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Optimality and Sustainability of Delayed Impulsive Harvesting

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Optimality and Sustainability of Delayed Impulsive Harvesting

by

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Abstract

Optimal and sustainable management of natural resources requires knowledge about the behaviour of mathematical models of harvesting under many different types of conditions. In this thesis, the effects of delays on the optimality and sustainability of harvesting models are studied, with a particular focus on delayed impulsive harvesting models. We begin by considering delays within a continuous harvesting model and derive sufficient conditions for stability of harvesting models with general growth and harvesting rate. We also derive maximum sustainable yields for models with both logistic and Gompertz growth, and show that they are delay dependent. Then we consider the main object of the thesis, a logistic differential equation subject to impulsive delayed harvesting, where the deduction information is a function of the population size at the time of one of the previous impulses. A close connection to the dynamics of high-order difference equations is used to conclude that while the inclusion of a delay in the impulsive condition does not impact the optimality of the yield, sustainability may be highly affected and is once again delay-dependent. Maximum and other types of yields are explored, and sharp stability tests are obtained for the model, as well as explicit sufficient conditions. It is also shown that persistence of the solution is not guaranteed for all positive initial conditions, and extinction in finite time is possible, as is illustrated in the simulations. The results of this thesis imply that delays within harvesting should be kept short to maintain the sustainability of resources.

Preface

This thesis contains material that has been submitted for publication. Chapter 3 is adapted from "Optimality and Sustainability of Delayed Impulsive Harvesting" which as of July 14, 2022 has been submitted for publication in Communications in Nonlinear Science and Numerical Simulation. In this paper I wrote the initial draft, with editing and additions done jointly between Professor Elena Braverman and myself.

Acknowledgments

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This research has been supported financially by the Natural Sciences and Engineering Research Council of Canada, the Government of Alberta, the University of Calgary and the family of Eric Milner. Thank you for allowing me the opportunity to conduct this research.

I would like to take this opportunity to acknowledge that this research has been conducted both on the traditional territories of the people of the Treaty 7 region in Southern Alberta, and on the treaty lands and territory of the Mississaugas of the Credit. The City of Calgary is also home to Métis Nation of Alberta, Region 3.

Now to the one who can do infinitely more than all I can ask or imagine according to his power that is working within me, to him be all the glory forever and ever amen.

This thesis is dedicated to the memory of John Patrick Cooper (1939 - 2022) and his love of sardines.

Contents

List of Symbols

$N(t)$ - size of population at time t.

$N(nT^+)$ - size of population immediately after a harvesting event occurring at $t = nT$.

$N(nT)$ - size of population immediately before a harvesting event occurring at $t = nT$.

$\Delta N(nT) = N(nT^+) - N(nT)$ - change in population during a harvesting event.

$N^* \in \mathbb{R}^+$ - positive equilibrium of continuous population model.

$N^*(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ - positive periodic solution of impulsive population model.

$K_c > 0$ - carrying capacity of population.

$r > 0$ - intrinsic growth rate of population.

$E > 0$ - harvesting effort.

E_{opt} - optimal harvesting effort.

$\tau > 0$ - harvesting delay.

$T > 0$ - time between successive harvesting events.

$C([a, b], [c, d])$ - space of continuous functions mapping $[a, b]$ to $[c, d]$.

$\phi(t)$ - initial data function.

$\mathbb{N} = 1, 2, 3, 4, ...$ - natural numbers.

$\mathbb{N}_0 = 0, 1, 2, 3, ...$ - natural numbers including zero.

$H(t) = \begin{cases} 0 & , t < 0 \\ 1/2 & , t = 0 \\ 1 & , t > 0 \end{cases}$ - Heaviside step function.

$Y(E)$ - yield.

SY - sustainable yield.

MY - maximum yield.

MSY - maximum sustainable yield.

MESY - expectation of the maximum sustainable yield.

DE - differential equation.

DDE - delay differential equation.

ODE - ordinary differential equation.

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Chapter 1

Introduction

1.1 Continuous Harvesting

Sustainable management of natural resources is a major issue currently facing our world. An essential part of the solution requires determining the behaviour and optimality of various types of harvesting models [9]. Harvesting can be represented as occurring either continuously or only during short-time periods. Continuous harvesting can be described as a continuous deduction term appearing in an ordinary Differential Equation (DE) describing population dynamics

$$\begin{cases} \frac{dN}{dt} = F(N(t)) - h(N(t)), \quad t > 0 \\ N(0) = N_0 \end{cases}$$

where F, h are the growth and harvesting functions, respectively. It assumes that harvesting occurs without any interruptions, whereas impulsive harvesting corresponds to part of the stock being removed at specific moments in time, with the duration of the harvesting event being negligible compared to the overall process time.

1.2 Impulsive Harvesting

Impulsive harvesting is represented by an impulsive DE

$$\begin{cases} \frac{dN}{dt} = F(N(t)), \quad t \neq nT, \quad n \in \mathbb{N} \\ \Delta N(nT) = h(N(nT)), \quad t = nT, \quad n \in \mathbb{N} \\ N(0) = N_0 \end{cases}$$

where F, h are the growth and harvesting functions respectively, $T > 0$ is the time between impulses. We assume that the function $N(t)$ is left continuous, and denote the size of the population after harvesting as $\lim_{h\to 0^+} N(nT + h) = N(nT^+)$, and the size of the population $N(t)$ before harvesting as $\lim_{h\to 0^-} N(nT + h) = N(nT)$. Then the change in population size after a harvesting event is denoted $\Delta N(nT) = N(nT^+) - N(nT)$.

In this model, the DE governs the behaviour of the system outside of the impulsive harvest moments, $t \neq nT$. At the impulsive moments $t = nT$, the impulsive condition $\Delta N(nT) = h(N(nT))$ takes control of the model and harvesting occurs. Impulsive DEs have many practical applications such as pest control [42], pulse vaccination strategies [20], and optimal harvesting in fisheries [11]. For more on the theory of impulsive DEs see the monograph [4].

The duration of an impulse is generally assumed to be negligible compared to the overall duration of the process. In cases where it is not, such as the administration and absorption of drugs into the body, non-instantaneous impulse theory is used [2]. In this thesis, it will be assumed that all impulses are instantaneous with negligible duration.

Even though continuous harvesting may be preferable from the point of view of both maximizing harvest and sustainability [7, 44], it is not always realistic or easily applicable. This is why investigation of impulsive harvesting models is important.

1.3 Common Growth and Harvesting Functions

Various growth and harvesting functions are considered in the literature. One of the most famous growth functions used in population models is the logistic growth function

$$F_1(N(t)) = rN(t)\left(1 - \frac{N(t)}{K_c}\right) \tag{1.1}$$

where $N(t)$ is the population size at time t, $r > 0$ is the intrinsic growth rate of the system, and $K_c > 0$ is the carrying capacity of the environment. Another commonly used function is the Gompertz growth function

$$F_2(N(t)) = rN(t)\ln\left(\frac{K_c}{N(t)}\right). \tag{1.2}$$

Others examples include the Smith, Gilpin-Ayala, and Nisbet-Gurney growth functions [5].

Similarly there are various forms of harvesting functions found in the literature. The simplest is a constant deduction

$$h_d(N) = d \tag{1.3}$$

where $d > 0$ is some constant that does not depend on $N(t)$. Since this type of harvesting is not dependent on $N(t)$, including a delay within harvesting would have no effect. The dynamics and optimal harvesting policies of (1.3) were investigated for continuous models in [9], and impulsive models in [47, 48]. A more commonly used harvesting function is

$$h(N) = EN(t) \tag{1.4}$$

where $E > 0$ is some constant that represents the harvesting effort. This type of linear harvesting can also be referred to as catch per unit effort harvesting (or the catch per unit effort hypothesis), since the yield is directly proportional to both the harvesting effort E, and the population size at time t. It will be the harvesting function that is predominantly used in this thesis. Other examples include non-linear harvesting functions similar to a Cobb-Douglas production function

$$h(N) = E^{\alpha} N(t)^{\beta} \tag{1.5}$$

where $\alpha > 0$ represents how the yield responds to changes in fishing effort, and $\beta > 0$ represents the sensitivity of the yield to changes in stock levels. Widely used in economics, the Cobb-Douglas function has been used in harvesting models where changes in population size do not immediately lead to changes in the yield. Pelagic fisheries, where fish tend to travel in large groups, are an example of where it has been applied [14, 21]. Since fishers are able to target large aggregations of fish for harvesting, yields may not decrease even though the fish stock and number of aggregations are decreasing.

We note that the harvesting effort E is interpreted differently between the continuous and impulsive harvesting models. In the continuous model, E is interpreted as a harvesting rate similar to how r is thought of as the growth rate, and can be any positive number $E > 0$. In contrast, in the impulsive harvesting model E is interpreted as the proportion of the population that is harvested, leading to the condition that $E \in (0, 1)$.

1.4 Delays

It is not always realistic to assume that the available population size data is completely up to date. Instead, it is more likely that all data will be at least somewhat out of date, leading to harvesting decisions that are made based off of old information, and introduces a delay into the harvesting functions of our models. In other studies, delays within harvesting have been included to represent targeted harvesting based on age [26].

It is well known that including delays within a DE model of population dynamics can lead to major changes to its behaviour, such as causing instability, oscillations, and extinction which are not observed in a corresponding ordinary DE model [36]. The famous Hutchinson equation is one such example. A logistic equation with no delay, has a carrying capacity equilibrium which is always a globally attractive equilibrium for all non-trivial positive solutions. Whereas the inclusion of a delay to form the Hutchinson equation can cause the carrying capacity equilibrium to become unstable for certain values of delay.

Stability of non-linear delay DEs (DDEs) is often studied by methods which are similar to those of ordinary DEs (ODEs), such as linearization.

In this method, an associated linear DDE is found by linearizing the non-linear DDE around an equilibrium with respect to both the function at time t, and to the function at the specified delay values $t - \tau$. We then look for solutions of the associated linear DDE which are in the form $N(t) = e^{\lambda t}$, and obtain a characteristic equation for the DDE in terms of λ. If all the roots of the characteristic equation have negative real part, then this indicates local asymptotic stability of the equilibrium. However, characteristic equations of DDEs are much more complex compared to those of ODEs. In general the characteristic equation that is obtained is transcendental, which makes analyzing the roots of the characteristic equation more difficult [36]. Take for example, one of the simplest DDEs

$$\frac{dN}{dt} = rN(t - \tau)$$

$r > 0$, $\tau > 0$. Assuming a solution of the form $N(t) = e^{\lambda t}$ gives the transcendental characteristic equation

$$\lambda = re^{-\lambda \tau}.$$

Other examples specific to the effects of delays on continuous harvesting models can be seen in [6, 26, 32, 45], where delay is incorporated into the harvesting function.

The incorporation of delays into impulsive conditions goes back to the 1990s [1], with recent progress summarized in [31]. Delayed impulsive harvesting of a logistic equation was considered in [34], with further discussion of bifurcations in [12, 13].

1.5 Optimal Harvesting Policies

If harvesting is restricted to only the surplus production of a population, then theoretically, harvesting should be able to continue indefinitely without drastically altering the stock levels. This is the idea behind the Maximum Sustainable Yield (MSY). A more precise characterization of the MSY will be given in Chapter 2 and Chapter 3 for the continuous and impulsive harvesting models respectively.

The concept of the MSY emerged in the 20th century, and since then has had a complicated history. It has been criticized for its contributions to the over-fishing and subsequent collapse of fisheries in recent years [29]. One reason for this is that catches have a tendency to exceed the MSY year over year. In addition, it has been shown that applying single species MSYs to ecosystems with a multi-level food chain, can cause extinction of some species [28, 30]. However, this does not mean that we have no use for the MSY. In [33] the author discusses the need to transition to an ecosystem based management strategy of fisheries, but emphasizes that single species indicators such as the MSY are still necessary since they can account for species specific traits. Instead, both [33, 38] discussed transitioning to seeing the MSY as a limiting factor instead of a goal to be achieved.

The MSY has been well studied for various types of models. The MSYs for logistic and Gompertz continuous linear harvesting models with no delay are given by $MSY_L = \frac{rK_c}{4}$ and $MSY_G = \frac{rK_c}{e}$, respectively [9, 24]. Optimal harvesting policies for impulsive harvesting models with no delay have been investigated for models with logistic growth [47] (the results of which will be summarized in Theorem 3.2.1), Gompertz growth [48], and for models with the addition of by catch mortality [7]. Moreover, the optimality of a stochastic impulsive harvesting Gompertz equation was considered in [43].

Since environments can change unpredictably, the addition of stochastic fluctuations into population models is worthy of attention. It represents an additional challenge in the evaluation of optimal harvesting policies, with the MSY often being replaced by the expectation of the MSY (MESY). For stochastic logistic equations with and without growth delays, the MESY is often dependent upon both the harvesting effort and the intensity of the environmental fluctuations [40, 46].

As has already been discussed, it is not feasible to only consider species existing in isolation. In reality, all species are part of an ecosystem and food chain where they coexist, compete and/or serve as prey. This has led to extensive literature on MSY harvesting of single or multiple populations within a food chain [3, 15, 23, 39].

To the best of our knowledge, there has been no discussion of optimal harvesting policies for delayed impulses in the literature. Our work in Chapter 3 aims to fill this gap. Furthermore, there has been no specific mention of optimal harvesting policies for delayed continuous harvesting with logistic or Gompertz growth in the literature. A MSY for these continuous models will be derived in Chapter 2 for the sake of comparison to optimal harvesting policies for delayed impulses.

1.6 Discrete vs. Continuous Models

Continuous models of population dynamics are closely connected to difference equations, which has led to extensive study of discrete population models [5]. A general difference equation is given by

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$$

with initial conditions $x_0, ..., x_{-k}$, and can be thought of as defining a recursive sequence via function iteration. While appearing simple, difference equations can exhibit complex dynamics, which underscores their importance within population modelling. Stability of non-linear equations is studied in much the same way as for continuous models, via linearization. One difference is that with continuous models, we assume a solution of the linear equation to be in the form $N(t) = e^{\lambda t}$ and look for roots of the characteristic equation which have negative real part to imply asymptotic stability. In contrast, for difference equations we assume a solution of the form $x_n = \lambda^n$ to obtain the characteristic equation, and have asymptotic stability if all roots lie inside the unit circle. For more on the theory of difference equations, see the monographs [16, 25].

Optimal harvesting policies for a discrete Beverton-Holt model were investigated in [8], while a discrete age structured model, which allows for selective harvesting based on age, was considered in [37]. The question of harvest timing within a season has been explored through the discrete Seno model [35] which is a convex combination of population growth before and after harvesting. Using this discrete model, [17] showed that for high enough harvesting intensities harvesting timing has no effect on the stability of the model. Further in [22] the authors showed that unless there is a cost to harvesting at the beginning or ending of a season, then it is generally optimal to harvest at the beginning of a season.

1.7 Overview

Our main goal in this thesis will be to investigate the optimality and sustainability of delayed impulsive harvesting, the majority of which is done in Chapter 3. In Chapter 2, we will begin by studying a general model with continuous delayed harvesting. We will derive sufficient conditions for the existence and uniqueness of positive equilibriums and their stability. We will then utilize the results to derive MSYs for both the logistic and Gompertz models with delayed continuous harvesting. The chapter will conclude with a brief discussion of the oscillation of the logistic equation with delayed harvesting, and numerical simulations. In Chapter 3 we study a logistic equation with impulsive delayed harvesting. We will begin by showing a connection between the continuous and discrete models, and derive sufficient conditions for the immediate extinction of the population. We then utilize the discrete model to derive stability conditions for the positive periodic solutions, and use those results to derive a MSY for the model. The results will show that while the inclusion of the delay in the impulses will not affect the optimality of the yield, it will greatly affect the sustainability. Sustainability of yields that are not maximal will also be discussed. The chapter will conclude with numerical results, including numerical simulations that show that the positive periodic solution is not globally attractive for all initial values. Finally, Chapter 4 will discuss some extensions of this research, and possible future work.

Chapter 2

Delayed Continuous Harvesting

2.1 Introduction

In population modelling, harvesting can be represented by a continuous deduction term. It is often assumed that population information used to make harvesting decisions is up to date and instantaneously available, leading to harvesting that is a function of the population at time t. Models of this type, such as the logistic and Gompertz differential equations with linear harvesting terms have been well studied [9, 24]. As was discussed in Chapter 1, it is natural to assume that any available population data will be at least somewhat out of date. When this is the case, the continuous deduction term representing harvesting in our models will be a function of a delayed estimate of the population, which introduces a delay into the model.

In this chapter we will consider the following general population dynamical system with delayed continuous harvesting

$$\begin{cases} \frac{dN}{dt} = F(N(t)) - h(N(t-\tau)), & t > 0 \\ N(t) = \phi(t), & t \in [-\tau, 0] \end{cases} \tag{2.1}$$

where $K_c > 0$ represents the carrying capacity of the population, $F(N) : [0, K_c] \to [0, \infty)$ represents the intrinsic combination of the birth and death rate of a population, $h(N) : [0, K_c] \to [0, K_c]$ is the harvesting function, $\tau > 0$ represents the harvesting delay, and $\phi(t) : [0, \infty) \to [0, \infty)$ gives the initial data of a system on $t \in [-\tau, 0]$. We begin by assuming the following about F and h.

H1. *F is continuously differentiable on $(0, K_c)$, $F(0) = F(K_c) = 0$, and $F'(0) > 0$.*

H2. *h is continuously differentiable on $(0, K_c)$, $h(0) = 0$, and $h'(N) > 0$ for all $N \in (0, K_c)$.*

As we will see, H1, H2 along with the condition that $F'(0) > h'(0)$ will guarantee existence of a positive equilibrium. But, under the stronger conditions H3, H4 we are also able to guarantee uniqueness.

H3. *F is continuously differentiable on $(0, K_c)$, $F(0) = F(K_c) = 0$, $F'(0) > 0$, and $F''(N) < 0$ $\forall N \in (0, K_c)$.*

H4. *h is continuously differentiable on $(0, K_c)$, $h(0) = 0$, $h'(N) > 0$ and $h''(N) \geq 0$ for all $N \in (0, K_c)$.*

Everywhere below, we assume that H3, H4 hold unless stated otherwise. H3 implies that F is "hump-shaped", while H4 implies that h is non-decreasing. We note that many of the most commonly used growth rates $F(\cdot)$ in the literature, such as logistic, Gompertz, Smith [5], satisfy H3. And furthermore many common harvesting terms $h(\cdot)$ satisfy H4 such as a linear harvesting function $h(N) = EN(t)$, or a Cobb Douglas harvesting function $h(N) = E^{\alpha}N(t)^{\beta}$, $\alpha > 0, \beta \geq 1$.

The chapter is organized as follows. Section 2.2 will state some needed background results on stability and oscillation of DDEs, as well as defining what is meant by a MSY for this model. Section 2.3 will show the existence and uniqueness of equilibriums of (2.1) under H3, H4, and derive conditions for stability of the positive equilibrium. These results will then be used to derive the MSY for both a logistic and Gompertz model with delayed harvesting. Section 2.4 will briefly explore oscillations of the logistic equation with delayed harvesting, and show that under specific conditions, solutions will be guaranteed to oscillate around the positive equilibrium. Finally, Section 2.5 will contain examples, illustrative numerical simulations, and concluding remarks.

2.2 Preliminaries

Our main focus of this chapter will be to derive a MSY for the delayed harvesting model under various growth rates. For the delayed continuous harvesting model, each yield $Y(E)$ will correspond to a constant solution of the DDE (2.1).

Definition 1. The *maximum yield (MY)* of (2.1) is a yield associated to an optimal constant solution such that no other solution will have a corresponding yield that will exceed the MY.

Definition 2. The optimal harvesting effort E_{opt} is the value of E which maximizes the yield $Y(E)$.

Since E_{opt} is the value of E which maximizes $Y(E)$, then $MY = Y(E_{opt})$.

Definition 3. A *maximum sustainable yield (MSY)* of (2.1) is a MY such that the optimal constant solution associated to the MY is at least locally asymptotically stable.

Definition 4. A solution N^* of (2.1) is said to be *stable* if for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that $\|\phi - N^*\| < \delta$ implies that $\|N(t) - N^*\| < \epsilon$, $t > 0$.

