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Effects Of A Three-Phase Circular Inclusion With Two Imperfect Interfaces On A Radial Crack

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Effects Of A Three-Phase Circular Inclusion With Two Imperfect Interfaces On A Radial Crack" submitted by Pan Gi Park in partial fulfilment of the requirements for the degree of Master of Science.

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ABSTRACT

Studies of material failure using fracture mechanics have been considered of significant practical importance to engineering applications for their ability to predict and prevent catastrophic failures. A popular model of fracture mechanics, the two-phase single inclusion model, has provided insight into the interaction between cracking and the inclusion. However, it ignores the profound effects of degraded bonding quality and other surrounding medium on the composite materials.

Thus, the author considers the three-phase inclusion model consisting of two circular regions, an infinitely extended region and the imperfect interfaces between the regions, which incorporates the profound effects of micromechanics, and thus, is of fundamental importance in understanding failure mechanisms and enhancing the performance of composite materials. In addition, its application can be extended to deal with functionally graded interphases and cemented implants embedded in bone. Although the three-phase model has recently been adapted to address imperfect adhesion and to provide fundamental solutions to a generalized self-consistent scheme, all of the previous works deal with a single imperfect interface. A comprehensive literature search found no research addressing the situation where both interfaces are imperfect.

In this study, a semi-analytic solution to the interaction between a pre-existing crack and a three-phase inclusion with spring-layer imperfect interfaces on both boundaries is presented. For the analytic approach, theories of linear elastic fracture mechanics (LEFM) and linear elasticity with complex variable techniques are applied to address how interfacial damage on both interfaces simultaneously affects the tendency for the crack to propagate during mode I loading.

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This dissertation is dedicated to
my Lord Jesus and Holy Mary.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1
Introduction

1.1 Composite Materials

A composite material is a material system comprised of two or more materials that form a new material with new individual properties or characteristics. This can be called a multiphase material. Within the multiphase, is a region that is very stiff and is called the *reinforcement*. The continuous and compliant phase surrounding the reinforcement is called the *matrix*. Applied loads are usually transmitted into the reinforcement material through the matrix phase, with a small amount of the load being sustained by the matrix. By restraining brittle cracks from propagating between reinforcements, the matrix provides a barrier to cracking in the composite materials [1]. In addition, an independent phase between the matrix and the reinforcement is occasionally considered and is called the *interphase* (see Figure 1.1). The interphase is the result of a chemical interaction or other processing effect.

Composite materials are prevalent in nature. Rocks are an aggregate of many different types of crystals and grains, and often include salt water or oil [2]. Bone and wood are the most common natural composites. From an engineering application standpoint, the composite is a multiphase material, which is artificially made to meet the specific performance requirements of everything from machinery, such as bicycles, automobiles and aircrafts, to biomechanical applications, such as carbon/epoxy artificial limbs and carbon/polymer stems. One simple scheme for the classification of engineering composite materials is shown in Figure 1.2. This includes particulate composites, fibre-reinforced composites and laminate composites.

Particulate composite materials are comprised of a continuous matrix phase and particles randomly embedded within the matrix as a reinforcement phase. In particulate composites, the matrix bears a majority of the applied load and the particles strengthen the composite by preventing motion of dislocation. Laminate composites consist of two or more layers bonded together, called *lamina*, which are thin, planar and unidirectional, and have a preferred high-strength direction. The orientation and combination of stacked lamina vary with the desired orientation of the high-strength reinforcement. Fibre-reinforced composites are the most dominant and important composite materials. With stiff and brittle fibres embedded in a soft and ductile matrix, fibre reinforcement composites dramatically increase strength, fatigue limits, resistance and stiffness. Stiffness of a material along the fibre direction is usually governed by the mechanical properties of the fibres, while enhancement in the overall strength of the composite is achieved through the properties of the matrix [3]. Since stiffness transverse to the fibres direction is much lower than stiffness normal to the fibres, combinations of both laminate and fibre-reinforced composites, which stack multi-layers of fibres in the different directions, have been developed.

Many new applications have been developed through the engineering of composite materials. Composite materials were first introduced in the form of steel-reinforced concrete in the nineteenth century and have been followed by applications such as the fibreglass boat in 1942, high-strength carbon fibre in the early 1960's and today's nano-composite materials [3]. The demand for such high performance materials which can overcome extreme and severe conditions—conditions which single-phase conventional materials cannot sustain—fuels the study of composite materials that have advantages and dominant features of each constituent combined. Consequently, the advent of many advanced composites and the significant progress made in materials science and analytical approaches, such as micromechanics and macromechanics which save time and costs associated with the expensive experiments necessary to test composites, considering the large number of permutations represented by composites, have attracted the attention of many scientists. Micromechanics is the analysis of the interactions between constituent materials, on the level of the individual phases and at the macromechanical level, to evaluate the equivalent overall response of a composite.

A composite is a multiphase material that consists of many inhomogeneities on a length scale much greater than the atomic scale, but which contain regions large enough to be considered locally continuous and essentially homogeneous at the macroscopic length scale [2, 4]. Also, the inhomogeneities bond with each other at interfaces. Thus, classical mechanics, such as linear elasticity and fracture mechanics can be applied to the material system without loosing generality, given certain assumptions. Namely, each phase, including the reinforcement and the matrix, shows linear elastic behaviour, and they bond together. The singular surface between the phases can be expressed in terms of special parameters such as the spring-type interface parameter, which connects the two elastic constituent materials together smoothly. Thus, this thesis is developed based on the premise that three different imperfectly bonded phases show linear elastic behaviour and spring-type interface parameters represent degradation of bonding. Linear elasticity and fracture mechanics are applied to the composite.

1.2 Applied Mechanics of Composite Materials

1.2.1 Introduction

The underlying studies of the theory and analysis of composite materials were introduced by Maxwell, Rayleigh and Einstein following Poisson's presentation of the theory of induced magnetism, in which a composite material consists of conducting spherical inclusions and a nonconducting matrix, in 1826 [2]. With the advent of fibres as reinforcement constituents in the 1960s, came the dramatic development of micro/macromechanics with respect to composite materials. A macromechanical approach determines the effective, or overall, parameters of composites and predicts their average response. A micromechanical analysis depicts the interaction and relationship between fibres and the matrix; thus, the analysis illustrates the behaviour of deformation and stress within each constituent and local failure, such as matrix/fibre cracking and bonding failure at the interface or the interphase layer [3]. For the study of strength and fracture toughness, which cannot be averaged, the micromechanical approach should be adapted. A well-known model of micromechanics, which considers three-phase system is the generalized self-consistent model (GSC) [5, 6].

Micromechanical analysis involves a representative volume element (RVE), which is defined as a sample volume having the same properties as the homogeneous composite. Making the assumption that an inclusion like a fibre within the RVE is far enough apart from the others so that the interaction between inclusions can be ignored, the RVE taken from composites is considered a single inclusion embedded within the matrix [7]. Hence, since it provides a much simpler approach to micromechanics, the single inclusion problem has attracted a great deal of attention, and has been considered a fundamental solution to the micromechanics analysis of composites.

The interaction between a single inclusion within a matrix and material flaws (such as a crack [8, 9, 10, 11, 12] or a dislocation [13, 14, 15, 16]) has received a considerable amount of attention in the area of fracture mechanics. Understanding crack behaviour within an inclusion (fibre cracking) or within the matrix (matrix cracking) leads to the ability to predict strengthening, hardening and sudden brittle failures of advanced composite materials. Thus, the solution to a single inclusion such as a fibre interacting with a crack or a dislocation in the vicinity of the inclusion is of fundamental importance in understanding the failure analysis of composite materials.

The widely accepted interface assumptions associated with micromechanics can be divided into the following categories: the interface is assumed perfect [5, 13, 16, 17] or imperfect [18, 19, 20, 21]; the imperfect interface is homogeneous [22, 23] or inhomeogeneous along its boundary [24, 25]; the imperfect interface is modeled as a spring layer [26, 27] or thin independent layer [28, 29].

1.2.2 Interface Conditions

The most simple and commonly made assumption regarding interface bonding conditions between constituent materials in composites is that the interface is perfectly bonded. It is an idealization of a complicated adhesion that assumes there is no bonding degradation or damage between the fibre and the surrounding matrix, such as weak adhesion and voids, or crack propagation along the interface. From an analytical point of view, the perfect interface means that tractions and displacements are continuous across the interface. Eshelby contributed to the development of this mechanical model [17]. Although Eshelby's model is inadequate for complicated modern applications, its simple mathematical advantage, contributing to the early development of micromechanics of composite materials, has been widely accepted [17, 29, 30, 31, 32].

Since the actual degradation of interface bonding is not taken into account with the perfect interface, imperfect interface is introduced to simulate intermediate states of bonding, from initial perfect bonding to complete debonding. The imperfect interface condition is also simulated as an interphase layer, which is a thin annular layer between a fibre and a matrix. In a micromechanic analysis, the intermediate zone is usually represented in two different ways: a finitely thin independent layer [29, 33, 34, 35] and a spring layer [18, 20, 21, 25, 26, 27].

As an independent phase, the mechanical properties of the interphase layer vary between the fibre and the surrounding matrix. The properties match those of the fibre on one side and the matrix on the other [33]. However, the interphase layer is considered a non-uniform, anisotropic region, with mechanical parameters distinct from both the fibre and the matrix [36]. Qaissaunee and Santare [35] investigated an edge dislocation interacting with a three-phase elliptical inclusion in which all the interphases were assumed perfectly bonded. They addressed the presence of an interfacial zone, which had significant influence on the stress field in the matrix and the inclusion. Three parameters, namely, two elastic constants and the thickness of the layer are required to elucidate the behaviour of the interphase layer. However, it is very difficult to determine the constants, and the experiments required are very expensive. To overcome this difficulty, a spring-layer model is often used.

The spring-layer model does not have an independent finite thickness layer. Thus, the three unknown parameters are reduced to two spring constant parameters: in other words, the two elastic constants and the thickness are incorporated into two spring constants defined along the whole length of the interphase [37, 38]. The mathematical representation of this model is based on the premise that tractions are continuous, and displacements are discontinuous across the interface. Displacement jumps across the interface are proportional, in terms of "spring-factor type" interface parameters, to their respective traction components at the boundary. Sudak [22] considered the interaction between a dislocation and a circular inhomogeneity subjected to antiplane shear, with the assumption of homogeneous spring-type interface parameters. It was found that a soft inclusion always attracts the dislocation, regardless of the state of adhesion at the interface. Thus, the equilibrium position of the dislocation cannot be obtained. Alternatively, a stiff inclusion causes an unstable equilibrium position that is affected by imperfect interface conditions and the shear modulus ratio. Amenyah et al. [26] investigated cracking within an fibre embedded in an infinite matrix subjected to thermal loadings. They incorporated the imperfect interface with spring-type interface parameters into the single inclusion (fibre)system and found that the imperfection of the interface had significant effects on the stress intensity factor (SIF). They also addressed the fact that the perfect interface model results in overestimation of the SIF by up to 100% in a hard inclusion, and 200% in a soft inclusion. Consequently, the imperfect interface condition is of significant importance in analysis. A relationship between an imperfect interface and crack propagation in the matrix is proposed by Liu et al. [27]. In contrast to the perfect bonding condition, the imperfect interface causes greater SIFs at the nearby crack tip than at the distant tip, even when the fibre is stiffer than the matrix. Also, the imperfectness of adhesion leads to crack propagation in the interface, which eventually results in debonding.

If the interface parameters are constant along the entire interface, it is called homogeneously imperfect, while it is non-homogeneously imperfect if it is not constant. The effects of an inhomogeneously imperfect interface on the stress fields induced within a fibre are addressed in [25]. The two spring-type interface parameters vary along normal and tangential directions, resulting in inhomogeneously imperfect interface conditions, which lead to average stresses different from those of a homogeneously imperfect interface.

Combining the two different methods mentioned above, authors have recently tried to derive a more descriptive and detailed interpretation of crack propagation interactions with respect to imperfect interfaces [39, 40, 41]. In this model, the interphase layer is considered an independent region with finite thickness. Inner interface bonding, between the region and a fibre, is considered imperfect and is depicted by two spring-type interface parameters. The outer interface, which is in contact with the surrounding matrix, is assumed perfectly bonded. Adapting a three-phase circular inclusion model with the imperfect interface between a fibre and interphase layer, Kim and Sudak indicate that the degradation of interface bonding quality and mechanical properties of the interphase layer alter SIFs of a radial matrix crack and their influence on micromechanical analysis of composite constituents should not be ignored [39]. The interaction between an inhomogeneity and a screw dislocation in the interphase layer with imperfect interface has been studied [41]. It has been demonstrated that the interphase thickness, imperfection of the inner interface, and ratio of material properties compete with each other to alter the equilibrium position and stability of the dislocation. Henceforth, both mechanical properties of the independent region and imperfect bonding conditions should be taken into account.

In this research, the interphase layer is assumed to be an independent region having imperfect bonding interfaces along both sides and the imperfection of the boundary is assumed to be homogeneously imperfect.

1.2.3 Two-Phase Micromechanics Scheme

In this section, a simplified two-phase composite materials model is briefly introduced. From the analytical point of view, a inclusion represents a void or reinforcement such as a fibre. One of the most significant contributors to the establishment and shaping of the conceptual framework of the two-phase micromechanics scheme is Eshelby. In his 1957 paper [17], Eshelby's developed a tensor to solve for an elastic field perturbation within an inclusion and the matrix, caused by the inclusion being perfectly embedded in the matrix. His tensor has provided a foundation for the study and application of linear elastostatic theories, such as the theory of dislocations and interactions between a crack and the matrix. In 1964 [13], Dundurs and Mura depicted the interaction between a circular inclusion embedded in an infinite elastic region and an edge dislocation in the neighbourhood of the inclusion by determining Airy's stress functions as a function of the Burger's vector. They addressed the fact that a stable equilibrium position exists in the matrix, even in the vicinity of a perfect interface. In addition, Poisson's ratios of the inclusion and the matrix and the orientation of the Burger's vector, with respect to the inclusion, are shown to have strong influences on matrix cracking. In 1965, Dundurs and Sendeckyj [14] turned their attention to fibre cracking, by solving a similar problem with a perfect interface, but which had a dislocation inside the inclusion. They found that the behaviours of the stable equilibrium position within the inclusion were similar. In 1967, Dundurs [42] evaluated screw dislocation interaction with an inclusion. The influence of inclusion stiffness on the behaviour of the screw dislocation were addressed. A stiff inclusion repels the screw dislocation, whereas a compliant inclusion attracts the dislocation. This result has been proven by many authors in different ways [8, 27, 39, 43].

The interaction between a circular inclusion and an external radial crack was defined in terms of SIFs by Tamate in 1968 [8]. He stated that a relatively stiff inclusion causes lower SIFs, while a softer inclusion results in higher SIFs. His findings were in agreement with the results of Dundurs' 1967 work. In addition, he showed that the near crack tip is more vulnerable to the effects of an inclusion's parameters than the distant crack tip. To overcome the limitations encountered in the form of the series expansion for complex stress potentials used by Tamate, Atkinson [9], in 1972, introduced a dislocation density method in which singular integral equations for the distribution of dislocations which form a crack, are set up and solved numerically. Integrating over the dislocation density yields the stress fields. The solution for a single dislocation interacting with an inclusion was addressed by Dundurs and Mura [13] in 1964. Their solution was generalized to an arbitrarily oriented line crack by Erdogan et al. in 1974 [16]. Over the last two decades, the condition in which an imperfect cohesion exists along an interface including an interphase layer has been incorporated into the two-phase model as described in the previous section [18, 21, 44, 45].

Research into the behaviours of a crack or dislocations interacting with an inclusion in the two-phase composite system has had a significant impact on the development of the fundamental theories of micromechanics. However, in considering the effects of an independent interphase layer; the influence of other inclusions surrounding an inhomogeneity, as in the GSC scheme; or other general three-phase cases such as cemented implant systems (implant/bone/PMMA); the three-phase micromechanics scheme is of great practical importance and interest.

1.2.4 Three-Phase Micromechanics Scheme

The three-phase model, known as the GSC model, was first introduced by Christensen and Lo in 1979 [5]. They addressed the fact that perturbation caused by neighbouring inclusions should be taken into consideration when determining the effective shear modulus of the composite materials. The three-phase model consists of two circular regions and an infinitely extended region. The advantage of this model is that it provides a mathematical simplification of the single inclusion problem and takes into account the effects of neighbouring inclusions and the matrix. Thus, it is a simplified and generalized composite model, which offers more precise determination of the effective parameters, and effects the local characteristics and fracture mechanism of many physical applications. From a classical micromechanics perspective, the internal circular region is considered part of the inclusion, the annulus a part of the matrix phase, and the infinitely extending outer region a part of the composite phase, which is affected by both the neighbouring inclusions and the matrix [5]. Thus, this model presents the fundamental solution for the GSC model analysis.

From an engineering application standpoint, a common physical example of the model is coated fibres embedded in a matrix. The internal component is the fibre, the intermediate region is the coating layer and the outer region is the surrounding matrix. The three-phase model is valuable for simulating the interphase layer associated with bonding in composite materials. Furthermore, the methodology addressed in the three-phase model can easily be expanded into a multi-layer composite material without losing generality [46] and can also be adapted to simulate particulate composite materials in two-dimensional micromechanical analysis [6].

Since Christensen and Lo [5] introduced the three-phase scheme in 1979, it has attracted the attention of many researchers interested in simulating the influence of both neighbouring inclusions and the surrounding matrix in the microstructure analysis of composite materials, and in representing an interphase layer. In 1989, Luo and Weng [47] addressed how stress and strain fields are uniform in a fibre when Eshelby's S-tensor is applied to the three-phase scheme with perfect bonding interfaces. Stress fields in a composite with a coated inclusion were subjected to thermoelastic loading and evaluated by Benveniste et al. in 1989 [28]. In 1991, Luo and Chen compared two-phase models and three-phase models by considering the effects of volume fibre fraction, and the thickness and material properties of an intermediate phase [48]. They demonstrated different results for the two models. By solving a problem of two concentric circles surrounded by unbounded medium, they addressed how the volume fibre fraction has a profound influence upon cracking in the matrix and how the stiffer surrounding medium, which is the infinitely extending outer region, suppresses crack propagation in the matrix. Also, they illustrated how the three-phase model shows a stable equilibrium position of the dislocation under much less stringent conditions, when compared with the two-phase model used by Dundurs and Mura [13], and how the trapping mechanism of the dislocation is more likely to take place in the three-phase model [49]. Of particular interest is how the stability of the dislocation is observed to be fundamentally independent of the orientation of its Burger's vector. These results are significantly different than the results seen using the two-phase model. In 1998, Ru provided the exact closed-form solution for stress fields within an inclusion surrounded by an interphase layer under thermal loads, showing that the interphase layer has strong effects on the local stress, but moderate influences on the mean stress values [50]. In 2002, Wang and Shen, further developed the model by considering imperfect interface such as the inner circumferentially homogeneous sliding interface between an inclusion and the surrounding matrix [51]. By investigating how crack behaviours in the matrix interact with the imperfect interface, they depicted how the sliding interface influences the SIFs—the degradation of the initial bonding condition plays a predominant role in altering SIFs.

1.3 Applications of the Three-Phase Model

1.3.1 Interphase Layer of Composite Materials

It is well known that the quality of adhesion along an interface between a fibre and the matrix has a profound influence on the performance and failure of composite materials, since the effective load transfer between the fibre and the matrix are compromised due to bonding defects, such as voids, impurities and cracking. The intermediate zone representing a state of adhesion between a fibre and matrix is known as the interphase layer. The interphase layer can be illustrated by a two-phase model having an imperfect interface, with the assumption that its thickness is very thin and there are three parameters, the thickness and two elastic constants, which are incorporated into two spring-type interface parameters. However, as addressed in Section 1.2.2, the three-phase model is preferred, since it provides a more precise analysis of the interphase layer's influence on the failure mechanism of composite materials.

The interphase layer is a chemical interaction by-product of the bonding between the constituents. However, it may also be introduced in the design stage to improve performance of the composite constituents. For example, fibres may be coated to improve the bonding between the fibres and the matrix. Xiao and Chen [52, 53] investigated the effect of coated fibre composites on bonding strength. They demonstrated that the thickness and mechanical properties of the coating materials compete with each other to alter the stress field around the circular inclusion. With a thick coating layer, the elastic properties of an inclusion have limited effects on the force of a dislocation, while the mechanical properties of both an inclusion and coating have substantial influence on the equilibrium position and the stability of the dislocation in the case of a thin coating layer.

The interphase layer can also include a thermal barrier coating in order to avoid thermal mismatch induced stresses in passivated interconnect lines in integrated circuits. Ru addressed how the interphase layer, with an intermediate thermal expansion coefficient between that of the matrix and the inclusion, should be implemented in order to reduce thermal stresses within both the inclusion and interphase layer [50]. In addition, in order to study stress and electric field concentrations in piezoelectric composites and devices, Sudak investigated the relationship between residual electroelastic stresses induced by electromechanical loadings, and lattice mismatch between buried active components and surrounding materials [54]. Considering the interphase layer as a continuum with finite thickness, the works mentioned above have adapted the three-phase model to investigate its effects on the behaviour of composite materials.

1.3.2 A Model of A Cemented Implant

According to the National Hospital Discharge Survey of 2003, 36,000 of the 217,000 total hip replacement (THR) surgeries and 33,000 of the 402,000 total knee replacement (TKR) operations performed in the United States underwent revision surgery. Cemented implants, usually PMMA, are among the most popular types of prostheses. The predominant failure of cemented implants is aseptic loosening of the stem from a host bone. Of the many factors causing aseptic loosening, damage to the mechanical interlocking along the interfaces is the primary reason for loosening of a stem from the bone cement.

It is important to understand the behaviours of each interface, namely, bone/cement and cement/implant, to predict failure of the cemented prosthesis system. First, the interface between an implant and the bone cement can have either a smooth polished surface, to facilitate implant insertion into the cement, or a porous surface, to increase interdigitation between the constituents.

The interface between bone cement and the host bone has strong interdigitation. The interdigitation in a well-fixed cementing keeps its initally strong mechanical interlocking and prohibits any relative shear slip. Compared to the adhesion between an implant and the bone cement, the mechanical interlocking and bonding along this interface is strong and less vulnerable to loosening. The three-phase model with two independently imperfect interfaces is of fundamental importance in predicting failure in an implant/cement/bone system.

1.4 Overview of the Current Study

Studying the behaviour of cracks or dislocations interacting with inclusions and the surrounding matrix is of fundamental importance in order to understand failure mechanisms and to enhance the performance of composite materials. A simplified scheme consisting of a single inclusion and a crack or a dislocation has been adopted for theoretical analyses by many researchers and has provided reasonable and reliable results. The application of the three-phase model includes bonding layers, any special coatings and cemented implants embedded into bone.

The majority of the research dealing with the three-phase model has assumed that both inner and outer interfaces are perfectly bonded or that only the inner boundary has imperfect adhesion. Although the imperfect interface has been considered a predominant parameter in micromechanical analysis and the degree of bonding degeneration at the interfaces has been known to dramatically alter stress fields in the composite constituents [26, 27], applying the imperfect interface condition to the three phase model is rare in the literature due to mathematical difficulties involved.

To illustrate this point, we will examine the dislocation density method.

The dislocation density method, also called the singular integral equation method, was introduced and developed by Atkinson [9] and Erdogan et al. [16] and has been the most popular and widely accepted for simulating crack-inclusion or dislocation-inclusion interactions. Although this procedure has contributed to the development of micromechanics of composite materials, the computational difficulties associated with solving the singular integral equations for imperfect interfaces are still extremely challenging. To overcome the difficulties, a relatively simple series method has been incorporated into a two-phase imperfect interface model by Amenyah et al. [26] and Liu et al. [27], and into a three-phase model with an imperfect inner interface by Sudak [41] and Kim [39]. Sudak studied interactions between a screw dislocation and an inclusion in anti-plane shear employing a three-phase model with an imperfect interface by determining unknown complex coefficients using the series method. By establishing coefficients of complex potentials, Kim and Sudak also solved the three-phase inclusion-crack interaction with general imperfect bonding conditions. Wang and Shen proposed the decoupled strategy, in which the dislocation density method and the series method were combined to show crack-inclusion interaction with sliding interface condition [51]. In this view, the three-phase model with imperfect bonding along both inner and outer interfaces is considered to be the preferred and more precise model. However, although the three-phase model with inner/outer imperfect interfaces is identified, to the best of my knowledge, there is no research in the literature dealing with the three-phase scheme having imperfect bonding along both inner and outer interfaces.

In the present work, both internal and external annular interfaces are considered imperfectly bonded and the series method is used to resolve the mathematical difficulties. Imperfectness of interfaces is modeled with dimensionless spring-type interface parameters and the effect of the parameters on the SIFs at crack tips is shown along crack locations. Thus, the aim of this research is to address how imperfections of both interfaces simultaneously affects propagation of a pre-existing radial crack in the annular region in plane elastostatics during the mechanical cracking processes.

This thesis is organized into six chapters. Following this introduction which reviewed the basis of the three-phase model with imperfect interfaces, Chapter 2 provides a derivation of the mathematical formulations of the boundary value problem. Specifically, a semi-analytical solution to plane elastic deformation of a single inclusion interacting with a pre-existing radial crack is presented. The boundary conditions of an imperfect bonding along two interfaces are expressed in terms of stress potentials, which are represented solely in series forms. In Chapter 3, a rigorous solution to the boundary value problem is derived from the two imperfect bonding interfaces and a set of general algebraic equations is obtained. Chapter 4 outlines the numerical analysis procedures used to solve the algebraic equations and lays a foundation for SIFs to describe the behaviour of a radial crack under mode I loading. For the verification of formulations, numerical results from the derived formulations with perfect interfaces are compared to those published. In particular, a new phenomenon, the so-called stable zone associated with debonding, is discussed in this chapter. Chapter 5 deals with another application example of this rigorous model, namely implant/bone cement/bone system. The semi-analytical solution to the three-phase model is applied to the cemented implant design to show how the non-slip surface conditions of the implant affect performance and failure of the prosthesis. Chapter 6 summarizes the new findings of this research and concludes that the three-phase model having imperfect bonding along both inner and outer interfaces is general and preferred from application perspective and the imperfectness of both interfaces should be taken into account for a more precise analysis. A brief insight into future work is presented at the end of the chapter.

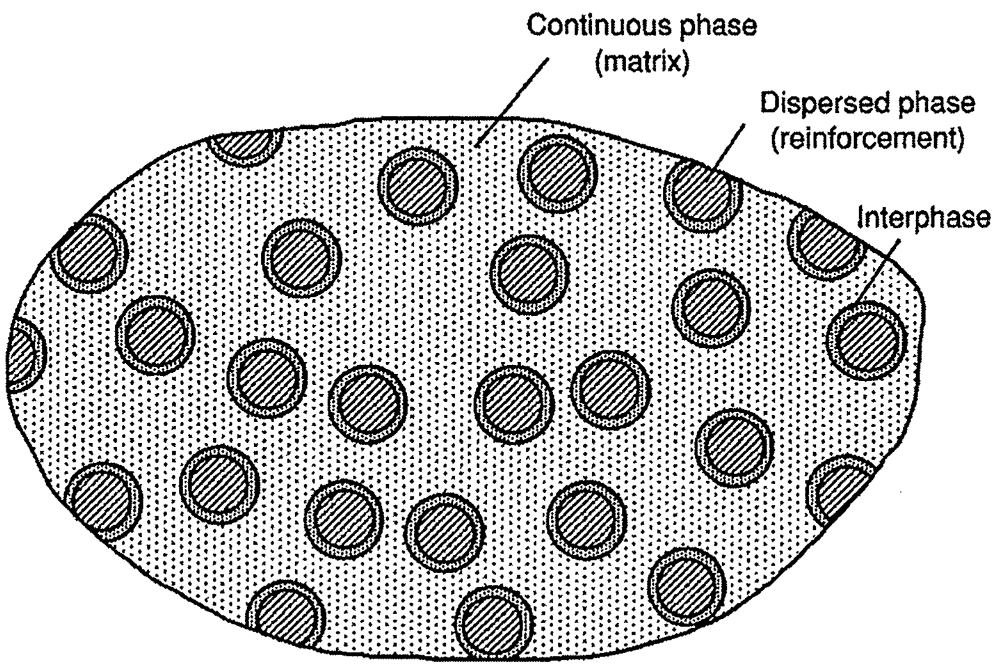
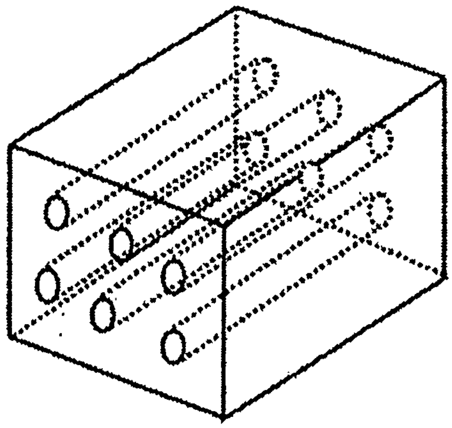
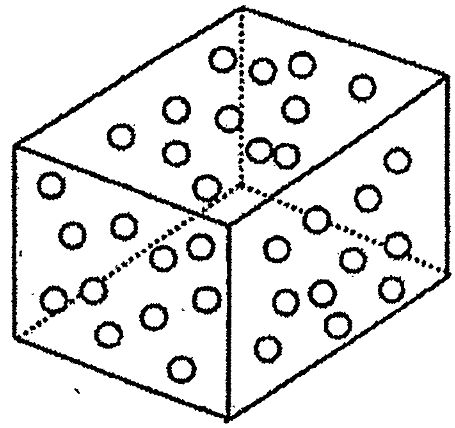


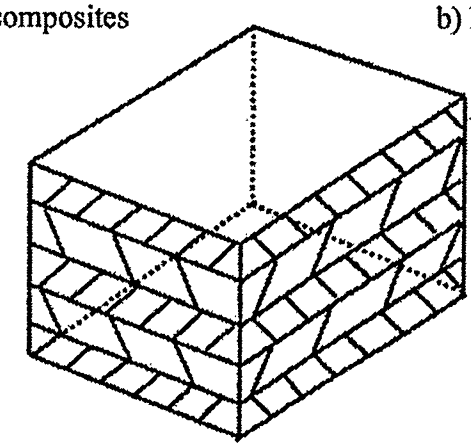
Figure 1.1 Phases of a composite materials (from Daniel et al. [3])



a) Fiber-reinforced composites



b) Particulate composites



c) Laminar composites

Figure 1.2 Classification of Composite Materials

CHAPTER 2
Analytical Formulation of A Boundary Value Problem

2.1 Introduction

We consider a domain in $\mathbb{R}^2$, infinite in extent, containing an inclusion, a matrix zone with a pre-existing radial crack and a composite phase (Figure 2.1). The inclusion, with center at the origin of the coordinate system and radius R_0 occupies a region denoted by S_0 and the matrix zone is modeled as an annulus with outer radius R_1 and represented by S_1. The surrounding composite phase is represented by S_2. All the materials occupying S_0, S_1 and S_2 are assumed to be homogeneous and isotropic. The inclusion/matrix interface and the matrix/composite phase interface are denoted by curves $\partial\Gamma_0$ and $\partial\Gamma_1$, respectively and the interfaces $\partial\Gamma_0$ and $\partial\Gamma_1$ are assumed to be homogeneously imperfect. The crack has length of $2l$ and unless otherwise stated, the subscripts 0, 1 and 2 will denote quantities in the domain S_0, S_1 and S_2, respectively.

2.2 Formulation

For plane deformation, the elastic stresses and their respective displacements in the polar coordinate system can be given in terms of two complex potentials $\varphi(z)$ and $\psi(z)$ as follows [55]:

$$\begin{aligned} 2\mu\left(u_r + iu_\theta\right) &= e^{-i\theta}\left[\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}\right], \\ \sigma_{rr} + \sigma_{\theta\theta} &= 2\left[\varphi'(z) + \overline{\varphi'(z)}\right], \\ \sigma_{rr} - i\sigma_{r\theta} &= \varphi'(z) + \overline{\varphi'(z)} - e^{2i\theta}\left[\overline{z}\varphi''(z) + \psi'(z)\right], \end{aligned} \tag{2.1}$$

where

$z = x + iy = re^{i\theta}$ is the complex coordinate,

$\mu =$ the shear modulus,

$\nu =$ Poisson's ratio,

$\kappa = 3 - 4\nu$ for plane strain, $(3 - \nu) / (1 + \nu)$ for plane stress.

Also, the resultant force acting on an arbitrary arc AB in an elastic body is given by [55]

$$F_x + iF_y = -i \left[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)} \right]_A^B \quad , \tag{2.2}$$

where $[f(*)]_A^B = f(B) - f(A)$.

Among the various mechanical descriptions of an imperfect interface, one of the most widely accepted models is based on the premise that tractions are continuous but displacements are discontinuous. Specifically, displacement jumps are proportional, in terms of 'spring-factor type' interface parameters, to their respective traction components. In view of this, let us assume that the inclusion is imperfectly bonded to the matrix along the circular curve $\partial\Gamma_0$ and the matrix zone is also imperfectly bonded to the composite phase along $\partial\Gamma_1$. Then, the boundary value equations describing the problem above can be formulated. The traction continuity and displacement jump condition along Γ_0 are given by [40]

$$\|\sigma_{rr} - i\sigma_{r\theta}\| = 0,$$

$$\sigma_{rr} = m_0 \|u_r\| - m_0 u_r^0 \quad , \quad \sigma_{r\theta} = n_0 \|u_\theta\| - n_0 u_\theta^0 \quad \partial\Gamma_0. \tag{2.3}$$

In the same way, the other boundary conditions along Γ_1 are given by

$$\|\sigma_{rr} - i\sigma_{r\theta}\| = 0,$$

$$\sigma_{rr} = m_1 \|u_r\| - m_1 u_r^0 \quad , \quad \sigma_{r\theta} = n_1 \|u_\theta\| - n_1 u_\theta^0 \quad \partial\Gamma_1, \tag{2.4}$$

where m and n are non-negative spring-factor type interface parameter representing the degree of interface damage. Physically, these parameters represent the mechanical properties of the interface such as strength, stiffness and the overall degree of adhesion along the interface [37]. The bracket expression $\|*\|$ denotes jump across the interface, namely $\|*\| = \|*\|_1 - \|*\|_0$ along Γ_0 and $\|*\| = \|*\|_2 - \|*\|_1$ along Γ_1, and u_r^0 and u_θ^0 are additional displacements induced by the uniform stress-free eigenstrains $(\varepsilon_x^0, \varepsilon_y^0, \varepsilon_z^0)$ prescribed within the inclusion and the bonding medium which might be induced by thermal mismatch between adjoining materials. The advantage of defining the interface parameters m in the normal direction and n in the tangential direction in (2.3) and (2.4) is that they allow representation of degree of the interface imperfectness from complete debonding to perfect bonding through intermediate states. It is shown that the parameters, $m = n = \infty$ in (2.3) and (2.4) lead the displacement jumps across the interfaces to zero, which corresponds to a perfect bonding condition. Interface tractions vanish as the value of the parameters approaches zero, which is associated with complete debonding. Any positive value of these parameters represents a degree of bonding inadequacy along the interfaces.

2.3 Boundary Value Problem

The boundary conditions along Γ_0 and Γ_1 in (2.3) and (2.4) need to be expressed in terms of analytic functions $\varphi(z)$ and $\psi(z)$. Each region such as S_0, S_1 and S_2 requires two stress potentials to describe its stress fields so that it is required that six analytic functions $\varphi_0(z)$, $\psi_0(z)$,$\varphi_1(z)$, $\psi_1(z)$,$\varphi_2(z)$ and $\psi_2(z)$ be defined. Attention should be given to the boundary conditions along Γ_0 and Γ_1 in region S_1 since a radial crack causes multi-valuedness. As a result, the analytic functions $\varphi_1(z)$ and $\psi_1(z)$ are divided into two pairs such as $\varphi_{11}(z)$, $\psi_{11}(z)$ and $\varphi_{12}(z)$, $\psi_{12}(z)$: thus, $\varphi_{11}(z)$ and $\psi_{11}(z)$ are defined on and outward from Γ_0 while $\varphi_{12}(z)$ and $\psi_{12}(z)$ are defined on and inward from Γ_1 (See section 2.4 for details).

The traction continuation along Γ_0 in $(2.3)_1$ leads to the following expression

$$(\sigma_{rr} - i\sigma_{r\theta})_1 = (\sigma_{rr} - i\sigma_{r\theta})_0 . \tag{2.5}$$

In view of $(2.1)_3$, the above equation is re-written in terms of analytic functions as

$$\varphi_1'(z)+\overline{\varphi_1'(z)}-e^{2i\theta}\left[\bar{z}\varphi_1''(z)+\psi_1'(z)\right]=\varphi_0'(z)+\overline{\varphi_0'(z)}-e^{2i\theta}\left[\bar{z}\varphi_0''(z)+\psi_0'(z)\right]. \tag{2.6}$$

Since $z\bar{z} = R_0^2$ and $z = R_0 e^{i\theta}$ at $\partial\Gamma_0$ and $\overline{\varphi'(\bar{z})} = \overline{\varphi}'(z)$ according to analytic continuation $(\overline{\varphi'(z)} = \overline{\varphi}'(\bar{z}))$, $\bar{z} = \frac{R_0^2}{z}$ and $e^{i\theta} = \frac{z}{R_0}$ are substituted into the equation (2.6) and it can be recast into the following form

$$\varphi_{11}'(z)+\overline{\varphi_{11}'}\left(\frac{R_0^2}{z}\right)-z\varphi_{11}''(z)-\frac{z^2}{R_0^2}\psi_{11}'(z)=\varphi_0'(z)+\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)-z\varphi_0''(z)-\frac{z^2}{R_0^2}\psi_0'(z) \quad \partial\Gamma_0. \tag{2.7}$$

It is noted that the stress potentials $\varphi_1$and ψ_1 are replaced with $\varphi_{11}$and ψ_{11}.

