&= -\tfrac{1}{2}d+\tfrac{1}{2}\ln d+O(m),
\end{aligned}$$

which in turn yields that

$$\ln\Gamma(\frac{d-m}{2}+1)=\tfrac{1}{2}d\ln d-\tfrac{1}{2}m\ln d-\tfrac{1}{2}(\ln 2+1)d+\tfrac{1}{2}\ln d+O(m).$$

Substituting $m=1$ and $m=j$ into the formula above results in

$$\begin{aligned}
\ln\Psi_3 &= \tfrac{1}{2}d\ln d-\tfrac{1}{2}\ln d-\tfrac{1}{2}(\ln 2+1)d+\tfrac{1}{2}\ln d\\
&\quad -\tfrac{1}{2}d\ln d+\tfrac{1}{2}j\ln d+\tfrac{1}{2}(\ln 2+1)d-\tfrac{1}{2}\ln d+O(j)\\
&= \tfrac{1}{2}j\ln d-\tfrac{1}{2}\ln d+O(j).
\end{aligned}$$

In order to evaluate Ψ_2, we consider two cases. First let $j\geq i/2$. Observe that $\Gamma(\frac{i}{2}+1)^2\leq\Gamma(i-j+1)\Gamma(j+1)$ by 1.3, and hence

$$\Psi_2\leq\frac{\Gamma(i+1)}{\Gamma(\frac{i}{2}+1)^2}\cdot\frac{1}{\Gamma(j+1)}.$$

Then by 4.2 and 1.5, we have

$$\ln\Psi_2=\frac{i}{2}\ln i-\frac{i}{2}\ln\frac{i}{2}+O(\frac{i}{2})-j\ln j+j-\tfrac{1}{2}\ln j+O(1).$$

The condition $j\leq i\leq 2j$ yields that

$$\frac{i}{2}\ln i-\frac{i}{2}\ln\frac{i}{2}=\tfrac{1}{2}(i\ln i-i\ln i+i\ln 2)=O(i)=O(j),$$

and also that $\ln i$ and $\ln j$ are $O(j)$. Note that $\ln j=\ln d^\beta=\beta\ln d$. Hence if $i/2\leq j\leq i$ then

$$\ln\Psi_2\leq -j\ln j+O(j)=-\beta j\ln d+O(j).$$

Now let $1 \leq j \leq i/2$. Then 4.2 and 1.5 yield that

$$\begin{aligned}\ln \Psi_2 &= j \ln i - j \ln j + O(j) - j \ln j + j - \tfrac{1}{2} \ln j + O(1) \\ &= j \ln i - 2j \ln j + O(j) = (\alpha - 2\beta) j \ln d + O(j).\end{aligned}$$

In order to make it easier to summerize the results, we repeat that

$$\ln \Psi_1 = (1-\alpha) \ln d + O(j) \quad \text{and} \quad \ln \Psi_3 = \tfrac{1}{2} j \ln d - \tfrac{1}{2} \ln d + O(j).$$

Adding the fomulae for $\ln \Psi_k$, $k = 1, 2, 3$, yields $\ln \Psi$. Recall that $\alpha > 1/3$ and $j \leq \frac{1}{2} d$. If $1 \leq j \leq i/2$ then

$$\ln \Psi(d, i, j) = (\tfrac{1}{2} + \alpha - 2\beta) j \ln d + (\tfrac{1}{2} - \alpha) \ln d + O(j), \tag{4.3}$$

or if $i/2 \leq j \leq i$ then

$$\begin{aligned}\ln \Psi(d, i, j) &\leq (\tfrac{1}{2} - \beta) j \ln d + (\tfrac{1}{2} - \alpha) \ln d + O(j) \\ &\leq (\tfrac{1}{2} + \alpha - 2\beta) j \ln d + O(j).\end{aligned} \tag{4.4}$$

For the last step we used that $\beta \leq \alpha$ and also that $\ln d = O(j)$ since $j \geq \frac{1}{2} i \geq \frac{1}{2} d^{1/3}$.

4.3.3 Upper bound for $B(d)$

First we prove a lower bound for $\Phi(d, j)$.

LEMMA 4.3.4 *Let $d \geq 2$ and $1 \leq j \leq \frac{1}{2} d$. Then*

$$\Phi(d, j) \geq \tfrac{1}{2}\, 2^j \, \frac{\Gamma(j+1)\Gamma(d-j+1)}{(d+1)\Gamma(d+1)}.$$

Proof: Recall 1.9, which states that

$$\Phi(d, j) = \int_0^1 e^{-j[h(\sqrt{\pi}(1/2-t))]^2} t^{d-j} \, dt,$$

and that $g(t) = e^{-[h(\sqrt{\pi}(1/2-t))]^2}$ is a concave function on $[0,1]$ with $g(\frac{1}{2}) = 1$ and $g(1) = 0$. It follows that $g(t) \geq 2(1-t)$ for $\frac{1}{2} \leq t \leq 1$, and hence

$$\Phi(d,j) \geq \int_{\frac{1}{2}}^{1} 2^j (1-t)^j t^{d-j}\, dt.$$

Observe that $\dfrac{(1-t)^j t^{d-j}}{t^j(1-t)^{d-j}} = \left(\dfrac{t}{1-t}\right)^{d-2j} \geq 1$ for $t \geq \frac{1}{2}$ and $j \leq \frac{1}{2}d$. Thus

$$\int_0^{\frac{1}{2}} (1-t)^j t^{d-j}\, dt = \int_{\frac{1}{2}}^{1} t^j (1-t)^{d-j}\, dt \leq \int_{\frac{1}{2}}^{1} (1-t)^j t^{d-j}\, dt,$$

which in turn yields that

$$\int_0^1 (1-t)^j t^{d-j}\, dt \leq 2 \cdot \int_{\frac{1}{2}}^{1} (1-t)^j t^{d-j}\, dt.$$

By 1.4, we deduce the lower bound

$$\Phi(d,j) \geq 2^j \cdot \tfrac{1}{2} \int_0^1 (1-t)^j t^{d-j}\, dt = \tfrac{1}{2}\, 2^j\, \frac{\Gamma(j+1)\Gamma(d-j+1)}{\Gamma(d+2)}.$$

Having this, $\Gamma(d+2) = (d+1)\Gamma(d+1)$ yields the lemma.

□

Let $\frac{1}{2} < \rho < 1$ be arbitrary and $\frac{1}{2} < \gamma < \rho$ have the property that $\gamma + \varepsilon < \rho$ for $\varepsilon = \frac{1}{6}(\gamma - \frac{1}{2})$. If d is large enough then both of the intervals $(d^{\gamma-\varepsilon}, d^{\gamma+\varepsilon})$ and $(d^{\varepsilon}, d^{3\varepsilon})$ contain some integers. So assume that $\gamma - \varepsilon < \log_d i = \alpha(d,i) < \gamma + \varepsilon$ and $\varepsilon < \log_d j = \beta(d,j) < 3\varepsilon$. Note that $j > d^{\varepsilon}$, and we may assume that $j < \frac{1}{2} i$.

Since $\ln(d+1) = o(d^{\varepsilon}) = O(j)$ and $j = d^{\beta}$, 4.2 and Lemma 4.3.4 yield that

$$\ln \Phi(d,j) \geq \ln \tfrac{1}{2} + j \ln 2 - \ln(d+1) - j \ln d + j \ln j + O(j) = (\beta - 1) j \ln d + O(j),$$

It follows by 4.3, that

$$\begin{aligned}\ln A(d,i,j) &= \ln\Psi(d,i,j)+\ln\Phi(d,j)\geq(\alpha-\tfrac{1}{2}-\beta)j\ln d+O(j)\\ &\geq (\gamma-\varepsilon-\tfrac{1}{2}-3\varepsilon)j\ln d+O(j)=\left(2\varepsilon+\frac{O(1)}{\ln d}\right)j\ln d.\end{aligned}$$

Therefore, if d is large enough then $\ln A(d,i,j)\geq\varepsilon j\ln d>0$, and hence $A(d,i,j)>1$. Note that consequently $\sum_{m=1}^{i}A(d,i,m)>1$. Now Lemma 4.3.2 and Lemma 4.3.3 yield that $B(d)\leq d^{\rho}$ for d large. By the arbitrariness of $\frac{1}{2}<\rho<1$,

$$\limsup_{d\to\infty}\frac{\ln B(d)}{\ln d}\leq\frac{1}{2}. \tag{4.5}$$

A lower bound for $\Phi(d,j)$ yielded an upper bound for $B(d)$, hence now we search for an upper bound for $\Phi(d,j)$.

4.3.4 Upper bound for $\ln\Phi(d,s)$

In this and the next section we use the symbol $o(1)$ for functions which depend solely on d.

Let $\zeta=\frac{1}{1+2\sqrt{\pi e}}=0.09401$ and $1\leq s\leq\zeta d$ be real number. The upper bound 1.10 yields that

$$\Phi(d,s)<\left(\frac{1}{2\sqrt{\pi}}\,\frac{d-s}{s}\right)^{-s}\left(\ln\frac{1}{2\sqrt{\pi}}\,\frac{d-s}{s}\right)^{s/2}<\left(\frac{1}{2\sqrt{\pi}}\,\frac{d-s}{s}\right)^{-s}(\ln d)^{s}.$$

Note that $-\ln(1-\frac{s}{d})\leq-\ln(1-\zeta)<\ln 2$ and let $\beta=\log_d s$. Hence, writing $d-s$ as $d(1-\frac{s}{d})$, we deduce that

$$\begin{aligned}\ln\Phi(d,i,s) &\leq -s(\ln\frac{1}{2\sqrt{\pi}}+\ln d+\ln(1-\frac{s}{d})-\ln s)+s\ln\ln d\\ &\leq s\ln s-s\ln d+s(\ln 2\sqrt{\pi}+\ln 2)+s\ln\ln d\\ &= (\beta-1+\frac{\ln 2\sqrt{\pi}+\ln 2+\ln\ln d}{\ln d})\,s\ln d\\ &= (\beta-1+o(1))\,s\ln d.\end{aligned} \tag{4.6}$$

This upper bound is not sufficient for us if s is small. In order to improve it we need some additional properties of $\Phi(d,s)$.