Definition 5. A solution N^* of (2.1) is *asymptotically stable* if it is stable and if there exists $b > 0$ such that whenever $\|\phi - N^*\| < b$ then $N(t) \to N^*$ as $t \to \infty$.

We cite the following result on the stability of linear delay differential equations from [36].

Lemma 2.2.1 ([36, Theorem 4.7, pg 53])**.** *Consider the linear delay differential equation*

$$u'(t) = Au(t) + Bu(t - \tau) \tag{2.2}$$

with $A, B \in \mathbb{R}$, $\tau > 0$. Then if:

(a) $A + B > 0$ then the zero solution of (2.2) *is unstable.*

(b) $A + B < 0$ and $B \geq A$ then the zero solution of (2.2) *is asymptotically stable.*

(c) $A+B<0$ and $B<A$ then there exists

$$\tau^* = \frac{\arccos(-A/B)}{\sqrt{B^2-A^2}}$$

such that the zero solution of (2.2) *is asymptotically stable for $0<\tau<\tau^*$ and unstable for $\tau>\tau^*$. Furthermore, when $\tau=\tau^*$ the characteristic equation has a pair of purely imaginary roots.*

Conditions for persistence and boundedness of solutions of the logistic equation with delayed impulsive harvesting have been studied in [6, 10]. While both papers assumed that initial data had the property $N_0 = \phi(0) > \phi(t)$ for $t \in [-\tau, 0)$, [6] had the additional condition that $N_0 < K_c$. The following is obtained as a simple corollary of results found in both papers.

Lemma 2.2.2 ([6, 10])**.** *Consider a logistic growth model with delayed harvesting,*

$$\begin{cases} \frac{dN}{dt} = rN(t)\left(1-\frac{N(t)}{K_c}\right) - EN(t-\tau) \\ N(t) = \phi(t), \quad t\in[-\tau,0] \end{cases} \tag{2.3}$$

Assume that the initial conditions satisfy $N_0 > \phi(t)$ for $t \in [-\tau, 0)$. Let

$$E\tau e^{(\lambda-1)r\tau} \leq \frac{1}{e}, \quad \lambda = \max\left\{1, \frac{N_0}{K_c}\right\} \tag{2.4}$$

hold, then if:

(i) $N_0 \leq K_c$, $0 < N(t) \leq K_c$

(ii) $N_0 > K_c$, there exists $T>0$ such that $0<N(t)\leq N_0$, $0\leq t<T$ and $0<N(t)\leq K_c$, $t\geq T$.

The following theorems focus on the linearized oscillation of delay differential equations.

Theorem 2.2.3 ([19, Theorem 1.5.1, pg 67])**.** *Consider*

$$\frac{dx}{dt} + \sum_{j=1}^{n} P_j(t)x(t-\tau_j) = 0; \ t \geq t_0 \tag{2.5}$$

where

$$P_j \in C([t_0, \infty), \mathbb{R}^+), \quad \lim_{t\to\infty} P_j(t) = p_j$$

$$\tau_j \in [0, \infty), \ j = 1, \ldots, n$$

If the characteristic equation

$$\lambda + \sum_{j=1}^{n} p_j e^{-\lambda \tau_j} = 0$$

associated with the limiting equation of (2.5) *has no real roots, then all the non-trivial solutions of* (2.5) *are oscillatory.*

2.3 Stability and MSY

Lemma 2.3.1. *The trivial equilibrium $N^* = 0$ of* (2.1) *always exists, and if $F'(0) > h'(0)$ where F, h satisfy H1, H2 respectively, then a positive equilibrium N^* also exists and is a solution of $F(N) = h(N)$ for $N \in (0, K_c)$.*

Proof. It is clear that the trivial equilibrium $N^* = 0$ always exists by H1, H2. Since h is strictly increasing $h(0) < h(K_c) \to F(K_c) < h(K_c)$ since $F(K_c) = 0$.

Define the continuous and differentiable function $G(N) = F(N) - h(N)$. Then $G(0) = 0$ and $G'(0) > 0$. By the limit definition of the derivative

$$G'(0) = \lim_{\delta \to 0^+} \frac{G(\delta) - G(0)}{\delta} = \frac{1}{\delta} \lim_{\delta \to 0^+} G(\delta) > 0.$$

Since G is continuous, this implies that for some neighbourhood around zero, $G(N) > 0$. Choose a point c in that neighbourhood, then $F(c) > h(c)$.

Since $F(K_c) < h(K_c)$, $F(c) > h(c)$ and F, h are continuous functions, then by The Intermediate Value Theorem $\exists N^* \in (0, K_c)$ such that $F(N^*) = h(N^*)$. $\square$

Note that under the assumptions H1, H2 the solution N^* is not necessarily unique.

Example 2.3.1.1. *Let*

$$F(N) = 0.6N \ln\left(\frac{20}{N}\right)$$

$$h(N) = \sin(N) + 1.11N.$$

Then $F'(0) > h'(0)$*,* $F(0) = F(20) = h(0) = 0$*, and* $h'(N) > 0 \ \forall N \in (0, 20)$*. Thus* F, h *satisfy H1, H2, however there are 3 solutions in* $(0, 20)$ *of* $F(N) = h(N)$ *given by* $N^* \approx 0.671, 3.137, 4.516$ *so the equilibrium is not unique.*

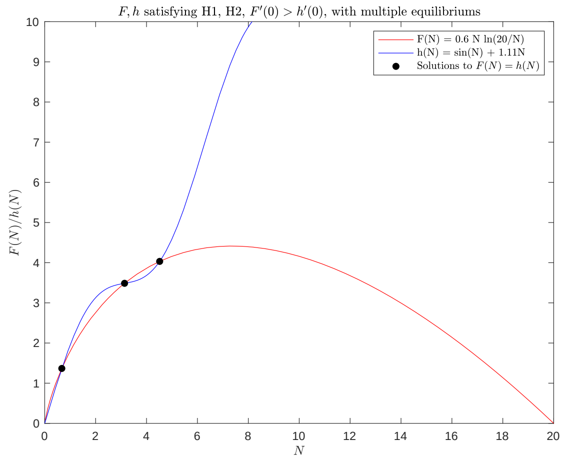

Figure 2.1: In this figure we see $F(N) = 0.6N \ln(20/N)$, $h(N) = \sin(N) + 1.11N$. Equilibriums N^* are solutions to $F(N) = h(N)$ for $N \in (0, 20)$. Since F, h satisfy H1, H2, $F'(0) > h'(0)$, (and not H3,H4) there are 3 equilibriums $N^* \approx 0.671, 3.137, 4.516$. See the appendix, MultipleEquils.m for MATLAB code.

To guarantee the uniqueness of the positive equilibrium we assume the stricter conditions H3, H4 on F, h.

Lemma 2.3.2. *The trivial equilibrium $N^* = 0$ of* (2.1) *always exists and if $F'(0) > h'(0)$ where F, h satisfy H3, H4 respectively, then a positive equilibrium N^* exists and is unique where N^* is a solution of $F(N^*) = h(N^*)$.*

Proof. By Lemma 2.3.1 we can see that both the trivial and positive equilibriums exist. Thus we only need to show uniqueness of the positive equilibrium.

Assume for the sake of contradiction $\exists N_1^*, N_2^* \in (0, K_c)$, $N_1^* < N_2^*$ such that $F(N_1^*) - h(N_1^*) = 0$, $F(N_2^*) - h(N_2^*) = 0$.

Let $G(N) = F(N) - h(N)$. By Rolle's Theorem $\exists b \in (0, N_1^*)$ such that $G'(b) = 0$. Similarly, by Rolle's Theorem $\exists c \in (N_1^*, N_2^*)$ such that

$G'(c) = 0$. Now since $F'' < 0$, $h''(N) \geq 0$ then $G'' < 0$ which implies that $G'(N)$ is strictly decreasing. Thus for $b < c$, $G'(b) > G'(c)$. However this would imply that $0 = G'(b) > G'(c) = 0$ which is a contradiction. Therefore N^* is unique. □

Lemma 2.3.3. *Let $F'(0) > h'(0)$, where H3, H4 hold. Then* (2.1) *has a non-zero, positive equilibrium N^*, and if:*

(a) $-h'(N^*) \geq F'(N^*)$ then the equilibrium is locally asymptotically stable.

(b) $-h'(N^*) < F'(N^*)$ then there exists

$$\tau^* := \frac{\arccos(F'(N^*)/h'(N^*))}{\sqrt{h'(N^*)^2 - F'(N^*)^2}} > 0 \tag{2.6}$$

such that the equilibrium is locally asymptotically stable if $0 < \tau < \tau^*$ and unstable for $\tau > \tau^*$.

Proof. By Lemma 2.3.2 the positive equilibrium exists, and is unique. Linearizing (2.1) around the positive equilibrium N^* we obtain the linearized equation

$$u'(t) = Au(t) + Bu(t - \tau)$$

with

$$A = \frac{\partial}{\partial N(t)}(F(N(t)) - h(N(t-\tau)))\bigg|_{(N^*,N^*)} = F'(N^*)$$
$$B = \frac{\partial}{\partial N(t-\tau)}(F(N(t)) - h(N(t-\tau)))\bigg|_{(N^*,N^*)} = -h'(N^*)$$

Whenever the zero solution of the linearized equation is asymptotically stable, by linearization theory the equilibrium is locally asymptotically stable. Whenever the zero solution of the linearized equation is unstable, then so is the equilibrium N^*.

While H3, H4 hold we will show that $F'(N^*) < h'(N^*)$. Thus case (a) of Lemma 2.2.1 will not apply, and we may always assume that $F'(N^*) < h'(N^*)$.

Once again let $G(N) = F(N) - h(N)$. Recall that the positive equilibrium N^* is defined as the solution of $F(N^*) = G(N^*)$. Thus $G(N^*) = 0$. Since $G(0) = 0$, $G(N^*) = 0$, by Rolle's Theorem $\exists b \in (0, N^*)$ such that $G'(b) = 0$.

Since $G''(N) < 0$, $\forall N \in (0, K_c)$ the derivative $G'(N)$ is strictly decreasing i.e. for $b < N^*$, $G'(b) > G'(N^*)$. Assume for the sake of contradiction that $F'(N^*) \geq h'(N^*)$ which implies $G'(N^*) \geq 0$. Then $0 = G'(b) > G'(N^*) \geq 0$ which is a contradiction.

Thus $F'(N^*) < h'(N^*)$, and we do not need to consider case (a) of Lemma 2.2.1.

The result then follows from letting $A = F'(N^*)$, $B = -h'(N^*)$ in parts (b) and (c) of Lemma 2.2.1. $\square$

Because of the general statement of Lemmata 2.3.1, 2.3.3 the results can be applied to any F satisfiying H3 such as (1.1),(1.2), and h satisfying H4 such as (1.4), (1.5) with $\alpha = 1$, $\beta \geq 1$. Below we will consider (2.1) with both a logistic (1.1) and a Gompertz (1.2) growth with linear harvesting term (1.4). The results will be used to derive a MSY for both models.

Logistic Growth

In this section we will derive a MSY for (2.1) with logistic growth rate and linear harvesting term.

Corollary 2.3.3.1. *Consider* (2.3)*. Let $r > E$, and*

$$\tau^* = \frac{\arccos(2 - r/E)}{\sqrt{E^2 - (2E - r)^2}}. \tag{2.7}$$

Then the positive equilibrium

$$N^* = \frac{(r - E)K_c}{r} \tag{2.8}$$

exists. If in addition:

(a) $r \geq 3E$ then N^* is locally asymptotically stable.

(b) $r < 3E$ and $0 \leq \tau < \tau^*$ then N^* is locally asymptotically stable.

(c) $r < 3E$ and $\tau > \tau^*$ then N^* is unstable.

Proof. By the definition of (1.1), (1.4), we compute

$$F_1'(N(t)) = r - \frac{2rN(t)}{K_c}, \quad h'(N(t - \tau)) = E.$$

The positive equilibrium exists and is equal to $N^* = \frac{(r-E)K_c}{r}$.

$$F_1'(N^*) = 2E - r, \quad h'(N^*) = E$$

Applying Lemma 2.3.3 part (b), if $r > E$ and $-E \geq 2E - r \leftrightarrow r \geq 3E$ then N^* is locally asymptotically stable. By Lemma 2.3.3 part (c) if $r > E$ and $r < 3E$ then if $0 \leq \tau < \tau^*$, with τ^* as defined in (2.7) created by substituting $F_1'(N^*), h'(N^*)$ into (2.6), then N^* is locally asymptotically stable, and if $\tau > \tau^*$ then N^* is unstable. Note that part (a) of Lemma 2.3.3 would only apply if $r < E$, as such it is not considered here. □

Lemma 2.3.4. *Consider* (2.3)*. Then the yield is given by*

$$Y(E) = EN^* = \frac{K_cE(r - E)}{r}. \tag{2.9}$$

It is increasing for $E \in (0, E_{opt})$ and decreasing for $E \in (E_{opt}, \infty)$ with $E_{opt} = r/2$. The maximum yield (MY) is given by

$$MY = Y(E_{opt}) = \frac{rK_c}{4} \tag{2.10}$$

and is associated to $N^* = K_c/2$.

Proof.

$$Y'(E) = K_c - \frac{2EK_c}{r}$$

$Y'(E) > 0$ for $E < r/2$, $Y'(E) < 0$ for $E > r/2$. Define $E_{opt} = r/2$, then $Y(E)$ is increasing for $E \in (0, E_{opt})$, decreasing for $E \in (E_{opt}, \infty)$ and is maximized when $E = E_{opt}$. Therefore the maximum yield $MY = Y(E_{opt})$ and this yield is associated to $N^*(E_{opt}) = \frac{(r-E_{opt})K_c}{r} = \frac{K_c}{2}$. $\square$

Theorem 2.3.5. *The MY* (2.10) *of* (2.3) *is a maximum sustainable yield (MSY) if* $0 \leq \tau < \pi/r$.

Proof. By Lemma 2.3.4 the yield is maximized when $E = E_{opt}$. When $E = E_{opt}$, then $3E_{opt} = 3(r/2) > r$ and $E_{opt} - r = (r/2) - r < 0$. So by Corollary 2.3.3.1 if $0 \leq \tau < \tau^*$ then $N^* = K_c/2$ is locally asymptotically stable. When $E = E_{opt}$, τ^* as given in (2.7) becomes,

$$\begin{aligned}
\tau^* &= \frac{\arccos(2 - \frac{r}{(r/2)})}{\sqrt{(r/2)^2 - (2(r/2) - r)^2}} \\
&= \frac{\arccos(0)}{\sqrt{r^2/4}} \\
&= \frac{\pi}{2} \cdot \frac{2}{r} \\
&= \frac{\pi}{r}
\end{aligned}$$

Therefore, if $0 \leq \tau < \pi/r$ then N^* is locally asymptotically stable, and the MY associated to N^* is sustainable. Therefore the MY becomes a MSY when $0 \leq \tau < \pi/r$. $\square$

Gompertz Growth

In this section we will derive a MSY for (2.1) with Gompertz growth rate and linear harvesting term.

Corollary 2.3.5.1. *Consider*

$$\begin{cases} \frac{dN}{dt} = rN(t)\ln\left(\frac{K_c}{N(t)}\right) - EN(t-\tau) \\ N(t) = \phi(t), \quad t \in [-\tau, 0]. \end{cases} \tag{2.11}$$

Let

$$\tau^* = \frac{\arccos(1 - r/E)}{\sqrt{2Er - r^2}}. \tag{2.12}$$

Then the positive equilibrium

$$N^* = K_c e^{-E/r} \tag{2.13}$$

exists. If in addition:

(a) $r \geq 2E$ then N^* is locally asymptotically stable.

(b) $r < 2E$ and $0 \leq \tau < \tau^*$ then N^* is locally asymptotically stable.

(c) $r < 2E$ and $\tau > \tau^*$ then N^* is unstable.

Proof. By the definition of (1.2), (1.4), we compute

$$F_2'(N(t)) = r\ln\left(\frac{K_c}{N(t)}\right) - r, \quad h'(N(t-\tau)) = E.$$

The positive equilibrium exists and is equal to $N^* = K_c e^{-E/r}$.

$$F_2'(N^*) = E - r, \quad h'(N^*) = E$$

Applying Lemma 2.3.3 part (b), if $-E \geq E - r \leftrightarrow r \geq 2E$ then N^* is locally asymptotically stable. By Lemma 2.3.3 part (c), if $r < 2E$

then if $0 \leq \tau < \tau^*$, with τ^* as defined in (2.12) found by substituting $F_2'(N^*), h'(N^*)$ into (2.6), then N^* is locally asymptotically stable. And, if $\tau > \tau^*$ then N^* is unstable. Note that Lemma 2.3.3 part (a) would only apply if $r < 0$, as such it is not considered here. $\square$

Lemma 2.3.6. *Consider* (2.11). *Then the yield is given by*

$$Y(E) = EN^* = EK_c e^{-E/r}. \tag{2.14}$$

It is increasing for $E \in (0, E_{opt})$ *and decreasing for* $E \in (E_{opt}, \infty)$ *with* $E_{opt} = r$. *The maximum yield (MY) is given by*

$$MY = Y(E_{opt}) = \frac{rK_c}{e} \tag{2.15}$$

and is associated to $N^* = K_c/e$.

Proof.

$$Y'(E) = K_c e^{-E/r}(1 - E/r)$$

$Y'(E) > 0$ for $E < r$, $Y'(E) < 0$ for $E > r$. Defining $E_{opt} = r$, $Y(E)$ is increasing for $E \in (0, E_{opt})$, decreasing for $E \in (E_{opt}, \infty)$ and is maximized when $E = E_{opt}$. Therefore the maximum yield $MY = Y(E_{opt})$ and this yield is associated to $N^*(E_{opt}) = K_c e^{-E_{opt}/r} = K_c/e$. $\square$

Theorem 2.3.7. *The MY* (2.15) *of* (2.11) *is a maximum sustainable yield (MSY) if* $0 \leq \tau < \pi/2r$.

Proof. By Lemma 2.3.6 the yield is maximized when $E = E_{opt}$. When $E = E_{opt}$ then $r < 2(r)$, so by Corollary 2.3.5.1 if $0 \leq \tau < \tau^*$ then $N^* = K_c/e$ is locally asymptotically stable. When $E = E_{opt}$, τ^* as given in (2.12) becomes

$$\begin{aligned}\tau^* &= \frac{\arccos(1 - r/(r))}{\sqrt{2(r)r - r^2}} \\ &= \frac{\arccos(0)}{r} \\ &= \frac{\pi}{2r}\end{aligned}$$

Therefore if $0 \leq \tau < \pi/r$ then N^* is locally asymptotically stable, and the MY associated to N^* is sustainable. Therefore the MY becomes a MSY when $0 \leq \tau < \pi/2r$. $\square$

2.4 Oscillation

A solution of (2.1) is said to oscillate around an equilibrium N^* if the function $N(t) - N^*$ is neither eventually positive, nor eventually negative. A DDE is said to be oscillatory if all of its solutions oscillate, and non-oscillatory if at least one solution does not oscillate. Many results on the oscillation of delay differential equations were given in the monographs [18, 19], one of which was included in Section 2.2.

Our goal for this section will be to show that under specific conditions, solutions of (2.3) oscillate around N^* under the additional assumption that if for some t^*, $N(t^*) \leq 0$ then $N(t) = 0$ for all $t > t^*$. We will say that a solution is eventually greater than a lower bound a, if $\exists t_1$ such that $\forall t > t_1$ $N(t) > a$, and eventually lower than an upper bound b, if $\exists t_2$ such that $\forall t > t_2$ $N(t) < b$.

The proof of the following Theorem will take place in two steps. The first step will be to show that there are no solutions of (2.3) which are eventually above N^*. This step will follow the proof of the linearized stability theorems found in [18, 19], but is included here for completeness. The second will be to show that there are no solutions which are eventually in $\left[\dfrac{EK_c}{r}, N^*\right)$. This step is similar to Step 1 and the proofs of linearized stability found in [18, 19], however, most linearized stability theorems assume that the non-linear functions $f_i(N(t) - N^*) < 0\ \forall N(t) \in (0, N^*)$ (f_i will be defined in the proof), which we will see is not the case here. Instead we restrict our attention to a smaller strip of possible $N(t)$ values, and show oscillation for solutions which are eventually in this strip.