The normal and tangential displacement jump condition along Γ_0 in $(2.3)_2$ is re-written based on $(2.3)_1$ as

$$\sigma_{rr} - i\sigma_{r\theta} = \left(m_0 \|u_r\| - m_0 u_r^0\right) - i\left(n_0 \|u_\theta\| - n_0 u_\theta^0\right), \qquad \partial\Gamma_0. \tag{2.8}$$

According to the assumption of traction continuation condition in $(2.3)_1$, the left-hand-side of (2.8) can be either $(\sigma_{rr} - i\sigma_{r\theta})_0 = (\sigma_{rr} - i\sigma_{r\theta})_1$ along Γ_0. Letting $\sigma_{rr} - i\sigma_{r\theta} = (\sigma_{rr} - i\sigma_{r\theta})_0$, we write the displacement jump condition in the following form (see Appendix 1 for detailed derivation)

$$(\sigma_{rr} - i\sigma_{r\theta})_0 = \left(\frac{m_0 - n_0}{2}\right) \|u_r + iu_\theta\| + \left(\frac{m_0 + n_0}{2}\right) \|u_r - iu_\theta\| - (m_0 u_r^0 - i n_0 u_\theta^0) \,. \tag{2.9}$$

The additional displacement induced by the eigenstrains prescribed in the inclusion can be written as [10]

$$u_r^0 = R\left(\varepsilon_x^0 \cos^2\theta + \varepsilon_y^0 \sin^2\theta + \varepsilon_{xy}^0 \sin 2\theta\right), \quad u_\theta^0 = R\left(\frac{\varepsilon_y^0 - \varepsilon_x^0}{2} \sin 2\theta + \varepsilon_{xy}^0 \cos 2\theta\right). \tag{2.10}$$

Sudak et.al.[25] have shown that the last term in (2.9), the displacement induced by the uniform eigenstrains can be written in the following form

$$(m_0 u_r^0 - i n_0 u_\theta^0) = m R_0 \varepsilon_1 + \left(\frac{m+n}{2R_0}\right)(\varepsilon_2 - i\varepsilon_3) z^2 + \left(\frac{m-n}{2z^2}\right) R_0^3 (\varepsilon_2 + i\varepsilon_3), \quad \partial\Gamma_0, \tag{2.11}$$

where $\varepsilon_1 = \frac{\varepsilon_x^0 + \varepsilon_y^0}{2}, \varepsilon_2 = \frac{\varepsilon_x^0 - \varepsilon_y^0}{2}$ and $\varepsilon_3 = \varepsilon_{xy}^0$. By substituting (2.11) into (2.9) and then using (2.1) and (2.2), we can express the displacement jump condition $(2.3)_2$ in the following form (see Appendix 1 for detailed derivation)

$$\begin{aligned} &(m_0 - n_0)(\kappa_1 + 1)\frac{R_0}{z}\varphi_{11}(z) + (m_0 + n_0)(\kappa_1 + 1)\frac{z}{R_0}\overline{\varphi_{11}}\left(\frac{R_0^2}{z}\right) \\ = \; & 4\mu_1\left[\varphi_0'(z) + \overline{\varphi_0'}\left(\frac{R_0^2}{z}\right) - z\varphi_0''(z) - \frac{z^2}{R_0^2}\psi_0'(z)\right] + (m_0 - n_0)(1 + \kappa_0\frac{\mu_1}{\mu_0})\frac{R_0}{z}\varphi_0(z) \end{aligned} \tag{2.12}$$

$$+(m_0-n_0)(1-\frac{\mu_1}{\mu_0})\left[R_0\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)+\frac{R_0}{z}\overline{\psi}_0\left(\frac{R_0^2}{z}\right)\right]$$
$$+(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})\frac{z}{R_0}\overline{\varphi_0}(\frac{R_0^2}{z})+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\left[R_0\varphi_0'(z)+\frac{z}{R_0}\psi_0(z)\right]$$
$$+4m_0R_0\mu_1\varepsilon_1+\frac{2\mu_1(m_0+n_0)(\varepsilon_2-i\varepsilon_3)}{R_0}z^2+\frac{2\mu_1(m_0-n_0)(\varepsilon_2+i\varepsilon_3)R_0^3}{z^2}$$

$, z\in\Gamma_0.$

In a similar fashion of (2.7), the traction continuation along Γ_1 in $(2.4)_1$ can be written as

$$\varphi_2'(z)+\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)-z\varphi_2''(z)-\frac{z^2}{R_1^2}\psi_2'(z)=\varphi_{12}'(z)+\overline{\varphi_{12}'}\left(\frac{R_1^2}{z}\right)-z\varphi_{12}''(z)-\frac{z^2}{R_1^2}\psi_{12}'(z) \quad (2.13)$$

$, z\in\Gamma_1$. It is noted that the stress potentials $\varphi_{12}$and ψ_{12} are used. Also, the displacement jump condition along Γ_1 in $(2.4)_2$ can be expressed in terms of stress potentials as follows (see Appendix 2 for detailed derivation)

$$\begin{aligned}
&(m_1-n_1)(\kappa_1+1)\frac{R_1}{z}\varphi_{12}(z)+(m_1+n_1)(\kappa_1+1)\frac{z}{R_1}\overline{\varphi_{12}}\left(\frac{R_1^2}{z}\right) \quad (2.14)\\
=\;&-4\mu_1\left[\varphi_2'(z)+\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)-z\varphi_2''(z)-\frac{z^2}{R_1^2}\psi_2'(z)\right]\\
&+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left[R_1\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)+\frac{R_1}{z}\overline{\psi_2}\left(\frac{R_1^2}{z}\right)\right]\\
&+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}\kappa_2\right)\frac{R_1}{z}\varphi_2(z)+(m_1+n_1)\frac{z}{R_1}\left(1+\frac{\mu_1}{\mu_2}\kappa_2\right)\overline{\varphi_2}\left(\frac{R_1^2}{z}\right)\\
&+(m_1+n_1)\frac{z}{R_1}\left(1-\frac{\mu_1}{\mu_2}\right)\left[\psi_2(z)+\overline{z}\varphi_2'(z)\right]
\end{aligned}$$

$, z\in\Gamma_1.$

In (2.14) there are no terms associated with the displacements induced by the uniform eigenstrains. Also, two different pairs of stress potentials in region S_1 are defined. The $\varphi_{11}(z)$ and $\psi_{11}(z)$ defined along $\partial\Gamma_0$ are incorporated into (2.7) and

(2.12): the other pair of analytic functions $\varphi_{12}(z)$ and $\psi_{12}(z)$ are used in (2.13) and (2.14) along $\partial\Gamma_1$. Thus, the boundary value problem describing the imperfect bonding condition along both inclusion/matrix and matrix/composite is expressed in the four equations such as (2.7), (2.12), (2.13) and (2.14) which are given solely in terms of stress potentials.

2.4 Series Representation for Stress Potentials

The stress potentials can be expressed solely in series representation. As addressed in Chapter 1, to overcome mathematical difficulties related to solving singular integral equations, this relatively simple series method is often adapted in the three-phase model. Now, the eight stress potentials shown in (2.7), (2.12), (2.13) and (2.14) are represented as series expansions in the variable of z within each domain.

Let's consider region S_0 first. The stress potentials $\varphi_0(z)$ and $\psi_0(z)$ are analytic within a inclusion and can be expanded into a Taylor series in S_0 as follows

$$\varphi_0(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad \psi_0(z) = \sum_{k=0}^{\infty} b_k z^k, \tag{2.15}$$

where a_k and b_k are undetermined complex coefficients.

Secondly, the infinitely expanded region S_2, namely the composite phase is considered. The stresses X_x, Y_y and X_y are assumed to exist and to be bounded and finite at infinity. In this view, the equations of $(2.1)_{2,3}$ lead to a conclusion that $\varphi_2(z)$ and $\psi_2(z)$ must have $O(1)$ as $|z| \to \infty$. Thus, the remote loading at intinite can be characterized by the uniform stress field

$$\varphi_2(z) = Az + O(1), \qquad \psi_2(z) = Bz + O(1), \qquad \text{as } |z| \to \infty, \tag{2.16}$$

where A is a given real number and B is a given complex number. Stresses in a Cartesian system [55] are given as

$$\begin{aligned} X_x + Y_y &= 2\left[\varphi_2'(z) + \overline{\varphi_2'(z)}\right], \\ Y_y - X_x + 2iX_y &= 2\left[\bar{z}\varphi_2''(z) + \psi_2'(z)\right]. \end{aligned} \tag{2.17}$$

By substituting (2.16) into (2.17) and solving the equations, we get a relationship between stresses and the constants in the following form

$$X_x + Y_y = 4A,\ Y_y + iX_y = 2A + B,\ X_x - iX_y = 2A - B. \tag{2.18}$$

Now, let X_x, Y_y and X_y denote principal stresses $(\sigma_x^\infty, \sigma_y^\infty, \sigma_{xy}^\infty)$ at $z = \infty$, respectively. We re-write (2.18) as

$$\sigma_x^\infty + \sigma_y^\infty = 4A,\ \sigma_y^\infty + i\sigma_{xy}^\infty = 2A + B,\ \sigma_x^\infty - i\sigma_{xy}^\infty = 2A - B. \tag{2.19}$$

In this dissertation, we consider the case of a uniaxial load normal to the radial crack, namely, $\sigma_x^\infty = 0$, $\sigma_y^\infty = \sigma^\infty$ and $\sigma_{xy}^\infty = 0$, and then the constants A and B are determined as

$$A = \frac{\sigma^\infty}{4}, B = \bar{B} = 2A = \frac{\sigma^\infty}{2}. \tag{2.20}$$

Thus, the stress potentials $\varphi_2(z)$ and $\psi_2(z)$ are expanded into a standard Laurent series in region S_2 in the following form

$$\varphi_2(z) = Az + \sum_{k=1}^{\infty} c_k z^{-k}, \qquad \psi_2(z) = Bz + \sum_{k=1}^{\infty} d_k z^{-k}, \tag{2.21}$$

where $A = \frac{\sigma^{\infty}}{4}$, $B = \overline{B} = \frac{\sigma^{\infty}}{2}$.

Finally, the stress potentials $\varphi_1(z)$ and $\psi_1(z)$ in region S_1 containing a pre-existing radial crack have been expanded as series forms. In the present problem, since the intermediate matrix region S_1 contains a crack, $\varphi_1(z)$ and $\psi_1(z)$ are not analytic in the region. To solve the difficulties, let us employ analytic continuation [55] to express $\varphi_1(z)$ and $\psi_1(z)$ in terms of two new functions that are analytic in a new domain D representing the intermediate region without the radial crack. Namely, $D = S_1 - 2l$. Thus, the stress potentials $\varphi_1(z)$ and $\psi_1(z)$ can be expanded into standard Laurent series in D. In view of (2.2) and traction free condition along the crack-face $2l$, we can get the following condition

$$F_x + iF_y = -i\left[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}\right]_A^B = 0. \tag{2.22}$$

The above condition is expressed along the crack face in the upper and lower half planes as follows

$$\begin{aligned} \varphi_1(z)^+ + z\overline{\varphi_1'(z)^+} + \overline{\psi_1(z)^+} &= 0, \qquad z \in 2l^+, \\ \varphi_1(z)^- + z\overline{\varphi_1'(z)^-} + \overline{\psi_1(z)^-} &= 0, \qquad z \in 2l^-. \end{aligned} \tag{2.23}$$

Let's consider the analytic continuation across the boundary which is the real axis. In particular, if the line of symmetry is the real axis, Schwarz's Reflection Principle can be employed into any analytic function f_1 in the upper plane D^+. f_2 defined in the lower plane D^- by $f_2(z) = \overline{f_1(\bar{z})}$ is the analytic continuation of f_1 to D^- [56] . Muskhelishvili [55] defines the $f_2(z)$ as a function to have the conjugate complex value of f_1 at the point $\bar{z}$, namely $\bar{f}(z) = \overline{f(\bar{z})}$. By analytic continuation,

$\left[\overline{\varphi_1'(z)}\right]^+ = \bar{\varphi}_1'(\bar{z}) = [\bar{\varphi}_1'(z)]^-$ and $\left[\overline{\psi_1(z)}\right]^+ = \bar{\psi}_1(\bar{z}) = [\bar{\psi}_1(z)]^-$. Thus,

$$\begin{aligned}\varphi_1(z)^+ + \left[z\overline{\varphi}_1'(z) + \bar{\psi}_1(z)\right]^- &= 0, \qquad z \in 2l, \\ \varphi_1(z)^- + \left[z\overline{\varphi}_1'(z) + \bar{\psi}_1(z)\right]^+ &= 0, \qquad z \in 2l.\end{aligned} \tag{2.24}$$

By adding and subtracting the above equations, we get

$$\begin{aligned}\varphi_1(z)^+ - \left[z\overline{\varphi}_1'(z) + \bar{\psi}_1(z)\right]^+ &= \varphi_1(z)^- - \left[z\overline{\varphi}_1'(z) + \bar{\psi}_1(z)\right]^-, \qquad z \in 2l, \\ \varphi_1(z)^+ + \left[z\overline{\varphi}_1'(z) + \bar{\psi}_1(z)\right]^+ &= -\left\{\varphi_1(z)^- + \left[z\overline{\varphi}_1'(z) + \bar{\psi}_1(z)\right]^-\right\}, \qquad z \in 2l.\end{aligned} \tag{2.25}$$

First, let's consider $(2.25)_1$. From the equation $(2.25)_1$, a new analytic function is defined as follows

$$X(z) \equiv \varphi_1(z) - \left[z\overline{\varphi}_1'(z) + \bar{\psi}_1(z)\right]. \tag{2.26}$$

In view of $(2.25)_1$, the following relation holds

$$X(z)^+ = G(z)X(z)^- + f(z), \qquad z \in 2l, \tag{2.27}$$

where $G(z) = 1$ and $f(z) = 0$. The above problem is called *the problem of linear relationship* or *the Hilbert Problem* [55]. Equation $(2.25)_1$ is expressed in terms of the new function such as $X(z)^+ = X(z)^-$, which means that $X(z)$ is continuous across the crack $2l$. Thus, $X(z)$ is analytic in D representing a domain S_1 minus $2l$ and across the crack face $2l$ and can be expanded into a Laurent series in D as follows

$$X(z) = \sum_{k=-\infty}^{\infty} e_k z^k, \qquad z \in D, \tag{2.28}$$

where e_k are unknown complex coefficients.

Secondly, $(2.25)_2$ shows that the value of $\varphi_1(z) + \left[z\overline{\varphi}_1'(z) + \bar{\psi}_1(z)\right]$ has opposite sign across the crack face and this is an another *Hilbert Problem* with $G(z) = -1$ and $f(z) = 0$. By Plemelj's function and a solution given by Muskhelishvili [55], we get

$$\varphi_1(z)^+ + \left[z\overline{\varphi}_1'(z) + \bar{\psi}_1(z)\right] = \frac{P(z)}{\sqrt{(z-a)(z-b)}}\,, \tag{2.29}$$

where $P(z)$ is an arbitrary polynomial and

$$\left[\sqrt{(z-a)(z-b)}\right]^+ = -\left[\sqrt{(z-a)(z-b)}\right]^-\,,\ z \in 2l = (a,b)\,.$$

For example, if the outer radius R_1 is extended to the infinity and the value of the above equation (2.29) is finite and bounded at infinity, the polynomial $P(z)$ would be taken as z. In the case where (2.29) holds in intermediate zone S_1, $P(z)$ is taken to be 1 for easy formulation. Now, let's define another new analytic function such as

$$Y(z) \equiv \sqrt{(z-a)(z-b)}\left[\varphi_1(z) + z\overline{\varphi}_1'(z) + \bar{\psi}_1(z)\right]. \tag{2.30}$$

Similarly to $X(z)$, $Y(z)$ is continuous across the crack $2l$ ($Y(z)^+ = Y(z)^-$) and is analytic in D. Hence, the new analytic function can be expanded into a standard Laurent series as follows

$$Y(z) = \sum_{k=-\infty}^{\infty} f_k z^k, \qquad z \in D, \tag{2.31}$$

where f_k are unknown complex coefficients. Consequently, from (2.26) and (2.30), the stress potentials $\varphi_1(z)$ and $\psi_1(z)$ defined in S_1 take the following forms

$$\varphi_1(z) = \frac{1}{2\sqrt{(z-a)(z-b)}}Y(z) + \frac{1}{2}X(z), \tag{2.32}$$

and

$$\psi_1(z) = -\overline{X}(z) + \overline{\varphi_1}(z) - z\varphi_1'(z). \tag{2.33}$$

It is noted that the aim of derivations of the stress potentials $\varphi_1(z)$ and $\psi_1(z)$ in S_1 is to investigate crack behaviours around crack tips, in other words, the inward crack propagation from crack tip a to the inner interface Γ_0 and the outward propagation from the other tip b to the outer interface Γ_1. In (2.32), the term of $\frac{1}{2\sqrt{(z-a)(z-b)}}$ is a multi-valued function across the crack face but is analytic within a circular domain $|z| < a$ including $\Gamma_0 \cup S_0$ and within an outward circular domain $|z| > b$ including $\Gamma_1 \cup S_2$. Let $F(z)$ denote $\frac{1}{2\sqrt{(z-a)(z-b)}}$ and expand it into standard series forms. Since two different domains are considered, two notations, $F(z)_{11}$ and $F(z)_{12}$ represent the value of $\frac{1}{2\sqrt{(z-a)(z-b)}}$ within a domain $|z| < a$ including $\Gamma_0 \cup S_0$ and $|z| > b$ including $\Gamma_1 \cup S_2$, respectively.

First, $F(z)_{11}$ in a domain $|z| < a$ is expanded into a Taylor series as follows

$$\begin{aligned} F_{11}(z) &= \frac{1}{2\sqrt{(z-a)(z-b)}} = \frac{1}{2}(z-a)^{-\frac{1}{2}}(z-b)^{-\frac{1}{2}} \\ &= \sum_{k=0}^{\infty} g_k z^k = g_0 + g_1 z + g_2 z^2 + g_3 z^3 + g_4 z^4 + g_5 z^5 + g_6 z^6 + g_7 z^7 + \dots, \end{aligned} \tag{2.34}$$

where g_k are coefficients which are determined in terms of the crack tip position a and b. For example, the first eight coefficients are the following (see Appendix 3 for detailed derivation)

$$
\begin{aligned}
g_0 &= \frac{1}{2\sqrt{ab}}, \\
g_1 &= \left(\frac{1}{4}\frac{1}{\sqrt{a^1b^3}} + \frac{1}{4}\frac{1}{\sqrt{a^3b^1}}\right), \\
g_2 &= \left(\frac{3}{16}\frac{1}{\sqrt{a^1b^5}} + \frac{1}{8}\frac{1}{\sqrt{a^3b^3}} + \frac{3}{16}\frac{1}{\sqrt{a^5b^1}}\right), \\
g_3 &= \left(\frac{5}{32}\frac{1}{\sqrt{a^1b^7}} + \frac{3}{32}\frac{1}{\sqrt{a^3b^5}} + \frac{3}{32}\frac{1}{\sqrt{a^5b^3}} + \frac{5}{32}\frac{1}{\sqrt{a^7b^1}}\right), \\
g_4 &= \left(\frac{35}{256}\frac{1}{\sqrt{a^1b^9}} + \frac{5}{64}\frac{1}{\sqrt{a^3b^7}} + \frac{9}{128}\frac{1}{\sqrt{a^5b^5}} + \frac{5}{64}\frac{1}{\sqrt{a^7b^3}} + \frac{35}{256}\frac{1}{\sqrt{a^9b^1}}\right), \\
g_5 &= \left(\begin{array}{c} \frac{63}{512}\frac{1}{\sqrt{a^1b^{11}}} + \frac{35}{512}\frac{1}{\sqrt{a^3b^9}} + \frac{15}{256}\frac{1}{\sqrt{a^5b^7}} + \frac{15}{256}\frac{1}{\sqrt{a^7b^5}} + \frac{35}{512}\frac{1}{\sqrt{a^9b^3}} \\ +\frac{63}{512}\frac{1}{\sqrt{a^{11}b^1}} \end{array}\right), \\
g_6 &= \left(\begin{array}{c} \frac{231}{2048}\frac{1}{\sqrt{a^1b^{13}}} + \frac{63}{1024}\frac{1}{\sqrt{a^3b^{11}}} + \frac{105}{2048}\frac{1}{\sqrt{a^5b^9}} + \frac{25}{512}\frac{1}{\sqrt{a^7b^7}} + \frac{105}{2048}\frac{1}{\sqrt{a^9b^5}} \\ +\frac{63}{1024}\frac{1}{\sqrt{a^{11}b^3}} + \frac{231}{2048}\frac{1}{\sqrt{a^{13}b^1}} \end{array}\right), \\
g_7 &= \left(\begin{array}{c} \frac{3003}{28672}\frac{1}{\sqrt{a^1b^{15}}} + \frac{231}{4096}\frac{1}{\sqrt{a^3b^{13}}} + \frac{189}{4096}\frac{1}{\sqrt{a^5b^{11}}} + \frac{175}{4096}\frac{1}{\sqrt{a^7b^9}} + \frac{175}{4096}\frac{1}{\sqrt{a^9b^7}} \\ +\frac{189}{4096}\frac{1}{\sqrt{a^{11}b^5}} + \frac{231}{4096}\frac{1}{\sqrt{a^{13}b^3}} + \frac{3003}{28672}\frac{1}{\sqrt{a^{15}b^1}} \end{array}\right).
\end{aligned}
$$

Now, let's expand $F(z)_{12}$ in a domain $|z| > b$ into Laurent series as follows

$$
\begin{aligned}
F_{12}(z) &= \frac{1}{2\sqrt{(z-a)(z-b)}} = \frac{1}{2z\sqrt{(1-\frac{a}{z})(1-\frac{b}{z})}} \\
&= \frac{1}{2z}(1-\frac{a}{z})^{-\frac{1}{2}}(1-\frac{b}{z})^{-\frac{1}{2}} = \sum_{k=1}^{\infty} h_{-k}z^{-k} \\
&= \left(\begin{array}{c} h_{-1}z^{-1} + h_{-2}z^{-2} + h_{-3}z^{-3} + h_{-4}z^{-4} + h_{-5}z^{-5} \\ +h_{-6}z^{-6} + h_{-7}z^{-7} + h_{-8}z^{-8} + \ldots \end{array}\right),
\end{aligned} \tag{2.35}
$$

where h_{-k} are also coefficients which are determined in terms of the crack tip position

a and b in the following form (see Appendix 3 for detailed derivation)

$$
\begin{aligned}
h_{-1} &= \frac{1}{2},\\
h_{-2} &= \frac{1}{2}\left(\frac{a+b}{2}\right),\\
h_{-3} &= \frac{1}{2}\left(\frac{3}{8}a^2+\frac{1}{4}ab+\frac{3}{8}b^2\right),\\
h_{-4} &= \frac{1}{2}\left(\frac{5}{16}a^3+\frac{3}{16}a^2b+\frac{3}{16}ab^2+\frac{5}{16}b^3\right),\\
h_{-5} &= \frac{1}{2}\left(\frac{35}{128}a^4+\frac{5}{32}a^3b+\frac{9}{64}a^2b^2+\frac{5}{32}ab^3+\frac{35}{128}b^4\right),\\
h_{-6} &= \frac{1}{2}\left(\frac{63}{256}a^5+\frac{35}{256}a^4b+\frac{15}{128}a^3b^2+\frac{15}{128}a^2b^3+\frac{35}{256}ab^4+\frac{63}{256}b^5\right),\\
h_{-7} &= \frac{1}{2}\left(\begin{array}{c}\frac{231}{1024}a^6+\frac{63}{512}a^5b+\frac{105}{1024}a^4b^2+\frac{25}{256}a^3b^3+\frac{105}{1024}a^2b^4\\ +\frac{63}{512}ab^5+\frac{231}{1024}b^6\end{array}\right),\\
h_{-8} &= \frac{1}{2}\left(\begin{array}{c}\frac{429}{2048}a^7+\frac{231}{2048}a^6b+\frac{189}{2048}a^5b^2+\frac{175}{2048}a^4b^3+\frac{175}{2048}a^3b^4\\ +\frac{189}{2048}a^2b^5+\frac{231}{2048}ab^6+\frac{429}{2048}b^7\end{array}\right).
\end{aligned}
$$

For approximation to $F(x)$, the first 8 coefficients of $F_{11}(z)$ in (2.34) are taken. Let $\varphi_{11}(z)$ and $\psi_{11}(z)$ denote stress potentials $\varphi_1(z)$ and $\psi_1(z)$ in a domain $|z| < a$ around the crack tip a. Substituting (2.28), (2.31) and $(2.34)_2$ into (2.32) for $\varphi_{11}(z)$ and (2.33) for $\psi_{11}(z)$ yields (see Appendix 3 for detailed derivation)

$$
\begin{aligned}
\varphi_{11}(z) &= p_7z^7+p_6z^6+p_5z^5+p_4z^4+p_3z^3+p_2z^2+p_1z^1+p_0\\
&\quad +p_{-1}z^{-1}+p_{-2}z^{-2}+p_{-3}z^{-3}+p_{-4}z^{-4},
\end{aligned} \tag{2.36}
$$

and

$$
\psi_{11}(z) = -6p_7z^7-5p_6z^6-4p_5z^5-(3p_4+\overline{e}_4)z^4-(2p_3+\overline{e}_3)z^3 \tag{2.37}
$$

$$
\begin{aligned}
&- (p_2 + \overline{e}_2)\, z^2 - \overline{e}_1 z^1 + (p_0 - \overline{e}_0) + (2p_{-1} - \overline{e}_{-1})\, z^{-1} \\
&+ (3p_{-2} - \overline{e}_{-2})\, z^{-2} + (4p_{-3} - \overline{e}_{-3})\, z^{-3} + (5p_{-4} - \overline{e}_{-4})\, z^{-4},
\end{aligned}
$$

where

$$
\begin{aligned}
p_6 &= (g_7 f_{-1} + g_6 f_0 + g_5 f_1 + g_4 f_2 + g_3 f_3 + g_2 f_4)\,, \\
p_5 &= (g_7 f_{-2} + g_6 f_{-1} + g_5 f_0 + g_4 f_1 + g_3 f_2 + g_2 f_3 + g_1 f_4)\,, \\
p_4 &= \left(g_7 f_{-3} + g_6 f_{-2} + g_5 f_{-1} + g_4 f_0 + g_3 f_1 + g_2 f_2 + g_1 f_3 + g_0 f_4 + \frac{1}{2} e_4\right), \\
p_3 &= \left(g_6 f_{-3} + g_5 f_{-2} + g_4 f_{-1} + g_3 f_0 + g_2 f_1 + g_1 f_2 + g_0 f_3 + \frac{1}{2} e_3\right), \\
p_2 &= \left(g_5 f_{-3} + g_4 f_{-2} + g_3 f_{-1} + g_2 f_0 + g_1 f_1 + g_0 f_2 + \frac{1}{2} e_2\right), \\
p_1 &= \left(g_4 f_{-3} + g_3 f_{-2} + g_2 f_{-1} + g_1 f_0 + g_0 f_1 + \frac{1}{2} e_1\right), \\
p_0 &= \left(g_3 f_{-3} + g_2 f_{-2} + g_1 f_{-1} + g_0 f_0 + \frac{1}{2} e_0\right), \\
p_{-1} &= \left(g_2 f_{-3} + g_1 f_{-2} + g_0 f_{-1} + \frac{1}{2} e_{-1}\right), \\
p_{-2} &= \left(g_1 f_{-3} + g_0 f_{-2} + \frac{1}{2} e_{-2}\right), \\
p_{-3} &= \left(g_0 f_{-3} + \frac{1}{2} e_{-3}\right), \\
p_{-4} &= \left(\frac{1}{2} e_{-4}\right).
\end{aligned}
$$

In the similar way, the first 8 coefficients of $F_{12}(z)$ in (2.35) are taken for approximation to $F(x)$ in a domain $|z| > b$. Now $\varphi_{12}(z)$ and $\psi_{12}(z)$ denoting stress potentials $\varphi_1(z)$ and $\psi_1(z)$ in a domain $|z| > b$ around a crack tip b are considered. Substituting (2.28), (2.31) and $(2.35)_2$ into (2.32) for $\varphi_{12}(z)$ and (2.33) for $\psi_{12}(z)$ leads to the following form (see Appendix 3 for detailed derivation)

$$\varphi_{12}(z) = q_3 z^3 + q_2 z^2 + q_1 z^1 + q_0 + q_{-1} z^{-1} + q_{-2} z^{-2} + q_{-3} z^{-3} + q_{-4} z^{-4} + q_{-5} z^{-5} + q_{-6} z^{-6}, \tag{2.38}$$

and

$$\begin{aligned} \psi_{12}(z) &= 7q_{-6} z^{-6} + 6q_{-5} z^{-5} + (5q_{-4} - \bar{e}_{-4})\, z^{-4} + (4q_{-3} - \bar{e}_{-3})\, z^{-3} \\ &\quad + (3q_{-2} - \bar{e}_{-2})\, z^{-2} + (2q_{-1} - \bar{e}_{-1})\, z^{-1} + (q_0 - \bar{e}_0) - \bar{e}_1 z^1 \\ &\quad + (-\bar{e}_2 - q_2)\, z^2 + (-\bar{e}_3 - 2q_3)\, z^3 - \bar{e}_4 z^4, \end{aligned} \tag{2.39}$$

where

$$\begin{aligned} q_3 &= \left(h_{-1} f_4 + \frac{1}{2} e_3 \right), \\ q_2 &= \left(h_{-2} f_4 + h_{-1} f_3 + \frac{1}{2} e_2 \right), \\ q_1 &= \left(h_{-3} f_4 + h_{-2} f_3 + h_{-1} f_2 + \frac{1}{2} e_1 \right), \\ q_0 &= \left(h_{-4} f_4 + h_{-3} f_3 + h_{-2} f_2 + h_{-1} f_1 + \frac{1}{2} e_0 \right), \\ q_{-1} &= \left(h_{-5} f_4 + h_{-4} f_3 + h_{-3} f_2 + h_{-2} f_1 + h_{-1} f_0 + \frac{1}{2} e_{-1} \right), \\ q_{-2} &= \left(h_{-6} f_4 + h_{-5} f_3 + h_{-4} f_2 + h_{-3} f_1 + h_{-2} f_0 + h_{-1} f_{-1} + \frac{1}{2} e_{-2} \right), \\ q_{-3} &= \left(h_{-7} f_4 + h_{-6} f_3 + h_{-5} f_2 + h_{-4} f_1 + h_{-3} f_0 + h_{-2} f_{-1} + h_{-1} f_{-2} + \frac{1}{2} e_{-3} \right), \\ q_{-4} &= \left(\begin{array}{c} h_{-8} f_4 + h_{-7} f_3 + h_{-6} f_2 + h_{-5} f_1 + h_{-4} f_0 + h_{-3} f_{-1} + h_{-2} f_{-2} \\ + h_{-1} f_{-3} + \frac{1}{2} e_{-4} \end{array} \right), \end{aligned}$$

$$q_{-5} = \begin{pmatrix} h_{-8}f_3 + h_{-7}f_2 + h_{-6}f_1 + h_{-5}f_0 + h_{-4}f_{-1} + h_{-3}f_{-2} + h_{-2}f_{-3} \\ +h_{-1}f_{-4} + \frac{1}{2}e_{-5} \end{pmatrix},$$

$$q_{-6} = \left(h_{-8}f_2 + h_{-7}f_1 + h_{-6}f_0 + h_{-5}f_{-1} + h_{-4}f_{-2} + h_{-3}f_{-3} + h_{-2}f_{-4} + \frac{1}{2}e_{-6}\right).$$

Finally, all the stress potentials $\varphi_0(z)$ and $\psi_0(z)$ in S_0, $\varphi_{11}(z)$ and $\psi_{11}(z)$ in a region $(R_0 \leq |z| < a) \subset S_1$, $\varphi_{12}(z)$ and $\psi_{12}(z)$ in a region $(b < |z| \leq R_1) \subset S_1$ and $\varphi_2(z)$ and $\psi_2(z)$ in S_2 are expressed solely in terms of finite series coefficients. Of the coefficients, a_k, b_k, c_k, d_k, e_k and f_k are unknown coefficients while g_k and h_k are known coefficients determined in terms of crack position a and b. In addition, p_k are functions of e_k, f_k and g_k, and q_k are determined by e_k, f_k and h_k. Consequently, this boundary problem is reduced to determining the unknown complex coefficients such that the boundary condition (2.7), (2.12), (2.13) and (2.14) along $\partial\Gamma_0$ and $\partial\Gamma_1$ hold.

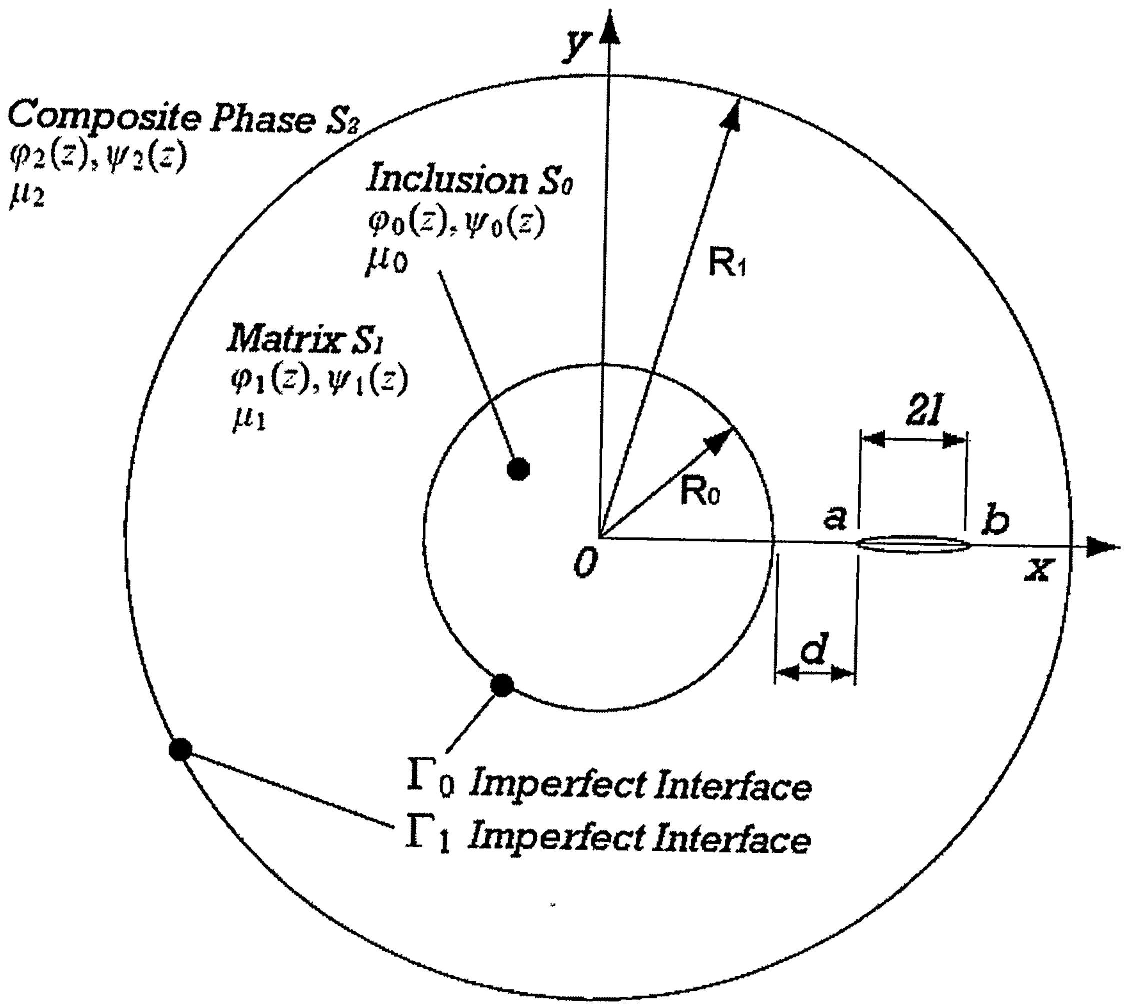


Figure 2.1 Three-phase circular inclusion model with a radial crack

CHAPTER 3
Algebraic Equations

3.1 Introduction

In this chapter, the eight stress potentials expressed in terms of finite series coefficients in Chapter 2 are substituted into the four boundary conditions (2.7), (2.12), (2.13) and (2.14). By comparing coefficients of powers of z, a set of 38 coupled linear algebraic equations are derived to determine 38 unknown coefficients of the stress potentials. In particular, uniaxial tensile loads are considered as the prescribed remote load normal to crack surface since Mode I stress intensity factor is the most critical factor leading composite material failure. Since a pre-existing crack is located and moving on the real axis, namely, the x-axis all the coefficients that we consider are assumed on the real axis as well. Henceforth, all the complex coefficients of stress potentials and their respective conjugates are the same. The imperfect interfaces are assumed homogeneous, which means the spring-factor type interface parameters do not vary along the interfaces.

3.2 General Algebraic Equations on Inner Interface Γ_0

There are two boundary conditions along the inner interface $\partial\Gamma_0$: one is the traction continuity and the other is the displacement jump across the interface. First, let's consider the traction continuity condition. Substituting the stress potentials $\varphi_0(z)$, $\psi_0(z)$, $\varphi_{11}(z)$ and $\psi_{11}(z)$ into (2.7) gives the following expression (See Appendix 4 for the details)

$$\frac{42}{R_0^2}p_7 z^8 + \frac{30}{R_0^2}p_6 z^7 + \left(-35p_7 + \frac{20}{R_0^2}p_5\right) z^6 \tag{3.1}$$

$$+\left(\frac{1}{R_0^2}\left(12p_4+4e_4\right)-24p_6-\frac{4p_{-4}}{R_0^{10}}\right)z^5+\left(\frac{1}{R_0^2}\left(6p_3+3e_3\right)-15p_5-\frac{3p_{-3}}{R_0^8}\right)z^4$$
$$+\left(\frac{1}{R_0^2}\left(2p_2+2e_2\right)-8p_4-\frac{2p_{-2}}{R_0^6}\right)z^3+\left(\frac{1}{R_0^2}e_1-3p_3-\frac{p_{-1}}{R_0^4}\right)z^2+(0)\,z$$
$$+\left(\frac{1}{R_0^2}\left(2p_{-1}-e_{-1}\right)+2p_1\right)$$
$$+\left(\frac{1}{R_0^2}\left(12p_{-3}-3e_{-3}\right)-3p_{-1}+3R_0^4p_3\right)z^{-2}$$
$$+\left(\frac{1}{R_0^2}\left(20p_{-4}-4e_{-4}\right)-8p_{-2}+4R_0^6p_4\right)z^{-3}+\left(5R_0^8p_5-15p_{-3}\right)z^{-4}$$
$$+\left(6R_0^{10}p_6-24p_{-4}\right)z^{-5}+7p_7R_0^{12}z^{-6}$$
$$=\ -\frac{7}{R_0^2}b_7z^8-\frac{6}{R_0^2}b_6z^7+\left(-35a_7-\frac{5}{R_0^2}b_5\right)z^6+\left(-24a_6-\frac{4}{R_0^2}b_4\right)z^5$$
$$+\left(-15a_5-\frac{3}{R_0^2}b_3\right)z^4+\left(-8a_4-\frac{2}{R_0^2}b_2\right)z^3+\left(-3a_3-\frac{1}{R_0^2}b_1\right)z^2+(0)\,z$$
$$+2a_1+2R_0^2a_2z^{-1}+3R_0^4a_3z^{-2}+4R_0^6a_4z^{-3}+5R_0^8a_5z^{-4}+6R_0^{10}a_6z^{-5}+7R_0^{12}a_7z^{-6}.$$

Also, deploying the stress potentials into (2.12) leads to the expression (See Appendix 4 for the details)

$$(m_0-n_0)(\kappa_1+1)R_0p_7z^6+\left((m_0-n_0)(\kappa_1+1)R_0p_6+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0^9}p_{-4}\right)z^5$$
$$+\left((m_0-n_0)(\kappa_1+1)R_0p_5+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0^7}p_{-3}\right)z^4 \qquad (3.2)$$
$$+\left((m_0-n_0)(\kappa_1+1)R_0p_4+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0^5}p_{-2}\right)z^3$$
$$+\left((m_0-n_0)(\kappa_1+1)R_0p_3+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0^3}p_{-1}\right)z^2$$
$$+\left((m_0-n_0)(\kappa_1+1)R_0p_2+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0}p_0\right)z$$
$$+2m_0(\kappa_1+1)R_0p_1+\left[(m_0-n_0)(\kappa_1+1)R_0p_0+(m_0+n_0)(\kappa_1+1)R_0^3p_2\right]z^{-1}$$
$$+\left[(m_0-n_0)(\kappa_1+1)R_0p_{-1}+(m_0+n_0)(\kappa_1+1)R_0^5p_3\right]z^{-2}$$
$$+\left[(m_0+n_0)(\kappa_1+1)R_0^7p_4+(m_0-n_0)(\kappa_1+1)R_0p_{-2}\right]z^{-3}$$

$$+\left[(m_0+n_0)(\kappa_1+1)R_0^9p_5+(m_0-n_0)(\kappa_1+1)R_0p_{-3}\right]z^{-4}$$
$$+\left[(m_0-n_0)(\kappa_1+1)R_0p_{-4}+(m_0+n_0)(\kappa_1+1)R_0^{11}p_6\right]z^{-5}$$
$$+(m_0+n_0)(\kappa_1+1)R_0^{13}p_7z^{-6}$$

$$\begin{aligned}
= \quad & \left(-4\mu_1\frac{7}{R_0^2}b_7+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_7\right)z^8 \\
& +\left(-4\mu_1\frac{6}{R_0^2}b_6+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_6\right)z^7 \\
& +\left(\begin{array}{c} 4\mu_1\left(-35a_7-\frac{5}{R_0^2}b_5\right)+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_7 \\ +7(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_7+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_5 \end{array}\right)z^6 \\
& +\left(\begin{array}{c} 4\mu_1\left(-24a_6-\frac{4}{R_0^2}b_4\right)+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_6 \\ +6(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_6+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_4 \end{array}\right)z^5 \\
& +\left(\begin{array}{c} 4\mu_1\left(-15a_5-\frac{3}{R_0^2}b_3\right)+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_5 \\ +5(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_5+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_3 \end{array}\right)z^4 \\
& +\left(\begin{array}{c} 4\mu_1\left(-8a_4-\frac{2}{R_0^2}b_2\right)+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_4 \\ +4(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_4+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_2 \end{array}\right)z^3 \\
& +\left(\begin{array}{c} 4\mu_1\left(-3a_3-\frac{1}{R_0^2}b_1\right)+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_3+\frac{2\mu_1(m_0+n_0)(\varepsilon_2-i\varepsilon_3)}{R_0} \\ +3(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_3+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_1 \end{array}\right)z^2 \\
& +\left(\begin{array}{c} (m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_2+(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})\frac{1}{R_0}a_0 \\ +(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_0+2(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_2 \end{array}\right)z \\
& +\left(\begin{array}{c} 8\mu_1a_1+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_1+R_0(m_0-n_0)(1-\frac{\mu_1}{\mu_0})a_1 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_1+4m_0R_0\mu_1\varepsilon_1+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_1 \end{array}\right) \\
& +\left(\begin{array}{c} 8\mu_1R_0^2a_2+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_0+2(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^3a_2 \\ +(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0b_0+(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^3a_2 \end{array}\right)z^{-1}
\end{aligned}$$

$$+\left(\begin{array}{c} 12\mu_1 R_0^4 a_3 + 3(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^5 a_3 + (m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^3 b_1 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^5 a_3 + 2\mu_1(m_0-n_0)(\varepsilon_2+i\varepsilon_3)R_0^3 \end{array}\right) z^{-2}$$

$$+\left(\begin{array}{c} 16\mu_1 R_0^6 a_4 + 4(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^7 a_4 + (m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^5 b_2 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^7 a_4 \end{array}\right) z^{-3}$$

$$+\left(\begin{array}{c} 20\mu_1 R_0^8 a_5 + 5(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^9 a_5 + (m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^7 b_3 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^9 a_5 \end{array}\right) z^{-4}$$

$$+\left(\begin{array}{c} 24\mu_1 R_0^{10} a_6 + 6(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^{11} a_6 + (m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^9 b_4 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^{11} a_6 \end{array}\right) z^{-5}$$

$$+\left(\begin{array}{c} 28\mu_1 R_0^{12} a_7 + 7(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^{13} a_7 + (m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^{11} b_5 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^{13} a_7 \end{array}\right) z^{-6}$$

$$+(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^{13} b_6 z^{-7} + (m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^{15} b_7 z^{-8}.$$

By comparing coefficients of the same power of z, from z^5 to z^{-4} in (3.1), we get the following set of coupled linear algebraic equations

$$\left(\frac{12}{R_0^2}p_4 + \frac{4}{R_0^2}e_4 - 24p_6 - \frac{4p_{-4}}{R_0^{10}}\right) z^5 = \left(-24a_6 - \frac{4}{R_0^2}b_4\right) z^5, \tag{3.3}$$

$$\left(\frac{6}{R_0^2}p_3 + \frac{3}{R_0^2}e_3 - 15p_5 - \frac{3p_{-3}}{R_0^8}\right) z^4 = \left(-15a_5 - \frac{3}{R_0^2}b_3\right) z^4, \tag{3.4}$$

$$\left(\frac{2}{R_0^2}p_2 + \frac{2}{R_0^2}e_2 - 8p_4 - \frac{2p_{-2}}{R_0^6}\right) z^3 = \left(-8a_4 - \frac{2}{R_0^2}b_2\right) z^3, \tag{3.5}$$

$$\left(\frac{1}{R_0^2}e_1 - 3p_3 - \frac{p_{-1}}{R_0^4}\right) z^2 = \left(-3a_3 - \frac{1}{R_0^2}b_1\right) z^2, \tag{3.6}$$

$$\left(\frac{2}{R_0^2}p_{-1} - \frac{1}{R_0^2}e_{-1} + 2p_1\right) = 2a_1, \tag{3.7}$$

$$\left(2R_0^2p_2+\frac{6}{R_0^2}p_{-2}-\frac{2}{R_0^2}e_{-2}\right)z^{-1}=2R_0^2a_2z^{-1}, \tag{3.8}$$

$$\left(\frac{12}{R_0^2}p_{-3}-\frac{3}{R_0^2}e_{-3}-3p_{-1}+3R_0^4p_3\right)z^{-2}=3R_0^4a_3z^{-2}, \tag{3.9}$$

$$\left(\frac{20}{R_0^2}p_{-4}-\frac{4}{R_0^2}e_{-4}-8p_{-2}+4R_0^6p_4\right)z^{-3}=4R_0^6a_4z^{-3}, \tag{3.10}$$

$$\left(5R_0^8p_5-15p_{-3}\right)z^{-4}=5R_0^8a_5z^{-4}. \tag{3.11}$$

It is noted that the coefficients of z^1 are 0 in both sides $(0*z^1=0*z^1)$.