Consider the functions

$$\alpha(t) = e^{-\frac{1}{2}p[h(\sqrt{\pi}(1/2-t))]^2}\, t^{\frac{1}{2}(d-p)}$$

$$\text{and} \qquad \beta(t) = e^{-\frac{1}{2}q[h(\sqrt{\pi}(1/2-t))]^2}\, t^{\frac{1}{2}(d-q)},$$

where p and q are positive numbers. According to Hölder's inequality, $(\int_0^1 \alpha(t)\cdot\beta(t)\,dt)^2 \leq \int_0^1 \alpha(t)^2\,dt \cdot \int_0^1 \beta(t)^2\,dt$. Substituting the definition of $\alpha(t)$ and $\beta(t)$ results in

$$\begin{aligned}&\left(\int_0^1 e^{-\frac{1}{2}(p+q)[h(\sqrt{\pi}(1/2-t))]^2}\, t^{d-\frac{1}{2}(p+q)}\,dt\right)^2\\ &\quad\leq \int_0^1 e^{-p[h(\sqrt{\pi}(1/2-t))]^2}\, t^{d-p}\,dt \cdot \int_0^1 e^{-q[h(\sqrt{\pi}(1/2-t))]^2}\, t^{d-q}\,dt.\end{aligned}$$

In terms of $\Phi(d,s)$, this yields that

$$\ln\Phi(d,\tfrac{1}{2}(p+q)) \leq \tfrac{1}{2}(\ln\Phi(d,p)+\ln\Phi(d,q)).$$

Since $\ln\Phi(d,s)$ is certainly continuous, it is also convex.

Let $1 \leq s \leq \ln d$. Recall 3.2, which says that $\Phi(d,1) \sim 2\sqrt{\pi}\frac{\sqrt{\ln d}}{d^2}$ as d tends to infinity. Thus if d is large enough then $\Phi(d,1) < \frac{\ln d}{d^2}$ and

$$\ln\Phi(d,1) < \ln\ln d - 2\ln d = \left(\frac{\ln\ln d}{\ln d} - 2\right)\ln d = [-2+o(1)]\ln d.$$

Note that $\beta(d,\ln d) = \frac{\ln\ln d}{\ln d} = o(1)$, and hence 4.6 yields that

$$\ln\Phi(d,\ln d) \leq [-1+o(1)]\ln d\cdot\ln d.$$

Define $0 \leq \lambda(d,s) \leq 1$ with the relation $s = (1-\lambda)\cdot 1 + \lambda\cdot\ln d$. Note that $s = 1+\lambda(\ln d - 1)$. Since $\ln\Phi(d,s)$ is convex in s, we have

$$\ln\Phi(d,s) \leq (1-\lambda)\Phi(d,1)+\lambda\Phi(d,\ln d) = (1-\lambda)[-2+o(1)]\ln d + \lambda[-1+o(1)](\ln d)^2.$$

Hence we deduce for $1 \leq s \leq \ln d$ that

$$\begin{aligned}\frac{\ln \Phi(d,s)}{\ln d} &\leq (1-\lambda)[-2+o(1)] + \lambda[-1+o(1)]\ln d \\ &= -2+o(1)+\lambda[2+o(1)+(-1+o(1))\ln d] \\ &= -2+o(1)+\lambda[-1+o(1)]\ln d \qquad (4.7)\end{aligned}$$

by $2+o(1) = o(1)\ln d$.

4.3.5 Proof of the asymptotic behavior of $B(d)$

Let $1/3 < \rho < 1/2$. We prove that $B(d) \geq d^\rho$ if d is large. Set $\gamma = \frac{1}{2}(\rho + \frac{1}{2})$ and $\varepsilon = \frac{1}{3}(\frac{1}{2} - \gamma)$. Note that $\rho < \gamma - \varepsilon$ and $\gamma + \varepsilon < 1/2$. If d is large enough then there exists an integer i with $\gamma - \varepsilon < \alpha = \log_d i < \gamma + \varepsilon$. Observe that $\frac{1}{2}\ln d > \ln i > \rho \ln d$.

LEMMA 4.3.5 *Let i be as above and $1 \leq j \leq i$. Then $\ln A(d,i,j) < -\frac{2}{3}\ln d$ for d sufficiently large.*

Proof: First assume $j \geq \ln d$. Hence 4.3 and 4.4 yield the existence of a constant $c > 0$ so that

$$\begin{aligned}\ln \Psi(d,i,j) &\leq (\tfrac{1}{2}+\alpha-2\beta)j\ln d + cj = (\tfrac{1}{2}+\alpha-2\beta+\frac{c}{\ln d})j\ln d \\ &= (\tfrac{1}{2}+\alpha-2\beta+o(1))j\ln d.\end{aligned}$$

Taking 4.6 also into account yields

$$\ln A(d,i,j) = \ln \Phi(d,j) + \ln \Psi(d,i,j) \leq [\alpha - \beta - \tfrac{1}{2} + o(1)]j\ln d.$$

If d is large enough then $\alpha - \beta - \frac{1}{2} + o(1) < \alpha - \frac{1}{2} + \varepsilon < \gamma + \varepsilon - \frac{1}{2} + \varepsilon = -\varepsilon$. Thus by $j \geq \ln d$, we have $\ln A(d,i,j) \leq -(\varepsilon \ln d)\cdot \ln d$. We may assume that d is large enough to ensure $\varepsilon \ln d > 2/3$, which in turn yields that $\ln A(d,i,j) < -\frac{2}{3}\ln d$.

Now assume that $1 \leq j \leq \ln d$. By $\beta \geq 0$, 4.3 yields a constant $c' > 0$ so that

$$\begin{aligned}\ln \Psi(d,i,j) &\leq (\tfrac{1}{2}+\alpha)j\ln d + (\tfrac{1}{2}-\alpha)\ln d + c'\cdot j\\ &= \left(\frac{1}{2}+\alpha+\frac{c'}{\ln d}\right) j\ln d + (\tfrac{1}{2}-\alpha)\ln d\\ &= [\tfrac{1}{2}+\alpha+o(1)]j\ln d + (\tfrac{1}{2}-\alpha)\ln d.\end{aligned}$$

Recall that $j = 1+\lambda(d,j)(\ln d - 1)$. It follows that

$$\begin{aligned}\frac{\ln \Psi(d,i,j)}{\ln d} &\leq [\tfrac{1}{2}+\alpha+o(1)][1+\lambda(\ln d-1)] + \tfrac{1}{2}-\alpha\\ &= 1+o(1)+\lambda[\tfrac{1}{2}+\alpha+o(1)](\ln d-1)\\ &\leq 1+o(1)+\lambda[\tfrac{1}{2}+\alpha+o(1)]\ln d.\end{aligned}$$

Combining this with 4.7 yields that

$$\frac{A(d,i,j)}{\ln d} = \frac{\Psi(d,i,j)}{\ln d} + \frac{\Phi(d,j)}{\ln d} \leq -1+o(1)+\lambda[\alpha-\tfrac{1}{2}+o(1)]\ln d.$$

If d is large enough then $-1+o(1) < -2/3$ and $\alpha-\frac{1}{2}+o(1) < \gamma+\varepsilon-\frac{1}{2}+\varepsilon = -\varepsilon < 0$, and hence again $\ln A(d,i,j) < \frac{2}{3}\ln d$.

□

Lemma 4.3.5 yields that $A(d,i,j) < d^{-2/3}$. As $i < d^{1/2}$, we deduce that

$$\sum_{j=1}^{i} A(d,i,j) < d^{1/2}\cdot d^{-2/3} = d^{-1/6} < 1$$

for d large, and hence $B(d) \geq i > d^{\rho}$. At the end, the arbitrariness of $1/3 < \rho < 1/2$ and 4.5 yield the theorem.

□

Let $d \geq 3$ and $1 \leq i \leq d$, and consider n-ball packings with $n \leq d+1$. In this and in the previous chapter we have investigated, for various i, the minimum properties of the *ith* intrinsic volume of the convex hull of the balls. The results indicate that, as i increases, the sausage arrangement becomes more optimal than the one determined by a regular simplex. We have not considered the case $3 < n < d+1$. The only known information about this is the result of [8], which states that $V(S_n + B^d) < V(T^{n-1} + B^d)$.

Chapter 5

About the sausage conjecture

After the intrinsic volumes of lower index, we consider the d-dimensional volume of the convex hull of n-ballpackings, mostly for $d \geq 5$. As a sharp contrast to the previous results for large n, most probably the minimal volume occurs if the centers are collinear; that is, the convex hull resembles a sausage, and this holds without any restriction on n. We prove that a minimal arrangement is not too far from the sausage-like one.

5.1 The story of the conjecture

We assume $d \geq 2$ for the whole chapter. Let $\mathcal{F}_n^d$ be the family of the convex hulls of the centers in n-ball packings, which is contained in the $\mathcal{G}_n^d$ of Chapter 2.

Let $\mathcal{F}_n^d = \{C_n, S_n, \ldots\}$. Let $C_n \in \mathcal{F}_n^d$. We define the density of the C_n-ballpackings (see Figure 5.1) as

$$\delta_n^d(C_n) = \frac{n\kappa_d}{V(C_n + B^d)}.$$

C_n is *reduced* if there is no packing of $n+1$ balls such that the centers are contained in C_n (hence $\nu(C_n) = n$, with the notation from Chapter 2). Next, C_n is *fat* if $r(C_n) > \frac{1}{2}n^{1/d}$.

Recall from Section 2.4, p.60 that $\mathcal{P}_{1,n}^d \in \mathcal{F}_n^d$ and $\mathcal{P}_{1,n}^d$ is reduced.

