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Singular Momentum Mappings, Presymplectic Dynamics and Gauge Groups

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled, "Singular Momentum Mappings, Presymplectic Dynamics and Gauge Groups" submitted by George W. Patrick in partial fulfillment of the requirements for the degree of Master of Science.

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ABSTRACT

Precise results are obtained on the existence and uniqueness problems for finite dimensional presymplectic systems. The second order problem for degenerate lagrangian dynamics is shown to be physically insignificant for a large class of lagrangians. This observation permits a proof of the equivalence of a degenerate lagrangian system and its canonical formulation. Hypotheses are provided which are sufficient to imply that the set of points of the canonical phase space which admit evolution is the zero level of a momentum mapping of the gauge group. The canonical evolution is displayed as a set of constrained hamiltonian evolutions. An example originating in the theory of Yang-Mills fields is discussed.

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INTRODUCTION

Let M be a Banach manifold. A presymplectic form on M is a closed, possibly degenerate, two form $\bar{\omega}$ on M. A presymplectic system is a triple $(M,\bar{\omega},\bar{H})$, where $\bar{\omega}$ is a presymplectic form and $\bar{H}$ is a smooth function on M. Any presymplectic system defines an evolution on M: points of M evolve along smooth curves c such that

$$\frac{dc}{dt} \lrcorner \bar{\omega} = d\bar{H} \circ c \ . \qquad 1$$

Typically, the set of points M_e of M through which there exists such a curve is proper; that is, not every point of M admits evolution. The determination of M_e is the existence problem for the evolution defined by equation (1). In several important examples, M_e is not a submanifold of M. If $m \in M_e$, there may be many essentially different solutions to equation (1) through m. Points of M that evolve concurrently from the same point may be considered physically equivalent. The smallest equivalence relation on M_e generated by this notion is called the gauge equivalence relation, and is denoted by R_g. The determination of R_g is the uniqueness question for the evolution defined by equation (1).

Dirac [1950] and Gotay-Nester-Hinds [1978] address the existence problem by defining an algorithm - the constraint algorithm - that proceeds under the assumption that certain intermediate constructs are imbedded submanifolds of M. If this algorithm may be applied, and if it terminates, then it constructs the subset M_e , and M_e is an imbedded submanifold of M by hypothesis. They do not directly attend to the case where M_e is not a submanifold of M. In Gotay-Nester [1979a] one finds an algorithm - the gauge vector field algorithm - which accepts certain vector fields whose flows respect the gauge relation R_g and purports to generate other vector fields with this same property. Absent from the literature is any result that might aid one in precisely determining the gauge relation of the evolution.

The primary goal of the first five chapters of this thesis is the construction of a formalism that can accomodate the singular nature of the subset M_e. The formalism identifies certain natural hypotheses that are sufficient to effect a resolution of the existence problem for the evolution. Attention is restricted to the consideration of systems that are first class in the sense of Dirac [1950], but this notion by itself is too weak to be of utility. Included are sufficient conditions under which the gauge vector field algorithm may be used to determine the gauge relation of the evolution. This latter analysis presumes that M_e is a submanifold of M, but chapter (7) shows one way to proceed when this is not the case. Except for preliminary material, the results apply only to finite dimensional systems.

To a large degree, classical physics is the study of lagrangian dynamics. In the language of modern differential geometry, lagrangian dynamics has the following form. The configuration space of the system is a smooth manifold Q, and the lagrangian is a smooth function L on TQ. The Legendre transformation is the map $FL : TQ \to T^*Q$ defined by

$$FL(v_q)w_q = \frac{d}{dt}\bigg|_{t=0} L(v_q + tw_q) .$$

The energy function E is defined by

$$E(v_q) = FL(v_q)v_q - L(v_q) .$$

The pull back by FL of the canonical symplectic form ω_0 on T^*Q is the Lagrange two form ω_L. Points of TQ evolve along smooth curves c which satisfy

$$\frac{dc}{dt} \lrcorner \; \omega_L = dE \circ c . \qquad 2$$

The lagrangian L is called regular if FL is a local diffeomorphism. In finite dimensions, this is equivalent to the condition that the matrix $\left[\dfrac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} \right]$, computed in any natural chart of TQ, is nondengenerate. Ignoring technical difficulties which might arise when considering infinite dimensional systems, if L is regular, then equation (2) implies a smooth, unique, well defined evolution on all of TQ. A curve in TQ is called second order if it is the derivative of its projection onto Q. Regularity of L is sufficient to imply that any curve in TQ which satisfies equation (2) is second order. L is called hyperregular if FL is a diffeomorphism. If L is hyperregular, one may construct the canonical formulation of the lagrangian system as follows. The hamiltonian is the function $H : T^*Q \to \mathbb{R}$ defined by $H = E \circ FL^{-1}$. Points of T^*Q evolve along smooth curves c which satisfy

$$\frac{dc}{dt} \lrcorner\, \omega_0 = dH \circ c\ . \qquad 3$$

The lagrangian system and its canonical formulation are equivalent in the sense that they are in bijective correspondence via the diffeomorphism FL.

The lagrangian formulation of certain field theories necessitates the consideration of lagrangians which are not regular. A weaker condition worthy of study is that of semiregularity: L is called semiregular if FL is a subimmersion and if the level sets of FL are connected. In finite dimensions, L is semiregular if the matrix $\left[\dfrac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\right]$ has constant rank on connected components of TQ, and the level sets of FL are connected. Semiregularity is sufficient to imply that the triple (TQ,ω_L,E) is a presymplectic system. Under the assumption that FL is an open or closed map onto its image, the canonical formulation of the lagrangian system is the presymplectic system $(M_0, i^*\omega_0, \bar{H})$, where $M_0 = \text{Image}(FL)$, $i : M_0 \to T^*Q$ is the inclusion map and $\bar{H} : M_0 \to \mathbb{R}$ is the unique smooth function on M_0 such that $E = \bar{H} \circ FL$.

Gotay-Nester [1979b] deal with the issue of the equivalence of a lagrangian system and its canonical formulation in the following way. The constraint algorithm is applied to each presymplectic system and is assumed to generate imbedded submanifolds $M_e \subseteq TQ$ and $M'_e \subseteq T^*Q$. Then $FL|M_e$ is a submersion from M_e to M'_e. It is shown that if X and Y are vector fields on TQ and T^*Q respectively, and if X and Y are FL related, then X satisfies the equation $X \lrcorner \omega_L = dE$ if and only if Y satisfies the equation $Y \lrcorner i^*\omega_0 = d\bar{H}$. In a companion paper (Gotay-Nester [1980]), they note that when L is not regular, there may exist solutions to equation (2) that are not second order. This observation gives rise to the second order problem; that is, the identification of those points of TQ which admit second order evolution. They proceed to give conditions that are sufficient to imply the existence of a submanifold of points that admit second order evolution.

When L is not regular, the Legendre transformation is not injective, and the question arises as to whether or not one is losing physically important information in passing to the canonical formulation. The first result of chapter (6) shows that this is not so when L is semiregular: points of TQ which admit evolution and lie in the same level set of FL are gauge equivalent. This gauge freedom is used to settle the second order problem by showing that any evolution curve is gauge equivalent to a second order evolution curve. Chapter (6) concludes with a proof of the exact equivalence of a lagrangian system and its canonical formulation: every curve in TQ which satisfies equation (2) is mapped by FL to a curve in T^*Q which satisfies equation (3), and every curve in T^*Q which satisfies equation (3) is the image by FL of some curve in TQ which satisfies equation (2).

The final chapter of this thesis is an analysis of the canonical formulation of lagrangian systems. The existence of a primary gauge group and a function H on T^*Q such that $H|M_0 = \bar{H}$ allows an extension of the canonical evolution to an evolution on all of T^*Q. This extension is discussed in the context of quadratic lagrangians. The methods of chapter (5) may be used to compute the gauge relation of the extended evolution. When the set of points $\bar{M}_e$ of T^*Q which admit canonical evolution is strongly first class, the gauge relation of the extended evolution and that of the canonical evolution coincide on $\bar{M}_e$. The gauge vector field algorithm is used to give a definition of the gauge group. Under mild nondegeneracy assumptions, $\bar{M}_e$ is shown to be the zero level of a momentum mapping of the gauge group. In a step which may be important for quantization, the canonical evolution is displayed as a set of hamiltonian evolutions. The chapter is concluded with an example that originates in the theory of Yang-Mills fields.

CHAPTER 1

Symplectic Geometry and Hamiltonian Dynamics

1 Symplectic geometry, and its presymplectic generalization, provides a context within which elements of the structure of many physical systems may be defined and discussed. The purpose of this chapter is to provide some of the basic definitions of symplectic geometry and hamiltonian dynamics. With few exceptions, the notation follows that of Abraham-Marsden [1978].

2 Let $\mathbb{E}$ be a Banach space, and $B : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ be bilinear. Define the linear map $B^\flat : \mathbb{E} \to \mathbb{E}^*$ by $B^\flat(e)f = B(e,f)$. Call B weakly nondegenerate if $B^\flat$ is an injection, and nondegenerate if $B^\flat$ is a bijection. If B is nondegenerate, then the inverse of $B^\flat$ is denoted by $B^\#$. B is called topologically closed if $B^\flat$ is a closed map. If B is weakly nondegenerate, $\mathbb{E}$ is reflexive and B is topologically closed, then an application of the Hahn-Banach theorem shows that B is nondegenerate.

3 Let B be antisymmetric. If $S \subseteq \mathbb{E}$, define the subspace

$$S^{B\perp} = \{e \in \mathbb{E} \; ; \; B(e,s) = 0 \text{ for all } s \in S\}.$$

Let $\mathbb{E}$ be reflexive and $\mathbb{F}$ be a closed subspace of $\mathbb{E}$. Then, if B is topologically closed,

$$\begin{aligned} B^\flat(\mathbb{F}) &= \mathrm{cl}(B^\flat(\mathbb{F})) \\ &= \{\alpha \in \mathbb{E}^* \ ; \ \alpha f = 0 \text{ for all } f \text{ such that } B^\flat(\mathbb{F})f = 0\} \\ &= \{\alpha \in \mathbb{E}^* \ ; \ \alpha f = 0 \text{ for all } f \text{ such that } B(\mathbb{F},f) = 0\} \\ &= \{\alpha \in \mathbb{E}^* \ ; \ \alpha f = 0 \text{ for all } f \in \mathbb{F}^{B\perp}\} \\ &= \{\alpha \in \mathbb{E}^* \ ; \ \alpha(\mathbb{F}^{B\perp}) = 0\}. \end{aligned}$$

4 A (weak) symplectic vector space is a pair $(\mathbb{E},\omega)$, where $\mathbb{E}$ is a Banach space and $\omega : \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ is bilinear, (weakly) nondegenerate and antisymmetric. If $\mathbb{F}$ is a closed subspace of $\mathbb{E}$ one says that:

1. $\mathbb{F}$ is isotropic if $\mathbb{F} \subseteq \mathbb{F}^{\omega\perp}$;
2. $\mathbb{F}$ is coisotropic if $\mathbb{F}^{\omega\perp} \subseteq \mathbb{F}$;
3. $\mathbb{F}$ is symplectic if $\mathbb{F} \cap \mathbb{F}^{\omega\perp} = 0$.

If $\mathbb{F}$ is symplectic, then ω restricted to $\mathbb{F} \times \mathbb{F}$ is weakly symplectic.

5 Let $(\mathbb{E},\omega)$ be a finite dimensional symplectic vector space. Let $S,T \subseteq \mathbb{E}$. The proof of the following facts may be found in Abraham-Marsden [1978: 403].

1. $S \subseteq T$ implies $T^{\omega\perp} \subseteq S^{\omega\perp}$.
2. $S^{\omega\perp} \cap T^{\omega\perp} = (S + T)^{\omega\perp} = (\mathrm{span}(S \cup T))^{\omega\perp}$.
3. $(\mathrm{span}(S) \cap \mathrm{span}(T))^{\omega\perp} = S^{\omega\perp} + T^{\omega\perp}$.
4. $(S^{\omega\perp})^{\omega\perp} = \mathrm{span}(S)$.
5. $\dim(\mathbb{E}) = \dim(\mathrm{span}(S)) + \dim(S^{\omega\perp})$.

6 Let P be a Banach manifold. A (weak) symplectic form on P is a closed two form ω such that $\omega(p)$ is (weakly) nondegenerate for all $p \in P$. A (weak) symplectic manifold is a pair (P,ω), where P is a Banach manifold and ω is a (weak) symplectic form. If (P,ω) is a

(weak) symplectic manifold, denote by $\omega^\flat$ the vector bundle monomorphism defined by $\omega^\flat(v_p) = (\omega(p))^\flat v_p$. If ω is a symplectic form, then $\omega^\flat$ is a vector bundle isomorphism, and its inverse is denoted by $\omega^\#$.

7 If (P,ω) and (P',ω') are two (weak) symplectic manifolds, then a smooth map $f : P \to P'$ is called a symplectomorphism if $f^*\omega' = \omega$. If P is finite dimensional, with dimension 2n, then $\omega^n = \omega \wedge \omega \wedge \ldots \wedge \omega$ (n times) is a volume form on P, so that any finite dimensional symplectic manifold is orientable. In finite dimensions, then, any symplectomorphism is a local diffeomorphism, since it will send a volume form to a volume form.

8 Let Q be a Banach manifold modelled on a Banach space $\mathbb{E}$. An important example of a weak symplectic manifold is the cotangent bundle of Q. Define the canonical one form θ_0 on T^*Q by

$$\theta_0(\alpha_q)v_{\alpha_q} = \alpha_q\left[T\tau_Q^*(v_{\alpha_q})\right] ,$$

where $\tau_Q^* : T^*Q \to Q$ is the canonical projection. It is easy to see that in a natural chart of T^*Q with range $U \times \mathbb{E}^*$,

$$\theta_0(u,\alpha)(u,\alpha,e,\beta) = \alpha(e) .$$

Define the canonical two form ω_0 on T^*Q by $\omega_0 = -d\theta_0$. A simple computation shows that

$$\omega_0(u,\alpha)[(u,\alpha,e_1,\beta_1),(u,\alpha,e_2,\beta_2)] = \beta_2(e_1) - \beta_1(e_2) ,$$

and it follows that ω_0 is weakly symplectic.

9 If $\mathbb{E}$ is reflexive then ω_0 is symplectic, since the expression

$$\omega_0^\flat(u,\alpha)(u,\alpha,e,\beta) = (u,\alpha,-\beta,e) \in U \times \mathbb{E}^* \times \mathbb{E}^* \times \mathbb{E}^{**}$$

shows that $\omega_0^\flat$ is a surjection if $\mathbb{E}$ is reflexive. If Q is finite dimensional with dimension n, then

$$\theta_0 = p_i dq^i$$
$$\omega_0 = dq^i \wedge dp_i \;,$$

in natural coordinates $(q^1,\ldots,q^n,p_1,\ldots,p_n)$ on T^*Q.

10 If (P,ω) is a (weakly) symplectic manifold, and $i : Q \to P$ is an immersion, then one says that Q is an isotropic (coisotropic, symplectic) immersed submanifold if $Ti(T_qQ)$ is an isotropic (coisotropic,symplectic) subspace of $(T_{i(q)}P,\omega(i(q)))$ for all $q \in Q$. The same terminology is used for imbedded submanifolds of P and vector bundles over submanifolds of P. If Q is symplectic, then $(Q,i^*\omega)$ provides another example of a weakly symplectic manifold.

11 In a natural chart of T^*Q, the canonical symplectic form ω_0 is constant. The theorem of Darboux shows that, for any symplectic manifold, there are charts about any point with this property. The proof is included here because a refinement of this result is needed in chapter (3).

12 <u>Theorem</u> (Darboux). Let (P,ω) be a symplectic manifold and let $p \in P$. Then there is a chart about p in which the local representative of ω is constant.

Proof

13 Let $U \subseteq \mathbb{E}$ be an open set containing 0, and let ω_0 and ω_1 be two symplectic forms such that $\omega_0(0) = \omega_1(0)$. It suffices to show that there is a smooth diffeomorphism $F : B \subseteq U \to \mathbb{E}$, defined on an open ball B containing 0, such that $F(0) = 0$ and $\omega_0 | B = F^*\omega_1$.

...14 Let $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$. Since ω_t is nondegenerate at $t = 0$, there is an open ball B about 0 on which ω_t is nondegenerate for all $t \in [0,1]$. By the Poincaré lemma, $\omega_1 - \omega_0 = d\alpha$ for some one form α on B, and one can assume that $\alpha(0) = 0$.

...15 Define a smooth, time dependant vector field X_t on B by $X_t \lrcorner \omega_t = -\alpha$. Since $X_t(0) = 0$, one may restrict the ball B so that the flow F_t of X_t is defined on B for time at least one. Then

$$\begin{aligned}
\left.\frac{d}{dt}\right|_{t_0} (F_t^*\omega_t) &= \left.\frac{d}{dt}\right|_{t_0} (F_{t_0}^*\omega_t) + \left.\frac{d}{dt}\right|_{t_0} F_t^*\omega_{t_0} \\
&= F_{t_0}^*(\omega_1 - \omega_0) + F_{t_0}^* L_{X_{t_0}} \omega_{t_0} \\
&= F_{t_0}^*(\omega_1 - \omega_0) + F_{t_0}^*(d(X_{t_0} \lrcorner\, \omega_{t_0}) + X_{t_0} \lrcorner\, d\omega_{t_0}) \\
&= F_{t_0}^*(\omega_1 - \omega_0) + F_{t_0}^*(-d\alpha) \\
&= 0 \,.
\end{aligned}$$

Therefore $F_1^*\omega_1 = F_0^*\omega_0 = \omega_0$, so F_1 satisfies the conditions of paragraph (13). ∎

16 If P is finite dimensional, of dimension 2n, then a symplectic chart of P is a chart with coordinates $q^1,\ldots,q^n,p_1,\ldots,p_n$ in which $\omega = dq^i \wedge dp_i$. After a linear transformation, the Darboux theorem shows that any point of P is contained in a symplectic chart.

17 If $f : P \to \mathbb{R}$ is a smooth function, then the hamiltonian vector field of f is the smooth vector field on P defined by

$$\begin{aligned} \text{hmlt\_vf}(f)(p) &= \text{hmlt\_vf}(f,p) \\ &= X_f(p) \\ &= \omega^{\#}(df(p)) \ . \end{aligned}$$

Thus, X_f is uniquely defined by the equation $X_f \lrcorner\, \omega = df$. Note that if P is connected, and f and g are smooth functions on P, then $X_f = X_g$ if and only if f and g differ by a constant. Indeed, if $X_f = X_g$ then

$$df = X_f \lrcorner\, \omega = X_g \lrcorner\, \omega = dg \ ,$$

so f and g differ by a constant. On the other hand, if f and g differ by a constant, then $df = dg$, so $X_f = X_g$.

18 If $H : P \to \mathbb{R}$ is smooth, then X_H defines a smooth flow on the phase space P. Points of P evolve along smooth curves c such that

$$\frac{dc}{dt} \lrcorner\, \omega = dH \circ c \ .$$

If P is finite dimensional, then in the symplectic chart of paragraph (16),

$$X_H = \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \ ,$$

so that c will satisfy Hamilton's equations:

$$\frac{d}{dt} (q^i \circ c(t)) = \frac{\partial H}{\partial p_i} \circ c(t) \ ,$$

$$\frac{d}{dt} (p_i \circ c(t)) = - \frac{\partial H}{\partial q^i} \circ c(t) \ .$$

H is conserved along c, since

$$\frac{d}{dt}(H \circ c(t)) = dH(c(t))\frac{dc}{dt}(t)$$
$$= dH(c(t))\; X_H(c(t))$$
$$= \omega(X_H(c(t)), X_H(c(t)))$$
$$= 0 \;.$$

One calls the triple (P,ω,H) a hamiltonian system.

19 If f and g are smooth functions on P, the Poisson bracket of f and g is defined by

$$\{f,g\} = \omega(X_f, X_g) = df(X_g) = -dg(X_f) \;.$$

The Poisson bracket is bilinear, antisymmetric and a derivation in each argument. In the symplectic chart of paragraph (16),

$$\{f,g\} = df(X_g) = \left[\frac{\partial f}{\partial q^i}\, dq^i + \frac{\partial f}{\partial p_i}\, dp_i\right]\left[\frac{\partial g}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial g}{\partial q^i}\frac{\partial}{\partial p_i}\right]$$
$$= \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i} \;.$$

20 The Poisson bracket satisfies two important identities. For the first, note that if f and g are smooth functions on P, then

$$[X_f, X_g] \lrcorner\, \omega = L_{X_f}(X_g \lrcorner\, \omega) - X_g \lrcorner\, (L_{X_f}\omega)$$
$$= L_{X_f}(dg) - X_g \lrcorner\, (d(X_f \lrcorner\, \omega) + X_f \lrcorner\, d\omega)$$
$$= L_{X_f}(dg) - X_g \lrcorner\, (d(df))$$
$$= d(L_{X_f} g)$$
$$= d(-\{f,g\}) \;,$$

so that,

$$[\mathrm{hmlt\_vf}(f)\ ,\ \mathrm{hmlt\_vf}(g)] = -\mathrm{hmlt\_vf}(\{f,g\})\ .$$

21 For the second identity, if h is another smooth function on P, then

$$\begin{aligned}
0 &= d\omega(X_f, X_g, X_h) \\
&= L_{X_f}(\omega(X_g,X_h)) - L_{X_g}(\omega(X_f,X_h)) + L_{X_h}(\omega(X_f,X_g)) \\
&\quad - \omega([X_f,X_g],X_h) + \omega([X_f,X_h],X_g) - \omega([X_g,X_h],X_f) \\
&= \{\{g,h\},f\} - \{\{f,h\},g\} + \{\{f,g\},h\} + \omega(\mathrm{hmlt_vf}(\{f,g\}),X_h) \\
&\quad - \omega(\mathrm{hmlt_vf}(\{f,h\}),X_g) + \omega(\mathrm{hmlt_vf}(\{g,h\}),X_f) \\
&= 2\{\{g,h\},f\} + 2\{\{f,g\},h\} + 2\{\{h,f\},g\}\ ,
\end{aligned}$$

so that,

$$\{f,\{g,h\}\} + \{h,\{f,g\}\} + \{g,\{h,f\}\} = 0\ .$$

22 Let the flow of the hamiltonian vector field of f be $F_t : D_t \to P$. D_t is an open subset of P, and hence inherits a natural symplectic structure, $\omega|D_t$. Then F_t is a symplectomorphism. Indeed,

$$\begin{aligned}
\frac{d}{ds}\bigg|_{s=t}\left[\omega\left[F_s(p)\right]\left[TF_s(v_p),TF_s(w_p)\right]\right] &= \frac{d}{ds}\bigg|_{s=t} F_s^*\omega(v_p,w_p) \\
&= F_t^* L_{X_f}\omega(v_p,w_p) \\
&= F_t^*(d(X_f \lrcorner\ \omega) + X_f \lrcorner\ d\omega)(v_p,w_p) \\
&= 0\ ,
\end{aligned}$$

so that

$$F_t^*\omega(v_p,w_p) = \omega(v_p,w_p)\ .$$

CHAPTER 2

Lie Groups and Momentum Mappings

1 With the exception of chapters (5) and (6), all manifolds considered in the remainder of this thesis will be finite dimensional, Hausdorff and second countable. Let G be a Lie group; that is, a manifold which is a group such that the operation of group multiplication is a smooth map from $G \times G$ to G. Denote by e the identity element of G, and by L_g and R_g the smooth diffeomorphisms of G which are the operations of left and right multiplication by $g \in G$, respectively.