Theorem 2.4.1. *Let $r - 2E > 0$. If $N(t)$ is a solution of* (2.3) *which is eventually greater than $\frac{EK_c}{r}$ and the equation*

$$\lambda + (r - 2E) + Ee^{-\lambda\tau} = 0 \tag{2.16}$$

has no real roots, then that solution oscillates around $N^* = \frac{K_c(r-E)}{r}$.

Proof. Assume that (2.16) has no real roots. Set $x(t) = N(t) - N^*$, then (2.3) becomes

$$\frac{dx}{dt} + \frac{r}{K_c}x(t)^2 + (r - 2E)x(t) + Ex(t - \tau) = 0 \tag{2.17}$$

and the condition that $N(t)$ is eventually greater than $\frac{EK_c}{r}$ is equivalent to the condition that $x(t)$ is eventually greater than $\frac{-K_c(r-2E)}{r}$. It is clear that the solution $N(t)$ of (2.3) oscillates around N^* if and only if $x(t)$ oscillates about 0. Now let,

$$p_1 = r - 2E, \quad p_2 = E$$

$$f_1(x) = \frac{r}{K_c(r - 2E)}x^2 + x, \quad f_2(x) = x$$

$$\tau_1 = 0, \quad \tau_2 = \tau$$

then we may re-write (2.17) as

$$\frac{dx}{dt} + \sum_{j=1}^{2} p_j f_j(x(t - \tau_j)) = 0$$

Step 1: No eventually positive solution of (2.17).

Suppose for the sake of contradiction that (2.17) has a non-oscillatory solution $x(t)$ which we shall assume is eventually positive. Since for some t_1, $x(t) > 0\ \forall t > t_1 + \tau$ then,

$$f_1(x(t)) > 0, \quad f_2(x(t - \tau)) > 0$$

$\forall t > t_1$. So eventually,

$$\frac{dx}{dt} = -\Big(p_1 f_1(x(t)) + p_2 f_2(x(t - \tau))\Big) < 0.$$

Thus,

$$\lim_{t\to\infty} x(t) = l \geq 0$$

exists. If $l > 0$ then $f_1(l) > 0, f_2(l) > 0$ and

$$\lim_{t\to\infty} \frac{dx}{dt} = -\Big(p_1 f_1(l) + p_2 f_2(l)\Big) < 0$$

This would imply that eventually $x(t)$ is always decreasing with no lower bound which is a contradiction since we have already assumed that $x(t)$ is eventually positive. Therefore $l = 0$ and

$$\lim_{t\to\infty} x(t) = 0.$$

Now let

$$P_j(t) = \frac{p_j f_j(x(t - \tau_j))}{x(t - \tau_j)} \tag{2.18}$$

so that (2.17) can be written as

$$\frac{dx}{dt} + \sum_{j=1}^{2} P_j(t)x(t-\tau_j) = 0, \tag{2.19}$$

where

$$\lim_{t\to\infty} P_1(t) = \lim_{t\to\infty}(r-2E)\left(\frac{r}{K_c(r-2E)}x(t)+1\right) = r-2E = p_1$$

$$\lim_{t\to\infty} P_2(t) = \lim_{t\to\infty} E = p_2.$$

Set $t_0 = t_1 + \tau$, then by Theorem 2.2.3 when the characteristic equation (2.16) has no real roots, all the non-trivial solutions of (2.17) oscillate. But this is a contradiction since we assumed $x(t)$ was eventually positive. Therefore, there are no eventually positive solutions of (2.17).

Step 2: No solutions which are eventually in $\left[\frac{-K_c(r-2E)}{r}, 0\right)$

Suppose for the sake of contradiction that (2.17) has a non-oscillatory solution $x(t)$ which is eventually in $\left[\frac{-K_c(r-2E)}{r}, 0\right)$. Then,

$$f_1(x(t)) \leq 0, \quad f_2(x(t-\tau)) < 0$$

$\forall t > t_2 + \tau$. So eventually,

$$\frac{dx}{dt} = -\Big(p_1 f_1(x(t)) + p_2 f_2(x(t-\tau))\Big) > 0$$

which implies that the function is increasing $\forall t > t_2 + \tau$.

Thus

$$\lim_{t\to\infty} x(t) = l, \quad l \in \left[\frac{-K_c(r-2E)}{r}, 0\right]$$

exists. However if $l \in \left[\frac{-K_c(r-2E)}{r}, 0\right)$ this would imply that l is an equilibrium point of (2.17). This is a contradiction since we can show that the only equilibriums of (2.17) are $x = 0, -N^*$. Thus

$$\lim_{t\to\infty} x(t) = 0.$$

We define the functions $P_j(t)$ as in (2.18) and re-write (2.17) as (2.19). Set $t_0 = t_2 + \tau$, then by Theorem 2.2.3 when the characteristic equation (2.16) has no real roots, all the non-trivial solutions of (2.17) oscillate. But this is a contradiction since we have assumed that $x(t)$ was eventually negative. Therefore there are no solutions which are eventually in $\left[\frac{-K_c(r-2E)}{r}, 0\right)$.

Since there are no solutions of (2.17) which are eventually above 0 or which are eventually in $\left[\frac{-K_c(r-2E)}{r}, 0\right)$ then we say that there are no solutions of (2.3) which are eventually above N^* or which are in $\left[\frac{EK_c}{r}, N^*\right)$. Thus if we have a solution which is eventually above $\frac{EK_c}{r}$ then this must mean that that solution oscillates around N^*. $\square$

It is important to note that Theorem 2.4.1 is not a full characterization of conditions for oscillation of (2.3). Perhaps more importantly, in Section 2.5 we will show that there exist solutions which do not oscillate around N^* even though the characteristic equation only has complex roots. Under the additional assumption that $N(t) = \max\{N(t), 0\}$ these solutions will become zero in finite time. This implies that for (2.3), oscillation of an associated linearized equation does not imply oscillation in all circumstances. It then makes sense that the Linearized Oscillation Theorems found in [18, 19], did not apply to our equation.

The real use of Theorem 2.4.1 comes when considering asymptotically stable equilibriums. Say we have an asymptotically stable equilibrium N^* of (2.3), and initial conditions are such that the equilibrium is attracting.

Then

$$\lim_{t \to \infty} N(t) = N^*.$$

We can then say that if the characteristic equation has no real roots, the solutions with initial conditions such that the equilibrium is attracting, will oscillate around N^* with decreasing amplitude.

2.5 Numerical Simulations and Discussion

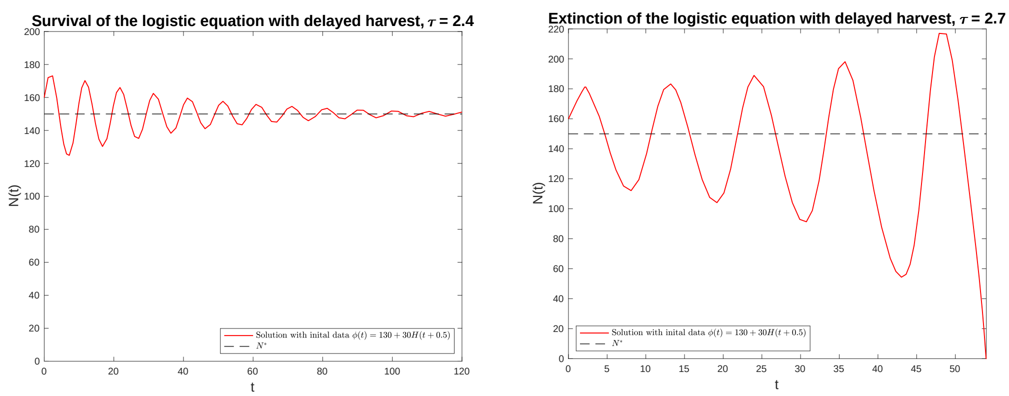

Figure 2.2: Solutions to (2.3), $r = 1.2$, $E_{opt} = 0.6$, $K_c = 300$, $\tau^* = 2.61$, $\phi(t) = 130 + 30H(t + 0.5)$. (Left) $\tau = 2.4 < 2.61$, thus $N^* = 150$ is locally asymptotically stable, and the solution converges to the equilibrium. (Right) $\tau = 2.7 > 2.61$, thus N^* is unstable and the solution oscillates around N^* with increasing amplitude before hitting extinction at $t \approx 55$.

In Figure. 2.2 stability of the positive equilibrium $N^* = 150$ of (2.3) ($r = 1.2$, $E = E_{opt} = 0.6$, $K_c = 300$ initial data $\phi(t) = 130 + 30H(t + 0.5)$, $t \in [-2.7, 0]$) is investigated for changing value of delay. The solutions were computed using the MATLAB function dde23 which solves delay differential equations for constant delays, with the additional stipulation that if for some t^*, $N(t^*) \leq 0$, then $N(t) = 0$ for all $t > t^*$. See the appendix, LogisticStable.m and LogisticUnStable.m for MATLAB code.

The function $H(t)$ refers to the Heaviside step function and is defined as

$$H(t) = \begin{cases} 0 & , t < 0 \\ \frac{1}{2} & , t = 0 \\ 1 & , t > 0 \end{cases}$$

Since $E = E_{opt}$, the harvesting is optimal, and the positive equilibrium $N^* = 150$ is associated to the maximum yield. By Theorem 2.3.5 the MY is sustainable, and by extension the positive equilibrium is locally asymptotically stable when $0 \leq \tau < 2.61799$. In Figure. 2.2 (left) $\tau = 2.4$ which means that N^* is locally asymptotically stable. We can see that the solution oscillates with decreasing amplitude before converging to the equilibrium. In Figure. 2.2 (right) $\tau = 2.7$ which means that N^* is unstable. Indeed we can see that while the initial history starts the solution close to N^*, the solution oscillates with increasing amplitude, eventually hitting zero in finite time.

In Figure. 2.3 stability of the positive equilibrium $N^* \approx 110.3638$ of (2.11) ($r = 0.6$, $E = E_{opt} = 0.6$, $K_c = 300$, initial data $\phi(t) = 90 + 30H(t + 0.5)$, $t \in [-2.7, 0]$) is investigated for changing value of delay. The solutions were once again computed using dde23, with the additional stipulation that if for some t^*, $N(t^*) \leq 0$, then $N(t) = 0$ for all $t > t^*$. See the appendix, GompertzStable.m and GompertzUnStable.m for MATLAB code.

Since $E = E_{opt}$ the harvesting is optimal, and the positive equilibrium

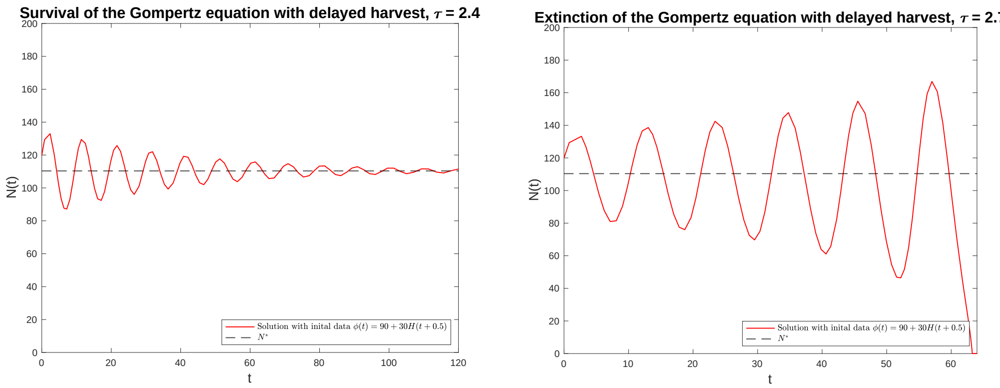

Figure 2.3: Solutions to (2.11) $r = 0.6$, $E_{opt} = 0.6$, $K_c = 300$, $\tau^* = 2.61$, $\phi(t) = 90 + 30H(t + 0.5)$. (Left) $\tau = 2.4 < 2.61$, thus $N^* \approx 110.3638$ is locally asymptotically stable, and the solution converges to the equilibrium. (Right) $\tau = 2.7 > 2.61$ and thus N^* is unstable, and the solution oscillates around N^* with increasing amplitude hitting extinction at some time $t \approx 62$.

N^* is associated to the maximum yield. By Theorem 2.3.7 the MY is sustainable, and by extension the positive equilibrium is locally asymptotically stable when $0 \leq \tau < 2.61799$. In the figure on the left $\tau = 2.4$ which means that N^* is locally asymptotically stable. We can see that the solution oscillates with decreasing amplitude before converging to the equilibrium. In the figure on the right $\tau = 2.7$ which means that N^* is unstable. Indeed we can see that while the initial history starts the solution close to N^*, the solution oscillates with increasing amplitude, eventually hitting zero in finite time.

In Figure. 2.4, solutions of (2.3) ($r = 1$, $E = 0.4$, $K_c = 300$, $\tau = 5$, with different initial data are compared. The solutions were computed using dde23, with the additional stipulation that if for some t^*, $N(t^*) \leq 0$, then $N(t) = 0$ for all $t > t^*$. See the appendix, Oscillation.m and NonOscillation.m for MATLAB code.

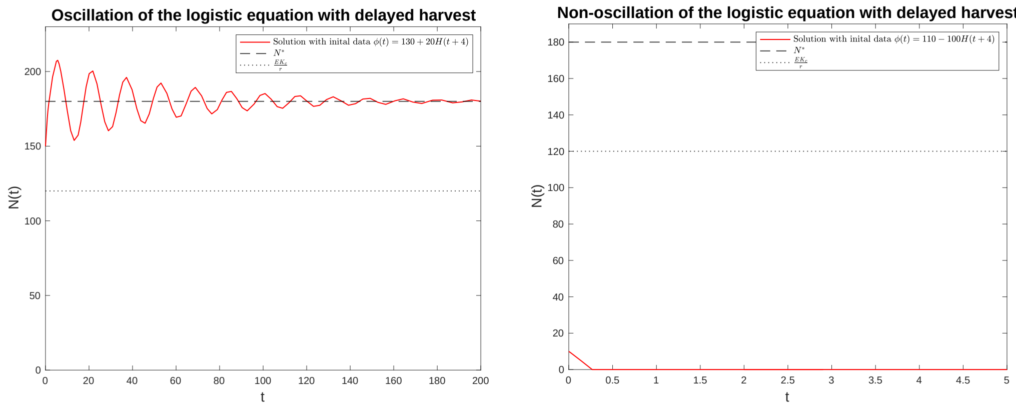

Figure 2.4: Solution to (2.3), $r = 1$, $E = 0.4$, $K_c = 300$, $\tau = 5 < \tau^* = 6.04$. Here $2E < r = 1 < 3E$, so N^* is locally asymptotically stable, and the roots of the characteristic equation are complex. (Left) Solution with $\phi(t) = 130 + 20H(t + 4)$ oscillating around N^*. (Right) Solution with $\phi(t) = 110 - 100H(t + 4)$ does not oscillate around N^*.

Since $2E < r < 3E$, and $\tau = 5 < \tau^* = 6.046$ by Corollary 2.3.3.1 the $N^* = 180$ equilibrium is locally asymptotically stable. If initial conditions are such that N^* is attractive, then

$$\lim_{t\to\infty} N(t) = N^*$$

and by Theorem. 2.4.1 the solution should oscillate around N^*.

In Figure. 2.4 (left), the solution with initial data $\phi(t) = 130 + 20H(t + 4)$ is shown oscillating around N^*, with decreasing amplitude. In contrast Figure. 2.4 (right) shows the solution with initial data $\phi(t) = 110 - 100H(t+4)$ going to zero without oscillating. The difference is that Figure. 2.4 (left) has initial data for which the equilibrium is attractive. Thus Theorem. 2.4.1 guarantees that the solution oscillates around N^*. While Figure. 2.4 (right) must have initial data for which the equilibrium is not attractive, and hence Theorem. 2.4.1 does not apply. This further illustrates how with the logistic model, oscillation of a corresponding linearized equation does not necessarily imply oscillation of the non-linear equation.

Conclusion

The results of this chapter can be summarized as follows:

1. Under appropriate assumptions on the growth and harvesting rates, we show the existence and uniqueness of a positive equilibrium for a general delayed harvesting model.

2. The inclusion of the delay into harvesting does not affect the maximum yield (optimality) of the logistic or Gompertz models. However it does affect the sustainability of both.

3. Oscillation of the logistic delayed harvesting model can be guaranteed for solutions which are eventually above a certain threshold when the characteristic equation has no real roots. This implies oscillation of solutions for asymptotically stable equilibriums, when the characteristic equation has no roots.

Furthermore, based on Theorems 2.3.5, 2.3.7, we see that delays should be kept small to avoid extinction of populations under optimal harvesting policies.

Representing harvesting by a continuous deduction in a differential equation is a simple model which allows for easy analysis, and has been used extensively. However, this type of model relies on the assumption that harvesting is constantly occurring. A more realistic harvesting model would be one that takes into account that harvesting does not occur without interruption and often a portion of the stock is removed at a single moment in time. Impulsive harvesting is one such example and will be investigated in the next chapter.

Chapter 3

Delayed Impulsive Harvesting

3.1 Introduction

Impulsive harvesting refers to when a large portion of the stock is removed at a single point in time and is modelled by impulsive DEs. As was discussed in chapter 1, whenever control is involved, it is natural to assume that the information available at the control implementation point is not up-to-date, leading to delayed impulsive conditions.

We consider the logistic equation with constant effort impulsive harvesting that is dependent upon delayed data. This assumes that the information used to determine hunting or fishery quotas is based on the on the population size collected during one of the previous harvesting events.

The main object of this chapter is the delayed impulsive harvesting model, given for a fixed $k \in \mathbb{N}$ as

$$\begin{cases} \dfrac{dN}{dt} = rN(t)\left(1 - \dfrac{N(t)}{K_c}\right), & t \neq nT,\ n \in \mathbb{N} \\ N(nT^+) = \max\{N(nT) - EN((n-k)T), 0\}, & t = nT,\ n \in \mathbb{N}, \\ N(0) = N_0, N(-T) = N_{-1}, ..., N(-kT) = N_{-k} \end{cases} \tag{3.1}$$

with prescribed initial conditions

$$N_i \in (0, \infty), \quad i = -k, ..., 0.$$

In this model, $N(t)$ represents a size or a biomass of the population at time t, $r > 0$ is the intrinsic growth rate, $K_c > 0$ is a carrying capacity of the environment, $T > 0$ is the time between two consecutive harvesting events, E is a harvesting effort and in contrast to Chapter 2, it is assumed that $E \in (0, 1)$ to avoid immediate extinction. In addition we assume that restocking does not occur. In general $k \in \mathbb{N}$, though references to the non-delayed model with $k = 0$ will also be given. We note that here we have dropped the notation $\Delta N(nT) = N(nT^+) - N(nT)$, and instead have rewritten the impulsive condition to emphasize that the population is always a maximum of the population and zero. This is to avoid the possibility of negative populations, which do not make biological sense.

The goal of this chapter is first to consider the sustainability of (3.1) under harvesting, which corresponds to the local asymptotic stability of a positive solution which will be described later, and second to explore the sustainable yield (SY) and the maximum sustainable yield (MSY) of (3.1). The chapter is structured to follow this purpose. After presenting relevant definitions and auxiliary results in Section 3.2, we explore stability of positive periodic solutions of (3.1) in Section 3.3. All the issues connected to SY and MSY, and relevant solutions of (3.1), are postponed to Section 3.4. We will show that while optimality is unaffected by the magnitude of delay, sustainability of the optimal solution is delay-dependent for $k \geq 2$. The analysis of the impact of the delay on local asymptotic stability of the positive solution is based on the results obtained in Section 3.3. Finally, Section 3.5 includes examples, numerical simulations, as well as discussion of the results and possible future directions.

3.2 Preliminaries

Definition 6. A solution $N^*(t)$ of (3.1) is said to be *stable* if for any $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that the inequalities $|N_j - N^*(jT)| < \delta$, $j = -k, \ldots, 0$, where N_j are the initial conditions, imply $|N(t) - N^*(t)| < \varepsilon$ for all $t > 0$.

Definition 7. A solution of the impulsive system (3.1) is said to be *locally asymptotically stable* if it is stable and there exists $\eta > 0$ such that $\lim\limits_{t\to\infty} |N(t) - N^*(t)| = 0$ for any $|N_j - N^*(jT)| < \eta$, $j = -k, \ldots, 0$, where N_j are the initial conditions.