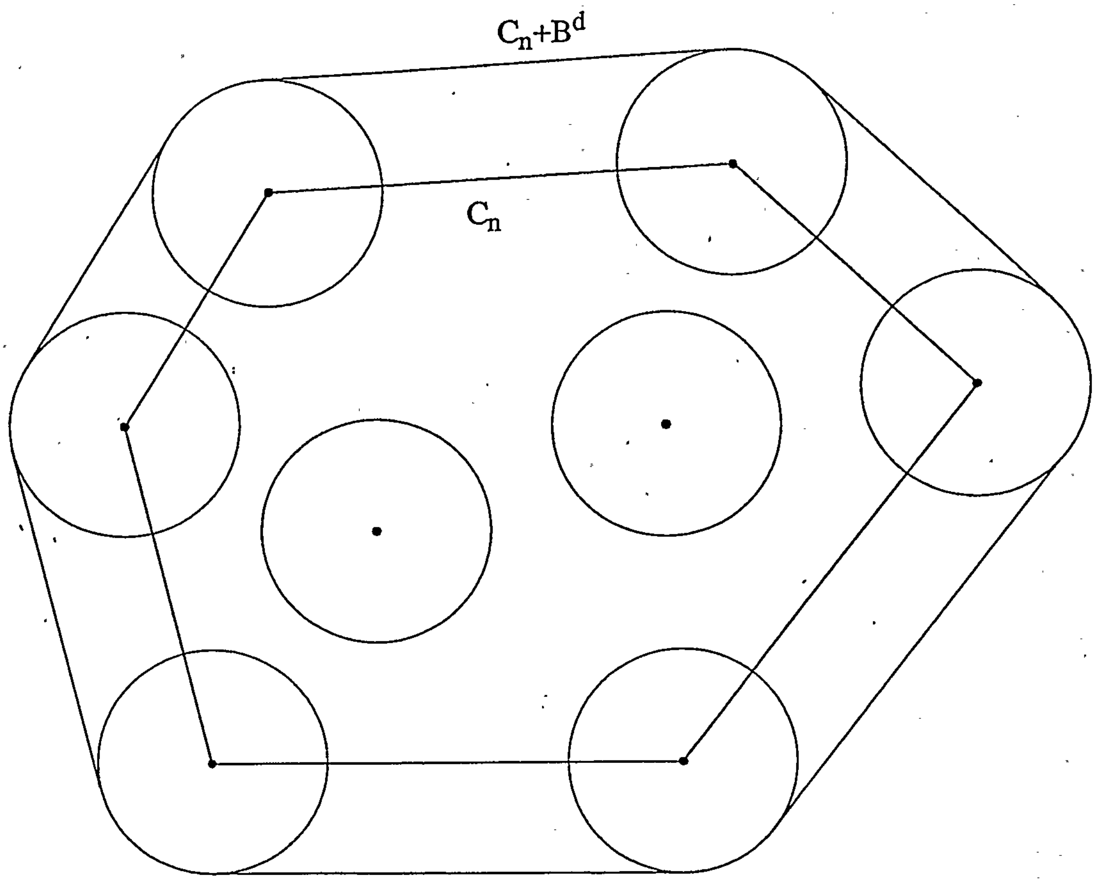

Figure 5.1

A C_n-ballpacking

Since $\mathcal{P}^d_{1,n} + B^d$ is contained in a ball of radius $R(\mathcal{P}^d_{1,n}) + 1$, Theorem 2.4.1 yields that

$$V_1(B^d) \cdot n^{1/d} \leq V_1(\mathcal{P}^d_{1,n} + B^d) \leq (R(\mathcal{P}^d_{1,n}) + 1)V_1(B^d).$$

Thus $R(\mathcal{P}^d_{1,n}) + 1 \geq n^{1/d}$, and if n is large enough then, by Corollary 2.5.2,

$$r(\mathcal{P}^d_{1,n}) \geq \frac{1}{2} n^{1/d}.$$

Hence $\mathcal{P}^d_{1,n}$ is also fat and there exists a sequence $\{C_n\}_{\{n>N\}}$ of reduced, fat bodies. By Lemma 2.1.1,

$$\lim_{n\to\infty} \delta^d_n(C_n) = \delta_d, \tag{5.1}$$

where δ_d is the packing density.

As a counterpart, the segment $S_n \in \mathcal{F}^d_n$, with length $2(n-1)$, is 'thin' and reduced. The density of the S_n-ballpacking (see Figure 5.2) is

$$\delta^d_n(S_n) = \frac{n\kappa_d}{2(n-1)\kappa_{d-1} + \kappa_d} > \frac{\kappa_d}{2\kappa_{d-1}}. \tag{5.2}$$

Here we use the fact that by 1.7,

$$\kappa_d < \sqrt{\frac{2\pi}{d}}\kappa_{d-1} < 2\kappa_{d-1} \quad \text{for } d \geq 2,$$

which in turn yields that

$$\begin{aligned} V(S_n + B^d) &= 2(n-1)\kappa_{d-1} + \kappa_d \\ &< 2(n-1)\kappa_{d-1} + 2\kappa_{d-1} = 2n\kappa_{d-1}. \end{aligned} \tag{5.3}$$

In 1975, L. Fejes Tóth baptised as a 'sausage', the convex hull $S_n + B^d$, *cf.* [11].

With regard to finite packings, L. Fejes Tóth proved in 1949 that in E^2 a finite packing of unit balls cannot be denser than the densest infinite packing.

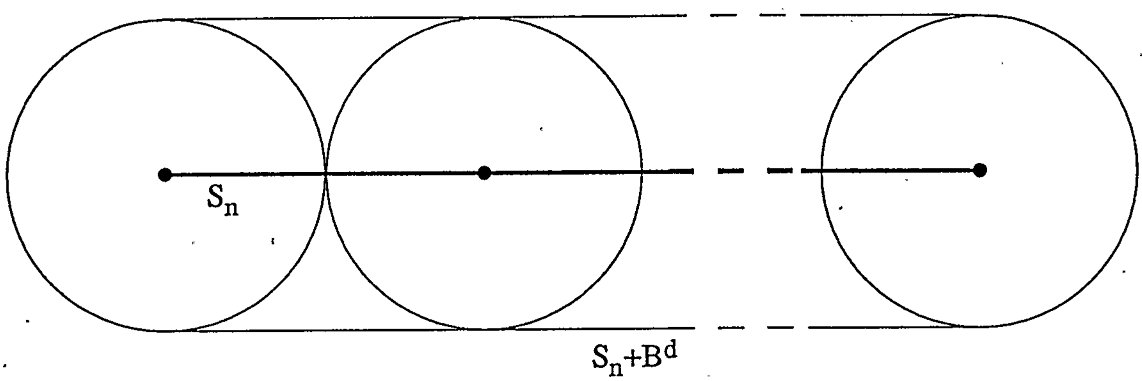

Figure 5.2

'The sausage S_n+B^d

As a point of interest, H. Groemer (*cf.* [15]) and G. Wegner (*cf.* [29]) showed that for $C_n \in \mathcal{F}_n^2$, the area $V(C_n + B^2)$ is minimal when the shape of C_n is hexagonal.

In dimensions three and four, the optimal arrangement is 'sausage' for n small, and probably 'fat' for n large (*cf.* [8] and [12]). To illustrate the latter statement, let n be large. Then $\delta_n^d(S_n) \approx \frac{\kappa_d}{2\kappa_{d-1}}$ and therefore

$$\delta_n^3(S_n) \approx \frac{2}{3} = 0.6666 \quad \text{and} \quad \delta_n^4(S_n) \approx \frac{3\pi}{16} = 0.5890.$$

On the other hand, we note that the existence of dense C_n-ballpackings (see 5.1) for reduced, fat $C_n \in \mathcal{F}_n^d$, $d \in \{3, 4\}$, yields that

$$\delta_n^3(C_n) > 0.7 \quad \text{and} \quad \delta_n^4(C_n) > 0.6.$$

Here we used the fact that $\delta_3 \geq 0.74080$ and $\delta_4 \geq 0.61685$.

For large d, the 'sausage' arrangement is better than the fat one. Since $\delta_d < 2^{-0.599d(1+o(1))}$ (*cf.* [19]), fixing d and letting n be large compared to d,

$$\delta_n^d(C_n) < 2^{-\frac{1}{2}d}$$

for reduced, fat $C_n \in \mathcal{F}_n^d$. Meanwhile for S_n,we have the lower bound

$$\delta_n^d(S_n) > \frac{\kappa_d}{2\kappa_{d-1}} > \sqrt{\frac{\pi}{2(d+1)}} > 2^{-d/2}.$$

This inequality uses 1.7 again.

In summary, as the dimension of the space increases, the density of the sausage arrangement becomes very large compared to the density of fat arrangements. These and other considerations convinced L. Fejes Tóth to postulate in 1975 (*cf.* [11]) his famous

CONJECTURE 5.1.1 (Sausage conjecture) *For $d \geq 5$ and $C_n \in \mathcal{F}_n^d$,*

$$\delta_n^d(C_n) \leq \delta_n^d(S_n)$$

with equality if and only if $C_n = S_n$.

The inequality above is equivalent to the 'Sausage Inequality'

$$V(C_n + B^d) \geq V(S_n + B^d). \tag{5.4}$$

There are a series of results supporting the conjecture, but it has not been proved yet in any dimension. We list some of those results under the assumption that $C_n \in \mathcal{F}_n^d$ and $d \geq 5$; and note that 5.4 has been verified in cases ii) to v).

i) $V(C_n + B^d) \geq (2 - \sqrt{3})\, V(S_n + B^d)$ [9],

ii) $\dim C_n \leq \frac{7}{12}(d-1)$ [3],

iii) $\dim C_n \leq 9$ and $d \geq \dim C_n + 1$ [2],

iv) $\dfrac{R(C_n)}{\sqrt{2}} + \sqrt{2} - 1 \leq r(C_n)$ for large d [8],

v) C_n is a regular simplex [8].

Assume that $\delta_d < \kappa_d / 2\kappa_{d-1}$ for a certain $d \geq 2$. By means of 2.3 and 5.2, there is a $\varrho > 0$ so that if $r(C_n) > \varrho$ then

$$\delta_n^d(C_n) < \delta_n^d(S_n).$$

This observation is the starting point of our investigation.

Let $1 \leq m \leq d$. Recall from Section 1.5 that the m-dimensional inner radius of C_n, $r_m(C_n)$ is the radius of the largest m-dimensional ball contained in C_n. The

m-dimensional outer radius of C_n, $R_m(C_n)$ is the minimum of $R_m \geq 0$ such that there exists a $(m-1)$-dimensional affine subspace g with $C_n \subset g + R_m B^d$. Then $\dim K < m$, $r_m(K) = 0$ and $R_m(K) = 0$ are equivalent statements.