Similarly, comparing coefficients of the same power of z, from z^5 to z^{-4} (3.2) gives another set of coupled linear algebraic equations as following

$$\begin{pmatrix} (m_0-n_0)(\kappa_1+1)R_0p_6 \\ +(m_0+n_0)(\kappa_1+1)\frac{1}{R_0^9}p_{-4} \end{pmatrix} z^5 = \begin{pmatrix} 4\mu_1\left(-24a_6-\frac{4}{R_0^2}b_4\right) \\ +(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_6 \\ +6(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_6 \\ +(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_4 \end{pmatrix} z^5, \tag{3.12}$$

$$\begin{pmatrix} (m_0-n_0)(\kappa_1+1)R_0p_5 \\ +(m_0+n_0)(\kappa_1+1)\frac{1}{R_0^7}p_{-3} \end{pmatrix} z^4 = \begin{pmatrix} 4\mu_1\left(-15a_5-\frac{3}{R_0^2}b_3\right) \\ +(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_5 \\ +5(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_5 \\ +(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_3 \end{pmatrix} z^4, \tag{3.13}$$

$$\begin{pmatrix} (m_0-n_0)(\kappa_1+1)R_0p_4 \\ +(m_0+n_0)(\kappa_1+1)\frac{1}{R_0^5}p_{-2} \end{pmatrix} z^3 = \begin{pmatrix} 4\mu_1\left(-8a_4-\frac{2}{R_0^2}b_2\right) \\ +(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_4 \\ +4(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_4 \\ +(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_2 \end{pmatrix} z^3, \tag{3.14}$$

$$\begin{pmatrix} (m_0-n_0)(\kappa_1+1)R_0p_3 \\ +(m_0+n_0)(\kappa_1+1)\frac{1}{R_0^3}p_{-1} \end{pmatrix} z^2 = \begin{pmatrix} 4\mu_1\left(-3a_3-\frac{1}{R_0^2}b_1\right)+\frac{2\mu_1(m_0+n_0)(\varepsilon_2-i\varepsilon_3)}{R_0} \\ +(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_3 \\ +3(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_3 \\ +(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_1 \end{pmatrix} z^2, \tag{3.15}$$

$$\begin{pmatrix} (m_0-n_0)(\kappa_1+1)R_0p_2 \\ +(m_0+n_0)(\kappa_1+1)\frac{1}{R_0}p_0 \end{pmatrix} z = \begin{pmatrix} (m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_2 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})\frac{1}{R_0}a_0 \\ +(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_0 \\ +2(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_2 \end{pmatrix} z, \tag{3.16}$$

$$2m_0(\kappa_1+1)R_0p_1 = \begin{pmatrix} 8\mu_1a_1+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_1 \\ +R_0(m_0-n_0)(1-\frac{\mu_1}{\mu_0})a_1 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_1 \\ +4m_0R_0\mu_1\varepsilon_1+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_1 \end{pmatrix}, \tag{3.17}$$

$$\begin{pmatrix} (m_0-n_0)(\kappa_1+1)R_0p_0 \\ +(m_0+n_0)(\kappa_1+1)R_0^3p_2 \end{pmatrix} z^{-1} = \begin{pmatrix} 8\mu_1R_0^2a_2+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_0 \\ +2(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^3a_2 \\ +(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0b_0 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^3a_2 \end{pmatrix} z^{-1}, \tag{3.18}$$

$$\begin{pmatrix} (m_0-n_0)(\kappa_1+1)R_0p_{-1} \\ +(m_0+n_0)(\kappa_1+1)R_0^5p_3 \end{pmatrix} z^{-2} = \begin{pmatrix} 12\mu_1R_0^4a_3+3(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^5a_3 \\ +(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^3b_1 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^5a_3 \\ +2\mu_1(m_0-n_0)(\varepsilon_2+i\varepsilon_3)R_0^3 \end{pmatrix} z^{-2}, \tag{3.19}$$

$$\begin{pmatrix} (m_0+n_0)(\kappa_1+1)R_0^7p_4 \\ +(m_0-n_0)(\kappa_1+1)R_0p_{-2} \end{pmatrix} z^{-3} = \begin{pmatrix} 16\mu_1R_0^6a_4+4(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^7a_4 \\ +(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^5b_2 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^7a_4 \end{pmatrix} z^{-3}, \tag{3.20}$$

$$\begin{pmatrix} (m_0+n_0)(\kappa_1+1)R_0^9p_5 \\ +(m_0-n_0)(\kappa_1+1)R_0p_{-3} \end{pmatrix} z^{-4} = \begin{pmatrix} 20\mu_1R_0^8a_5+5(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^9a_5 \\ +(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^7b_3 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^9a_5 \end{pmatrix} z^{-4}. \tag{3.21}$$

3.3 General Algebraic Equations on Outer Interface Γ_1

Along the outer interface Γ_1, there are also two boundary conditions as addressed in the previous section. The traction continuity condition is considered first. Substituting the stress potentials $\varphi_2(z)$, $\psi_2(z)$, $\varphi_{12}(z)$ and $\psi_{12}(z)$ into (2.13) we have the following expression (See Appendix 4 for the details)

$$-\frac{8}{R_1^{18}}c_8z^9-\frac{7}{R_1^{16}}c_7z^8-\frac{6}{R_1^{14}}c_6z^7-\frac{5}{R_1^{12}}c_5z^6-\frac{4}{R_1^{10}}c_4z^5-\frac{3}{R_1^{8}}c_3z^4 \tag{3.22}$$

$$
\begin{aligned}
&-\frac{2}{R_1^6}c_2z^3+\left(-\frac{1}{R_1^4}c_1-\frac{1}{R_1^2}B\right)z^2+\left(2A+\frac{1}{R_1^2}d_1\right)+\frac{2}{R_1^2}d_2z^{-1}\\
&+\left(-3c_1+\frac{3}{R_1^2}d_3\right)z^{-2}+\left(-8c_2+\frac{4}{R_1^2}d_4\right)z^{-3}+\left(-15c_3+\frac{5}{R_1^2}d_5\right)z^{-4}\\
&+\left(-24c_4+\frac{6}{R_1^2}d_6\right)z^{-5}+\left(-35c_5+\frac{7}{R_1^2}d_7\right)z^{-6}+\left(-48c_6+\frac{8}{R_1^2}d_8\right)z^{-7}\\
&-63c_7z^{-8}-80c_8z^{-9}\\
=\;&\left(-6q_{-6}\frac{1}{R_1^{14}}z^7\right)+\left(-5q_{-5}\frac{1}{R_1^{12}}z^6\right)+\left(4\overline{e}_4\frac{1}{R_1^2}-4q_{-4}\frac{1}{R_1^{10}}\right)z^5\\
&+\left(-3\left(-\overline{e}_3-2q_3\right)\frac{1}{R_1^2}-3q_{-3}\frac{1}{R_1^8}\right)z^4+\left(-2\left(-\overline{e}_2-q_2\right)\frac{1}{R_1^2}-2q_{-2}\frac{1}{R_1^6}\right)z^3\\
&+\left(\overline{e}_1\frac{1}{R_1^2}-q_{-1}\frac{1}{R_1^4}+3q_3-6q_3\right)z^2+\left(2q_2-2q_2\right)z\\
&+\left(q_1+q_1+\left(2q_{-1}-\overline{e}_{-1}\right)\frac{1}{R_1^2}\right)+\left(2q_2R_1^2+2\left(3q_{-2}-\overline{e}_{-2}\right)\frac{1}{R_1^2}\right)z^{-1}\\
&+\left(3q_3R_1^4-q_{-1}-2q_{-1}+3\left(4q_{-3}-\overline{e}_{-3}\right)\frac{1}{R_1^2}\right)z^{-2}\\
&+\left(-2q_{-2}+4\left(5q_{-4}-\overline{e}_{-4}\right)\frac{1}{R_1^2}-6q_{-2}\right)z^{-3}\\
&+\left(-3q_{-3}+30q_{-5}\frac{1}{R_1^2}-12q_{-3}\right)z^{-4}+\left(-4q_{-4}+42q_{-6}\frac{1}{R_1^2}-20q_{-4}\right)z^{-5}\\
&+\left(-5q_{-5}-30q_{-5}\right)z^{-6}+\left(-6q_{-6}-42q_{-6}\right)z^{-7}.
\end{aligned}
$$

Finally, with $\varphi_2(z)$, $\psi_2(z)$, $\varphi_{12}(z)$ and $\psi_{12}(z)$ substituted into (2.14), the last boundary condition (2.14) yields the following expression (See Appendix 4 for the details)

$$
\begin{aligned}
&\left(m_1+n_1\right)\left(k_1+1\right)q_{-6}\frac{1}{R_1^{13}}z^7+\left(m_1+n_1\right)\left(k_1+1\right)q_{-5}\frac{1}{R_1^{11}}z^6\\
&+\left(m_1+n_1\right)\left(k_1+1\right)q_{-4}\frac{1}{R_1^9}z^5+\left(m_1+n_1\right)\left(k_1+1\right)q_{-3}\frac{1}{R_1^7}z^4\\
&+\left(m_1+n_1\right)\left(k_1+1\right)q_{-2}\frac{1}{R_1^5}z^3\\
&+\left(\left(m_1-n_1\right)\left(k_1+1\right)R_1q_3+\left(m_1+n_1\right)\left(k_1+1\right)q_{-1}\frac{1}{R_1^3}\right)z^2
\end{aligned}
\tag{3.23}
$$

$$+\left((m_1-n_1)(k_1+1)R_1q_2+(m_1+n_1)(k_1+1)\frac{1}{R_1}q_0\right)z+2m_1(k_1+1)R_1q_1$$
$$+\left((m_1-n_1)(k_1+1)R_1q_0+(m_1+n_1)(k_1+1)R_1^3q_2\right)z^{-1}$$
$$+\left((m_1-n_1)(k_1+1)R_1q_{-1}+(m_1+n_1)(k_1+1)R_1^5q_3\right)z^{-2}$$
$$+(m_1-n_1)(k_1+1)R_1q_{-2}z^{-3}+(m_1-n_1)(k_1+1)R_1q_{-3}z^{-4}$$
$$+(m_1-n_1)(k_1+1)R_1q_{-4}z^{-5}+(m_1-n_1)(k_1+1)R_1q_{-5}z^{-6}$$
$$+(m_1-n_1)(k_1+1)R_1q_{-6}z^{-7}$$
$$=\left(4\mu_1\frac{4}{R_1^{10}}c_4-(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\frac{4}{R_1^9}c_4+(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)c_4\frac{1}{R_1^9}\right)z^5$$
$$+\begin{pmatrix} 4\mu_1\frac{3}{R_1^8}c_3+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_5\frac{1}{R_1^9}-\frac{3}{R_1^7}c_3\right) \\ +(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)c_3\frac{1}{R_1^7}\end{pmatrix}z^4$$
$$+\begin{pmatrix} 4\mu_1\frac{2}{R_1^6}c_2+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_4\frac{1}{R_1^7}-\frac{2}{R_1^5}c_2\right) \\ +(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)c_2\frac{1}{R_1^5}\end{pmatrix}z^3$$
$$+\begin{pmatrix} 4\mu_1\left(\frac{1}{R_1^4}c_1+\frac{1}{R_1^2}B\right)+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_3\frac{1}{R_1^5}-\frac{1}{R_1^3}c_1\right) \\ +(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)c_1\frac{1}{R_1^3}+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)B\frac{1}{R_1}\end{pmatrix}z^2$$
$$+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)d_2\frac{1}{R^3}z$$
$$-4\mu_1\left(2A+\frac{1}{R_1^2}d_1\right)+(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)AR_1$$
$$+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(AR_1+d_1\frac{1}{R_1}\right)$$
$$+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(AR_1+d_1\frac{1}{R_1}\right)+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)AR_1$$
$$+\left(-4\mu_1\frac{2}{R_1^2}d_2+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)d_2\frac{1}{R_1}\right)z^{-1}$$
$$+\begin{pmatrix} -4\mu_1\left(-3c_1+\frac{3}{R_1^2}d_3\right)+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)BR_1^3 \\ +(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)R_1c_1 \\ +(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_3\frac{1}{R_1}-R_1c_1\right)\end{pmatrix}z^{-2}$$

$$+\left(\begin{array}{c} -4\mu_1\left(-8c_2+\frac{4}{R_1^2}d_4\right)+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)R_1c_2 \\ +(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_4\frac{1}{R_1}-2R_1c_2\right) \end{array}\right)z^{-3}$$
$$+\left(\begin{array}{c} -4\mu_1\left(-15c_3+\frac{5}{R_1^2}d_5\right)+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)R_1c_3 \\ +(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_5\frac{1}{R_1}-3R_1c_3\right) \end{array}\right)z^{-4}$$
$$+\left(\begin{array}{c} -4\mu_1\left(-24c_4+\frac{6}{R_1^2}d_6\right)+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)R_1c_4 \\ -(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)4R_1c_4 \end{array}\right)z^{-5}.$$

By comparing coefficients of the same power of z, from z^5 to z^{-4} in (3.22), we get the following set of coupled linear algebraic equations

$$-\frac{4}{R_1^{10}}c_4z^5=\left(4e_4\frac{1}{R_1^2}-4q_{-4}\frac{1}{R_1^{10}}\right)z^5, \tag{3.24}$$

$$-\frac{3}{R_1^8}c_3z^4=\left(+3e_3\frac{1}{R_1^2}+6q_3\frac{1}{R_1^2}-3q_{-3}\frac{1}{R_1^8}\right)z^4, \tag{3.25}$$

$$-\frac{2}{R_1^6}c_2z^3=\left(+2e_2\frac{1}{R_1^2}+2q_2\frac{1}{R_1^2}-2q_{-2}\frac{1}{R_1^6}\right)z^3, \tag{3.26}$$

$$\left(-\frac{1}{R_1^4}c_1-\frac{1}{R_1^2}B\right)z^2=\left(e_1\frac{1}{R_1^2}-q_{-1}\frac{1}{R_1^4}-3q_3\right)z^2, \tag{3.27}$$

$$\left(2A+\frac{1}{R_1^2}d_1\right)=\left(+2q_1+2q_{-1}\frac{1}{R_1^2}-e_{-1}\frac{1}{R_1^2}\right), \tag{3.28}$$

$$\frac{2}{R_1^2}d_2z^{-1}=\left(2q_2R_1^2+6q_{-2}\frac{1}{R_1^2}-2e_{-2}\frac{1}{R_1^2}\right)z^{-1}, \tag{3.29}$$

$$\left(-3c_1+\frac{3}{R_1^2}d_3\right)z^{-2}=\left(3q_3R_1^4-3q_{-1}+12q_{-3}\frac{1}{R_1^2}-3e_{-3}\frac{1}{R_1^2}\right), \tag{3.30}$$

$$\left(-8c_2+\frac{4}{R_1^2}d_4\right)z^{-3}=\left(4\left(5q_{-4}-e_{-4}\right)\frac{1}{R_1^2}-8q_{-2}\right)z^{-3}, \tag{3.31}$$

$$\left(-15c_3+\frac{5}{R_1^2}d_5\right)z^{-4}=\left(+30q_{-5}\frac{1}{R_1^2}-15q_{-3}\right)z^{-4}. \tag{3.32}$$

Similar to the case of traction continuation along Γ_0, the coefficients of z^1 are 0 in both sides ($0 * z^1 = 0 * z^1$).

Finally, the last set of coupled linear algebraic equations is obtained by comparing coefficients of the same power of z, from z^5 to z^{-4} in (3.23)

$$(m_1+n_1)(k_1+1)\,q_{-4}\frac{1}{R_1^9}z^5 = \begin{pmatrix} 4\mu_1\frac{4}{R_1^{10}}c_4 - (m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\frac{4}{R_1^9}c_4 \\ +(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)c_4\frac{1}{R_1^9} \end{pmatrix} z^5, \quad (3.33)$$

$$(m_1+n_1)(k_1+1)\,q_{-3}\frac{1}{R_1^7}z^4 = \begin{pmatrix} 4\mu_1\frac{3}{R_1^8}c_3 + (m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_5\frac{1}{R_1^9}-\frac{3}{R_1^7}c_3\right) \\ +(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)c_3\frac{1}{R_1^7} \end{pmatrix} z^4; \tag{3.34}$$

$$(m_1+n_1)(k_1+1)\,q_{-2}\frac{1}{R_1^5}z^3 = \begin{pmatrix} 4\mu_1\frac{2}{R_1^6}c_2 + (m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_4\frac{1}{R_1^7}-\frac{2}{R_1^5}c_2\right) \\ +(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)c_2\frac{1}{R_1^5} \end{pmatrix} z^3, \tag{3.35}$$

$$\begin{pmatrix} (m_1-n_1)(k_1+1)\,R_1q_3 \\ +(m_1+n_1)(k_1+1)\,q_{-1}\frac{1}{R_1^3} \end{pmatrix} z^2 = \begin{pmatrix} 4\mu_1\left(\frac{1}{R_1^4}c_1+\frac{1}{R_1^2}B\right) \\ +(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_3\frac{1}{R_1^5}-\frac{1}{R_1^3}c_1\right) \\ +(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)c_1\frac{1}{R_1^3} \\ +(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)B\frac{1}{R_1} \end{pmatrix} z^2, \tag{3.36}$$

$$\left((m_1-n_1)(k_1+1)R_1q_2+(m_1+n_1)(k_1+1)\frac{1}{R_1}q_0\right)z=(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)d_2\frac{1}{R^3}z, \tag{3.37}$$

$$2m_1(k_1+1)R_1q_1=\begin{pmatrix} 4\mu_1\left(2A+\frac{1}{R_1^2}d_1\right)+(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)AR_1 \\ +(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(AR_1+d_1\frac{1}{R_1}\right) \\ +(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(AR_1+d_1\frac{1}{R_1}\right) \\ +(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)AR_1 \end{pmatrix}, \tag{3.38}$$

$$\begin{pmatrix} (m_1-n_1)(k_1+1)R_1q_0 \\ +(m_1+n_1)(k_1+1)R_1^3q_2 \end{pmatrix}z^{-1}=\left(-4\mu_1\frac{2}{R_1^2}d_2+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)d_2\frac{1}{R_1}\right)z^{-1}, \tag{3.39}$$

$$\begin{pmatrix} (m_1-n_1)(k_1+1)R_1q_{-1} \\ +(m_1+n_1)(k_1+1)R_1^5q_3 \end{pmatrix}z^{-2}=\begin{pmatrix} -4\mu_1\left(-3c_1+\frac{3}{R_1^2}d_3\right) \\ +(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)BR_1^3 \\ +(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)R_1c_1 \\ +(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_3\frac{1}{R_1}-R_1c_1\right) \end{pmatrix}z^{-2}, \tag{3.40}$$

$$(m_1-n_1)(k_1+1)R_1q_{-2}z^{-3}=\begin{pmatrix} -4\mu_1\left(-8c_2+\frac{4}{R_1^2}d_4\right) \\ +(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)R_1c_2 \\ +(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_4\frac{1}{R_1}z^{-3}-2R_1c_2\right) \end{pmatrix}z^{-3}, \tag{3.41}$$

$$(m_1 - n_1)(k_1 + 1) R_1 q_{-3} z^{-4} = \begin{pmatrix} -4\mu_1 \left(-15c_3 + \frac{5}{R_1^2} d_5\right) \\ +(m_1 - n_1)\left(1 + \frac{\mu_1}{\mu_2} k_2\right) R_1 c_3 \\ +(m_1 + n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\left(d_5 \frac{1}{R_1} - 3R_1 c_3\right) \end{pmatrix} z^{-4}. \tag{3.42}$$

Now, 38 coupled linear algebraic equations have been derived from the four boundary conditions along imperfect interfaces Γ_0 and Γ_1. With e_k and f_k encapsulated in p_k and q_k, the 38 equations contain 38 unknown coefficients of a_k, b_k, c_k, d_k, e_k and f_k for 8 stress potentials.

3.4 Homogeneous Imperfect Interfaces

In this present research, we consider the homogeneous imperfect interface: namely, the normal (m) and tangential (n) spring-factor type interface parameters are constant along the interface Γ_0 and Γ_1. For convenience, let us introduce non-dimensional parameter $M_0 = \frac{m_0+n_0}{2\mu_1} R_0$, $M_1 = \frac{m_1+n_1}{2\mu_1} R_1$ characterizing the degree of damage along the interface. In this view, a very small value of M (say $M = 0.01$) represents complete debonding and a large value of M (say $M = 100$) corresponds to the case of perfect bonding condition. The other values between 0.01 and 100 are assumed to characterize the state of imperfect adhesion.

Also, the uniform eigenstrains $(\varepsilon_x^0, \varepsilon_y^0, \varepsilon_z^0)$ prescribed in the inclusion and the displacements induced by them are assumed zero since their effects on crack propagations under the Model I load are negligible.

Substituting p_k and q_k into (3.3) thru (3.21) and (3.24) thru (3.42) and deploying M_0, and M_1 into the equations we can write the following 38 coupled linear algebraic equations (see Appendix 5 for detail calculations):

$$\begin{aligned}0 &= 24a_6 + \frac{4}{R_0^2}b_4 - \frac{4}{R_0^{10}}\frac{1}{2}e_{-4} + \frac{10}{R_0^2}e_4 + \frac{12}{R_0^2}g_7 f_{-3} + \frac{12}{R_0^2}g_6 f_{-2} \\ &+ \left(\frac{12}{R_0^2}g_5 - 24g_7\right) f_{-1} + \left(\frac{12}{R_0^2}g_4 - 24g_6\right) f_0 + \left(\frac{12}{R_0^2}g_3 - 24g_5\right) f_1 \\ &+ \left(\frac{12}{R_0^2}g_2 - 24g_4\right) f_2 + \left(\frac{12}{R_0^2}g_1 - 24g_3\right) f_3 + \left(\frac{12}{R_0^2}g_0 - 24g_2\right) f_4,\end{aligned} \tag{3.43}$$

$$\begin{aligned}0 &= 15a_5 + \frac{3}{R_0^2}b_3 - \frac{3}{R_0^8}\frac{1}{2}e_{-3} + \frac{6}{R_0^2}e_3 + \left(\frac{6}{R_0^2}g_6 - \frac{3}{R_0^8}g_0\right) f_{-3} \\ &+ \left(\frac{6}{R_0^2}g_5 - 15g_7\right) f_{-2} + \left(\frac{6}{R_0^2}g_4 - 15g_6\right) f_{-1} + \left(\frac{6}{R_0^2}g_3 - 15g_5\right) f_0 \\ &+ \left(\frac{6}{R_0^2}g_2 - 15g_4\right) f_1 + \left(\frac{6}{R_0^2}g_1 - 15g_3\right) f_2 + \left(\frac{6}{R_0^2}g_0 - 15g_2\right) f_3 - 15g_1 f_4,\end{aligned} \tag{3.44}$$

$$\begin{aligned}0 &= 8a_4 + \frac{2}{R_0^2}b_2 - \frac{1}{R_0^6}e_{-2} + \frac{3}{R_0^2}e_2 - 4e_4 + \left(\frac{2}{R_0^2}g_5 - 8g_7 - \frac{2}{R_0^6}g_1\right) f_{-3} \\ &+ \left(\frac{2}{R_0^2}g_4 - 8g_6 - \frac{2}{R_0^6}g_0\right) f_{-2} + \left(\frac{2}{R_0^2}g_3 - 8g_5\right) f_{-1} + \left(\frac{2}{R_0^2}g_2 - 8g_4\right) f_0 \\ &+ \left(\frac{2}{R_0^2}g_1 - 8g_3\right) f_1 + \left(\frac{2}{R_0^2}g_0 - 8g_2\right) f_2 - 8g_1 f_3 - 8g_0 f_4,\end{aligned} \tag{3.45}$$

$$\begin{aligned}0 &= 3a_3 + \frac{1}{R_0^2}b_1 - \frac{1}{2}\frac{1}{R_0^4}e_{-1} + \frac{1}{R_0^2}e_1 - 3\frac{1}{2}e_3 + \left(-3g_6 - \frac{1}{R_0^4}g_2\right) f_{-3} \\ &+ \left(-3g_5 - \frac{1}{R_0^4}g_1\right) f_{-2} + \left(-3g_4 - \frac{1}{R_0^4}g_0\right) f_{-1} \\ &-3g_3 f_0 - 3g_2 f_1 - 3g_1 f_2 - 3g_0 f_3,\end{aligned} \tag{3.46}$$

$$\begin{aligned}0 &= -2a_1 + e_1 + \left(2g_4 + \frac{2}{R_0^2}g_2\right) f_{-3} \\ &+ \left(2g_3 + \frac{2}{R_0^2}g_1\right) f_{-2} + \left(2g_2 + \frac{2}{R_0^2}g_0\right) f_{-1} + 2g_1 f_0 + 2g_0 f_1,\end{aligned} \tag{3.47}$$

$$0 = -2R_0^2 a_2 + \frac{1}{R_0^2} e_{-2} + R_0^2 e_2 + \left(2R_0^2 g_5 + \frac{6}{R_0^2} g_1\right) f_{-3} + \left(2R_0^2 g_4 + \frac{6}{R_0^2} g_0\right) f_{-2} + 2R_0^2 g_3 f_{-1} + 2R_0^2 g_2 f_0 + 2R_0^2 g_1 f_1 + 2R_0^2 g_0 f_2, \tag{3.48}$$

$$0 = -3R_0^4 a_3 + \frac{3}{R_0^2} e_{-3} - \frac{3}{2} e_{-1} + 3R_0^4 \frac{1}{2} e_3 + \left(3R_0^4 g_6 - 3g_2 + \frac{12}{R_0^2} g_0\right) f_{-3} + \left(3R_0^4 g_5 - 3g_1\right) f_{-2} + \left(3R_0^4 g_4 - 3g_0\right) f_{-1} + 3R_0^4 g_3 f_0 + 3R_0^4 g_2 f_1 + 3R_0^4 g_1 f_2 + 3R_0^4 g_0 f_3, \tag{3.49}$$

$$0 = -4R_0^6 a_4 + \frac{6}{R_0^2} e_{-4} - 4e_{-2} + 2R_0^6 e_4 + \left(4R_0^6 g_7 - 8g_1\right) f_{-3} + \left(4R_0^6 g_6 - 8g_0\right) f_{-2} + 4R_0^6 g_5 f_{-1} + 4R_0^6 g_4 f_0 + 4R_0^6 g_3 f_1 + 4R_0^6 g_2 f_2 + 4R_0^6 g_1 f_3 + 4R_0^6 g_0 f_4, \tag{3.50}$$

$$0 = -5R_0^8 a_5 - 15\frac{1}{2} e_{-3} - 15 g_0 f_{-3} + 5R_0^8 g_7 f_{-2} + 5R_0^8 g_6 f_{-1} + 5R_0^8 g_5 f_0 + 5R_0^8 g_4 f_1 + 5R_0^8 g_3 f_2 + 5R_0^8 g_2 f_3 + 5R_0^8 g_1 f_4, \tag{3.51}$$

$$0 = \left(48\frac{1}{M_0} - \frac{(m_0 - n_0)}{(m_0 + n_0)}(1 + \kappa_0 \frac{\mu_1}{\mu_0}) - 6(1 - \frac{\mu_1}{\mu_0})\right) a_6 + \left(-(1 - \frac{\mu_1}{\mu_0})\frac{1}{R_0^2} + \frac{8}{M_0}\frac{1}{R_0^2}\right) b_4 + (\kappa_1 + 1)\frac{1}{2R_0^{10}} e_{-4} + \frac{(m_0 - n_0)}{(m_0 + n_0)}(\kappa_1 + 1) g_7 f_{-1} + \frac{(m_0 - n_0)}{(m_0 + n_0)}(\kappa_1 + 1) g_6 f_0 + \frac{(m_0 - n_0)}{(m_0 + n_0)}(\kappa_1 + 1) g_5 f_1 + \frac{(m_0 - n_0)}{(m_0 + n_0)}(\kappa_1 + 1) g_4 f_2 + \frac{(m_0 - n_0)}{(m_0 + n_0)}(\kappa_1 + 1) g_3 f_3 + \frac{(m_0 - n_0)}{(m_0 + n_0)}(\kappa_1 + 1) g_2 f_4, \tag{3.52}$$

$$
\begin{aligned}
0 \;=\; & \left(30\frac{1}{M_0} - \frac{(m_0-n_0)}{(m_0+n_0)}(1+\kappa_0\frac{\mu_1}{\mu_0}) - 5(1-\frac{\mu_1}{\mu_0})\right) a_5 \qquad (3.53)\\
& + \left(-(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0^2} + \frac{1}{M_0}\frac{6}{R_0^2}\right) b_3 + (\kappa_1+1)\frac{1}{2R_0^8}e_{-3}\\
& +(\kappa_1+1)\frac{1}{R_0^8}g_0 f_{-3} + \frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_7 f_{-2} + \frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_6 f_{-1}\\
& +\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_5 f_0 + \frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_4 f_1 + \frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_3 f_2\\
& +\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_2 f_3 + \frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_1 f_4,
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\; & \left(16\frac{1}{M_0} - \frac{(m_0-n_0)}{(m_0+n_0)}(1+\kappa_0\frac{\mu_1}{\mu_0}) - 4(1-\frac{\mu_1}{\mu_0})\right) a_4 \qquad (3.54)\\
& + \left(-(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0^2} + \frac{1}{M_0}\frac{4}{R_0^2}\right) b_2 + (\kappa_1+1)\frac{1}{R_0^6}\frac{1}{2}e_{-2} + \frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)\frac{1}{2}e_4\\
& + \left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_7 + (\kappa_1+1)\frac{1}{R_0^6}g_1\right) f_{-3}\\
& + \left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_6 + (\kappa_1+1)\frac{1}{R_0^6}g_0\right) f_{-2} + \frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_5 f_{-1}\\
& +\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_4 f_0 + \frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_3 f_1\\
& +\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_2 f_2 + \frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_1 f_3 + \frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_0 f_4,
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\; & \left(6\frac{1}{M_0} - \frac{(m_0-n_0)}{(m_0+n_0)}(1+\kappa_0\frac{\mu_1}{\mu_0}) - 3(1-\frac{\mu_1}{\mu_0})\right) a_3 \qquad (3.55)\\
& + \left(-(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0^2} + \frac{1}{M_0}\frac{2}{R_0^2}\right) b_1 + (\kappa_1+1)\frac{1}{R_0^4}\frac{1}{2}e_{-1}\\
& +\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)\frac{1}{2}e_3 + \left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_6 + (\kappa_1+1)\frac{1}{R_0^4}g_2\right) f_{-3}\\
& + \left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_5 + (\kappa_1+1)\frac{1}{R_0^4}g_1\right) f_{-2}\\
& + \left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_4 + (\kappa_1+1)\frac{1}{R_0^4}g_0\right) f_{-1}
\end{aligned}
$$

$$+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_3f_0+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_2f_1$$
$$+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_1f_2+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_0f_3,$$

$$\begin{aligned}0 \;=\; & \left(-\frac{(m_0-n_0)}{(m_0+n_0)}(1+\kappa_0\frac{\mu_1}{\mu_0})R_0-2(1-\frac{\mu_1}{\mu_0})R_0\right)a_2-(1+\kappa_0\frac{\mu_1}{\mu_0})\frac{1}{R_0}a_0 \quad (3.56)\\ & -(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_0+(\kappa_1+1)\frac{1}{R_0}\frac{1}{2}e_0+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)R_0\frac{1}{2}e_2\\ & +\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)R_0g_5+(\kappa_1+1)\frac{1}{R_0}g_3\right)f_{-3}\\ & +\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)R_0g_4+(\kappa_1+1)\frac{1}{R_0}g_2\right)f_{-2}\\ & +\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)R_0g_3+(\kappa_1+1)\frac{1}{R_0}g_1\right)f_{-1}\\ & +\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)R_0g_2+(\kappa_1+1)\frac{1}{R_0}g_0\right)f_0\\ & +\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)R_0g_1f_1+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)R_0g_0f_2,\end{aligned}$$

$$\begin{aligned}0 \;=\; & \left(\begin{array}{c}-4\frac{1}{M_0}-\frac{(m_0-n_0)}{(m_0+n_0)}(1+\kappa_0\frac{\mu_1}{\mu_0})-\frac{(m_0-n_0)}{(m_0+n_0)}(1-\frac{\mu_1}{\mu_0})\\ -(1+\kappa_0\frac{\mu_1}{\mu_0})-(1-\frac{\mu_1}{\mu_0})\end{array}\right)a_1 \quad (3.57)\\ & +\frac{m_0(\kappa_1+1)}{(m_0+n_0)}e_1+\frac{2m_0(\kappa_1+1)}{(m_0+n_0)}g_4f_{-3}+\frac{2m_0(\kappa_1+1)}{(m_0+n_0)}g_3f_{-2}\\ & +\frac{2m_0(\kappa_1+1)}{(m_0+n_0)}g_2f_{-1}+\frac{2m_0(\kappa_1+1)}{(m_0+n_0)}g_1f_0+\frac{2m_0(\kappa_1+1)}{(m_0+n_0)}g_0f_1,\end{aligned}$$

$$\begin{aligned}0 \;=\; & -\frac{(m_0-n_0)}{(m_0+n_0)}(1+\kappa_0\frac{\mu_1}{\mu_0})a_0+\left(\begin{array}{c}-4\frac{1}{M_0}R_0^2-\frac{(m_0-n_0)}{(m_0+n_0)}2(1-\frac{\mu_1}{\mu_0})R_0^2\\ -(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^2\end{array}\right)a_2 \quad (3.58)\\ & -\frac{(m_0-n_0)}{(m_0+n_0)}(1-\frac{\mu_1}{\mu_0})b_0+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)\frac{1}{2}e_0+(\kappa_1+1)R_0^2\frac{1}{2}e_2\end{aligned}$$

$$+\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_3+(\kappa_1+1)R_0^2g_5\right)f_{-3}$$
$$+\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_2+(\kappa_1+1)R_0^2g_4\right)f_{-2}$$
$$+\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_1+(\kappa_1+1)R_0^2g_3\right)f_{-1}$$
$$+\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_0+(\kappa_1+1)R_0^2g_2\right)f_0+(\kappa_1+1)R_0^2g_1f_1+(\kappa_1+1)R_0^2g_0f_2,$$

$$\begin{aligned}
0 \;=\; & \left(-\frac{6}{M_0}R_0^4-\frac{(m_0-n_0)}{(m_0+n_0)}3(1-\frac{\mu_1}{\mu_0})R_0^4-(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^4\right)a_3 \qquad (3.59)\\
& -\frac{(m_0-n_0)}{(m_0+n_0)}(1-\frac{\mu_1}{\mu_0})R_0^2b_1+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)\frac{1}{2}e_{-1}\\
& +(\kappa_1+1)R_0^4\frac{1}{2}e_3+\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_2+(\kappa_1+1)R_0^4g_6\right)f_{-3}\\
& +\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_1+(\kappa_1+1)R_0^4g_5\right)f_{-2}\\
& +\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_0+(\kappa_1+1)R_0^4g_4\right)f_{-1}+(\kappa_1+1)R_0^4g_3f_0\\
& +(\kappa_1+1)R_0^4g_2f_1+(\kappa_1+1)R_0^4g_1f_2+(\kappa_1+1)R_0^4g_0f_3,
\end{aligned}$$

$$\begin{aligned}
0 \;=\; & \left(-8\frac{1}{M_0}R_0^6-\frac{(m_0-n_0)}{(m_0+n_0)}4(1-\frac{\mu_1}{\mu_0})R_0^6-(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^6\right)a_4 \qquad (3.60)\\
& -\frac{(m_0-n_0)}{(m_0+n_0)}(1-\frac{\mu_1}{\mu_0})R_0^4b_2+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)\frac{1}{2}e_{-2}\\
& +(\kappa_1+1)R_0^6\frac{1}{2}e_4+\left((\kappa_1+1)R_0^6g_7+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_1\right)f_{-3}\\
& +\left((\kappa_1+1)R_0^6g_6+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_0\right)f_{-2}+(\kappa_1+1)R_0^6g_5f_{-1}\\
& +(\kappa_1+1)R_0^6g_4f_0+(\kappa_1+1)R_0^6g_3f_1+(\kappa_1+1)R_0^6g_2f_2\\
& +(\kappa_1+1)R_0^6g_1f_3+(\kappa_1+1)R_0^6g_0f_4,
\end{aligned}$$

$$0 = \left(-10\frac{1}{M_0}R_0^8 - \frac{(m_0-n_0)}{(m_0+n_0)}5(1-\frac{\mu_1}{\mu_0})R_0^8 - (1+\kappa_0\frac{\mu_1}{\mu_0})R_0^8\right)a_5 \tag{3.61}$$
$$-\frac{(m_0-n_0)}{(m_0+n_0)}(1-\frac{\mu_1}{\mu_0})R_0^6 b_3 + \frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)\frac{1}{2}e_{-3}$$
$$+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_0 f_{-3} + (\kappa_1+1)R_0^8 g_7 f_{-2} + (\kappa_1+1)R_0^8 g_6 f_{-1}$$
$$+(\kappa_1+1)R_0^8 g_5 f_0 + (\kappa_1+1)R_0^8 g_4 f_1 + (\kappa_1+1)R_0^8 g_3 f_2$$
$$+(\kappa_1+1)R_0^8 g_2 f_3 + (\kappa_1+1)R_0^8 g_1 f_4,$$

$$0 = +\frac{4}{R_1^{10}}c_4 + 4\frac{1}{R_1^2}e_4 - \frac{2}{R_1^{10}}e_{-4} - \frac{4}{R_1^{10}}h_{-1}f_{-3} - \frac{4}{R_1^{10}}h_{-2}f_{-2} \tag{3.62}$$
$$-\frac{4}{R_1^{10}}h_{-3}f_{-1} - \frac{4}{R_1^{10}}h_{-4}f_0 - \frac{4}{R_1^{10}}h_{-5}f_1 - \frac{4}{R_1^{10}}h_{-6}f_2$$
$$-\frac{4}{R_1^{10}}h_{-7}f_3 - \frac{4}{R_1^{10}}h_{-8}f_4,$$