2 A vector field X on G is left invariant if $L_g^* X = X$ for all $g \in G$; that is, for all $g,h \in G$, $X(gh) = TL_g X(h)$. Denote by $L(G)$ the tangent space to G at the identity. If $\xi \in L(G)$, the left invariant vector field generated by ξ is defined by

$$\begin{aligned} \text{lin\_vf}(\xi,g) &= \text{lin\_vf}(\xi)(g) \\ &= TL_g \xi \ . \end{aligned}$$

This correspondence between elements of $L(G)$ and vector fields on G allows one to define a Lie bracket on $L(G)$ by

$$\begin{aligned} [\xi,\eta] &= \mathrm{ad}_\xi \eta \\ &= [\text{lin\_vf}(\xi),\text{lin\_vf}(\eta)](e) \ , \end{aligned}$$

if $\xi,\eta \in L(G)$.

3 Left invariance shows that the flow of lin_vf(ξ), F_t^{ξ}, is complete, and linearity of lin_vf in its first argument shows that $F_t^{s\xi} = F_{st}^{\xi}$ for all $s,t \in \mathbb{R}$. Define the exponential mapping, exp : $L(G) \to G$ by $\exp(\xi) = F_1^{\xi}(e)$. The fundamental existence theorem of flows shows that exp is smooth. Clearly, $T_e\exp : L(G) \to L(G)$ is the identity, so exp is a local diffeomorphism at the identity, and in particular, the image of exp contains an open neighbourhood of the identity. Examples (Abraham-Marsden [1978: 257]) show that, in general, exp is not onto the connected component of G which contains the identity. The flow of lin_vf(ξ) is

$$\begin{aligned} F_t^{\xi}(g) &= g\cdot\exp(t\xi) \\ &= R_{\exp(t\xi)}(g) \ . \end{aligned}$$

4. Reference will be made later to the following standard result. For the proof, see Abraham-Marsden [1978: 259].

5 Theorem. Let H be a subgroup of G which is a closed subset of G. Then H is an imbedded submanifold of G, and in particular, H is a Lie group.

6 Let M be a manifold. A smooth left action of G on M is a smooth map $\phi : G \times M \to M$ such that $\phi(e,m) = m$ for all $m \in M$ and $\phi(g,\phi(h,m)) = \phi(gh,m)$ for all $g,h \in G$ and $m \in M$. One often denotes $\phi(g,m)$ by gm, so $em = m$ and $g(hm) = (gh)m$. The action is called free if for any $m \in M$, $gm = m$ only if $g = e$. A subset S of M is called ϕ invariant if $gs \in S$ for all $g \in G$ and $s \in S$.

7 The fundamental vector field on M generated by $\xi \in L(G)$ is defined by

$$
\begin{aligned}
\text{fund\_vf}(\phi,\xi)(m) &= \text{fund\_vf}(\phi,\xi,m) \\
&= \text{fund\_vf}(\xi,m) \\
&= \text{fund\_vf}(\xi)(m) \\
&= \xi(m) \\
&= \left.\frac{d}{dt}\right|_{t=0} \phi(\exp t\xi,m) \\
&= T\phi_m \xi \ .
\end{aligned}
$$

Clearly then, the flow of fund_vf(ϕ,ξ) is $(m,t) \to \phi(\exp t\xi,m)$. The action ϕ is called infinitesimally free if, for any $m \in M$, fund_vf(ϕ,ξ,m) is zero only if $\xi = 0$.

8 The map $\xi \to$ fund_vf(ϕ,ξ) is a Lie algebra antihomomorphism:

$$
\begin{aligned}
\text{fund\_vf}([\xi,\eta],m) &= T\phi_m\ [\text{lin\_vf}(\xi),\text{lin\_vf}(\eta)](e) \\
&= T\phi_m \left.\frac{d}{dt}\right|_{t=0} \left[R^*_{\exp(t\xi)}\ \text{lin\_vf}(\eta)\right](e) \\
&= \left.\frac{d}{dt}\right|_{t=0} T\phi_m\ TR_{\exp(-t\xi)}\ \text{lin\_vf}(\eta,\exp(t\xi)) \\
&= \left.\frac{d}{dt}\right|_{t=0} T\phi_{\exp(-t\xi)m}\ TL_{\exp(t\xi)}\ \eta \\
&= \left.\frac{d}{dt}\right|_{t=0} T\phi_{\exp(t\xi)}\ T\phi_{\exp(-t\xi)m}\ \eta \\
&= \left.\frac{d}{dt}\right|_{t=0} T\phi_{\exp(t\xi)}\ \text{fund\_vf}(\phi,\eta,\exp(-t\xi)m) \\
&= \left.\frac{d}{dt}\right|_{t=0} \phi^*_{\exp(-t\xi)}\ \text{fund\_vf}(\phi,\eta,m) \\
&= -\ [\text{fund\_vf}(\phi,\xi),\text{fund\_vf}(\phi,\eta)](m)\ .
\end{aligned}
$$

9 If $g \in G$, define the homomorphism $I_g : G \to G$ by $I_g(h) = ghg^{-1}$. The adjoint action of G on L(G) is defined by

$$\mathrm{Ad}_g\xi = \mathrm{Ad}(g,\xi) = TI_g\xi \ .$$

The fundamental vector fields of the Ad action are easy to compute:

$$\begin{aligned}
\mathrm{fund_vf}(\mathrm{Ad},\xi,\eta) &= \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}(\exp(t\xi),\eta) \\
&= \left.\frac{d}{dt}\right|_{t=0} TR_{\exp(-t\xi)}\, TL_{\exp(t\xi)}\, \eta \\
&= \left.\frac{d}{dt}\right|_{t=0} TR_{\exp(-t\xi)}\, \mathrm{lin_vf}(\eta, R_{\exp(t\xi)}\, e) \\
&= \left.\frac{d}{dt}\right|_{t=0} \left[R^*_{\exp(t\xi)}\, \mathrm{lin_vf}(\eta)\right](e) \\
&= [\mathrm{lin_vf}(\xi),\mathrm{lin_vf}(\eta)](e) \\
&= [\xi,\eta] \ .
\end{aligned}$$

10 The fundamental vector fields of the action satisfy the identity

$$\mathrm{fund\_vf}(\phi,\mathrm{Ad}_g\xi) = \phi^*_{g^{-1}}\, \mathrm{fund\_vf}(\phi,\xi) \ .$$

Indeed,

$$\begin{aligned}
\mathrm{fund_vf}(\mathrm{Ad}_g\xi,m) &= T\phi_m\, \mathrm{Ad}_g\xi \\
&= T\phi_m\, TR_{g^{-1}}\, TL_g\xi \\
&= T\phi_g\, T\phi_{g^{-1}m}\,\xi \\
&= T\phi_g\, \mathrm{fund_vf}(\phi,\xi,g^{-1}m) \\
&= \left[\phi^*_{g^{-1}}\, \mathrm{fund_vf}(\phi,\xi)\right](m) \ .
\end{aligned}$$

11 The dual of the adjoint action of G on $L(G)$, called the co-adjoint action of G on $L(G)^*$ is defined by

$$Ad^*(g,\mu) = \mu \circ Ad_{g^{-1}} .$$

Its fundamental vector fields are also easily computed:

$$\begin{aligned}
\text{fund_vf}(Ad^*,\xi,\mu)\eta &= \left[\frac{d}{dt}\bigg|_{t=0} Ad^*(\exp(t\xi),\mu)\right]\eta \\
&= \frac{d}{dt}\bigg|_{t=0} \mu\left[Ad_{\exp(-t\xi)}\ \eta\right] \\
&= \mu(-[\xi,\eta]) \\
&= -\mu \circ ad_{\xi}\ \eta ,
\end{aligned}$$

so,

$$\text{fund\_vf}(Ad^*,\xi,\mu) = -\mu \circ ad_{\xi} .$$

12 Let $\bar{\phi} : G \times M \to M \times M$ be the map $\bar{\phi}(g,m) = (m,gm)$. The action ϕ is called proper if $\bar{\phi}$ is a proper map; that is, if the inverse image by $\bar{\phi}$ of a compact subset of $M \times M$ is a compact subset of $G \times M$. If $m \in M$, then the isotropy group of m is the subgroup $I_m = \{g \in G ; gm = m\}$. The isotropy group of m is a Lie subgroup of G, since it follows from

$$I_m = \{g ; (g,m) \in (\bar{\phi})^{-1}(m,m)\}$$

that I_m is closed. The Lie algebra of I_m is

$$L(I_m) = \{\xi \in L(G) ; \text{fund\_vf}(\phi,\xi,m) = 0\} .$$

The orbit of m is the subset $G\cdot m = \{gm ; g \in G\}$. The proof of the following theorem may be found in Abraham-Marsden [1978: 265].

13 Theorem. Let ϕ be a smooth, proper left action of a Lie group G on M. If $m \in M$, then $G\cdot m$ is a closed, imbedded submanifold of M such that the map $G \to G\cdot m$ by $g \to gm$ is a submersion. If $m' \in G\cdot m$, then

$$T_{m'}(G\cdot m) = \{\mathrm{fund\_vf}(\phi,\xi,m') \; ; \; \xi \in L(G)\}$$
$$= \mathrm{fund\_vf}(\phi,L(G),m') \; .$$

14 Let (P,ω) be a connected symplectic manifold. Let ϕ be the symplectic left action of a Lie group G on P; that is, for all $g \in G$, $\phi_g^*\omega = \omega$. A momentum mapping for the action ϕ is a map $J : P \to L(G)^*$ such that

$$\mathrm{fund\_vf}(\phi,\xi) \lrcorner\, \omega = dJ(\xi) = dJ_\xi \; .$$

The quadruple (P,ω,ϕ,J) is called a hamiltonian G-space. J is called Ad^* equivariant if, for all $g \in G$,

$$J(gp) = Ad_g^* J(p) \; .$$

15 For example, let Q be a manifold and let ϕ be an action of G on Q. Then G also acts on TQ and T^*Q by

$$\phi^T(g,v_q) = T\phi_g \, v_q$$
$$\phi^{T^*}(g,\alpha_q) = \alpha_q \circ T\phi_{g^{-1}}$$

respectively. The action ϕ^{T^*} is symplectic on (T^*Q,ω_0); indeed, this action preserves the canonical one form θ_0:

$$\left[\phi_g^{T^*}\right]^* \theta_0(w_{\alpha_q}) = \theta_0\left[T\phi^{T^*}_{g^{-1}}\ w_{\alpha_g}\right]$$

$$= \phi^{T^*}_{g^{-1}}(\alpha_q)\left[T\tau_Q^*\ T\phi^{T^*}_{g^{-1}}\ w_{\alpha_q}\right]$$

$$= \alpha_q\left[T\phi_g\left[T\left[\tau_Q^* \circ \phi^{T^*}_{g^{-1}}\right]w_{\alpha_q}\right]\right]$$

$$= \alpha_q\left[T\phi_g\ T\phi_{g^{-1}}\ T\tau_Q^*\ w_{\alpha_q}\right]$$

$$= \theta_0(w_{\alpha_q}) \ .$$

16 An Ad^* equivariant momentum mapping for the action of ϕ^{T^*} on T^*Q is given by J_ξ = fund_vf$(\phi^{T^*},\xi) \lrcorner\ \theta_0$. Thus,

$$J_\xi(\alpha_q) = \theta_0\left[\text{fund\_vf}\left[\phi^{T^*},\xi,\alpha_q\right]\right]$$

$$= \alpha_q\left[T\tau_Q^*\ \text{fund\_vf}\left[\phi^{T^*},\xi,\alpha_q\right]\right]$$

$$= \alpha_q\left[T\tau_Q^*\ T\phi^{T^*}_{\alpha_q}\ \xi\right]$$

$$= \alpha_q\left[T\left[\tau_Q^* \circ \phi^{T^*}_{\alpha_q}\right]\xi\right]$$

$$= \alpha_q(T\phi_q\ \xi)$$

$$= \alpha_q(\text{fund\_vf}(\phi,\xi,q)) \ .$$

That J is a momentum mapping can be seen by noting that $L_\xi\theta_0 = 0$ since θ_0 is ϕ^{T^*} invariant, so

$$dJ_\xi = d(\xi \lrcorner\ \theta_0) = L_\xi\theta_0 - \xi \lrcorner\ d\theta_0$$
$$= \xi \lrcorner\ \omega_0\ .$$

J is Ad^* equivariant since

$$J_\xi(\phi_g^{T^*}\ \alpha_q) = \alpha_q\left[T\phi_{g^{-1}}\ \text{fund\_vf}(\phi,\xi,gq)\right]$$
$$= \alpha_q\left[\left[\phi_g^*\ \text{fund\_vf}(\phi,\xi)\right](q)\right]$$
$$= \alpha_q\left[\text{fund\_vf}\left[\phi, Ad^*_{g^{-1}}\xi, q\right]\right]$$
$$= \left[Ad_g^* J(q)\right]\ \xi\ .$$

17 If (P,ω,ϕ,J) is a hamiltonian G-space, then, for each $g \in G$, $Ad_g^*J \circ \phi_{g^{-1}}$ is another momentum mapping for ϕ:

$$d\left[Ad_g^* J \circ \phi_{g^{-1}}\right]\xi = \phi^*_{g^{-1}}\ dJ_{Ad_{g^{-1}}\xi}$$
$$= \phi^*_{g^{-1}}(\text{fund\_vf}(\phi, Ad_{g^{-1}}\xi) \lrcorner\ \omega)$$
$$= \phi^*_{g^{-1}}\ \text{fund\_vf}(\phi, Ad_{g^{-1}}\xi) \lrcorner\ \phi^*_{g^{-1}}\omega$$
$$= \text{fund\_vf}(\phi,\xi) \lrcorner\ \ \omega\ .$$

Therefore, the function

$$(g,p) \to Ad_g^*\ J(g^{-1}p) - J(p)$$

is independent of p.

18 This observation has an important consequence: suppose that, for some $p_0 \in P$, J is constant along the orbit $G \cdot p_0$. Let $\mu_0 = J(p_0)$. Then $(P,\omega,\phi,J - \mu_0)$ is another hamiltonian G-space, and $J - \mu_0$ is Ad^* equivariant. Clearly, $J - \mu_0$ is a momentum mapping, and

$$\begin{aligned}(J - \mu_0)(gp) &= J(gp) - \mu_0 \\ &= Ad_g^* J(p) - \left[Ad_g^* J(p) - J(gp)\right] - \mu_0 \\ &= Ad_g^* J(p) - \left[Ad_g^* J(g^{-1}(gp)) - J(gp)\right] - \mu_0 \\ &= Ad_g^* J(p) - \left[Ad_g^* J(g^{-1}p_0) - J(p_0)\right] - \mu_0 \\ &= Ad_g^* J(p) - \left[Ad_g^* \mu_0 - \mu_0\right] - \mu_0 \\ &= Ad_g^*(J(p) - \mu_0) .\end{aligned}$$

19 If J is Ad^* equivariant, then, for any $\xi,\eta \in L(G)$,

$$\{J_\xi,J_\eta\} = J_{[\xi,\eta]} .$$

By hypothesis,

$$J_\xi(\exp(t\eta)p) = Ad^*(\exp(t\eta),J_\xi(p)) .$$

The derivative of the left side at t = 0 is

$$\begin{aligned}\left.\frac{d}{dt}\right|_{t=0} J_\xi(\exp(t\eta)p) &= dJ_\xi\left[\left.\frac{d}{dt}\right|_{t=0} \exp(t\eta)p\right] \\ &= dJ_\xi(\text{fund\_vf}(\phi,\eta,p)) \\ &= dJ_\xi(\text{hmlt\_vf}(J_\eta,p)) \\ &= \{J_\xi,J_\eta\}(p) ,\end{aligned}$$

and the derivative of the right side is

$$\left.\frac{d}{dt}\right|_{t=0} Ad^*(\exp(t\eta), J_\xi(p)) = \mathrm{fund\_vf}(Ad^*, \eta, J(p))(\xi)$$

$$= -J(p) \circ ad_\eta(\xi)$$

$$= J(p)[\xi,\eta]$$

$$= J_{[\xi,\eta]}(p) \ .$$

20 The final result of this chapter is the following conservation law: if H is a smooth, ϕ invariant function on P, then J is a conserved quantity of the flow of X_H. Indeed, if c(t) is an integral curve of X_H, then

$$\frac{d}{dt}(J_\xi \circ c(t)) = dJ_\xi\left[\frac{dc}{dt}(t)\right]$$

$$= dJ_\xi\left[X_H(c(t))\right]$$

$$= \xi \lrcorner\ \omega\left[X_H(c(t))\right]$$

$$= -\ X_H \lrcorner\ \omega\left[\xi(c(t))\right]$$

$$= -\ dH\left[\xi(c(t))\right]$$

$$= -\left.\frac{\partial}{\partial s}\right|_{s=0} H(\exp(s\xi)c(t))$$

$$= -\left.\frac{\partial}{\partial s}\right|_{s=0} H(c(t))$$

$$= 0 \ .$$

CHAPTER 3

Singular Momentum Mappings

1 Let (P,ω,ϕ,J) be a hamiltonian G-space. This section is an analysis of the set $J^{-1}(0)$ in the absence of the hypothesis that 0 is a regular value for J, using the methods of Arms-Marsden-Moncrief [1981]. If $p \in P$, an infinitesimal symmetry at p is a vector $\xi \in L(G)$ such that $\mathrm{fund\_vf}(\phi,\xi,p) = 0$. Thus, the set of infinitesimal symmetries at p is exactly $L(I_p)$. The first step in the analysis of $J^{-1}(0)$ is the following basic link between the analytic notion of a regular point of J and the geometric notion of an infinitesimal symmetry of the action.

2 <u>Theorem</u>. Let (P,ω,ϕ,J) be a hamiltonian G-space. Then $p \in P$ is a regular point of J if and only if there are no infinitesimal symmetries at p. In fact,

$$\mathrm{Image}(dJ(p)) = \mathrm{ann}(L(I_p)) .$$

<u>Proof</u>

3 It is sufficient to show the second statement. If $\mu = dJ(p)v$ for some $v \in T_pP$ then, for any $\xi \in L(I_p)$,

$$\begin{aligned}\mu(\xi) &= (dJ(p)v)(\xi) \\ &= \text{fund\_vf}(\xi,p) \lrcorner\ \omega(v) \\ &= 0\ ,\end{aligned}$$

so $\mu \in \text{ann}(L(I_p))$.

...4 Let $\mu \in \text{ann}(L(I_p))$. Since $L(I_p)$ is the kernel of the linear map $\xi \to \text{fund\_vf}(\xi,p)$, there is an $\alpha \in (T_pP)^*$ such that $\mu(\xi) = \alpha(\text{fund\_vf}(\xi,p))$ for all $\xi \in L(G)$. Let $v = -\omega^{\#}(\alpha) \in T_pP$. Then

$$\begin{aligned}(dJ(p)v)(\xi) &= (\text{fund\_vf}(\xi,p) \lrcorner\ \omega)(v) \\ &= -\omega(v,\text{fund\_vf}(\xi,p)) \\ &= \alpha(\text{fund\_vf}(\xi,p)) \\ &= \mu(\xi)\ ,\end{aligned}$$

so that $dJ(p)v = \mu$, and hence $\mu \in \text{Image}(dJ(p))$. ∎

5 The next result uses some slightly nonstandard definitions. Let M be a manifold and S be a subset of M. Given $m \in S$, $v \in T_mM$ is tangent to S at m if there exists a smooth curve $\gamma : [0,1] \to M$ such that $\gamma([0,1]) \subseteq S$, $m = \gamma(0)$ and

$$v = \left.\frac{d}{dt}\right|_{t=0} \gamma\ .$$

Denote by T_mS the set of all tangent vectors of S at m, and by TS the set of all tangent vectors of S. If S is a imbedded submanifold of M, then TS is the usual tangent bundle of S. In general, for each $m \in S$, T_mS is a cone in T_mP, in the sense that, for each $v \in T_mS$ and $a \geq 0$, $av \in T_mS$.

6 The set S is said to be locally diffeomorphic to its tangent bundle at $m \in S$ if there is a smooth diffeomorphism Ψ from some open neighbourhood of m to some open neighbourhood of zero in T_mM such that $\Psi(m) = 0$, $T_m\Psi : T_mM \to T_mM$ is the identity, and $\Psi(S) = \text{Image}(\Psi) \cap T_mS$. S is locally diffeomorphic to its tangent bundle if S is locally diffeomorphic to its tangent bundle at m for each $m \in S$. If S is a submanifold of M, then S is locally diffeomorphic to its tangent bundle. Figure (7) shows a set which is not locally diffeomorphic to its tangent bundle.

7

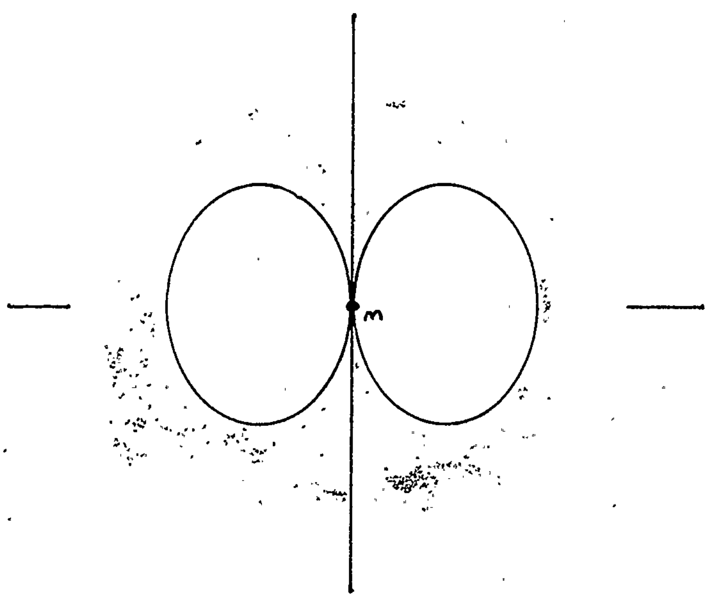

Figure 7: A subset of $\mathbb{R}^2$ which is not locally diffeomorphic to its tangent bundle at m.