Define

$$\psi(d) = \begin{cases} \min\{d, 10\} & \text{if } 5 \leq d \leq 18 \\ [\frac{7}{12}(d-1)] + 1 & \text{if } d \geq 19, \end{cases}$$

where $[b]$ is the largest integer not greater than b. If $\dim C_n < \psi(d)$ then it is easy to verify that C_n satisfies the Sausage Inequality 5.4 by ii) and iii). On the other hand, by iv) the same conclusion holds if C_n is a very fat convex body. Our aim is to narrow the gap between the two types of results. We prove that for $m = \dim C_n$ or $m = \psi(d)$, C_n satisfies the Sausage Inequality if the m-dimensional inner radius of C_n is not very small (and hence similar conclusion holds for $R_m(C_n)$).

All the considerations in this chapter are based on a method of Blichfeldt, which is reviewed in the second section. Let $\dim C_n = m \geq \psi(d)$. We show that C_n satisfies the Sausage Inequality if (Section 5.3) $R_{\psi(d)}(C_n)$ has order of at least $\ln d$, or if (Section 5.5) $\Delta_m(C_n)$ has order of at least $\frac{\ln m}{\sqrt{m}}$.

We determine lower bounds for $r_{\psi(d)}(C_n)$ and $r_m(C_n)$, which in turn yield lower bounds for $R_{\psi(d)}$ and $\Delta_m(C_n)$.

5.2 The method of Blichfeldt

Let us consider a packing $\{x_1 + B^d, \ldots, x_n + B^d\}$ with $C_n = \mathrm{conv}\{x_1, \ldots, x_n\}$. We know that $V(C_n + B^d)$ is greater than $n\kappa_d$, the integral over E^d of the sum of the characteristic functions of the unit balls. As that sum is zero on a large part of the convex hull of the balls (on the space among the balls), the resulting integral is a very bad estimation for $V(C_n + B^d)$. Blichfeldt found a function $\rho(x)$ such that translating it to the center of each ball, the sum of the resulting functions would fill much more evenly the space among the unit balls.

Define

$$\rho(x) = \begin{cases} 2 - |x|^2 & \text{if } |x| \leq \sqrt{2} \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$\Phi^d = \int_{E^d} \rho(x)dx = \int_{\sqrt{2}B^d} 2 - |x|^2 dx.$$

Since the balls $x_i + B^d$, $i = 1, \ldots, n$, form a packing, the function ρ satisfies the inequality

$$\mu(x) = \sum_{i=1}^{n} \rho(x - x_i) \leq 2 \tag{5.5}$$

for any x in E^d. This property yields a lower bound for $V(C_n + \sqrt{2}B^d)$ because

$$\int_{E^d} \rho(x)dx = \int_{\sqrt{2}B^d} \rho(x)dx = \int_{x_i+\sqrt{2}B^d} \rho(x - x_i)dx$$

for $i = 1, \ldots, n$ by the definition of ρ, and

$$\begin{aligned} n\,\Phi^d &= n\int_{E^d} \rho(x)dx = \sum_{i=1}^{n} \int_{E^d} \rho(x - x_i)dx \\ &= \int_{C_n+\sqrt{2}B^d} \mu(x)dx \leq 2V(C_n + \sqrt{2}B^d). \end{aligned} \tag{5.6}$$

We denote $< u, u >$ by u^2 for $u \in E^d$. In order to prove 5.5, we may assume that $x = 0$. Since $\rho(x_i) = 0$ for $|x_i| \geq \sqrt{2}$, it may be also assumed that $|x_i| \leq \sqrt{2}$ for any i=1,...,n. The centers of the balls satisfy

$$(x_i - x_j)^2 \geq 4,\ 1 \leq i < j \leq n$$

because the unit balls form a packing. In the family of the $\binom{n}{2} = \frac{n(n-1)}{2}$ inequalities above, writing them in the form

$$x_i^2 + x_j^2 - 2x_i x_j \geq 4,$$

the term x_k^2 appears $n-1$ times for any k. Hence summing these inequalities results in

$$(n-1)\sum_{i=1}^{n} x_i^2 - 2 \sum_{1\leq i<j\leq n} x_i x_j \geq 2n(n-1).$$

Adding and substracting $\sum_{i=1}^{n} x_i^2$ from the left hand side yields

$$n\sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2 \geq n \cdot 2n - 2n,$$

which is equivalent to

$$2n \geq n\sum_{i=1}^{n}(2 - x_i^2) + (\sum_{i=1}^{n} x_i)^2.$$

Since $x = 0$ and $\rho(x_i) = \rho(-x_i)$, dividing the last inequality by n establishes 5.5.

By means of 1.2,

$$\begin{aligned}
\Phi^d &= d\,\kappa_d \int_0^{\sqrt{2}} (2 - r^2) r^{d-1} dr = d\,\kappa_d \int_0^{\sqrt{2}} 2r^{d-1} - r^{d+1} dr \\
&= d\,\kappa_d \left(\frac{2 \cdot \sqrt{2}^d}{d} - \frac{\sqrt{2}^{d+2}}{d+2} \right) = d\,\kappa_d \cdot 2^{(d+2)/2} \cdot \frac{2}{d(d+2)} \\
&= \frac{4\kappa_d}{d+2}\, 2^{d/2}. \qquad (5.7)
\end{aligned}$$

Finally, 5.6 yields Blichfeldt's estimate

LEMMA 5.2.1 *Let $d \geq 2$, $n \geq 1$ and $C_n \in \mathcal{F}_n^d$. Then*

$$V(C_n + \sqrt{2}B^d) \geq \frac{2\kappa_d}{d+2}\, 2^{d/2} \cdot n. \qquad (5.8)$$

Rankin [25] modified Blichfeldt's function. Since Rankin's method is much more complicated than that of Blichfeldt, and the improvement is significant only for small dimensions, we prefer Blichfeldt's approach. The only exeption is if $d = 6$. In this case Blichfeldt's estimate does not provide any information with respect to the Sausage Conjecture as opposed to Rankin's slightly better estimate.

So let $d = 6$, $\{x_1+B^6, \ldots, x_n+B^6\}$ be a packing in E^6, and $C_n = \mathrm{conv}\{x_1, \ldots, x_n\}$. Rankin defined a function $\sigma : E^6 \to R$ so that $\sigma(x) = 0$ for $x \notin \sqrt{2}B^6$ and

$$\sum_{i=1}^{n} \sigma(x - x_i) \leq 1$$

for any $x \in E^6$. Hence for

$$\Psi^6 = \int_{E^6} \sigma(x)\, dx,$$

one can derive the analog of 5.6; that is,

$$n \cdot \Psi^6 \leq V(C_n + \sqrt{2}B^6). \tag{5.9}$$

We note that $\Psi^6 = 10.54305$ by the formula given in [25].

5.3 About the fatness of a packing

For $d \geq 5$, define

$$\varphi(d) = \begin{cases} \min\{d, 10\} & \text{if } 5 \leq d \leq 17 \\ 11 & \text{if } d = 18 \\ \left[\frac{7}{12}(d-1)\right] + 1 & \text{if } d \geq 19. \end{cases}$$

We remark that $\varphi(d) = \psi(d)$ if $d \neq 18$ and also $\varphi(d) > \frac{7}{12}d - \frac{7}{12}$.

It follows from ii) and iii) of the previous section that if $d \geq 5$, $d \neq 18$ and $r_{\varphi(d)}(C_n) = 0$ for $C_n \in \mathcal{F}_n^d$, then C_n satisfies the Sausage Inequality 5.4. Hence 5.4

holds if C_n is not 'too far' from S_n. Our approach is from the other direction; that is, how far can C_n be from being a d-dimensional ball (which was considered basically in iv)) and still satisfy the Sausage Inequality. It turns out that for large d, even a quite small value of $r_{\varphi(d)}$ is sufficient but unfortunately $r_{\varphi(d)}$ can not be arbitrarily small. Note that Blichfeldt's method gives the estimate $\frac{1}{2}n\Phi^d$ for $V(C_n + \sqrt{2}B^d)$, independent of the shape of C_n. If $r_{\varphi(d)}(C_n)$ is large then the ratio $V(C_n + B^d)$ over $V(C_n + \sqrt{2}B^d)$ is large enough to yield

$$V(C_n + B^d) \geq \frac{V(C_n + B^d)}{V(C_n + \sqrt{2}B^d)} \cdot \frac{1}{2}n\Phi^d > V(S_n + B^d),$$

where the first inequality follows by 5.6. In other words, the fact that similar estimates hold for $V(C_n + B^d)$ and $V(C_n + \sqrt{2}B^d)$ can correct the error coming from Blichfeldt's method. If $r_{\varphi(d)}(C_n)$ is arbitrarily small then $V(C_n + B^d)$ is much smaller than $V(C_n + \sqrt{2}B^d)$ and the second inequality above does not hold any more.

In order to prove our main theorem, we need the following lemma. For a fixed $r > 0$ and $1 \leq m \leq d$, let T_o be the linear transformation with the diagonal matrix

$$\begin{array}{c} \text{m}\left\{ \vphantom{\begin{matrix} a \\ b \\ c \end{matrix}} \right. \\ \text{d-m}\left\{ \vphantom{\begin{matrix} a \\ b \\ c \end{matrix}} \right. \end{array} \begin{bmatrix} \frac{1}{r+1} & & & & & \\ & \ddots & & & 0 & \\ & & \frac{1}{r+1} & & & \\ & & & 1 & & \\ & 0 & & & \ddots & \\ & & & & & 1 \end{bmatrix}.$$

Denote by E^m the Euclidean space spanned by the first m coordinate axes, and by B^m, the unit ball in E^m centered at the origin.