$$0 = +\frac{3}{R_1^8}c_3 + \frac{6}{R_1^2}e_3 + \left(\frac{6}{R_1^2}h_{-1} - \frac{3}{R_1^8}h_{-7}\right)f_4 \tag{3.63}$$
$$-\frac{3}{R_1^8}h_{-6}f_3 - \frac{3}{R_1^8}h_{-5}f_2 - \frac{3}{R_1^8}h_{-4}f_1 - \frac{3}{R_1^8}h_{-3}f_0$$
$$-\frac{3}{R_1^8}h_{-2}f_{-1} - \frac{3}{R_1^8}h_{-1}f_{-2} - \frac{1}{R_1^8}\frac{3}{2}e_{-3},$$

$$0 = \frac{2}{R_1^6}c_2 + \frac{3}{R_1^2}e_2 - \frac{1}{R_1^6}e_{-2} + \left(\frac{2}{R_1^2}h_{-2} - \frac{2}{R_1^6}h_{-6}\right)f_4 \tag{3.64}$$
$$+\left(\frac{2}{R_1^2}h_{-1} - \frac{2}{R_1^6}h_{-5}\right)f_3 - \frac{2}{R_1^6}h_{-4}f_2 - \frac{2}{R_1^6}h_{-3}f_1 - \frac{2}{R_1^6}h_{-2}f_0 - \frac{2}{R_1^6}h_{-1}f_{-1},$$

$$-\frac{1}{R_1^2}B = +\frac{1}{R_1^4}c_1 - \frac{3}{2}e_3 + \frac{1}{R_1^2}e_1 - \frac{1}{R_1^4}\frac{1}{2}e_{-1} + \left(-3h_{-1} - \frac{1}{R_1^4}h_{-5}\right)f_4 \tag{3.65}$$
$$-\frac{1}{R_1^4}h_{-4}f_3 - \frac{1}{R_1^4}h_{-3}f_2 - \frac{1}{R_1^4}h_{-2}f_1 - \frac{1}{R_1^4}h_{-1}f_0,$$

$$2A = -\frac{1}{R_1^2}d_1 + e_1 + \left(2h_{-3} + \frac{2}{R_1^2}h_{-5}\right) f_4 + \left(2h_{-2} + \frac{2}{R_1^2}h_{-4}\right) f_3 \qquad (3.66)$$
$$+ \left(2h_{-1} + \frac{2}{R_1^2}h_{-3}\right) f_2 + \frac{2}{R_1^2}h_{-2}f_1 + \frac{2}{R_1^2}h_{-1}f_0,$$

$$0 = -\frac{2}{R_1^2}d_2 + R_1^2 e_2 + \frac{1}{R_1^2}e_{-2} + \left(2R_1^2 h_{-2} + \frac{6}{R_1^2}h_{-6}\right) f_4 \qquad (3.67)$$
$$+ \left(2R_1^2 h_{-1} + \frac{6}{R_1^2}h_{-5}\right) f_3 + \frac{6}{R_1^2}h_{-4}f_2 + \frac{6}{R_1^2}h_{-3}f_1 + \frac{6}{R_1^2}h_{-2}f_0 + \frac{6}{R_1^2}h_{-1}f_{-1},$$

$$0 = +3c_1 - \frac{3}{R_1^2}d_3 + R_1^4\frac{3}{2}e_3 - \frac{3}{2}e_{-1} + \frac{3}{R_1^2}e_{-3} \qquad (3.68)$$
$$+ \left(\frac{12}{R_1^2}h_{-7} + 3R_1^4 h_{-1} - 3h_{-5}\right) f_4 + \left(\frac{12}{R_1^2}h_{-6} - 3h_{-4}\right) f_3 + \left(\frac{12}{R_1^2}h_{-5} - 3h_{-3}\right) f_2$$
$$+ \left(\frac{12}{R_1^2}h_{-4} - 3h_{-2}\right) f_1 + \left(\frac{12}{R_1^2}h_{-3} - 3h_{-1}\right) f_0 + \frac{12}{R_1^2}h_{-2}f_{-1} + \frac{12}{R_1^2}h_{-1}f_{-2},$$

$$0 = +8c_2 - \frac{4}{R_1^2}d_4 - 4e_{-2} + \frac{6}{R_1^2}e_{-4} + \left(\frac{20}{R_1^2}h_{-8} - 8h_{-6}\right) f_4 \qquad (3.69)$$
$$+ \left(\frac{20}{R_1^2}h_{-7} - 8h_{-5}\right) f_3 + \left(\frac{20}{R_1^2}h_{-6} - 8h_{-4}\right) f_2 + \left(\frac{20}{R_1^2}h_{-5} - 8h_{-3}\right) f_1$$
$$+ \left(\frac{20}{R_1^2}h_{-4} - 8h_{-2}\right) f_0 + \left(\frac{20}{R_1^2}h_{-3} - 8h_{-1}\right) f_{-1} + \frac{20}{R_1^2}h_{-2}f_{-2} + \frac{20}{R_1^2}h_{-1}f_{-3},$$

$$0 = 15c_3 - \frac{5}{R_1^2}d_5 - \frac{15}{2}e_{-3} + \frac{15}{R_1^2}e_{-5} - 15h_{-7}f_4 \qquad (3.70)$$
$$+ \left(\frac{30}{R_1^2}h_{-8} - 15h_{-6}\right) f_3 + \left(\frac{30}{R_1^2}h_{-7} - 15h_{-5}\right) f_2$$
$$+ \left(\frac{30}{R_1^2}h_{-6} - 15h_{-4}\right) f_1 + \left(\frac{30}{R_1^2}h_{-5} - 15h_{-3}\right) f_0$$
$$+ \left(\frac{30}{R_1^2}h_{-4} - 15h_{-2}\right) f_{-1} + \left(\frac{30}{R_1^2}h_{-3} - 15h_{-1}\right) f_{-2} + \frac{30}{R_1^2}h_{-2}f_{-3} + \frac{30}{R_1^2}h_{-1}f_{-4},$$

$$
\begin{aligned}
0 = & \left(\frac{1}{M_1}\frac{8}{R_1^9} - \frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)\frac{4}{R_1^9} + \left(1+\frac{\mu_1}{\mu_2}k_2\right)\frac{1}{R_1^9}\right)c_4 \\
& -(k_1+1)\frac{1}{R_1^9}h_{-8}f_4 - (k_1+1)\frac{1}{R_1^9}h_{-7}f_3 - (k_1+1)\frac{1}{R_1^9}h_{-6}f_2 \\
& -(k_1+1)\frac{1}{R_1^9}h_{-5}f_1 - (k_1+1)\frac{1}{R_1^9}h_{-4}f_0 - (k_1+1)\frac{1}{R_1^9}h_{-3}f_{-1} \\
& -(k_1+1)\frac{1}{R_1^9}h_{-2}f_{-2} - (k_1+1)\frac{1}{R_1^9}h_{-1}f_{-3} - (k_1+1)\frac{1}{R_1^9}\frac{1}{2}e_{-4},
\end{aligned}
\tag{3.71}
$$

$$
\begin{aligned}
0 = & \left(\frac{1}{M_1}\frac{6}{R_1^7} - \frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)\frac{3}{R_1^7} + \left(1+\frac{\mu_1}{\mu_2}k_2\right)\frac{1}{R_1^7}\right)c_3 \\
& +\frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^9}d_5 - \frac{(k_1+1)}{R_1^7}\frac{1}{2}e_{-3} - \frac{(k_1+1)}{R_1^7}h_{-7}f_4 \\
& -\frac{(k_1+1)}{R_1^7}h_{-6}f_3 - \frac{(k_1+1)}{R_1^7}h_{-5}f_2 - \frac{(k_1+1)}{R_1^7}h_{-4}f_1 - \frac{(k_1+1)}{R_1^7}h_{-3}f_0 \\
& -\frac{(k_1+1)}{R_1^7}h_{-2}f_{-1} - \frac{(k_1+1)}{R_1^7}h_{-1}f_{-2},
\end{aligned}
\tag{3.72}
$$

$$
\begin{aligned}
0 = & \left(\frac{1}{M_1}\frac{4}{R_1^5} - \frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)\frac{2}{R_1^5} + \left(1+\frac{\mu_1}{\mu_2}k_2\right)\frac{1}{R_1^5}\right)c_2 \\
& +\frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^7}d_4 - \frac{(k_1+1)}{R_1^5}\frac{1}{2}e_{-2} - \frac{(k_1+1)}{R_1^5}h_{-6}f_4 - \frac{(k_1+1)}{R_1^5}h_{-5}f_3 \\
& -\frac{(k_1+1)}{R_1^5}h_{-4}f_2 - \frac{(k_1+1)}{R_1^5}h_{-3}f_1 - \frac{(k_1+1)}{R_1^5}h_{-2}f_0 - \frac{(k_1+1)}{R_1^5}h_{-1}f_{-1},
\end{aligned}
\tag{3.73}
$$

$$
\begin{aligned}
& \left(-\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1} - \frac{1}{M_1}\frac{2}{R_1}\right)B \\
= & \left(\frac{1}{M_1}\frac{2}{R_1^3} - \frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^3} + \left(1+\frac{\mu_1}{\mu_2}k_2\right)\frac{1}{R_1^3}\right)c_1 \\
& +\frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^5}d_3 - \frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)R_1\frac{1}{2}e_3 \\
& -\frac{(k_1+1)}{R_1^3}\frac{1}{2}e_{-1} + \left(-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)R_1h_{-1} - \frac{(k_1+1)}{R_1^3}h_{-5}\right)f_4 \\
& -\frac{(k_1+1)}{R_1^3}h_{-4}f_3 - \frac{(k_1+1)}{R_1^3}h_{-3}f_2 - \frac{(k_1+1)}{R_1^3}h_{-2}f_1 - \frac{(k_1+1)}{R_1^3}h_{-1}f_0,
\end{aligned}
\tag{3.74}
$$

$$\begin{aligned}
0 \;=\; & \frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R^3}d_2 - \frac{(m_1-n_1)}{2(m_1+n_1)}(k_1+1)R_1e_2 \\
& -(k_1+1)\frac{1}{R_1}\frac{1}{2}e_0 + \left(-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)R_1h_{-2} - (k_1+1)\frac{1}{R_1}h_{-4}\right)f_4 \\
& + \left(-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)R_1h_{-1} - (k_1+1)\frac{1}{R_1}h_{-3}\right)f_3 \\
& -(k_1+1)\frac{1}{R_1}h_{-2}f_2 - (k_1+1)\frac{1}{R_1}h_{-1}f_1,
\end{aligned} \tag{3.75}$$

$$\begin{aligned}
& \frac{4A}{M_1} - \frac{2m_1A}{(m_1+n_1)}\left(1+\frac{\mu_1}{\mu_2}k_2\right) - \frac{2m_1A}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right) \\
= & \left(-\frac{2}{R_1^2}\frac{1}{M_1} + \left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^2} + \frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^2}\right)d_1 \\
& -\frac{m_1(k_1+1)}{(m_1+n_1)}e_1 - \frac{2m_1(k_1+1)}{(m_1+n_1)}h_{-3}f_4 - \frac{2m_1(k_1+1)}{(m_1+n_1)}h_{-2}f_3 - \frac{2m_1(k_1+1)}{(m_1+n_1)}h_{-1}f_2,
\end{aligned} \tag{3.76}$$

$$\begin{aligned}
0 \;=\; & \left(-\frac{4}{R_1}\frac{1}{M_1} + \left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1}\right)d_2 - (k_1+1)R_1^3\frac{1}{2}e_2 \\
& -\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)R_1\frac{1}{2}e_0 + \left(-(k_1+1)R_1^3h_{-2} - \frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)R_1h_{-4}\right)f_4 \\
& + \left(-(k_1+1)R_1^3h_{-1} - \frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)R_1h_{-3}\right)f_3 \\
& -\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)R_1h_{-2}f_2 - \frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)R_1h_{-1}f_1,
\end{aligned} \tag{3.77}$$

$$\begin{aligned}
& -\frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)BR_1^2 \\
= & \left(\frac{6}{M_1} + \frac{(m_1-n_1)}{(m_1+n_1)}\left(1+\frac{\mu_1}{\mu_2}k_2\right) - \left(1-\frac{\mu_1}{\mu_2}\right)\right)c_1 \\
& + \left(-\frac{6}{R_1^2}\frac{1}{M_1} + \left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^2}\right)d_3 - (k_1+1)R_1^4\frac{1}{2}e_3 - \frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)\frac{1}{2}e_{-1} \\
& + \left(-(k_1+1)R_1^4h_{-1} - \frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-5}\right)f_4 - \frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-4}f_3 \\
& -\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-3}f_2 - \frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-2}f_1 - \frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-1}f_0,
\end{aligned} \tag{3.78}$$

$$\begin{aligned}
0 \;=\;& \left(\frac{16}{M_1} + \frac{(m_1-n_1)}{(m_1+n_1)}\left(1+\frac{\mu_1}{\mu_2}k_2\right) - 2\left(1-\frac{\mu_1}{\mu_2}\right)\right)c_2 \\
&+ \left(-\frac{1}{R_1^2}\frac{8}{M_1} + \left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^2}\right)d_4 - \frac{(m_1-n_1)(k_1+1)}{2(m_1+n_1)}e_{-2} \\
&- \frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-6}f_4 - \frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-5}f_3 \\
&- \frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-4}f_2 - \frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-3}f_1 \\
&- \frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-2}f_0 - \frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-1}f_{-1},
\end{aligned} \tag{3.79}$$

$$\begin{aligned}
0 \;=\;& \left(\frac{30}{M_1} + \frac{(m_1-n_1)}{(m_1+n_1)}\left(1+\frac{\mu_1}{\mu_2}k_2\right) - 3\left(1-\frac{\mu_1}{\mu_2}\right)\right)c_3 \\
&+ \left(-\frac{1}{R_1^2}\frac{10}{M_1} + \left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^2}\right)d_5 - \frac{(m_1-n_1)(k_1+1)}{(m_1+n_1)}h_{-7}f_4 \\
&- \frac{(m_1-n_1)(k_1+1)}{(m_1+n_1)}h_{-6}f_3 - \frac{(m_1-n_1)(k_1+1)}{(m_1+n_1)}h_{-5}f_2 - \frac{(m_1-n_1)(k_1+1)}{(m_1+n_1)}h_{-4}f_1 \\
&- \frac{(m_1-n_1)(k_1+1)}{(m_1+n_1)}h_{-3}f_0 - \frac{(m_1-n_1)(k_1+1)}{(m_1+n_1)}h_{-2}f_{-1} \\
&- \frac{(m_1-n_1)(k_1+1)}{(m_1+n_1)}h_{-1}f_{-2} - \frac{(m_1-n_1)(k_1+1)}{2(m_1+n_1)}e_{-3}.
\end{aligned} \tag{3.80}$$

CHAPTER 4
Numerical Analysis Results / Discussion

4.1 Introduction

In this chapter, the numerical analysis of 38 coupled linear algebraic equations derived in Chapter 3 is presented to determine the 38 unknown coefficients. The unknown coefficients vary according to mechanical properties of the composites such as Poisson's ratio and crack locations. Based on the resulting coefficients, stress intensity factors at the crack tips under Mode I are derived to investigate crack behaviour around the tips. Thus, this chapter first presents derivations of stress intensity factors at crack tips a and b in terms of the resulting coefficients of stress potentials.

The equations of (3.43) thru (3.80) can be expressed in a matrix form such as $\{b\} = [A]\{x\}$ where $[A]$ is a 38 x 38 coefficient matrix, $\{b\}$ corresponds to a load vector and $\{x\}$ is the solution vector representing the resulting coefficients. Using MATLAB, the inverse matrix of $[A]$ is calculated and the solution vector $\{x\}$ is readily calculated by $\{x\}=[A]^{-1}\{b\}$. For the verification of the formulation, a comparison of the results corresponding to a three-phase model with the perfect interfaces reported in the literature is made. Finally, some numerical results such as effects of inclusion stiffness, interphase stiffness and crack length on cracking are addressed.

4.2 Stress Fields around Crack Tips

In many engineering applications, a study of the behaviour of materials under certain loads in the presence of material flaws such as internal voids, minor cracks and different materials embedded inside is quite important to prevent their failures. In particular, it is well known that failures associated with sudden crack propagation in solid materials take place even at the far lower than the failure limits of the stress. Eventually, to evaluate the effects of flaw size, fracture toughness and applied stress on material failures, the early frame works of fracture mechanics has been intensively addressed and developed by Inglis, Griffith and Westergaard. One of the most dominant researchers in 1950s is G.R. Irwin leading the Naval Research Laboratory. He further developed the early works and introduced a new idea that stresses and displacements in the neighbourhood of crack tips could be characterized by a single constant, called the stress intensity factor (SIF). The stress intensity factor represents the driving force to cause fracture in the vicinity of the crack tips.

There are three types of loadings that a crack can experience: Mode I represents a crack opening deformation, Mode II an in-plane shear and Mode III an out-of-plane deformation. A system with a crack can undergo any one of these modes, or arbitrary combinations of two or three modes. However, Mode I is known as the most dominant failure scenario [38] compared to the other modes. Furthermore, under uniform remote tensile stress, a singularity-dominated zone is defined as a region where the singularity $1/\sqrt{r}$ (r: distance from a crack tip) dominates the stress fields and the stress intensity factor in the zone defines the amplitude of the crack-tip singularity [57]. Thus, in this study, stress intensity factor under Mode I loading near crack tip a and b, namely within singularity dominated zone, is investigated. Also, it is assumed that there is no crack growth at the tips so that the Model I stress intensity factor K_I is smaller than the critical stress intensity factor K_{IC}.

Using the Cartesian form of (2.1), we can re-write it as

$$\sigma_{xx} + \sigma_{yy} = 2\left[\varphi'(z) + \overline{\varphi'(z)}\right], \tag{4.1}$$

$$\sigma_{xx} - i\sigma_{xy} = \varphi'(z) + \overline{\varphi'(z)} - \left[\bar{z}\varphi''(z) + \psi'(z)\right]. \tag{4.2}$$

Subtracting the conjugate of (4.2) from (4.1) leads

$$\sigma_{yy} - i\sigma_{xy} = \varphi'(z) + \overline{\varphi'(z)} + z\overline{\varphi''(z)} + \overline{\psi'(z)}. \tag{4.3}$$

It is noted that $\mathrm{Re}(z) = \mathrm{Re}\,(\bar{z})$ and $\mathrm{Im}(z) = -\,\mathrm{Im}(\bar{z})$. Thus, we get the stress field from (4.2) and (4.3) as follows

$$\begin{aligned} \sigma_{xx} &= \mathrm{Re}\left[\varphi'(z) + \overline{\varphi'(z)} - z\overline{\varphi''(z)} - \overline{\psi'(z)}\right], \\ \sigma_{yy} &= \mathrm{Re}\left[\varphi'(z) + \overline{\varphi'(z)} + z\overline{\varphi''(z)} + \overline{\psi'(z)}\right], \\ \sigma_{xy} &= \mathrm{Im}\left[\varphi'(z) + \overline{\varphi'(z)} - z\overline{\varphi''(z)} - \overline{\psi'(z)}\right]. \end{aligned} \tag{4.4}$$

Let's express $\psi(z)$ in terms of $\varphi(z)$ and $X(z)$ by using (2.33).

$$\begin{aligned} \psi_1'(z) &= \frac{d}{dz}\left[-\overline{X}(z) + \overline{\varphi_1}(z) - z\varphi_1'(z)\right] \\ &= \overline{\varphi_1'}(z) - \varphi_1'(z) - z\varphi_1''(z) - \overline{X'}(z). \end{aligned} \tag{4.5}$$

Taking conjugate of (4.5) gives

$$\overline{\psi_1'(z)} = \varphi_1'(\bar{z}) - \overline{\varphi_1'(z)} - \bar{z}\overline{\varphi_1''(z)} - X'(\bar{z}). \tag{4.6}$$

Substituting (4.5) and (4.6) into (4.4), we get the stress fields in region 1 as follows

$$\begin{aligned} \sigma_{xx} &= \mathrm{Re}\left[\varphi_1'(z) - \varphi_1'(\bar{z}) + 2\overline{\varphi_1'(z)} + (\bar{z} - z)\,\overline{\varphi_1''(z)} + X'(\bar{z})\right], \\ \sigma_{yy} &= \mathrm{Re}\left[\varphi_1'(z) + \varphi_1'(\bar{z}) + [z - \bar{z}]\,\overline{\varphi_1''(z)} - X'(\bar{z})\right], \\ \sigma_{xy} &= \mathrm{Im}\left[\varphi_1'(z) - \varphi_1'(\bar{z}) + 2\overline{\varphi_1'(z)} + (\bar{z} - z)\,\overline{\varphi_1''(z)} + X'(\bar{z})\right]. \end{aligned} \tag{4.7}$$

Considering leading order terms of each stress potential shown on (4.7), the explicit expressions of the stress fields in the neighbourhood of the crack tips a are given as (See Appendix 6 for detail derivation)

$$\begin{aligned}
\sigma_{yy} &= \frac{1}{\sqrt{r_1}\sqrt{2l}}\left(-\frac{5}{8}\sin\frac{\theta_1}{2}+\frac{1}{8}\sin\frac{5\theta_1}{2}\right)\left[\sum_{k=-\infty}^{\infty} k f_k a^{k-1}\right]+O\left(r_1\right), \qquad (4.8)\\
\sigma_{xx} &= \frac{1}{\sqrt{r_1}\sqrt{2l}}\left(-\frac{3}{8}\sin\frac{\theta_1}{2}-\frac{1}{8}\sin\frac{5\theta_1}{2}\right)\left[\sum_{k=-\infty}^{\infty} k f_k a^{k-1}\right]+O\left(r_1\right),\\
\sigma_{xy} &= \frac{1}{\sqrt{r_1}\sqrt{2l}}\left(-\frac{1}{8}\cos\frac{\theta_1}{2}+\frac{1}{8}\cos\frac{5\theta_1}{2}\right)\left[\sum_{k=-\infty}^{\infty} k f_k a^{k-1}\right]+O\left(r_1\right).
\end{aligned}$$

where $2l = b - a$ and $(z-a) = r_1 e^{i\theta_1}$ $(0 \leq \theta_1 \leq 2\pi)$.

Also, the stresses near crack tip b are determined in the following forms

$$\begin{aligned}
\sigma_{yy} &= \frac{1}{\sqrt{r_2}\sqrt{2l}}\left(\frac{5}{8}\cos\frac{\theta_2}{2}-\frac{1}{8}\cos\frac{5\theta_2}{2}\right)\left[\sum_{k=-\infty}^{\infty} k f_k b^{k-1}\right]+O\left(r_2\right), \qquad (4.9)\\
\sigma_{xx} &= \frac{1}{\sqrt{r_2}\sqrt{2l}}\left(\frac{3}{8}\cos\frac{\theta_2}{2}+\frac{1}{8}\cos\frac{5\theta_2}{2}\right)\left[\sum_{k=-\infty}^{\infty} k f_k b^{k-1}\right]+O\left(r_2\right),\\
\sigma_{xy} &= \frac{1}{\sqrt{r_2}\sqrt{2l}}\left(-\frac{1}{8}\sin\frac{\theta_2}{2}+\frac{1}{8}\sin\frac{5\theta_2}{2}\right)\left[\sum_{k=-\infty}^{\infty} k f_k b^{k-1}\right]+O\left(r_2\right).
\end{aligned}$$

where $b - a = 2l$ and $(z-b) = r_2 e^{i\theta_2}$ $(-\pi \leq \theta_2 \leq \pi)$.

It should be noted that the remote loading term is not shown in (4.8) and (4.9) since the load is not directly applied to the crack in the matrix but transferred to the domain through imperfect interface Γ_1 which alters the effect of the load on the crack. The undetermined coefficients f_k play a dominant role in transferring the load from a infinite point to the crack tips through the interface and the square bracketed terms represent the influence of the imperfect bonding condition.

4.3 Stress Intensity Factor

The normalized Mode I stress intensity factor in the vicinity of tips is defined as

$$\frac{K_I}{K_{I\ w/o\ inclusion}}, \tag{4.10}$$

where $K_I = \sigma_{yy}\sqrt{2\pi r}$ and $K_{I\ w/o\ inclusion} = \sigma_\infty\sqrt{\pi l}$. K_I represents Mode I stress intensity factor in the intermediate matrix zone under uni-axial loading while $K_{I\ w/o\ inclusion}$ denotes the Mode I SIF for the same crack in the homogeneous matrix without an inclusion. It shows the pertubation in stress fields caused by mechanical properties of an inclusion, matrix, composite phase and imperfect interfaces.

In view of (4.8), the normalized SIF at tip a where θ_1 is π is

$$\begin{aligned} & \left[\frac{K_I}{K_{I\ w/o\ inclusion}}\right]_{at\ a} \\ = & \frac{\sqrt{2\pi r}}{\sigma_\infty\sqrt{\pi l}}\left\{\frac{1}{\sqrt{r_1}\sqrt{2l}}\left(-\frac{5}{8}\sin\frac{\theta_1}{2}+\frac{1}{8}\sin\frac{5\theta_1}{2}\right)\left(\sum_{k=-\infty}^{\infty} kf_kz^{k-1}\right)+O\left(r_1^0\right)\right\}_{\theta_1=\pi} \\ = & \frac{1}{\sqrt{l}}\left\{\frac{1}{\sqrt{l}}\left(-\frac{5}{8}+\frac{1}{8}\right)\left(\sum_{k=-\infty}^{\infty} kf_kz^{k-1}\right)+O\left(r_1\right)\right\} \\ = & -\frac{1}{2l}\left(-3f_{-3}a^{-4}-2f_{-2}a^{-3}-f_{-1}a^{-2}+f_1+2f_2a+3f_3a^2+4f_4a^3+\ldots\right)+O\left(r_1\right). \end{aligned} \tag{4.11}$$

In the similar way, the normalized SIF at tip b where θ_2 is 0 is

$$\begin{aligned} & \left[\frac{K_I}{K_{I\ w/o\ inclusion}}\right]_{at\ b} \\ = & \frac{\sqrt{2\pi r}}{\sigma_\infty\sqrt{\pi l}}\left\{\frac{1}{\sqrt{r_2}\sqrt{2l}}\left(\frac{5}{8}\cos\frac{\theta_2}{2}-\frac{1}{8}\cos\frac{\theta_2}{2}\right)\left(\sum_{k=-\infty}^{\infty} kf_kz^{k-1}\right)+O\left(r_2^0\right)\right\}_{\theta_2=0} \\ = & \frac{1}{\sqrt{l}}\left\{\frac{1}{\sqrt{l}}\left(\frac{5}{8}-\frac{1}{8}\right)\left(\sum_{k=-\infty}^{\infty} kf_kz^{k-1}\right)+O\left(r_2^0\right)\right\} \end{aligned} \tag{4.12}$$

$$= \frac{1}{2l}\left(-3f_{-3}b^{-4} - 2f_{-2}b^{-3} - f_{-1}b^{-2} + f_1 + 2f_2b + 3f_3b^2 + 4f_4b^3 + \ldots\right) + O\left(r_2\right).$$

4.4 Numerical Results

4.4.1 Verification of Formulae

For convenience, the imperfect interface parameters m and n introduced in Chapter 2 are expressed in the terms of new dimensionless interface parameter M, defined by $M_0 = \frac{m_0+n_0}{2\mu_1}R_0$ on Γ_0 and $M_1 = \frac{m_1+n_1}{2\mu_1}R_1$ on Γ_0. As mentioned before, these parameters represent the effectiveness of the bonding at the interface in transferring load thru interface. A small value of M or m (say, 0.01) represents debonding between adjoining materials while a large value of M or m (say, 100) simulates the perfect bonding condition. The varying values of M , n and m from 0.01 to 100 correspond to intermediate states between perfect bonding and complete debonding. These figures just represent the relative degree of bonding deterioration at the interfaces, not absolute.

The comparison method is to simulate the three phase perfect interface condition with a radial crack between the two interfaces as evaluated by Luo and Chen [48]. Taking M_0 and M_1 equal to 100, it is shown that the current formulation is in good agreement with those of Luo (see Figure 4.1). There are small differences in the actual values around interfaces and the differences seem to be from the number of unknown coefficients taken in the series expansions. Therefore, the good agreement with published results of crack behaviour confirms the validity of the equations. Expansion of the terms from z^{-4} to z^5 represents the minimum of terms in this analysis. Addition of more terms makes the numerical results converge to the exact solution addressed by Luo and Chen.

4.4.2 Influence of Imperfect Interface Γ_0 & Γ_1

The effect of an imperfect bonding condition at the inclusion-matrix and matrix-composite interfaces on SIFs at crack tips is evaluated for stiffer inclusion and softer than the matrix phase composites (Figure 4.2 and Figure 4.3). With respect to the GSC scheme, the softer composite phase results in mass damage in the surrounding matrix and inclusions outside the RVE. This case also represents the bonding medium that is stiffer than the matrix. In the case of a perfect bonding at the composite-matrix interface (M_1=100), the remote load is assumed to be fully transferred into the matrix and the inclusion-matrix interface. In Figure 4.2, it is clearly shown that the greater the degradation of the inclusion-matrix bonding Γ_0, the higher the stress intensity factor at the crack tips in the vicinity of the interface; hence, the degeneration of the inclusion-matrix interface fixation accelerates propagation from both crack tips adjacent to the interface in both directions. It shall be noted that the influence of the deterioration plays a dominant role in increasing SIFs, not only near tip a but also distant tip b. Cracks located around the interface Γ_0 easily propagate in both directions, deteriorating the integrity of the interface fixation, resulting in increased degradation, which accelerates the cracking process further. Another interesting phenomenon is the existence of a stable zone associated with debonding in which the SIFs corresponding to debonding are lower than those of the initial perfect bonding condition. As the distance between a crack and the inclusion-matrix interface increases, the SIFs corresponding to the imperfect interfaces decrease up to a certain crack location and increase beyond the minimum point. The SIFs associated with the degraded bonding are lower within the zone and higher beyond this zone than those with perfect bonding. In addition, within the zone, the debonding of the matrix-inclusion interface suppresses matrix cracking. In the vicinity of the matrix-composite interface, the influence of an imperfect matrix-inclusion interface increases SIFs at crack tip a and suppresses crack propagation at crack tip b. At the initial adhesion, when both interfaces are perfect, the stiff inclusion repels crack propagation from tip a and b around the inclusion-matrix interface and from tip a around the matrix-composite. It increases SIFs at crack tip b as the crack approaches the matrix-composite phase interface.

With respect to the influence of the composite-matrix interface, as the fixation at the composite-matrix interface Γ_1 deteriorates, the stable zone associated with debonding becomes wider and deeper, and the influence of both interfaces diminishes, since the load transferred into the matrix thru the matrix-composite interface decreases, Figure 4.3. The outer interface does not directly affect the crack propagation at tip a and b, but indirectly shifts the SIFs downward in conjunction with the influence of the inner matrix-inclusion interface. SIFs at both crack tips go to zero, regardless of crack locations, as the matrix becomes totally debonded (M_1=0.01) from the composite phase, indicating no load is being transferred thru this interface (Figure 4.4). The matrix-inclusion bonding condition is more critical for matrix cracking than matrix-composite adhesion quality. Liu and Kim addressed how imperfect bonding plays a dominant role in cracking outside an inclusion [27, 39]. Based on this work, it can be concluded that loosened matrix-composite phase fixation affects crack propagation outside the matrix, resulting in cracking or damage in the composite phase, namely altering the influence of neighbouring inclusions and the matrix. This type of cracking in the composite phase will be investigated in a future study.

4.4.3 Influence of Inclusion Stiffness μ_0 on Cracking

Figure 4.5 shows the influence of inclusion stiffness and bonding at the matrix-inclusion interface on SIFs of a crack fixed at d/R_0=0.4 for a soft composite phase. SIFs at both crack tips dramatically increase when the matrix-inclusion interface fixation becomes imperfect, M_0 close to zero. Stiffer inclusions lead to lower SIFs than compliant inclusions, although SIFs corresponding to stiff inclusions are still greater than 1. This phenomenon agrees with reported results that stiff inclusions repel cracks from the interface [27, 39]. Increasing inclusion stiffness beyond 25 times stiffer than the matrix, does not have a significant impact on suppressing crack propagation. Thus, determining proper stiffness of inclusions, up to the critical ratio of inclusion to matrix stiffness, is recommended to discourage crack propagation. In cases involving a stiffer composite phase -2 times harder than the matrix- the overall behaviour of the SIFs are similar, however, their values are less than 1. Thus, a stiffer composite phase repels crack propagation in the matrix; keeping the composite phase harder than the matrix is the best way to suppress crack propagation. When the outer interface degenerates, all of the SIFs shift downward. This phenomenon is illustrated in Figure 4.3.

4.4.4 Influence of Interphase Stiffness μ_1 on Cracking

Figure 4.6 presents the effect of matrix stiffness and degeneration of the matrix-inclusion fixation on cracking in the matrix. Similar to the results realized for inclusion stiffness, the deterioration of matrix-inclusion bonding accelerates matrix cracking in the vicinity of the interface. Contrary to the case of inclusion stiffness, the normalized SIF values increase as the stiffness of the matrix increases. Stiff matrix accelerates the cracking process in the matrix, while soft matrix suppresses crack propagation.

Comparing the magnitude of the changes in SIFs in Figure 4.6 with those in Figure 4.5, it can be seen that the ratio of matrix stiffness to the composite phase is more critical than that of the ratio between inclusions and the matrix. However, it should be noted that the soft matrix suppresses matrix cracking on itself, but may cause damage to the neighbouring matrix and inclusions in the vicinity of the matrix-composite phase interface. Thus, the advantages and disadvantages of the soft matrix properties should be weighed with regard to the total damage experienced by the inclusion-matrix-composite phase system. SIFs corresponding to matrix softer than the composite phase shift downward much more readily than SIFs corresponding to hard matrix as the matrix-composite interface starts to loosen. The effect of the loosening of the outer interface is similar to the effects mentioned above; however, its influence on a soft matrix is stronger than on a hard matrix.

4.4.5 Influence of Crack Length on Cracking

Interactions between imperfect matrix-inclusion bonding, SIFs and the length of a crack fixed in the vicinity of the matrix-inclusion interface (d/R_0=0.4) are presented in Figure 4.7. SIFs at crack tip a dramatically increase when fixation is highly degenerated: imperfection parameter M_0 goes to zero. Under poor bonding conditions, namely for $M_0 < 5$, the influence of the crack length on the propagation at tip a is not significant. However, with $M_0 > 10$, the longer crack results in higher SIFs at tip a than the shorter crack; therefore, under certain conditions involving adhesion damage, longer cracks are more vulnerable to cracking than shorter cracks at tip a. On the contrary, the behaviour of crack tip b is quite different from cracks at tip a. A Shorter crack's SIFs at tip b corresponding to highly imperfect bonding, $M_0 < 5$, are dramatically higher than those of a longer crack. The location of a longer crack's tip b is further away from the interface Γ_0 than that of a shorter crack and the degeneration at the interface has less influence on a longer crack's tip b. Microcracks are more vulnerable to cracking at tip b. With $M_0 > 10$, the effect of crack length on the propagation at tip b diminishes. Consequently, as the initial perfect bonding is damaged, long cracks are more vulnerable to propagating at tip a than short cracks, the propagation would cause further damage to the interface, and beyond certain damage, namely $M_0 < 5$, microcracks become more susceptible to proceeding from tip b than long cracks.

4.4.6 The Stable Equilibrium Position

The stable equilibrium position, called the trapping mechanism by Dundurs and Mura [13], is confirmed in this present work. As Luo and Chen [49] mentioned, the stable equilibrium position is more likely to occur in a three-phase model than a two-phase model and is dependent on material properties such as those addressed by Poisson's ratio. Luo and Chen explain that when an inclusion and a composite phase are stiffer than the matrix ($\mu_0 = \mu_2 = 1.25 * \mu_1$) with $v_0 = v_2 = 0.2$ & $v_1 = 0.4$, the dislocation does not have a stable equilibrium position. A simulation of this case was conducted as part of this study and the findings agree with their results. Given the values of Poisson's ratio and stiffness, all the SIFs at tip a are greater than 1, which means that there is no stable equilibrium position. Around tip b, all of the SIFs are between 0.94 and 1.06. With the assumption that interfaces are perfectly bonded, Luo and Chen addressed how stable equilibrium positions exist for any combination of Poisson's ratios where min $(\mu_0/\mu_1, \mu_2/\mu_1) > \sqrt{3}$. With $\mu_0 = \mu_2 = \sqrt{3.1} * \mu_1$ and the same Poisson's ratio as in the previous case, SIFs at tip a corresponding to relatively good bonding quality ($M_0 = 100$ & 10) and SIFs at tip b are less than 1, Figure 4.8.

These results show that there is a stable equilibrium position, which agrees with Luo and Chen's analysis. It is noted that the loosening of the inner interface is likely to decrease the range of the stable equilibrium position ($M_0 = 0.01$ to 1). When $\mu_0 = \mu_2 = 3\mu_1$, all the SIFs at tip a and b are less than 1. Thus, the interface damage and Poisson's ratio affect the trapping mechanism, but the stiffness of the composite phase and inclusion play a significant role in crack propagation. Luo and Chen did not identify the location of the stable equilibrium position within the intermediate region. However, in this study, the actual locations of the position are shown to be along the real axis.

4.4.7 The Stable Zone

Figures 4.9 and 4.10 show the effect of a stiffer composite phase on the stable zone associated with debonding. This new phenomenon called The Stable Zone is introduced in section 4.4.2. All of the conditions in Figure 4.9 and Figure 4.10 are the same as those in Figure 4.2 and 4.3, except for the stiffness of the composite phase. Compared to Figure 4.2 and 4.3, the hard composite phase definitely suppresses the SIFs at tip a for both M_1=100 and M_1=5. The stable zone near tip a with a compliant composite phase, as shown in Figure 4.2, does not take place for the stiffer composite phase shown in Figure 4.9. Regarding crack tip b, the stable zone does not take place with the soft composite phase (Figure 4.2). In Figure 4.9, shifting all the SIFs at tip b downward, the stiffer composite phase causes a stable zone to occur again. However, once the adhesion on the outer interface deteriorates (M_1: from 100 to 5), the stable zone goes away (Figure 4.10). Figure 4.10 shows that the stiffer composite phase changes the stable zone, resulting in a more narrow zone for M_1=5. It is important to note that the stiffer composite phase decreases or removes the effect of the zone and the debonding of the matrix-composite diminishes the effect of a stiff composite and causes the stable zone associated with debonding. Both the stiffness and the outer bonding condition M_1 affect the existence and shape of the stable zone. Consequently, the outer interface bonding and the stiffness of the outer composite phase are leading factors in the development of a stable zone associated with debonding.

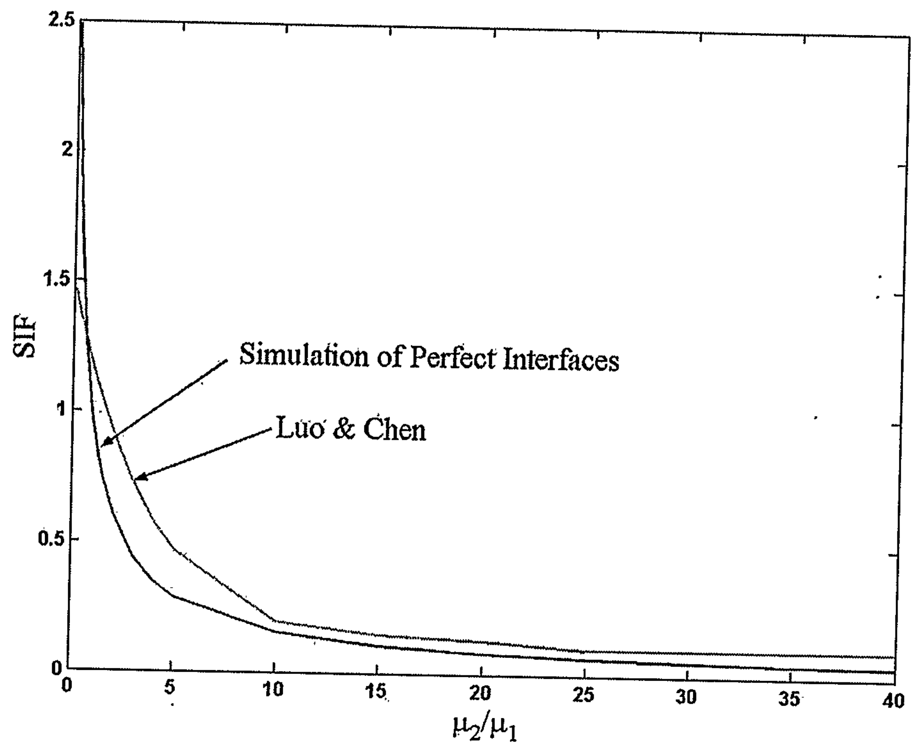


Figure 4.1 Comparison with Luo's results with perfect interfaces along Γ_0 and Γ_1.