8 Let X be a vector field on M, and suppose $m \in M$ is such that $X(m) = 0$. The linearization of X at M is the linear vector field $dX(m) : T_mM \to T_mM$ defined by

$$dX(m)v = \frac{d}{dt}\bigg|_{t=0} T_mF_tv = -L_XV(m) ,$$

where F_t is the flow of X, and V is any vector field such that $V(m) = v$. If M is modelled on a vector space $\mathbb{E}$, then in a chart with domain $U \subseteq \mathbb{E}$, $X : U \to \mathbb{E}$ and $dX(u) = DX(u)$.

9 Let f be a smooth function on M, and suppose $m \in M$ is such that $df(m) = 0$. If $v_1, v_2 \in T_mM$, define

$$d^2f(m)(v_1,v_2) = d(V_1 \lrcorner\, df)V_2(m) ,$$

where V_1 and V_2 are any two vector fields on M such that $V_1(m) = v_1$ and $V_2(m) = v_2$. This expression is obviously independent of the vector field V_2, and

$$\begin{aligned} d(V_2 \lrcorner\, df)V_1(m) &= [V_1 \lrcorner\, d(L_{V_2}f)](m) \\ &= [V_1 \lrcorner\, L_{V_2}df](m) \\ &= [L_{V_2}(V_1 \lrcorner\, df) + [V_2,V_1] \lrcorner\, df](m) \\ &= d(V_2 \lrcorner\, df)V_1(m) , \end{aligned}$$

so the expression is independent of the choice of V_1 as well, and is symmetric in v_1 and v_2. In the chart of paragraph (8), $d^2f(u)(v_1,v_2)$ $= D^2f(u)(v_1,v_1)$.

10 The next theorem determines the local structrue of the set $J^{-1}(0)$ near a fixed point of the action.

11 Theorem. Let (P,ω,ϕ,J) be a hamiltonian G-space, $(P,\langle,\rangle)$ be a reimannian manifold and suppose ϕ is an isometric action on P. Suppose that $J(p) = 0$ and that p is a fixed point of ϕ. Then paragraph (2.18) shows that J is Ad^* equivariant, and

1. $T_pJ^{-1}(0) = \{v \in T_pP \; ; \; d^2J(p)(v,v) = 0\}$,
2. $J^{-1}(0)$ is locally diffeomorphic to its tangent bundle at p.
3. Furthermore, let F be the set of fixed points of $\bar{\phi}$. Then $F \cap J^{-1}(0)$ is the union of connected components of F, $F \cap J^{-1}(0)$ is a closed, symplectic submanifold of P, and

$$T_p(F \cap J^{-1}(0)) = \bigcap_{\xi\in L(G)} \ker(d\xi(p))$$

$$= \ker(d^2J(p))$$

$$= \{v \in T_pP \; ; \; d^2J(p)(v,w) = 0 \; \forall \; w \in T_pP\} ,$$

$$T_p(F \cap J^{-1}(0))^{\omega\perp} = \bigcup_{\xi\in L(G)} \mathrm{Image}(d\xi(p)) .$$

Proof

12 Since p is a fixed point of ϕ, one can define the linearized action $T\phi$ of G on T_pP by $T\phi(g,v) = T\phi_g v$. The pair $(T_pP,\omega(p))$ is a linear symplectic space, and $T\phi$ acts symplectically: if $v,w \in T_pP$, then

$$\begin{aligned}\omega(p)(T\phi_g v,T\phi_g w) &= \omega(gp)(T\phi_g v,T\phi_g w)\\ &= (\phi_g^*\omega)(p)(v,w)\\ &= \omega(p)(v,w) .\end{aligned}$$

Also,

$$\text{fund\_vf}(T\phi,\xi,v) = \left.\frac{d}{dt}\right|_{t=0} (T\phi_{\exp(t\xi)}\ v)$$

$$= d(\text{fund\_vf}(\phi,\xi))(p)v$$

$$= d\xi(p)v\ .$$

...13 Let $v,w \in T_pP$ and choose vector fields V and W such that $V(p) = v$ and $W(p) = w$. Then, if $\xi \in L(G)$,

$$\begin{aligned} d^2J_\xi(v,w) &= d(V \lrcorner\ dJ_\xi)W(p) \\ &= d(-\xi \lrcorner\ (V \lrcorner\ \omega))W(p) \\ &= -L_W(\xi \lrcorner\ (V \lrcorner\ \omega))(p) \\ &= (-[W,\xi] \lrcorner\ (V \lrcorner\ \omega) - \xi \lrcorner\ L_W(V \lrcorner\ \omega))(p) \\ &= -\omega(v,[W,\xi](p)) - L_W(V \lrcorner\ \omega)\xi(p) \\ &= \omega(-L_\xi W(p),v) \\ &= \omega(p)(d\xi(p)w,v) \\ &= \omega(p)(\text{fund\_vf}(T\phi,\xi,w),v)\ . \end{aligned}$$

Therefore, $v \to d^2J(p)(v,v)$ is a momentum mapping for the linear action $T\phi$ of G on T_pP; that is $(T_pP,\omega(p),T\phi,d^2J(p))$ is a linear hamiltonian G-space.

...14 Let $\exp_p$ be the exponential mapping of the reimannian manifold $(P,\langle,\rangle)$ at p. Then $\exp_p$ is a diffeomorphism from some ball B about the origin to some open neighbourhood U of p. Since ϕ is an isometric action, for any $g \in G$ and $v \in T_pP$, $\phi_g \circ \exp_p(vt)$ is a geodesic which starts at p. Therefore, $\phi_g \circ \exp_p(vt) = \exp_p(T\phi_g(vt))$, since

$$\left.\frac{d}{dt}\right|_{t=0} \phi_g \circ \exp_p(vt) = T\phi_g v\ .$$

Thus, $\exp_p$ is an equivariant diffeomorphism: $\phi_g \circ \exp_p = T\phi_g \circ \exp_p$ for all $g \in G$. A similar argument to paragraph (12) shows that $T\phi$ is an isometric action on the linear metric space $(T_pP,\langle,\rangle)$, so that B is $T\phi$ invariant.

...15 Refer to the proof of the Darboux theorem as applied to the two symplectic forms $\omega_0 = \omega(p)$ and $\omega_1 = \exp_p^*\omega$ on B. By the Poincaré lemma (Abraham-Marsden [1978: 118]), if

$$\alpha(v)w = \int_0^1 t(\omega_1 - \omega_0)(tv)(v,w)dt ,$$

then $d\alpha = \omega_1 - \omega_0$. This particular α is $T\phi$ invariant, since $T\phi$ is a linear action and acts symplectically on both symplectic spaces $(T_pP,\exp_p^*\omega)$ and $(T_pP,\omega(p))$: if $g \in G$, then

$$\begin{aligned}
\alpha(T\phi_g v)(T(T\phi_g)w) &= \int_0^1 t(\omega_1 - \omega_0)(tT\phi_g v)(T\phi_g v, T\phi_g w)dt \\
&= \int_0^1 t(\omega_1 - \omega_0)(T\phi_g tv)(T\phi_g v, T\phi_g w)dt \\
&= \int_0^1 tT\phi_g^*(\omega_1 - \omega_0)(tv)(v,w)dt \\
&= \int_0^1 t(\omega_1 - \omega_0)(tv)(v,w)dt \\
&= \alpha(v)w .
\end{aligned}$$

...16 By further restriction of B, the $T\phi$ invariant forms $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$ and α define a $T\phi$ invariant time dependent vector field X_t by $X_t \lrcorner\, \omega_t = -\alpha$, and the time one flow F_1 of X_t exists on B and satisfies $F_1(0) = 0$ and $\omega_0|B = F_1^*\omega_1$. $T\phi$ invariance of X_t serves to show that F_1 is $T\phi$ equivariant: $T\phi_g \circ F_1 = F_1 \circ T\phi_g$ for all $g \in G$. The linear map $T_0F_1 : T_pP \to T_pP$ is also $T\phi$ equivariant, since $T\phi$ is a linear action. Also, since $\omega_1(0) = \omega_2(0) = \omega(p)$, $\omega_0|B = F_1^*\omega_1$ shows that $T_0F_1^*\omega(p) = \omega(p)$. Let $V = T_0F_1^{-1}(B)$, and define $\Psi : V \subseteq T_pP \to U \subseteq P$ by $\Psi = \exp_p \circ F_1 \circ T_0F_1^{-1}$. Then Ψ is a symplectomorphism from $(V,\omega(p))$ to $(U,\omega|U)$ such that $\Psi \circ T\phi_g = \phi_g \circ \Psi$ for all $g \in G$, and $T_p\Psi$ is the identity.

...17 These considerations suffice to show that Ψ^*J is a momentum mapping for the action $T\phi$ of G on V. But $v \to d^2J(v,v)$ is also a momentum mapping on V for $T\phi$, so these functions differ by a constant on V. Since both have value zero at the origin, $\Psi^*J(v) = d^2J(v,v)$ for all $v \in V$. Therefore, Ψ^{-1} maps $U \cap J^{-1}(0)$ to $\{v \in V \; ; \; d^2J(p)(v,v) = 0\}$, and the proof of statements (11.1) and (11.2) will be complete if

$$T_p\{v \in V \; ; \; d^2J(p)(v,v) = 0\} = \{v \in T_pP \; ; \; d^2J(p)(v,v) = 0\} ,$$

since $T_p\Psi^{-1}$ is the identity.

...18 In fact, if $v \in V$ is such that $d^2J(p)(v,v) = 0$, then $\gamma(t) = vt$ is a curve contained in V for small t such that $d^2J(p)(\gamma(t),\gamma(t)) = 0$ and

$$v = \left.\frac{d}{dt}\right|_{t=0} \gamma ,$$

so that $v \in T_p\{v \in V \; ; \; d^2J(p)(v,v) = 0\}$. On the other hand, let $\gamma(t)$ be a curve such that $d^2J(p)(\gamma(t),\gamma(t)) = 0$ and $\gamma(0) = 0$. Then

$$\begin{aligned} 0 &= \frac{d}{dt}\left(d^2J(p)(\gamma(t),\gamma(t)\right) \\ &= 2d^2J(p)\left[\gamma(t),\frac{d\gamma}{dt}(t)\right] , \end{aligned}$$

so that,

$$0 = \frac{d}{dt}\bigg|_{t=0} d^2J(p)\left[\gamma(t), \frac{d\gamma}{dt}(t)\right]$$

$$= d^2J(p)\left[\frac{d\gamma}{dt}(0), \frac{d\gamma}{dt}(0)\right] + d^2J\left[\gamma(0), \frac{d^2\gamma}{dt}(0)\right]$$

$$= d^2J(p)\left[\frac{d\gamma}{dt}(0), \frac{d\gamma}{dt}(0)\right] ,$$

and

$$\frac{d}{dt}\bigg|_{t=0} \gamma(t) \in \{v \in T_pP \; ; \; d^2J(p)(v,v) = 0\} .$$

...19 Let F' be the set of fixed points of the action $T\phi$ of G on T_pP. Then the inclusion

$$F' \subseteq \{v \in T_pP \; ; \; \text{fund\_vf}(T\phi, \xi, v) = 0 \; \forall \; \xi \in L(G)\}$$

is obvious. The reverse inclusion is also true: let $v \in T_pP$ be such that fund_vf($T\phi$,ξ,v) = 0 for all $\xi \in L(G)$. Then $T\phi(\mathrm{expt}\xi, v) = v$ for all t, so I_v, the isotropy group of v, contains an open neighbourhood of the identity. By left translation, I_v is open, and since I_v is closed as well, connectivity of G implies that $I_v = G$, that is, $v \in F'$. By paragraph (13), then,

$$F' = \{v \in T_pP \; ; \; \text{fund\_vf}(T\phi, \xi, v) = 0 \; \forall \; \xi \in L(G)\}$$

$$= \bigcap_{\xi \in L(G)} \ker(d\xi(p))$$

$$= \{v \; ; \; d^2J(v,w) = 0 \; \forall \; w \in T_pP\}.$$

...20 In particular, $v \to d^2J(p)(v,v)$ vanishes on F', so equivariance of Ψ shows that

$$\begin{aligned}\Psi^{-1}(F \cap J^{-1}(0)) &= \Psi^{-1}(F) \cap \Psi^{-1}(J^{-1}(0)) \\ &= (F' \cap V) \cap \{v \in V \ ; \ d^2J(p)(v,v) = 0\} \\ &= F' \cap V \\ &= V \cap \bigcap_{\xi \in L(G)} \ker(d\xi(p)) \ .\end{aligned}$$

Thus, Ψ serves as a submanifold chart for $F \cap J^{-1}(0)$ near p, and

$$T_p(F \cap J^{-1}(0)) = \bigcap_{\xi \in L(G)} \ker(d\xi(p)) \ .$$

...21 The equality $\Psi^{-1}(F \cap J^{-1}(0)) = F' \cap V$ shows that $F \cap J^{-1}(0) \cap U = F \cap U$, so $F \cap J^{-1}(0)$ contains an open neighbourhood of p in the relative topology of F. Since p is arbitrary, $F \cap J^{-1}(0)$ is open in F, and is clearly closed in F. Therefore, $F \cap J^{-1}(0)$ is the union of connected components of F. Since

$$G \times F = (\bar{\phi})^{-1}((q,q) \ ; \ q \in P) \ ,$$

F is a closed subset of M, so $J^{-1}(0) \cap F$ is a closed submanifold of P.

...22 Since $T\phi$ is symplectic, the linear maps $d\xi(p)$ are infinitesimally symplectic; that is, for $v,w \in T_pP$,

$$\omega(p)(d\xi(p)v,w) = -\omega(p)(v,d\xi(p)w) \ .$$

Thus, $\mathrm{Image}(d\xi(p)) = \ker(d\xi(p))^{\omega\perp}$ for all $\xi \in L(G)$, so

$$\begin{aligned}T_p(F \cap J^{-1}(0))^{\omega\perp} &= \left[\bigcap_{\xi \in L(G)} \ker(d\xi(p))\right]^{\omega\perp} \\ &= \sum_{\xi \in L(G)} \mathrm{Image}(d\xi(p))\end{aligned}$$

$$= \bigcup_{\xi \in L(G)} \mathrm{Image}(d\xi(p)) \ .$$

Similarly, since $T\phi$ acts isometrically, the linear maps $d\xi(p)$ are skew symmetric: for $v, w \in T_pP$,

$$\langle d\xi(p)v, w\rangle = -\langle v, d\xi(p)w\rangle \ .$$

The same computation shows that

$$T_p(F \cap J^{-1}(0))^{\perp} = \bigcup_{\xi \in L(G)} \mathrm{Image}(d\xi(p))$$
$$= T_p(F \cap J^{-1}(0))^{\omega\perp} \ .$$

Therefore, $T_p(F \cap J^{-1}(0))^{\omega\perp} \cap T_p(F \cap J^{-1}(0)) = \{0\}$, so $F \cap J^{-1}(0)$ is a symplectic submanifold of P. ■

23 The restriction that p is a fixed point of the action places a severe limitation on the utility of theorem (11). This theorem may be applied to obtain more useful resuts, however, essentially by moding out by the nontrivial part of the action at the point in question, as the proof of the following theorem shows.

24 <u>Theorem</u>. Let (P, ω, ϕ, J) be a hamiltonian G-space, $(P, \langle , \rangle)$ be a reimannian manifold and suppose that ϕ is a proper, isometric action on G. Let L(G) admit an Ad invariant metric and let J be Ad^* equivariant. If $p \in P$, define $D_p = T_p(G\cdot p)^{\perp} \cap T_p(G\cdot p)^{\omega\perp}$. If $J(p) = 0$, then D_p is symplectic, and

1. $T_pJ^{-1}(0) = T_p(G\cdot p) \oplus \{v \in D_p \ ; \ d^2J(p)(v,v) = 0\}$,
2. $J^{-1}(0)$ is locally diffeomorphic to its tangent bundle at p.
3. Furthermore, let G_p be the identity component of the

isotropy group of p and N_p be the set of points in P with the same continuous symmetry type as p; that is, $q \in N_p$ if and only if G_q is conjugate to G_p. Then $N_p \cap J^{-1}(0)$ is an imbedded submanifold of P, and

$$\begin{aligned} T_p(N_p \cap J^{-1}(0)) &= T_p(G\cdot p) \oplus \{v \in D_p \ ; \ d\xi(p)v = 0 \ \forall \xi \in L(G_p)\} \\ &= T_p(G\cdot p) \oplus \bigcap_{\xi \in L(G_p)} D_p \cap \ker(d\xi(p)) \\ &= T_p(G\cdot p) \oplus \{v \in D_p \ ; \ d^2J(p)(v,w) = 0 \ \forall \ \xi \in L(G_p) \\ &\qquad \text{and } w \in D_p\} \ , \end{aligned}$$

$$T_p(N_p \cap J^{-1}(0))^{\omega\perp} = T_p(G\cdot p) \oplus \bigcup_{\xi \in L(G_p)} d\xi(p)D_p \ ,$$

$$T_p(N_p \cap J^{-1}(0)) + T_p(N_p \cap J^{-1}(0))^{\omega\perp} = T_p(G\cdot p) \oplus D_p \ ,$$

$$T_p(N_p \cap J^{-1}(0)) \cap T_p(N_p \cap J^{-1}(0))^{\omega\perp} = T_p(G\cdot p) \ .$$

Proof

25 There is a submanifold S_p, containing p, called a slice at p, and a map χ from a neighbourhood U of p in $G\cdot p$ to G such that:

1. $gp = p$ implies that $gS_p = S_p$.
2. If $gS_p \cap S_p \neq \phi$ then $gp = p$.
3. $\chi(p) = e$, $\chi(u)p = u$ for all $u \in U$, and the map $\Psi_1 : U \times S_p \to P$ by $\Psi_1(u,p) = \chi(u)p$ is a diffeomorphism from $U \times S_p$ to some open neighbourhood of p such that $T_{(p,p)}(v,w) = v + w$ for all $v \in T_p(G\cdot p)$ and $w \in T_pS_p$.

...26 Let B' be a ball in T_pP such that $\exp_p$ is a diffeomorphism from B' to some open neighbourhood of p. Define

$$S'_p = \exp_p(B' \cap T_p(G\cdot p)^{\perp}) \ .$$

Then S'_p is I_p invariant, since G acts isometrically, B' is a ball, and $T_p(G\cdot p)^{\perp}$ is $T\Phi_g$ invariant if $g \in I_p$: if $\xi \in L(G)$ and $v \in T_p(G\cdot p)^{\perp}$ then

$$\begin{aligned}
\langle T\Phi_g v, \xi(p)\rangle &= -\ \langle v, T\Phi_{g^{-1}}\ \xi(p)\rangle \\
&= -\ \langle v, \text{fund_vf}(\phi, \mathrm{Ad}_g\xi, p)\rangle \\
&= 0 \ .
\end{aligned}$$

...27 The assumption that ϕ is proper implies that $G\cdot p$ is an imbedded submanifold of P and the canonical map $\kappa : G \to G\cdot p$ by $\kappa(g) = gp$ is a submersion. Since $\kappa(e) = p$, the implicit function theorem may be used to obtain a local left inverse $\chi : U \subseteq G\cdot p \to G$ of κ such that $\chi(p) = e$. Compute the derivative of the map $\Psi_1 : U \times S'_p \to P$ by $\Psi_1(u,q) = \chi(u)q$: if $v \in T_p(G\cdot p)$ and $w \in T_pS'_p$, then

$$\begin{aligned}
T\Psi_1(v,w) &= T(u \to \Psi_1(u,p))v + T(q \to \Psi_1(p,q))w \\
&= T(u \to \chi(u)p)v + T(q \to \chi(p)q)w \\
&= T(u \to \kappa \circ \chi(u))v + T(q \to eq)w \\
&= v + w \ .
\end{aligned}$$

Obviously, $T_p\Psi_1$ is a bijection, so Ψ_1 is a diffeomorphism from some neighbourhood of (p,p) in $U \times S'_p$ to some neighbourhood of p in P. By further restricting B' and U, one may assume that Ψ_1 is a diffeomorphism on $U \times S'_p$.

...28 The set $K = \{g \in G \ ; \ gS'_p \cap S'_p \neq \phi \Longrightarrow g \in I_p\}$ contains an open neighbourhood of I_p; in fact,

$$I_p = \kappa^{-1}(p) \subseteq \kappa^{-1}(U) \subseteq K \ .$$

Let $g \in \kappa^{-1}(U)$ and suppose $gq = q'$ for $q, q' \in S'_p$. Then, $\kappa(g) \in U$, so $\kappa(g) = gp = u = \chi(u)p$ for some $u \in U$. Thus, $\chi(u)^{-1}g \in I_p$, so $\chi(u)^{-1}gq \in S'_p$. But

$$\begin{aligned} \Psi_1(u, \chi(u)^{-1}gq) &= \chi(u)\chi(u)^{-1}gq \\ &= gq \\ &= q' \\ &= \Psi_1(p, q') \ , \end{aligned}$$

and since Ψ_1 is a diffeomorphism, $p = u$. Therefore, $gp = p$, so $g \in I_p$.

...29 Consider the following subset A of G:

$$\begin{aligned} A &= \{g \in G \ ; \ g\cdot cl(S'_p) \cap S'_p \neq \phi\} \\ &= \{g \in G \ ; \ \exists \ q \in cl(S'_p) \text{ such that } \bar{\Phi}(g,q) \in cl(S'_p) \times cl(S'_p)\} \ . \end{aligned}$$

The since $\bar{\Phi}$ is proper, A is compact, so $B = A \cap (P - \kappa^{-1}(U))$ is also compact. If $g \in B$, then $g \notin \kappa^{-1}(U)$, so $gp \notin U$, and, in particular, $gp \neq p$. Continuity of ϕ implies that there are open neighbourhoods A_g of g, U_g of p and V_g of gp such that $\phi(A_g, U_g) \subseteq V_g$ and $U_g \cap V_g = \phi$. Since B is compact, finitely many of the sets A_g cover B, say $A_{g_1}, \ldots, A_{g_n}$. Let $U_0 = \cap U_{g_i}$; U_0 has the property that if $g \in B$, then $gU_0 \cap U_0 = \phi$. Indeed, if $g \in B$, then $g \in A_{g_i}$ for some i, so

$$gU_0 \subseteq gU_{g_i} \subseteq \phi(A_{g_i}, U_{g_i}) \subseteq V_{g_i} ,$$

and $V_{g_i} \cap U_0 = \phi$ for all i.