LEMMA 5.3.1 *For $K \in \mathcal{K}^d$, $1 \leq m \leq d$ and $r = r_m(K)$,*

$$V(K + B^d) \geq \left(\frac{r+1}{r+\sqrt{2}}\right)^m 2^{(m-d)/2}\, V(K + \sqrt{2}B^d). \tag{5.10}$$

Proof: Notice that any $x \in B^d$ can be written as x=y+z with $y \in B^m$, $\langle y, z\rangle = 0$ and $|y|^2 + |z|^2 \leq 1$. Then

$$\begin{aligned} T_0^{-1}x &= T_0^{-1}y + T_0^{-1}z = (r+1)y + z \\ &= ry + x \in rB^m + B^d. \end{aligned}$$

This yields that $T_0^{-1}(B^d) \subset rB^m + B^d$, or

$$B^d \subset T_0(rB^m + B^d).$$

For simplicity, we assume that $rB^m \subset K$. Then $B^d \subset T_0(K + B^d)$, and it follows that

$$\begin{aligned} K + \sqrt{2}B^d &= K + B^d + (\sqrt{2}-1)B^d \\ &\subset K + B^d + (\sqrt{2}-1)T_0(K + B^d) \\ &= (I + (\sqrt{2}-1)T_0)(K + B^d). \end{aligned}$$

Denote the linear transformation $I + (\sqrt{2}-1)T_0$ by U. Note the relation

$$1 + (\sqrt{2}-1)\frac{1}{r+1} = \frac{r+\sqrt{2}}{r+1},$$

which in turn yields that

$$\det(U) = \left(\frac{r+\sqrt{2}}{r+1}\right)^m \cdot \sqrt{2}^{d-m}.$$

In summary, we deduce that

$$\begin{aligned} V(K+\sqrt{2}B^d) &\leq V(U(K+B^d)) = \det(U)\,V(K+B^d) \\ &= \left(\frac{r+\sqrt{2}}{r+1}\right)^m \cdot 2^{(d-m)/2} V(K+B^d). \end{aligned}$$

□

THEOREM 5.3.2 *Let $d \geq 5$.*

a) There exists a function $c(d)$ such that if $r_{\varphi(d)}(C_n) \geq c(d) \ln d/d$ for $C_n \in \mathcal{F}_n^d$, then $V(C_n + B^d) \geq V(S_n + B^d)$.

b) $c(d) \sim \dfrac{(2+\sqrt{2})18}{7}\left(1 + O\left(\dfrac{\ln d}{d}\right)\right) = 8.7794\left(1 + O\left(\dfrac{\ln d}{d}\right)\right).$

Proof: First let $d = 5$, and hence $\varphi(d) = 5$, and let $C_n \in \mathcal{F}_n^5$ for some $n \geq 1$. Note that according to the table in [5], p.15.,

$$\frac{\delta_5}{\kappa_5} \leq 0.09987,$$

which in turn yields that $\kappa_5/\delta_5 > 10$. By means of Corollary 2.1.2, there exist a $\varrho > 0$ so that if $r(C_n) \geq \varrho$ then

$$V(C_n + B^5) > 10 \cdot n.$$

On the other hand, by 5.3,

$$V(S_n + B^5) < 2\,\kappa_4 \cdot n = 9.86960 \cdot n,$$

and hence $r(C_n) \geq \varrho$ yields the Sausage Inequality 5.4. Therefore we may set $c(5) = 5\,\varrho/\ln 5$. Unfortunately, we have no estimate for ϱ.

We consider the case $d = 6$ later, so let $d \geq 7$. By 5.3, the volume of the 'sausage' is

$$V(S_n + B^d) < 2n\kappa_{d-1}.$$

For a $C_n \in \mathcal{F}_n^d$, combining Lemma 5.2.1 with Lemma 5.3.1 results in

$$\begin{aligned} V(C_n + B^d) &\geq V(C_n + \sqrt{2}B^d)\left(\frac{r+1}{r+\sqrt{2}}\right)^{\varphi} \cdot 2^{(\varphi-d)/2} \\ &\geq \frac{2\kappa_d}{d+2} 2^{d/2} n \cdot \left(\frac{r+1}{r+\sqrt{2}}\right)^{\varphi} \cdot 2^{(\varphi-d)/2} \\ &= \left(\frac{r+1}{r+\sqrt{2}}\right)^{\varphi} \frac{2\kappa_d}{d+2} \cdot 2^{\varphi/2} \cdot n, \end{aligned}$$

where $\varphi = \varphi(d)$, $C_n \in \mathcal{F}_n^d$ and $r = r_\varphi(C_n)$. The Sausage Inequality 5.4 holds if

$$\left(\frac{r+1}{r+\sqrt{2}}\right)^{\varphi} \cdot 2^{\varphi/2} \cdot \frac{2\kappa_d}{d+2} n \geq 2\kappa_{d-1} n, \tag{5.11}$$

which is equivalent to

$$\left(\frac{\sqrt{2}r+\sqrt{2}}{r+\sqrt{2}}\right)^{\varphi} \geq \frac{\kappa_{d-1}}{\kappa_d}(d+2). \tag{5.12}$$

Define the function $h(r)$ for non-negative r by

$$h(r) = \frac{\sqrt{2}r+\sqrt{2}}{r+\sqrt{2}}.$$

This function is monotonically increasing and satisfies $h(0) = 1$ and $\lim_{r\to\infty} h(r) = \sqrt{2}$. Since by 1.7,

$$\frac{\kappa_{d-1}}{\kappa_d}(d+2) > \sqrt{\frac{d}{2\pi}}(d+2) > 1$$

for $d \geq 7$, 5.12 can be extended to

$$(\sqrt{2})^{\varphi} > \left(\frac{\sqrt{2}r+\sqrt{2}}{r+\sqrt{2}}\right)^{\varphi} \geq \frac{\kappa_{d-1}}{\kappa_d}(d+2) > 1. \tag{5.13}$$

It follows right away that r must be positive. On the other hand, the existence of a positive r satisfying 5.13 is dependent on whether

$$\frac{\kappa_{d-1}}{\kappa_d}(d+2) < (\sqrt{2})^{\varphi}. \tag{5.14}$$

This inequality does not hold if $(d,\varphi) = (5,5)$, $(6,6)$ or $(18,10)$. If $(d,\varphi) = (6,6)$ then using 5.9 instead of 5.6 yields the corresponding version of 5.11; that is, the Sausage Inequality follows from

$$\left(\frac{r+1}{r+\sqrt{2}}\right)^6 \cdot \Psi^6 \cdot n \geq 2\,\kappa_5 \cdot n,$$

which inequality is equivalent to

$$\left(\frac{r+1}{r+\sqrt{2}}\right)^6 \geq \frac{2\,\kappa_5}{\Psi^6}. \tag{5.15}$$

We note that

$$\frac{2\,\kappa_5}{\Psi^6} = 0.99853 < 1,$$

and hence there is a $r(6) \in R$ so that the Sausage Inequality holds if $r(C_n) \geq r(6)$. In the cases $(d,\varphi) = (5,5)$ or $(18,10)$ even Rankin's improved estimate is unable to provide a suitable lower bound for $r(C_n)$.

Define $r(d)$ as the value of r satisfying the equality in 5.15 if $d = 6$, or in 5.12 if $d = 7,\ldots,48$, and let $c(d) = \frac{r(d)d}{\ln d}$ for $d = 6,\ldots,48$. Since $h(r)$ is monotonically increasing, in order to prove that any $r \geq r(d)$ satisfies 5.15 if $d = 6$, or 5.13 if $d = 7,\ldots,48$, it is sufficient to consider the case $r = r(d)$. We did this with the help of a computer, and the resulting values are contained in Table 5.1.

d	$\phi=\phi(d)$	lower bound for r_ϕ	lower bound for c	lower bound for R_ϕ
6	6	1690.3813	5660.5187	4183.4804
7	7	19.6265	70.6022	51.9268
8	8	9.2130	35.4443	26.2208
9	9	6.0494	24.7789	18.1483
10	10	4.5203	19.6318	14.3541
11	10	5.7206	26.2427	62.9273
12	10	7.3818	35.6482	81.2007
13	10	9.8417	49.8810	108.2589
14	10	13.8717	73.5887	152.5895
15	10	21.7058	120.2297	238.7647
16	10	43.6246	251.7484	479.8715
17	10	462.5658	2775.5124	5088.2239
18	11	15.2755	95.1293	183.3061
19	11	21.4220	138.2330	257.0645
20	12	9.2274	61.6040	119.9569
21	12	10.9718	75.6799	142.6342
22	13	6.6048	47.0089	92.4679
23	13	7.3840	54.1643	103.3760
24	14	5.1437	38.8443	77.1558
25	15	3.9635	30.7836	63.4168
26	15	4.2139	33.6280	67.4236
27	16	3.3995	27.8492	57.7916
28	16	3.5710	30.0070	60.7080
29	17	2.9767	25.6362	53.5810
30	17	3.1003	27.3462	55.8059
31	18	2.6483	23.9078	50.3188
32	19	2.3154	21.3787	46.3081
33	19	2.3861	22.5204	47.7229
34	20	2.1157	20.3992	44.4305
35	20	2.1720	21.3820	45.6124
36	21	1.9482	19.5722	42.8617
37	22	1.7681	18.1177	40.6676
38	22	1.8058	18.8645	41.5339
39	23	1.6516	17.5825	39.6399
40	23	1.6832	18.2520	40.3977
41	24	1.5498	17.1113	38.7465
42	24	1.5766	17.7167	39.4162
43	25	1.4601	16.6933	37.9642
44	26	1.3605	15.8196	36.7349
45	26	1.3805	16.3199	37.2747
46	27	1.2919	15.5224	36.1748
47	27	1.3094	15.9844	36.6635
48	28	1.2301	15.2526	35.6737

Table 5.1

The lower bound for R_φ is defined as

$$R_\varphi = \begin{cases} \sqrt{d}\, r_\varphi & \text{if } d = 7, 9 \\ \frac{d+1}{\sqrt{d+2}}\, r_\varphi & \text{if } d = 6, 8, 10 \\ (\varphi + 1)\, r_\varphi & \text{if } d \geq 11, \end{cases}$$

where r_φ is taken from the third column. The actual use of R_φ will be apparent later in this section.

We observe that the minimum value of $r_\varphi = r_{\varphi(d)}$ is considerably larger for $d = 6$ and $d = 17$ than for other values of d. If $d = 6$ then the reason is that the lower bound provided by Rankin's method for $V(C_n + B^6)$, $C_n \in \mathcal{F}^6$, is just slightly greater then $V(S_n + B^6)$ for n large. With respect to the other anomaly, we note that as $d < 17$ approaches 17, $\varphi(d) = 10$ by definition and $\frac{\kappa_{d-1}}{\kappa_d}(d+2)$ approaches $\sqrt{2}^{10}$. Thus $(\frac{\sqrt{2}r + \sqrt{2}}{r + \sqrt{2}})^{10}$ is necessarily close to $\sqrt{2}^{10}$, and this forces r to be large. This forcing is not present for $10 \leq d < 17$ because φ is still 10, and for $d > 17$ because φ now increases as well, and the ratio of φ over d is more or less constant, around $\frac{7}{12}$.