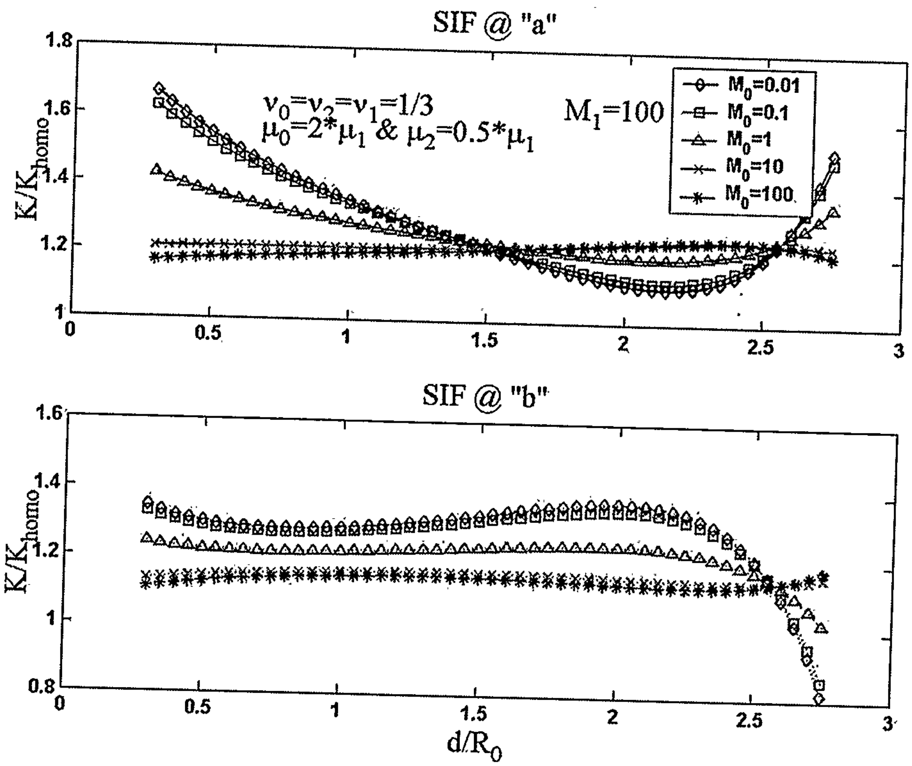


Figure 4.2 Influence of imperfect Γ_0 with perfect Γ_1 ($M_1 = 100$). Normalized SIFs of crack tip "a" & "b" vs. crack locations. $\mu_0 = 2\mu_1$, $\mu_2 = 0.5\mu_1$.

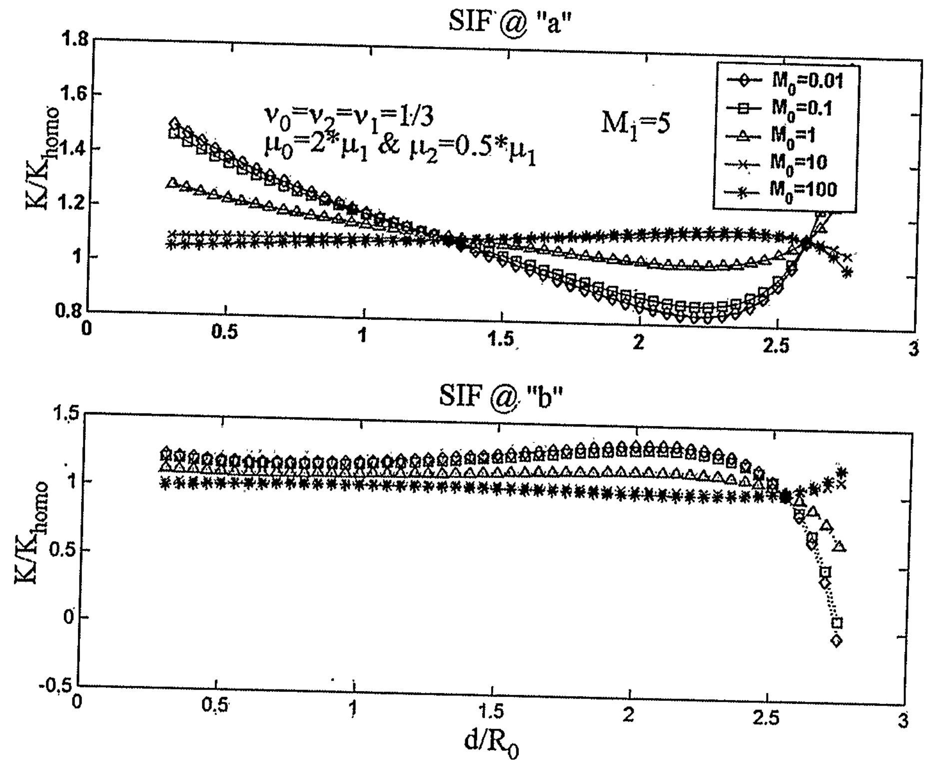


Figure 4.3 Influence of imperfect Γ_0 with imperfect Γ_1 ($M_1 = 5$). Normalized SIFs of crack tip "a" & "b" vs. crack locations. $\mu_0 = 2\mu_1$, $\mu_2 = 0.5\mu_1$.

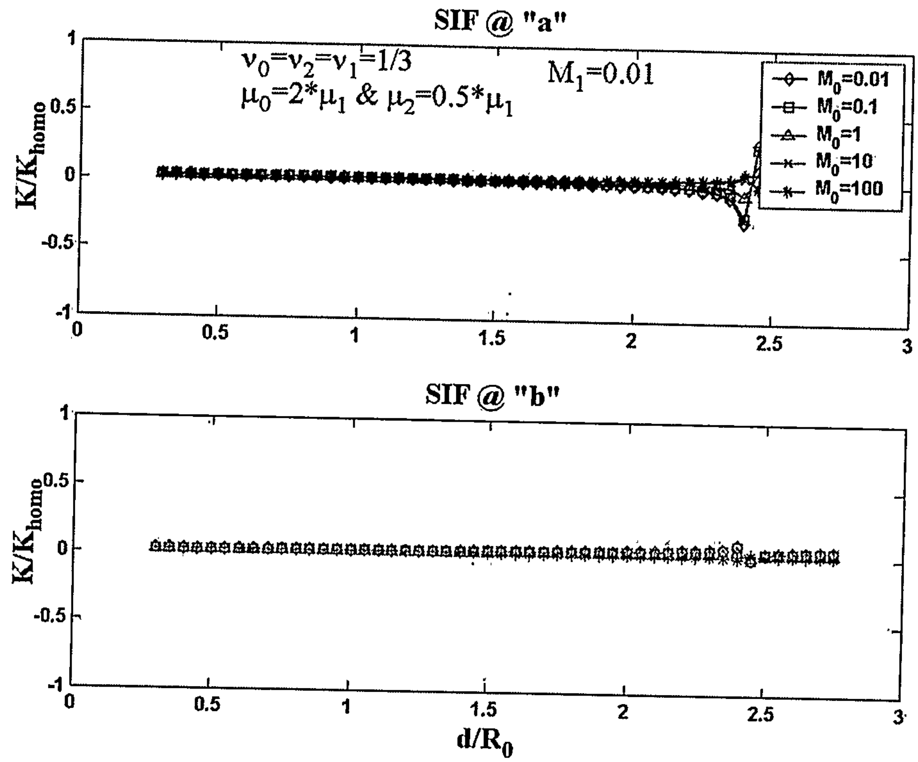


Figure 4.4 Debonding ($M_1 = 0.01$)

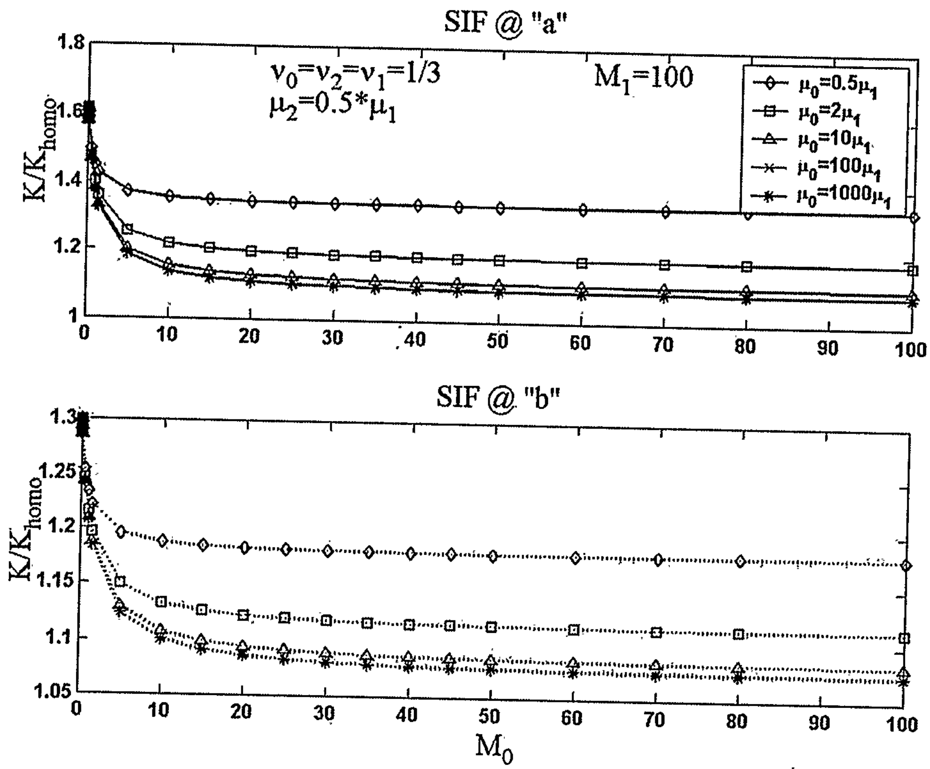


Figure 4.5 Influence of stiffness of an inclusion. Normalized SIFs of crack tip "a" & "b" vs. Γ_0 imperfect parameter M_0 at the fixed crack location. $\mu_2 = 0.5\mu_1$ and $d/R_0 = 0.4$.

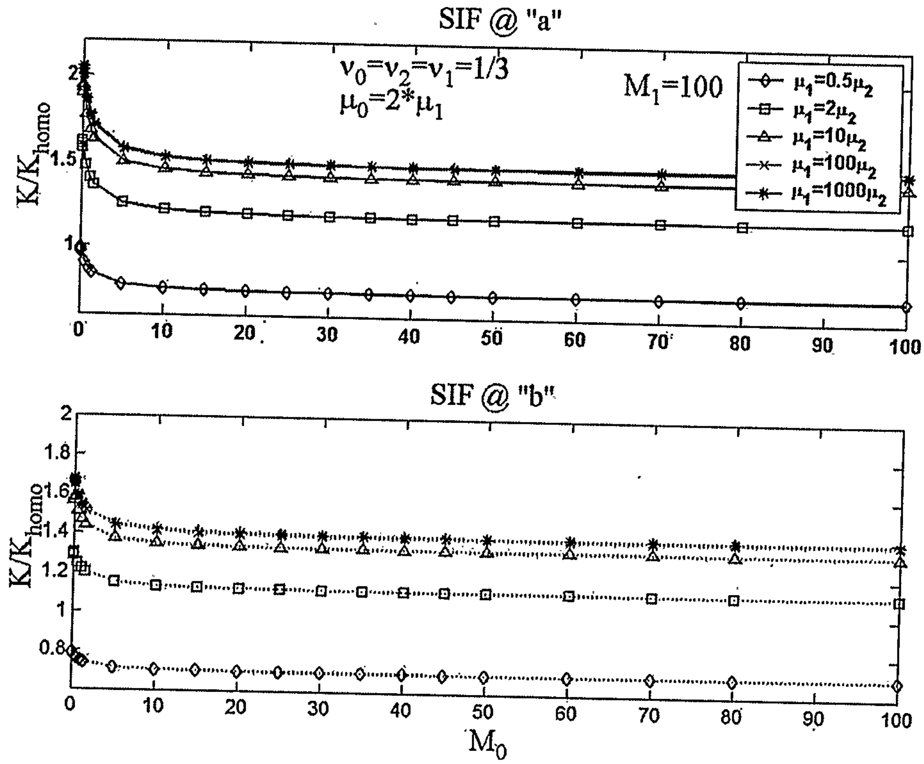


Figure 4.6 Influence of stiffness of the matrix phase. Normalized SIFs of crack tip "a" vs.Γ_0 imperfect parameter M_0 at the fixed crack location. $\mu_0 = 2\mu_1$ and $d/R_0 = 0.4$.

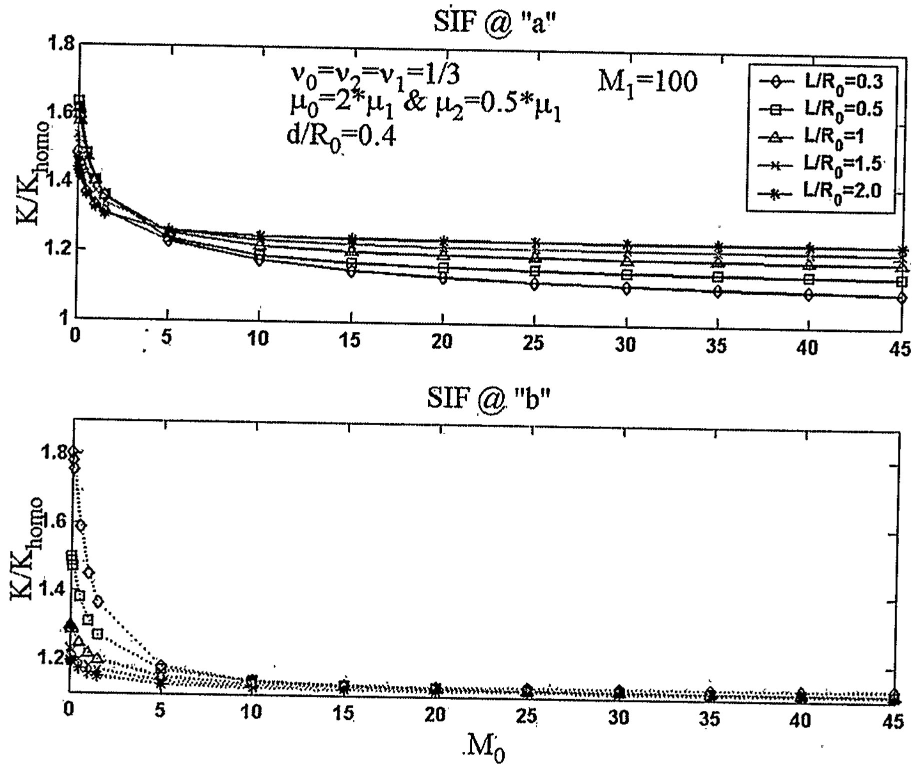


Figure 4.7 Influence of the crack length. Normalized SIFs of crack tip "a" vs. Γ_0 imperfect parameter M_0 at the fixed crack location. $\mu_0 = 2\mu_1$, $\mu_2 = 0.5\mu_1$ and $d/R_0 = 0.4$.

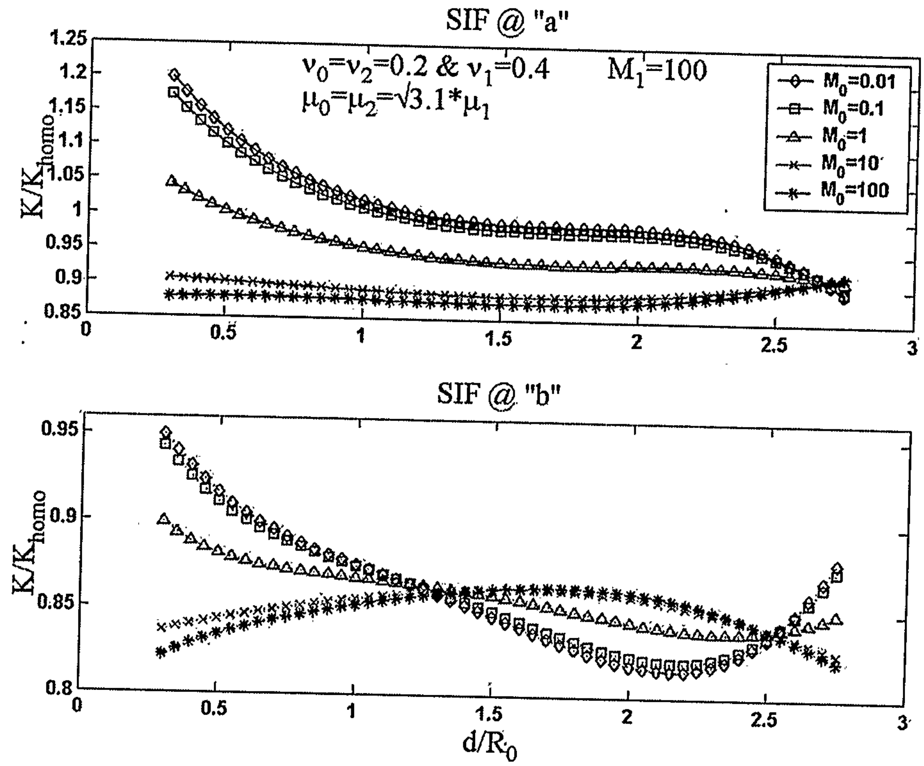


Figure 4.8 Influence of imperfect Γ_0 with perfect Γ_1 ($M_1 = 100$). Normalized SIFs of crack tip "a" & "b" vs. Crack locations. $\mu_0 = \mu_2 = \sqrt{3}\mu_1$.

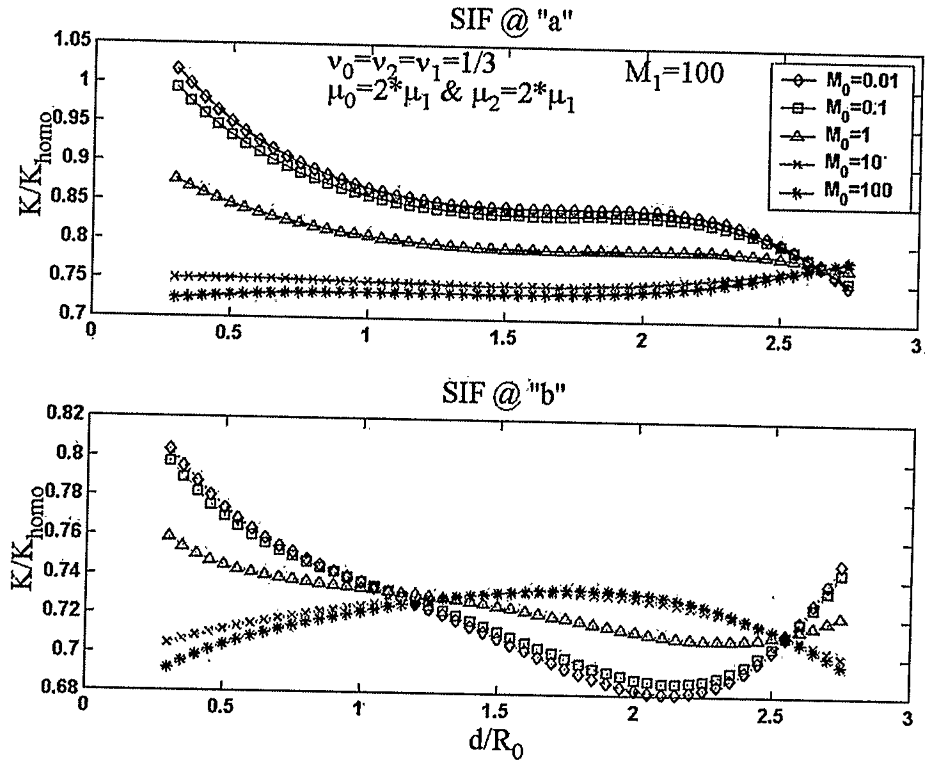


Figure 4.9 Influence of stiffer composite phase on the stable zone ($M_1 = 100$).

$\mu_0 = 2\mu_1$, $\mu_2 = 2\mu_1$.

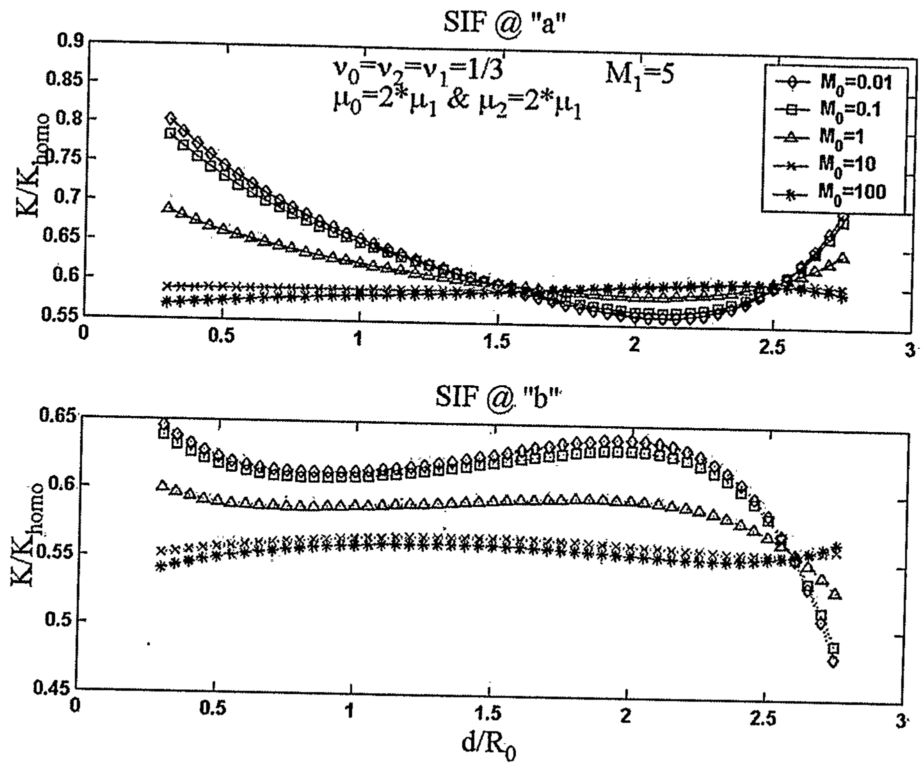


Figure 4.10 Influence of stiffer composite phase on the stable zone ($M1 = 5$).

$\mu_0 = 2\mu_1$, $\mu_2 = 2\mu_1$.

CHAPTER 5
An Example of Application - Bone/Implant Modeling

5.1 Introduction

According to the National Hospital Discharge Survey of 2003, approximately 217,000 and 402,000 patients in the U.S. underwent Total Hip Replacement (THR) and Total Knee Replacement (TKR) operations, respectively, and $24.7 billion dollars were spent in hospitalization related to these replacement surgeries. In addition, there were 36,000 revision hip replacements and 33,000 revision knee replacements. However, the revision rate of joint replacement is greater for those who are heavier or more active, such as young patients. Many surgeons and biomechanical scientists have asked the question whether femoral/tibial component fixation of total hip/knee arthroplasty shows better performance with or without bone cement and the answer is still quite controversial. With respect to fixation, performance of cemented prostheses was reported to be better than that of cementless protheses [58]. However, some researchers have demonstrated no differences between these two techniques [59]. Currently, the cemented surgery is more widely performed and several mechanisms of debonding of cemented implants from bone have been suggested.

Biologically, wear particles and debris from bone cement and implants spread into tissue within the vicinity of the interface and provoke osteolysis. Another important factor leading to debonding is the mechanical properties of cement and cementing techniques used during surgery. Although the dramatic improvements made in cementing techniques from the earlier hand-packed cementing to centrifugation and vacuum mixing have decreased the revision surgery rate [60, 61], poor mechanical properties of the bone cement and inadequate interdigitation adjacent to the bone cement interface are still being addressed as primary failure factors initiating the aseptic loosening of cemented prosthetics [62, 63, 64]. In addition, variables such as a surgeon's experience, patient weight and implant type are also implicated as factors leading to poor mechanical properties in the mantle [65]. Through such a multi-functional failure mechanism, the majority of popular cemented implants continue to migrate over time even though the fixation is perfect at the initial stage [58] and the loosening in the cement mantle is believed inevitable.

Interdigitation between bone and cement in well-fixed cemented implants keeps its mechanical interlocking long after implantation [66]. Poor interdigitation or permeation at the cement-bone interface [67], and deteriorated resistance at the cement-implant boundary, result in imperfect interface bonding. However, from the standpoint of sliding motion, the strong interdigitation between the cement and cancellous bone prevents relative slip at the interface and provides strong mechanical interlocking, which is referred to as a no-slip interface. The no-slip bonding enables shear loads to be sustained along the interface even when tractions fail in the normal direction. This no-slip condition is of considerable practical interest and is suggested as an adequate model of cement-bone interface [68, 69]. In contrast, implant-cement interface is either smooth for polished implants or mechanically interlocked for rough surface implants. The rough surface implants were introduced to provide mechanical interlocking over the whole interface and prevent the prosthesis from subsiding. It was reported that rough implants increased friction between cement and an implant and reduced its subsidence and the effects of debonding at the interface [70]. It is noted that this mechanical interlocking is not as strong as interlocking on a cement-bone interface. In contrast to the longevity of cement-bone interdigitation, mechanical interlocking between a rough stem and cement vanishes once loosening of the stem progresses.

Of the many factors causing aseptic loosening, mechanical loading is the most commonly accepted as the leading failure factor [71, 72]. Repeated mechanical loads, in conjunction with other factors mentioned earlier, have resulted in the continuing migration in the cement mantle and finally loosening of prosthetic components. This multifunctional mechanism has attracted the attention of many researchers. The aseptic loosening of cemented implants is most frequently a long-term complication [73, 74]. To evaluate the influence of the loads, researchers have investigated the stress field in the region adjacent to the implant and cracking at the cement-bone and cement-implant interfaces under cyclic loads.

With finite element analysis, interactions and relationships between stress magnitude in the cement mantle and failure have been reported [75]. However, experimental investigations have addressed how the predicted magnitude of the peak stress is not a dominant factor leading to clinical failure [76]. Damage accumulation theory was suggested as a failure scenario of joint replacement [77]. The propagation process of cracks in the cement mantle was investigated experimentally, and pre-load and load-initiated cracks have been reported to play a determinant role in progress of failure [78]. Furthermore, it has been demonstrated that failure of the cement mantle caused by radial cracks in the mantle initiated loosening of cemented implants [79] and many radial cracks generated adjacent to the cement-implant interface were found by SEM micrographs after stem failure [70]. Also, the bonding conditions of cement-bone and cement-implant interfaces are believed to affect aseptic loosening. Deterioration of the interface condition between cancellous bone and implants is considered to be a significant failure mechanism, which results in mechanical fracture and debonding of the prosthesis from the bone [80, 81, 82].

The interface between an implant and the cement has also been attracting the attention of researchers. Different surface finishing techniques, designed to improve the strength and fixation at the interface, have been suggested [83]. Repeated cyclic stress imposed on artificial implants results in loosening between the prosthesis and the bone, and the development of load-initiated cracks in the vicinity of the cement-bone interface [78]. Debonding and resistance of the cement-implant interface was investigated with respect to residual stress developed during cement curing [84]. A theoretical mechanical failure process of metal-cement interface, with respect to the cohesive zone, was proposed [85]. Whether the cracks and debonding begin first from the cement-implant interface or from the cement mantle has not yet been clearly determined [86, 87, 88]. Gharpuray developed a hypothesis that the initial cracks were emanating from the perfectly bonded interfaces, and consequently, they accelerated degradation of the prosthesis fixation, micro-motions between bone and implant, and bone resorption around the implant [71]. Gharpuray proposed a theoretical model to show cracking around a void and a circular elastic inclusion in the vicinity of the cement-implant interface and demonstrated how the cracking caused loosening of the fixation. A majority of this research has assumed perfect bonding in which displacement compatibility at the cement-implant and cement bone interface holds. However, with respect to the unavoidable degradation of the fixation, this study demonstrates that the imperfectness of the interface resistance should be considered and the inevitable interaction between deterioration in the interface bonding quality and crack behaviours within the cement mantle, or local region adjacent to the interfaces, causes catastrophic failure of joint replacement arthroplasty.

As addressed in Chapter 1, imperfect bonding between two materials with different mechanical properties has been known to be a dominant factor affecting crack propagation located outside the inclusion in the vicinity of the interface [27]. Furthermore, this thesis provides a general three-phase model, which considers imperfect bonding conditions along both interfaces. Thus, the proposed model is well suited to the bone cement cracking interacting with an implant and the bone matrix under damaged adhesion conditions along both boundaries. This research provides, for the first time, analytical rationale for the possible failure mechanism of a cemented implant system in the full range of bone cement. This study addresses how the imperfectness of both the cement-bone and cement-implant interface simultaneously affects radial crack propagation in bone cement during the mechanical cracking process. Specifically, a no-slip condition on cement-bone and cement-stem interfaces are taken into account. The procedures addressed in the previous chapters are applied to this specific application. Thus, the imperfectness of the interfaces is modeled with interface parameters and the effect of the parameters on SIFs at crack tips is evaluated along a radial crack location from implant-cement interface to cement-bone interface.

5.2 Formulation

Figure 5.1 shows a simplified model of cemented joint arthroplasty. We consider a domain of $\mathbb{R}^2$, infinite in extent, containing an implant, bone cement PMMA and bone matrix with a radial crack in cement. The circular implant, with center at the origin of the coordinate system and radius of R_0, occupies a region denoted by the domain S_0 and the cement is also modeled as a circle with radius of R_1 represented by the domain S_1. The surrounding bone matrix is represented by S_2. All the materials are assumed to be homogeneous and isotropic with shear moduli μ_0, μ_1 and μ_2, respectively. Interfaces Γ_0 and Γ_1 are imperfect and represent implant-cement and cement-bone interface, respectively and the adhesion imperfectness in the radial and the tangential direction is represented by the non-negative interface parameter m and n, respectively. All the notations, symbols and boundary conditions are same as ones specified in Chapter 2.

5.3 Discussion

The crack length is fixed as $l = R_0$ and the other mechanical properties used during numerical calculation are shown in Table 5.1. Linear elastic fracture mechanics is applicable to brittle materials which do not have a considerable plastic zone near the crack tips. PMMA, the most common bone cement material, is an amorphous brittle material with very high mechanical strength. Thus, our approach to applying linear elastic fracture mechanics to this brittle PMMA is reasonable. Although the bone cement contains some inclusions introduced during its formation, such as beads and voids which cause less isotropic and homogeneous properties locally, the influences of the inclusions on cement properties are small and can be ignored. Therefore, the bone cement is considered isotropic and homogeneous. While the cement-implant interface is clearly shown physically, interface between cement and bone can not be simply defined, due to interdigitation and permeation. However, the simple model of the interfaces as a single layer is enough to give us insight into the cracking process initiated by imperfect bonding. Uniaxial compression is the most likely loading in bone cement; however, uniaxial tension is occasionally applied to bone cement. As addressed by Gharpuray, the uniaxial compression case can also be obtained as the negative of the tension value [71].

The result of the no-slip conditions on both interfaces is shown on Figure 5.2. These conditions can be achieved by using rough surface implants. No-slip interfaces dramatically reduce the effects of loosening in the normal direction on SIFs at the crack tips. Tractions in the tangential direction transfer to the cement and then to an implant through interfaces, and compensates for reduction of tractions in the normal direction. Thus, imperfect bonding in the normal direction has little influence on the SIFs at the crack tips. SIFs at tip a continuously increase as the crack approaches the outer interface. Since the outer bone matrix is much more compliant than cement, it attracts crack propagation and increases stress intensity in the vicinity of its interface. Similarly, SIFs at tip b have a maximum value near the outer bone cement interface. It is noticed that the stiff implant - 26 times the cement shear modulus - does not have a noticeable effect on the SIFs at both tips in the region adjacent to the cement-implant interface, while the change in shear modulus across the cement-bone interface dramatically alters the magnitude of SIFs near the outer interface. The cement-bone interface is the first barrier which transforms the tractions initiated from remote loading conditions and has greater influence on crack behaviour than the cement-implant interface. Preventing slip on the cement-implant interface results in very stable behaviour of tip b except at the region adjacent to the cement-bone interface.

Figure 5.3 shows crack behaviour with no-slip and imperfect bonding in the normal direction on the cement-bone interface. This is a case in which a polished implant has been embedded into cement and the cement is interdigitated with cancellous bone. A rough implant can be modeled with this type of debonding, when bonding at the cement-implant interface deteriorates and mechanical interlocking in tangential direction is no longer sustained. The imperfect cement-implant interface attracts propagation at tip a and accelerates opening of crack tip b in the vicinity of the cement-implant interface. As for tip a behaviour, debonding at the cement-implant interface causes higher SIFs than perfect bonding up to a certain point; beyond this point, the effect of the debonding decreases, since the crack is far from the interface. Soft bone matrix attracts crack propagation at tip a as long as the cement-implant bonding is strong enough, and attracts propagation of cracks at tip b near the cement-bone interface at the initial perfect bonding. SIFs increase up to a critical point around the cement-bone interface as the degeneration of cement-implant bonding worsens. Debonding at the cement-implant interface results in failure of load transfer from the remote load to the implant. Thus, the SIFs at the tip close to the cement-bone interface dramatically decrease beyond the critical point. The interesting behaviours at tip b are the existence of a suppression region where the normalized SIF is below 1. At the initial stage, when integrity of the bonding is perfect, there is a stable region for crack tip b which disappears as the implant-cement interface starts to loosen. Thus, debonding at the cement-implant interface accelerates crack propagation at tip b and promotes crack opening at both tips in the region adjacent to the cement-implant interface.

While the cement-cancellous bone interface has strong interdigitation and no-slip bonding, mechanical interlocking of the cement-implant interface in the tangential direction is weaker than interlocking of the cement-cancellous interface. Figure 5.4 shows the effect of the weaker, no-slip cement-implant interface on crack behaviour. Parameter n in the tangential direction on the cement-implant interface is reduced to 10 to simulate the weakened no-slip bonding condition. The result shows that this is an intermediate stage between Figure 5.2 and 5.3. While cement-implant bonding in the normal direction degrades from 100 to 0.01, the constant bonding parameter n in the tangential direction continues to carry loads and reduces the effect of the degeneration of bonding. Overall SIFs are lower than those in Figure 5.3.

The behaviours of crack tips near both interfaces are of considerable practical interest, since crack propagation in the vicinity of the interfaces accelerates degradation of interface bonding. The existence of mechanical interlocking in the tangential direction on the cement-implant interface reduces SIFs within the region adjacent to the interfaces. Specifically, it repels crack propagation near the cement-implant interface by decreasing the effect of imperfect bonding. One outstanding advantage of a no-slip cement-stem interface is that it prevents further progress of cement cracking from a good initial condition by maintaining the initial pattern of crack behaviour. Furthermore, from the standpoint of damage accumulation theory, a no-slip cement-stem interface suppresses overall SIFs in the cement mantle; hence, it decreases damage accumulation throughout the mantle. However, longevity and durability of a no-slip cement-implant interface is still questionable, since the rough implants do not provide strong mechanical interlocking and can result in breaking of the interdigitation, inappropriate permeation and breaking debris. Further design features to prevent slip on the interface are required.

In this present study, a titanium alloy implant is used during numerical calculations. Crack behaviours with stiffer alloys such as Cr/Co and stainless steel alloys produce similar results as those with titanium alloy. It is shown that changes in implant properties do not have a substantial influence on crack propagation. A similar result was reached by a study that noted how the elastic modulus of an implant does not alter the load transfer mechanism between the cement and the prosthesis, since the elastic modulus of the implant material is much higher than that of the cement and bone [79].

Material	Poisson's Ratio, ν	Shear Modulus(GPa), μ
Titanium	0.3	41
PMMA	0.3	1.54
Bone	0.3	0.13

Table 5.1 Properties of materials for orthopaedic arthroplasty

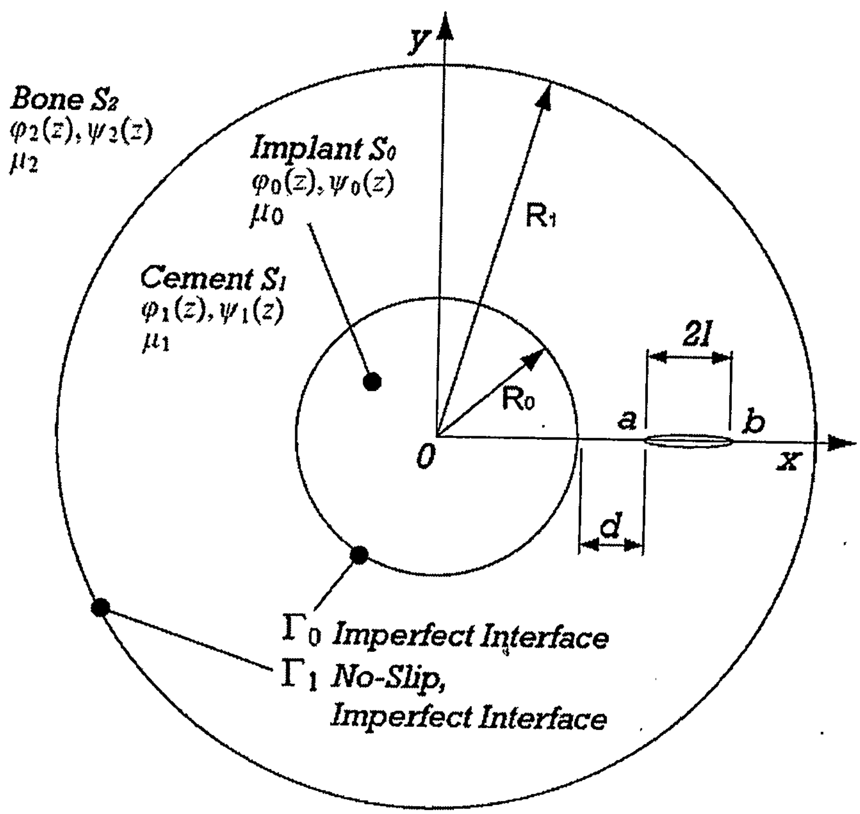


Figure 5.1 Three-phase stem-cement-bone system.

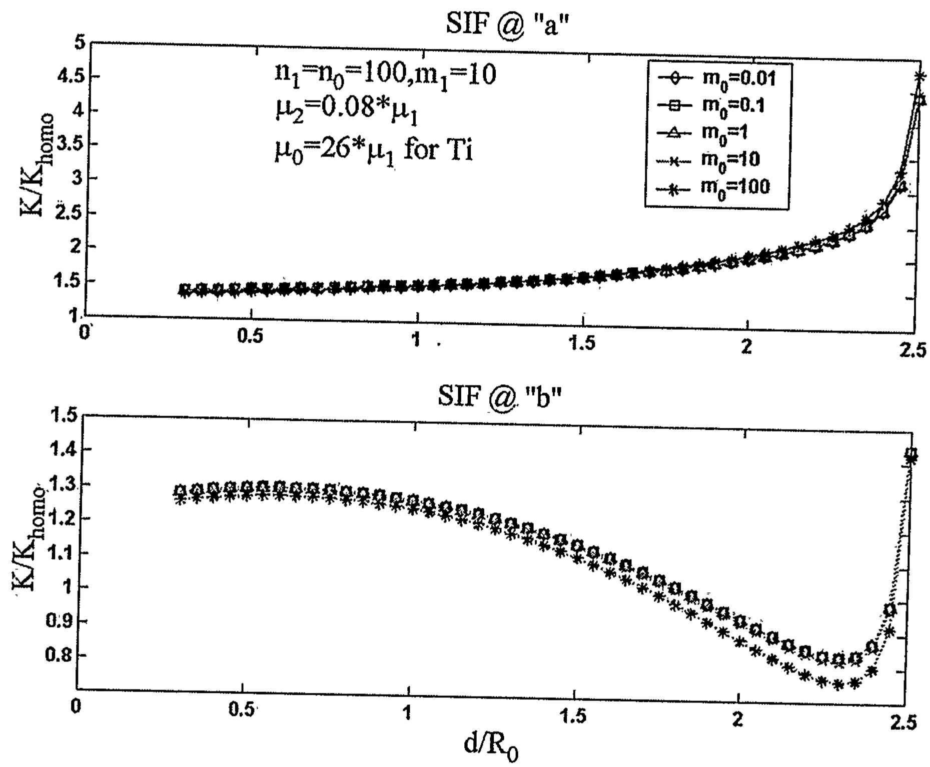


Figure 5.2 Normalized SIFs of crack tip "*a*" and "*b*" vs. Crack locations. No-slip imperfect stem-cement/ cement-bone bonding ($m_1 = 10$)

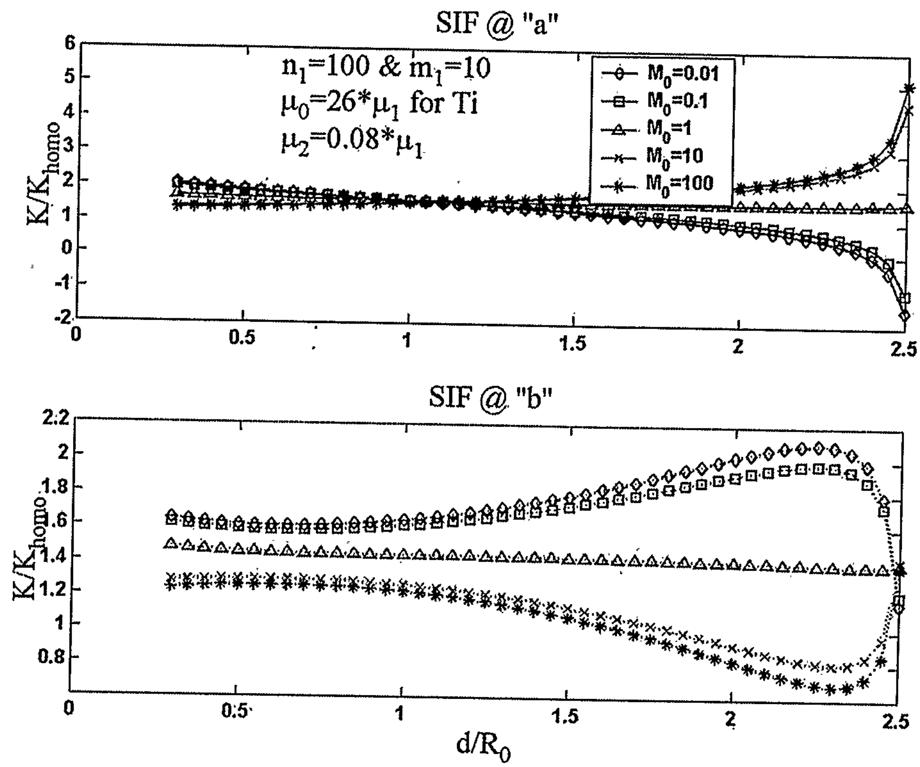


Figure 5.3 Normalized SIFs of crack tip "a" and "b" vs. Crack locations. No-slip imperfect cement-bone bonding ($m_1 = 10$).