Definition 8. A solution of the impulsive system (3.1) is said to be *globally asymptotically stable* if it is stable and if $\forall N_0, ..., N_{-k} > 0$, $\lim\limits_{t\to\infty} |N(t) - N^*(t)| = 0$ (i.e. $\eta = \infty$).

A general difference equation is given by

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \tag{3.2}$$

with the initial conditions $x_0, ..., x_{-k}$.

Definition 9. A solution $x_n \equiv x^*$ of (3.2) is *stable* if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that $\max\{|x_0 - x^*|, ..., |x_{-k} - x^*|\} < \delta$ implies $|x_n - x^*| < \varepsilon$ for any $n \geq 0$.

Definition 10. A solution x^* of (3.2) is *locally asymptotically stable* if it is stable and if there exists $\eta > 0$ such that if $\max\{|x_0 - x^*|, ..., |x_{-k} - x^*|\} < \eta$ then $\lim_{n\to\infty} |x_n - x^*| = 0$.

Definition 11. A solution x^* of (3.2) is said to be *globally asymptotically stable* if it is stable and if $\forall x_0, ..., x_{-k} > 0$, $\lim_{n\to\infty} |x_n - x^*| = 0$.

Since in (3.1) we will be considering T-periodic solutions, there is a slight change in the definition of the MSY. In chapter 2 our yields corresponded to constant solutions of the continuous harvesting model, whereas for the delayed impulsive harvesting model our yields will be associated to T-periodic solutions of (3.1). The interpretation of E_{opt} as the value of E which maximizes the yield remains the same.

Definition 12. The *maximum yield (MY)* of (3.1) is a yield associated to an optimal T-periodic solution of (3.1) such that no other solution will have a corresponding yield that will exceed the MY.

Definition 13. A *maximum sustainable yield (MSY)* of (3.1) is a MY such that the optimal T-periodic solution associated to the MY is at least locally asymptotically stable.

In [47], the authors considered a MSY for (3.1) with $k = 0$. The results are summarized in the following.

Lemma 3.2.1. *[47] Consider* (3.1) *for* $k = 0$

$$\begin{cases} \dfrac{dN}{dt} = rN(t)\Big(1 - \dfrac{N(t)}{K_c}\Big), \ t \neq nT, \quad n \in \mathbb{N} \\ N(nT^+) = \max\{(1-E)N(nT), 0\}, \ t = nT, \quad n \in \mathbb{N} \\ N(0) = N_0. \end{cases} \tag{3.3}$$

Then the optimal harvesting effort is

$$E_{opt} = 1 - e^{-rT/2}. \tag{3.4}$$

and the MSY is given by

$$MSY = \frac{K_c(e^{rT/2} - 1)}{T(e^{rT/2} + 1)} \tag{3.5}$$

The optimal positive periodic solution $N^*(t)$ *of* (3.3) *corresponding to the MSY and* E_{opt} *is globally attracting with*

$$N^*(nT^+) = \frac{K_c}{e^{rT/2} + 1}. \tag{3.6}$$

In [47] analysis of a non-delayed impulsive model (3.3) is reduced to a non-linear difference equation which is first order. We also intensively exploit this connection between the difference and the impulsive equations. Although when a delay is incorporated in the impulsive condition such as in (3.1), the difference equation becomes higher order. We recall that for difference equations, the roots of the characteristic equation of an associated linearized model should lie inside the unit circle for local asymptotic stability. This is in contrast to differential equations where the real parts of the roots have to be negative. Some auxiliary results regarding difference equations are listed below.

Lemma 3.2.2 ([16, Theorem 2.37, pg. 95]). *The zeros of the characteristic polynomial with $p_0 > 0$, $p_1 > 0$*

$$p(\lambda) = \lambda^2 - p_0\lambda + p_1$$

lie inside the unit circle if and only if $p_0 - 1 < p_1 < 1$.

The result of [25, Theorem 1.1.1, Part f, pg. 7] describes conditions when a root of a quadratic equation lies on the unit disc.

Lemma 3.2.3 ([25, Theorem 1.1.1, Part f, pg. 7]). *A necessary and sufficient condition for a root of the characteristic equation of*

$$p(\lambda) = \lambda^2 - p_0\lambda + p_1$$

with $p_0, p_1 \in \mathbb{R}$ to have a root satisfying $|\lambda| = 1$ is that either

$$|p_0| = |1 + p_1|$$

or

$$p_1 = 1 \text{ and } |p_0| \leq 2.$$

In this case the equilibrium is called a non-hyperbolic point.

The following result is cited from [16, Theorem 5.10, pg. 253].

Lemma 3.2.4 ([16, Theorem 5.10, pg. 253]). *If $\sum_{i=0}^{k} |p_i| < 1$ then the zero solution of the difference equation*

$$x_{n+k+1} + p_0 x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0$$

is asymptotically stable.

The following result can be found in [16, Theorem 5.3, P. 249].

Lemma 3.2.5 ([16, Theorem 5.3, pg. 249])**.** *Let $p_0 > 0$, $p_k \in \mathbb{R}$ be arbitrary, and $k \in \mathbb{N}$. The zero solution of*

$$x_{n+1} - p_0 x_n + p_k x_{n-k} = 0 \tag{3.7}$$

is asymptotically stable if and only if $|p_0| < (k+1)/k$ and

(i) $|p_0| - 1 < p_k < (p_0^2 + 1 - 2|p_0|\cos(\theta^*))^{1/2}$ if k is odd or

(ii) $|p_k - p_0| < 1$ and $|p_k| < (p_0^2 + 1 - 2|p_0|\cos(\theta^*))^{1/2}$ if k is even,

where θ^* is the solution of the equation

$$\frac{\sin(k\theta)}{\sin((k+1)\theta)} = \frac{1}{|p_0|}, \quad \theta \in \left(0, \frac{\pi}{k+1}\right). \tag{3.8}$$

However, we do not need the general form of Lemma 3.2.5, since in our model $0 < p_k < p_0$. When this is the case, the left inequality in both (i) and (ii) becomes $p_0 < p_k + 1$, while the right inequalities coincide.

Corollary 3.2.5.1. *Let $0 < p_k < p_0$. Then* (3.7) *is asymptotically stable if and only if the following two inequalities hold:*

$$p_0 < \min\left\{p_k + 1, \frac{k+1}{k}\right\}, \quad p_k < \sqrt{p_0^2 + 1 - 2p_0\cos(\theta^*)}, \tag{3.9}$$

where θ^* is the solution of (3.8)*.*

A generalization of Lemma 3.2.6 is considered in [16, Theorem 5.2, pg. 248].

Lemma 3.2.6 ([16, Theorem 5.2, pg. 248])**.** *Let $q \in (0, 2)$. The zero solution of the equation*

$$x_{n+1} = x_n - q x_{n-k}$$

is asymptotically stable for $k = 1$. It is asymptotically stable for $k \geq 2$ if and only if in addition

$$q < 2\cos\left(\frac{k\pi}{2k+1}\right).$$

After stating the above results, we are in a position to proceed to analysis of sustainability under delayed harvesting.

3.3 Stability

First, we reduce the dynamics of a differential equation with delayed impulsive harvesting to those of a difference equation.

Lemma 3.3.1. *The solution of* (3.1) *on the interval* $t \in (nT, (n+1)T)$, $n \in \mathbb{N}_0$ *is*

$$N(t) = \frac{K_c e^{r(t-nT)} N(nT^+)}{K_c + N(nT^+)(e^{r(t-nT)} - 1)}. \tag{3.10}$$

The solution of (3.1) *with* $N(nT^+) = x_n$ *at the point just after harvesting* $t = nT$ *satisfies the difference equation*

$$\begin{aligned} x_{n+1} &= \max\left\{ \frac{K_c x_n e^{rT}}{K_c + x_n(e^{rT} - 1)} - E\frac{K_c x_{n-k} e^{rT}}{K_c + x_{n-k}(e^{rT} - 1)}, 0 \right\} \\ &= \max\{f(x_n, x_{n-k}), 0\}, \quad n \in \mathbb{N}_0, \end{aligned} \tag{3.11}$$

where $x_{-k}, ..., x_0$, *are given by the initial data*

$$x_i = N_i, i = -k, ..., 0.$$

Proof. For $t \in (nT, (n+1)T)$, $n \in \mathbb{N}_0$, the impulsive model is non-delayed, and the solution of the differential equation exists and is monotone on the interval. To find the solution on $t \in (nT, (n+1)T)$ we solve the differential equation and obtain

$$N(t) = \frac{K_c e^{r(t-nT)} N(nT^+)}{N(nT^+)(e^{r(t-nT)} - 1) + K_c}, \quad t \in (nT, (n+1)T).$$

The size of $N(t)$ at the end of the nth time period and before harvesting is given by,

$$N((n+1)T) = \frac{K_c N(nT^+) e^{rT}}{K_c + N(nT^+)(e^{rT} - 1)}.$$

Using the value of $N(t)$ before harvesting as a function of $N(nT^+)$, combined with the definition of the impulsive condition in (3.1), gives

$$N((n+1)T^+) = \max\left\{\frac{K_c N(nT^+)e^{rT}}{K_c + N(nT^+)(e^{rT}-1)} - E\frac{K_c N((n-k)T^+)e^{rT}}{K_c + N((n-k)T^+)(e^{rT}-1)}, 0\right\}.$$

Letting $N(nT^+) = x_n$, which is the size of the population after a harvesting event, we obtain difference equation (3.11) for $n \in \mathbb{N}_0$, with prescribed initial conditions $x_i = N(iT^+) = N_i$ for $i = -k, ..., 0$. □

After the reduction to a difference equation, let us justify that stability of (3.1) can be deduced from that of (3.11).

Lemma 3.3.2. x^* *is a locally asymptotically stable equilibrium of the difference equation* (3.11) *if and only if the solution to* (3.1) *with* $N(nT^+) = x^*$, *is locally asymptotically stable as well.*

Proof. It is easy to see that asymptotic stability of a solution to (3.1) implies asymptotic stability of the associated difference equation. First, let us note that the function

$$g(a,x) = \frac{K_c a x}{K_c + x(a-1)} \tag{3.12}$$

for a fixed $a = e^{rs} > 1$ and any non-negative x, has the derivative in x

$$\frac{\partial g(s,x)}{\partial x} = \frac{K_c^2 e^{rs}}{(K_c + x(e^{rs}-1))^2}, \quad \left|\frac{\partial g(s,x)}{\partial x}\right| \le e^{rs} \le e^{rT}.$$

Let us assume that the solution x^* of (3.11) is stable. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that, once all $|x_{-j} - x^*| < \delta$, $j = 0, \dots, k$, we get $|x_n - x^*| < \varepsilon e^{-rT}$. The solution N^* corresponding to x^* on interval $(nT, (n+1)T)$ is $N^*(nT^+) = x^*$ with

$$N^*(t) = \frac{K_c e^{r(t-nT)} x^*}{x^*(e^{r(t-nT)} - 1) + K_c}.$$

Following (3.10), we note that on $(nT, (n+1)T)$, there exist ζ between x_n and x^*, $s \in [0, T]$ such that

$$\begin{aligned} |N(t) - N^*(t)| &= \left| \frac{K_c e^{r(t-nT)} x_n}{x_n(e^{r(t-nT)} - 1) + K_c} - \frac{K_c e^{r(t-nT)} x^*}{x^*(e^{r(t-nT)} - 1) + K_c} \right| \\ &= \left| \frac{\partial g(s, \zeta)}{\partial x} \right| |x_n - x^*| \le e^{rT} \varepsilon e^{-rT} = \varepsilon, \end{aligned}$$

thus N^* is stable.

Finally, let the solution x^* of (3.11) be locally asymptotically stable. If n_0 is such a number that $|x_n - x^*| < \varepsilon e^{-rT}$ for $n \ge n_0$, as above,

$$|N(t) - N^*(t)| \le e^{rT} \varepsilon e^{-rT} = \varepsilon, \quad t \ge nT, \quad n \ge n_0,$$

thus N^* is both stable and attractive, and therefore is locally asymptotically stable, which concludes the proof. □

Next, let us describe solution bounds for a harvested population.

Lemma 3.3.3. *Let $E \in (0, 1)$. Then for any non-negative initial values and $x_0 > 0$ there exists $n_0 \in \mathbb{N}$ such that the solution x_n to (3.11) is in $[0, K_c]$ for $n \ge n_0$.*

Proof. First, let us note that if for any $n \in \mathbb{N}_0$, $x_n \le K_c$, then $x_{n+1} \le K_c$. As we will see, this also holds true for the initial conditions. If at least one $x_j \le K_c$ for $j = -k, ..., -1$ then at the latest $x_{j+k+1} \le K_c$, and the sequence is less than K_c for all further $n \ge j + k + 1$. Therefore we only have to exclude the case $x_n, x_{n-k} > K_c$ for any $n \geq 0$. Assuming $n > k$, $x_n, x_{n-k} > K_c$, we get

$$\begin{aligned} x_{n+1} &= \frac{K_c x_n e^{rT}}{K_c + x_n(e^{rT}-1)} - E\frac{K_c x_{n-k} e^{rT}}{K_c + x_{n-k}(e^{rT}-1)} \\ &< x_n - E\frac{K_c^2 e^{rT}}{K_c + K_c(e^{rT}-1)} \\ &= x_n - EK_c, \end{aligned}$$

thus after $j = \left[\dfrac{x_n - K_c}{EK_c}\right] + 1$ steps, where $[y]$ is the integer part of y, we get $x_{n+j} \leq K_c$, which concludes the proof. □

Difference equation (3.11) has the trivial equilibrium $x^* = 0$, $\forall rT > 0$, and when $rT > -\ln(1-E)$ it has a single positive equilibrium

$$x^* = \frac{((1-E)e^{rT}-1)K_c}{e^{rT}-1}. \tag{3.13}$$

If $rT \leq -\ln(1-E)$ then only the trivial equilibrium exists, and as we will show in Lemma 3.3.4 the solutions of (3.11), and hence (3.1) will inevitably go to extinction.

Lemma 3.3.4. *If*

$$rT \leq -\ln(1-E), \tag{3.14}$$

all solutions of (3.11) *tend to zero.*

Proof. Let $rT \leq -\ln(1-E)$. By Lemma 3.3.3 we can simply consider $x_n \in [0, K_c]$ for n large enough. If for some $n \in \mathbb{N}$, $x_n = 0$, $x_{n+j} = 0$ $\forall j \in \mathbb{N}$ in the solution of (3.11), implying convergence of the sequence to zero, so we restrict ourselves to only considering positive sequences $\{x_n\}$.

Let $\{x_n\}$ be an eventually monotone sequence, then it has a limit d. If $d = 0$, we get that the sequence has converged to zero; if $d > 0$, we let

$n \to \infty$ in

$$x_{n+1} = \frac{K_c x_n e^{rT}}{K_c + x_n(e^{rT} - 1)} - E\frac{K_c x_{n-k} e^{rT}}{K_c + x_{n-k}(e^{rT} - 1)}.$$

This implies that d is a positive equilibrium solution of (3.11), which is a contradiction when $rT \leq -\ln(1 - E)$. Thus, we have only to consider sequences $\{x_n\}$ that are neither eventually non-increasing nor eventually non-decreasing.

Before we handle this case, let us notice, first, that the function

$$g(a, x) := \frac{K_c x a}{K_c + x(a - 1)} = K_c - \frac{K_c(K_c - x)}{K_c + x(a - 1)} \tag{3.15}$$

is strictly increasing in both x and a for $K_c > 0$ and $a > 1$ (here $a = e^{rT} > 1$ for $rT > 0$).

Let $k = 1$, since we are only considering sequences that are not non-decreasing, this implies that there exists some n such that $x_n < x_{n-1}$. This leads us to the following,

$$\begin{aligned}
x_{n+1} &= \frac{K_c x_n e^{rT}}{K_c + x_n(e^{rT} - 1)} - E\frac{K_c x_{n-1} e^{rT}}{K_c + x_{n-1}(e^{rT} - 1)} \\
&< \frac{(1 - E)K_c x_n e^{rT}}{K_c + x_n(e^{rT} - 1)} \\
&\leq \frac{(1 - E)K_c x_n \frac{1}{1-E}}{K_c + x_{n-k}(\frac{1}{1-E} - 1)} \\
&= \frac{K_c x_n}{K_c + x_n \frac{E}{1-E}} \\
&< x_n.
\end{aligned}$$

By induction we get that $\{x_j\}$ is a monotonically decreasing sequence starting with $j = n$, and thus it converges to zero, as justified above.

Next, consider $k \geq 2$. If there exists n such that

$$x_n = \min\{x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}, x_n\},$$

then since $g(e^{rT}, x_n) \leq g(e^{rT}, x_{n-k})$ and by (3.14), $e^{rT} \leq \dfrac{1}{1-E}$, we get

$$\begin{aligned}
x_{n+1} &= \frac{K_c x_n e^{rT}}{K_c + x_n(e^{rT}-1)} - E\frac{K_c x_{n-k} e^{rT}}{K_c + x_{n-k}(e^{rT}-1)} \\
&\leq \frac{(1-E)K_c x_n e^{rT}}{K_c + x_n(e^{rT}-1)} \\
&\leq \frac{(1-E)K_c x_n \frac{1}{1-E}}{K_c + x_{n-k}(\frac{1}{1-E}-1)} \\
&= \frac{K_c x_n}{K_c + x_n \frac{E}{1-E}} \\
&< x_n
\end{aligned}$$

as above. Thus $x_{n+1} < x_n$ and

$$x_{n+1} = \min\{x_{n-k+1}, x_{n-k+2}, \ldots, x_n, x_{n+1}\},$$

which yields that $x_{n+2} < x_{n+1}$. The same argument implies by induction that $\{x_n\}$ is monotonically decreasing and thus converges to zero.

Finally, let us consider the case when

$$x_n > \min\{x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}\}$$

for any n. Then there exists $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$

$$m := \liminf_{n\to\infty} x_n \geq \min\{x_{n_0-k}, ..., x_{n_0}\}. \tag{3.16}$$

Introducing the function A through g as defined in (3.15) and using inequality (3.14), we get

$$\begin{aligned}
A(x) &:= x - (1-E)g(e^{rT}, x) \\
&\geq x - (1-E)g\left(\frac{1}{1-E}, x\right) \\
&= x - \frac{K_c x}{K_c + x\frac{E}{1-E}} =: a(x).
\end{aligned} \tag{3.17}$$

Note that

$$a(x) = x\left(1 - \frac{K_c}{K_c + \frac{E}{1-E}x}\right)$$

is monotone increasing for $x \in [0, K_c]$ (as a product of two non-negative increasing functions) from $a(0) = 0$ to $a(K_c) = EK_c > 0$. Let us choose $\varepsilon > 0$ such that $a(x) > \varepsilon$ for $x \in [\frac{m}{2}, K_c]$. From (3.17) we get $A(x) > \varepsilon$, $x \in [\frac{m}{2}, K_c]$ as well. Define a positive $\delta \in \left(0, \frac{\varepsilon}{4}\right)$ satisfying

$$\delta < \min\left\{\frac{m}{2}, K_c - \frac{m}{2}\right\}$$

such that for any $x, y \in [0, K_c]$, the inequality $|x - y| \le \delta$ implies

$$\left|g\left(e^{rT}, x\right) - g\left(e^{rT}, y\right)\right| \le \frac{\varepsilon}{2}.$$

Such $\delta > 0$ exists, as $g(e^{rT}, x)$ defined in (3.15) is continuous and thus uniformly continuous for $x \in [0, K_c]$.