Now we turn to the case $d > 48$. Let $\tau = \frac{(\sqrt{2}-1)r}{r+\sqrt{2}}$. Then

$$1 + \tau = \frac{\sqrt{2}r + \sqrt{2}}{r + \sqrt{2}} < \sqrt{2},$$

and r is non-negative if and only if $0 \leq \tau < \sqrt{2} - 1$. Let $0 \leq \tau < \sqrt{2} - 1$. Then

$$r = \frac{\sqrt{2}\,\tau}{\sqrt{2} - 1 - \tau} = (2 + \sqrt{2})\,\tau\,\frac{1}{1 - (\sqrt{2} + 1)\tau}. \tag{5.16}$$

Since $\frac{\kappa_{d-1}}{\kappa_d} < \sqrt{\frac{d+1}{2\pi}}$ by 1.7, the inequality 5.13 follows if

$$(\sqrt{2})^\varphi > (1 + \tau)^\varphi \geq \sqrt{\frac{d+1}{2\pi}}(d+2). \tag{5.17}$$

We solve 5.17 and determine r by 5.16, instead of solving 5.13 directly, because $\frac{\kappa_{d-1}}{\kappa_d}(d+2)$ is difficult to express as a simple function of d.

Recall that $1+\tau<\sqrt{2}$, and $\varphi(d)>\frac{7}{12}d-\frac{7}{12}$. Hence

$$(1+\tau)^{\varphi}>(1+\tau)^{\frac{7}{12}d-\frac{7}{12}}>(1+\tau)^{\frac{7}{12}d}\sqrt{2}^{-\frac{7}{12}},$$

and 5.17 follows if

$$(1+\tau)^{\frac{7}{12}d}>\sqrt{2}^{\frac{7}{12}}\frac{\sqrt{d+1}}{\sqrt{2\pi}}(d+2)=0.4883\sqrt{d+1}(d+2).$$

For $d>48$, we choose

$$(1+\tau)^{\frac{7}{12}d}=d^{\frac{3}{2}}>0.4883\sqrt{d+1}(d+2),$$

which then has the solution

$$\begin{aligned}\tau(d) &= e^{\frac{18}{7}\ln d/d}-1=\left(1+\frac{18\ln d}{7d}+O\left(\left(\frac{18\ln d}{7d}\right)^2\right)\right)-1\\ &= \frac{18}{7}\frac{\ln d}{d}\left(1+O\left(\frac{\ln d}{d}\right)\right).\end{aligned}$$

Notice that $\ln d/d$, and also τ are decreasing functions of d. Hence for $d>48$,

$$\tau(d)<e^{\frac{18\ln 48}{7\cdot 48}}-1=0.23045<\sqrt{2}-1$$

and τ yields a solution for 5.13.

It follows by

$$\frac{1}{1-(\sqrt{2}+1)\tau(d)}=1+O(\tau(d)),$$

and by 5.16 that

$$\begin{aligned}r(d) &= (2+\sqrt{2})\frac{18\ln d}{7d}\left(1+O\left(\frac{\ln d}{d}\right)\right)(1+O(\tau))\\ &= (2+\sqrt{2})\frac{18\ln d}{7d}\left(1+O\left(\frac{\ln d}{d}\right)\right)\left(1+O\left(\frac{\ln d}{d}\right)\right)\\ &= 8.7794\,\frac{\ln d}{d}\left(1+O\left(\frac{\ln d}{d}\right)\right).\end{aligned}$$

Thus $c(d) = 8.7794\left(1 + O\left(\frac{\ln d}{d}\right)\right)$.

□

In simple terms, Theorem 5.3.2 states that $C_n \in \mathcal{F}_n^d$ satisfies the sausage conjecture if it has a certain interior property. Next, we wish to show that $C_n \in \mathcal{F}_n^d$ satisfies the sausage conjecture if it has a certain exterior property.

The following lemma is a variant of Theorem 1.5.1 for centrally symmetric convex, compact sets. We denote by $\tilde{E}^k$ the k-dimensional subspace spanned by the last k coordinate axes, and let $\tilde{B}^k = \tilde{E}^k \cap B^d$.

LEMMA 5.3.3 *Let $K \in \mathcal{K}^d$ be centrally symmetric, and $1 \leq m \leq d$. Then*

$$R_m(K) \leq \sqrt{d}\, r_m(K).$$

Proof: We may assume $\dim K = k \geq m$, otherwise the inequalities readily hold. Consider first the special case of the k-dimensional ellipsoid M with the equation

$$\frac{(x^1)^2}{a_1^2} + \cdots + \frac{(x^k)^2}{a_k^2} \leq 1, \quad x^{k+1} = \ldots = x^d = 0, \quad a_1 \geq \ldots \geq a_k > 0.$$

Then $a_m B^m \subset M$ and $r_m(M) \geq a_m$. On the other hand,

$$M \subset \{(x^1, \ldots, x^d) |\, (x^m)^2 + \cdots + (x^d)^2 \leq a_m^2\} = E^{m-1} + a_m \tilde{B}^{d-m+1},$$

which in turn yields $R_m(M) \leq a_m$, and hence $r_m(M) = R_m(M)$ by $R_m(M) \geq r_m(M) \geq a_m$.

We recall that by Theorem 1.5.4, there is a point x and a k-dimensional ellipsoid M such that

$$x + M \subset K \subset x + \sqrt{k}\, M.$$

It follows that

$$r_m(K) \geq r_m(x + M) = r_m(M),$$
$$R_m(K) \leq R_m(x + \sqrt{k}\, M) = R_m(\sqrt{k}\, M) \subset R_m(\sqrt{d}\, M),$$

which in turn yield that

$$R_m(K) \leq \sqrt{d}\, R_m(M) = \sqrt{d}\, r_m(M) \leq \sqrt{d}\, r_m(K).$$

$\square$

Remark: M. Henk proved recently that if a $K \in \mathcal{K}^d$ is centrally symmetric then

$$R_m(K) \leq \min\{m, d+1-m\} \cdot r_m(K).$$

His bound is better than ours if either m or $d+1-m$ is small. If m/d is bounded from 0 and 1, which is our case, then our estimate is better.

Since by Lemma 5.3.3, a lower bound for $R_{\varphi(d)}$ forces a lower bound for $r_{\varphi(d)}$, Theorem 1.5.1 and Theorem 5.3.2 yield

THEOREM 5.3.4 *Let $d \geq 5$ and $c(d)$ be the function in Theorem 5.3.2. Then there exist a bounded function $c_0(d)$ so that if $C_n \in \mathcal{F}_n^d$ and either*

a) $R_{\varphi(d)}(C_n) \geq c_0(d) \ln d$, *or*

b) C_n *is centrally symmetric and* $R_{\varphi(d)}(C_n) \geq c(d) \ln d/\sqrt{d}$, *then*

$$V(C_n + B^d) \geq V(S_n + B^d).$$

By Theorem 1.5.1, we may define $c_0(d) = \frac{\varphi(d)+1}{d} c(d)$ for $d \geq 11$. Then b) of Theorem 5.3.2 and $\varphi(d) \sim \frac{7}{12} d$ yield that

$$c_0(d) \sim \frac{3(2+\sqrt{2})}{2}\left(1 + O\left(\frac{\ln d}{d}\right)\right) = 5.1213\left(1 + O\left(\frac{\ln d}{d}\right)\right).$$

Notice that the lower bound in b) approaches 0 when $d \to \infty$. In Table 5.1, we considered the case of general C_n. In order to calculate R_φ, we used the maximum value of Δ_d by Theorem 1.5.3 and $2R_d(K) = \Delta_d(K)$ for $d = 6, \ldots, 10$, and Theorem 1.5.1 for $d = 11, \ldots, 48$. The anomalies around $d = 6$ and $d = 17$ are caused by the behavior of $r_{\varphi(d)}$.

5.4 Modifying Blichfeldt's function $\rho(x)$

The notions and the results below will be needed in the fifth section.

Denote by D^d_m, for $m = 1, \ldots, d$, the ellipsoid with equation

$$\frac{(x^1)^2}{2} + \cdots + \frac{(x^m)^2}{2} + (x^{m+1})^2 + \cdots + (x^d)^2 \leq 1,$$

and we define the function

$$\rho^d_m(x) = \begin{cases} 2 - x^2 & \text{if } x \in D^d_m, \\ 0 & \text{otherwise.} \end{cases}$$

In addition, set

$$\Phi^d_m = \int_{E^d} \rho^d_m(x)dx = \int_{D^d_m} \rho^d_m(x)dx.$$

If $d = m$, then D^d_m, ρ^d_m and Φ^d_m are $\sqrt{2}B^d$, ρ and Φ^d, respectively. Fix a $1 \leq m \leq d$ and a packing $\{x_1 + B^d, \ldots, x_n + B^d\}$ with $C_n = \operatorname{conv}\{x_1, \ldots, x_n\}$.