...30 Let S_p be the image of $B \cap T_p(G\cdot p)^{\perp}$ under $\exp_p$, where B is a restriction of B' such that $\exp_p(B) \subseteq U_0$. Then S_p clearly has property (25.1). It also has property (25.2): if $gS_p \cap S_p \neq \phi$, then $g \in A$. Since $S_p \subseteq U_0$, $gU_0 \cap U_0 \neq \phi$, so $g \notin B$. Thus, $g \notin P - \kappa^{-1}(U)$, so $g \in \kappa^{-1}(U)$, and paragraph (28) shows that $g \in I_p$. Since property (25.3) holds for χ and S_p', and $S_p \subseteq S_p'$, one may ensure that S_p and χ have the property (25.3) by restricting the domain of χ.

...31 Let $\mathbb{P} : L(G)^* \to \mathrm{Image}(dJ(p))$ be the orthogonal projection arising from the Ad invariant metric on L(G). Then $\mathrm{Ad}_g^* \circ \mathbb{P} = \mathbb{P} \circ \mathrm{Ad}_g^*$ for any $g \in G$. Clearly, $J(q) = 0$ if and only if $\mathbb{P}J(q) = 0$ and

$$J(q) \in \mathrm{Image}(dJ(q)) = \mathrm{ann}(L(I_p)) = \mathrm{ann}(L(G_p)) .$$

Denoting by $i : L(G_p) \to L(G)$ the canonical inclusion, it follows that $J(q) = 0$ if and only if $\mathbb{P}J(q) = 0$ and $i^*J(q) = 0$.

...32 Clearly, $\mathbb{P}J$ is a submersion at p, so that $(PJ)^{-1}(0)$ is a submanifold near p, with tangent bundle

$$\begin{aligned} Tp((\mathbb{P}J)^{-1}(0)) &= \{v \in T_pP \; ; \; \mathbb{P}dJ(p)v = 0\} \\ &= \{v \in T_pP \; ; \; dJ(p)v = 0\} \\ &= \{v \in T_pP \; ; \; \omega(p)(\xi(p), v) = 0 \; \forall \; \xi \in L(G)\} \\ &= T_p(G\cdot p)^{\omega\perp} . \end{aligned}$$

Ad^* equivariance of J shows that $J(\exp(t\xi)p) = 0$ for all $\xi \in L(G)$, and so

$$0 = \left.\frac{d}{dt}\right|_{t=0} J(\exp(t\xi)p)$$

$$= dJ(p)\xi(p) \ .$$

Therefore, $T_p(G\cdot p) \subseteq \ker(dJ(p)) = T_p((\mathbb{P}J)^{-1}(0))$. Since $T_pS_p = T_p(G\cdot p)^{\perp}$, $(\mathbb{P}J)^{-1}(0)$ and S_p are transversal at p. By choosing S_p small enough, then, $(\mathbb{P}J)^{-1}(0) \cap S_p$ is a submanifold of P, with tangent bundle

$$\begin{aligned} T_p((\mathbb{P}J)^{-1}(0) \cap S_p) &= T_p((\mathbb{P}J)^{-1}(0)) \cap T_pS_p \\ &= T_p(G\cdot p)^{\omega\perp} \cap T_p(G\cdot p)^{\perp} \\ &= D_p \ . \end{aligned}$$

...33 I claim that

$$\begin{aligned} T_p(G\cdot p) &= T_p(G\cdot p)^{\omega\perp} \cap \left[T_p(G\cdot p) + (T_p(G\cdot p)^{\perp})^{\omega\perp}\right] \\ &= T_p(G\cdot p)^{\omega\perp} \cap D_p^{\ \omega\perp} \ . \end{aligned}$$

Once this fact is established, it is easy to see that D_p is symplectic:

$$\begin{aligned} D_p \cap D_p^{\ \omega\perp} &= (T_p(G\cdot p)^{\omega\perp} \cap T_p(G\cdot p)^{\perp}) \cap D_p^{\ \omega\perp} \\ &= T_p(G\cdot p) \cap T_p(G\cdot p)^{\perp} = 0 \ . \end{aligned}$$

Let $v = v_1 + v_2$ be such that $v \in T_p(G\cdot p)^{\omega\perp}$, $v_1 \in T_p(G\cdot p)$ and $v_2 \in (T_p(G\cdot p)^{\perp})^{\omega\perp}$. Then $v_2 = v - v_1 \in T_p(G\cdot p)^{\omega\perp}$, since $T_p(G\cdot p) \subseteq T_p(G\cdot p)^{\omega\perp}$. Thus

$$\begin{aligned} v_2 &\in (T_p(G\cdot p)^{\perp})^{\omega\perp} \cap T_p(G\cdot p)^{\omega\perp} \\ &= (T_p(G\cdot p)^{\perp} + T_p(G\cdot p))^{\omega\perp} \\ &= 0\ . \end{aligned}$$

Thus, $v = v_1$, so $v \in T_p(G\cdot p)$, so $T_p(G\cdot p)^{\omega\perp} \cap D_p^{\omega\perp} \subseteq T_p(G\cdot p)$. For the reverse inclusion,

$$\begin{aligned} T_p(G\cdot p) &= (T_p(G\cdot p)^{\omega\perp})^{\omega\perp} \\ &= (T_p(G\cdot p)^{\omega\perp} + D_p)^{\omega\perp} \\ &\subseteq (T_p(G\cdot p) + D_p)^{\omega\perp} \\ &= T_p(G\cdot p)^{\omega\perp} \cap D_p^{\omega\perp}\ . \end{aligned}$$

It follows that one may choose S_p small enough so that $(\mathbb{P}J)^{-1}(0) \cap S_p$ is a symplectic submanifold of P.

…34 If $g \in G_p$, then ϕ_g maps the symplectic manifold $(\mathbb{P}J)^{-1}(0) \cap S_p$ into itself: ϕ_g sends S_p to itself by property (25.1) and the fact that $G_p \subseteq I_p$, and ϕ_g sends $(\mathbb{P}J)^{-1}(0)$ to itself by Ad^* equivariance of J and the fact that $\mathbb{P}$ commutes with Ad_g^*. Let ϕ' be the action of G_p on $(\mathbb{P}J)^{-1}(0) \cap S_p$ and let $j : (\mathbb{P}J)^{-1}(0) \cap S_p \to P$ be the canonical inclusion. Then ϕ' is a symplectic action on $((\mathbb{P}J)^{-1}(0) \cap S_p, j^*\omega)$, and $((\mathbb{P}J)^{-1}(0) \cap S_p, j^*\omega, \phi', (i^*J) \circ j)$ is a hamiltonian G_p-space. Let Ψ_2 be the diffeomorphism from some open neighbourhood of p in $\mathbb{P}J^{-1}(0) \cap S_p$ to some open neighbourhood of 0 in D_p constructed in the proof of theorem (11). Using a submanifold chart of $\mathbb{P}J^{-1}(0) \cap S_p$ at p in S_p, and by further restriction of S_p, one may extend Ψ_2 to a diffeomorphism Ψ_3 of some neighbourhood of p in S_p to $T_pS_p = T_p(G\cdot p)^{\perp}$

such that $T_p\Psi_3$ is the identity. A restriction of the set U of paragraph (25) will permit the construction of a diffeomorphism Ψ_4 from U to some neighbourhood of 0 in $T_p(G\cdot p)$ such that $T_p\Psi_4$ is the identity. Let Ψ_5 be defined by the following compositions:

$$\Psi_5 : \mathrm{domain}(\Psi_1^{-1}) \subseteq P \xrightarrow{\Psi_1^{-1}} U \times S_p \xrightarrow{\Psi_4\times\Psi_3} T_p(G\cdot p) \times T_p(G\cdot p)^{\perp} \xrightarrow{+} T_pP .$$

...35 By the definition of Ψ_1,

$$\begin{aligned} J \circ \Psi_1(u,q) &= J(\chi(u)q) \\ &= \mathrm{Ad}^*\chi(u)\ J(q) \ , \end{aligned}$$

so that $J \circ \Psi_1(u,q) = 0$ if and only if $J(q) = 0$. Therefore,

$$\begin{aligned} \Psi_1^{-1}J^{-1}(0) &= U \times J^{-1}(0) \cap S_p \\ &= U \times ((i^*J)^{-1}(0) \cap S_p) \cap ((\mathbb{P}J)^{-1}(0) \cap S_p) \ , \\ &= U \times ((i^*J) \circ j)^{-1}(0) \ , \end{aligned}$$

and since Ψ_3 agrees with Ψ_2 on $(\mathbb{P}J)^{-1}(0) \cap S_p$,

$$(\Psi_4 \times \Psi_3)(\Psi_1^{-1}J^{-1}(0)) = \mathrm{Image}(\Psi_4) \times \{v \in \mathrm{Image}(\Psi_2) \ ; \ d^2J(p)(v,v) = 0\}.$$

Therefore,

$$\Psi_5(J^{-1}(0)) = \mathrm{Image}(\Psi_4) \oplus \{v \in \mathrm{Image}(\Psi_3) \cap D_p, d^2J(p)(v,v) = 0\} .$$

...36 Clearly,

$$T_p(\Psi_5 J^{-1}(0)) = T_p(\mathrm{Image}(\Psi_4)) \oplus T_p\{v \in \mathrm{Image}(\Psi_3) \cap D_p \; ; \; d^2J(p)(v,v) = 0\}$$

$$= T_p(G\cdot p) \oplus \{v \in D_p \; ; \; d^2J(p)(v,v) = 0\} \;,$$

and Ψ_5 maps $J^{-1}(0)$ to a neighbourhood of zero of this set. Since $T\Psi_1 : T_p(G\cdot p) \times T_p(G\cdot p)^{\perp} \to T_pP$ is simply addition, $T_p\Psi_1{}^{-1} : T_pP \to T_p(G\cdot p) \times T_p(G\cdot p)^{\perp}$ is orthogonal decomposition. The fact that $T_p\Psi_4$ and $T_p\Psi_3$ are both the identity then implies that $T_p\Psi_5$ is the identity. This completes the proof of statements (24.1) and (24.2).

...37 Let $q \in S_p$. Then, for any $u \in U$,

$$\begin{aligned} G_{\chi(u)q} &= \text{identity component of } \{h \in G \; ; \; h\chi(u)q = \chi(u)q\} \\ &= \text{identity component of } \{h \in G \; ; \; \chi(u)^{-1}h\chi(u)q = q\} \\ &= \text{identity component of } \chi(u)\{h \in G, hq = q\}\; \chi(u)^{-1} \\ &= \chi(u)G_q\chi(u)^{-1} \;, \end{aligned}$$

so that $\chi(u)q \in N_q$ for all $q \in S_p$. Therefore,

$$\Psi_1^{-1}(N_p) = U \times (N_p \cap S_p) \;.$$

But if $q \in N_p \cap S_p$, G_q is a subgroup of I_p, by property (25.2), so G_q is a subgroup of the isotropy group of p which is conjugate to G_p. Connectivity of G_p implies that $G_q = G_p$, and it follows that

$$\begin{aligned} \Psi_1^{-1}(N_p \cap J^{-1}(0)) &= \Psi_1^{-1}(N_p) \cap \Psi_1^{-1}(J^{-1}(0)) \\ &= U \times \{q \in (\mathbb{P}J)^{-1}(0) \cap S_p \; ; \; q \text{ is fixed under the action of } G_p\}. \end{aligned}$$

Therefore,

$$\Psi_5(N_p \cap J^{-1}(0)) = \text{Image}(\Psi_4) \oplus \text{Image}(\Psi_3) \cap \bigcap_{\xi \in L(G_p)} D_p \cap \ker(d\xi(p)) \;,$$

so Ψ_5 serves as a submanifold chart for $N_p \cap J^{-1}(0)$ at p.

...38 The identification of $T_p(N_p \cap J^{-1}(0))$ is a trivial consequence of the fact that $T_p\Psi_5$ is the identity. The symplectic complement of $T_p(N_p \cap J^{-1}(0))$ is easily computed:

$$\begin{aligned} T_p(N_p \cap J^{-1}(0)) &= T_p(G\cdot p)^{\omega\perp} \cap \left[D_p^{\omega\perp} \oplus \bigcup_{\xi \in L(G_p)} d\xi(p)D_p \right] \\ &= T_p(G\cdot p)^{\omega\perp} \cap D_p^{\omega\perp} \oplus \bigcup_{\xi \in L(G_p)} d\xi(p)D_p \;, \end{aligned}$$

since $\bigcup_{\xi \in L(G_p)} d\xi(p)D_p \subseteq D_p \subseteq T_p(G\cdot p)^{\omega\perp}$. But paragraph (33) implies that $T_p(G\cdot p)^{\omega\perp} \cap D_p^{\omega\perp} = T_p(G\cdot p)$, so

$$T_p(N_p \cap J^{-1}(0)) = T_p(G\cdot p) \oplus \bigcup_{\xi \in L(G_p)} d\xi(p)D_p \;.$$

...39 Finally, since the two spaces

$$\bigcup_{\xi \in L(G_p)} d\xi(p)D_p \;, \quad \bigcap_{\xi \in L(G_p)} \ker(d\xi(p)) \cap D_p$$

are symplectic in D_p, and symplectic complements of each other in D_p,

$$
\begin{aligned}
T_p(N_p \cap J^{-1}(0)) &+ T_p(N_p \cap J^{-1}(0))^{\omega\perp} \\
&= (T_p(G\cdot p) \oplus \bigcap_{\xi\in L(G_p)} D_p \cap \ker(d\xi(p))) \\
&\quad + (T_p(G\cdot p) \oplus \bigcup_{\xi\in L(G_p)} d\xi(p)D_p) \\
&= T_p(G\cdot p) \oplus D_p \ ,
\end{aligned}
$$

and,

$$
\begin{aligned}
T_p(N_p \cap J^{-1}(0)) &\cap T_p(N_p \cap J^{-1}(0))^{\omega\perp} \\
&= (T_p(N_p \cap J^{-1}(0)) \oplus T_p(N_p \cap J^{-1}(0))^{\omega\perp})^{\omega\perp} \\
&= T_p(G\cdot p)^{\omega\perp} \cap D_p^{\omega\perp} \\
&= T_p(G\cdot p) \ . \qquad \blacksquare
\end{aligned}
$$

CHAPTER 4

The Dirac Theory of Constraints

1 In Chapter (6) it will be shown that the following situation is common: (P,ω) is a symplectic manifold; the points of P represent the classical states of the system under consideration. The classical states that admit evolution form a subset M of P. The space P is called the extended phase space of the system, and M is called the constraint set. The analysis of M is the subject of the Dirac theory of constraints. This chapter develops the aspects of this theory which are relevant to this thesis, closely following Sniatycki [1981], with special attention to the case where M is the zero level of a momentum mapping on P.

2 A smooth function f on P is called a constraint if f vanishes on M. The set of all constraints is an ideal of the associative algebra of smooth functions on P. The development of the theory will presuppose that M is a closed subset of P. Then, by the smooth version of Urysohn's lemma, M is completely determined by the constraints; that is, $p \in M$ if and only if $f(p) = 0$ for all constraints f.

3 A constraint f is first class if, for each constraint f', the Poisson bracket $\{f,f'\}$ is a constraint. The constraints that are not first class are called second class constraints. The set of first class constraints forms an ideal of the associative algebra and a Lie subalgebra of the Poisson algebra of smooth functions on P. Moreover, if f is any constraint, then f^2 is a first class constraint, so the constraint set is determined by the first class constraints: $p \in M$ if and only if $f(p) = 0$ for all first class constraints f. The set M itself is called first class if each constraint is first class. An observable is a smooth function g on P such that, for each first class constraint f, the Poisson bracket $\{g,f\}$ is a constraint.

4 Unfortunately, at this level of generality, the terminology above is somewhat deficient. For example, one would like the hamiltonian vector field X_g of an observable g to be tangent to M; that is, $X_g(M) \subseteq TM$. A typical argument proceeds as follows: for each $m \in M$ and constraint f,

$$df(m)X_g(m) = -\{g,f\}(m) = 0 \ ,$$

and therefore, the integral curve of X_g starting at some point in M is contained in the zero level set of f. Of course, the difficulty here is that, while $X_g \lrcorner\, df = 0$ on some open neighbourhood of M would imply the desired conclusion, $X_g \lrcorner\, df = 0$ on M does not.

5 I have in mind the following example: $P = \mathbb{R}^4$ with coordinates (q^1, q^2, p_1, p_2) and symplectic form $\omega = dq^i \wedge dp_i$. Let M be the subset of the $p_1 = p_2 = 0$ plane depicted in Figure (6). It is apparent from the figure that, at any point $m \in M$,

$$\operatorname{span}(T_m M) = \operatorname{span}\left[\frac{\partial}{\partial q^1}(m) \ , \ \frac{\partial}{\partial q^2}(m)\right] .$$

If f and f' are any two constraints, then

$$df(\operatorname{span}(T_m M)) = df'(\operatorname{span}(T_m M)) = 0 ,$$

so,

$$\frac{\partial f}{\partial q^1}(m) = \frac{\partial f}{\partial q^2}(m) = \frac{\partial f'}{\partial q^1}(m) = \frac{\partial f'}{\partial q^2}(m) = 0 ,$$

and paragraph (1.19) shows that $\{f,f'\}(m) = 0$. Therefore, M is first class. So $p_1 + p_2$ is a first class constraint, and hence an observable, but

$$\text{hmlt\_vf}(p_1 + p_2) = \frac{\partial}{\partial q^1} + \frac{\partial}{\partial q^2} ,$$

which is not tangent to M at the origin, and the upper boundary points of M.

6

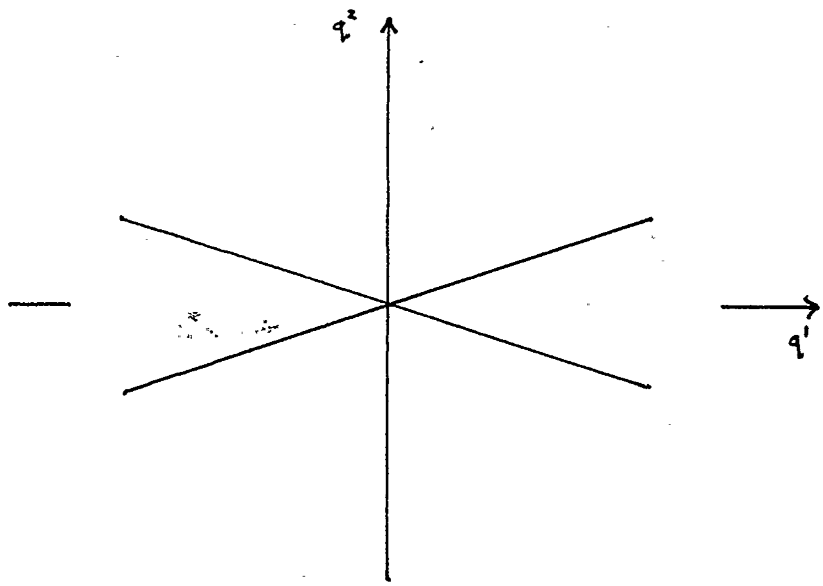

Figure 6: A first class subset of $\mathbb{R}^4$ which admits an observable whose hamiltonian vector field is not tangent to the subset everywhere.

7 Owing to these pathologies, call a subset M of P strongly first class if M is first class and the hamiltonian vector field of each observable of M is tangent to M. If M is strongly first class and g is an observable, then M is an invariant set for the flow of X_g; that is, integral curves c of X_g such that $c(0) \in M$ have the property that $c(t) \in M$ for all $t \in \mathrm{domain}(c)$. According to Abraham-Marsden [1978: 97], one need only verify that

$$\lim_{h \to 0^+} \frac{d(m \pm hX_g(m), M)}{h} = 0 \,, \qquad 1$$

where d is the distance function implied by the norm provided by some chart containing m.

8 Let $\gamma : [0,1] \to P$ be any smooth curve such that $\gamma(0) = m$, $\gamma([0,1]) \subseteq M$ and

$$\left.\frac{d}{dt}\right|_{t=0} \gamma(t) = X_g(m) \ .$$

Then,

$$\begin{aligned} \lim_{h\to 0^+} \frac{d(m + hX_g(m),M)}{h} &\leq \lim_{h\to 0^+} \frac{\|m + hX_g(m) - \gamma(h)\|}{h} \\ &= \lim_{h\to 0^+} \frac{\|\gamma(h) - \gamma(m) + hX_g(m)\|}{h} \\ &= 0 \ . \end{aligned}$$

Since $-g$ is an observable if g is, and $X_{-g} = -X_g$, it is also true that

$$\lim_{h\to 0^+} \frac{\|m - hX_g(m),M\|}{h} = 0 \ ,$$

so equation (7.1) is verified.

9 As in the submanifold case, denote by $TM^{\omega\perp}$ the union of the vector spaces $T_mM^{\omega\perp}$ over $m \in M$, and call M coisotropic if $TM^{\omega\perp} \subseteq TM$.

10 <u>Proposition</u>.

1. If M is strongly first class and locally diffeomorphic to its tangent bundle, then M is coisotropic.
2. If M is coisotropic, then M is first class.

Proof

11 For the first statement, let $v \in T_m M^{\omega\perp}$, and let $\alpha = v \lrcorner\, \omega(m)$. One may use the local diffeomorphism guaranteed by hypothesis, and a bump function, to construct a constraint f such that $df(m) = \alpha$. Then X_f is tangent to M and $v = X_f(m)$, so $v \in T_m M$. Therefore, M is coisotropic.

12 For the second statement, let f and f' be any two constraints. Then $X_{f'}$ is tangent to M, since for any $v \in T_m M$,

$$\omega(X_{f'}(m), v) = df'(m)v = 0 \ ,$$

so $X_{f'}(m) \in T_m M^{\omega\perp} \subseteq T_m M$. Therefore,

$$\{f, f'\}(m) = df(m) X_{f'}(m) = 0 \ ,$$

so that M is first class. ∎

13 The null set of M is defined by

$$\mathrm{null}(M) = TM \cap TM^{\omega\perp} \ .$$

The next proposition illustrates the importance of the null set in the analysis of constraints.