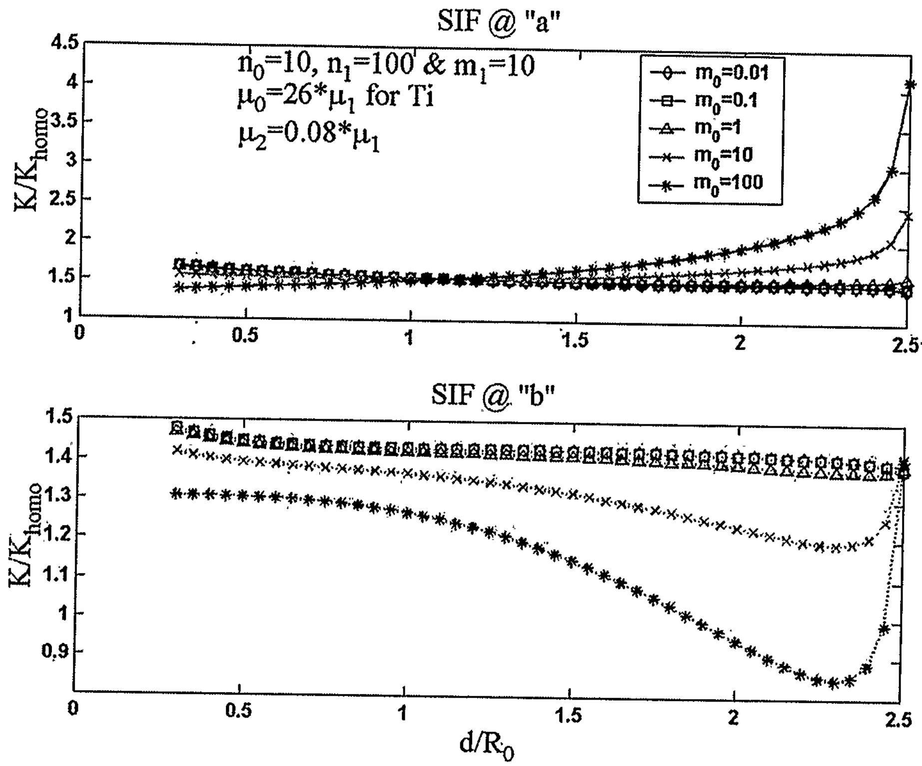


Figure 5.4 Normalized SIFs of crack tip "*a*" and "*b*" vs. Crack locations. Weakened no-slip imperfect stem-cement; No-slip imperfect cement-bone bonding.

CHAPTER 6
Conclusions and Future Works

6.1 Conclusions

The most commonly used model of classical micromechanics is a two-phase single inclusion surrounded by an infinite elastic medium, called matrix phase, with perfect interface. This simple model has been providing insight into the interaction between cracking, or fracture mechanism, and an inclusion. However, since the presence of the interphase layer has been identified as a major parameter in altering stress fields around the inclusion, the study of the three-phase model, in which the interphase layer is considered, has become of practical and theoretical interest. Also, the three-phase model is quite general and is applicable to the generalized self-consistent and implant/cement/bone system. Furthermore, many researchers have shown that perturbation in the stress fields around the inclusion is caused by interface bonding conditions, which draws attention to the importance of the imperfect bonding. They have also addressed how there are significant differences between previous perfect bonding and imperfect bonding in micromechanics. There are a few studies which consider the three-phase model with imperfect bonding along one interface. However, due to the cumbersome mathematical problems involved, there has been no research in the literature to address the imperfectness of both interfaces using the three-phase model.

In this study, a semi-analytic solution to the interaction between a pre-exiting crack and a three-phase inclusion with imperfect bonding along both boundaries is presented. For the analytic approach, linear elasticity with complex variable techniques and linear elastic fracture mechanics are applied to the system. Also, to overcome the mathematical difficulties regarding imperfect interfaces, analytical continuations and series methods are employed. With the fundamental theories applied to the three-phase single inclusion system, the problem reduces to a boundary value problem. By solving 38 coupled linear algebraic equations numerically, unknown coefficients of stress potentials representing stress fields near crack tips have been determined. The coefficients reflect the effects of damaged interfaces, Poisson's ratio and material stiffness on SIFs at each of the radial crack tips. An interpretation of the numerical results is provided in Chapter 4 and an analytical solution to the implant/bone/cement system is presented as an application example in Chapter 5. For verification of the formulae derived in this dissertation, comparisons are drawn with corresponding cases published in the literature.

New findings established through the numerical analysis results presented in Chapter 4 are as follows:

-. The proper stiffness of fibres as an inclusion must be determined. Benefit is seen up to a critical ratio of fibre to the matrix; increasing fibre stiffness beyond this critical ratio has little effect on crack propagation. Also, keeping the composite phase harder than the matrix is the best way to suppress crack propagation.

-. Outer interface degeneration shifts all of the SIFs at both crack tips downward.

-. Ratio of composite phase stiffness to matrix phase stiffness is more critical than the ratio for stiffness between inclusions and the matrix. SIFs corresponding to a softer than the matrix composite phase, shift downward more readily than those of a harder matrix, as the matrix-composite interface starts to loosen.

-. Regarding crack lengths, longer cracks under certain damaged adhesion conditions are more vulnerable to cracking than short cracks at near tip a, while microcracks are more vulnerable to cracking at tip b. Consequently, as the initial perfect bonding becomes damaged, long cracks are more vulnerable to propagating at tip a, and the propagation can result in further damage on the inner interface M_0. Beyond certain damages, microcracks are more susceptible to proceeding from tip b.

-. The stable equilibrium position, called the trapping mechanism by Dundurs and Mura, is known to be affected by other factors, such as the interface condition, Poisson's ratios and stiffness. In their analysis, they demonstrated that the stable equilibrium position does not take place under certain conditions that lead to inner interface loosening. Thus, the damage of interface, Poisson's ratio and stiffness play a pronounced role in the trapping mechanism.

-. A new phenomenon, called the *stable zone*, which is associated with debonding, is introduced. In this zone, SIFs corresponding to debonding are lower than those of an initial perfect adhesion. This zone is affected by Poisson's ratio, material stiffness and crack location. The degradation of adhesion accelerates cracking overall; however, in a certain region, it depresses crack propagation. Stiffer composite phase decreases or removes the effect of the zone and the debonding of the matrix-composite diminishes the effect on the stable zone.

Thus, cracking at the matrix is affected by various factors such as degree of imperfect interface bonding, crack length, mechanical properties of the matrix and inclusion, and crack location. Of the parameters, the bonding condition along the inclusion and the matrix has a profound effect on the failure mechanism in the vicinity of the interface, regardless of other factors.

In addition, Chapter 5 addresses how the mechanical interlocking in the tangential direction on the cement-stem interface substantially suppresses crack development throughout the cement mantle and debonding on the interface accelerates cement cracking in the vicinity of the implant. The design of special stems with strong interdigitation, namely non-slip interface, is of practical importance. It is noted that changing implant materials does not have a substantial impact on crack propagation within the bone cement mantle.

This research contributes to the understanding of the interaction between matrix cracking and homogeneously imperfect interfaces, and introduces a new phenomenon, called the stable zone, associated with debonding. In addition, the interaction between cement crack propagation and the influence of non-slip implant-cement interface, namely rough implant interface, is explored and guidance for the design of implants is presented.

6.2 Future Works

This research deals with single inclusion embedded in an infinite homogeneous isotropic elastic medium that interacts with an external pre-existing crack and imperfect adhesion interfaces. To further the knowledge gained through this dissertation, the author anticipates the following future works:

-. The schematic study of the effect of inhomogeneous, imperfect bonding along both interfaces in the three-phase model. Homogeneous imperfection along the interfaces is an idealization of inhomogeneous bonding. Incorporating the inhomogeneous imperfect bonding case into the three-phase model will provide a more realistic solution.

-. The schematic study of cracking within the outer infinite phase. Kim and Sudak [39] studied this matrix cracking in the three-phase model by employing inner imperfect interface only, which might underestimate the perturbation effect caused by the outer bonding condition. This future study, I believe, will provide helpful information regarding the bone damage process which is caused by damage to the implant/cement and cement/bone bonding.

-. The schematic study of interaction between two cracks within the intermediate matrix phase. In real life situations, there are many microcracks and voids in the intermediate layer, such as bone cements, and it is well known that the aggregation of the near cracks/voids causes catastrophic failures in implant arthroplasty and composite materials. The study of the behaviour of cracks interacting with each other under imperfect interface conditions will undoubtedly provide a foundation for exploring the relationships between crack aggregation and fracture mechanisms.

Hence, deploying parameters governing fracture mechanisms, such as mechanical properties (stiffness and Poisson's ratio) and physical conditions (bonding condition, crack length and multiple cracks) in a micromechanics analysis, and working to understand the fracture mechanism analytically, will lead to the development of a foundation through which failures in many engineering applications can be predicted and prevented.

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APPENDIX 1
Derivation of Boundary Condition $\partial\Gamma_0$

A1.1 Derivation of Equation (2.9)

$$\begin{aligned}
(\sigma_{rr} - i\sigma_{r\theta})_0 &= \left(m_0 \|u_r\| - m_0 u_r^0\right) - i\left(n_0 \|u_\theta\| - n_0 u_\theta^0\right) \qquad \text{(A1.1)}\\
&= \left(\frac{m_0}{2}\|u_r\| + \frac{m_0}{2}\|u_r\|\right) - \left(i\frac{n_0}{2}\|u_\theta\| + i\frac{n_0}{2}\|u_\theta\|\right) - \left(m_0 u_r^0 - i n_0 u_\theta^0\right)\\
&= \left(\frac{m_0}{2}\|u_r\| + \frac{m_0}{2}\|u_r\|\right) - \left(i\frac{n_0}{2}\|u_\theta\| + i\frac{n_0}{2}\|u_\theta\|\right) - \left(m_0 u_r^0 - i n_0 u_\theta^0\right)\\
&\quad + \left(\frac{n_0}{2}\|u_r\| - \frac{n_0}{2}\|u_r\|\right) + \left(i\frac{m_0}{2}\|u_\theta\| - i\frac{m_0}{2}\|u_\theta\|\right)\\
&= \left(\frac{m_0}{2}\|u_r\| - \frac{n_0}{2}\|u_r\| + \frac{m_0}{2}\|u_r\| + \frac{n_0}{2}\|u_r\|\right) - \left(m_0 u_r^0 - i n_0 u_\theta^0\right)\\
&\quad - \left(i\frac{n_0}{2}\|u_\theta\| - i\frac{m_0}{2}\|u_\theta\| + i\frac{n_0}{2}\|u_\theta\| + i\frac{m_0}{2}\|u_\theta\|\right)\\
&= \left(\frac{m_0 - n_0}{2}\right)\|u_r\| + \left(\frac{m_0 + n_0}{2}\right)\|u_r\| - \left(m_0 u_r^0 - i n_0 u_\theta^0\right)\\
&\quad + \left(\frac{m_0 - n_0}{2}\right) i\|u_\theta\| - \left(\frac{m_0 + n_0}{2}\right) i\|u_\theta\|\\
&= \left(\frac{m_0 - n_0}{2}\right)\|u_r + i u_\theta\| + \left(\frac{m_0 + n_0}{2}\right)\|u_r - i u_\theta\| - \left(m_0 u_r^0 - i n_0 u_\theta^0\right).
\end{aligned}$$

A1.2 Derivation of Equation (2.12)

From equation (2.9).

$$(\sigma_{rr} - i\sigma_{r\theta})_0 = \left(\frac{m_0 - n_0}{2}\right)\|u_r + i u_\theta\| + \left(\frac{m_0 + n_0}{2}\right)\|u_r - i u_\theta\| - \left(m_0 u_r^0 - i n_0 u_\theta^0\right). \qquad \text{(A1.2)}$$

By taking the right-hand-side in (2.7), the left-hand-side of (A1.2) is

$$(\sigma_{rr} - i\sigma_{r\theta})_0 = \varphi_0'(z) + \overline{\varphi_0'}\left(\frac{R_0^2}{z}\right) - z\varphi_0''(z) - \frac{z^2}{R_0^2}\psi_0'(z). \tag{A1.3}$$

By $(2.1)_1$,

$$\begin{aligned} u_r + iu_\theta &= \frac{e^{-i\theta}}{2\mu}\left[\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}\right] \\ &= \frac{R_0}{z2\mu}\left[\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}\right]. \end{aligned} \tag{A1.4}$$

Thus,

$$\begin{aligned} \|u_r + iu_\theta\| &= \frac{R_0}{z}\frac{1}{2\mu_1}\left[\kappa_1\varphi_{11}(z) - z\overline{\varphi_{11}'(z)} - \overline{\psi_{11}(z)}\right] \\ &\quad -\frac{R_0}{z}\frac{1}{2\mu_0}\left[\kappa_0\varphi_0(z) - z\overline{\varphi_0'(z)} - \overline{\psi_0(z)}\right]. \end{aligned} \tag{A1.5}$$

Taking conjugate of (A1.4) leads to

$$\begin{aligned} u_r - iu_\theta &= \frac{e^{i\theta}}{2\mu}\left[\kappa\overline{\varphi(z)} - \overline{z}\varphi'(z) - \psi(z)\right] \\ &= \frac{z}{R_0 2\mu}\left[\kappa\overline{\varphi(z)} - \overline{z}\varphi'(z) - \psi(z)\right]. \end{aligned} \tag{A1.6}$$

Thus,

$$\begin{aligned} \|u_r - iu_\theta\| &= \frac{z}{2R_0}\frac{1}{\mu_1}\left[\kappa_1\overline{\varphi_{11}(z)} - \overline{z}\varphi_{11}'(z) - \psi_{11}(z)\right] \\ &\quad -\frac{z}{2R_0}\frac{1}{\mu_0}\left[\kappa_0\overline{\varphi_0(z)} - \overline{z}\varphi_0'(z) - \psi_0(z)\right]. \end{aligned} \tag{A1.7}$$

Substituting (A1.5), (A1.7) and (2.11) into (2.9), we can express the right-hand-side in the following form

$$
\begin{aligned}
&\frac{m_0-n_0}{2}\left\|u_r+iu_\theta\right\|+\frac{m_0+n_0}{2}\left\|u_r-iu_\theta\right\|-\left(m_0u_r^0-in_0u_\theta^0\right) \qquad \text{(A1.8)}\\
=\;&\frac{m_0-n_0}{2}\frac{R_0}{2z}\left\{\begin{array}{l}\frac{1}{\mu_1}\left[\kappa_1\varphi_{11}(z)-z\overline{\varphi_{11}'(z)}-\overline{\psi_{11}(z)}\right]\\ -\frac{1}{\mu_0}\left[\kappa_0\varphi_0(z)-z\overline{\varphi_0'(z)}-\overline{\psi_0(z)}\right]\end{array}\right\}\\
&+\frac{m_0+n_0}{2}\frac{z}{2R_0}\left\{\begin{array}{l}\frac{1}{\mu_1}\left[\kappa_1\overline{\varphi_{11}(z)}-\overline{z}\varphi_{11}'(z)-\psi_{11}(z)\right]\\ -\frac{1}{\mu_0}\left[\kappa_0\overline{\varphi_0(z)}-\overline{z}\varphi_0'(z)-\psi_0(z)\right]\end{array}\right\}\\
&-m_0R_0\varepsilon_1-\frac{(m_0+n_0)(\varepsilon_2-i\varepsilon_3)}{2R_0}z^2-\frac{(m_0-n_0)(\varepsilon_2+i\varepsilon_3)R_0^3}{2z^2}\\
=\;&\frac{m_0-n_0}{2}\frac{R_0}{2z}\left\{\begin{array}{l}\frac{1}{\mu_1}\left[k_1\varphi_{11}(z)-z\overline{\varphi_{11}'}\left(\frac{R_0^2}{z}\right)-\overline{\psi_{11}}\left(\frac{R_0^2}{z}\right)\right]\\ -\frac{1}{\mu_0}\left[k_0\varphi_0(z)-z\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)-\overline{\psi_0}\left(\frac{R_0^2}{z}\right)\right]\end{array}\right\}\\
&+\frac{m_0+n_0}{2}\frac{z}{2R_0}\left\{\begin{array}{l}\frac{1}{\mu_1}\left[k_1\overline{\varphi_{11}}\left(\frac{R_0^2}{z}\right)-\overline{z}\varphi_{11}'(z)-\psi_{11}(z)\right]\\ -\frac{1}{\mu_0}\left[k_0\overline{\varphi_0}\left(\frac{R_0^2}{z}\right)-\overline{z}\varphi_0'(z)-\psi_0(z)\right]\end{array}\right\}\\
&-m_0R_0\varepsilon_1-\frac{(m_0+n_0)(\varepsilon_2-i\varepsilon_3)}{2R_0}z^2-\frac{(m_0-n_0)(\varepsilon_2+i\varepsilon_3)R_0^3}{2z^2}.
\end{aligned}
$$

Let's consider resultant forces expressed in (2.2) along a contour Γ_0. The contour is a circle: namely, a point "B" on the arc coincides with the starting point "A", which means the resultant force between "A" and "B" is zero. Let "A" be a reference point which is 0 and "B" any point along the circle. Thus, (2.2) is

$$F_x+iF_y=-i\left[\varphi_0(z)+z\overline{\varphi_0'(z)}+\overline{\psi_0(z)}\right]_0^B=-i\left[\varphi_{11}(z)+z\overline{\varphi_{11}'(z)}+\overline{\psi_{11}(z)}\right]_0^B, \qquad \text{(A1.9)}$$

and

$$z\overline{\varphi_{11}'(z)}+\overline{\psi_{11}(z)} = -\varphi_{11}(z)+\varphi_0(z)+z\overline{\varphi_0'(z)}+\overline{\psi_0(z)}, \qquad \text{(A1.10)}$$

$$z\overline{\varphi_{11}'}\left(\frac{R_0^2}{z}\right)+\overline{\psi_{11}}\left(\frac{R_0^2}{z}\right) = -\varphi_{11}(z)+\varphi_0(z)+z\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)+\overline{\psi_0}\left(\frac{R_0^2}{z}\right).$$

Taking conjugate of (A1.10) leads to

$$\begin{aligned}\bar{z}\varphi_{11}'(z)+\psi_{11}(z) &= -\overline{\varphi_{11}(z)}+\overline{\varphi_0(z)}+\bar{z}\varphi_0'(z)+\psi_0(z), \\ \bar{z}\varphi_{11}'(z)+\psi_{11}(z) &= -\overline{\varphi_{11}}\left(\frac{R_0^2}{z}\right)+\overline{\varphi_0}\left(\frac{R_0^2}{z}\right)+\bar{z}\varphi_0'(z)+\psi_0(z).\end{aligned} \tag{A1.11}$$

By substituting (A1.10) and (A1.11) into (A1.8), and then (A1.8) and (A1.3) into (2.9), we obtain an another notation of (2.9) in the following form

$$\begin{aligned}&\varphi_0'(z)+\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)-z\varphi_0''(z)-\frac{z^2}{R_0^2}\psi_0'(z) \\ &= \frac{m-n}{2}\frac{R_0}{2z}\left\{\begin{array}{c}\frac{1}{\mu_1}\left[k_1\varphi_{11}(z)-\left(-\varphi_{11}(z)+\varphi_0(z)+z\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)+\overline{\psi_0}\left(\frac{R_0^2}{z}\right)\right)\right] \\ -\frac{1}{\mu_0}\left[k_0\varphi_0(z)-z\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)-\overline{\psi_0}\left(\frac{R_0^2}{z}\right)\right]\end{array}\right\} \\ &+\frac{m+n}{2}\frac{z}{2R_0}\left\{\begin{array}{c}\frac{1}{\mu_1}\left[k_1\overline{\varphi_{11}}\left(\frac{R_0^2}{z}\right)-\left(-\overline{\varphi_{11}}\left(\frac{R_0^2}{z}\right)+\overline{\varphi_0}\left(\frac{R_0^2}{z}\right)+\bar{z}\varphi_0'(z)+\psi_0(z)\right)\right] \\ -\frac{1}{\mu_0}\left[k_0\overline{\varphi_0}\left(\frac{R_0^2}{z}\right)-\bar{z}\varphi_0'(z)-\psi_0(z)\right]\end{array}\right\} \\ &-m_0R_0\varepsilon_1-\frac{(m_0+n_0)(\varepsilon_2-i\varepsilon_3)}{2R_0}z^2-\frac{(m_0-n_0)(\varepsilon_2+i\varepsilon_3)R_0^3}{2z^2}.\end{aligned} \tag{A1.12}$$

Multiplying $4\mu_1$ to both side yields

$$\begin{aligned}&4\mu_1\left[\varphi_0'(z)+\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)-z\varphi_0''(z)-\frac{z^2}{R_0^2}\psi_0'(z)\right] \\ &= (m-n)\frac{R_0}{z}\left\{\begin{array}{c}\left[k_1\varphi_{11}(z)+\varphi_{11}(z)-\varphi_0(z)-z\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)-\overline{\psi_0}\left(\frac{R_0^2}{z}\right)\right] \\ -\frac{\mu_1}{\mu_0}\left[k_0\varphi_0(z)-z\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)-\overline{\psi_0}\left(\frac{R_0^2}{z}\right)\right]\end{array}\right\} \\ &+(m+n)\frac{z}{R_0}\left\{\begin{array}{c}\left[k_1\overline{\varphi_{11}}\left(\frac{R_0^2}{z}\right)+\overline{\varphi_{11}}\left(\frac{R_0^2}{z}\right)-\overline{\varphi_0}\left(\frac{R_0^2}{z}\right)-\bar{z}\varphi_0'(z)-\psi_0(z)\right] \\ -\frac{\mu_1}{\mu_0}\left[k_0\overline{\varphi_0}\left(\frac{R_0^2}{z}\right)-\bar{z}\varphi_0'(z)-\psi_0(z)\right]\end{array}\right\} \\ &-4\mu_1m_0R_0\varepsilon_1-4\mu_1\frac{(m_0+n_0)(\varepsilon_2-i\varepsilon_3)}{2R_0}z^2-4\mu_1\frac{(m_0-n_0)(\varepsilon_2+i\varepsilon_3)R_0^3}{2z^2}.\end{aligned} \tag{A1.13}$$

By making arrangement of left-hand side for φ_{11} and right-hand-side for φ_0 and ψ_0, we get (2.12)

$$
\begin{aligned}
& (m_0-n_0)(\kappa_1+1)\frac{R_0}{z}\varphi_{11}(z)+(m_0+n_0)(\kappa_1+1)\frac{z}{R_0}\overline{\varphi_{11}}\left(\frac{R_0^2}{z}\right) \\
= \; & 4\mu_1\left[\varphi_0'(z)+\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)-z\varphi_0''(z)-\frac{z^2}{R_0^2}\psi_0'(z)\right]+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})\frac{R_0}{z}\varphi_0(z) \\
& +(m_0-n_0)(1-\frac{\mu_1}{\mu_0})\left[R_0\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)+\frac{R_0}{z}\overline{\psi}_0\left(\frac{R_0^2}{z}\right)\right] \\
& +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})\frac{z}{R_0}\overline{\varphi_0}(\frac{R_0^2}{z})+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\left[R_0\varphi_0'(z)+\frac{z}{R_0}\psi_0(z)\right] \\
& +4m_0R_0\mu_1\varepsilon_1+\frac{2\mu_1(m_0+n_0)(\varepsilon_2-i\varepsilon_3)}{R_0}z^2+\frac{2\mu_1(m_0-n_0)(\varepsilon_2+i\varepsilon_3)R_0^3}{z^2}
\end{aligned}
$$

$, z \in \Gamma_1.$

APPENDIX 2
Derivation of Boundary Condition $\partial\Gamma_1$

A2.1 Derivation of equation (2.14)

In the similar fashion of (2.9), the displacement jump condition $(2.4)_2$ is

$$(\sigma_{rr} - i\sigma_{r\theta})_{1\ or\ 2} = \frac{m_1 - n_1}{2}\|u_r + iu_\theta\| + \frac{m_1 + n_1}{2}\|u_r - iu_\theta\|. \tag{A2.1}$$

Let's consider stresses at region S_2 in (2.13) and the left-hand-side of (A2.1) is

$$(\sigma_{rr} - i\sigma_{r\theta})_2 = \varphi_2'(z) + \overline{\varphi_2'}\left(\frac{R_1^2}{z}\right) - z\varphi_2''(z) - \frac{z^2}{R_1^2}\psi_2'(z). \tag{A2.2}$$

The right-hand-side of (A2.1) can be expressed as follows

$$\begin{aligned}
&\frac{m_1 - n_1}{2}\|u_r + iu_\theta\| + \frac{m_1 + n_1}{2}\|u_r - iu_\theta\| \qquad\qquad (A2.3)\\
=\ & \frac{m_1 - n_1}{2}\frac{R_1}{2z}\left\{\begin{array}{c} \frac{1}{\mu_2}\left[k_2\varphi_2(z) - z\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right) - \overline{\psi_2}\left(\frac{R_1^2}{z}\right)\right] \\ -\frac{1}{\mu_1}\left[k_1\varphi_{12}(z) - z\overline{\varphi_{12}'}\left(\frac{R_1^2}{z}\right) - \overline{\psi_{12}}\left(\frac{R_1^2}{z}\right)\right] \end{array}\right\} \\
& + \frac{m_1 + n_1}{2}\frac{z}{2R_1}\left\{\begin{array}{c} \frac{1}{\mu_2}\left[k_2\overline{\varphi_2}\left(\frac{R_1^2}{z}\right) - \overline{z}\varphi_2'(z) - \psi_2(z)\right] \\ -\frac{1}{\mu_1}\left[k_1\overline{\varphi_{12}}\left(\frac{R_1^2}{z}\right) - \overline{z}\varphi_{12}'(z) - \psi_{12}(z)\right] \end{array}\right\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\varphi_2'(z) + \overline{\varphi_2'}\left(\frac{R_1^2}{z}\right) - z\varphi_2''(z) - \frac{z^2}{R_1^2}\psi_2'(z) \qquad\qquad (A2.4)\\
=\ & \frac{m - n}{2}\frac{R_1}{2z}\left\{\begin{array}{c} \frac{1}{\mu_2}\left[k_2\varphi_2(z) - z\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right) - \overline{\psi_2}\left(\frac{R_1^2}{z}\right)\right] \\ -\frac{1}{\mu_1}\left[k_1\varphi_{12}(z) - z\overline{\varphi_{12}'}\left(\frac{R_1^2}{z}\right) - \overline{\psi_{12}}\left(\frac{R_1^2}{z}\right)\right] \end{array}\right\}
\end{aligned}$$

$$+\frac{m+n}{2}\frac{z}{2R_1}\left\{\begin{array}{c}\frac{1}{\mu_2}\left[k_2\overline{\varphi_2}\left(\frac{R_1^2}{z}\right)-\overline{z}\varphi_2'(z)-\psi_2(z)\right]\\-\frac{1}{\mu_1}\left[k_1\overline{\varphi_{12}}\left(\frac{R_1^2}{z}\right)-\overline{z}\varphi_{12}'(z)-\psi_{12}(z)\right]\end{array}\right\}.$$

In addition, consider forces along the boundary Γ_1,

$$\begin{aligned}F_x+iF_y &= -i\left[\varphi_2(z)+z\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)+\overline{\psi_2}\left(\frac{R_1^2}{z}\right)\right]_0^B \\ &= -i\left[\varphi_{12}(z)+z\overline{\varphi_{12}'}\left(\frac{R_0^2}{z}\right)+\overline{\psi_{12}}\left(\frac{R_0^2}{z}\right)\right]_0^B,\end{aligned}\tag{A2.5}$$

and

$$\varphi_2(z)+z\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)+\overline{\psi_2}\left(\frac{R_1^2}{z}\right)=\varphi_{12}(z)+z\overline{\varphi_{12}'}\left(\frac{R_1^2}{z}\right)+\overline{\psi_{12}}\left(\frac{R_1^2}{z}\right).\tag{A2.6}$$

Thus,

$$\begin{aligned}z\overline{\varphi_{12}'}\left(\frac{R_1^2}{z}\right)+\overline{\psi_{12}}\left(\frac{R_1^2}{z}\right) &= -\varphi_{12}(z)+\varphi_2(z)+z\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)+\overline{\psi_2}\left(\frac{R_1^2}{z}\right) \quad \text{(A2.7)}\\ \overline{z}\varphi_{12}'(z)+\psi_{12}(z) &= -\overline{\varphi_{12}}\left(\frac{R_0^2}{z}\right)+\overline{\varphi_2}\left(\frac{R_1^2}{z}\right)+\overline{z}\varphi_2'(z)+\psi_2(z).\end{aligned}$$

Substituting these equations into (A2.4) gives

$$\begin{aligned}&\varphi_2'(z)+\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)-z\varphi_2''(z)-\frac{z^2}{R_1^2}\psi_2'(z) \qquad \text{(A2.8)}\\ =&\ \frac{m-n}{2}\frac{R_1}{2z}\left\{\begin{array}{c}\frac{1}{\mu_2}\left[k_2\varphi_2(z)-z\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)-\overline{\psi_2}\left(\frac{R_1^2}{z}\right)\right]\\-\frac{1}{\mu_1}\left[k_1\varphi_{12}(z)+\varphi_{12}(z)-\varphi_2(z)-z\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)-\overline{\psi_2}\left(\frac{R_1^2}{z}\right)\right]\end{array}\right\}\\ &+\frac{m+n}{2}\frac{z}{2R_1}\left\{\begin{array}{c}\frac{1}{\mu_2}\left[k_2\overline{\varphi_2}\left(\frac{R_1^2}{z}\right)-\overline{z}\varphi_2'(z)-\psi_2(z)\right]\\-\frac{1}{\mu_1}\left[k_1\overline{\varphi_{12}}\left(\frac{R_1^2}{z}\right)+\overline{\varphi_{12}}\left(\frac{R_0^2}{z}\right)-\overline{\varphi_2}\left(\frac{R_1^2}{z}\right)-\overline{z}\varphi_2'(z)-\psi_2(z)\right]\end{array}\right\}.\end{aligned}$$

Multiplying $4\mu_2$ to both sides leads to the following expression

$$
\begin{aligned}
& 4\mu_2\left[\varphi_2'(z)+\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)-z\varphi_2''(z)-\frac{z^2}{R_1^2}\psi_2'(z)\right] \qquad \text{(A2.9)}\\
= & \ (m-n)\frac{R_1}{z}\left\{\begin{array}{c} k_2\varphi_2(z)-z\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)-\overline{\psi_2}\left(\frac{R_1^2}{z}\right) \\ -\frac{\mu_2}{\mu_1}\left[k_1\varphi_{12}(z)+\varphi_{12}(z)-\varphi_2(z)-z\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)-\overline{\psi_2}\left(\frac{R_1^2}{z}\right)\right]\end{array}\right\} \\
& +(m+n)\frac{z}{R_1}\left\{\begin{array}{c} k_2\overline{\varphi_2}\left(\frac{R_1^2}{z}\right)-\overline{z}\varphi_2'(z)-\psi_2(z) \\ -\frac{\mu_2}{\mu_1}\left[k_1\overline{\varphi_{12}}\left(\frac{R_1^2}{z}\right)+\overline{\varphi_{12}}\left(\frac{R_1^2}{z}\right)-\overline{\varphi_2}\left(\frac{R_1^2}{z}\right)-\overline{z}\varphi_2'(z)-\psi_2(z)\right]\end{array}\right\}.
\end{aligned}
$$

By making arrangement of the left-hand-side for $\varphi_{12}(z)$ and the right-hand-side for $\varphi_2(z)$ and $\psi_2(z)$, we have

$$
\begin{aligned}
& (m-n)\frac{\mu_2}{\mu_1}(k_1+1)\frac{R_1}{z}\varphi_{12}(z)+(m+n)\frac{z}{R_1}\frac{\mu_2}{\mu_1}(k_1+1)\overline{\varphi_{12}}\left(\frac{R_1^2}{z}\right) \qquad \text{(A2.10)}\\
= & -4\mu_2\left[\varphi_2'(z)+\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)-z\varphi_2''(z)-\frac{z^2}{R_1^2}\psi_2'(z)\right] \\
& +(m-n)\left(\frac{\mu_2}{\mu_1}-1\right)\left[R_1\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)+\frac{R_1}{z}\overline{\psi_2}\left(\frac{R_1^2}{z}\right)\right] \\
& +(m-n)\left(\frac{\mu_2}{\mu_1}+k_2\right)\frac{R_1}{z}\varphi_2(z)+(m+n)\frac{z}{R_1}\left(\frac{\mu_2}{\mu_1}+k_2\right)\overline{\varphi_2}\left(\frac{R_1^2}{z}\right) \\
& +(m+n)\frac{z}{R_1}\left(\frac{\mu_2}{\mu_1}-1\right)\left[\psi_2(z)+\overline{z}\varphi_2'(z)\right].
\end{aligned}
$$

Finally, by multiplying $\frac{\mu_1}{\mu_2}$ to both sides, we get (2.14)

$$
\begin{aligned}
& (m_1-n_1)(k_1+1)\frac{R_1}{z}\varphi_1(z)+(m_1+n_1)(k_1+1)\frac{z}{R_1}\overline{\varphi_1}\left(\frac{R_1^2}{z}\right) \\
= & -4\mu_1\left[\varphi_2'(z)+\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)-z\varphi_2''(z)-\frac{z^2}{R_1^2}\psi_2'(z)\right] \\
& +(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left[R_1\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)+\frac{R_1}{z}\overline{\psi_2}\left(\frac{R_1^2}{z}\right)\right] \\
& +(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)\frac{R_1}{z}\varphi_2(z)+(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)\frac{z}{R_1}\overline{\varphi_2}\left(\frac{R_1^2}{z}\right)
\end{aligned}
$$

$$+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left[R_1\varphi_2'(z)+\frac{z}{R_1}\psi_2(z)\right].$$

APPENDIX 3
Stress Potentials

A3.1 Coefficients g_k of $F_{11}(z)$ (2.34)

$$\begin{aligned} F_{11}(z) &= \frac{1}{2\sqrt{(z-a)(z-b)}} = \frac{1}{2}(z-a)^{-\frac{1}{2}}(z-b)^{-\frac{1}{2}} \\ &= \sum_{k=0}^{\infty} g_k z^k = g_0 + g_1 z + g_2 z^2 + g_3 z^3 + g_4 z^4 + g_5 z^5 + g_6 z^6 + g_7 z^7. \end{aligned} \tag{A3.1}$$

Each term is expanded into Taylor series as follows

$$\begin{aligned} &(z-a)^{-\frac{1}{2}} \\ = &(-a)^{-\frac{1}{2}} + \left(-\frac{1}{2}\right)(-a)^{-\frac{3}{2}} z + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-a)^{-\frac{5}{2}} z^2 \\ &+\frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-a)^{-\frac{7}{2}} z^3 + \ldots \\ &+\frac{1}{n!}\left(-\frac{1}{2}\right)^n [1*3*5*\ldots*(2n-1)](-a)^{-\frac{2n+1}{2}} z^n + \ldots \\ = &\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{2}\right)^n [1*3*5*\ldots*(2n-1)] \frac{1}{(-a)^{\frac{2n+1}{2}}} z^n \\ = &\frac{1}{(-a)^{\frac{1}{2}}} + \frac{1}{2(-a)^{\frac{3}{2}}} z + \frac{3}{8(-a)^{\frac{5}{2}}} z^2 + \frac{5}{16(-a)^{\frac{7}{2}}} z^3 + \frac{35}{128(-a)^{\frac{9}{2}}} z^4 + \frac{63}{256(-a)^{\frac{11}{2}}} z^5 \ldots, \end{aligned} \tag{A3.2}$$

and

$$\begin{aligned} &(z-b)^{-\frac{1}{2}} \\ = &(-b)^{-\frac{1}{2}} + \left(-\frac{1}{2}\right)(-b)^{-\frac{3}{2}} z + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-b)^{-\frac{5}{2}} z^2 \\ &+\frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-b)^{-\frac{7}{2}} z^3 + \ldots \\ &+\frac{1}{n!}\left(-\frac{1}{2}\right)^n [1*3*5*\ldots*(2n-1)](-b)^{-\frac{2n+1}{2}} z^n + \ldots \end{aligned} \tag{A3.3}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{2}\right)^n [1*3*5*\ldots*(2n-1)] \frac{1}{(-b)^{\frac{2n+1}{2}}} z^n \\
&= \frac{1}{(-b)^{\frac{1}{2}}} + \frac{1}{2\,(-b)^{\frac{3}{2}}} z + \frac{3}{8\,(-b)^{\frac{5}{2}}} z^2 + \frac{5}{16\,(-b)^{\frac{7}{2}}} z^3 + \frac{35}{128\,(-b)^{\frac{9}{2}}} z^4 + \frac{63}{256\,(-b)^{\frac{11}{2}}} z^5 \ldots .
\end{aligned}
$$