By the definition of m in (3.16) for any $\delta > 0$ there is $n_0 \in \mathbb{N}$ such that $x_n > m - \frac{\delta}{2}$ for any $n \ge n_0 - k$. Now, we also know that for any n_0 there is $n > n_0$ such that $x_n < m + \frac{\delta}{2}$. Based on this, we know that there must exist some $m - \frac{\delta}{2} < x_{n-k}$ and some $x_n < m + \frac{\delta}{2}$, and since we have assumed that $x_n > \min\{x_{n-k}, x_{n-k+1}, \ldots, x_{n-1}\}$ then there must exist $x_{n-k} < x_n$ which means that for n sufficiently large $m - \frac{\delta}{2} < x_{n-k} < x_n < m + \frac{\delta}{2}$ and thus for any $\delta > 0$ there exists $n_0 \in \mathbb{N}$ such that $|x_n - x_{n-k}| < \delta$. Using this we obtain,

$$
\begin{aligned}
x_{n+1} &= g\left(e^{rT}, x_n\right) - Eg\left(e^{rT}, x_{n-k}\right) \\
&= g\left(e^{rT}, x_n\right) - Eg\left(e^{rT}, x_n\right) + E\Bigg[g\left(e^{rT}, x_n\right) - g\left(e^{rT}, x_{n-k}\right)\Bigg] \\
&= x_n - A(x_n) + E\Bigg[g\left(e^{rT}, x_n\right) - g\left(e^{rT}, x_{n-k}\right)\Bigg] \\
&\le x_n - a(x_n) + E\Bigg|g\left(e^{rT}, x_n\right) - g\left(e^{rT}, x_{n-k}\right)\Bigg| \\
&< x_n - \varepsilon + E\frac{\varepsilon}{2} \\
&< x_n - \varepsilon + \frac{\varepsilon}{2} \\
&= x_n - \frac{\varepsilon}{2} \\
&< x_n - 2\delta \le m + \delta - 2\delta \\
&= m - \delta \\
&< m - \frac{\delta}{2},
\end{aligned}
$$

which contradicts to the assumption that $x_n > m - \delta/2$ for any $n \ge n_0 - k$. Thus the scenario described by (3.16) is impossible, and the solution either coincides with zero, starting at some x_j, or tends to zero. □

Theorem 3.3.5. *If* (3.14) *holds, all solutions of* (3.1) *tend to zero.*

Proof. By Lemma 3.14, if (3.14) holds then all solutions of the difference equation tend to zero. This implies that $N(nT^+) \to 0$, and by (3.10) the solution of (3.1) $N(t)$ on $(nT, (n+1)T)$ also tends to zero. Therefore all solutions of (3.1) tend to zero. □

By Theorem 3.3.5, it is sufficient to consider only the case

$$rT > -\ln(1 - E). \tag{3.18}$$

If this condition is not satisfied, all solutions tend to zero. Everywhere below we assume that (3.18) holds.

Next, let us focus on the behaviour of the difference equation when the harvesting is delayed by a single time period $k = 1$. This will lead to the second-order difference equation, and allow us to apply necessary and sufficient results such as Lemma 3.2.2 and 3.2.3 to obtain explicit conditions for stability of the difference equation.

Lemma 3.3.6. *Let $k = 1$. If $E \in (0, 1/2]$ then there exists a positive equilibrium of* (3.11) *which is locally asymptotically stable.*

For $E \in (1/2, 1)$, if

$$rT < -\ln\left(\frac{(1-E)^2}{E}\right) \tag{3.19}$$

then the positive equilibrium of (3.11) *is locally unstable, while if*

$$rT > -\ln\left(\frac{(1-E)^2}{E}\right), \tag{3.20}$$

the positive equilibrium of (3.11) *is locally asymptotically stable.*

Proof. As $rT > -\ln(1-E)$, there exists a unique positive equilibrium x^*.

Let $k = 1$, then (3.11) has the form $x_{n+1} = \max\{f(x_n, x_{n-1}), 0\}$. Linearizing around x^* we get the corresponding linearized equation,

$$u_{n+1} = p_0 u_n - p_1 u_{n-1}$$

where,

$$p_0 = \frac{\partial f}{\partial x_n}(x^*, x^*) = \left.\frac{K^2 e^{rT}}{(K + x_n(e^{rT}-1))^2}\right|_{(x^*,x^*)} = \frac{e^{-rT}}{(1-E)^2}$$

$$p_1 = -\frac{\partial f}{\partial x_{n-1}}(x^*, x^*) = E\left.\frac{K^2 e^{rT}}{(K + x_{n-1}(e^{rT}-1))^2}\right|_{(x^*,x^*)} = \frac{Ee^{-rT}}{(1-E)^2}.$$

The characteristic equation of the linearized equation is $\lambda^2 - p_0\lambda + p_1 = 0$ and we wish to find explicit conditions describing when these roots lie inside the unit disc.

By applying Lemma 3.2.2, for the roots of the characteristic equation to lie inside the unit disc, we only require that $p_1 < 1$ (the left inequality $p_0 - 1 < p_1$ is automatically satisfied for all $rT > -\ln(1-E)$). This inequality is equivalent to,

$$rT > -\ln\left(\frac{(1-E)^2}{E}\right),$$

which coincides with (3.20).

If $E \in (0, 1/2]$, then $-\ln((1-E)^2/E) \le -\ln(1-E)$, and therefore the positive equilibrium x^* is locally asymptotically stable as (3.20) holds automatically when (3.18) holds.

If $E \in (1/2, 1)$, (3.20) implies that the positive equilibrium x^* exists and is locally asymptotically stable.

Using Lemma 3.2.3, we analyze instability case. When $E \in (1/2, 1)$, if $-\ln(1-E) < rT \le -\ln((1-E)^2/E)$, the positive equilibrium exists and the roots of the characteristic equation satisfy $\max\{|\lambda_1|, |\lambda_2|\} \ge 1$. By Lemma 3.2.3 we can show that a root of the characteristic equation has $|\lambda| = 1$ if and only if $rT = -\ln((1-E)^2/E)$ on this interval. Therefore for $-\ln(1-E) < rT < -\ln((1-E)^2/E)$ which corresponds to (3.19), $\max\{|\lambda_1|, |\lambda_2|\} > 1$ and by linearization the equilibrium is unstable. $\square$

Lemmata 3.3.6 and 3.3.2 immediately imply

Theorem 3.3.7. *Let k=1. If* $rT > -\ln\left(\frac{(1-E)^2}{E}\right)$*, then there exists a unique positive periodic solution* $N^*(t)$ *of* (3.1) *with*

$$N^*(nT^+) = x^* = \frac{((1-E)e^{rT}-1)K_c}{e^{rT}-1}\,, \tag{3.21}$$

and this solution is locally asymptotically stable.

While Theorem 3.3.7 is similar in many ways to the result of [47] cited in Lemma 3.2.1, here we do not observe attractivity of the solution for all initial conditions, implying that the solution is attractive but not globally attractive, as is illustrate in Figure 3.5

Remark 1. *Even for* $k = 1$ *and any* x_{-1}*, there is a domain of initial values* x_0 *guaranteeing immediate extinction. Since,* $g(x, e^{rT})$ *as defined in* (3.12) *is strictly increasing in both arguments,* $g(e^{rT}, 0) = 0$*, there are values of* $x_0 < x_{-1}$ *such that*

$$\frac{K_c x_0 e^{rT}}{K_c + x_0(e^{rT}-1)} \leq \frac{EK_c x_{-1} e^{rT}}{K_c + x_{-1}(e^{rT}-1)},$$

leading to $x_1 = x_2 = \cdots = 0$.

The following condition is more general than Lemma 3.3.6 in that it is valid for all $k \in \mathbb{N}$. However it will in general be difficult to use it to obtain explicit conditions for stability like the ones in Lemma 3.3.6.

Lemma 3.3.8. *The positive equilibrium* x^* *of difference equation* (3.11) *exists. It is asymptotically stable if and only if both inequalities hold*

$$\begin{aligned} \frac{e^{-rT}}{(1-E)^2} &< \frac{k+1}{k} \\ \cos(\theta^*) &< \frac{e^{-rT}(1+E)}{2(1-E)} + \frac{(1-E)^2 e^{rT}}{2}, \end{aligned} \tag{3.22}$$

where θ^* is a solution of

$$\frac{\sin(k\theta)}{\sin((k+1)\theta)} = (1-E)^2 e^{rT}, \quad \theta \in \left(0, \frac{\pi}{k+1}\right). \tag{3.23}$$

Proof. As (3.18) holds, there exists a positive equilibrium x^* of (3.11). Linearizing the difference equation $x_{n+1} = f(x_n, x_{n-k})$ as given in (3.11) around the positive equilibrium x^* given in (3.13), we get the corresponding linearized equation

$$u_{n+1} = p_0 u_n - p_k u_{n-k}, \quad p_0 = \frac{e^{-rT}}{(1-E)^2}, \quad p_k = \frac{Ee^{-rT}}{(1-E)^2}. \tag{3.24}$$

By Corollary 3.2.5.1, the zero solution of the linearized equation is asymptotically stable if and only if the inequalities in (3.9) hold, or equivalently,

$$\begin{aligned} &\frac{e^{-rT}}{(1-E)^2} < \frac{k+1}{k}, \\ &\frac{Ee^{-rT}}{(1-E)^2} < \sqrt{\frac{e^{-2rT}}{(1-E)^4} + 1 - 2\frac{e^{-rT}}{(1-E)^2}\cos(\theta^*)}. \end{aligned}$$

Note that the inequality $p_0 < p_k + 1 \Leftrightarrow e^{-rT} < Ee^{-rT} + (1-E)^2$ is equivalent to $e^{-rT} < 1 - E$, which is satisfied due to (3.18). Thus the first inequality is the same as in (3.22), while computing the squares in the second gives

$$\frac{E^2 e^{-2rT}}{(1-E)^4} < \frac{e^{-2rT}}{(1-E)^4} + 1 - \frac{2e^{-rT}}{(1-E)^2}\cos(\theta^*).$$

After rearranging, the desired result is acquired. □

Applied to (3.1), Lemma 3.3.8 immediately gives a sharp asymptotic stability result.

Theorem 3.3.9. *There exists a unique positive periodic solution $N^*(t)$ of* (3.1) *given by* (3.21) *which is locally asymptotically stable if and only if both inequalities in* (3.22) *hold, where θ^* is a solution of* (3.23).

The following stability condition is delay-independent, but it is only sufficient, meaning there may still exist (and indeed do exist) values of $rT < \ln(E+1) - 2\ln(1-E)$ such that the equilibrium is locally asymptotically stable.

Theorem 3.3.10. *If*

$$rT > \ln(E+1) - 2\ln(1-E) \tag{3.25}$$

then the positive periodic solution $N^*(t)$ *of* (3.1) *exists as given in* (3.21)*, and this solution is locally asymptotically stable.*

Proof. If (3.25) holds, then so does (3.18) and the positive equilibrium of (3.11) exists. Reducing (3.1) to difference equation (3.11), we once again linearize (3.11) around x^* to obtain (3.24), and require

$$|p_0| + |p_k| = \frac{1+E}{(1-E)^2} e^{-rT} < 1.$$

By Lemma 3.2.4, if the above inequality which is equivalent to (3.25) holds, then the zero solution of the linearized equation is asymptotically stable. Thus if (3.25) holds, x^* is locally asymptotically stable. By Lemma 3.3.2 the solution (3.21) is locally asymptotically stable if (3.25) holds. $\square$

3.4 MSY

Next, we proceed with the analysis of a maximum yield (MY) and a maximum sustainable yield (MSY). We recall that a yield is said to be sustainable if it corresponds to a solution that is at least locally asymptotically stable.

Lemma 3.4.1. *The yield of* (3.1) *is associated to solution* (3.21) *and is given by*

$$Y(E) = \frac{K_c E}{(1-E)T}\left(\frac{(1-E)e^{rT}-1}{e^{rT}-1}\right). \tag{3.26}$$

This yield is an increasing function for $E \in (0, E_{opt})$ *and decreasing for* $E \in (E_{opt}, 1)$.

Proof. For an optimal T-periodic solution of (3.1), we get $N^*(nT+) = N^*((n+1)T^+) = x^*$ where (3.21) holds. Then the associated yield is

$$\begin{aligned} Y(E) &= \frac{EN(nT)}{T} \\ &= \frac{E}{1-E} \cdot \frac{N(nT^+)}{T} \\ &= \frac{E}{1-E} \cdot \frac{K_c}{T}\left(\frac{(1-E)e^{rT}-1}{e^{rT}-1}\right). \end{aligned}$$

Its derivative in E

$$\begin{aligned} Y'(E) &= \frac{K_c}{T(e^{rT}-1)}\frac{d}{dE}\left[Ee^{rT}+1-\frac{1}{1-E}\right] \\ &= \frac{K_c}{T(e^{rT}-1)}\left[e^{rT}-\frac{1}{(1-E)^2}\right], \end{aligned}$$

satisfies $Y'(E) > 0$ for $(1-E)^2 > e^{-rT}$, which is equivalent to $E \in (0, E_{opt})$, and $Y'(E) < 0$ for $E \in (E_{opt}, 1)$. $\square$

Lemma 3.4.2. *The maximal yield (MY) for the delayed impulsive harvesting model* (3.1) $k \in \mathbb{N}$ *is equal to the MY for the non-delayed impulsive harvesting model* (3.5), *with* $k = 0$. *The optimal harvesting effort is* $E_{opt} = 1 - e^{-rT/2}$ *and the MY is associated to the solution* (3.6).

Proof. By the proof of Lemma 3.4.1 the maximum yield is attained at $E = E_{opt}$ and has the value (yield per time)

$$MY_{delayed} = Y(E_{opt}) = \frac{K_c}{T}\left(\frac{e^{rT/2}-1}{e^{rT/2}+1}\right) = MY_{non-delayed}.$$

with $MY_{non-delayed}$ given in (3.5). In addition, the periodic solution (3.21) when $E = E_{opt}$ becomes $N^*(nT^+) = \dfrac{K_c}{e^{rT/2}+1}$, as in (3.6). $\square$

Theorem 3.4.3. *The MY of* (3.1) *is a MSY if either $k = 1$ or $k \geq 2$ and*

$$rT < -2\ln\left(1 - 2\cos\left(\frac{k\pi}{2k+1}\right)\right). \tag{3.27}$$

Proof. By Lemma 3.3.1 the solutions to (3.1) satisfy (3.11). Linearizing the difference equation $x_{n+1} = f(x_n, x_{n-k})$ around the positive equilibrium x^* given in (3.13) with

$$f(x_n, x_{n-k}) = \frac{K_c x_n e^{rT}}{K_c + x_n(e^{rT}-1)} - E\frac{K_c x_{n-k} e^{rT}}{K_c + x_{n-k}(e^{rT}-1)},$$

we get

$$u_{n+1} = \frac{e^{-rT}}{(1-E)^2}u_n - \frac{Ee^{-rT}}{(1-E)^2}u_{n-k}.$$

When $E = E_{opt}$ by Lemma 3.4.2, the yield is maximal and the linearized difference equation becomes

$$u_{n+1} = u_n - (1 - e^{-rT/2})u_{n-k}$$

with the equilibrium (3.6). In addition (3.18) is equivalent to $rT > 0$. Then by Lemma 3.2.6, for $k = 1$ the zero solution of the linearized equation is asymptotically stable for any $rT > 0$. For $k \geq 2$ the zero solution of the linearized equation is locally asymptotically stable if and only if

$$1 - e^{-rT/2} < 2\cos\left(\frac{k\pi}{2k+1}\right)$$

which is equivalent to (3.27). Finally, by Lemma 3.3.2, a solution of (3.1) satisfying (3.6) is locally asymptotically stable, once (3.27) is satisfied. By definition, a unique positive periodic solution $N^*(t)$ given by (3.6), for either $k = 1$ or both $k \geq 2$ and rT satisfying (3.27), leads to MSY. $\square$

Unlike the non-delayed case, there is a possibility that the maximum yield is not sustainable. If E is different from E_{opt}, then to avoid extinction the choice of harvesting efforts should still be among those leading to a sustainable yield. The set of such efforts is non-empty, as the following statement guarantees.

Theorem 3.4.4. *Let $k \geq 2$ and*

$$E^* = \frac{2 + e^{-rT} - \sqrt{e^{-rT}(e^{-rT} + 8)}}{2}. \tag{3.28}$$

Then $E^* < E_{opt}$, and for any $E \in (0, E^*)$ the yield as given in (3.26) is sustainable.

Proof. First, let us note that, first, E^* defined in (3.28) is positive and, as $4e^{-rT/2} > 4e^{-rT}$, we have $\sqrt{e^{-rT}(e^{-rT} + 8)} > e^{-rT} + 2e^{-rT/2}$ leading to $E^* < E_{opt}$.

For a fixed $E \in (0, 1)$, we get a solution (3.21) $N(nT^+) = x^*$, corresponding to a yield as given in (3.26). As justified earlier, $Y(E)$ is an increasing function of E for $E \in (0, E_{opt})$.

By Theorem 3.3.10 and Lemma 3.3.2 the solution is locally asymptotically stable for any k if $e^{rT} > \dfrac{E+1}{(1-E)^2}$, which is equivalent to $E^2 - (2 + e^{-rT})E + 1 - e^{-rT} > 0$. The quadratic inequality is also satisfied if $E \in (0, E^*)$, meaning that for any $E \in (0, E^*)$ the yield is sustainable, which concludes the proof. $\square$

Lemma 3.4.5. *If for some choice of $E_s \in (0, E_{opt}]$ the associated yield is sustainable, then for any $E \in (0, E_s]$ the yield associated with E is also sustainable.*

Proof. If the yield associated with E_s is sustainable then by definition this implies that the associated solution with $N^*(nT^+) = x^*$ as given by (3.21) is locally asymptotically stable.

By Lemma 3.3.8 this implies that both inequalities in (3.22) must hold for E_s, where θ^* is a root of (3.23).

Since for any $E \leq E_{opt}$,

$$\frac{e^{-rT}}{(1-E)^2} \leq \frac{e^{-rT}}{(1-E_{opt})^2} = 1 < \frac{k+1}{k},$$

it is clear that the first inequality in (3.22) is satisfied for any $E \in (0, E_s]$. Thus we can turn our attention to the second inequality. Denote the right hand side of the second inequality in (3.22) for a fixed rT as h_1

$$h_1(E) := \frac{(1+E)e^{-rT}}{2(1-E)} + \frac{(1-E)^2 e^{rT}}{2} \quad \text{for some fixed rT} \tag{3.29}$$

and the left hand side in (3.23) as

$$h_2(\theta(E)) := \frac{\sin(k\theta(E))}{\sin((k+1)\theta(E))}, \quad \theta \in I = (0, \pi/(k+1)). \tag{3.30}$$

We have from (3.23),

$$h_2'(\theta(E))\frac{d\theta}{dE} = \frac{d}{dE}[(1-E)^2 e^{rT}] = -2(1-E)e^{rT} < 0.$$

Also,

$$\begin{aligned} h_2'(\theta(E)) &= \frac{k\cos(k\theta)\sin((k+1)\theta) - (k+1)\cos((k+1)\theta)\sin(k\theta)}{\sin((k+1)\theta)^2} \\ &= k\left(\frac{\sin((k+1)\theta)\cos(k\theta) - \sin(k\theta)\cos((k+1)\theta)}{\sin((k+1)\theta)^2}\right) \\ &\qquad - \frac{\cos((k+1)\theta)\sin(k\theta)}{\sin((k+1)\theta)^2} \\ &= k\left(\frac{\sin(\theta)}{\sin((k+1)\theta)^2}\right) - \frac{\cos((k+1)\theta)\sin(k\theta)}{\sin((k+1)\theta)^2} \\ &= \frac{\sin(\theta)}{\sin((k+1)\theta)^2}\left(k - \frac{\cos((k+1)\theta)\sin(k\theta)}{\sin(\theta)}\right) \end{aligned}$$

Now, $\sin(\theta)/\sin((k+1)\theta)^2 > 0 \; \forall \theta \in I$, and we will show that $k\sin(\theta) - \cos((k+1)\theta)\sin(k\theta) > 0$ leading to $h_2'(\theta(E)) > 0$. Since $\cos((k+1)\theta) \leq 1$ then

$$k\sin(\theta) - \cos((k+1)\theta)\sin(k\theta) > k\sin(\theta) - \sin(k\theta) := H_1(\theta).$$

Now $H_1'(\theta) = k(\cos(\theta) - \cos(k\theta)) > 0$ since $\cos(\theta)$ is decreasing for all $\theta \in (0,\pi)$ and for $\theta \in I$ both $\theta, k\theta \in (0,\pi)$. Thus since $H_1(0) = 0$, then $H_1(\theta) > 0$ and $k\sin(\theta) - \cos((k+1)\theta)\sin(k\theta) > 0$ for $\theta \in I$.