Since $\rho^d_m(y) \leq \rho(y)$ for any y, 5.5 yields that

$$\sum_{i=1}^{n} \rho^d_m(x - x_i) \leq 2$$

for any x. Using this inequality in the same way as 5.5 was used in 5.6 and replacing $\sqrt{2}B^d$ with D^d_m, results in

$$n \cdot \Phi^d_m \leq 2V(C_n + D^d_m). \tag{5.18}$$

LEMMA 5.4.1 *For m=1,...,d,*

$$\Phi_m^d = \frac{d-m+4}{d+2}\kappa_d 2^{m/2}.$$

Proof: For $m = d$, the statement is the same as 5.7, and hence assume $1 \leq m < d$. Then every $x \in D_m^d$ can be written as $x = (u, v)$, with $u \in E^m$ and $v \in \tilde{E}^{d-m}$. In terms of u and v, the equation of D_m^d is

$$u^2 + 2v^2 \leq 2,$$

with $u \in \sqrt{2}B^m$ and $v \in \frac{1}{\sqrt{2}}\sqrt{2-u^2}\tilde{B}^{d-m}$. Defining $p = \sqrt{2-u^2}$ for a $u \in \sqrt{2}B^m$ and using 1.2,

$$\begin{aligned}\int_{u+\tilde{E}^{d-m}} \rho_m^d(x)\,dx &= \int_{\frac{1}{\sqrt{2}}p\tilde{B}^{d-m}} p^2 - v^2\,dv \\ &= (d-m)\kappa_{d-m}\int_0^{\frac{p}{\sqrt{2}}}(p^2-s^2)s^{d-m-1}\,ds.\end{aligned}$$

The Fundamental Theorem of Calculus yields that

$$\begin{aligned}\int_{u+\tilde{E}^{d-m}} \rho_m^d(x)\,dx &= (d-m)\kappa_{d-m}\left(\frac{p^2}{d-m}\left(\frac{p}{\sqrt{2}}\right)^{d-m} - \frac{1}{d-m+2}\left(\frac{p}{\sqrt{2}}\right)^{d-m+2}\right) \\ &= (d-m)\kappa_{d-m}\frac{p^{d-m+2}}{\sqrt{2}^{d-m+2}}\left(\frac{2}{d-m} - \frac{1}{d-m+2}\right) \\ &= \kappa_{d-m}(2-u^2)^{\frac{d-m+2}{2}} \cdot 2^{\frac{m-d-2}{2}}\,\frac{d-m+4}{d-m+2}.\end{aligned}$$

We recall that by Fubini's theorem, for any integrable $f: E^d \to R$,

$$\int_{E^d} f(x)\,dx = \int_{E^m}\int_{\tilde{E}^{d-m}} f(u,v)\,dv\,du.$$

It follows by 1.2, that

$$
\begin{aligned}
\Phi_m^d &= \int_{\sqrt{2}B^m} \kappa_{d-m}(2-u^2)^{\frac{d-m+2}{2}} \cdot 2^{\frac{m-d-2}{2}} \frac{d-m+4}{d-m+2} du \\
&= \frac{d-m+4}{d-m+2}\kappa_{d-m} 2^{\frac{m-d}{2}-1} \cdot m\,\kappa_m \int_0^{\sqrt{2}} (2-s^2)^{\frac{d-m+2}{2}} s^{m-1}\, ds \\
&= \frac{d-m+4}{d-m+2}\kappa_{d-m}\, 2^{\frac{m-d}{2}-1} \cdot m\,\kappa_m \int_0^1 (2-2t)^{\frac{d-m+2}{2}} (2t)^{\frac{m-2}{2}}\, dt \\
&= \frac{m(d-m+4)}{d-m+2}\kappa_{d-m}\kappa_m\, 2^{\frac{m}{2}-1} \int_0^1 (1-t)^{\frac{d-m+2}{2}} t^{\frac{m}{2}-1}\, dt.
\end{aligned}
$$

We used the substitution $s^2 = 2t$, which satisfies $sds = dt$. Both the volume of the unit ball and the integral in t in the last line can be expressed with the help of the Γ function (see 1.4). With the basic identity $t\Gamma(t) = \Gamma(t+1)$,

$$
\begin{aligned}
\Phi_m^d &= \frac{m(d-m+4)}{d-m+2} \cdot \frac{\pi^{(d-m)/2}}{\Gamma(\frac{d-m}{2}+1)} \cdot \frac{\pi^{m/2}}{\Gamma(\frac{m}{2}+1)} \cdot 2^{\frac{m}{2}-1} \frac{\Gamma(\frac{d-m+2}{2}+1)\Gamma(\frac{m}{2})}{\Gamma(\frac{d}{2}+2)} \\
&= \frac{m(d-m+4)}{d-m+2} \cdot \frac{\pi^{d/2}}{(\frac{d}{2}+1)\Gamma(\frac{d}{2}+1)} \cdot \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}+1)} \cdot \frac{\Gamma(\frac{d-m+2}{2}+1)}{\Gamma(\frac{d-m+2}{2})} \cdot 2^{\frac{m}{2}-1} \\
&= \frac{m(d-m+4)}{d-m+2} \cdot \frac{2\kappa_d}{d+2} \cdot \frac{2}{m} \cdot \frac{d-m+2}{2} \cdot 2^{\frac{m}{2}-1} \\
&= \frac{d-m+4}{d+2}\, \kappa_d\, 2^{\frac{m}{2}}.
\end{aligned}
$$

□

Combining the previous lemma with 5.18 yields the following version of Blichfeldt's result:

LEMMA 5.4.2 *For m=1,...,d and* $C_n \in \mathcal{F}_n^d$,

$$
V(C_n + D_m^d) \geq n\, \frac{d-m+4}{2(d+2)}\, \kappa_d\, 2^{m/2}. \tag{5.19}
$$

5.5 The relative width of C_n

Let $C_n \in \mathcal{F}_n^d$. In Theorem 5.3.4, to ensure that C_n satisfy the Sausage Inequality 5.4, we had to assume that $R_{\varphi(d)}(C_n)$ has order of at least $\ln d$. Unfortunately, $\ln d$ is large for large d. So we verify 5.4 by considering instead the relative width of C_n for which we give a much smaller lower bound. In other words, if C_n would serve as a counterexample for the Sausage Conjecture then it must be very flat in its affine hull. The method of Blichfeldt has again the central role, only with $\rho_m^d(x)$. The modification has no effect on the order of the lower bound. For $m \sim \varphi(d)$ and large d, our estimate is about the half of the lower bound which we would obtain from $\rho(x)$.

For $1 \leq m \leq d$, let T_1 be the linear transformation with diagonal matrix

$$\begin{array}{r l} \text{m}\left\{\begin{array}{l} \\ \\ \\ \end{array}\right. & \left[\begin{array}{cccccc} 1 & & & & & \\ & \ddots & & & 0 & \\ & & 1 & & & \\ & & & \frac{1}{\sqrt{2}} & & \\ & 0 & & & \ddots & \\ & & & & & \frac{1}{\sqrt{2}} \end{array}\right] . \\ \text{d-m}\left\{\begin{array}{l} \\ \\ \\ \end{array}\right. & \end{array}$$

LEMMA 5.5.1 *Let $1 \leq m \leq d$ and $K \in \mathcal{K}^d$ with $K \subset E^m$ and $r = r_m(K)$. Then*

$$V(K + B^d) \geq \left(\frac{r+1}{r+\sqrt{2}}\right)^m V(K + D_m^d). \tag{5.20}$$

Proof: Notice that $T_1(K) = K$ and $D_m^d = T_1(\sqrt{2}B^d)$ imply

$$K + D_m^d = T_1(K) + T_1(\sqrt{2}B^d) = T_1(K + \sqrt{2}B^d).$$

Then Lemma 5.3.1 yields that

$$\begin{aligned} V(K+D_m^d) &= \det T_1\, V(K+\sqrt{2}B^d) \\ &\le \left(\frac{1}{\sqrt{2}}\right)^{d-m} 2^{\frac{d-m}{2}} \left(\frac{r+\sqrt{2}}{r+1}\right)^m V(K+B^d) \\ &= \left(\frac{r+\sqrt{2}}{r+1}\right)^m V(K+B^d). \end{aligned}$$

□

The last lemma we need is a rather technical one.

LEMMA 5.5.2 *If $m \ge 10$ and $m \le d \le 2m-1$, then*

$$\frac{\kappa_{m-1}}{\kappa_m}(m+2) \ge \frac{\kappa_{d-1}}{\kappa_d}\frac{4(d+2)}{d-m+4}.$$

Proof: If $m = d$ then the assertion is true and hence assume that $m+1 \le d$. By means of 1.7, it sufficient to prove that

$$\sqrt{\frac{m}{2\pi}}(m+2) > \sqrt{\frac{d+1}{2\pi}}\frac{4(d+2)}{d-m+4},$$

which is equivalent to

$$\frac{d-m+4}{4} > \sqrt{\frac{d+1}{m}} \cdot \frac{d+2}{m+2}.$$

Defining $k = d-m$, the last inequality may be written as

$$1+\frac{k}{4} > \sqrt{1+\frac{k+1}{m}}\left(1+\frac{k}{m+2}\right),$$

where $1 \le k \le m-1$ and $m \ge 10$. If $1 \le k \le 7$, then as

$$\sqrt{1+t} < 1+\frac{t}{2}$$

for positive t, we have that

$$\begin{aligned}
\sqrt{1+\frac{k+1}{m}}\left(1+\frac{k}{m+2}\right) &\le \sqrt{1+\frac{2k}{m}}\left(1+\frac{k}{m+2}\right) < \left(1+\frac{k}{10}\right)\left(1+\frac{k}{12}\right)\\
&\le 1+\frac{k}{10}+\frac{k}{12}+\frac{7}{10}\cdot\frac{k}{12} = 1+\frac{29}{120}k\\
&< 1+\frac{k}{4}.
\end{aligned}$$

If $k \ge 8$ then

$$\sqrt{1+\frac{k+1}{m}}\left(1+\frac{k}{m+2}\right) \le \sqrt{2}\cdot 2 = 2.8284 < 1+\frac{k}{4}.$$

□

Recall that for $d \ge 5$,

$$\psi(d) = \begin{cases} \min\{d, 10\} & \text{if } 5 \le d \le 18 \\ \left[\frac{7}{12}(d-1)\right]+1 & \text{if } d \ge 19, \end{cases}$$

and that ii) and iii) from the first section state that if $r_{\psi(d)}(C_n) = 0$ for $C_n \in \mathcal{F}_n^d$ then C_n satisfies the Sausage Inequality. The preparations above enable us to prove

THEOREM 5.5.3 *Let $m \ge 5$, $d \ge 5$ and $\psi(d) \le m \le d$.*

a) There exists a function $\tilde{c}(m)$ such that for $C_n \in \mathcal{F}_n^d$ with $m = dimC_n$, if $r_m(C_n) \ge \tilde{c}(m)\ln m/m$ then $V(C_n+B^d) \ge V(S_n+B^d)$.

b) $\tilde{c}(m) = (3+1.5\sqrt{2})\left(1+O\left(\frac{\ln m}{m}\right)\right) = 5.1213\left(1+O\left(\frac{\ln m}{m}\right)\right).$

Remark: The very same reason, as quoted before Theorem 5.3.2 does not allow us to fill the gap between this Theorem and the conditions ii) and iii). Now the ratio $V(C_n+B^d)$ over $V(C_n+D_m^d)$ becomes too small if $r_m(C_n)$ is arbitrarily small.