Consequently, (A3.1) is re-written as

$$
\begin{aligned}
&\frac{1}{2}(z-a)^{-\frac{1}{2}}(z-b)^{-\frac{1}{2}} \qquad\qquad \text{(A3.4)} \\
&= \frac{1}{2}\left[\begin{array}{c} \frac{1}{(-a)^{\frac{1}{2}}} + \frac{1}{2(-a)^{\frac{3}{2}}} z + \frac{3}{8(-a)^{\frac{5}{2}}} z^2 + \frac{5}{16(-a)^{\frac{7}{2}}} z^3 \\ + \frac{35}{128(-a)^{\frac{9}{2}}} z^4 + \frac{63}{256(-a)^{\frac{11}{2}}} z^5 \ldots \end{array}\right] \\
&\quad * \left[\begin{array}{c} \frac{1}{(-b)^{\frac{1}{2}}} + \frac{1}{2(-b)^{\frac{3}{2}}} z + \frac{3}{8(-b)^{\frac{5}{2}}} z^2 + \frac{5}{16(-b)^{\frac{7}{2}}} z^3 \\ + \frac{35}{128(-b)^{\frac{9}{2}}} z^4 + \frac{63}{256(-b)^{\frac{11}{2}}} z^5 \ldots \end{array}\right] \\
&= \frac{1}{2\sqrt{ab}} + \left(\frac{1}{4}\frac{1}{\sqrt{a^1b^3}} + \frac{1}{4}\frac{1}{\sqrt{a^3b^1}}\right) z + \left(\frac{3}{16}\frac{1}{\sqrt{a^1b^5}} + \frac{1}{8}\frac{1}{\sqrt{a^3b^3}} + \frac{3}{16}\frac{1}{\sqrt{a^5b^1}}\right) z^2 \\
&\quad + \left(\frac{5}{32}\frac{1}{\sqrt{a^1b^7}} + \frac{3}{32}\frac{1}{\sqrt{a^3b^5}} + \frac{3}{32}\frac{1}{\sqrt{a^5b^3}} + \frac{5}{32}\frac{1}{\sqrt{a^7b^1}}\right) z^3 \\
&\quad + \left(\frac{35}{256}\frac{1}{\sqrt{a^1b^9}} + \frac{5}{64}\frac{1}{\sqrt{a^3b^7}} + \frac{9}{128}\frac{1}{\sqrt{a^5b^5}} + \frac{5}{64}\frac{1}{\sqrt{a^7b^3}} + \frac{35}{256}\frac{1}{\sqrt{a^9b^1}}\right) z^4 \\
&\quad + \left(\begin{array}{c} \frac{63}{512}\frac{1}{\sqrt{a^1b^{11}}} + \frac{35}{512}\frac{1}{\sqrt{a^3b^9}} + \frac{15}{256}\frac{1}{\sqrt{a^5b^7}} + \frac{15}{256}\frac{1}{\sqrt{a^7b^5}} \\ + \frac{35}{512}\frac{1}{\sqrt{a^9b^3}} + \frac{63}{512}\frac{1}{\sqrt{a^{11}b^1}} \end{array}\right) z^5 \\
&\quad + \left(\begin{array}{c} \frac{231}{2048}\frac{1}{\sqrt{a^1b^{13}}} + \frac{63}{1024}\frac{1}{\sqrt{a^3b^{11}}} + \frac{105}{2048}\frac{1}{\sqrt{a^5b^9}} + \frac{25}{512}\frac{1}{\sqrt{a^7b^7}} \\ + \frac{105}{2048}\frac{1}{\sqrt{a^9b^5}} + \frac{63}{1024}\frac{1}{\sqrt{a^{11}b^3}} + \frac{231}{2048}\frac{1}{\sqrt{a^{13}b^1}} \end{array}\right) z^6 \\
&\quad + \left(\begin{array}{c} \frac{3003}{28672}\frac{1}{\sqrt{a^1b^{15}}} + \frac{231}{4096}\frac{1}{\sqrt{a^3b^{13}}} + \frac{189}{4096}\frac{1}{\sqrt{a^5b^{11}}} + \frac{175}{4096}\frac{1}{\sqrt{a^7b^9}} \\ + \frac{175}{4096}\frac{1}{\sqrt{a^9b^7}} + \frac{189}{4096}\frac{1}{\sqrt{a^{11}b^5}} + \frac{231}{4096}\frac{1}{\sqrt{a^{13}b^3}} + \frac{3003}{28672}\frac{1}{\sqrt{a^{15}b^1}} \end{array}\right) z^7 \\
&= g_0 + g_1 z + g_2 z^2 + g_3 z^3 + g_4 z^4 + g_5 z^5 + g_6 z^6 + g_7 z^7.
\end{aligned}
$$

A3.2 Coefficients h_k of $F_{12}(z)$ (2.35)

$$
\begin{aligned}
F_{12}(z) &= \frac{1}{2\sqrt{(z-a)(z-b)}} = \frac{1}{2z\sqrt{(1-\frac{a}{z})(1-\frac{b}{z})}} \\
&= \frac{1}{2z}(1-\frac{a}{z})^{-\frac{1}{2}}(1-\frac{b}{z})^{-\frac{1}{2}} = \sum_{k=1}^{\infty} h_{-k} z^{-k} \\
&= \left(\begin{array}{c} h_{-1}z^{-1} + h_{-2}z^{-2} + h_{-3}z^{-3} + h_{-4}z^{-4} + h_{-5}z^{-5} \\ +h_{-6}z^{-6} + h_{-7}z^{-7} + h_{-8}z^{-8} \end{array} \right).
\end{aligned}
\tag{A3.5}
$$

Each term is expanded into Taylor series

$$
\begin{aligned}
& (\frac{a}{z}-1)^{-\frac{1}{2}} \\
&= (-1)^{-\frac{1}{2}} + \left(-\frac{1}{2}\right)(-1)^{-\frac{3}{2}}\left(\frac{a}{z}\right)^1 + \left(\frac{1}{2!}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-1)^{-\frac{5}{2}}\left(\frac{a}{z}\right)^2 \\
&\quad + \left(\frac{1}{3!}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-1)^{-\frac{7}{2}}\left(\frac{a}{z}\right)^3 \\
&\quad \dots + \left(\frac{1}{n!}\right)\left(-\frac{1}{2}\right)^n [1*3*5*\dots*(2n-1)](-1)^n(-1)^{-\frac{1}{2}}\left(\frac{a}{z}\right)^n \dots \\
&= \sum_{n=0}^{\infty}(-1)^{-\frac{1}{2}}\left(\frac{1}{n!}\right)\left(\frac{1}{2}\right)^n [1*3*5*\dots*(2n-1)]\left(\frac{a}{z}\right)^n \\
&= (-1)^{-\frac{1}{2}} + (-1)^{-\frac{1}{2}}\frac{1}{2}\left(\frac{a}{z}\right) + (-1)^{-\frac{1}{2}}\frac{3}{8}\left(\frac{a}{z}\right)^2 + (-1)^{-\frac{1}{2}}\frac{5}{16}\left(\frac{a}{z}\right)^3 \\
&\quad +(-1)^{-\frac{1}{2}}\frac{35}{128}\left(\frac{a}{z}\right)^4 + (-1)^{-\frac{1}{2}}\frac{63}{256}\left(\frac{a}{z}\right)^5 + (-1)^{-\frac{1}{2}}\frac{231}{1024}\left(\frac{a}{z}\right)^6 \\
&\quad +(-1)^{-\frac{1}{2}}\frac{429}{2048}\left(\frac{a}{z}\right)^7 + (-1)^{-\frac{1}{2}}\frac{6435}{32768}\left(\frac{a}{z}\right)^8 \dots,
\end{aligned}
\tag{A3.6}
$$

and

$$
\begin{aligned}
& (\frac{b}{z}-1)^{-\frac{1}{2}} \\
&= (-1)^{-\frac{1}{2}} + \left(-\frac{1}{2}\right)(-1)^{-\frac{3}{2}}\left(\frac{b}{z}\right)^1 + \left(\frac{1}{2!}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(-1)^{-\frac{5}{2}}\left(\frac{b}{z}\right)^2
\end{aligned}
\tag{A3.7}
$$

$$
\begin{aligned}
& +\left(\frac{1}{3!}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(-1)^{-\frac{7}{2}}\left(\frac{b}{z}\right)^{3} \\
& \ldots+\left(\frac{1}{n!}\right)\left(-\frac{1}{2}\right)^{n}[1*3*5*\ldots*(2n-1)]\,(\frac{1}{i})^{2n+1}\left(\frac{b}{z}\right)^{n}\ldots \\
= & \sum_{n=0}^{\infty}(-1)^{-\frac{1}{2}}\left(\frac{1}{n!}\right)\left(\frac{1}{2}\right)^{n}[1*3*5*\ldots*(2n-1)]\left(\frac{b}{z}\right)^{n} \\
= & (-1)^{-\frac{1}{2}}+(-1)^{-\frac{1}{2}}\frac{1}{2}\left(\frac{b}{z}\right)+(-1)^{-\frac{1}{2}}\frac{3}{8}\left(\frac{b}{z}\right)^{2}+(-1)^{-\frac{1}{2}}\frac{5}{16}\left(\frac{b}{z}\right)^{3} \\
& +(-1)^{-\frac{1}{2}}\frac{35}{128}\left(\frac{b}{z}\right)^{4}+(-1)^{-\frac{1}{2}}\frac{63}{256}\left(\frac{b}{z}\right)^{5}+(-1)^{-\frac{1}{2}}\frac{231}{1024}\left(\frac{b}{z}\right)^{6} \\
& +(-1)^{-\frac{1}{2}}\frac{429}{2048}\left(\frac{b}{z}\right)^{7}+(-1)^{-\frac{1}{2}}\frac{6435}{32768}\left(\frac{b}{z}\right)^{8}\ldots .
\end{aligned}
$$

Consequently, (A3.5) is re-written as

$$
\begin{aligned}
& \frac{1}{2z}(1-\frac{a}{z})^{-\frac{1}{2}}(1-\frac{b}{z})^{-\frac{1}{2}} \qquad \text{(A3.8)} \\
= & \frac{1}{2z}\left[\begin{array}{c}
(-1)^{-\frac{1}{2}}+(-1)^{-\frac{1}{2}}\frac{1}{2}\left(\frac{a}{z}\right)+(-1)^{-\frac{1}{2}}\frac{3}{8}\left(\frac{a}{z}\right)^{2}+(-1)^{-\frac{1}{2}}\frac{5}{16}\left(\frac{a}{z}\right)^{3} \\
+(-1)^{-\frac{1}{2}}\frac{35}{128}\left(\frac{a}{z}\right)^{4}+(-1)^{-\frac{1}{2}}\frac{63}{256}\left(\frac{a}{z}\right)^{5}+(-1)^{-\frac{1}{2}}\frac{231}{1024}\left(\frac{a}{z}\right)^{6} \\
+(-1)^{-\frac{1}{2}}\frac{429}{2048}\left(\frac{a}{z}\right)^{7}+(-1)^{-\frac{1}{2}}\frac{6435}{32768}\left(\frac{a}{z}\right)^{8}\ldots
\end{array}\right] \\
& *\left[\begin{array}{c}
(-1)^{-\frac{1}{2}}+(-1)^{-\frac{1}{2}}\frac{1}{2}\left(\frac{b}{z}\right)+(-1)^{-\frac{1}{2}}\frac{3}{8}\left(\frac{b}{z}\right)^{2}+(-1)^{-\frac{1}{2}}\frac{5}{16}\left(\frac{b}{z}\right)^{3} \\
+(-1)^{-\frac{1}{2}}\frac{35}{128}\left(\frac{b}{z}\right)^{4}+(-1)^{-\frac{1}{2}}\frac{63}{256}\left(\frac{b}{z}\right)^{5}+(-1)^{-\frac{1}{2}}\frac{231}{1024}\left(\frac{b}{z}\right)^{6} \\
+(-1)^{-\frac{1}{2}}\frac{429}{2048}\left(\frac{b}{z}\right)^{7}+(-1)^{-\frac{1}{2}}\frac{6435}{32768}\left(\frac{b}{z}\right)^{8}\ldots
\end{array}\right] \\
= & \frac{1}{2}\frac{1}{z}+\frac{1}{2}\left(\frac{a+b}{2}\right)\frac{1}{z^{2}}+\frac{1}{2}\left(\frac{3}{8}a^{2}+\frac{1}{4}ab+\frac{3}{8}b^{2}\right)\frac{1}{z^{3}} \\
& +\frac{1}{2}\left(\frac{5}{16}a^{3}+\frac{3}{16}a^{2}b+\frac{3}{16}ab^{2}+\frac{5}{16}b^{3}\right)\frac{1}{z^{4}} \\
& +\frac{1}{2}\left(\frac{35}{128}a^{4}+\frac{5}{32}a^{3}b+\frac{9}{64}a^{2}b^{2}+\frac{5}{32}ab^{3}+\frac{35}{128}b^{4}\right)\frac{1}{z^{5}} \\
& +\frac{1}{2}\left(\frac{63}{256}a^{5}+\frac{35}{256}a^{4}b+\frac{15}{128}a^{3}b^{2}+\frac{15}{128}a^{2}b^{3}+\frac{35}{256}ab^{4}+\frac{63}{256}b^{5}\right)\frac{1}{z^{6}}
\end{aligned}
$$

$$+\frac{1}{2}\left(\begin{array}{c}\frac{231}{1024}a^6+\frac{63}{512}a^5b+\frac{105}{1024}a^4b^2+\frac{25}{256}a^3b^3+\frac{105}{1024}a^2b^4\\+\frac{63}{512}ab^5+\frac{231}{1024}b^6\end{array}\right)\frac{1}{z^7}$$
$$+\frac{1}{2}\left(\begin{array}{c}\frac{429}{2048}a^7+\frac{231}{2048}a^6b+\frac{189}{2048}a^5b^2+\frac{175}{2048}a^4b^3+\frac{175}{2048}a^3b^4\\+\frac{189}{2048}a^2b^5+\frac{231}{2048}ab^6+\frac{429}{2048}b^7\end{array}\right)\frac{1}{z^8}$$
$$=\left(\begin{array}{c}h_{-1}z^{-1}+h_{-2}z^{-2}+h_{-3}z^{-3}+h_{-4}z^{-4}+h_{-5}z^{-5}\\+h_{-6}z^{-6}+h_{-7}z^{-7}+h_{-8}z^{-8}\end{array}\right).$$

A3.3 Coefficients p_k of $\varphi_{11}(z)$ and $\psi_{11}(z)$

$$\varphi_1(z)=\frac{1}{2\sqrt{(z-a)(z-b)}}Y(z)+\frac{1}{2}X(z), \tag{2.32}$$

$$\psi_1(z)=-\overline{X}(z)+\overline{\varphi_1}(z)-z\varphi_1'(z). \tag{2.33}$$

From (2.32), $\varphi_{11}(z)$ corresponding to $\varphi_1(z)$ in a domain $|z|<a$ near a crack tip a is derived as follows

$$\begin{aligned}
&\varphi_{11}(z) \qquad\qquad \text{(A3.9)}\\
=\;&\left[g_0+g_1z+g_2z^2+g_3z^3+g_4z^4+g_5z^5+g_6z^6+g_7z^7\right]\\
&\times\left[...f_{-4}z^{-4}+f_{-3}z^{-3}+f_{-2}z^{-2}+f_{-1}z^{-1}+f_0+f_1z^1+f_2z^2+f_3z^3+f_4z^4...\right]\\
&+\frac{1}{2}\left[...e_{-4}z^{-4}+e_{-3}z^{-3}+e_{-2}z^{-2}+e_{-1}z^{-1}+e_0+e_1z^1+e_2z^2+e_3z^3+e_4z^4...\right]\\
=\;&\left(f_0g_7+f_1g_6+f_2g_5+f_3g_4+f_4g_3\right)z^7\\
&+\left(f_{-1}g_7+f_0g_6+f_1g_5+f_2g_4+f_3g_3+f_4g_2\right)z^6\\
&+\left(f_{-2}g_7+f_{-1}g_6+f_0g_5+f_1g_4+f_2g_3+f_3g_2+f_4g_1\right)z^5\\
&+\left(f_{-3}g_7+f_{-2}g_6+f_{-1}g_5+f_0g_4+f_1g_3+f_2g_2+f_3g_1+f_4g_0+\frac{1}{2}e_4\right)z^4\\
&+\left(f_{-4}g_7+f_{-3}g_6+f_{-2}g_5+f_{-1}g_4+f_0g_3+f_1g_2+f_2g_1+f_3g_0+\frac{1}{2}e_3\right)z^3
\end{aligned}$$

$$
\begin{aligned}
&+\left(f_{-4}g_6+f_{-3}g_5+f_{-2}g_4+f_{-1}g_3+f_0g_2+f_1g_1+f_2g_0+\frac{1}{2}e_2\right)z^2\\
&+\left(f_{-4}g_5+f_{-3}g_4+f_{-2}g_3+f_{-1}g_2+f_0g_1+f_1g_0+\frac{1}{2}e_1\right)z^1\\
&+\left(f_{-4}g_4+f_{-3}g_3+f_{-2}g_2+f_{-1}g_1+f_0g_0+\frac{1}{2}e_0\right)z^0\\
&+\left(f_{-4}g_3+f_{-3}g_2+f_{-2}g_1+f_{-1}g_0+\frac{1}{2}e_{-1}\right)z^{-1}\\
&+\left(f_{-4}g_2+f_{-3}g_1+f_{-2}g_0+\frac{1}{2}e_{-2}\right)z^{-2}\\
&+\left(f_{-4}g_1+f_{-3}g_0+\frac{1}{2}e_{-3}\right)z^{-3}\\
&+\left(f_{-4}g_0+\frac{1}{2}e_{-4}\right)z^{-4}\\
=\;& p_7z^7+p_6z^6+p_5z^5+p_4z^4+p_3z^3+p_2z^2+p_1z^1+p_0\\
&+p_{-1}z^{-1}+p_{-2}z^{-2}+p_{-3}z^{-3}+p_{-4}z^{-4}.
\end{aligned}
$$

Also, from (2.33), $\psi_{11}(z)$ corresponding to $\psi_1(z)$ in a domain $|z|<a$ around a crack tip a is expressed as follows

$$
\begin{aligned}
&\psi_{11}(z) \qquad\qquad \text{(A3.10)}\\
=\;& -\overline{X}(z)+\overline{\varphi_1}(z)-z\varphi_1'(z)\\
=\;& -\left(\ldots\overline{e}_{-4}z^{-4}+\overline{e}_{-3}z^{-3}+\overline{e}_{-2}z^{-2}+\overline{e}_{-1}z^{-1}+\overline{e}_0+\overline{e}_1z^1+\overline{e}_2z^2+\overline{e}_3z^3+\overline{e}_4z^4\ldots\right)\\
&+\left(\begin{array}{c}\overline{p}_7z^7+\overline{p}_6z^6+\overline{p}_5z^5+\overline{p}_4z^4+\overline{p}_3z^3+\overline{p}_2z^2+\overline{p}_1z^1+\overline{p}_0+\overline{p}_{-1}z^{-1}\\ +\overline{p}_{-2}z^{-2}+\overline{p}_{-3}z^{-3}+\overline{p}_{-4}z^{-4}\end{array}\right)\\
&-\left(\begin{array}{c}7p_7z^7+6p_6z^6+5p_5z^5+4p_4z^4+3p_3z^3+2p_2z^2+p_1z^1-p_{-1}z^{-1}\\ -2p_{-2}z^{-2}-3p_{-3}z^{-3}-4p_{-4}z^{-4}\end{array}\right)\\
=\;& -6p_7z^7-5p_6z^6-4p_5z^5-(3p_4+\overline{e}_4)z^4-(2p_3+\overline{e}_3)z^3-(p_2+\overline{e}_2)z^2\\
&-\overline{e}_1z^1+(p_0-\overline{e}_0)+(2p_{-1}-\overline{e}_{-1})z^{-1}+(3p_{-2}-\overline{e}_{-2})z^{-2}
\end{aligned}
$$

$$+ (4p_{-3} - \bar{e}_{-3})\, z^{-3} + (5p_{-4} - \bar{e}_{-4})\, z^{-4}.$$

A3.4 Coefficients q_k of $\varphi_{12}(z)$ and $\psi_{12}(z)$

From (2.32), $\varphi_{12}(z)$ corresponding to $\varphi_1(z)$ in a domain $|z| > b$ near a crack tip b takes the following form

$$\begin{aligned}
&\varphi_{12}(z) \qquad\qquad (\text{A3.11})\\
= &\left[\begin{array}{c} h_{-1}z^{-1} + h_{-2}z^{-2} + h_{-3}z^{-3} + h_{-4}z^{-4} + h_{-5}z^{-5} \\ +h_{-6}z^{-6} + h_{-7}z^{-7} + h_{-8}z^{-8} \end{array}\right] \\
&\times \left[\begin{array}{c} ...f_{-4}z^{-4} + f_{-3}z^{-3} + f_{-2}z^{-2} + f_{-1}z^{-1} + f_0 + f_1 z^1 \\ +f_2 z^2 + f_3 z^3 + f_4 z^4 ... \end{array}\right] \\
&+\frac{1}{2}\left[\begin{array}{c} ...e_{-4}z^{-4} + e_{-3}z^{-3} + e_{-2}z^{-2} + e_{-1}z^{-1} + e_0 + e_1 z^1 \\ +e_2 z^2 + e_3 z^3 + e_4 z^4 ... \end{array}\right] \\
= &\left(h_{-1}f_4 + \frac{1}{2}e_3\right) z^3 + \left(h_{-2}f_4 + h_{-1}f_3 + \frac{1}{2}e_2\right) z^2 \\
&+ \left(h_{-3}f_4 + h_{-2}f_3 + h_{-1}f_2 + \frac{1}{2}e_1\right) z^1 \\
&+ \left(h_{-4}f_4 + h_{-3}f_3 + h_{-2}f_2 + h_{-1}f_1 + \frac{1}{2}e_0\right) z^0 \\
&+ \left(h_{-5}f_4 + h_{-4}f_3 + h_{-3}f_2 + h_{-2}f_1 + h_{-1}f_0 + \frac{1}{2}e_{-1}\right) z^{-1} \\
&+ \left(h_{-6}f_4 + h_{-5}f_3 + h_{-4}f_2 + h_{-3}f_1 + h_{-2}f_0 + h_{-1}f_{-1} + \frac{1}{2}e_{-2}\right) z^{-2} \\
&+ \left(\begin{array}{c} h_{-7}f_4 + h_{-6}f_3 + h_{-5}f_2 + h_{-4}f_1 + h_{-3}f_0 + h_{-2}f_{-1} \\ +h_{-1}f_{-2} + \frac{1}{2}e_{-3} \end{array}\right) z^{-3} \\
&+ \left(\begin{array}{c} h_{-8}f_4 + h_{-7}f_3 + h_{-6}f_2 + h_{-5}f_1 + h_{-4}f_0 + h_{-3}f_{-1} + h_{-2}f_{-2} \\ +h_{-1}f_{-3} + \frac{1}{2}e_{-4} \end{array}\right) z^{-4}
\end{aligned}$$

$$
\begin{aligned}
&+\begin{pmatrix} h_{-8}f_3+h_{-7}f_2+h_{-6}f_1+h_{-5}f_0+h_{-4}f_{-1}+h_{-3}f_{-2}+h_{-2}f_{-3} \\ +h_{-1}f_{-4}+\frac{1}{2}e_{-5} \end{pmatrix} z^{-5} \\
&+\begin{pmatrix} h_{-8}f_2+h_{-7}f_1+h_{-6}f_0+h_{-5}f_{-1}+h_{-4}f_{-2}+h_{-3}f_{-3} \\ +h_{-2}f_{-4}+\frac{1}{2}e_{-6} \end{pmatrix} z^{-6} \\
= \;& q_3z^3+q_2z^2+q_1z^1+q_0+q_{-1}z^{-1}+q_{-2}z^{-2}+q_{-3}z^{-3}+q_{-4}z^{-4} \\
&+q_{-5}z^{-5}+q_{-6}z^{-6}.
\end{aligned}
$$

From (2.33), $\psi_{12}(z)$ corresponding to $\psi_1(z)$ in a domain $|z|>b$ near a crack tip b can be written in the following form

$$
\begin{aligned}
&\psi_{12}(z) \qquad\qquad (\text{A3.12}) \\
= \;& -\overline{X}(z)+\overline{\varphi_{12}}(z)-z\varphi_{12}'(z) \\
= \;& -\left(\ldots\overline{e}_{-4}z^{-4}+\overline{e}_{-3}z^{-3}+\overline{e}_{-2}z^{-2}+\overline{e}_{-1}z^{-1}+\overline{e}_0+\overline{e}_1z^1+\overline{e}_2z^2+\overline{e}_3z^3+\overline{e}_4z^4\ldots\right) \\
&+\begin{pmatrix} q_3z^3+q_2z^2+q_1z^1+q_0+q_{-1}z^{-1}+q_{-2}z^{-2}+q_{-3}z^{-3}+q_{-4}z^{-4} \\ +q_{-5}z^{-5}+q_{-6}z^{-6} \end{pmatrix} \\
&-\begin{pmatrix} 3q_3z^3+2q_2z^2+q_1z^1-q_{-1}z^{-1}-2q_{-2}z^{-2}-3q_{-3}z^{-3}-4q_{-4}z^{-4} \\ -5q_{-5}z^{-5}-6q_{-6}z^{-6} \end{pmatrix} \\
= \;& 7q_{-6}z^{-6}+6q_{-5}z^{-5}+\left(5q_{-4}z^{-4}-\overline{e}_{-4}z^{-4}\right)+\left(4q_{-3}z^{-3}-\overline{e}_{-3}z^{-3}\right) \\
&+\left(3q_{-2}z^{-2}-\overline{e}_{-2}z^{-2}\right)+\left(2q_{-1}z^{-1}-\overline{e}_{-1}z^{-1}\right)+\left(q_0-\overline{e}_0\right)-\overline{e}_1z^1 \\
&+\left(-\overline{e}_2z^2-q_2z^2\right)+\left(-\overline{e}_3z^3-2q_3z^3\right)-\overline{e}_4z^4.
\end{aligned}
$$

APPENDIX 4
General Algebraic Equations

A4.1 Derivation of Equation (3.1)

The traction continuation condition along the Γ_0 (2.7) is

$$\varphi_{11}'(z)+\overline{\varphi_{11}'}\left(\frac{R_0^2}{z}\right)-z\varphi_{11}''(z)-\frac{z^2}{R_0^2}\psi_{11}'(z)=\varphi_0'(z)+\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)-z\varphi_0''(z)-\frac{z^2}{R_0^2}\psi_0'(z), \quad (2.7)$$

and the respective stress components defined in Chapter 2 are

$$\begin{aligned}\varphi_{11}(z) &= p_7z^7+p_6z^6+p_5z^5+p_4z^4+p_3z^3+p_2z^2\\ &\quad +p_1z^1+p_0+p_{-1}z^{-1}+p_{-2}z^{-2}+p_{-3}z^{-3}+p_{-4}z^{-4},\end{aligned}$$

$$\begin{aligned}\psi_{11}(z) &= -6p_7z^7-5p_6z^6-4p_5z^5-(3p_4+e_4)\,z^4\\ &\quad -(2p_3+e_3)\,z^3-(p_2+e_2)\,z^2-e_1z^1+(p_0-e_0)\\ &\quad +(2p_{-1}-e_{-1})\,z^{-1}+(3p_{-2}-e_{-2})\,z^{-2}+(4p_{-3}-e_{-3})\,z^{-3}\\ &\quad +(5p_{-4}-e_{-4})\,z^{-4},\end{aligned}$$

$$\varphi_0(z)=\sum_{k=0}^{\infty}a_kz^k=a_0+a_1z+a_2z^2+a_3z^3+a_4z^4+a_5z^5+a_6z^6+a_7z^7+\dots,$$

$$\psi_0(z)=\sum_{k=0}^{\infty}b_kz^k=b_0+b_1z+b_2z^2+b_3z^3+b_4z^4+b_5z^5+b_6z^6+b_7z^7+\dots.$$

Let's derive each item required to be substituted:

$$\begin{aligned}\varphi_{11}'(z) &= 7p_7z^6+6p_6z^5+5p_5z^4+4p_4z^3+3p_3z^2+2p_2z^1\\ &\quad +p_1-p_{-1}z^{-2}-2p_{-2}z^{-3}-3p_{-3}z^{-4}-4p_{-4}z^{-5},\end{aligned} \quad \text{(A4.1)}$$

$$\overline{\varphi'_{11}}\left(\frac{R_0^2}{z}\right) = 7p_7\left(\frac{R_0^2}{z}\right)^6 + 6p_6\left(\frac{R_0^2}{z}\right)^5 + 5p_5\left(\frac{R_0^2}{z}\right)^4 + 4p_4\left(\frac{R_0^2}{z}\right)^3 \tag{A4.2}$$
$$+3p_3\left(\frac{R_0^2}{z}\right)^2 + 2p_2\left(\frac{R_0^2}{z}\right)^1 + p_1 - p_{-1}\left(\frac{R_0^2}{z}\right)^{-2} - 2p_{-2}\left(\frac{R_0^2}{z}\right)^{-3}$$
$$-3p_{-3}\left(\frac{R_0^2}{z}\right)^{-4} - 4p_{-4}\left(\frac{R_0^2}{z}\right)^{-5}$$
$$= 7p_7R_0^{12}z^{-6} + 6p_6R_0^{10}z^{-5} + 5p_5R_0^8z^{-4} + 4p_4R_0^6z^{-3} + 3p_3R_0^4z^{-2}$$
$$+2p_2R_0^2z^{-1} + p_1 - \frac{p_{-1}}{R_0^4}z^2 - \frac{2p_{-2}}{R_0^6}z^3 - \frac{3p_{-3}}{R_0^8}z^4 - \frac{4p_{-4}}{R_0^{10}}z^5,$$

$$z\varphi''_{11}(z) = 42p_7z^6 + 30p_6z^5 + 20p_5z^4 + 12p_4z^3 + 6p_3z^2 + 2p_2z \tag{A4.3}$$
$$+2p_{-1}z^{-2} + 6p_{-2}z^{-3} + 12p_{-3}z^{-4} + 20p_{-4}z^{-5},$$

$$\frac{z^2}{R_0^2}\psi'_{11}(z) \tag{A4.4}$$
$$= \frac{z^2}{R_0^2}\left(\begin{array}{c} -42p_7z^6 - 30p_6z^5 - 20p_5z^4 - 4\left(3p_4+e_4\right)z^3 - 3\left(2p_3+e_3\right)z^2 \\ -2\left(p_2+e_2\right)z^1 - e_1 - \left(2p_{-1}-e_{-1}\right)z^{-2} - 2\left(3p_{-2}-e_{-2}\right)z^{-3} \\ -3\left(4p_{-3}-e_{-3}\right)z^{-4} - 4\left(5p_{-4}-e_{-4}\right)z^{-5} \end{array}\right)$$
$$= \begin{array}{c} -\frac{42}{R_0^2}p_7z^8 - \frac{30}{R_0^2}p_6z^7 - \frac{20}{R_0^2}p_5z^6 - \frac{1}{R_0^2}\left(12p_4+4e_4\right)z^5 - \frac{1}{R_0^2}\left(6p_3+3e_3\right)z^4 \\ -\frac{1}{R_0^2}\left(2p_2+2e_2\right)z^3 - e_1\frac{1}{R_0^2}z^2 - \frac{1}{R_0^2}\left(2p_{-1}-e_{-1}\right) \\ -\frac{1}{R_0^2}\left(6p_{-2}-2e_{-2}\right)z^{-1} - \frac{1}{R_0^2}\left(12p_{-3}-3e_{-3}\right)z^{-2} - \frac{1}{R_0^2}\left(20p_{-4}-4e_{-4}\right)z^{-3}, \end{array}$$

$$\varphi'_0(z) = \left(a_0 + a_1z + a_2z^2 + a_3z^3 + a_4z^4 + a_5z^5 + a_6z^6 + a_7z^7\right)' \tag{A4.5}$$
$$= 7a_7z^6 + 6a_6z^5 + 5a_5z^4 + 4a_4z^3 + 3a_3z^2 + 2a_2z + a_1,$$

$$\overline{\varphi'_0}\left(\frac{R_0^2}{z}\right) = 7R_0^{12}a_7z^{-6} + 6R_0^{10}a_6z^{-5} + 5R_0^8a_5z^{-4} + 4R_0^6a_4z^{-3} \tag{A4.6}$$

$$+3R_0^4a_3z^{-2}+2R_0^2a_2z^{-1}+a_1,$$

$$\begin{aligned} z\varphi_0''(z) &= z\left(7a_7z^6+6a_6z^5+5a_5z^4+4a_4z^3+3a_3z^2+2a_2z+a_1\right)' \qquad \text{(A4.7)} \\ &= 42a_7z^6+30a_6z^5+20a_5z^4+12a_4z^3+6a_3z^2+2a_2z, \end{aligned}$$

$$\begin{aligned} \frac{z^2}{R_0^2}\psi_0'(z) &= \frac{z^2}{R_0^2}\left(b_0+b_1z+b_2z^2+b_3z^3+b_4z^4+b_5z^5+b_6z^6+b_7z^7\right)' \qquad \text{(A4.8)} \\ &= \frac{1}{R_0^2}b_1z^2+\frac{2}{R_0^2}b_2z^3+\frac{3}{R_0^2}b_3z^4+\frac{4}{R_0^2}b_4z^5+\frac{5}{R_0^2}b_5z^6+\frac{6}{R_0^2}b_6z^7+\frac{7}{R_0^2}b_7z^8. \end{aligned}$$

Thus, by substituting (A4.1) thru (A4.4) into (2.7), the L.H.S. of (2.7) becomes

$$\varphi_{11}'(z)+\overline{\varphi_{11}'}\left(\frac{R_0^2}{z}\right)-z\varphi_{11}''(z)-\frac{z^2}{R_0^2}\psi_{11}'(z) \qquad \text{(A4.9)}$$

$$\begin{aligned} &= \left(\begin{array}{c} 7p_7z^6+6p_6z^5+5p_5z^4+4p_4z^3+3p_3z^2+2p_2z^1+p_1-p_{-1}z^{-2}-2p_{-2}z^{-3} \\ -3p_{-3}z^{-4}-4p_{-4}z^{-5} \end{array}\right) \\ &+\left(\begin{array}{c} 7p_7R_0^{12}z^{-6}+6p_6R_0^{10}z^{-5}+5p_5R_0^8z^{-4}+4p_4R_0^6z^{-3}+3p_3R_0^4z^{-2}+2p_2R_0^2z^{-1} \\ +p_1-\frac{p_{-1}}{R_0^4}z^2-\frac{2p_{-2}}{R_0^6}z^3-\frac{3p_{-3}}{R_0^8}z^4-\frac{4p_{-4}}{R_0^{10}}z^5 \end{array}\right) \\ &-\left(\begin{array}{c} 42p_7z^6+30p_6z^5+20p_5z^4+12p_4z^3+6p_3z^2+2p_2z+2p_{-1}z^{-2}+6p_{-2}z^{-3} \\ +12p_{-3}z^{-4}+20p_{-4}z^{-5} \end{array}\right) \\ &-\left(\begin{array}{c} -\frac{42}{R_0^2}p_7z^8-\frac{30}{R_0^2}p_6z^7-\frac{20}{R_0^2}p_5z^6-\frac{1}{R_0^2}\left(12p_4+4e_4\right)z^5-\frac{1}{R_0^2}\left(6p_3+3e_3\right)z^4 \\ -\frac{1}{R_0^2}\left(2p_2+2e_2\right)z^3-e_1\frac{1}{R_0^2}z^2-\frac{1}{R_0^2}\left(2p_{-1}-e_{-1}\right) \\ -\frac{1}{R_0^2}\left(6p_{-2}-2e_{-2}\right)z^{-1}-\frac{1}{R_0^2}\left(12p_{-3}-3e_{-3}\right)z^{-2}-\frac{1}{R_0^2}\left(20p_{-4}-4e_{-4}\right)z^{-3} \end{array}\right) \\ &= \frac{42}{R_0^2}p_7z^8+\frac{30}{R_0^2}p_6z^7+\left(-35p_7+\frac{20}{R_0^2}p_5\right)z^6+\left(\frac{1}{R_0^2}\left(12p_4+4e_4\right)-24p_6-\frac{4p_{-4}}{R_0^{10}}\right)z^5 \\ &+\left(\frac{1}{R_0^2}\left(6p_3+3e_3\right)-15p_5-\frac{3p_{-3}}{R_0^8}\right)z^4+\left(\frac{1}{R_0^2}\left(2p_2+2e_2\right)-8p_4-\frac{2p_{-2}}{R_0^6}\right)z^3 \end{aligned}$$

$$+\left(\frac{1}{R_0^2}e_1 - 3p_3 - \frac{p_{-1}}{R_0^4}\right)z^2 + (0)\,z + \left(\frac{1}{R_0^2}\left(2p_{-1} - e_{-1}\right) + 2p_1\right)$$
$$+\left(2R_0^2p_2 + \frac{1}{R_0^2}\left(6p_{-2} - 2e_{-2}\right)\right)z^{-1} + \left(\frac{1}{R_0^2}\left(12p_{-3} - 3e_{-3}\right) - 3p_{-1} + 3R_0^4p_3\right)z^{-2}$$
$$+\left(\frac{1}{R_0^2}\left(20p_{-4} - 4e_{-4}\right) - 8p_{-2} + 4R_0^6p_4\right)z^{-3} + \left(5R_0^8p_5 - 15p_{-3}\right)z^{-4}$$
$$+\left(6R_0^{10}p_6 - 24p_{-4}\right)z^{-5} + 7p_7R_0^{12}z^{-6}.$$

Similarly, substituting (A4.5) thru (A5.8) into R.H.S. of (2.7) yields

$$\varphi_0'(z) + \overline{\varphi_0'}\left(\frac{R_0^2}{z}\right) - z\varphi_0''(z) - \frac{z^2}{R_0^2}\psi_0'(z) \tag{A4.10}$$
$$= \left(7a_7z^6 + 6a_6z^5 + 5a_5z^4 + 4a_4z^3 + 3a_3z^2 + 2a_2z + a_1\right)$$
$$+\left(7R_0^{12}a_7z^{-6} + 6R_0^{10}a_6z^{-5} + 5R_0^8a_5z^{-4} + 4R_0^6a_4z^{-3} + 3R_0^4a_3z^{-2} + 2R_0^2a_2z^{-1} + a_1\right)$$
$$-\left(42a_7z^6 + 30a_6z^5 + 20a_5z^4 + 12a_4z^3 + 6a_3z^2 + 2a_2z\right)$$
$$-\left(\frac{1}{R_0^2}b_1z^2 + \frac{2}{R_0^2}b_2z^3 + \frac{3}{R_0^2}b_3z^4 + \frac{4}{R_0^2}b_4z^5 + \frac{5}{R_0^2}b_5z^6 + \frac{6}{R_0^2}b_6z^7 + \frac{7}{R_0^2}b_7z^8\right)$$
$$= -\frac{7}{R_0^2}b_7z^8 - \frac{6}{R_0^2}b_6z^7 + \left(-35a_7 - \frac{5}{R_0^2}b_5\right)z^6 + \left(-24a_6 - \frac{4}{R_0^2}b_4\right)z^5$$
$$+\left(-15a_5 - \frac{3}{R_0^2}b_3\right)z^4 + \left(-8a_4 - \frac{2}{R_0^2}b_2\right)z^3 + \left(-3a_3 - \frac{1}{R_0^2}b_1\right)z^2$$
$$+(0)\,z + 2a_1 + 7R_0^{12}a_7z^{-6} + 6R_0^{10}a_6z^{-5} + 5R_0^8a_5z^{-4} + 4R_0^6a_4z^{-3}$$
$$+3R_0^4a_3z^{-2} + 2R_0^2a_2z^{-1}.$$

Thus, (2.7) is re-written in the following form

$$\frac{42}{R_0^2}p_7z^8 + \frac{30}{R_0^2}p_6z^7 + \left(-35p_7 + \frac{20}{R_0^2}p_5\right)z^6 \tag{3.1}$$
$$+\left(\frac{1}{R_0^2}\left(12p_4 + 4e_4\right) - 24p_6 - \frac{4p_{-4}}{R_0^{10}}\right)z^5 + \left(\frac{1}{R_0^2}\left(6p_3 + 3e_3\right) - 15p_5 - \frac{3p_{-3}}{R_0^8}\right)z^4$$
$$+\left(\frac{1}{R_0^2}\left(2p_2 + 2e_2\right) - 8p_4 - \frac{2p_{-2}}{R_0^6}\right)z^3 + \left(\frac{1}{R_0^2}e_1 - 3p_3 - \frac{p_{-1}}{R_0^4}\right)z^2 + (0)\,z$$

$$
\begin{aligned}
&+\left(\frac{1}{R_0^2}\left(2p_{-1}-e_{-1}\right)+2p_1\right)\\
&+\left(\frac{1}{R_0^2}\left(12p_{-3}-3e_{-3}\right)-3p_{-1}+3R_0^4p_3\right)z^{-2}\\
&+\left(\frac{1}{R_0^2}\left(20p_{-4}-4e_{-4}\right)-8p_{-2}+4R_0^6p_4\right)z^{-3}+\left(5R_0^8p_5-15p_{-3}\right)z^{-4}\\
&+\left(6R_0^{10}p_6-24p_{-4}\right)z^{-5}+7p_7R_0^{12}z^{-6}\\
= &-\frac{7}{R_0^2}b_7z^8-\frac{6}{R_0^2}b_6z^7+\left(-35a_7-\frac{5}{R_0^2}b_5\right)z^6+\left(-24a_6-\frac{4}{R_0^2}b_4\right)z^5\\
&+\left(-15a_5-\frac{3}{R_0^2}b_3\right)z^4+\left(-8a_4-\frac{2}{R_0^2}b_2\right)z^3+\left(-3a_3-\frac{1}{R_0^2}b_1\right)z^2+(0)\,z\\
&+2a_1+2R_0^2a_2z^{-1}+3R_0^4a_3z^{-2}+4R_0^6a_4z^{-3}+5R_0^8a_5z^{-4}+6R_0^{10}a_6z^{-5}+7R_0^{12}a_7z^{-6}.
\end{aligned}
$$

A4.2 Derivation of Equation (3.2)

The displacement jump condition along the Γ_0 (2.12) is

$$
\begin{aligned}
&(m_0-n_0)(\kappa_1+1)\frac{R_0}{z}\varphi_{11}(z)+(m_0+n_0)(\kappa_1+1)\frac{z}{R_0}\overline{\varphi_{11}}\left(\frac{R_0^2}{z}\right) \qquad (2.12)\\
= &\;4\mu_1\left(\varphi_0'(z)+\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)-z\varphi_0''(z)-\frac{z^2}{R_0^2}\psi_0'(z)\right)+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})\frac{R_0}{z}\varphi_0(z)\\
&+(m_0-n_0)(1-\frac{\mu_1}{\mu_0})\left(R_0\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)+\frac{R_0}{z}\overline{\psi}_0\left(\frac{R_0^2}{z}\right)\right)\\
&+(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})\frac{z}{R_0}\overline{\varphi_0}(\frac{R_0^2}{z})+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\left(R_0\varphi_0'(z)+\frac{z}{R_0}\psi_0(z)\right)\\
&+4m_0R_0\mu_1\varepsilon_1+\frac{2\mu_1(m_0+n_0)(\varepsilon_2-i\varepsilon_3)}{R_0}z^2+\frac{2\mu_1(m_0-n_0)(\varepsilon_2+i\varepsilon_3)R_0^3}{z^2}.
\end{aligned}
$$