Since $h_2'(\theta(E)) > 0$, the inequality $h_2'(\theta(E))\dfrac{d\theta}{dE} < 0$ implies $\dfrac{d\theta}{dE} < 0$. Thus $\theta(E)$ decreases in E and $\cos(\theta(E))$ increases in E. Further we can show that,

$$h_1'(E) = \frac{e^{-rT}}{(1-E)^2} - (1-E)e^{rT} < 0$$

for $E < 1 - e^{-2rT/3}$. Since we have assumed that $E \leq E_{opt} < 1 - e^{-2rT/3}$ then $h_1'(E) < 0$ and we can see $h_1(E)$ decreases in E.

Since the yield associated with E_s is sustainable, we have

$$\cos(\theta(E_s)) < h_1(E_s).$$

Since $\cos(\theta(E))$ decreases and $h_1(E)$ increases for decreasing E, then for any $E \leq E_s$

$$\cos(\theta(E)) < h_1(E)$$

and the second inequality in (3.22) is satisfied.

Since both inequalities in (3.22) are satisfied for $E \leq E_s$, the solution associated to E is locally asymptotically stable and thus the yield is sustainable. $\square$

Corollary 3.4.5.1. *Let $k = 1$, then for any $E \in (0, E_{opt}]$ the associated yield is sustainable.*

Proof. Follows immediately from Lemma 3.4.5, and Theorem 3.4.3. □

Corollary 3.4.5.2. *Let $k \geq 2$, and $rT < -2\ln\left(1 - 2\cos\left(\frac{k\pi}{2k+1}\right)\right)$. Then for any $E \in (0, E_{opt}]$ the associated yield is sustainable.*

Proof. Follows immediately from Lemma 3.4.5 and Theorem 3.4.3. □

Theorem 3.4.6 determines the maximum bound on E which guarantees a sustainable yield when rT does not satisfy (3.27).

Theorem 3.4.6. *Let $k \geq 2$, and $rT \geq -2\ln\left(1 - 2\cos\left(\frac{k\pi}{2k+1}\right)\right)$. Then there exists $E^{**} \in (0, E_{opt}]$ and $\theta^{**} \in \left(0, \frac{\pi}{k+1}\right)$ such that (E^{**}, θ^{**}) is a solution of*

$$\begin{cases} \cos(\theta) = \frac{(1+E)e^{-rT}}{2(1-E)} + \frac{(1-E)^2 e^{rT}}{2} \\ \dfrac{\sin(k\theta)}{\sin((k+1)\theta)} = (1-E)^2 e^{rT}. \end{cases} \tag{3.31}$$

For any $E \in (0, E^{**})$ the yield is sustainable, while for $E \in [E^{**}, 1)$ the yield is unsustainable.

Proof. Let us prove that E^{**} exists. By Theorem 3.4.3 when $k \geq 2$, $rT \geq -2\ln\left(1-2\cos\left(\frac{k\pi}{2k+1}\right)\right)$ and $E = E_{opt}$ then the solution associated to the yield is unstable and $\cos(\theta(E_{opt})) \geq h_1(E_{opt})$. On the other hand, when $E \to 0^+$ we get in (3.22) $\cos(\theta(0^+)) < \cosh(rT) = h_1(0^+)$. Since $\cos(\theta(0^+)) < h_1(0^+)$ and $\cos(\theta(E_{opt})) \geq h_1(E_{opt})$ then by the continuity of $\cos(\theta(E))$ and the Intermediate Value Theorem, there exists a solution $(\theta(E^{**}), E^{**})$ of (3.31) with $E^{**} \in (0, E_{opt}]$

Since by Lemma 3.4.1 the yield $Y(E)$ is increasing for $E \in (0, E_{opt}]$, then $Y(E^{**})$ is maximal among the possible sustainable yields, although because it does not satisfy (3.22), it itself is not sustainable.

However for $E = (E^{**})^{-} = \lim_{E \to (E^{**})^{-}} E$, by the argument that as E decreases, $\cos(\theta(E))$ decreases and $h_1(E)$ increases, $(E^{**})^{-}$ will satisfy the conditions in (3.22) and the associated yield will be sustainable. Then by Lemma 3.4.5, for any $E \in (0, E^{**})$ the associated yield will be sustainable.

$\square$

3.5 Numerical Simulations and Discussion

Let us illustrate sustainability of the optimal yield with simulations.

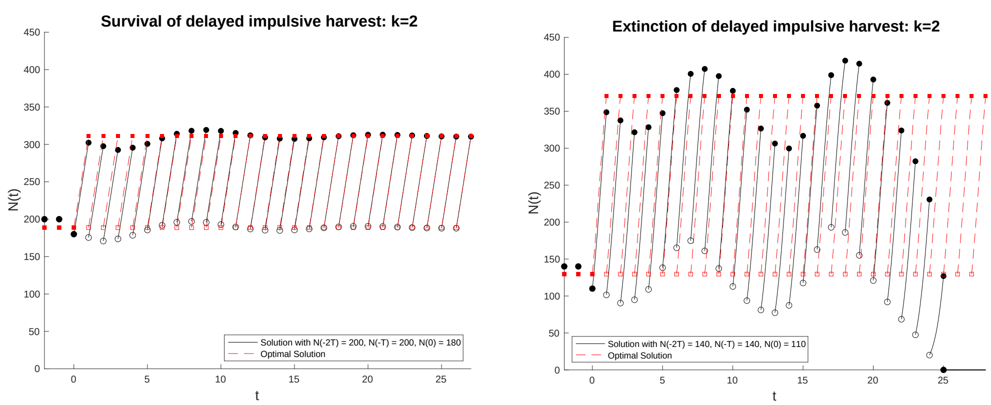

Figure 3.1: Solution to (3.1) with $k = 2$, $K_c = 500$, and $E = E_{opt} = 1 - e^{-rT/2}$. The figure on the left shows the optimal solution to (3.1) (red), and the solution to (3.1) with $r = 1$, $T = 1$ and initial conditions $N(-2T) = 200$, $N(-T) = 200$, $N(0) = 180$ (black). Since $0 < rT < 1.9248$, the optimal solution is locally asymptotically stable. The figure on the right shows the optimal solution to (3.1) (red), and the solution to (3.1) with $r = 2.1$, $T = 1$ and initial conditions $N(-2T) = 140$, $N(-T) = 140$, $N(0) = 110$ (black). Since $rT > 1.9248$, the optimal solution is unstable.

In Fig. 3.1, stable and unstable optimal solutions to (3.1) with $k = 2$ are compared. Solutions were computed until either the relative error of $N((n+1)T^{+})$ and $N(nT^{+})$ was less than 10^{-4}, $n = 100$ had been reached, or the population reached extinction i.e. $N(nT^+) = 0$. See the appendix, OptimalImpulsiveHarvest.m for MATLAB code.

Since $k = 2 > 1$, Theorem 3.4.3 states that the optimal solution is locally asymptotically stable if and only if $0 < rT < 1.9248$. The left figure has $rT = 1$, hence by Theorem 3.4.3 the optimal solution (3.6) is locally asymptotically stable. Therefore the solution $N(t)$ converges to the optimal positive periodic solution (3.6), and is said to survive. The figure on the right has $rT = 2.1$, thus by Theorem 3.4.3 the optimal solution (3.6) is unstable. Therefore, the solution $N(t)$ goes to zero, and the population is said to be extinct at time $t = 25$.

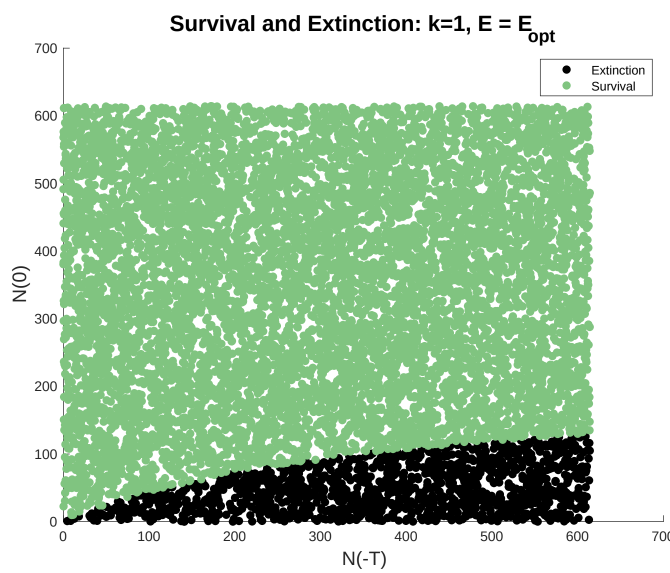

Figure 3.2: Solutions to (3.1) $k = 1$, $r = 1.37$, $T = 1$, $K_c = 307.16$, $E_{opt} \approx 0.49$. The positive T-periodic solution with $N^*(nT^+) = 102.78$ is locally asymptotically stable. The dots in this figure show whether the population survives (green) or goes extinct (black) given the initial conditions.

In the above figure, each dot represents a solution to (3.1)($k = 1$), with a set of initial values $N(0)$ and $N(-T)$. For each solution, both

$N(0)$ and $N(-T)$ were chosen from a uniform distribution of numbers in $[0, 2K_c] = [0, 614.3218]$. If after 10500 iterations $N(nT^+)$ was within ± 10 of the positive equilibrium solution $N^*(nT^+) = 102.7816$, then the population was said to have survived, and the dot was coloured green. If at any point within the 10500 iterations the size of the population after harvesting was less than 10^{-3} then the population was said to have gone to extinction, and the dot was coloured black. See the appendix, montecarlo.m for MATLAB code. By Theorem 3.4.3 (see also Theorem 3.4.4) the positive equilibrium solution with $N^*(nT^+) = 102.7816$ is locally asymptotically stable. Indeed for a wide range of initial values the population does converge to the equilibrium solution and the population survives. However for a range of initial values far from the positive equilibrium the population does not survive and goes to extinction. This highlights the lack of global attractivity of the equilibrium even for the optimal harvesting model which is locally asymptotically stable for all $rT > 0$.

We have shown that asymptotic stability of the difference equation which corresponds to the optimal solution of impulsive model (3.1) is k-dependent. In fact, we can show that as $k \to \infty$, the range of possible rT values that allow a locally asymptotically stable equilibrium shrinks. The upper bound of the range of values plotted in Fig. 3.3, is

$$b(k) = -2 \ln \left(1 - 2 \cos \left(\frac{\pi k}{2k+1} \right) \right).$$

Conclusion

The results of this chapter can be summarized as follows:

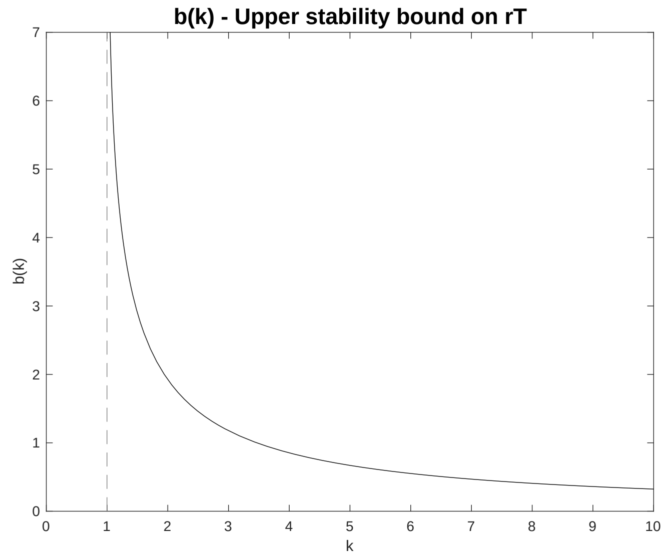

Figure 3.3: The function $b(k)$ is monotonically decreasing since $b'(k) < 0$ for $k > 1$, and $\lim\limits_{k\to\infty} b(k) = 0$.

1. With delayed impulsive harvesting, positivity of a solution with non-negative non-trivial initial conditions is not guaranteed, extinction in finite time is possible.

2. The delay does not influence the maximum yield (optimality) but can influence its sustainability.

3. With sharp local stability conditions, the optimal solution associated with the maximum yield is locally asymptotically stable in both non-delay [47] and one-step-delay cases. For longer delays, there are bounds on rT to attain MSY: the longer the delay is, the more frequent impulses should be.

Chapter 4

Conclusions and Future Work

4.1 Conclusions

In this thesis we have shown that the inclusion of a delay in the harvesting terms of both continuous and impulsive harvesting models does not affect the maximum yield but can highly affect sustainability. In general, we observe that sustainability of harvesting is dependent on the delay.

Chapter 2 focused on continuous models of delayed harvesting. There we derived sufficient conditions for the stability of positive equilibriums for a general harvesting model under appropriate assumptions on growth and harvesting functions F, h. We then derived the MSY for the logistic and Gompertz models and found that while the MY of the delayed models were equal to the MSY of the models with no delay, the sustainability was guaranteed only under additional assumptions on model parameters, unlike the model with no delay. For example, the MSY of the logistic equation was found to only be valid when $0 \leq \tau < \pi/r$, or equivalently $0 \leq r < \pi/\tau$. This implies that as the delay increases, the range of possible parameters r which will give a sustainable yield will become smaller and smaller, leading to a higher chance that the MY will be unsustainable. This is similar to what is observed later on in Chapter 3 where as k increases the range of sustainable parameters rT decreases. Further in this chapter, we looked at the oscillation of solutions of the logistic equation with delayed harvesting, and saw that the non-existence of real roots of the characteristic equation did not immediately imply oscillation of solutions around the positive equilibrium.

In Chapter 3 we considered a logistic equation subject to delayed impulsive harvesting. We saw that for this model, when model parameters satisfy $rT \leq -\ln(1-E)$, or equivalently $E \geq 1 - e^{-rT}$, then solutions of the delayed impulsive harvesting model go to 0. This is particularly interesting since this extinction condition is completely independent of the delay. Then we derived a MSY for the model and saw that like the continuous model, the sustainability of the MY was subject to additional conditions on model parameters. In particular, the conditions were delay dependent, and as was seen in Figure 3.3 the upper bound of sustainable rT values was monotonically decreasing for increasing k. This is significant since it shows that the delays in information should be taken into consideration when making harvesting decisions. If a delay is present in population data, but harvesting decisions are made using an MSY from the non-delayed model, then the population could go to extinction unexpectedly.

Now let us consider what happens when rT is outside of the bounds for MY sustainability. When this is the case, there is two main courses of action which would maintain the sustainability of harvesting. The first is that the time between harvesting events T should be reduced until rT is within (3.27). However, this assumes an ability to change T, which may not be possible if T represents a seasonal harvesting event. The second option is that the harvesting effort should be reduced to below E^{**} as given in Theorem 3.4.6. This would give a yield that is sustainable, although care should be taken to not meet or exceed E^{**} since yields associated with $E \geq E^{**}$ will not be sustainable.

Finally in Chapter 3 we saw that the positive periodic solutions, which correspond to persistence of the population, are not globally attracting for all initial values. In particular if the delayed population estimate is very high and the true population size very low, then immediate extinction of the population can occur. This means that attention needs to be paid to historical data and external factors when making harvesting decisions. For example, if past historical data indicates a high population, but directly before harvesting an adverse event such as an epidemic is observed to destroy a large proportion of the stock, then harvesting should be reduced below the optimal levels found in this thesis or completely stopped to avoid immediate extinction. While specific ranges of initial values which guarantee persistence of the solution were not obtained, the method shown in Figure 3.5 allows for initial examination of possible ranges.

4.2 Future Work

Chapter 2 concluded with a brief discussion on when we could guarantee oscillation of the logistic equation with delayed continuous harvesting

(2.3). It was noted that because this non-linear equation did not fulfill the hypotheses of the linearized oscillation Theorems found in [18, 19], then oscillation of an associated linear equation did not necessarily imply oscillation of the non-linear equation (see Figure 2.4). While we were able to derive some results, a full characterization of conditions for the oscillation of solutions has yet to be obtained. As we have seen in the proof, the lower bound $\frac{EK_c}{r}$ plays an important role in the oscillation conditions for this equation. By this fact, and by further numerical simulations, we consider it likely that if $\phi(0) > \frac{EK_c}{r}$, and the characteristic equation (2.16) has no real roots, then solutions of (2.3) will oscillate around N^*.

In Chapter 3, it was noted that the positive periodic solution of the delayed impulsive harvesting model is not globally attracting for all initial values. Thus, as was illustrated in Figure 3.5, even though the solution $N(t)$ may be locally asymptotically stable (and the associated yield sustainable), there exists a range of initial values where the population goes to extinction. Indeed we have shown that for initial values where the delayed estimate of the population size is very high, and the current population size is very low we can have immediate extinction of the population. It would be advantageous to obtain ranges of initial values for which the positive solution is guaranteed to be globally attracting. Following [25], a good strategy for approaching this problem is to only consider domains where the right hand side of the difference equation is either monotonically increasing or decreasing in all of its arguments. In addition, we note that the domains obtained by the method for Figure 3.5, change with changing rT, K_c values, which implies that the domain of attractive initial values will depend in some way on rT, K_c.

In the delayed impulsive harvesting model, it was assumed that all harvesting delays were a multiple $k \in \mathbb{N}$ of the time period T. It would be useful to extend this research to more general impulse delays, such as S-type delays which encompass both discrete and distributed delays, and were used in [40]. This would allow investigation of harvesting which is dependent on population data collected at a time that is not a multiple of T, or where the timestamps of the population data are non-constant and are chosen from a probability distribution.

The approach of [7] where a deduction which does not contribute to the yield is incorporated in each impulsive harvest, can also be incorporated into the delayed impulsive harvesting model. This deduction could represent the presence of bycatch mortality, where harvesting is associated with some harm to the population. It is expected that like the results seen in Chapters 2,3, a harvesting delay will not change the maximum yield in the presence of bycatch mortality. Whether or not the maximum yield will still be sustainable is an open question.

Another extension would be including a delay during times of continuous dynamics controlled by the DE. Doing so would change this problem from a logistic, to a Hutchinson's equation type model

$$\begin{cases} \frac{dN}{dt} = rN(t)\left(1 - \frac{N(t-\rho)}{K_c}\right), \quad t \neq nT \\ N(nT^+) = \max\{N(nT) - EN((n-k)T), 0\}, \quad t = nT \end{cases}$$

for some $\rho > 0$ and initial conditions $N(t) = \phi(t)$ for $t \in [-\max\{\tau, kT\}, 0]$. Unless $\rho = jT$ for some $j \in \mathbb{N}$, it is unlikely that the solution to the DE will be able to be computed analytically on some interval. This would mean that we would not be able to find a corresponding difference equation, and would require a different method of analysis. Some options could include reducing the delayed impulsive equation to a system of non-delayed impulsive equations as in [12] or other methods found in [31]. Again we anticipate that the maximum yield will not change. But the additional delay is expected to further contribute to instability of positive periodic solutions.

Finally, in Chapter 1 it was mentioned that it is not realistic to assume that species exist in complete isolation from one another. While single species harvesting policies are still useful, it is necessary to expand our thinking and model food chains and ecosystems since applying a single species policy to a multi-species ecosystem could cause extinction of one or more of the individual species [28, 30]. Some work has been done to find optimal harvesting policies for non-delayed impulsive systems with populations which interact with one another [27, 41]. In our context, this would mean investigating the effect of delayed impulses on a multi-level predator-prey model, and deriving an optimal harvesting policy.