14 Proposition. Let M be coisotropic and locally diffeomorphic to its tangent bundle. Then:

1. The null set of M is spanned by the Hamiltonian vector fields of the constraints.

2. A smooth function g on P is an observable if and only if $dg(\text{null}(M)) = 0$.

<u>Proof</u>

15 For any constraint f and $v \in T_m M$,

$$\omega(X_f(m),v) = df(m)v = 0 ,$$

so X_f takes values in $TM^{\omega\perp} = \text{null}(M)$. Conversely, if $v \in T_m M^{\omega\perp}$, then let $\alpha = v \lrcorner \omega(m)$. Then α vanishes on $T_m M$, so there is a constraint f such that $df(m) = \alpha$. Therefore, $v = X_f(m)$, and the proof of statement (14.1) is complete. Statement (14.2) is an immediate consequence of statement (14.1) and the identity $dg(m)X_f(m) = \{g,f\}(m)$ for smooth functions f and g on P. ∎

16 The next proposition shows that the pathologies of paragraph (5) do not arise when M is an imbedded submanifold of P.

17 <u>Proposition</u>. Let (P,ω) be a symplectic manifold and let M be an imbedded submanifold of P. Then the following are equivalent:

1. M is strongly first class.
2. M is coisotropic.
3. M is first class.

In the case that these statements hold, a smooth function g on P is an observable if and only if the hamiltonian vector field of g is tangent to M.

Proof

18 The implications (17.1) $\Longrightarrow$ (17.2) and (17.2) $\Longrightarrow$ (17.3) have already been shown. Let M be first class and let g be an observable. Let $m \in M$ and suppose $\alpha \in (T_m P)^*$ is such that $\alpha(T_m M) = 0$. Choose a constraint f such that $df(m) = \alpha$, possible since M is an imbedded submanifold of P. Then, if g is an observable,

$$\alpha X_g(m) = df(m) X_g(m) = \{f,g\}(m) = 0 \, .$$

Since $T_m M$ is a closed subspace of $T_m P$, this shows that $X_g(m) \in T_m M$. Therefore, the hamiltonian vector field of any observable is tangent to M, so M is strongly first class. For the last statement, note that if g is a smooth function on P such that X_g is tangent to M, then for any constraint f,

$$\{f,g\} = -X_g \lrcorner\, df = 0 \, ,$$

so that g is an observable. ∎

19 Let (P,ω,J,ϕ) be a hamiltonian G-space. In chapter (7) it will be shown that the constraint set $J^{-1}(0)$ is of interest. If J is Ad^* equivariant, then paragraph (2.19) shows that the functions J_ξ form a Lie subalgebra of the Poisson algebra of smooth functions on P, so one might suspect that $J^{-1}(0)$ is first class. That this is true when 0 is a regular value of J is part of the content of the next theorem.

20 Theorem. Let (P,ω,J,ϕ) be a hamiltonian G-space. Let J be Ad^* equivariant and let 0 be a regular value of J. Then:

1. $J^{-1}(0)$ is a coisotropic, imbedded submanifold of P.
2. $\mathrm{null}(J^{-1}(0)) = \mathrm{fund\_vf}(L(G), J^{-1}(0))$.

3. A smooth function f on P is an observable if and only if f is invariant under the action of the connected component of the identity of G on P.

Proof

21 Obviously, $J^{-1}(0)$ is an imbedded submanifold of P by hypothesis, and if $J(p) = 0$, then

$$\begin{aligned} T_pJ^{-1}(0) &= \{v \in T_pP \ ; \ dJ(p)v = 0\} \\ &= \{v \in T_pP \ ; \ dJ_\xi(p)v = 0 \ \forall \ \xi \in L(G)\} \\ &= \{v \in T_pP \ ; \ \omega(p)(\xi(p),v) = 0 \ \forall \ \xi \in L(G)\} \\ &= \text{fund\_vf}(L(G),p)^{\omega\perp} \ . \end{aligned}$$

Since J is Ad^* equivariant, $T_p(G\cdot p) \subseteq T(G\cdot p)^{\omega\perp}$, so

$$\begin{aligned} T_pJ^{-1}(0)^{\omega\perp} &= \text{fund\_vf}(L(G),p) \\ &= T_p(G\cdot p) \subseteq T_p(G\cdot p)^{\omega\perp} \\ &= T_pJ^{-1}(0) \ , \end{aligned}$$

so J is coisotropic. This also shows that

$$\begin{aligned} \text{null}(J^{-1}(0)) &= T(J^{-1}(0))^{\omega\perp} \\ &= \text{fund\_vf}(L(G),J^{-1}(0)) \ . \end{aligned}$$

...22 If follows from proposition (14) that a smooth function f on P is an observable if and only if

$$df(\text{fund\_vf}(L(G),J^{-1}(0))) = 0 \ . \tag{1}$$

If f is invariant under the action of the connected component of the identity on G, then $f((\exp(t\xi))p) = f(p)$ for all $t \in \mathbb{R}$, so f will satisfy equation (22.1), so that f is an observable. Conversely, if f satisfies equation (22.1), define

$$A = \{g \in G \ ; \ f(gp) = f(p) \ \forall \ p \in J^{-1}(0)\} \ .$$

Then,

$$\begin{aligned} A \times J^{-1}(0) &= (\bar{\Phi})^{-1}\{(p,q) \ ; \ p,q \in J^{-1}(0) \text{ and } f(p) = f(q)\} \\ &= (\bar{\Phi})^{-1}\left[J^{-1}(0) \times J^{-1}(0) \cap (f \times f)^{-1}\{(t,t) \ ; \ t \in \mathbb{R}\}\right] \ , \end{aligned}$$

so $A \times J^{-1}(0)$ is closed, which imples that A is closed. But A contains the image of the exponential map, since f satisfies equation (22.1). Therefore, A contains an open neighbourhood of the identity, so A is open by left translation. If follows that A contains the connected component of the identity, so f is invariant under the action of the connected component of the identity. ∎

23 If 0 is not a regular value for J, then the conclusions of theorem (20) are false without further hypothesis. Consider $p = \mathbb{R}^4$ with coordinates (q^1,q^2,p_1,p_2), $\omega = dq^i \wedge dp_i$, Φ the action of the circle S^1 by rotations in the q^1,p_1 variables, and $J = (q^1)^2 + (p_1)^2$. Then $J^{-1}(0)$ is an imbedded submanifold of P, but it is symplectic, not coisotropic. This pathology arises from the fact that, if $J(p) = 0$, then there are directions of the action arbitrarily close to p which do not arise from the action within $J^{-1}(0)$ (see figure (24)); indeed, in this example, the action of S^1 on $J^{-1}(0)$ is trivial. The next theorem shows that when this behaviour is eliminated by hypothesis,

the analysis of the zero level of J of chapter (3) may be used to obtain results similar to those of theorem (20).

24

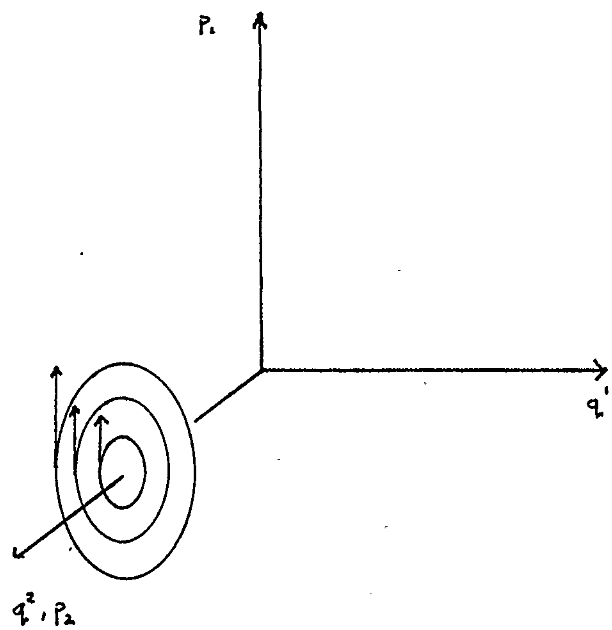

Figure 24: Directions of the action which do not arise from the action within $J^{-1}(0)$.

25 <u>Theorem</u>. Let the hypothesis of theorem (3.24) hold. Then the following are equivalent:

1. $D_p \subseteq \mathrm{span}(T_pJ^{-1}(0))$.

2. $$\bigcup_{\xi \in L(G_p)} d\xi(p) D_p = \bigcup_{\xi \in L(G_p)} d\xi(p)(\mathrm{span}(T_pJ^{-1}(0) \cap D_p)).$$

3. $\bigcup_{\xi \in L(G_p)} d\xi(p)D_p \subseteq \mathrm{span}(T_pJ^{-1}(0))$.

If these conditions hold for all $p \in J^{-1}(0)$, the call 0 a quasi-regular value of J. If 0 is a quasi-regular value of J then:

4. $J^{-1}(0)$ is strongly first class.
5. $\mathrm{null}(J^{-1}(0)) = \mathrm{fund\_vf}(L(G),J^{-1}(0))$.
6. A smooth function f on P is an observable if and only if f is invariant under the action of the connected component of G on $J^{-1}(0)$.
7. A smooth function f on P is an observable if and only if X_f is tangent to $N_p \cap J^{-1}(0)$ for all $p \in J^{-1}(0)$.

Proof

26 Obviously, statement (25.1) implies statement (25.2). Suppose statement (25.2) holds, and let

$$v \in \bigcup_{\xi \in L(G_p)} d\xi(p)D_p \ .$$

Then there are vectors $u_i \in T_pJ^{-1}(0) \cap D_p$ such that

$$v = d\xi(p) \sum_i u_i \ .$$

I claim that, for each u_i, $d\xi(p)u_i \in \mathrm{span}(T_pJ^{-1}(0))$. This suffices to show that $v \in \mathrm{span}\ (T_pJ^{-1}(0))$, and completes the proof of the implication (21.2) $\Longrightarrow$ (21.3).

...27 Indeed, since the flow of ξ is $m \to (\exp(t\xi))m$,

$$d\xi(p)u_i = \frac{d}{dt}\bigg|_{t=0} T\phi_{\exp(t\xi)}\, u_i$$

$$= \lim_{h\to 0} \frac{T\phi_{\exp(h\xi)}\, u_i - u_i}{h} .$$

If $s \to \gamma(s)$ is a curve in $J^{-1}(0)$ with $\gamma(0) = p$ and

$$\frac{d}{dt}\bigg|_{t=0} \gamma(t) = u_i ,$$

then $s \to (\exp(t\xi))\gamma(s)$ is another curve in $J^{-1}(0)$, since J is Ad^* equivariant. The derivative of this curve at $s = 0$ is $T\phi_{\exp(t\xi)}\, u_i$, so that $T\phi_{\exp(t\xi)}\, u_i \in T_pJ^{-1}(0)$. It follows that $d\xi(p)u_i$ is the limit of vectors in $\mathrm{span}(T_pJ^{-1}(0))$, which shows that $d\xi(p)u_i \in \mathrm{span}(T_pJ^{-1}(0))$.

...28 For the implication (21.3) $\Longrightarrow$ (21.1), note that if (21.3) holds, then

$$T_p(N_p \cap J^{-1}(0))^{\omega\perp} = T_p(G\cdot p) \oplus \bigcup_{\xi\in L(G_p)} d\xi(p)D_p \subseteq \mathrm{span}(T_pJ^{-1}(0)) ,$$

$$\begin{aligned} T_p(G\cdot p) \oplus D_p &= T_p(N_p \cap J^{-1}(0)) + T_p(N_p \cap J^{-1}(0))^{\omega\perp} \\ &\subseteq \mathrm{span}(T_pJ^{-1}(0)). \end{aligned}$$

Therefore, $D_p \subseteq \mathrm{span}(T_pJ^{-1}(0))$.

...29 If 0 is a quasi-regular value for J, then

$$\begin{aligned} \mathrm{span}(T_pJ^{-1}(0)) &= T_p(G\cdot p) \oplus T_pD_p \\ &= T_p(G\cdot p)^{\omega\perp} , \end{aligned}$$

by paragraph (3.33). Therefore,

$$T_pJ^{-1}(0)^{\omega\perp} = (\mathrm{span}(T_pJ^{-1}(0)))^{\omega\perp}$$
$$= T_p(G\cdot p) \subseteq T_pJ^{-1}(0) \ ,$$

and hence $J^{-1}(0)$ is coisotropic. Obviously

$$\mathrm{null}(J^{-1}(0)) = T(J^{-1}(0))^{\omega\perp} = \mathrm{fund\_vf}(L(G),J^{-1}(0)) \ ,$$

and this, along with the argument of paragraph (22), the fact that $J^{-1}(0)$ is locally diffeomorphic to its tangent bundle, and proposition (14), proves statement (25.6).

...30 Let f be a smooth function on P which is invariant under the action of the connected component of the identity on $J^{-1}(0)$. Let $u \in T_pJ^{-1}(0)$, γ be a curve within $J^{-1}(0)$ such that $\gamma(0) = p$, and

$$\left.\frac{d}{dt}\right|_{t=0} \gamma(t) = u \ .$$

Then, for any $\xi \in L(G_p)$ and $s,t \in \mathbb{R}$,

$$f((\exp(s\xi))\gamma(t)) = 0 \ .$$

Therefore,

$$\left.\frac{\partial}{\partial t}\right|_{t=0} f \circ ((\exp(s\xi))\gamma(t)) = df(p)T\Phi_{\exp(s\xi)}\, u = 0 \ ,$$

so,

$$\left.\frac{\partial}{\partial s}\right|_{s=0} df(p)\ T\Phi_{\exp(s\xi)}\, u = df(p)(d\xi(p)u) = 0 \ .$$

It follows that

$$df(p)\left[T_p(N_p \cap J^{-1}(0))^{\omega\perp}\right]$$

$$= df(p)\left[T_p(G\cdot p) \oplus \bigcup_{\xi\in L(G_p)} d\xi(p)D_p\right]$$

$$= df(p)\left[T_p(G\cdot p) \oplus \bigcup_{\xi\in L(G_p)} d\xi(p)\left[D_p \cap \operatorname{span}(T_pJ^{-1}(0))\right]\right]$$

$$= 0 \ ,$$

and hence $X_f(p)$ is tangent to $N_p \cap J^{-1}(0)$ at p. If X_f is tangent to $N_p \cap J^{-1}(0)$, this same computation shows that

$$df(p)(T_p(G\cdot p)) = 0 \ ,$$

so that f is an observable by the proof of statement (25.6). Statement (25.7) implies that any observable is tangent to $J^{-1}(0)$, so that $J^{-1}(0)$ is strongly first class. This completes the proof of theorem (25). ∎

31 The last result of this chapter is a verification of the hypotheses of the previous theorem for the case of the total angular momentum of n particles in $\mathbb{R}^3$. Let $Q = (\mathbb{R}^3)^n$ and ϕ be the natural action of SO(3) on Q. Then L(SO(3)) is $\mathbb{R}^3$ with the cross product as Lie bracket, and the standard metric on $\mathbb{R}^3$ is Ad invariant. The action ϕ provides the hamiltonian G-space $(T^*Q,\omega_0,\phi^{T^*},J)$, as defined in paragraph (2.15). Identifying $\mathbb{R}^3$ with its dual using the standard metric, one easily computes that

$$T^*Q = (\mathbb{R}^3)^n \times (\mathbb{R}^3)^n$$

$$\omega((e_i,f_i),(e_i',f_i')) = \sum_{i=1}^{n} (f_i' \cdot e_i - f_i \cdot e_i')$$

$$\text{fund\_vf}(\phi^{T^*},\xi,(q^i,p_i)) = (\xi \times q^i, \xi \times p_i)$$

$$J(q^i,p_i) = \sum_{i=1}^{n} q^i \times p_i \ .$$

Since SO(3) is compact, ϕ^{T^*} is a proper action.

32 The points of $J^{-1}(0)$ with nontrivial isotropy group have the form

$$p = (a_1\hat{n},\ldots,a_n\hat{n},b_1\hat{n},\ldots,b_n\hat{n}) = (a_i\hat{n},b_i\hat{n}) \ ,$$

for some $\hat{n} \in \mathbb{R}^3$. The first step in the verification is the following computation of D_p:

$$\begin{aligned} T_p(G\cdot p) &= \text{fund\_vf}(L(SO(3)),p) \\ &= \{(a_i\xi \times \hat{n}, b_i\xi \times \hat{n}) \ ; \ \xi \in \mathbb{R}^3\} \\ &= \{(a_i e, b_i e) \ ; \ e \in \text{span}\{\hat{n}\}^\perp\} \ . \end{aligned}$$

Since,

$$(e_i,f_i)\cdot(a_i e,b_i e) = e \cdot \sum_{i=1}^{n} (a_i e_i + b_i f_i) \ ,$$

$$\omega((e_i,f_i),(a_i e,b_i e)) = e \cdot \sum_{i=1}^{n} (b_i e_i - a_i f_i) \ ,$$

it follows that

$$D_p = T_p(G\cdot p)^{\perp} \cap T_p(G\cdot p)^{\omega\perp}$$

$$= \left\{ (e_i, f_i) \ , \ \sum_{i=1}^{n} a_i e_i + b_i f_i = c_1 \hat{n} \text{ and } \sum_{i=1}^{n} b_i e_i - a_i f_i = c_2 \hat{n}, \text{ for some } c_1, c_2 \in \mathbb{R} \right\} .$$

33 If $p = 0$, then $L(G_p) = \mathbb{R}^3$, while if some of the a_i or b_i are nonzero, $L(G_p) = \{c\hat{n} \ ; \ c \in \mathbb{R}\}$. In the first case, $D_p = (\mathbb{R}^3)^n \times (\mathbb{R}^3)^n$, so that

$$\bigcup_{\xi \in L(G_p)} d\xi(p) D_p = \bigcup_{\xi \in \mathbb{R}^3} \xi \times D_p = (\mathbb{R}^3)^n \times (\mathbb{R}^3)^n ,$$

while in the second,

$$\bigcup_{\xi \in L(G_p)} d\xi(p) D_p = \left\{ (e_i, f_i) \ ; \ \sum_{i=1}^{n} a_i e_i + b_i f_i = 0 \ , \ \sum_{i=1}^{n} b_i e_i - a_i f_i = 0 \ , \text{ where } e_i, f_i \in \operatorname{span}\{n\}^{\omega\perp} \right\} .$$

Let this vector space be A_p. Then the hypothesis of theorem (25) will be verified if it is shown that $A_p \subseteq \operatorname{span}(T_p J^{-1}(0))$.

34 Consider the curve $t \to (a_i \hat{n}, b_i \hat{n}) + t(e_i, f_i)$ for $(e_i, f_i) \in A_p$. Then

$$
\begin{aligned}
J(a_i\hat{n} + te_i, b_i\hat{n} + tf_i) &= \sum_{i=1}^{n} (a_i\hat{n} + te_i) \times (b_i\hat{n} + tf_i) \\
&= \sum_{i=1}^{n} (ta_i\hat{n} \times f_i - tb_i\hat{n} \times e_i + t^2 e_i \times f_i) \\
&= -t\hat{n} \times \sum_{i=1}^{n} (b_i e_i - a_i f_i) + t^2 \sum_{i=1}^{n} c_i \times f_i \\
&= t^2 \sum_{i=1}^{n} e_i \times f_i \ .
\end{aligned}
$$

Therefore, this curve will lie within $J^{-1}(0)$ whenever

$$\sum_{i=1}^{n} e_i \times f_i = 0 \ . \qquad 1$$

35 If $i = 1,2,\ldots,n$, denote by e^i the vector $(0,\ldots,0,e,0,\ldots,0)$ in $(\mathbb{R}^3)^n$ with e in the i^{th} position. The previous paragraph shows that, in order to prove that $A_p \subseteq \operatorname{span}(T_p J^{-1}(0))$, it suffices to prove that A_p is spanned by vectors in D_p which satisfy equation (34.1). If $p = 0$, this is a triviality, since vectors of the form $(e^i,0)$ and $(0,f^i)$ are contained in D_p and satisfy equation (34.1). By relabeling the particles, one may assume, then, that one of either a_1 or b_1 is nonzero.

36 Elementary operations on the equations

$$
\begin{aligned}
a_1 e_1 + b_1 f_1 + \sum_{i=2}^{n} (a_i e_i + b_i f_i) &= 0 \\
b_1 e_1 - a_1 f_1 + \sum_{i=2}^{n} (b_i e_i - a_i f_i) &= 0 \ ,
\end{aligned}
$$

show that

$$e_1 = L(0,e_2,\ldots,e_n,0,f_2,\ldots,f_n)$$

$$= \frac{-1}{a_1{}^2+b_1{}^2} \sum_{i=2}^{n} \left[(a_1a_i + b_1b_i)e_i + (a_1b_i - b_1a_i)f_i\right]$$

$$f_1 = L(0,f_2,\ldots,f_n,0,-e_2,\ldots,e_n)$$

$$= \frac{-1}{a_1{}^2+b_1{}^2} \sum_{i=2}^{n} \left[(b_1a_i - a_1b_i)e_i + (b_1b_i - a_1a_i)f_i\right] .$$

Then A_p is spanned by the vectors

$$L((e^i,0))^1 + L((0,-e^i))^n + (e^i,0) \qquad i = 2,\ldots,n \qquad 1$$

$$L((0,e^i))^1 + L((e^i,0))^n + (0,e^i) \qquad i = 2,\ldots,n\ , \qquad 2$$

where $e \in \text{span}\{n\}^{\perp}$. But it is clear that each of these vectors satisfies equation (34.1). Indeed, (34.1) evaluated on the vectors of the form (36.1) yields

$$L(e^i) \times L(-e^{i+n}) + e^i \times 0 = (a_1a_i + b_1b_i)e \times (a_1b_i - b_1a_i)e$$
$$= 0\ ,$$

and similarly with the vectors of the form (36.2).

CHAPTER 5

Presymplectic Dynamics

1 For some physical systems, notably field theories and certain finite dimensional lagrangian systems derived from them, the symplectic formalism of chapter (1) is too restrictive. These systems can be accomodated by relaxing the restriction that the symplectic form be nondegenerate, which brings the system into a presymplectic context. This chapter defines presymplectic systems, and analyzes the existence and uniqueness questions for the evolution that they define.

2 A presymplectic manifold is a pair $(M,\bar{\omega})$, where M is a Banach manifold, and $\bar{\omega}$ is a closed two form on M. A presymplectic system is a triple $(M,\bar{\omega},\bar{H})$, where $\bar{H} : M \to \mathbb{R}$ is a smooth function. Any such presymplectic system defines an evolution on M by decreeing that points of M evolve along smooth curves c such that

$$\frac{dc}{dt} \lrcorner \; \bar{\omega} = d\bar{H} \circ c \, . \qquad 1$$

3 In contrast to the hamiltonian systems of chapter (1), some points of M may not admit evolution, and for those that do, this evolution may not be unique. For example, if $m \in M$ is such that

$d\bar{H}(m) \notin \bar{\omega}^{\flat}(T_mM)$, then equation (2.1) is inconsistent at m, so m cannot admit evolution. Furthermore, if $d\bar{H}(m) \in \bar{\omega}^{\flat}(T_mM)$, the solutions to the equation

$$v \lrcorner\, \bar{\omega}(m) = d\bar{H}(m)$$

are undetermined up to vectors in $\ker(\bar{\omega}(m))$, which allows the possibility of a nonunique evolution.