Proof: If $d = 5, \ldots, 10$, and hence $m = d$, then we may choose $\tilde{c}(m) = c(m)$, where $c(m)$ is the function of Theorem 5.3.2. So we assume $d \geq 11$. It follows from Lemma 5.4.2 and Lemma 5.5.1 that for $r = r_m(C_n)$,

$$\begin{aligned} V(C_n + B^d) &\geq \left(\frac{r+1}{r+\sqrt{2}}\right)^m V(C_n + D^d_m) \\ &\geq \left(\frac{r+1}{r+\sqrt{2}}\right)^m \frac{d-m+4}{2(d+2)} \kappa_d 2^{m/2} n \\ &= \left(\frac{\sqrt{2}r+\sqrt{2}}{r+\sqrt{2}}\right)^m \frac{d-m+4}{2(d+2)} \kappa_d \cdot n. \end{aligned}$$

Since $V(S_n + B^d) < 2\kappa_{d-1} n$, the inequality

$$\left(\frac{\sqrt{2}r+\sqrt{2}}{r+\sqrt{2}}\right)^m \frac{d-m+4}{2(d+2)} \kappa_d \geq 2\kappa_{d-1}$$

yields $V(C_n + B^d) \geq V(S_n + B^d)$. The inequality is equivalent to

$$\left(\frac{\sqrt{2}r+\sqrt{2}}{r+\sqrt{2}}\right)^m \geq \frac{\kappa_{d-1}}{\kappa_d} \frac{4(d+2)}{d-m+4}. \tag{5.21}$$

If $d \geq 19$ then

$$\psi(d) = \left[\frac{7}{12}(d-1)\right] + 1 = \left[\frac{7}{12}d + \frac{5}{12}\right].$$

We note that

$$\begin{aligned} 2\left[\frac{7}{12}d + \frac{5}{12}\right] - 1 &\geq 2\left(\frac{7}{12}d + \frac{5}{12}\right) - 2 - 1 = \frac{7d}{6} - \frac{13}{6} \\ &= d + \frac{d}{6} - \frac{13}{6} > d, \end{aligned}$$

and so $\psi(d) \leq m \leq d$ yields that $d \leq 2\psi(d) - 1 \leq 2m - 1$ for $d \geq 19$. Since $10 = \psi(d) \leq m$ for $d = 11, \ldots, 18$, we have $m \leq d \leq 2m - 1$ for $d \geq 11$. By Lemma 5.5.2, for $d \geq 11$ the inequality 5.21 follows from

$$\left(\frac{\sqrt{2}r+\sqrt{2}}{r+\sqrt{2}}\right)^m \geq \frac{\kappa_{m-1}}{\kappa_m}(m+2); \tag{5.22}$$

that is, from an inequality which contains only m as a parameter.

Recall the function

$$h(r) = \frac{\sqrt{2}r + \sqrt{2}}{r + \sqrt{2}}$$

from the proof of Theorem 5.3.2. It is monotonic for non-negative r and $h(0) = 1$, $\lim_{r\to\infty} h(r) = \sqrt{2}$. In addition, 1.7 and $m \geq 7$ yield

$$\frac{\kappa_{m-1}}{\kappa_m}(m+2) > \sqrt{\frac{m}{2\pi}}(m+2) > 1.$$

Hence the existence of a non-negative r satisfying 5.22 is equivalent to the inequality

$$(\sqrt{2})^m > \frac{\kappa_{m-1}}{\kappa_m}(m+2). \tag{5.23}$$

Let $\tilde{c}(m) = r(m)\frac{m}{\ln m}$. For $m = 10, \ldots, 48$, we check the condition above by computer, search for the minimal r satisfying 5.22 and calculate $\tilde{c}$. The results are contained in Table 5.2 (we refer later to the value Δ_m which is the maximum allowed by Theorem 1.5.3 for given r_m). If $m = 6, \ldots, 10$ then we used the corresponding values from Table 5.1.

Assume now $m > 48$ and define

$$\tau(m) = e^{\frac{3\ln m}{2m}} - 1 = \frac{3}{2}\frac{\ln m}{m}\left(1 + O\left(\frac{\ln m}{m}\right)\right).$$

This function satisfies

$$(1+\tau)^m = m^{3/2} > \sqrt{\frac{m+1}{2\pi}}(m+2) > \frac{\kappa_{m-1}}{\kappa_m}(m+2)$$

by $m > 48$ and 1.7. On the other hand, since $\ln m/m$ is monotonically decreasing,

$$1 + \tau(m) < e^{\frac{3\ln 48}{2\cdot 48}} = 1.12859 < \sqrt{2}.$$

m	lower bound for r_m	lower bound for $\tilde{c}$	lower bound for Δ_m
6	1690.3813	5660.5187	8366.9608
7	19.6265	70.6022	103.8537
8	9.2130	35.4443	52.4416
9	6.0494	24.7789	36.2966
10	4.5203	19.6318	28.7083
11	3.6191	16.6023	24.0067
12	3.0247	14.6069	21.0182
13	2.6030	13.1931	18.7709
14	2.2882	12.1388	17.1617
15	2.0440	11.3222	15.8333
16	1.8491	10.6707	14.8185
17	1.6897	10.1388	13.9339
18	1.5569	9.6962	13.2297
19	1.4446	9.3219	12.5940
20	1.3482	9.0013	12.0731
21	1.2647	8.7235	11.5913
22	1.1915	8.4804	11.1879
23	1.1268	8.2658	10.8083
24	1.0692	8.0750	10.4851
25	1.0176	7.9041	10.1769
26	0.9711	7.7502	9.9111
27	0.9290	7.6109	9.6549
28	0.8906	7.4841	9.4315
29	0.8555	7.3682	9.2146
30	0.8233	7.2619	9.0236
31	0.7935	7.1640	8.8370
32	0.7660	7.0735	8.6713
33	0.7405	6.9896	8.5087
34	0.7168	6.9117	8.3633
35	0.6947	6.8390	8.2200
36	0.6740	6.7711	8.0910
37	0.6546	6.7075	7.9635
38	0.6363	6.6478	7.8482
39	0.6192	6.5916	7.7338
40	0.6030	6.5387	7.6298
41	0.5877	6.4888	7.5265
42	0.5732	6.4415	7.4321
43	0.5595	6.3968	7.3382
44	0.5465	6.3544	7.2520
45	0.5341	6.3141	7.1661
46	0.5223	6.2758	7.0870
47	0.5111	6.2393	7.0080
48	0.5003	6.2045	6.9351

Table 5.2

Thus the properties of $h(r)$ yield $r(m)$ with

$$\tau(m) = \frac{(\sqrt{2}-1)r(m)}{r(m)+\sqrt{2}},$$

and if $r \geq r(m)$ for a fixed m, then it satisfies 5.22. With the help of 5.16, one can write

$$\begin{aligned} r(m) &= (2+\sqrt{2})\frac{3\ln m}{2m}\left(1+O\left(\frac{\ln m}{m}\right)\right)(1+O(\tau)) \\ &= (3+1.5\sqrt{2})\frac{\ln m}{m}\left(1+O\left(\frac{\ln m}{m}\right)\right)\left(1+O\left(\frac{\ln m}{m}\right)\right) \\ &= 5.1213\left(1+O\left(\frac{\ln m}{m}\right)\right)\frac{\ln m}{m}, \end{aligned}$$

and thus $\tilde{c} = 5.1213\left(1+O\left(\frac{\ln m}{m}\right)\right)$.

$\square$

COROLLARY 5.5.4 *Let $d \geq 5$, $\psi(d) \leq m \leq d$ and $C_n \in \mathcal{F}_n^d$ with $dim C_n = m$, and $\tilde{c}(m)$ as the function of the previous theorem. If C_n satisfies $\Delta_m(C_n) \geq 2.1\tilde{c}(m)\frac{\ln m}{\sqrt{m}}$ then $V(C_n+B^d) \geq V(S_n+B^d)$.*

Proof: According to Theorem 1.5.3,

$$r_m(C_n) \geq \begin{cases} \frac{\sqrt{m+2}}{2(m+1)}\Delta_m(C_n) & \text{if m is even,} \\ \frac{1}{2\sqrt{m}}\Delta_m(C_n) & \text{if m is odd.} \end{cases}$$

For even $m \geq 6$,

$$\begin{aligned} \frac{\sqrt{m+2}}{2(m+1)}\cdot 2.1\frac{\ln m}{\sqrt{m}} &= \frac{2.1}{2}\sqrt{\frac{m^2+2m}{m^2+2m+1}}\,\frac{lnm}{m} \\ &\geq \frac{2.1}{2}\sqrt{\frac{6^2+2\cdot 6}{6^2+2\cdot 6+1}}\,\frac{lnm}{m} \\ &> \frac{\ln m}{m}. \end{aligned}$$

Thus if $\Delta_m(C_n) \geq 2.1\tilde{c}(m)\frac{\ln m}{\sqrt{m}}$ then $r_m(C_n) \geq \tilde{c}(m)\frac{\ln m}{m}$, and the corollary follows from the Theorem above.

□

5.6 Conclusion

Assume that $C_n \in \mathcal{F}_n^d$ provides a counterexample for the Sausage Conjecture. It was known, according to ii) and iii) of the first section, that the dimension of C_n can not be too small with respect to d; that is, it is at least $\psi(d)$ for $d \geq 5$. We have strengthened the counterpart in iv) in the following manner: The relative width of C_n should be at most $O(\ln m/m)$ for $d \geq 5$, where $m = \dim C_n$. In addition if $d \neq 18$, then $R_{\psi(d)} < O(\ln d)$, which means that the shape of C_n is not very far from being at most $\psi(d) - 1$ dimensional for large n.

Probably the method used in this Chapter can not be streched much further. The case where we do not succeed is if C_n is 'very thin'. In this case, the quotient of the volume of $C_n + B^d$ over the volume of $C_n + \sqrt{2}B^d$ (or $C_n + \sqrt{2}D_m^d$) is so small that it can not overcome the otherwise not very significant error of Blichfeldt's estimate (see also the explanation at the beginning of the third section). The improvements on the function of Blichfeldt are not significant enough to improve even on the asymptotic behavior of our estimates. Recently, with the help of code theory, a breakthrough took place with respect to infinite packings. Unfortunately, those considerations do not seem to yield any information on 'non-fat' finite packings.

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