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Appendix

MATLAB Code - Continuous Delayed Harvesting

Conditions for Multiple Equilibriums

MultipleEquils.m

```
% Script MultipleEquils.m
%
% This script produces Figure 2.1 which illustrates how
% for F,h under the assumptions H1, H2 (without H3,H4)
% it is possible to have multiple positive equilibriums,
% where equilibriums are solutions of F(N) = h(N)
%
% F(N) = 0.6*N*(log(20/N))
% h(N) = sin(N) + 1.11*N
%
% Plot F
fplot(@(N) 0.6*N*(log(20/N)),[0 20],...
    'r','DisplayName','F(N) = 0.6 N ln(20/N)')
hold on
```

```
% Plot h
fplot(@(N) sin(N) + 1.11*N,[0 20],...
     'b','DisplayName','h(N) = sin(N) + 1.11N')
title('$F,h$ satisfying H1, H2, ...
    $F^{\prime}(0)>h^{\prime}(0)$, with multiple ...
    equilibriums','Interpreter','latex')
xlabel('$N$','Interpreter','latex')
ylabel('$F(N)/h(N)$','Interpreter','latex')
% Structure to hold the zeros of F(N) = h(N)
x=zeros(1,3);
y=zeros(1,3);
% Use fzero to find the 3 zeros of F(N) = h(N)
x(1) = fzero(@(N) 0.6*N*(log(20/N)) - (sin(N) + ...
    1.11*N),[0.5,0.7]);
x(2) = fzero(@(N) 0.6*N*(log(20/N)) - (sin(N) + ...
    1.11*N),[3,3.4]);
x(3) = fzero(@(N) 0.6*N*(log(20/N)) - (sin(N) + ...
    1.11*N),[4.2,4.7]);
y(1) = 0.6*x(1)*(log(20/x(1)));
y(2) = 0.6*x(2)*(log(20/x(2)));
y(3) = 0.6*x(3)*(log(20/x(3)));
% Plot solutions of F(N) = h(N)
scatter(x,y, 'ok','filled','DisplayName','Solutions to ...
    $ F(N) = h(N)$')
legend('Interpreter','latex')
ylim([0 10])
```

Stable and Unstable Solutions with Logistic Growth Rate

LogisticStable.m

```
% function LogisticStable
%
% This function produces a graph which shows a solution
% of the logistic equation with delayed harvesting
% converging to the positive equilibrium as seen
% in Figure 2.2.
%
% Model parameters are chosen so that harvesting is optimal
% the equilibrium is locally asymptotically stable
% according to conditions derived in Theorem 2.3.5.
%
% Note that there is no input or output of this
% function. This function simply displays a graph.
%
% Input:
%
% Output:
%
function LogisticStable
% Get model parameters
[r,E,K,lag]=params;
% Solve the solution from t = 0 to t = 120 using dde23
sol1 = dde23(@rhs,lag,@hist,[0, 120]);
l = length(sol1.y);
x = zeros(1,l);
```

```
y = zeros(1,l);
x = sol1.x;
y = sol1.y;
% If population has hit zero at some time then the
% population is zero for all further times
for i=1:l
   if y(i)≤0
       y(i) = 0;
   end
end
figure;
% Plot solution of logistic equation with delayed
% harvesting
plot(x,y,'-r','LineWidth',1,'DisplayName','Solution ...
    with inital data $\phi(t)=130+30H(t+0.5)$');
title('Survival of the logistic equation with delayed ...
    harvest, \tau = 2.4','FontSize',16);
xlabel('t','FontSize',14);
ylabel('N(t)','FontSize',14);
hold on
% Plot equilibrium line N^* = K/2
yline(K/2,'--k','LineWidth',1,'DisplayName','$N^*$');
legend('Interpreter','Latex','Location','Southeast');
xlim([0,120])
ylim([0,200])
hold off

% function s = hist(t)
%
% Initial history function - needed for dde23
% See dde23 for more documentation about what is needed in
```

```
% this history function. See ddex1, ddex2 for more
% examples.
%
% Input:
%   t — time
%
% Output:
%   s — value of the initial data at t
%
function s = hist(t)
s = 130 + heaviside(t+0.5)*(160—130);

% function dNdt = rhs(t,N,Z)
%
% Function defining the right hand side of the DDE
% See dde23 for more documentation about what is needed in
% the DE function. See ddex1, ddex2 for more examples.
%
% Input:
%   t — time
%   N — solution
%   Z — approximates the value of the solution at t = t—tau
%
% Output:
%   dNdt — change in population size at time t
function dNdt = rhs(t,N,Z)
[r,E,K,lag]=params;
Nlag = Z(:,1);
dNdt = (r)*N*(1—(N/(K))) — (E)*Nlag;

% function [r,E,K,tau] = params
```

```
%
% Manually sets model parameters.
%
% Input:
%
% Output:
%   r — intrinsic growth rate
%   E — harvesting effort
%   tau — harvesting delay
%   K — carrying capacity
function [r,E,K,tau] = params
r = 1.2;
E = r/2; % optimal harvesting effort
% 2.4 < pi/1.2 thus the equilibrium is locally
% asymptotically stable
tau = 2.4;
K = 300;
```

LogisticUnStable.m

```
% function LogisticUnStable
%
% This function produces a graph which shows a solution
% of the logistic equation with delayed harvesting going
% to zero as seen in Figure 2.2.
%
% Model parameters are chosen so that harvesting is
% optimal the equilibrium is unstable according to
% conditions derived in Theorem 2.3.5.
% Note that there is no input or output of this function.
% This function simply displays a graph.
```

```
%
% Input:
%
% Output:
%
function LogisticUnStable
% Get model parameters
[r,E,K,lag]=params;
% Solve the solution from t = 0 to t = 60 using dde23
sol1 = dde23(@rhs,lag,@hist,[0, 60]);
l = length(sol1.y);
x = zeros(1,l);
y = zeros(1,l);
x = sol1.x;
y = sol1.y;
% If population has hit zero at some time then the
% population is zero for all further times
for i=1:l
    if y(i)≤0
        y(i) = 0;
    end
end
figure;
% Plot solution of logistic equation with
% delayed harvesting
plot(x,y,'-r','Linewidth',1,'DisplayName','Solution ...
    with inital data $\phi(t)=130+30H(t+0.5)$');
title('Extinction of the logistic equation with delayed ...
    harvest, \tau = 2.7','FontSize',16);
xlabel('t','FontSize',14);
ylabel('N(t)','FontSize',14);
```

```
hold on
% Plot equilibrium line N^* = K/2
yline(K/2,'--k','LineWidth',1,'DisplayName','$N^*$');
legend('Interpreter','Latex','Location','Southwest');
xlim([0,54])
ylim([0,220])
hold off

% function s = hist(t)
%
% Initial history function
% See dde23 for more documentation about what is needed
% in this history function. See ddex1, ddex2 for more
% examples.
%
% Input:
%   t - time
%
% Output:
%   s - value of the initial data at t
%
function s = hist(t)
s = 130 + heaviside(t+0.5)*(160-130);

% function dNdt = rhs(t,N,Z)
%
% Function defining the right hand side of the DDE
% See dde23 for more documentation about what is
% needed in the DE function. See ddex1, ddex2
% for more examples.
%
```

```
% Input:
%   t — time
%   N — solution
%   Z — approximates the value of the
%        solution with at t = t—tau
%
% Output:
%   dNdt — change in population size at time t
function dNdt = rhs(t,N,Z)
[r,E,K,lag]=params;
Nlag = Z(:,1);
dNdt = (r)*N*(1−(N/(K))) − (E)*Nlag;

% function [r,E,K,tau] = params
%
% Manually sets model parameters.
%
% Input:
%
% Output:
%   r — intrinsic growth rate
%   E — harvesting effort
%   tau — harvesting delay
%   K — carrying capacity
function [r,E,K,tau] = params
r = 1.2;
E = r/2; % optimal harvesting E_{opt} = r/2
tau = 2.7; % 2.7 > pi/1.2 thus the equilibrium is unstable
K = 300;
```

Stable and Unstable Solutions with Gompertz Growth Rate

GompertzStable.m

```
% function GompertzStable
%
% This function produces a graph which shows a solution
% of the Gompertz equation with delayed harvesting
% converging to the positive equilibrium as seen in
% Figure 2.3.
%
% Model parameters are chosen so that harvesting is
% optimal and the equilibrium is locally asymptotically
% stable according to conditions derived in Theorem 2.3.7.
%
% Note that there is no input or output of this function.
% This function simply displays a graph.
%
% Input:
%
% Output:
%
function GompertzStable
% Get model parameters
[r,E,K,lag]=params;
% Solve the solution from t = 0 to t = 120 using dde23
sol1 = dde23(@rhs,lag,@hist,[0, 120]);
l = length(sol1.y);
x = zeros(1,l);
```

```
y = zeros(1,l);
x = sol1.x;
y = sol1.y;
% If population has hit zero at some time then the
% population is zero for all further times
for i=1:l
    if y(i)≤0
        y(i) = 0;
    end
end
figure;
% Plot solution of Gompertz equation with delayed
% harvesting
plot(x,y,'-r','Linewidth',1,'DisplayName','Solution ...
    with inital data $\phi(t)=90+30H(t+0.5)$');
title('Survival of the Gompertz equation with delayed ...
    harvest, \tau = 2.4','FontSize',16);
xlabel('t','FontSize',14);
ylabel('N(t)','FontSize',14);
hold on
% Plot equilibrium line N^* = K/e
yline(K/exp(1),'--k','LineWidth',1,'DisplayName','$N^*$');
legend('Interpreter','Latex','Location','Southeast');
xlim([0,120])
ylim([0,200])
hold off

% function s = hist(t)
%
% Initial history function - needed for dde23
% See dde23 for more documentation about what is needed
```

```
% in this history function. See ddex1, ddex2 for more
% examples.
%
% Input:
%   t - time
%
% Output:
%   s - value of the initial data at t
%
function s = hist(t)
s = 90 + heaviside(t+0.5)*(160-130);

% function dNdt = rhs(t,N,Z)
%
% Function defining the right hand side of the DDE
% See dde23 for more documentation about what is needed
% in the DE function. See ddex1, ddex2 for more examples.
%
% Input:
%   t - time
%   N - solution
%   Z - approximates the value of the solution at t = t-tau
%
% Output:
%   dNdt - change in population size at time t
function dNdt = rhs(t,N,Z)
[r,E,K,lag]=params;
Nlag = Z(:,1);
dNdt = (r)*N*(log(K/N)) - (E)*Nlag;

% function [r,E,K,tau] = params
```

```
%
% Manually sets model parameters.
%
% Input:
%
% Output:
%   r — intrinsic growth rate
%   E — harvesting effort
%   tau — harvesting delay
%   K — carrying capacity
function [r,E,K,tau] = params
r = 0.6;
E = r; % optimal harvesting effort E_{opt} = r
% 2.4 < pi/2*(0.6) thus the equilibrium is locally
% asymptotically stable
tau = 2.4;
K = 300;
```

GompertzUnStable.m

```
% function GompertzUnStable
%
% This function produces a graph which shows a solution
% of the Gompertz equation with delayed harvesting going
% to zero as seen in Figure 2.3.
%
% Model parameters are chosen so that harvesting is optimal
% and the equilibrium is unstable according to conditions
% derived in Theorem 2.3.7.
%
% Note that there is no input or output of this function.
```

```
% This function simply displays a graph.
%
% Input:
%
% Output:
%
function GompertzUnStable
% Get model parameters
[r,E,K,lag]=params;
% Solve the solution from t = 0 to t = 120 using dde23
sol1 = dde23(@rhs,lag,@hist,[0, 120]);
l = length(sol1.y);
x = zeros(1,l);
y = zeros(1,l);
x = sol1.x;
y = sol1.y;
% If population has hit zero at some time then the
% population is zero for all further times
for i=1:l
    if y(i)≤0
        y(i) = 0;
    end
end
figure;
% Plot solution of Gompertz equation with delayed
% harvesting
plot(x,y,'-r','Linewidth',1,'DisplayName','Solution ... with inital data $\phi(t)=90+30H(t+0.5)$');
title('Extinction of the Gompertz equation with delayed ... harvest, \tau = 2.7','FontSize',16);
xlabel('t','FontSize',14);
```

```
ylabel('N(t)','FontSize',14);
hold on
% Plot equilibrium line N^* = K/e
yline(K/exp(1),'--k','LineWidth',1,'DisplayName','$N^*$');
legend('Interpreter','Latex','Location','Southeast')
xlim([0,64])
ylim([0,200])
hold off

% function s = hist(t)
%
% Initial history function
% See dde23 for more documentation about what is needed
% in this history function. See ddex1, ddex2 for more
% examples.
%
% Input:
%    t - time
%
% Output:
%    s - value of the initial data at t
%
function s = hist(t)
s = 90 + heaviside(t+0.5)*(160-130);

% function dNdt = rhs(t,N,Z)
%
% Function defining the right hand side of the DDE
% See dde23 for more documentation about what is needed
% in the DE function. See ddex1, ddex2 for more examples.
%
```

```
% Input:
%   t - time
%   N - solution
%   Z - approximates the value of the solution at t = t-tau
%
% Output:
%   dNdt - change in population size at time t
function dNdt = rhs(t,N,Z)
[r,E,K,lag]=params;
Nlag = Z(:,1);
dNdt = (r)*N*(log(K/N)) - (E)*Nlag;

% function [r,E,K,tau] = params
%
% Manually sets model parameters.
%
% Input:
%
% Output:
%   r - intrinsic growth rate
%   E - harvesting effort
%   tau - harvesting delay
%   K - carrying capacity
function [r,E,K,tau] = params
r = 0.6;
E = r; % optimal harvesting effort E_{opt} = r
% 2.7 > pi/2*(0.6) thus the equilibrium is unstable
tau = 2.7;
K = 300;
```

Oscillation and Non-oscillation of the Logistic Equation

Oscillation.m

```
% function Oscillation
%
% This function produces a graph which shows a solution
% oscillating around the equilibrium of the logistic
% equation with delayed harvesting when the roots of the
% characteristic equation are complex as seen in
% Figure 2.4.
% Note that there is no input or output of this function.
% This function simply displays a graph.
%
% Input:
%
% Output:
%
%
function Oscillation
% Check that the roots of the characteristic equation are
% complex.
flag = checkRoots;
if flag==1
    error('Check model parameters')
end
% Get model parameters
[r,E,K,lag]=params;
% Calculate important values
```

```
[equil,permBound] = importantVals;
% Solve the solution from t = 0 to t = 200 using dde23
sol1 = dde23(@rhs,lag,@hist,[0, 200]);
% Create solution structures
l = length(sol1.y);
x = zeros(1,l);
y = zeros(1,l);
x = sol1.x;
y = sol1.y;
% If population has hit zero at some time then the
% population is zero for all further times
for i=1:l
   if y(i)≤0
       y(i) = 0;
   end
end
figure;
% Plot solution of logistic equation with delayed
% harvesting
plot(x,y,'-r','Linewidth',1,'DisplayName','Solution ...
    with inital data $\phi(t)=130+20H(t+4)$');
title('Oscillation of the logistic equation with ...
    delayed harvest','FontSize',16);
xlabel('t','FontSize',14);
ylabel('N(t)','FontSize',14);
hold on
% Plot equilibrium line
yline(equil,'--k','LineWidth',1,'DisplayName','$N^*$');
% Plot EK_c/r line
yline(permBound,':k','LineWidth',1,'DisplayName','$\frac{E ...
    K_c}{r}$');
```

```
legend('Interpreter','Latex','Location','Northeast');
ylim([0,230])


% function s = hist(t)
%
% Initial history function
% See dde23 for more documentation about what is needed
% in this history function. See ddex1, ddex2 for more
% examples.
%
% Input:
%   t - time
%
% Output:
%   s - value of the initial data at t
%
function s = hist(t)
% Initial data is above EK_c/r
s = 130 + heaviside(t+4)*(150-130);

% function dNdt = rhs(t,N,Z)
%
% Function defining the right hand side of the DDE
% See dde23 for more documentation about what is needed
% in the DE function. See ddex1, ddex2 for more examples.
%
% Input:
%   t - time
%   N - solution
%   Z - approximates the value of the solution at t = t-tau
```

```
%
% Output:
%   dNdt - change in population size at time t
function dNdt = rhs(t,N,Z)
% Get model parameters
[r,E,K,lag]=params;
Nlag = Z(:,1);
% Set RHS
dNdt = (r)*N*(1-(N/(K))) - (E)*Nlag;

% function [equil,permBound] = importantVals
%
% Calculate some important values for use later
%
% Input:
%
% Output:
%   equil - positive equilibrium of the logistic equation
%           with delayed harvesting
%   permBound - lower bound EK_c/r
function [equil,permBound] = importantVals
% Get model parameters
[r,E,K,lag]=params;
equil = (r-E)*K/r;
permBound = E*K/r;

% function [r,E,K,tau] = params
%
% Manually sets model parameters.
%
% Input:
```

```
%
% Output:
%   r — intrinsic growth rate
%   E — harvesting effort
%   tau — harvesting delay
%   K — carrying capacity
function [r,E,K,tau] = params
r = 1;
E = 0.4;
tau = 5;
K = 300;

% function [flag] = checkRoots
%
% This function checks to make sure that the roots of the
% characteristic equation are complex, as well as r>2E.
% If the roots are all complex then flag = 0.
% If there is a real root of the characteristic equation
% then flag = 1 and an error message is shown. If 2E>r
% then flag = 1 and error message is shown.
%
% If an error message is shown, change the model parameters
% in function params.
%
% Input:
%
% Output:
% flag — 0 if model parameters are okay, 1 otherwise.
function [flag] = checkRoots
[r,E,K,lag]=params;
if lag*exp((r-2*E)*lag)>1/(exp(1)*E)
```

```
    flag = 0;
else
    flag = 1;
    error('chosen parameters have real root of char eqn')
end
if 2*E>r
    flag = 1;
    error('r must be greater than 2E')
end
```

NonOscillation.m

```
% function NonOscillation
%
% This function produces a graph which shows a solution
% not oscillating around the equilibrium of the logistic
% equation with delayed harvesting when the roots of the
% characteristic equation are complex as seen in
% Figure 2.4.
% Note that there is no input or output of this function.
% This function simply displays a graph.
%
% Input:
%
% Output:
%
%
function NonOscillation
% Check that the roots of the characteristic equation are
% complex.
flag = checkRoots;
```

```
if flag==1
    error('Check model parameters')
end
% Get model parameters
[r,E,K,lag]=params;
% Calculate important values
[equil,permBound] = importantVals;
% Solve the solution from t = 0 to t = 100 using dde23
sol1 = dde23(@rhs,lag,@hist,[0, 100]);
% Create solution structures
l = length(sol1.y);
x = zeros(1,l);
y = zeros(1,l);
x = sol1.x;
y = sol1.y;
% If population has hit zero at some time then the
% population is zero for all further times
for i=1:l
   if y(i)≤0
       y(i) = 0;
   end
end
figure;
% Plot solution of logistic equation with delayed
% harvesting
plot(x,y,'-r','Linewidth',1,'DisplayName','Solution ...
    with inital data $\phi(t)=110-100H(t+4)$');
title('Non-oscillation of the logistic equation with ...
    delayed harvest','FontSize',16);
xlabel('t','FontSize',14);
ylabel('N(t)','FontSize',14);
```

```
hold on
% Plot equilibrium line
yline(equil,'--k','LineWidth',1,'DisplayName','$N^*$');
% Plot EK_c/r line
yline(permBound,':k','LineWidth',1,'DisplayName','$\frac{E ...
    K_c}{r}$');
legend('Interpreter','Latex','Location','Northeast');
ylim([0,190])
% dde23 only shows the solution up until around t= 2.9.
% This simply extends the graph to an even number.
h1 = fplot(@(x) 0, [2,5], '-r', 'Linewidth', 1);
h1.Annotation.LegendInformation.IconDisplayStyle = 'off';

% function s = hist(t)
%
% Initial history function
% See dde23 for more documentation about what is needed
% in this history function.
% See ddex1, ddex2 for more examples.
%
% Input:
%   t - time
%
% Output:
%   s - value of the initial data at t
%
function s = hist(t)
[equil,permBound] = importantVals;
% Initial data begins below EK_c/r
s = permBound - 10 + heaviside(t+4)*(10-(permBound-10));
```

```
% function dNdt = rhs(t,N,Z)
%
% Function defining the right hand side of the DDE
% See dde23 for more documentation about what is needed
% in the DE function. See ddex1, ddex2 for more examples.
%
% Input:
%   t - time
%   N - solution
%   Z - approximates the value of the solution at t = t-tau
%
% Output:
%   dNdt - change in population size at time t
function dNdt = rhs(t,N,Z)
% Get model parameters
[r,E,K,lag]=params;
Nlag = Z(:,1);
% Set RHS
dNdt = (r)*N*(1-(N/(K))) - (E)*Nlag;

% function [equil,permBound] = importantVals
%
% Calculate some important values for use later
%
% Input:
%
% Output:
%   equil - positive equilibrium of the logistic equation
%           with delayed harvesting
%   permBound - lower bound EK_c/r
function [equil,permBound] = importantVals
```

```
% Get model parameters
[r,E,K,lag]=params;
equil = (r−E)*K/r;
permBound = E*K/r;

% function [r,E,K,tau] = params
%
% Manually sets model parameters.
%
% Input:
%
% Output:
%   r − intrinsic growth rate
%   E − harvesting effort
%   tau − harvesting delay
%   K − carrying capacity
function [r,E,K,tau] = params
r = 1;
E = 0.4;
tau = 5;
K = 300;

% function [flag] = checkRoots
%
% This function checks to make sure that the roots of the
% characteristic equation are complex, as well as r>2E.
% If the roots are all complex then flag = 0. If there ...
    is a
% real root of the characteristic equation then flag = 1
% and an error message is shown. If 2E>r then flag = 1 and
% error message is shown.
```

```
%
% If an error message is shown, change the model parameters
% in function params.
%
% Input:
%
% Output:
% flag - 0 if model parameters are okay, 1 otherwise.
function [flag] = checkRoots
[r,E,K,lag]=params;
if lag*exp((r-2*E)*lag)>1/(exp(1)*E)
    flag = 0;
else
    flag = 1;
    error('chosen parameters have real root of char eqn')
end
if 2*E>r
    flag = 1;
    error('r must be greater than 2E')
end
```