4 Denote by M_e the set of points of M that admit evolution. The condition that $\bar{\omega}$ is degenerate implies symmetries of the phase space, in the sense that points of M_e which evolve concurrently from the same point may be considered physically equivalent. Specifically, if $m_1, m_2 \in M_e$ are such that there are curves c_1 and c_2 which satisfy equation (2.1) and such that $c_1(0) = c_2(0)$, $c_1(t) = m_1$, $c_2(t) = m_2$ for some t, then write $m_1\mathring{R}_g m_2$. Let R_g be the smallest equivalence relation on M_e containing $\mathring{R}_g$, and call m_1 and m_2 gauge equivalent if $m_1R_gm_2$. The gauge equivalence relation measures the extent to which the evolution defined by equation (2.1) fails to be unique.

5 It can happen that there are curves c_1 and c_2 with common domain such that $c_1(t)R_gc_2(t)$ for all t but such that c_1 satisfies equation (2.1) and c_2 does not. In this case, the evolution defined by equation (2.1) is regarded as inadequate to represent the evolution of the system, and the evolution of the system is augmented by all such curves c_2.

6 If $S \subseteq M$, define the statement $P(S)$ by

$P(S)$: for all $m \in S$, there is a vector $v \in T_m S$ such that

$$v \lrcorner \bar{\omega}(m) = d\bar{H}(m).$$

Obviously, if $S_1, S_2 \subseteq M$ and $P(S_1)$, $P(S_2)$ are true, then $P(S_1 \cup S_2)$ is true. It follows that

$$M_f = \cup \{S \subseteq M \; ; \; P(S)\}$$

is the unique maximal element of $\{S \; ; \; P(S)\}$. The set M_f is called the final constraint set. Clearly, $P(M_e)$ is true, so that $M_e \subseteq M_f$. It is often the case that $M_e = M_f$, but a proof is unavailable at this level of generality. The existence question for the evolution defined by equation (2.1) is approached by finding M_f and showing, by example specific methods, that $M_e = M_f$. For instance, one might attempt to find a smooth vector field X on M such that $X \lrcorner \bar{\omega} = d\bar{H}$ on M and such that M_f is an invariant set for the flow of X.

7 In many cases, one can compute M_f by a finite number of iterations of the Dirac algorithm, in a formulation due to Gotay-Nester-Hinds [1978]. The algorithm generates a sequence of subsets M_i of M defined as follows:

$$M_0 = M$$

$$M_{i+1} = \{m \in M_i \; ; \; \exists \; v \in T_m M_i \text{ such that } v \lrcorner \bar{\omega}(m) = d\bar{H}(m)\}.$$

The algorithm terminates if $M_n = M_{n+1}$ for some n, in which case it is obvious that $M_n = M_f$. The intermediate construct M_i is called the i^{th}

secondary constraint set. When M is reflexive, $\bar{\omega}$ is topologically closed, and M_i is an imbedded submanifold of M, then paragraph (1.3) shows that

$$M_{i+1} = \{m \in M_i \ ; \ d\bar{H}(m)(T_m M_i)^{\bar{\omega}\perp} = 0\}.$$

8 In chapter (7), the following situation will be of interest: M is a coisotropic, imbedded submanifold of a finite dimensional symplectic manifold (P,ω), $\bar{\omega} = i^*\omega$ where $i : M \to P$ is the inclusion map and there is a function H on P such that $H|M = \bar{H}$. Gotay [1980] implies that many presymplectic systems may be so realized. Suppose that M_f is a strongly first class subset of (P,ω). If $m \in M_f$, then

$$\begin{aligned} T_m M_f^{\bar{\omega}\perp} &= \{v \in T_m M_f \ ; \ \bar{\omega}(v, T_m M_f) = 0\} \\ &= \{v \in T_m M_f \ ; \ \omega(v, T_m M_f) = 0\} \\ &= T_m M_f \cap T_m M_f^{\omega\perp} . \end{aligned}$$

Since M_f is strongly first class, M_f is coisotropic, so that

$$\mathrm{null}(M_f) = TM_f^{\omega\perp} = T_m M_f^{\bar{\omega}\perp} .$$

By the definition of M_f, if $m \in M_f$ then there is a $v \in T_m M_f$ such that $v \lrcorner \bar{\omega}(m) = d\bar{H}(m)$, and it follows that

$$dH(\mathrm{null}(M_f)) = d\bar{H}\left[T_m M_f^{\bar{\omega}\perp}\right] = 0 .$$

Therefore, H is an observable, so X_H is tangent to M_f. Hence, each point of M_f admits evolution, and $M_e = M_f$.

9 With the same context as the previous paragraph, suppose that M_f is an imbedded submanifold of (P,ω). Suppose that $TM^{\omega\perp}|M_f$ is generated by a set of vector fields G_0, and write $TM^{\omega\perp}|M_f = \mathrm{dstb}(G_0)$. Consider the following gauge vector field algorithm (Gotay-Nester [1979a]):

$$G_{i+1} = G_i \cup [G_i,G_i] \cup [X_H,G_i] \ .$$

The algorithm terminates if $G_n = G_{n+1}$, the common value of which is denoted by G_f. The span of G_f is the smallest Lie subalgebra of vector fields on M_f which contains G_0 and is mapped to itself under the action of L_{X_H}. The next part of this chapter is devoted to proving the following fact: if G_f is a finite set, the connected components of the equivalence classes of the gauge relation are exactly the maximal integral submanifolds of $\mathrm{dstb}(G_f)$.

10 Note first that any piecewise smooth curve c which satisfies equation (2.1) is locally an integral curve of a time dependent vector field of the form $X_H + f_iY^i$, where the f_i are smooth functions on $M_f \times \mathbb{R}$ and the Y^i are vector fields in G_0. Thus, any evolution curve may be constructed by concatenating the integral curves of such vector fields. Indeed, if $t_0 \in \mathrm{domain}(c)$, choose $Y^i \in G_0$ such that the vectors $Y^i(c(t_0))$ form a basis of $T_{c(t_0)}M^{\omega\perp}$. As $TM^{\omega\perp}|M_f$ is a subbundle of TM_f, there is an $\epsilon > 0$ such that $Y^i(c(t))$ forms a basis of $T_{c(t)}M^{\omega\perp}$, and therefore, there are smooth functions f_i defined on a neighbourhood of $c(t_0) \times (t_0 - \epsilon, t_0 + \epsilon)$ such that

$$\frac{dc}{dt} - X_H(c(t)) = f_i(c(t),t)Y^i \;, \qquad 1$$

since the left hand side takes values in $TM^{\omega\perp}|M_f$:

$$\omega\left[\frac{dc}{dt}(t) - X_H(c(t)),v\right] = d\bar{H}(c(t))v - dH(c(t))v = 0,$$

for all $v \in T_mM$. Choosing some $\epsilon' < \epsilon$, one may extend the functions f_i to all of $M_f \times \mathbb{R}$ in such a manner that equation (10.1) holds on $(t_0 - \epsilon', t_0 + \epsilon')$.

11 Denote by D_1, D_2 the vector fields on $M_f \times \mathbb{R}$ of the following form:

> D_1 : $(m,s) \to (X_H(m) + Y(m,s),s,1)$ where Y is a smooth, time dependent vector field on $M_f \times \mathbb{R}$ which takes values in $TM^{\omega\perp}$,
>
> D_2 : $(m,s) \to (G(m),s,0)$, where $G \in G_f$, and $(m,s) \to (X_H(m),s,1)$.

Let $D_3 = D_1 \cup D_2$.

12 Define the D_i reachability relation as follows: $x_1 D_i x_2$ if there are vector fields $X_1,\ldots,X_n \in D_i$ and real numbers $t_1,\ldots,t_n$ such that $x_1 = F^{X_1}_{t_1} \circ \ldots \circ F^{X_n}_{t_n}(x_2)$, where $F^{X_k}_{t_k}$ denotes the flow of X_k. The D_i reachability relation is an equivalence relation, and the D_i reachability equivalence class of $x \in M_f \times \mathbb{R}$ is denoted by $[x]_{D_i}$.

13 The set of vector fields D_3 is locally of finite type; that is, if $x \in M \times \mathbb{R}$, then there are vector fields $X^1,\ldots,X^n$ on $M \times \mathbb{R}$ such that:

1. $X^1(x),\ldots,X^n(x)$ span $\mathrm{dstb}(D_3)$.

2. If $X \in D_3$, then there is some open neighbourhood U of x and smooth functions f^i_j on U such that

$$[X,X^i]|U = \sum_j f^i_j(X^j|u) \ .$$

One simply takes X^i to be the vector fields in D_2 and verifies the following: for $G \in G_f$, $Y^i \in G_0$, and f_i smooth functions on $M_f \times \mathbb{R}$,

$$\Big[(m,s) \to X_H(m) + (f_i(m,s)Y^i(m),s,1),(m,s) \to (G(m),s,0)\Big]$$
$$= (m,s) \to ([X_H,G](m),s,0) + \Big[-\Big[L_G f^s_i\Big](m)Y^i(m),s,0\Big]$$
$$+ (f(m,s)[Y^i,G](m),s,0) \ ,$$

where $f^s_i : M \to \mathbb{R}$ by $f^s_i(m) = f_i(m,s)$, and,

$$\Big[(m,s) \to X_H(m) + (f_i(m,s)Y^i(m),s,1),(m,s) \to (X_H(m),s,1)\Big]$$
$$= (m,s) \to \left[\left[\frac{\partial f_i}{\partial s}(m,s) - L_{X_H} f^s_i\right] Y_i(m),s,0\right]$$
$$+ \Big[f_i(m,s)[Y^i,X_H](m),s,0\Big] \ .$$

14 Let $x \in M_f \times \mathbb{R}$. The results of Sussman [1973] imply that $[x]_{D_3}$ is an immersed submanifold and maximal integral submanifold of $\mathrm{dstb}(D_3)$. The same reference shows that $[x]_{D_1}$ is an immersed submanifold and maximal integral submanifold of some distribution, say E. Obviously, $[x]_{D_1} \subseteq [x]_{D_3}$, so $E \subseteq \mathrm{dstb}(D_3)$. Taking Lie brackets of

vector fields in D_1 and considering the form of G_f and the fact that E is involutive, it is apparent that $\mathrm{dstb}(D_3) \subseteq E$, so $\mathrm{dstb}(D_3) = E$. Thus, $[x]_{D_1} = [x]_{D_3}$, by the uniqueness of maximal integral submanifolds.

15 Let $p_1 : M_f \times \mathbb{R} \to M_f$ be the projection onto the first factor. It is obvious that the gauge equivalence class $[m]_g$ of $m \in M$ is

$$[m]_g = p_1\left[[m,0]_{D_1} \cap M \times \{0\}\right] .$$

That is, $[m]_g$ is the set of points reachable from m in total time zero. If $m' \in M_f$, then

$$T_{(m',0)}[(m,0)]_{D_1} = \mathrm{dstb}(D_3)_{(m',0)}$$

contains the vector $(X_H(m'),0,1)$. Therefore, $[(m,0)]_{D_1}$ is transversal to $M \times \{0\}$, so that $[m]_g$ is an immersed submanifold of M.

16 If $m' \in [m]_g$, then

$$\begin{aligned} T_{m'}[m]_g &= Tp_1\left[T_{(m',0)}[(m,0)]_{D_1} \cap T_{(m',0)}M \times \{0\}\right] \\ &= Tp_1\left[\mathrm{dstb}(D_3)_{(m',0)} \cap T_{(m',0)}M \times \{0\}\right] \\ &= Tp_1\left[\mathrm{span}(X(m',0)\ ;\ X \in D_2) \cap T_{(m',0)}M \times \{0\}\right] , \end{aligned}$$

by property (13.1) of the set of vector fields D_3. Thus,

$$T_{m'}[m]_g = Tp_1\Big[(\Sigma\ a_iG_i(m') + (bX_H(m),s,b)\ ;\ G_i \in G_f, a_i, b \in \mathbb{R}) \cap T_{(m',0)}M \times \{0\}\Big]$$

$$= Tp_1\Big[(\Sigma\ a_iG_i(m'),s,0)\ ;\ G_i \in G_f, a_i \in \mathbb{R}\Big]$$

$$= \mathrm{dstb}(G_f)_{m'}\ ,$$

so that $[m]_g$ is an integral submanifold of $\mathrm{dstb}(G_f)$.

17 The set of vector fields G_f is clearly of locally finite type. Thus, if $m \in M$, the G_f reachability class of m, $[m]_{G_f}$, is a maximal integral submanifold of $\mathrm{dstb}(G_f)$. Since the vector fields

$$(m,s) \to (G(m),s,0)\ ,$$

where $G \in G_f$, are contained in D_3,

$$[m]_{G_f} \times \{0\} \subseteq [(m,0)]_{D_3} \cap M \times \{0\} = [m]_g \times \{0\}\ ,$$

so that $[m]_{G_f} \subseteq [m]_g$. Therefore, the connected components of $[m]_g$ are the maximal integral submanifolds of $\mathrm{dstb}(G_f)$.

18 It is useful to note the following fact: the gauge relation defined by the evolution generated by the time dependent vector fields of paragraph (10) and by the gauge relation generated by the time independant vector fields $X_H + Y$, $Y \in G_0$ are identical. For the proof, let D_4 be the set of vector fields on $M_f \times \mathbb{R}$ of the form

$$(m,s) \to (X_H(m) + Y(m),s,1)\ ,$$

where $Y \in G_0$. Then $D_4 \cup D_2$ is locally of finite type for the same reason that D_3 is, and the same argument as paragraph (14) shows that, if $x \in M_f \times \mathbb{R}$, $[x]_{D_4} = [x]_{D_4 \cup D_2}$. Since $\mathrm{dstb}(D_4 \cup D_2) = \mathrm{dstb}(D_3)$, the uniqueness of maximal integral submanifolds shows that $[x]_{D_4} = [x]_{D_3} = [x]_{D_1}$, which implies that the two gauge relations are the same.

19 This fact may be used to show that if the vector fields $X_H + Y$, $Y \in G_0$ are complete on M_f, then the gauge equivalence classes of the evolution defined by equation (2.1) are connected, and hence are exactly the maximal integral submanifolds of $\mathrm{dstb}(G_f)$. Indeed, if $m_1 R_g m_2$, then there are vector fields $Y^i \in G_0$ and real numbers t_i such that $\Sigma\, t_i = 0$ and

$$m_1 = F_{t_1}^{X_H+Y^1} \circ \ldots \circ F_{t_n}^{X_H+Y^n} (m_2) .$$

But completeness of the vector fields implies that the curve

$$\gamma : s \to F_{st_1}^{X_H+Y^1} \circ \ldots \circ F_{st_n}^{X_H+Y^n} (m_2)$$

is well defined, and $\gamma(1) = m_1$, $\gamma(0) = m_2$. Thus, any two points of a gauge equivalence class may be connected by a smooth curve.

CHAPTER 6

Lagrangian Systems

1 Many physical systems have a natural lagrangian formulation: there is given a Banach manifold Q, called the configuration space, and a smooth function $L : TQ \to \mathbb{R}$ called the lagrangian. The Legendre transformation is the smooth, fiber preserving map $FL : TQ \to T^*Q$ defined by taking the fiberwise derivative of L:

$$\begin{aligned} FL(v_q)w_q &= D(L|T_qQ)(v_q)w_q \\ &= \frac{d}{dt}\bigg|_{t=0} L(v_q + tw_q) . \end{aligned}$$

Define the Lagrange one and two forms, and the energy function, by

$$\begin{aligned} \theta_L &= FL^*\theta_0 \\ \omega_L &= FL^*\omega_0 \\ E(v_q) &= FL(v_q)v_q - L(v_q) . \end{aligned}$$

Points of TQ evolve along smooth curves c such that

$$\frac{dc}{dt} \lrcorner\, \omega_L = dE \circ c \quad . \tag{1}$$

2 Define the smooth, fiber preserving map $F^2L : Q \to T^0_2Q$ by

$$F^2L(v_q)(w_q,w_{q'}) = D^2(L|T_qQ)(v_q)(w_q,w'_q) .$$

F^2L has image in the symmetric elements of T^0_2Q. The Lagrangian L is called (weakly) regular if F^2L has image in the (weakly) nondegenerate elements of T^0_2Q. In a natural chart of TQ with range $U \times \mathbb{E}$, $FL(u,e) = (u, D_2L(u,e))$, so

$$TFL(u,e,e_1,e_2) = (u, D_2L(u,e), e_1, D_1D_2L(u,e)e_1 + D_2D_2L(u,e)e_2) \ , \qquad 1$$

and also,

$$F^2L(u,e)(e_1,e_2) = D_2^2L(u,e)(e_1,e_2) \ . \qquad 2$$

An examination of these equations shows that L is (weakly) regular if and only if FL is an (immersion) local diffeomorphism. If FL is a diffeomorphism, then L is called hyperregular. If L is regular, and Q is reflexive, then ω_L is symplectic, and the evolution defined by equation (1.1) is given by the flow of the hamiltonian vector field of E.

3 The following theorem gives an important property of curves which satisfy equation (1.1), and displays the connection between such curves and the classical Euler-Lagrange equations. A curve c in TQ is called second order if it is the derivative of its projection to Q; that is, if

$$\frac{d}{dt}(\tau_Q \circ c) = c \ .$$

4 <u>Theorem</u>. Let Q be a Banach manifold and L be a smooth function on TQ.

1. If a smooth curve c in TQ satisfies equation (1.1), then it

also satisfies

$$F^2L(c(t))\left[c(t) - \frac{d}{dt}(\tau_Q \circ c)(t), w\right] = 0 \ ,$$

for all $w \in T_{\tau_Q \circ c(t)}Q$. Thus, if L is weakly regular, any curve satisfying equation (1.1) is second order.

2. In a natural chart of TQ with range $U \times \mathbb{E}$, a second order curve

$$c(t) = \left[u(t), \frac{du}{dt}(t)\right]$$

satisfies equation (1.1) if and only if it satisfies Lagrange's equations in this chart:

$$\frac{d}{dt}(D_2L(c(t))f) = D_1L(c(t))f$$

for all $f \in \mathbb{E}$. In finite dimensions, these are equivalent to the classical Euler-Lagrange equations

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^K}\,(q^i(t), \dot{q}^i(t))\right] = \frac{\partial L}{\partial q^K}\,(q^i(t), \dot{q}^i(t)) \ ,$$

using coordinates $q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n$ on TQ.

<u>Proof</u>

5 Paragraph (1.8) and equation (2.1) show that

$$\begin{aligned} &\omega_L(u,e)((u,e,e_1,e_2),(u,e,e_1{}',e_2{}')) \\ &\quad = \omega_0(FL(u,e))(TFL(u,e,e_1,e_2), TFL(u,e,e_1{}',e_2{}')) \\ &\quad = D_1D_2L(u,e)(e_1{}',e_1) - D_1D_2L(u,e)(e_1,e_1{}') \\ &\qquad + D_2D_2L(u,e)(e_2{}',e_1) - D_2D_2L(u,e)(e_2,e_1{}') \ . \end{aligned}$$

Also, $E(u,e) = D_2L(u,e)e - L(u,e)$, so

$$\begin{aligned} dE(u,e)(u,e,e_1',e_2') &= D_1D_2L(u,e)(e_1',e) + D_2D_2L(u,e)(e_2',e) \\ &\quad + D_2L(u,e)e_2' - D_1L(u,e)e_1' - D_2L(u,e)e_2' \\ &= D_1D_2L(u,e)(e_1',e) + D_2D_2L(u,e)(e_2',e) \\ &\quad - D_1L(u,e)e_1' \ . \end{aligned}$$

…6 Collecting terms, a vector (u,e,e_1,e_2) will satisfy

$$(u,e,e_1,e_2) \lrcorner\, \omega_L = dE(u,e)$$

if and only if

$$\begin{aligned} &D_2D_2L(u,e)(e_2',e - e_1) \\ &+ D_1D_2L(u,e)(e_1',e_1 - e) \\ &+ D_1D_2L(u,e)(e_1,e_1') + D_2D_2L(u,e)(e_2,e_1') - D_1L(u,e)e_1' = 0 \ , \end{aligned}$$

for all $e_1',e_2' \in \mathbb{E}$. Letting $e_1' = 0$ and $e_2' = 0$ separately, these equations are equivalent to the following two equations:

$$D_2^2L(u,e)(f,e - e_1) = 0 \qquad 1$$

$$D(D_2L(u,e)f)(e_1,e_2) - D_1L(u,e)f = -D_1D_2L(u,e)(e_1 - e) \ . \qquad 2$$

…7 If $c(t) = (u(t),e(t))$ satisfies equation (1.1), then substituting

$$\left[u(t),\ e(t),\ \frac{du}{dt}(t),\ \frac{d^2u}{dt^2}(t)\right]$$

for (u,e,e_1,e_2) in equation (6.1) and observing equation (2.2) yields the first statement of the theorem. Substituting

$$\left[u(t),\ \frac{du}{dt}(t),\ \frac{du}{dt}(t),\ \frac{d^2u}{dt^2}(t)\right]$$

for (u,e,e_1,e_2), it is apparent that equation (6.1) is statisfied identically, and that equation (6.2) yields Lagrange's equations in the chart. ∎

8 If L is hyperregular, one may define a smooth function H on T^*Q by $H = E \circ FL^{-1}$. The hamiltonian systems (TQ,ω_L,E) and (T^*Q,ω_0,H) are in bijective correspondence via the symplectomorphism FL. One calls the system (T^*Q,ω_0,H) the canonical formulation of the lagrangian system. It is possible to construct a lagrangian system from a hamiltonian system on T^*Q, under conditions similar to hyperregularity (Abraham-Marsden [1978: 221]).

9 A lagrangian $L : TQ \to \mathbb{R}$ is called semiregular if FL is a subimmersion (Abraham-Marsden-Ratiu [1983: 171]) and the level sets of FL are connected.

10 Theorem. Let Q be a Banach manifold and $L : T_qQ \to \mathbb{R}$ be semiregular. Let $v_q \in T_qQ$ admit evolution; that is, there is a smooth curve c in TQ that satisfies equation (1.1) and $c(0) = v_q$. If $v'_q \in T_qQ$ is such that $FL(v_q) = FL(v'_q)$, then there is a smooth curve c' in TQ that satisfies equation (1.1), $c'(0) = v'_q$, and $FL \circ c'$ is a restriction of $FL \circ c$. Furthermore, v_q and v'_q are gauge equivalent.