MATLAB Code - Impulsive Delayed Harvesting

Stable and Unstable Solutions with Logistic Growth Rate

OptimalImpulsiveHarvest.m

```
% Script OptimalImpulsiveHarvest.m
%
% This script creates two graphs which illustrate how
% changing rT value can affect stability of the optimal
% solution when k > 1 as shown in Figure
% 3.1.
%
% The harvesting in this model will be delayed by k=2
% harvesting periods and be optimal.
% i.e. E = 1 - e^{-rT/2}.
% The first graph will have 0<rT<1.9248 which means the
% harvesting will be stable, and the population will
% survive. The second graph will have rT> 1.9248 which
% means the harvesting will be unstable and the population
% will go to extinction.
%
% Set unchanging parameters
% K - carrying capacity
% iters - max numbers of iterations
% TOL - tolerance
% delay - number of time intervals data has been
```

```
%          delayed by (k)
K=500;
iters = 100;
TOL = 1e-4;
delay = 2;
%
% Figure 1 - Stable Oscillations
%
% r - intrinsic growth rate
% T - time between harvesting intervals
r=1;
T=1;
% Calculate optimal harvesting effort
E = 1 - exp(-r*T/2);
% Calculate optimal N(nT^+)
N0 = calculateN0(r,K,T,E);
h = @(t)200+heaviside(t+T/2)*(180-200);
% Calculate the value of the solution at t = nT^+, N(nT^+)
[t1,x1] = ImpulsiveHarvestSequence3(r,E,K,T,h,iters,TOL);
figure(1)
% Using the solution values N(nT^+) plot these points and
% plot the solution
% trajectory between these points as determined by the
% differential equation.
plotSolution(t1,x1,r,T,K,h,delay,'k','k','-k','Solution ...
    with N(-2T) = 200, N(-T) = 200, N(0) = 180')
hold on
title('Survival of delayed impulsive harvest: ...
    k=2','FontSize',16)
xlabel('t','FontSize',14)
ylabel('N(t)','FontSize',14)
```

```
% Create optimal solution with N(nT^+) = N0 for all n.
tO = t1;
xO = zeros(1,length(tO));
for i=1:length(tO)
   xO(i) = N0;
end
% Plot optimal solution trajectory for t != nT^+
plotSolution(tO,xO,r,T,K,h,delay,'r','rs','--r','Optimal ...
    Solution')
ylim([0,450])
legend('Location','southeast')
hold off
%
% Figure 2 - Unstable Oscillations
%
r=2.1;
T=1;
% Calculate optimal harvesting effort
E = 1 - exp(-r*T/2);
% Calculate optimal N(nT^+)
N0=calculateN0(r,K,T,E);
h = @(t) 140 +heaviside(t+T/2)*(110-140);
% Calculate the value of the solution at t = nT^+, N(nT^+)
[t2,x2] = ImpulsiveHarvestSequence3(r,E,K,T,h,iters,TOL);
figure(2)
% Using the solution values N(nT^+) plot these points and
% plot the solution trajectory between these points as
% determined by the differential equation.
plotSolution(t2,x2,r,T,K,h,delay,'k','k','-k','Solution ...
    with N(-2T) = 140, N(-T) = 140, N(0) = 110')
hold on
```

```
title('Extinction of delayed impulsive harvest: ...
    k=2','FontSize',16)
xlabel('t','FontSize',14)
ylabel('N(t)','FontSize',14)
% Reshape tO
% NOTE this variable is t "oh" not t "zero"
tO = zeros(1,length(t2)+2);
for i=1:length(t2)
   tO(i) = t2(i);
end
tO(i+1)=tO(i)+T;
tO(i+2)=tO(i)+2*T;
% Create optimal solution
xO2 = zeros(1,length(t2)+2);
for i=1:length(tO)
   xO2(i) = N0;
end
% Plot optimal solution trajectory for t != nT^+
plotSolution(tO,xO2,r,T,K,h,delay,'r','rs','--r','Optimal ...
    Solution')
ylim([0,450])
xlim([t2(1),t2(end)+3*T])
legend('Location','southwest')
hold off
```

ImpulsiveHarvestSequence3.m

```
% function [t,x] = ...
    ImpulsiveHarvestSequence3(r,E,K,T,h,iters,TOL)
%
% This function calculates the value of solutions to the
```

```
% logistic equation with delayed impulsive harvesting at
% t=nT^+. Here the harvesting is delayed by a two time
% periods.
% N(nT^+) = x_n is defined by iterates of the third order
% non-linear difference equation.
%
% k = 2
%
% Input:
%   r - growth rate for the system
%   K - carrying capacity
%   T - period for impulsive harvesting (also the delay
%       value)
%   h - function defining the history of the system when
%       -T<t<0
%   E - harvesting effort
%   iters - max number of iterations desired
%   TOL - desired relative error for stopping
%
% Output:
%   t - time points starting at -T, and repeating every
%       T units, up to the
%   last time point corresponding to the last iteration
%   x - iterations corresponding to N((n-2)T+,0,N_0)
%
function [t,x] = ...
    ImpulsiveHarvestSequence3(r,E,K,T,h,iters,TOL)
% creates an array for the max size that could possibly
% needed as defined by iters. Most of x1 will not be needed
% and will be discarded later.
x1 = zeros(1,iters+3);
```

```
% define x0 and x1 the two initial values as described by
% the initial data function h
x1(1) = h(-2*T);
x1(2) = h(-T);
x1(3) = h(0);
ERR = 100000;
% Create iterations from the definition of the difference
% equation
for n=4:(iters+3)
    % Calculate iterates
    x1(n) = K*x1(n-1)*exp(r*T)/(K+x1(n-1)*(exp(r*T) - ...
        1))+(-E)*(K*x1(n-3)*exp(r*T))/(K+x1(n-3)*(exp(r*T)-1));
    % Calculate relative error
    ERR = abs(x1(n) - x1(n-1))/x1(n);
    % If iterates are less than or equal to zero, then all
    % further iterates
    % are zero (no negative populations)
    if x1(n)≤0
        x1(n) = 0;
        break;
    end
    % If two succesive iterations are sufficiently close
    % then stop creating new iterations.
    if (ERR < TOL)
         break;
    end
end
% Defines the result vector x by the iterations calculated,
% discards the rest of the unused x1
x = zeros(1,n);
t = zeros(1,n);
```

```
for i=1:n
    x(i) = x1(i);
    t(i) = -2*T + (i-1)*T;
end
end
```

plotSolution.m

```
% function [tV,xV] = ...
    plotSolution(tV,xV,r,T,K,h,delay,color,marker,linestyle,title)
%
% This function takes in the value of the solution at
% N(nT^+) and produces the a graph defining the evolution
% of the function over all t values as defined by the
% impulsive equation.
%
% Input:
%
%   tV - time vector running back to -delay*T
%   xV - values of the solution at N(nT^+) going back
%        to -delay*T
%   r - intrinsic growth rate
%   T - length of time between harvesting events
%   K - carrying capacity of the population
%   h - initial data function
%   delay - number of intervals by which the harvesting is
%           delayed
%   color - string indicating the color plot should be
%           (may not be used - enter random color e.g. 'r')
%   marker - string indicating type of marker to indicate
%            endpoints e.g.'rs' give a red square
```

```
%   linestyle — string indicating linestyle and color
%                e.g. '--r' gives a
%   red dotted line
%   title — string containing legend display name for graph
%
% Output:
%   tV — time vector running back to -delay*T
%   xV — values of the solution at N(nT^+) going back
%        to -delay*T
%
function [tV,xV] = ...
    plotSolution(tV,xV,r,T,K,h,delay,color,marker,linestyle,title)
% # of intervals covered by tV, xV
d = length(tV);
% # of intervals that require a solution plotted between
% N(nT^+) values
m = d - (delay + 1);
% Create structures to hold N(nT)
endpoints = zeros(1,d);
tendpoints = [tV,(d-delay)*T];
% Initial data points are considered to be N(nT) values
for i=1:delay+1
    endpoints(i)=xV(i);
end
hold on
% Plot the solution on each sub interval
for n=0:m
    N = @(t) createSol(t,tV,xV,r,K,T,delay,n);
    % Calculate N(nT) values
    endpoints(n+delay+2) = N((n+1)*T);
    if n==0
```

```
        fplot(N,[n*T,(n+1)*T],linestyle,'DisplayName',title);
    else
        h = ...
            fplot(N,[n*T,(n+1)*T],linestyle,'DisplayName','');
        % Since there are m separate plots, we only need
        % legend information for the first one.
        h.Annotation.LegendInformation.IconDisplayStyle ...
            = 'off';
    end
end
xlim([tV(1),tendpoints(end)])
% The next two plots illustrate the left continuous
% nature of our solution
% This plot puts open dots at N(nT^+)
h = scatter(tV,xV,marker);
h.Annotation.LegendInformation.IconDisplayStyle = 'off';
% This plot puts closed dots at N(nT)
if endpoints(end) == 0
    % The population has become zero at some point.
    % Do not keep plotting solution values.
    h = ...
        scatter(tendpoints(1:d),endpoints(1:d),marker,'filled','DisplayName','');
    h.Annotation.LegendInformation.IconDisplayStyle = ...
        'off';
    h = ...
        scatter([tV(end)],[xV(end)],marker,'filled','DisplayName','');
    h.Annotation.LegendInformation.IconDisplayStyle = ...
        'off';
    h = fplot(@(t) 0, ...
        [tV(end),tV(end)+4*T],linestyle,'Linewidth',1);
    h.Annotation.LegendInformation.IconDisplayStyle = ...
```

```
        'off';
else
    h = ...
        scatter(tendpoints,endpoints,marker,'filled','DisplayName','');
    h.Annotation.LegendInformation.IconDisplayStyle = ...
        'off';
end
hold off
end
```

createSol.m

```
% function f = createSol(t,tV,xV,r,K,T,delay,n)
%
% This givees the solution to the logistic DE over the
% interval [nT,(n+1)T]
%
% Input:
%   t - time variable running from nT to (n+1)T
%   tV - vector of time stamps running from -delay*T
%        forwards
%   xV - vector of the solution values at N(nT^+) running
%        from -delay*T forwards
%   r - intrinsic growth rate
%   K - carrying capacity
%   T - length of time between harvesting intervals
%   delay - number of harvesting intervals that the
%           harvesting has been delayed by
%   n - refers to which interval you are plotting over
%
% Output:
```

```
%   f - value of the solution of the logistic DE at ...
    time t
%
function f = createSol(t,tV,xV,r,K,T,delay,n)
    f = ...
        K*exp(r*(t-n*T))*xV(n+delay+1)/(xV(n+delay+1)*(exp(r*(t-n*T))-1)+K);
end
```

calculateN0.m

```
% function N0 = calculateN0(r,K,T,E)
%
% This function calculates the positive equilibrium
% xStar of the difference equation, and the value
% of the T-periodic solution at t=nT^+
% i.e. N(nT^+) = xStar.
%
% Input:
%   r - intrinsic growth rate
%   K - carrying capacity
%   T - time between harvesting moments
%   E - harvesting effort
%
% Output:
%   xStar - positive equilibrium of the non-linear
%           difference equation
%
function xStar = calculateN0(r,K,T,E)
xStar=K*((1-E)*exp(r*T)-1)/(exp(r*T)-1);
end
```

Non-Global Attractivity of the Optimal Solution

montecarlo.m

```
% Script montecarlo.m
%
% Runs a Monte Carlo like simulation for testing
% attractivity of the equilibrium as shown in Figure 3.2.
%
% Begin with simulations for logistic impulsive equation
% with impulse delay k = 1.
%
% This code will compute 10000 solutions of the logistic
% equation with delayed impulsive harvesting at t=nT^+.
%
% Solutions will be computed until either the relative
% error of N(nT^+) and N((n+1)T^+) is within 1e-4, or
% until n = 10500.
%
iters = 10500;
err = 1e-4;
% N - number of solutions to be calculated
N = 10000;
%
% Set model parameters. Since E = E_{opt}, k=1, by
% Theorem 3.4.3 the solution calculated will be locally
% asymptotically stable for every
% choice of rT.
%
% r - intrinsic growth rate
% K - carrying capacity
```

```
% T - time between harvesting moments
% E - harvesting effort
%
r = 1.3747;
K = 307.1609;
T = 1;
E = 1 - exp(-r*T/2);
%
% For each solution N data1 stores information about that
% solution. data1(N,1) = N(-T), data1(N,2) = N(0),
% data1(N,3) = 0,1,-1.
%
% If data1(N,3) = 0 then either there exists N(nT^+) = 0
% or N(nT^+)< 10*err
% If data1(N,3) = 1 then the solution has converged to
% the positive T-periodic solution.
% If data1(N,3) = -1 then after 10500 iterations that
% solution has neither converged to zero, nor come within
% 10 units of the positive T-periodic solution.
% These data points are not displayed
data1 = zeros(N,3);
% Choose initial conditions from uniform distribution
% between 0 and twice the carrying capacity
%
low = 0;
high = 2*K;
IC1 = (high - low).*rand(N,1) + low;
IC2 = (high - low).*rand(N,1) + low;
% Begin calculating solutions
for i = 1:N
    % Use heaviside step function to create a continuous
```

```
    % initial data function for passing to
    % ImpulsiveHarvestSequence2
    h = @(t)IC1(i) + heaviside(t+(T/2))*(IC2(i) - IC1(i));
    % Call to ImpulsiveHarvestSequence2
    % t1 - vector containing time points T, 2T, 3T, etc.
    % x1 - vector containing solution values at N(nT^+)
    [t1,x1] = ...
        ImpulsiveHarvestSequence2(r,E,K,T,h,iters,err);
    % Put initial conditions into data1 structure
    data1(i,1) = IC1(i);
    data1(i,2) = IC2(i);
    % Call to calculateN0 which calculates the value of
    % the optimal T-periodic solution at t=nT^+
    N0 = calculateN0(r,K,T,E);
    % if after 10500 iterations the N(nT^+) is within 10
    % units of the optimal T-periodic solution then the
    % solution is said to have converged.
    if (x1(end)≥ N0 - 10)&&(x1(end)≤ N0 + 10)
        data1(i,3) = 1;
    % if after 10500 iterations N(nT^+) is less than
    % 10*err then the solution is said to have gone extinct
    elseif (x1(end)≤ 10*err)
        data1(i,3) = 0;
    elseif x1(end)==0
        data1(i,3) = 0;
    % solution has neither converged nor gone to zero
    % after 10500 iterations
    else
        data1(i,3) = -1;
    end
end
```

```
figure(1)
% Sort row of data1 so that all solutions which went
% extinct are grouped together, as are solutions which
% survived and inconlusive solutions.
B = sortrows(data1,3);
% Calculate the number of solutions which did not converge.
num3 = sum(B(:,3)==-1);
% Calculate the number of solutions which went extinct.
num2 = sum(B(:,3)==0);
% Calculate the number of solutions which survived.
num1 = sum(B(:,3)==1);
% Begin creating the figure
hold on
if num2> 0
    % plot the initial conditions from all the
    % solutions which went extinct. Color the dots black.
    % All extinct solutions are held within rows num3+1
    % to num2+num3 of the structure B.
    scatter(B(num3+1:num2+num3,1),B(num3+1:num2+num3,2),'filled','black','Displa
end
if num1>0
    % plot the initial conditions from all the solutions
    % which survived. Color the dots green.
    % All survived solutions are held within rows
    % num2 + num3 + 1 to num3 + num2 + num1 of the
    % structure B.
    scatter(B(num3+num2+1:num3+num2+num1,1),B(num3+num2+1:num3+num2+num1,2),'fil
        0.77, 0.5],'DisplayName','Survival')
end
% Create legend and title for the graph
legend
```

```
xlabel('N(-T)','FontSize',14)
ylabel('N(0)','FontSize',14)
title('Survival and Extinction: k=1, E = ...
    E_{opt}','FontSize',16)
hold off
```

ImpulsiveHarvestSequence2.m

```
% function [t,x] = ...
    ImpulsiveHarvestSequence2(r,E,K,T,h,iters,TOL)
%
% This function calculates the value of solutions to the
% logistic equation with delayed impulsive harvesting at
% t=nT^+. Here the harvesting is delayed by a single
% time period.
%
% k = 1
%
% N(nT^+) = x_n is defined by iterates of the second order
% non-linear difference equation.
%
% Input:
%   r - intrinsic growth rate
%   K - carrying capacity
%   T - time between harvesting moments (also equal to
%       the delay when k=1)
%   h - function defining the history of the system when
%       -T<t<0
%   E - harvesting effort
%   iters - max number of iterations desired
%   TOL - desired relative error for stopping
```

```
%
% Output:
%   t - time points starting at -T, and repeating every
%       T units, up to the
%   last time point corresponding to the last iteration
%   x - iterations of the difference equation
%       x(n) = N(nT^+)
%
function [t,x] = ...
    ImpulsiveHarvestSequence2(r,E,K,T,h,iters,TOL)
% Creates an array for the max size that could possibly
% needed as defined by iters. Most of x1 will not be needed
% and will be discarded later.
x1 = zeros(1,iters+2);
% Define x0 and x1 the two initial values as described by
% the initial data
% function h
x1(1) = h(-T);
x1(2) = h(0);
ERR = 100000;
% Create iterations from the definition of the difference
% equation
for n=3:(iters+2)
    % Calculate iterates
    x1(n) = K*x1(n-1)*exp(r*T)/(K+x1(n-1)*(exp(r*T) - ...
        1))+(-E)*(K*x1(n-2)*exp(r*T))/(K+x1(n-2)*(exp(r*T)-1));
    % Calculate relative error
    ERR = abs(x1(n) - x1(n-1))/x1(n);
    % If iterates are less than or equal to zero, then all
    % further iterates are zero (no negative populations)
    if x1(n)<=0
```

```
            x1(n) = 0;
            break;
        end
        % If the two succesive iterations are sufficiently
        % close then stop creating new iterations
        if (ERR < TOL)
             break;
        end
    end
    % Defines the result vector x by the iterations calculated,
    % discards the rest of the unused x1
    x = zeros(1,n);
    t = zeros(1,n);
    for i=1:n
        x(i) = x1(i);
        t(i) = -T + (i-1)*T;
    end
    end
```

Upper Bound

UpperBound.m

```
% Script UpperBound.m
%
% Code to create Figure 3.3 illustrating how the ...
    function b(k) goes to zero
% monotonically as k becomes large
%
% plot b(k)
fplot(@(k)-2*log(1-2*cos(pi*k/(2*k+1))),[0,10],'k')
hold on
% Create title, x,y labels and force y limits to start ...
    at zero
title('b(k) - Upper stability bound on rT','FontSize',16)
xlabel('k')
ylabel('b(k)')
ylim([0,7])
```

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Re: Copyright Permission

Elena Braverman [redacted]
Thu 7/14/2022 1:22 PM
To: Jenny Lawson [redacted]

Hello Jenny,

You certainly have my permission to include all the material from our joint submitted paper.

However, I believe no special permission is required for this.

Concerning the attached form, I believe you have to sign it first.

Best,
Elena