Proof

11 Let $\gamma : [0,1] \to TQ$ be a smooth curve such that $FL \circ \gamma$ is a constant, say α_q. Then γ lies within the fiber T_qQ of TQ, since FL is

fiber preserving. E is constant along γ:

$$\begin{aligned}\frac{d}{dt}(E \circ \gamma(t)) &= \frac{d}{dt}\Big[FL(\gamma(t))\gamma(t) - L \circ \gamma(t)\Big] \\ &= \frac{d}{dt}\Big[\alpha_q(\gamma(t)) - L \circ \gamma(t)\Big] \\ &= \alpha_q \frac{d\gamma}{dt}(t) - FL(\gamma(t)) \frac{d\gamma}{dt}(t) \\ &= 0 .\end{aligned}$$

As FL is a subimmersion, $FL^{-1}(\alpha_q)$ is an imbedded submanifold of TQ, so $FL^{-1}(\alpha_q)$ is smoothly pathwise connected. Thus, E is constant on the level sets of FL.

...12 Let U and V be open subsets of TQ and $\alpha : U \to V$ be a diffeomorphism such that $FL \circ \alpha = FL$. If c is a curve in U satisfying equation (1.1), then $\alpha \circ c$ also satisfies equation (1.1):

$$\begin{aligned}\Big[\frac{d}{dt}(\alpha \circ c)(t) \lrcorner\, \omega_L\Big](w) &= \omega_L(\alpha \circ c(t))\Big[\frac{d}{dt}(\alpha \circ c)(t), w\Big] \\ &= \omega_0(FL \circ \alpha \circ c(t))\Big[TFL\Big[\frac{d}{dt}(\alpha \circ c)(t)\Big], TFLw\Big] \\ &= \omega_0(FL \circ c(t))\Big[TFL\Big[\frac{dc}{dt}(t)\Big], TFL(T\alpha T\alpha^{-1}w)\Big] \\ &= \omega_0(FL \circ c(t))\Big[TFL\Big[\frac{dc}{dt}(t)\Big], TFL(T\alpha^{-1}w)\Big] \\ &= \frac{dc}{dt}(t) \lrcorner\, \omega_L(T\alpha^{-1}w) \\ &= dE(T\alpha^{-1}w) \\ &= d(E \circ \alpha^{-1})w \\ &= dE(w) ,\end{aligned}$$

since E is constant along the level sets of FL.

...13 Let $w_0 \in TQ$. Since FL is a subimmersion, there is a neighbourhood U of w, a convex neighbourhood V of 0 in $\ker(T_w FL)$, and a smooth map $p : U \to V$ such that $p(w_0) = 0$, FL(U) is an imbedded submanifold of T^*Q, and the map $\Psi : U \to FL(U) \times V$ by $\Psi(w) = (FL(w),p(w))$ is a diffeomorphism. I claim that the theorem is true when restricted to points and curves in U.

...14 Let c be a smooth curve in U which satisfies equation (1.1) and let $c(0) = v_q$. Suppose $v'_q \in U$ is such that $FL(v_q) = FL(v'_q)$. The translation $\tau_1 : (x,y) \to (x,y + p(v'_q) - p(v_q))$ maps an open neigbourhood $W_1 \subseteq FL(U) \times V$ of $\Psi(v_q)$ to an open neighbourhood $W_2 \subseteq FL(U) \times V$ of $\Psi(v'_q)$, since

$$\begin{aligned}\tau_1(\Psi(v_q)) &= \tau_1(FL(v_q),p(v_q)) \\ &= (FL(v_q),p(v'_q)) \\ &= (FL(v'_q),p(v'_q)) \\ &= \Psi(v'_q) \ .\end{aligned}$$

Thus, $\tau_2 : \Psi^{-1}(W_1) \to \Psi^{-1}(W_2)$ by $\tau_2 = \Psi^{-1} \circ \tau_1 \circ \Psi$ is a diffeomorphism such that $\tau_2(v_q) = v_{q'}$ and $FL \circ \tau_2 = \tau_2$. For some $a_1 > 0$, $c((-a_1,a_1)) \subseteq \Psi^{-1}(W_1)$, so $c' = \tau_2 \circ c|(-a_1,a_1)$ is a curve which satisfies equation (1.1), by paragraph (12), $c'(0) = v'_q$ and $FL \circ c'$ agrees with $FL \circ c$ on $(-a_1,a_1)$.

...15 Suppose that $\left.\frac{d}{dt}\right|_0 FL \circ c(t) \neq 0$. Then $FL \circ c$ admits a local left inverse at $t = 0$; that is, a map $\beta : A \subseteq FL(U) \to (-a_2,a_2)$ where A is an open set containing $c((-a_2,a_2))$, $\beta(v_q) = 0$ and β satisfies $\beta \circ FL \circ c(t) = t$ for all $t \in (-a_2,a_2)$. One may assume that $a_2 < a_1$.

Let $f : \mathbb{R} \to [0,1]$ be a smooth function which is 0 on $(-\infty, a_2/2]$ and is 1 on $[0,\infty)$. Define the map $\tau_3 : A \times V \to A \times \ker(T_w FL)$ by

$$\tau_3(x,y) = (x, y + (f \circ \beta(x)) \cdot (p \circ c' \circ \beta(x) - p \circ c \circ \beta(x)) .$$

If $t \in (-a_2, a_2)$, then

$$\begin{aligned}\tau_3(\Psi \circ c(t)) = (FL(c(t)),\ & p \circ c(t) + f(t) \\ & \cdot (p \circ c'(t) - p \circ c(t))) ,\end{aligned}$$

so τ_3 maps $\Psi \circ c((-a_2, a_2))$ into $A \times V$, since V is convex. It follows that τ_3 is a diffeomorphism from some open neighbourhood $W_3 \subseteq FL(U) \times V$ of $\Psi \circ c((-a_2, a_2))$ to some open set $W_4 \subseteq FL(U) \times V$. Define the diffeomorphism $\tau_4 : \Psi^{-1}(W_3) \to \Psi^{-1}(W_4)$ by $\tau_4 = \Psi \circ \tau_3 \circ \Psi^{-1}$. Then $c((-a_2, a_2)) \subseteq W_3$ and τ_4 satisfies $FL \circ \tau_4 = FL$, so that $c'' = \tau_4 \circ (c|(-a_2, a_2))$ is a curve that satisfies equation (1.1). But $c''(-a_2/2) = c(-a_2/2)$, $c(0) = v_q$ and $c''(0) = v_q'$, so $v_q R_g v_q'$.

...16 If $\left.\frac{d}{dt}\right|_0 FL \circ c(t) = 0$, then it is clear that $dE(v_q) = 0$, since c satisfies equation (1.1) and $\omega_L = FL^*\omega_0$. If $w \in U$ is such that $FL(w) = FL(v_q)$, the argument of paragraph (14) yields a diffeomorphism τ_5 from some open neighbourhood of v_q to some open neighbourhood of w such that $\tau_5(v_q) = w$ and $FL \circ \tau_5 = FL$. Then $E \circ \tau_5 = E$, so

$$\begin{aligned} dE(v_q) &= dE(\tau_5(w)) \\ &= dE(\tau_5(w)) T_{v_q} \tau_5 \circ T_w(\tau_5^{-1}) \\ &= (\tau_5^* dE) \circ T_w(\tau_5^{-1}) \\ &= dE \circ T_w(\tau_5^{-1}) \\ &= 0 . \end{aligned}$$

Therefore, dE is zero on $FL^{-1}(FL(v_q)) \cap U$. The following two curves are contained in $FL^{-1}(FL(v_q)) \cap U$ for some open interval I of $\mathbb{R}$ containing 0 and 1:

$$k_1(t) = \Psi^{-1}\left[FL(v_q), \frac{1}{2}(v_q + v_q') + \frac{t}{2}(v_q - v_q')\right]$$

$$k_2(t) = \Psi^{-1}\left[FL(v_q), \frac{1}{2}(v_q + v_q') - \frac{t}{2}(v_q - v_q')\right] .$$

Since $k_i(I)$ is contained in a level set of FL, $i = 1,2$,

$$\frac{dk_i}{dt}(t) \lrcorner \omega_L = 0 = dE \circ k_i(t)$$

on I. But $k_1(0) = k_2(0)$, $k_1(1) = v_q$ and $k_2(1) = v_q'$, so $v_q R_g v_q'$. This completes the proof that the theorem is true when restricted to points and curves in U.

…17 Let $v_q \in T_qQ$, $FL(v_q) = \alpha_q$ and suppose that there is a smooth curve c in TQ that satisfies equation (1.1) and $c(0) = v_q$. Consider the set

$S = \{w \in FL^{-1}(\alpha_q)$; there is a smooth curve c' in TQ which satisfies equation (1.1), $c'(0) = w$ and $FL \circ c'$ is a restriction of $FL \circ c\}$.

If $w \in S$, then there is an open neighbourhood U of w such that the theorem is true when restricted to points and curves in U. Therefore $U \cap FL^{-1}(\alpha_q) \subseteq S$. This shows that S is an open subset of $FL^{-1}(\alpha_q)$. An identical argument shows that the complement of S in $FL^{-1}(\alpha_q)$ is open. Since $FL^{-1}(\alpha_q)$ is connected, $S = FL^{-1}(\alpha_q)$. A similar argument shows that the gauge equivalence class of v_q contains $FL^{-1}(\alpha_q)$. ∎

18 Finding the points of TQ which admit evolution curves that are second order is the second order problem for lagrangian systems (Gotay-Nester [1980]). The next theorem casts doubt on the physical significance of the second order problem by showing that, in the semiregular case, every evolution curve is gauge equivalent to a second order evolution curve.

19 <u>Theorem</u>. Let Q be a Banach manifold and $L : TQ \to \mathbb{R}$ be semiregular. Let c be a smooth curve in TQ that satisfies equation (1.1).

1. If c' is any other smooth curve in TQ such that $FL \circ c = FL \circ c'$, then c also satisfies equation (1.1)
2. $FL \circ c = FL \circ \frac{d}{dt}(\tau_Q \circ c)$.

<u>Proof</u>

20 Let c be a smooth curve in TQ that satisfies equation (1.1) and suppose that c' is another smooth curve in TQ such that $FL \circ c = FL \circ c'$. Let $t_0 \in \text{domain}(c) = \text{domain}(c')$. After some translations of $\mathbb{R}$, theorem (10) implies that there is a smooth curve c" in TQ that satisfies equation (1.1), $c''(t_0) = c'(t_0)$ and $FL \circ c''$ is a restriction of $FL \circ c = FL \circ c'$. Therefore,

$$
\begin{aligned}
TFL\left[\left.\frac{d}{dt}\right|_{t_0} c'(t) - \left.\frac{d}{dt}\right|_{t_0} c''(t)\right]
&= \left.\frac{d}{dt}\right|_{t_0} FL \circ c'(t) - \left.\frac{d}{dt}\right|_{t_0} FL \circ c''(t) \\
&= 0 \ ,
\end{aligned}
$$

so that,

$$\left.\frac{d}{dt}\right|_{t_0} c'(t) \lrcorner\, \omega \;=\; \left.\frac{d}{dt}\right|_{t_0} c''(t) \lrcorner\, \omega_L = dE(c''(t_0))$$

$$= dE(c'(t_0)) \;.$$

As t_0 is arbitrary, this shows that c' satisfies equation (1.1).

...21 For the second statement, let $t_0 \in$ domain(c), $\alpha_q = FL(c(t_0))$ and

$$a = \left.\frac{d}{dt}\right|_{t_0} (\tau_Q \circ c(t)) \;.$$

Let X be the vector field on T_qQ defined by $X(v_q) = a - v_q$. If $v_q \in FL^{-1}(\alpha_q)$, then there is a curve c' in TQ which satisfies equation (1.1) $c'(0) = v_q$, and $FL \circ c'$ is a restriction of $FL \circ c$. By statement (4.1),

$$\begin{aligned} D(FL|T_qQ)(v_q)((v_q - a, v_q') &= F^2L(v_q)(v_q - a, v_q') \\ &= F^2L(v_q)\left[c'(0) - \left.\frac{d}{dt}\right|_{t_0} (\tau_Q \circ c(t)), v_q'\right] \\ &= F^2L(v_q)\left[c'(0) - \left.\frac{d}{dt}\right|_{t_0} (\tau_Q \circ c'(t)), v_q'\right] \\ &= 0 \;, \end{aligned}$$

for any $v_q' \in T_qQ$. Therefore, X is tangent to the closed, imbedded submanifold $FL^{-1}(\alpha_q)$. The curve

$$\gamma(t) = a + (c(t_0) - a)e^{-t}$$

is an integral curve of X with initial condition $\gamma(0) = c(t_0) \in FL^{-1}(\alpha_q)$, so γ is a curve in $FL^{-1}(\alpha_q)$. Then

$$\begin{aligned} FL(a) &= FL\left[\lim_{t\to\infty} \gamma(t)\right] \\ &= \lim_{t\to\infty} FL(\gamma(t)) \\ &= \alpha_q \\ &= c(t_0) . \end{aligned}$$

As t_0 was arbitrary, this completes the proof of statement (19.2). ∎

22 Suppose that L is semiregular, and an open or closed map onto its image. Then M_0 = Image(FL) is an imbedded submanifold of T^*Q. Since E is constant along the level sets of FL and FL is a submersion onto M_0, there is a smooth function H on M_0 such that $E = \bar{H} \circ FL$. Let $i : M_0 \to T^*Q$ be the inclusion. The presymplectic system $(M_0, i^*\omega_0, \bar{H})$ is called the canonical formulation of the lagrangian system.

23 The evolution on M_0 defined by the presymplectic system $(M_0, i^*\omega_0, \bar{H})$ is the image under FL of the lagrangian evolution on TQ. Indeed, a straightforward computation shows that if c' is any smooth curve in TQ that satisfies equation (1.1), and $c = FL \circ c'$, then

$$\frac{dc}{dt} \lrcorner\, i^*\omega_0 = d\bar{H} \circ c .$$

On the other hand, let c be a curve in T^*Q that satisfies this equation, and let $t_0 \in$ domain(c). Choose $w_0 \in TQ$ such that $FL(w_0) = c(t_0)$. Using a neighbourhood U of w_0 as in paragraph (13), one may find a curve $c' : (t_0 - \epsilon, t_0 + \epsilon) \to U$ such that $FL \circ c' = c|(t_0 - \epsilon, t_0 + \epsilon)$. A straightforward computation shows that c' satisfies equation (1.1). Therefore

$$\frac{d}{dt}\left[\tau_Q^* \circ c|(t_0 - \epsilon, t_0 + \epsilon)\right] = \frac{d}{dt}\left[\tau_Q \circ c'|(t_0 - \epsilon, t_0 + \epsilon)\right]$$

also satisfies equation (1.1), and

$$\begin{aligned} FL \circ \frac{d}{dt}\left[\tau_Q^* \circ c|(t_0 - \epsilon, t_0 + \epsilon)\right] &= FL \circ c'|(t_0 - \epsilon, t_0 + \epsilon) \\ &= c|(t_0 - \epsilon, t_0 + \epsilon) \ . \end{aligned}$$

As t_0 is arbitrary, this shows that the curve

$$t \to \frac{d}{dt}\left[\tau_Q^* \circ c\right](t)$$

satisfies equation (1.1) and is mapped by FL to c. Thus, c is the image of a lagrangian evolution curve.

CHAPTER 7

The Extended Canonical Formalism and the Gauge Group

1 Further analysis of the lagrangian evolution of chapter (6) might proceed by a study of the presymplectic system $(M_0, i^*\omega, \bar{H})$. This presymplectic approach has the advantage of requiring no additional data for its implementation, but has some drawbacks. The familiar notions of Poisson bracket and momentum mapping, for example, are difficult or impossible to define in a presymplectic context. The problem of quantization motivates the attempt to realize the evolution on M_0 as a set of constrained hamiltonian evolutions, providing a cogent reason for retaining the symplectic structure of T^*Q from the outset.

2 For this program, one needs additional structure on the phase space T^*Q. Namely, assume the following:

1. There is a smooth function H on T^*Q such that $\bar{H} = H|M_0$.
2. $(T^*Q, \omega_0, \phi^0, J^0)$ is a hamiltonian G^0 space such that the action ϕ^0 is infinitesimally free and $M_0 = (J^0)^{-1}(0)$.

G^0 is called the primary gauge group. This structure serves to extend the evolution on M_0 to an evolution on all of T^*Q: one decrees that points of T^*Q evolve along smooth curves c in T^*Q such that

$$\left[\frac{dc}{dt} \lrcorner\ \omega_0 = dH \circ c\right] \cdot \text{fund\_vf}(\phi^0, L(G^0))^{\omega_0\perp} . \qquad 1$$

That is,

$$\left[\frac{dc}{dt}(t) \lrcorner\ \omega_0\right] v = d\dot{H}(c(t))v ,$$

for all $v \in \text{fund\_vf}(\phi, L(G), c(t))^{\omega_0\perp}$.

3 Consider the lagrangian $L : TQ \to \mathbb{R}$ by

$$L(v_q) = {}^1/_2\ B(q)(v_q, v_q) + \beta(q)v_q - V(q) ,$$

where B is a smooth section of T^0_2Q of constant rank and image in the symmetric elements of T^0_2Q, β is a smooth one form on Q, and V is a smooth function on Q. One easily shows that

$$FL(\dot{v}_q)w_q = B(q)(v_q, w_q) + \beta(q)w_q ,$$
$$E(v_q) = {}^1/_2\ B(v_q, v_q) + V(q) ,$$
$$F^2L(v_q) = B(q) .$$

Thus, F^2L is of constant rank, so FL is a subimmersion. Also, if $\alpha_q \in \text{Image}(FL)$, then $FL^{-1}(\alpha_q)$ is a translation of a subspace of T_qQ, and hence is connected, so L is semiregular. Since $B^\flat$ is a vector bundle morphism of constant rank, it is an open mapping onto its image. As FL is $B^\flat$ followed by the diffeomorphism of TQ which is addition by β, FL is also an open mapping onto its image.

4 Suppose that the distribution $\ker(B)$ on TQ is spanned by the fundamental vector fields of an infinitesimally free action ϕ of a Lie group G^0. Assume that the map

$$q \to \beta(q)(\mathrm{fund\_vf}(\phi,\xi,q))$$

has constant value μ_0 on Q. The action ϕ provides the hamiltonian G^0 space $(T^*Q, \omega_0, \phi^{T^*}, J^0)$, as defined in paragraph (2.15).

5 The expression $FL = B^\flat + \beta \circ \tau_Q$ shows that

$$\begin{aligned} M_0 &= \mathrm{Image}(FL) \\ &= \mathrm{ann}(\ker(B)) + \beta \,. \end{aligned}$$

Also,

$$\begin{aligned} & \alpha_q \in (J^0)^{-1}(\mu_0) \\ \Longleftrightarrow\quad & J^0_\xi(\alpha_q) - \mu_0(\xi) = 0 \quad \text{for all} \quad \xi \in L(G) \\ \Longleftrightarrow\quad & (\alpha_q - \beta(q))(\mathrm{fund\_vf}(\phi,\xi,q)) = 0 \quad \text{for all} \quad \xi \in L(G) \\ \Longleftrightarrow\quad & \alpha_q - \beta(q) \in \mathrm{ann}(\ker(B)) \\ \Longleftrightarrow\quad & \alpha_q \in \mathrm{ann}(\ker(B)) + \beta \,, \end{aligned}$$

so $(J^0)^{-1}(\mu_0) = M_0$. If μ_0 is fixed under the Ad^* action of G^0 on $L(G^0)^*$, then M_0 is coisotropic, and M_0 is the zero level of the Ad^* equivariant momentum mapping $J^0 - \mu_0$ for the ϕ^{T^*} action of G^0 on T^*Q. This is obviously the case when G^0 is commutative. Another sufficient condition is that β is a G^0 invariant one form on Q, since in this case,

$$\begin{aligned}
Ad^*_g\mu_0(\xi) &= \mu_0\left[Ad_{g^{-1}}\xi\right] \\
&= B(q)\left[\mathrm{fund_vf}\left[\phi, Ad_{g^{-1}}\xi, q\right]\right] \\
&= \beta(q)((\phi^*_g \mathrm{fund_vf}(\phi,\xi)(q)) \\
&= \left[\phi^*_{g^{-1}}\beta\right](q)(\mathrm{fund_vf}(\phi,\xi,q)) \\
&= \mu_0 \ .
\end{aligned}$$

6 Regardless of whether μ_0 is a fixed point of the Ad^* action or not, one may extend $\bar{H}$ by choosing a complement E to ker(B). If Q is a riemannian manifold, then one may take E to be the orthogonal complement of ker(B), so the entire extension process depends only on the choice of action ϕ. The splitting $TQ = E \oplus \ker(B)$ gives rise to the splitting $T^*Q = \mathrm{ann}(E) \oplus \mathrm{ann}(\ker(B))$ and the projections $p_1 : TQ \to E$ and $p_1^* : TQ \to \mathrm{ann}(\ker(B))$. The map $\pi : T^*Q \to M_0$ by

$$\pi(\alpha_q) = p_1^*(\alpha_q - \beta(q)) + \beta(q)$$

is a projection of T^*Q onto M_0 : if $\alpha_q \in \mathrm{ann}(\ker(B))$ then

$$\begin{aligned}
\pi(\alpha_q + \beta(q)) &= p_1^*(\alpha_q) + \beta(q) \\
&= \alpha_q + \beta(q) \ .
\end{aligned}$$

Thus, one may define $H = \bar{H} \circ \pi$. If M_0 is zero, then $\beta \in \mathrm{ann}(\ker(B))$ and $M_0 = \mathrm{ann}(\ker(B))$. In this case, $H(\alpha_q^1 + \alpha_q^2) = \bar{H}(\alpha_q^2)$, where $\alpha_q^1 \in \mathrm{ann}(E)$ and $\alpha_q^2 \in \mathrm{ann}(\ker(B))$.

7 Let $\mu \in L(G^0)^*$ and consider the evolution on $(J^0)^{-1}(\mu)$ defined by equation (2.1). It is of interest to determine if this evolution arises from some lagrangian evolution. Of course, if $M = M_0$, this evolution is the image under FL of the lagrangian evolution implied by L, as shown in paragraph (6.23). Denote by $\bar{\mu}$ the unique, smooth one form on Q that satisfies the conditions

$$\bar{\mu}(q)(\text{fund\_vf}(\phi,\xi,q)) = \mu(\xi) - \mu_0(\xi), \quad \text{and}$$

$$\bar{\mu}(q)(v_q) = 0 \quad \text{if} \quad v_q \in E.$$

The evolution defined by equation (2.1) on $(J^0)^{-1}(\mu)$ corresponds to the lagrangian evolution defined by the lagrangian $\bar{L} = L + \bar{\mu}$. All that needs to be verified is that $\mathrm{Im}(F\bar{L}) = (J^0)^{-1}(\mu)$ and that $H \circ F\bar{L} = \bar{E}$, where $\bar{E}$ is the energy function of $\bar{L}$. As $\bar{L}$ satisfies all the conditions of the previous analysis on L, and

$$\begin{aligned}(\beta(q) + \bar{\mu})(\text{fund\_vf}(\phi,\xi,q)) &= \mu_0(\xi) + \mu(\xi) - \mu_0(\xi) \\ &= \mu(\xi) \ ,\end{aligned}$$

for all $\xi \in L(G^0)$, $\mathrm{Im}(F\bar{L}) = (J\ )^{-1}(\mu)$. Since E is independant of β anyway, $E = \bar{E}$ is clear. But then

$$\begin{aligned}H \circ F\bar{L}(v_q) &= H(FL(v_q) + \bar{\mu}(q)) \\ &= \bar{H}(p_1^*(FL(v_q) + \bar{\mu}(q) - \beta(q)) + \beta(q)) \\ &= \bar{H}(p_1^*(FL(v_q) - \beta(q)) + \beta(q)) \\ &= H \circ FL(v_q)\end{aligned}$$

$$= E(v_q)$$

$$= \bar{E}(v_q).$$

8 I return now to the more general context of paragraph (2). The following algorithm generates a sequence of subsets of vector fields on T^*Q:

$$E_0 = \text{fund\_vf}(\phi, L(G^0))$$

$$E_{i+1} = E_i \cup [X_H, E_i] \cup [E_i, E_i] \;.$$

As in chapter (5), if this algorithm terminates at a finite subset E_f, then the connected components of the gauge relation defined by the extended evolution on T^*Q are the maximal integral submanifolds of dstb(E_f). In what follows, suppose that the gauge equivalance classes of the extended evolution are connected, and postulate the existence of a gauge group: a connected Lie group G with symplectic action ϕ on T^*Q such that:

1. G^0 is a closed subgroup of G.
2. If $g \in G^0$, then $\phi(g,m) = \phi^0(g,m)$.
3. fund_vf(ϕ,L(G)) = span(E_f).

It follows that the orbits of G are exactly the gauge eqiuvalence classes of the extended evolution. One may assume that $J_\xi = J^0_\xi$ for all $\xi \in L(G^0)$.

9 Suppose that the final constraint set $M_f \subseteq M_0$ is strongly first class. Then H is an observable, and each point of M_f admits evolution. I claim that J is constant on M_f. The proof is by

induction on the following statement:

P(i): If $\xi \in L(G)$ is such that $\text{fund\_vf}(\phi,\xi) \in E_i$, then J_ξ is constant on M_f.

P(0) is true: if $\xi \in L(G)$ is such that $\text{fund\_vf}(\phi,\xi) = \text{fund\_vf}(\phi,\xi')$ for some $\xi' \in L(G^0)$, then J_ξ and $J_{\xi'}$ differ by a constant on T^*Q. By hypothesis, $J_{\xi'}$ is zero on M_0, and since $M_f \subseteq M_0$, this shows that J_ξ is constant on M_f.

Suppose $i \geq 0$ is some integer for which P(i) is true. Let $\xi \in L(G)$ be such that $\text{fund\_vf}(\phi,\xi) \in E_{i+1}$. Then one of the following three statements is true.

1. $\text{fund\_vf}(\phi,\xi) \in E_i$.
2. $\text{fund\_vf}(\phi,\xi) = [\text{fund\_vf}(\phi,\xi_1),\text{fund\_vf}(\phi,\xi_2)]$ for some $\xi_j \in L(G)$ such that $\text{fund\_vf}(\phi,\xi_j) \in E_i$, $j = 1,2$.
3. $\text{fund\_vf}(\phi,\xi) = [X_H,\text{fund\_vf}(\phi,\xi')]$ for some $\xi' \in L(G)$ such that $\text{fund\_vf}(\phi,\xi') \in E_i$.

In the first case, J_ξ is constant on M_f by the induction hypothesis directly. For the second and third cases, it suffices to find a function f such that $X_f = \text{fund\_vf}(\phi,\xi)$ and f is constant on M_f. If the second statement is true, then one may take $f = \{J_{\xi_1},J_{\xi_2}\}$, by paragraph (1.20). Suppose the third statement is true, and let the constant value of $J_{\xi'}$ on M_f be c. Then $J_{\xi'} - c$ is a constraint, so $\{H,J_{\xi'}\} = \{H,J_{\xi'} - c\}$ is also a constraint, and one may take $f = \{H,J_{\xi'}\}$.

10 These hypotheses are sufficient to show that M_f is G invariant: Let $\xi \in L(G)$ and suppose J has value μ on M_f. Since M_f is strongly first class, $J_\xi - \mu(\xi)$ is a constraint on M_f, and

$$\begin{aligned} \text{hmlt\_vf}(J_\xi - \mu(\xi)) &= \text{hmlt\_vf}(J_\xi) \\ &= \text{fund\_vf}(\phi,\xi) \ , \end{aligned}$$

M_f is invariant under the flow of fund_vf(ϕ,ξ). Let $S \subseteq G$ be defined by

$$S = \{g \in G \ ; \ gm \in M_f \text{ for all } m \in M_f\}.$$

Then S contains the image of the exponential mapping, and so contains an open neighbourhood of the identity. By right translation, S is open. Since $A \times M_f = (\bar{\phi})^{-1}(M_f \times M_f)$, A is closed as well. As G is connected, A = G, so M_f is G invariant. By paragraph (2.18), one may assume that M_f is contained in the zero level set of J, J is Ad^* equivariant and the conditions of paragraph (8) continue to hold.

11 Let Q be a riemannian manifold, let the hypotheses of theorem (3.24) hold and suppose 0 is a quasiregular value of J. Then I claim that $M_f = J^{-1}(0)$. Since $M_f \subseteq J^{-1}(0)$, it suffices to show that each point of $J^{-1}(0)$ admits evolution. By theorem (4.25), this will be true if $\{H,J_\xi\}$ vanishes on $J^{-1}(0)$ for all $\xi \in L(G)$. If $\xi \in L(G)$, then

$$[X_H, \text{fund\_vf}(\phi,\xi)] = \text{fund\_vf}(\phi,\xi')$$

for some $\xi' \in L(G)$. Therefore, $\{H,J_\xi\}$ and $J_{\xi'}$ differ by a constant, so $\{H,J_\xi\}$ is constant on $J^{-1}(0)$. But $\{H,J_\xi\}$ vanishes on $M_f \subseteq J^{-1}(0)$, so $\{H,J_\xi\}$ vanishes on $J^{-1}(0)$. In rough terms, then, the following statement is true:

If a gauge group for the evolution exists and admits a sufficiently regular momentum mapping, and if the final constraint set is strongly first class, then the final constraint set is the zero level of a momentum mapping of the gauge group.

12 The most general evolution on M_0 is given piecewise by smooth curves of the form $gc(t) = g(t)c(t)$, where c is an integral curve of X_H and g is a smooth curve in G. The final theoretical result of this chapter is a proof that gc is the integral curve of the (time dependent) hamiltonian vector field of $H + J_{\xi(t)}$ for some piecewise smooth curve ξ in L(G). Thus, the evolution on M_0 is displayed as a set of hamiltonian evolutions.

13 The derivative of the curve gc(t) is

$$\begin{aligned}\frac{d}{dt}(gc(t)) &= \frac{d}{dt}(\Phi(g(t),c(t)) \\ &= T\Phi_{g(t)}\frac{dc}{dt}(t) + T\Phi_{c(t)}\frac{dg}{dt}(t) \\ &= T\Phi_{g(t)}X_H(c(t)) + T\Phi_{c(t)}\frac{dg}{dt}(t)\ .\end{aligned} \tag{1}$$

Let $v \in T_{g(t)c(t)}J^{-1}(0)$, so that

$$v = \left.\frac{d}{ds}\right|_{s=0} \gamma(s)$$

for some curve γ in $J^{-1}(0)$. Then

$$\omega_0(T\phi_{g(t)}X_H(c(t)),v) = \omega_0\left[X_H(c(t)),T\phi_{g(t)^{-1}}v\right]$$

$$= dH(c(t))T\phi_{g(t)^{-1}}v$$

$$= \frac{\partial}{\partial s}\bigg|_{s=0} H \circ \phi_{g(t)^{-1}} \circ \gamma(s)$$

$$= \frac{\partial}{\partial s}\bigg|_{s=0} H \circ \gamma(s)$$

$$= dH(g(t)c(t))v\ ,$$

since H is G invariant on $J^{-1}(0)$. Therefore,

$$\omega_0\left[T\phi_{g(t)}X_H(c(t)) - X_H(g(t)c(t)),v\right]$$
$$= dH(g(t)c(t))v - dH(g(t)c(t))v$$
$$= 0\ ,$$

for all $v \in T_{g(t)c(t)}J^{-1}(0)$. Thus, the curve

$$t \to T\phi_{g(t)}X_H(c(t)) - X_H(g(t)c(t))$$

takes values in fund_vf(ϕ,L(G),g(t)c(t)). Let $t_0 \in$ domain(c) and $m_0 = c(t_0)$. The curve c(t) lies in the manifold $N_{m_0} \cap J^{-1}(0)$ for some open interval of $\mathbb{R}$ about t_0, since X_H is tangent to this manifold. Therefore, the curve g(t)c(t) lies in the manifold $N_{m_0} \cap J^{-1}(0)$ for some open interval of $\mathbb{R}$ about t_0. But

$$\text{fund\_vf}(\phi,N_{m_0} \cap J^{-1}(0))$$

is a smooth distribution of constant rank on $N_{m_0} \cap J^{-1}(0)$. Therefore, this is a smooth curve ξ_1 in L(G), with domain an open interval in $\mathbb{R}$, such that

$$T\phi_{g(t)}X_H(c(t)) - X_H(g(t)c(t)) = \text{fund\_vf}(\phi,\xi_1(t),g(t)c(t)) \ ,$$

for all $t \in$ domain (ξ_1). Looking at the second summand of equation (13.1),

$$\begin{aligned} T\phi_{c(t)} \frac{dg}{dt}(t) &= T\phi_{c(t)} TR_{g(t)} TR_{g(t)^{-1}} \frac{dg}{dt}(t) \\ &= T\phi_{g(t)c(t)} TR_{g(t)^{-1}} \frac{dg}{dt}(t) \\ &= \text{fund\_vf}\left[\phi, TR_{g(t)^{-1}} \frac{dg}{dt}(t), g(t)c(t)\right] \ . \end{aligned}$$

Let

$$\xi(t) = \xi_1(t) + TR_{g(t)^{-1}} \frac{dg}{dt}(t) \ ,$$

defined on domain (ξ_1). Then,

$$\begin{aligned} \frac{d}{dt}(g(t)c(t)) &= X_H(g(t)c(t)) + \text{fund\_vf}(\phi,\xi_1(t),g(t)c(t)) \\ &\qquad + \text{fund\_vf}\left[\phi, TR_{g(t)^{-1}} \frac{dg}{dt}(t), g(t)c(t)\right] \\ &= X_H(g(t)c(t)) + \text{fund\_vf}(\phi,\xi(t),gc(t)) \\ &= \text{hmlt\_vf}(H,gc(t)) + \text{hmlt\_vf}(J_{\xi(t)},gc(t)) \\ &= \text{hmlt\_vf}(H + J_{\xi(t)},gc(t)) \ , \end{aligned}$$

for $t \in$ domain (ξ_1).

14 This chapter is concluded with an example which has its origins in the theory of Yang-Mills fields (Harnad-Shnider-Vinet [1979]).

Specifically, $Q = \mathbb{R}^3 \times \mathbb{R}^4$ with coordinates y^b and x_i^a, and the lagrangian is given by

$$F_{ij}^a = -\epsilon_{ijk}x_k^a + \epsilon_{abc}x_i^b x_j^c$$

$$F_{ok}^a = \dot{x}_k^a + \epsilon_{abc}y^b x_k^c$$

$$L = {}^1/_2\ F_{ok}^a F_{ok}^a - {}^1/_4\ F_{ij}^a F_{ij}^a \ . \tag{1}$$

One may organize this lagrangian in a more transparent form by writing $\mathbb{R}^9 = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$; that is, by considering x_i^a as the three tuplet (x_1^a, x_2^a, x_3^a) of vectors in $\mathbb{R}^3$. Then

$$F_{ij}^a = -\epsilon_{ijk}(x_k)^a + (x_i \times x_j)^a$$

$$F_{ok}^a = (\dot{x}_k)^a + (y \times x_k)^a \ .$$

The last term of equation (14.1) becomes

$$\begin{aligned} F_{ij}^a F_{ij}^a &= \sum_{i,j} \|x_i \times x_j - \epsilon_{ijk}x_k\|^2 \\ &= \|x_1 \times x_2 - x_3\|^2 + \|x_1 \times x_3 + x_2\|^2 + \|x_2 \times x_1 + x_3\|^2 \\ &\quad + \|x_2 \times x_3 - x_1\|^2 + \|x_3 \times x_1 + x_2\|^2 + \|x_3 \times x_2 + x_1\|^2 \\ &= 2(\|x_1 \times x_2 - x_3\|^2 + \|x_3 \times x_1 - x_2\|^2 + \|x_2 \times x_3 - x_1\|^2) \ , \end{aligned}$$

so that

$$\begin{aligned} L = \frac{1}{2}\sum_k \|\dot{x}_k + y \times x_k\|^2 - \frac{1}{2}(\|x_1 \times x_2 - x_3\|^2 + \|x_3 \times x_1 - x_2\|^2 \\ + \|x_2 \times x_3 - x_1\|^2) \ . \end{aligned}$$

Note that $F_{ij}^a F_{ij}^a$ has the peculiar property of being zero if and only if either all of the vectors x_1, x_2 and x_3 are zero or if the vectors x_1, x_2, x_3 form a right handed orthonormal set.

16 This lagrangian is of the form of paragraph (3), with

$$B(y,x_i)\left[\dot{y}\,\frac{\partial}{\partial\dot{y}} + \dot{x}_i\,\frac{\partial}{\partial\dot{x}_i}\ ,\ \dot{y}'\,\frac{\partial}{\partial\dot{y}} + \dot{x}_i'\,\frac{\partial}{\partial\dot{x}_i}\right] = \sum_i \dot{x}_i \cdot \dot{x}_i'$$

$$\beta(y,x_i)\left[\dot{y}\,\frac{\partial}{\partial\dot{y}} + \dot{x}_i\,\frac{\partial}{\partial\dot{x}_i}\right] = \sum_i (y \times x_i) \cdot \dot{x}_i$$

$$\beta(y,x_i) = \sum_i (y \times x_i) \cdot dx_i$$

$$V(y,x_i) = \frac{1}{2}(\|x_1 \times x_2 - x_3\|^2 + \|x_3 \times x_1 - x_2\|^2 + \|x_2 \times x_3 - x_1\|^2 - \sum_i \|y \times x_i\|^2) .$$

Clearly, B is of constant rank; in fact,

$$\ker(B) = \left\{\dot{y}\,\frac{\partial}{\partial\dot{y}}\ ;\ \dot{y} \in \mathbb{R}^3\right\} .$$

This observation provides a natural choice of the primary gauge group: $G^0 = \mathbb{R}^3$ with action ϕ^0 on $Q = \mathbb{R}^3 \times (\mathbb{R}^3)^3$ by addition in the y variables:

$$\phi^0(t,(y,x_i)) = (y + t,x_i) .$$

The action ϕ^0 is free (and hence infinitesimally free), $L(G^0) = \mathbb{R}^3$ and if $\xi \in \mathbb{R}^3$

$$\text{fund\_vf}(\phi\ ,\xi,(y,x_i)) = \xi\,\frac{\partial}{\partial y}\bigg|_{(y,x_i)} ,$$

so ker(B) is spanned by the fundamental vector fields of ϕ^0.

Obviously,

$$\beta(y,x_i)(\text{fund\_vf}(\phi^0,\xi,(y,x_i)) = \sum_i (y \times x_i) \cdot 0$$
$$= 0 \ ,$$

so $\mu_0 = 0$. Identifying T^*Q with $(\mathbb{R}^3 \times (\mathbb{R}^3)^3)^2$, and using coordinates (y,x_i,p,p_i), one computes

$$J^0_\xi(y,x_i,p,p_i) = (pdy + p_i dx_i)\left[\text{fund\_vf}(\phi^0,\xi,(y,x_i))\right]$$
$$= p\cdot\xi \ ,$$

and so paragraph (5) implies that

$$M_0 = \text{Image}(FL)$$
$$= (J^0)^{-1}(0)$$
$$= \{(y,x_i,p,p_i) \ ; \ p_i = 0\} \ .$$

The function $\bar{H}$ is easily computed:

$$E(y,x_i,\dot{y},\dot{x}_i) = \frac{1}{2}\sum_i \|\dot{x}_i\|^2 + V(y,x_i)$$

$$FL(y,x,\dot{y},\dot{x}_i) = (y,x_i,0,\dot{x}_i + y \times x_i) \ ,$$

so,

$$\bar{H}(y,x_i,0,p_i) = \frac{1}{2}\sum_i \|p_i - y \times x_i\|^2 + V(y,x_i) \ .$$

The configuration space Q is naturally a riemannian manifold, and

$$\ker(B)^\perp = \left\{\dot{x}_i \frac{\partial}{\partial x_i} \ ; \ \dot{x}_i \in \mathbb{R}^3\right\} .$$

Then $p_1^* : T^*Q \to \text{ann}(\ker(B))$ is given by

$$\mathrm{pi}^{*}(y,x_i,p,p_i) = (y,x_i,0,p_i) \ ,$$

so that

$$\begin{aligned}\pi(y,x_i,p,p_i) &= \mathrm{pi}^{*}((y,x_i,p,p_i - y \times x_i)) + (y,x_i,0,y \times x_i) \\ &= (y,x_i,0,p_i) \ ,\end{aligned}$$

$$\begin{aligned}H(y,x_i,p,p_i) &= \bar{H} \circ \pi(y,x_i,p,p_i) \\ &= \bar{H}(y,x_i,0,p_i) \\ &= \frac{1}{2} \sum_i \|p_i - y \times x_i\|^2 + V(y,x_i) \\ &= \frac{1}{2} \sum_i \|p_i\|^2 - \sum_i y \cdot (x_i \times p_i) \\ &\quad + \frac{1}{2} (\|x_1 \times x_2 - x_3\|^2 + \|x_3 \times x_1 - x_2\|^2 \\ &\qquad + \|x_2 \times x_3 - x_1\|^2) \ .\end{aligned}$$

This completes the construct of the extended canonical formalism.

17 The gauge vector field algorithm proceeds as follows:

$$E_0 = \left\{ \xi \frac{\partial}{\partial y} \ ; \ \xi \in \mathbb{R}^3 \right\} .$$

Any two vector fields in E_0 commute, and

$$\begin{aligned}\left[X_H, \xi \frac{\partial}{\partial y}\right] &= \mathrm{hmlt\_vt}(-\{H,J_\xi\}) \\ &= \mathrm{hmlt\_vf}\left[-dH\left[\xi \frac{\partial}{\partial y}\right]\right] \\ &= \mathrm{hmlt\_vf}\left[-\xi^i \frac{\partial H}{\partial y^i}\right] \\ &= \mathrm{hmlt\_vf}\left[\sum_i \xi^i \cdot (x_i \times p_i)\right] .\end{aligned}$$

Thus, if one considers the natural action ϕ^1 of SO(3) on the x_i variables as in paragraph (4.31), then

$$E_2 = \text{fund\_vf}(\phi^0, L(G^0)) \cup \text{fund\_vf}(\phi^1, L(SO(3))) .$$

Obviously, the Lie bracket of any two vector fields in E_2 is again in E_2. Although a direct computation will show that, if $\xi \in L(SO(3))$,

$$[X_H, \text{fund\_vf}(\phi^1, \xi)] = 0 ,$$

this equality is a trivial consequence of the fact that H is invariant under the action of SO(3) on T^*Q. Therefore, the gauge vector field algorithm terminates at E_2, and one can take $g = \mathbb{R}^3 \times SO(3)$ as the gauge group, with product action

$$\phi((t,A),(y,x_i,p,p_i)) = (y + t, Ax_i, p, Ap_i) .$$

18 I claim that the evolution vector fields hmlt_vf($H + J_\xi$) are complete. It suffices to show that if c(t) is an integral curve of the vector field hmlt_vf($H + J_\xi$), and $(t_1,t_2) \subseteq$ domain(c), then $c((t_1,t_2))$ is contained in a compact set of T^*Q. From the hamiltonian directly,

$$\frac{d}{dt} (p \circ c(t)) = \sum_i x_i \times p_i .$$

As $H + J_\xi$ is invariant under the action of SO(3) on T^*Q, $\Sigma\, x_i \times p_i$ is a conserved quantity of the flow of hmlt_vf($H + J_\xi$). Let its value on the curve c be a. Then, for some $b \in \mathbb{R}^3$,

$$p \circ c(t) = at + b .$$

Another converved quantity is $H + J_\xi$, so that

$$(H + J_\xi) \circ c(t) = H \circ c(t) + \xi\cdot(at + b) = k_1 ,$$

for some constant k_1. Let $\xi\cdot(at + b)$ have minimum value k_2 on $[t_1,t_2]$. The $H \circ c(t) \subseteq k_1 - k_2$. An observation of H shows that the set

$$\{(y,x_i,p,p_i) \;;\; H(y,x_i,0,p_i) \le k_1 - k_2\}$$

is compact. Since the image of $[t_1,t_2]$ under $p \circ c$ is also compact, $c(t_1,t_2)$ is contained in the product of two compact sets. It follows from paragraph (5.19) that the gauge equivalence classes of the extended evolution are the orbits of the action of G on T^*Q.

19 Finally, $M_f = J^{-1}(0)$: it is clear that $J^{-1}(0) \subseteq M_f$, since 0 is a quasiregular value of J, $J^{-1}(0) \subseteq M$, and H is G invariant on $J^{-1}(0)$. But also, $M_f \subseteq J^{-1}(0)$, since the first secondary constraint set is

$$\begin{aligned} M_1 &= \left\{\alpha_q \;;\; dH(\alpha_q)(T_{\alpha_q} M_0)^{\omega_0\perp} = 0\right\} \\ &= \{\alpha_q \;;\; dH(\alpha_q)(\text{fund\_vf}(\phi^0,L(G^0),\alpha_q)) = 0\} \\ &= \left\{(y,x_i,0,p_i) \;;\; \sum_i \xi^i\cdot(x_i \times p_i) = 0\right\} \\ &= J^{-1}(0) . \end{aligned}$$

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