Let's derive each item required to be substituted:

$$
\begin{aligned}
\frac{R_0}{z}\varphi_{11}(z) &= \frac{R_0}{z}\begin{pmatrix} p_7z^7+p_6z^6+p_5z^5+p_4z^4+p_3z^3+p_2z^2+p_1z^1\\ +p_0+p_{-1}z^{-1}+p_{-2}z^{-2}+p_{-3}z^{-3}+p_{-4}z^{-4}\end{pmatrix} \qquad (A4.11)\\
&= R_0p_7z^6+R_0p_6z^5+R_0p_5z^4+R_0p_4z^3+R_0p_3z^2+R_0p_2z+R_0p_1\\
&\quad +R_0p_0z^{-1}+R_0p_{-1}z^{-2}+R_0p_{-2}z^{-3}+R_0p_{-3}z^{-4}+R_0p_{-4}z^{-5},
\end{aligned}
$$

$$\frac{z}{R_0}\overline{\varphi_{11}}\left(\frac{R_0^2}{z}\right) = \frac{z}{R_0}\left(\begin{array}{c} p_7\left(R_0^2z^{-1}\right)^7+p_6\left(R_0^2z^{-1}\right)^6+p_5\left(R_0^2z^{-1}\right)^5 \\ +p_4\left(R_0^2z^{-1}\right)^4+p_3\left(R_0^2z^{-1}\right)^3+p_2\left(R_0^2z^{-1}\right)^2 \\ +p_1\left(R_0^2z^{-1}\right)^1+p_0+p_{-1}\left(R_0^2z^{-1}\right)^{-1} \\ +p_{-2}\left(R_0^2z^{-1}\right)^{-2}+p_{-3}\left(R_0^2z^{-1}\right)^{-3}+p_{-4}\left(R_0^2z^{-1}\right)^{-4} \end{array}\right) \quad \text{(A4.12)}$$

$$= p_7R_0^{13}z^{-6}+p_6R_0^{11}z^{-5}+p_5R_0^9z^{-4}+p_4R_0^7z^{-3}+p_3R_0^5z^{-2}+p_2R_0^3z^{-1}$$
$$+p_1R_0+\frac{1}{R_0}p_0z+\frac{1}{R_0^3}p_{-1}z^2+\frac{1}{R_0^5}p_{-2}z^3+\frac{1}{R_0^7}p_{-3}z^4+\frac{1}{R_0^9}p_{-4}z^5,$$

$\varphi_0'(z), \overline{\varphi_0'}\left(\frac{R_0^2}{z}\right), z\varphi_0''(z)$, and $\frac{z^2}{R_0^2}\psi_0'(z)$ are derived in the previous condition (A4.5 thru A4.8).

$$\frac{R_0}{z}\varphi_0(z) = R_0a_0z^{-1}+R_0a_1+R_0a_2z+R_0a_3z^2+R_0a_4z^3+R_0a_5z^4+R_0a_6z^5+R_0a_7z^6, \quad \text{(A4.13)}$$

$$R_0\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right) = 7R_0^{13}a_7z^{-6}+6R_0^{11}a_6z^{-5}+5R_0^9a_5z^{-4}+4R_0^7a_4z^{-3} \quad \text{(A4.14)}$$
$$+3R_0^5a_3z^{-2}+2R_0^3a_2z^{-1}+R_0a_1,$$

$$\frac{R_0}{z}\overline{\psi_0}\left(\frac{R_0^2}{z}\right) = \frac{R_0}{z}\left(\begin{array}{c} b_0+b_1\left(\frac{R_0^2}{z}\right)+b_2\left(\frac{R_0^2}{z}\right)^2+b_3\left(\frac{R_0^2}{z}\right)^3 \\ +b_4\left(\frac{R_0^2}{z}\right)^4+b_5\left(\frac{R_0^2}{z}\right)^5+b_6\left(\frac{R_0^2}{z}\right)^6+b_7\left(\frac{R_0^2}{z}\right)^7 \end{array}\right) \quad \text{(A4.15)}$$
$$= R_0b_0z^{-1}+R_0^3b_1z^{-2}+R_0^5b_2z^{-3}+R_0^7b_3z^{-4}+R_0^9b_4z^{-5}+R_0^{11}b_5z^{-6}$$
$$+R_0^{13}b_6z^{-7}+R_0^{15}b_7z^{-8},$$

$$\frac{z}{R_0}\overline{\varphi_0}\left(\frac{R_0^2}{z}\right) = \frac{1}{R_0}a_0z+R_0a_1+R_0^3a_2z^{-1}+R_0^5a_3z^{-2} \quad \text{(A4.16)}$$
$$+R_0^7a_4z^{-3}+R_0^9a_5z^{-4}+R_0^{11}a_6z^{-5}+R_0^{13}a_7z^{-6},$$

$$R_0\varphi_0'(z) = 7R_0a_7z^6 + 6R_0a_6z^5 + 5R_0a_5z^4 \tag{A4.17}$$
$$+4R_0a_4z^3 + 3R_0a_3z^2 + 2R_0a_2z + R_0a_1,$$

$$\frac{z}{R_0}\psi_0(z) = \frac{1}{R_0}b_0z + \frac{1}{R_0}b_1z^2 + \frac{1}{R_0}b_2z^3 + \frac{1}{R_0}b_3z^4 \tag{A4.18}$$
$$+\frac{1}{R_0}b_4z^5 + \frac{1}{R_0}b_5z^6 + \frac{1}{R_0}b_6z^7 + \frac{1}{R_0}b_7z^8.$$

Let's substitute (A4.11) and (A4.12) into L.H.S. of (2.12). Thus, L.H.S. of (2.12) is

$$(m_0 - n_0)(\kappa_1 + 1)\frac{R_0}{z}\varphi_{11}(z) + (m_0 + n_0)(\kappa_1 + 1)\frac{z}{R_0}\overline{\varphi_{11}}\left(\frac{R_0^2}{z}\right) \tag{A4.19}$$

$$= (m_0 - n_0)(\kappa_1 + 1)\begin{pmatrix} R_0p_7z^6 + R_0p_6z^5 + R_0p_5z^4 + R_0p_4z^3 \\ +R_0p_3z^2 + R_0p_2z + R_0p_1 + R_0p_0z^{-1} \\ +R_0p_{-1}z^{-2} + R_0p_{-2}z^{-3} + R_0p_{-3}z^{-4} + R_0p_{-4}z^{-5} \end{pmatrix}$$
$$+(m_0 + n_0)(\kappa_1 + 1)\begin{pmatrix} p_7R_0^{13}z^{-6} + p_6R_0^{11}z^{-5} + p_5R_0^9z^{-4} + p_4R_0^7z^{-3} \\ +p_3R_0^5z^{-2} + p_2R_0^3z^{-1} + p_1R_0 + \frac{1}{R_0}p_0z \\ +\frac{1}{R_0^3}p_{-1}z^2 + \frac{1}{R_0^5}p_{-2}z^3 + \frac{1}{R_0^7}p_{-3}z^4 + \frac{1}{R_0^9}p_{-4}z^5 \end{pmatrix}$$
$$= (m_0 - n_0)(\kappa_1 + 1)R_0p_7z^6$$
$$+ \left((m_0 - n_0)(\kappa_1 + 1)R_0p_6 + (m_0 + n_0)(\kappa_1 + 1)\frac{1}{R_0^9}p_{-4}\right)z^5$$
$$+ \left((m_0 - n_0)(\kappa_1 + 1)R_0p_5 + (m_0 + n_0)(\kappa_1 + 1)\frac{1}{R_0^7}p_{-3}\right)z^4$$
$$+ \left((m_0 - n_0)(\kappa_1 + 1)R_0p_4 + (m_0 + n_0)(\kappa_1 + 1)\frac{1}{R_0^5}p_{-2}\right)z^3$$
$$+ \left((m_0 - n_0)(\kappa_1 + 1)R_0p_3 + (m_0 + n_0)(\kappa_1 + 1)\frac{1}{R_0^3}p_{-1}\right)z^2$$
$$+ \left((m_0 - n_0)(\kappa_1 + 1)R_0p_2 + (m_0 + n_0)(\kappa_1 + 1)\frac{1}{R_0}p_0\right)z$$
$$+2m_0(\kappa_1 + 1)R_0p_1$$
$$+ \left((m_0 - n_0)(\kappa_1 + 1)R_0p_0 + (m_0 + n_0)(\kappa_1 + 1)R_0^3p_2\right)z^{-1}$$

$$+\left((m_0-n_0)(\kappa_1+1)R_0p_{-1}+(m_0+n_0)(\kappa_1+1)R_0^5p_3\right)z^{-2}$$
$$+\left((m_0+n_0)(\kappa_1+1)R_0^7p_4+(m_0-n_0)(\kappa_1+1)R_0p_{-2}\right)z^{-3}$$
$$+\left((m_0+n_0)(\kappa_1+1)R_0^9p_5+(m_0-n_0)(\kappa_1+1)R_0p_{-3}\right)z^{-4}$$
$$+\left((m_0-n_0)(\kappa_1+1)R_0p_{-4}+(m_0+n_0)(\kappa_1+1)R_0^{11}p_6\right)z^{-5}$$
$$+(m_0+n_0)(\kappa_1+1)R_0^{13}p_7z^{-6}.$$

By deploying (A4.13) thru (A4.18) into (2.12), the R.H.S. of (2.12) becomes

$$\begin{aligned}
&4\mu_1\left(\varphi_0'(z)+\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)-z\varphi_0''(z)-\frac{z^2}{R_0^2}\psi_0'(z)\right)+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})\frac{R_0}{z}\varphi_0(z)\\
&+(m_0-n_0)(1-\frac{\mu_1}{\mu_0})\left(R_0\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)+\frac{R_0}{z}\overline{\psi}_0\left(\frac{R_0^2}{z}\right)\right) \qquad \text{(A4.20)}\\
&+(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})\frac{z}{R_0}\overline{\varphi_0}(\frac{R_0^2}{z})+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\left(R_0\varphi_0'(z)+\frac{z}{R_0}\psi_0(z)\right)\\
&+4m_0R_0\mu_1\varepsilon_1+\frac{2\mu_1(m_0+n_0)(\varepsilon_2-i\varepsilon_3)}{R_0}z^2+\frac{2\mu_1(m_0-n_0)(\varepsilon_2+i\varepsilon_3)R_0^3}{z^2}\\
=\;&4\mu_1\left(\begin{array}{c}
-\frac{7}{R_0^2}b_7z^8-\frac{6}{R_0^2}b_6z^7+\left(-35a_7-\frac{5}{R_0^2}b_5\right)z^6+\left(-24a_6-\frac{4}{R_0^2}b_4\right)z^5\\
+\left(-15a_5-\frac{3}{R_0^2}b_3\right)z^4+\left(-8a_4-\frac{2}{R_0^2}b_2\right)z^3+\left(-3a_3-\frac{1}{R_0^2}b_1\right)z^2\\
+(0)\,z+2a_1+7R_0^{12}a_7z^{-6}+6R_0^{10}a_6z^{-5}+5R_0^8a_5z^{-4}\\
+4R_0^6a_4z^{-3}+3R_0^4a_3z^{-2}+2R_0^2a_2z^{-1}
\end{array}\right)\\
&+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})\left(\begin{array}{c}
R_0a_0z^{-1}+R_0a_1+R_0a_2z+R_0a_3z^2+R_0a_4z^3\\
+R_0a_5z^4+R_0a_6z^5+R_0a_7z^6
\end{array}\right)\\
&+(m_0-n_0)(1-\frac{\mu_1}{\mu_0})\left(\begin{array}{c}
7R_0^{13}a_7z^{-6}+6R_0^{11}a_6z^{-5}+5R_0^9a_5z^{-4}+4R_0^7a_4z^{-3}\\
+3R_0^5a_3z^{-2}+2R_0^3a_2z^{-1}+R_0a_1+R_0b_0z^{-1}\\
+R_0^3b_1z^{-2}+R_0^5b_2z^{-3}+R_0^7b_3z^{-4}+R_0^9b_4z^{-5}\\
+R_0^{11}b_5z^{-6}+R_0^{13}b_6z^{-7}+R_0^{15}b_7z^{-8}
\end{array}\right)
\end{aligned}$$

$$+(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})\left(\begin{array}{c}\frac{1}{R_0}a_0z+R_0a_1+R_0^3a_2z^{-1}+R_0^5a_3z^{-2}+R_0^7a_4z^{-3}\\+R_0^9a_5z^{-4}+R_0^{11}a_6z^{-5}+R_0^{13}a_7z^{-6}\end{array}\right)$$

$$+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\left(\begin{array}{c}7R_0a_7z^6+6R_0a_6z^5+5R_0a_5z^4+4R_0a_4z^3\\+3R_0a_3z^2+2R_0a_2z+R_0a_1+\frac{1}{R_0}b_0z\\+\frac{1}{R_0}b_1z^2+\frac{1}{R_0}b_2z^3+\frac{1}{R_0}b_3z^4+\frac{1}{R_0}b_4z^5\\+\frac{1}{R_0}b_5z^6+\frac{1}{R_0}b_6z^7+\frac{1}{R_0}b_7z^8\end{array}\right)$$

$$+4m_0R_0\mu_1\varepsilon_1+\frac{2\mu_1(m_0+n_0)(\varepsilon_2-i\varepsilon_3)}{R_0}z^2+\frac{2\mu_1(m_0-n_0)(\varepsilon_2+i\varepsilon_3)R_0^3}{z^2}$$

$$\begin{aligned}=\;&\left(-4\mu_1\frac{7}{R_0^2}b_7+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_7\right)z^8\\
&+\left(-4\mu_1\frac{6}{R_0^2}b_6+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_6\right)z^7\\
&+\left(\begin{array}{c}4\mu_1\left(-35a_7-\frac{5}{R_0^2}b_5\right)+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_7\\+7(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_7+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_5\end{array}\right)z^6\\
&+\left(\begin{array}{c}4\mu_1\left(-24a_6-\frac{4}{R_0^2}b_4\right)+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_6\\+6(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_6+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_4\end{array}\right)z^5\\
&+\left(\begin{array}{c}4\mu_1\left(-15a_5-\frac{3}{R_0^2}b_3\right)+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_5\\+5(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_5+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_3\end{array}\right)z^4\\
&+\left(\begin{array}{c}4\mu_1\left(-8a_4-\frac{2}{R_0^2}b_2\right)+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_4\\+4(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_4+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_2\end{array}\right)z^3\\
&+\left(\begin{array}{c}4\mu_1\left(-3a_3-\frac{1}{R_0^2}b_1\right)+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_3\\+3(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_3+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_1+\frac{2\mu_1(m_0+n_0)(\varepsilon_2-i\varepsilon_3)}{R_0}\end{array}\right)z^2\\
&+\left(\begin{array}{c}(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_2+(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})\frac{1}{R_0}a_0\\+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_0+2(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_2\end{array}\right)z\end{aligned}$$

$$
\begin{aligned}
&+\left(\begin{array}{c} 8\mu_1 a_1 + (m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0 a_1 + R_0(m_0-n_0)(1-\frac{\mu_1}{\mu_0})a_1 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0 a_1 + 4m_0 R_0\mu_1\varepsilon_1 + (m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0 a_1 \end{array}\right) \\
&+\left(\begin{array}{c} 8\mu_1 R_0^2 a_2 + (m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0 a_0 + 2(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^3 a_2 \\ +(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0 b_0 + (m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^3 a_2 \end{array}\right) z^{-1} \\
&+\left(\begin{array}{c} 12\mu_1 R_0^4 a_3 + 3(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^5 a_3 + (m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^3 b_1 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^5 a_3 + 2\mu_1(m_0-n_0)(\varepsilon_2+i\varepsilon_3)R_0^3 \end{array}\right) z^{-2} \\
&+\left(\begin{array}{c} 16\mu_1 R_0^6 a_4 + 4(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^7 a_4 + (m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^5 b_2 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^7 a_4 \end{array}\right) z^{-3} \\
&+\left(\begin{array}{c} 20\mu_1 R_0^8 a_5 + 5(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^9 a_5 + (m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^7 b_3 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^9 a_5 \end{array}\right) z^{-4} \\
&+\left(\begin{array}{c} 24\mu_1 R_0^{10} a_6 + 6(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^{11} a_6 + (m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^9 b_4 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^{11} a_6 \end{array}\right) z^{-5} \\
&+\left(\begin{array}{c} 28\mu_1 R_0^{12} a_7 + 7(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^{13} a_7 + (m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^{11} b_5 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^{13} a_7 \end{array}\right) z^{-6} \\
&+(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^{13} b_6 z^{-7} + (m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^{15} b_7 z^{-8}.
\end{aligned}
$$

Consequently, (2.12) is expressed in terms of complex coefficients as follows

$$
\begin{aligned}
&(m_0-n_0)(\kappa_1+1)R_0 p_7 z^6 + \left((m_0-n_0)(\kappa_1+1)R_0 p_6 + (m_0+n_0)(\kappa_1+1)\frac{1}{R_0^9}p_{-4}\right) z^5 \\
&+\left((m_0-n_0)(\kappa_1+1)R_0 p_5 + (m_0+n_0)(\kappa_1+1)\frac{1}{R_0^7}p_{-3}\right) z^4 \qquad (3.2) \\
&+\left((m_0-n_0)(\kappa_1+1)R_0 p_4 + (m_0+n_0)(\kappa_1+1)\frac{1}{R_0^5}p_{-2}\right) z^3 \\
&+\left((m_0-n_0)(\kappa_1+1)R_0 p_3 + (m_0+n_0)(\kappa_1+1)\frac{1}{R_0^3}p_{-1}\right) z^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left((m_0-n_0)(\kappa_1+1)R_0p_2+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0}p_0\right)z \\
&+2m_0(\kappa_1+1)R_0p_1+\left[(m_0-n_0)(\kappa_1+1)R_0p_0+(m_0+n_0)(\kappa_1+1)R_0^3p_2\right]z^{-1} \\
&+\left[(m_0-n_0)(\kappa_1+1)R_0p_{-1}+(m_0+n_0)(\kappa_1+1)R_0^5p_3\right]z^{-2} \\
&+\left[(m_0+n_0)(\kappa_1+1)R_0^7p_4+(m_0-n_0)(\kappa_1+1)R_0p_{-2}\right]z^{-3} \\
&+\left[(m_0+n_0)(\kappa_1+1)R_0^9p_5+(m_0-n_0)(\kappa_1+1)R_0p_{-3}\right]z^{-4} \\
&+\left[(m_0-n_0)(\kappa_1+1)R_0p_{-4}+(m_0+n_0)(\kappa_1+1)R_0^{11}p_6\right]z^{-5} \\
&+(m_0+n_0)(\kappa_1+1)R_0^{13}p_7z^{-6}
\end{aligned}
$$

$$
\begin{aligned}
= \;& \left(-4\mu_1\frac{7}{R_0^2}b_7+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_7\right)z^8 \\
&+\left(-4\mu_1\frac{6}{R_0^2}b_6+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_6\right)z^7 \\
&+\left(\begin{array}{c} 4\mu_1\left(-35a_7-\frac{5}{R_0^2}b_5\right)+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_7 \\ +7(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_7+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_5 \end{array}\right)z^6 \\
&+\left(\begin{array}{c} 4\mu_1\left(-24a_6-\frac{4}{R_0^2}b_4\right)+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_6 \\ +6(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_6+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_4 \end{array}\right)z^5 \\
&+\left(\begin{array}{c} 4\mu_1\left(-15a_5-\frac{3}{R_0^2}b_3\right)+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_5 \\ +5(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_5+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_3 \end{array}\right)z^4 \\
&+\left(\begin{array}{c} 4\mu_1\left(-8a_4-\frac{2}{R_0^2}b_2\right)+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_4 \\ +4(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_4+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_2 \end{array}\right)z^3 \\
&+\left(\begin{array}{c} 4\mu_1\left(-3a_3-\frac{1}{R_0^2}b_1\right)+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_3+\frac{2\mu_1(m_0+n_0)(\varepsilon_2-i\varepsilon_3)}{R_0} \\ +3(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_3+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_1 \end{array}\right)z^2
\end{aligned}
$$

$$
\begin{aligned}
&+\left(\begin{array}{c} (m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_2+(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})\frac{1}{R_0}a_0 \\ +(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_0+2(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_2 \end{array}\right)z \\
&+\left(\begin{array}{c} 8\mu_1a_1+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_1+R_0(m_0-n_0)(1-\frac{\mu_1}{\mu_0})a_1 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_1+4m_0R_0\mu_1\varepsilon_1+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0a_1 \end{array}\right) \\
&+\left(\begin{array}{c} 8\mu_1R_0^2a_2+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_0+2(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^3a_2 \\ +(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0b_0+(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^3a_2 \end{array}\right)z^{-1} \\
&+\left(\begin{array}{c} 12\mu_1R_0^4a_3+3(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^5a_3+(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^3b_1 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^5a_3+2\mu_1(m_0-n_0)(\varepsilon_2+i\varepsilon_3)R_0^3 \end{array}\right)z^{-2} \\
&+\left(\begin{array}{c} 16\mu_1R_0^6a_4+4(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^7a_4+(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^5b_2 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^7a_4 \end{array}\right)z^{-3} \\
&+\left(\begin{array}{c} 20\mu_1R_0^8a_5+5(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^9a_5+(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^7b_3 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^9a_5 \end{array}\right)z^{-4} \\
&+\left(\begin{array}{c} 24\mu_1R_0^{10}a_6+6(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^{11}a_6+(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^9b_4 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^{11}a_6 \end{array}\right)z^{-5} \\
&+\left(\begin{array}{c} 28\mu_1R_0^{12}a_7+7(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^{13}a_7+(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^{11}b_5 \\ +(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^{13}a_7 \end{array}\right)z^{-6} \\
&+(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^{13}b_6z^{-7}+(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^{15}b_7z^{-8}.
\end{aligned}
$$

A4.3 Derivation of Equation (3.22)

The traction continuation condition along the Γ_1 (2.13) is

$$
\varphi_2'(z)+\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)-z\varphi_2''(z)-\frac{z^2}{R_1^2}\psi_2'(z)=\varphi_{12}'(z)+\overline{\varphi_{12}'}\left(\frac{R_1^2}{z}\right)-z\varphi_{12}''(z)-\frac{z^2}{R_1^2}\psi_{12}'(z),
$$

(2.13)

and the stress potentials associated with (2.13) are defined in Chapter 2 in the following form

$$\varphi_2(z) = Az+\sum_{k=1}^{\infty} c_k z^{-k} = Az+c_1z^{-1}+c_2z^{-2}+c_3z^{-3}+c_4z^{-4}+c_5z^{-5}+c_6z^{-6}+c_7z^{-7}+c_8z^{-8},$$

$$\psi_2(z) = Bz+\sum_{k=1}^{\infty} d_k z^{-k} = Bz+d_1z^{-1}+d_2z^{-2}+d_3z^{-3}+d_4z^{-4}+d_5z^{-5}+d_6z^{-6}+d_7z^{-7}+d_8z^{-8},$$

$$\varphi_{12}(z) = q_3z^3+q_2z^2+q_1z^1+q_0+q_{-1}z^{-1}+q_{-2}z^{-2}+q_{-3}z^{-3}+q_{-4}z^{-4}+q_{-5}z^{-5}+q_{-6}z^{-6},$$

$$\begin{aligned}\psi_{12}(z) &= 7q_{-6}z^{-6}+6q_{-5}z^{-5}+(5q_{-4}-\bar{e}_{-4})\,z^{-4}+(4q_{-3}-\bar{e}_{-3})\,z^{-3}+(3q_{-2}-\bar{e}_{-2})\,z^{-2}\\ &\quad+(2q_{-1}-\bar{e}_{-1})\,z^{-1}+(q_0-\bar{e}_0)-\bar{e}_1z^1+(-\bar{e}_2-q_2)\,z^2+(-\bar{e}_3-2q_3)\,z^3-\bar{e}_4z^4.\end{aligned}$$

Let's derive the items on the R.H.S:

$$\varphi_{12}'(z) = 3q_3z^2+2q_2z^1+q_1-q_{-1}z^{-2}-2q_{-2}z^{-3}-3q_{-3}z^{-4}-4q_{-4}z^{-5}-5q_{-5}z^{-6}-6q_{-6}z^{-7},$$

(A4.21)

$$\begin{aligned}\overline{\varphi_{12}'}\left(\frac{R_1^2}{z}\right) &= 3q_3\left(R_1^2z^{-1}\right)^2+2q_2\left(R_1^2z^{-1}\right)^1+q_1-q_{-1}\left(R_1^2z^{-1}\right)^{-2} \qquad \text{(A4.22)}\\ &\quad-2q_{-2}\left(R_1^2z^{-1}\right)^{-3}-3q_{-3}\left(R_1^2z^{-1}\right)^{-4}-4q_{-4}\left(R_1^2z^{-1}\right)^{-5}\\ &\quad-5q_{-5}\left(R_1^2z^{-1}\right)^{-6}-6q_{-6}\left(R_1^2z^{-1}\right)^{-7}\\ &= 3q_3R_1^4z^{-2}+2q_2R_1^2z^{-1}+q_1-q_{-1}\frac{1}{R_1^4}z^2-2q_{-2}\frac{1}{R_1^6}z^3-3q_{-3}\frac{1}{R_1^8}z^4\\ &\quad-4q_{-4}\frac{1}{R_1^{10}}z^5-5q_{-5}\frac{1}{R_1^{12}}z^6-6q_{-6}\frac{1}{R_1^{14}}z^7,\end{aligned}$$

$$-z\varphi_{12}''(z) = -z\begin{pmatrix} 3q_3z^2+2q_2z^1+q_1-q_{-1}z^{-2}-2q_{-2}z^{-3} \\ -3q_{-3}z^{-4}-4q_{-4}z^{-5}-5q_{-5}z^{-6}-6q_{-6}z^{-7} \end{pmatrix}' \tag{A4.23}$$
$$= -6q_3z^2-2q_2z-2q_{-1}z^{-2}-6q_{-2}z^{-3}-12q_{-3}z^{-4}-20q_{-4}z^{-5}$$
$$-30q_{-5}z^{-6}-42q_{-6}z^{-7},$$

$$-\frac{z^2}{R_1^2}\psi_{12}'(z) = -\frac{z^2}{R_1^2}\begin{pmatrix} 7q_{-6}z^{-6}+6q_{-5}z^{-5}+(5q_{-4}-\overline{e}_{-4})z^{-4} \\ +(4q_{-3}-\overline{e}_{-3})z^{-3}+(3q_{-2}-\overline{e}_{-2})z^{-2} \\ +(2q_{-1}-\overline{e}_{-1})z^{-1}+(q_0-\overline{e}_0)-\overline{e}_1z^1 \\ +(-\overline{e}_2-q_2)z^2+(-\overline{e}_3-2q_3)z^3-\overline{e}_4z^4 \end{pmatrix}' \tag{A4.24}$$
$$= -\frac{z^2}{R_1^2}\begin{pmatrix} -42q_{-6}z^{-7}-30q_{-5}z^{-6}-4(5q_{-4}-\overline{e}_{-4})z^{-5} \\ -3(4q_{-3}-\overline{e}_{-3})z^{-4}-2(3q_{-2}-\overline{e}_{-2})z^{-3} \\ -(2q_{-1}-\overline{e}_{-1})z^{-2}-\overline{e}_1 \\ +2(-\overline{e}_2-q_2)z+3(-\overline{e}_3-2q_3)z^2-4\overline{e}_4z^3 \end{pmatrix}$$
$$= 42q_{-6}\frac{1}{R_1^2}z^{-5}+30q_{-5}\frac{1}{R_1^2}z^{-4}+4(5q_{-4}-\overline{e}_{-4})\frac{1}{R_1^2}z^{-3}$$
$$+3(4q_{-3}-\overline{e}_{-3})\frac{1}{R_1^2}z^{-2}+2(3q_{-2}-\overline{e}_{-2})\frac{1}{R_1^2}z^{-1}+(2q_{-1}-\overline{e}_{-1})\frac{1}{R_1^2}$$
$$+\overline{e}_1\frac{z^2}{R_1^2}-2(-\overline{e}_2-q_2)\frac{1}{R_1^2}z^3-3(-\overline{e}_3-2q_3)\frac{1}{R_1^2}z^4+4\overline{e}_4\frac{1}{R_1^2}z^5.$$

Also, let's derive the items on the L.H.S:

$$\varphi_2'(z) = A-c_1z^{-2}-2c_2z^{-3}-3c_3z^{-4}-4c_4z^{-5}-5c_5z^{-6}-6c_6z^{-7}-7c_7z^{-8}-8c_8z^{-9}, \tag{A4.25}$$

$$\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right) = A-c_1\left(R_1^2z^{-1}\right)^{-2}-2c_2\left(R_1^2z^{-1}\right)^{-3}-3c_3\left(R_1^2z^{-1}\right)^{-4} \tag{A4.26}$$

$$-4c_4\left(R_1^2z^{-1}\right)^{-5}-5c_5\left(R_1^2z^{-1}\right)^{-6}-6c_6\left(R_1^2z^{-1}\right)^{-7}$$
$$-7c_7\left(R_1^2z^{-1}\right)^{-8}-8c_8\left(R_1^2z^{-1}\right)^{-9}$$
$$= A-\frac{1}{R_1^4}c_1z^2-\frac{2}{R_1^6}c_2z^3-\frac{3}{R_1^8}c_3z^4-\frac{4}{R_1^{10}}c_4z^5-\frac{5}{R_1^{12}}c_5z^6$$
$$-\frac{6}{R_1^{14}}c_6z^7-\frac{7}{R_1^{16}}c_7z^8-\frac{8}{R_1^{18}}c_8z^9,$$

$$\begin{aligned} z\varphi_2''(z) &= 2c_1z^{-2}+6c_2z^{-3}+12c_3z^{-4}+20c_4z^{-5} \\ &\quad +30c_5z^{-6}+42c_6z^{-7}+56c_7z^{-8}+72c_8z^{-9}, \end{aligned} \tag{A4.27}$$

$$\begin{aligned} \frac{z^2}{R_1^2}\psi_2'(z) &= \frac{z^2}{R_1^2}\begin{pmatrix} Bz+d_1z^{-1}+d_2z^{-2}+d_3z^{-3}+d_4z^{-4} \\ +d_5z^{-5}+d_6z^{-6}+d_7z^{-7}+d_8z^{-8} \end{pmatrix}' \\ &= \frac{1}{R_1^2}Bz^2-\frac{1}{R_1^2}d_1-\frac{2}{R_1^2}d_2z^{-1}-\frac{3}{R_1^2}d_3z^{-2}-\frac{4}{R_1^2}d_4z^{-3}-\frac{5}{R_1^2}d_5z^{-4} \\ &\quad -\frac{6}{R_1^2}d_6z^{-5}-\frac{7}{R_1^2}d_7z^{-6}-\frac{8}{R_1^2}d_8z^{-7}. \end{aligned} \tag{A4.28}$$

Thus, by substituting (A4.25) thru (A4.28) into (2.13), the L.H.S. of (2.13) becomes

$$\begin{aligned} &\varphi_2'(z)+\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)-z\varphi_2''(z)-\frac{z^2}{R_1^2}\psi_2'(z) \\ = &\left(A-c_1z^{-2}-2c_2z^{-3}-3c_3z^{-4}-4c_4z^{-5}-5c_5z^{-6}-6c_6z^{-7}-7c_7z^{-8}-8c_8z^{-9}\right) \\ &+\begin{pmatrix} A-\frac{1}{R_1^4}c_1z^2-\frac{2}{R_1^6}c_2z^3-\frac{3}{R_1^8}c_3z^4-\frac{4}{R_1^{10}}c_4z^5-\frac{5}{R_1^{12}}c_5z^6 \\ -\frac{6}{R_1^{14}}c_6z^7-\frac{7}{R_1^{16}}c_7z^8-\frac{8}{R_1^{18}}c_8z^9 \end{pmatrix} \\ &-\begin{pmatrix} 2c_1z^{-2}+6c_2z^{-3}+12c_3z^{-4}+20c_4z^{-5}+30c_5z^{-6}+42c_6z^{-7} \\ +56c_7z^{-8}+72c_8z^{-9} \end{pmatrix} \\ &-\begin{pmatrix} \frac{1}{R_1^2}Bz^2-\frac{1}{R_1^2}d_1-\frac{2}{R_1^2}d_2z^{-1}-\frac{3}{R_1^2}d_3z^{-2}-\frac{4}{R_1^2}d_4z^{-3} \\ -\frac{5}{R_1^2}d_5z^{-4}-\frac{6}{R_1^2}d_6z^{-5}-\frac{7}{R_1^2}d_7z^{-6}-\frac{8}{R_1^2}d_8z^{-7} \end{pmatrix} \end{aligned} \tag{A4.29}$$

$$
\begin{aligned}
= & -\frac{8}{R_1^{18}}c_8z^9 - \frac{7}{R_1^{16}}c_7z^8 - \frac{6}{R_1^{14}}c_6z^7 - \frac{5}{R_1^{12}}c_5z^6 - \frac{4}{R_1^{10}}c_4z^5 - \frac{3}{R_1^{8}}c_3z^4 \\
& -\frac{2}{R_1^{6}}c_2z^3 + \left(-\frac{1}{R_1^4}c_1z^2 - \frac{1}{R_1^2}Bz^2\right) + \left(2A + \frac{1}{R_1^2}d_1\right) + \frac{2}{R_1^2}d_2z^{-1} \\
& + \left(-3c_1z^{-2} + \frac{3}{R_1^2}d_3z^{-2}\right) + \left(-8c_2z^{-3} + \frac{4}{R_1^2}d_4z^{-3}\right) \\
& + \left(-15c_3z^{-4} + \frac{5}{R_1^2}d_5z^{-4}\right) + \left(-24c_4z^{-5} + \frac{6}{R_1^2}d_6z^{-5}\right) \\
& + \left(-35c_5z^{-6} + \frac{7}{R_1^2}d_7z^{-6}\right) + \left(-48c_6z^{-7} + \frac{8}{R_1^2}d_8z^{-7}\right) - 63c_7z^{-8} - 80c_8z^{-9}.
\end{aligned}
$$

Also, substituting (A4.21) thru (A4.24) into the R.H.S. of (2.13) yields

$$
\begin{aligned}
& \varphi_{12}'(z) + \overline{\varphi_{12}'}\left(\frac{R_1^2}{z}\right) - z\varphi_{12}''(z) - \frac{z^2}{R_1^2}\psi_{12}'(z) \qquad \text{(A4.30)} \\
= & \left(\begin{array}{c} 3q_3z^2 + 2q_2z^1 + q_1 - q_{-1}z^{-2} - 2q_{-2}z^{-3} - 3q_{-3}z^{-4} \\ -4q_{-4}z^{-5} - 5q_{-5}z^{-6} - 6q_{-6}z^{-7} \end{array}\right) \\
& + \left(\begin{array}{c} 3q_3R_1^4z^{-2} + 2q_2R_1^2z^{-1} + q_1 - q_{-1}\frac{1}{R_1^4}z^2 - 2q_{-2}\frac{1}{R_1^6}z^3 \\ -3q_{-3}\frac{1}{R_1^8}z^4 - 4q_{-4}\frac{1}{R_1^{10}}z^5 - 5q_{-5}\frac{1}{R_1^{12}}z^6 - 6q_{-6}\frac{1}{R_1^{14}}z^7 \end{array}\right) \\
& -6q_3z^2 - 2q_2z - 2q_{-1}z^{-2} - 6q_{-2}z^{-3} - 12q_{-3}z^{-4} - 20q_{-4}z^{-5} \\
& -30q_{-5}z^{-6} - 42q_{-6}z^{-7} + 42q_{-6}\frac{1}{R_1^2}z^{-5} + 30q_{-5}\frac{1}{R_1^2}z^{-4} \\
& +4\left(5q_{-4} - \overline{e}_{-4}\right)\frac{1}{R_1^2}z^{-3} + 3\left(4q_{-3} - \overline{e}_{-3}\right)\frac{1}{R_1^2}z^{-2} \\
& +2\left(3q_{-2} - \overline{e}_{-2}\right)\frac{1}{R_1^2}z^{-1} + \left(2q_{-1} - \overline{e}_{-1}\right)\frac{1}{R_1^2} + \overline{e}_1\frac{z^2}{R_1^2} \\
& -2\left(-\overline{e}_2 - q_2\right)\frac{1}{R_1^2}z^3 - 3\left(-\overline{e}_3 - 2q_3\right)\frac{1}{R_1^2}z^4 + 4\overline{e}_4\frac{1}{R_1^2}z^5 \\
= & \left(-6q_{-6}\frac{1}{R_1^{14}}z^7\right) + \left(-5q_{-5}\frac{1}{R_1^{12}}z^6\right) + \left(4\overline{e}_4\frac{1}{R_1^2}z^5 - 4q_{-4}\frac{1}{R_1^{10}}z^5\right) \\
& + \left(-3\left(-\overline{e}_3 - 2q_3\right)\frac{1}{R_1^2}z^4 - 3q_{-3}\frac{1}{R_1^8}z^4\right) + \left(-2\left(-\overline{e}_2 - q_2\right)\frac{1}{R_1^2}z^3 - 2q_{-2}\frac{1}{R_1^6}z^3\right) \\
& + \left(\overline{e}_1\frac{z^2}{R_1^2} - q_{-1}\frac{1}{R_1^4}z^2 + 3q_3z^2 - 6q_3z^2\right) + \left(2q_2z^1 - 2q_2z\right)
\end{aligned}
$$

$$+\left(q_1+q_1+(2q_{-1}-\overline{e}_{-1})\frac{1}{R_1^2}\right)+\left(2q_2R_1^2z^{-1}+2(3q_{-2}-\overline{e}_{-2})\frac{1}{R_1^2}z^{-1}\right)$$
$$+\left(3q_3R_1^4z^{-2}-q_{-1}z^{-2}-2q_{-1}z^{-2}+3(4q_{-3}-\overline{e}_{-3})\frac{1}{R_1^2}z^{-2}\right)$$
$$+\left(-2q_{-2}z^{-3}+4(5q_{-4}-\overline{e}_{-4})\frac{1}{R_1^2}z^{-3}-6q_{-2}z^{-3}\right)$$
$$+\left(-3q_{-3}z^{-4}+30q_{-5}\frac{1}{R_1^2}z^{-4}-12q_{-3}z^{-4}\right)$$
$$+\left(-4q_{-4}z^{-5}+42q_{-6}\frac{1}{R_1^2}z^{-5}-20q_{-4}z^{-5}\right)$$
$$+\left(-5q_{-5}z^{-6}-30q_{-5}z^{-6}\right)+\left(-6q_{-6}z^{-7}-42q_{-6}z^{-7}\right).$$

Thus, (2.13) is determined solely in terms of the complex coefficients

$$\begin{aligned}
&-\frac{8}{R_1^{18}}c_8z^9-\frac{7}{R_1^{16}}c_7z^8-\frac{6}{R_1^{14}}c_6z^7-\frac{5}{R_1^{12}}c_5z^6-\frac{4}{R_1^{10}}c_4z^5-\frac{3}{R_1^8}c_3z^4 \qquad (3.22)\\
&-\frac{2}{R_1^6}c_2z^3+\left(-\frac{1}{R_1^4}c_1-\frac{1}{R_1^2}B\right)z^2+\left(2A+\frac{1}{R_1^2}d_1\right)+\frac{2}{R_1^2}d_2z^{-1}\\
&+\left(-3c_1+\frac{3}{R_1^2}d_3\right)z^{-2}+\left(-8c_2+\frac{4}{R_1^2}d_4\right)z^{-3}+\left(-15c_3+\frac{5}{R_1^2}d_5\right)z^{-4}\\
&+\left(-24c_4+\frac{6}{R_1^2}d_6\right)z^{-5}+\left(-35c_5+\frac{7}{R_1^2}d_7\right)z^{-6}+\left(-48c_6+\frac{8}{R_1^2}d_8\right)z^{-7}\\
&-63c_7z^{-8}-80c_8z^{-9}\\
=\;&\left(-6q_{-6}\frac{1}{R_1^{14}}z^7\right)+\left(-5q_{-5}\frac{1}{R_1^{12}}z^6\right)+\left(4\overline{e}_4\frac{1}{R_1^2}-4q_{-4}\frac{1}{R_1^{10}}\right)z^5\\
&+\left(-3(-\overline{e}_3-2q_3)\frac{1}{R_1^2}-3q_{-3}\frac{1}{R_1^8}\right)z^4+\left(-2(-\overline{e}_2-q_2)\frac{1}{R_1^2}-2q_{-2}\frac{1}{R_1^6}\right)z^3\\
&+\left(\overline{e}_1\frac{1}{R_1^2}-q_{-1}\frac{1}{R_1^4}+3q_3-6q_3\right)z^2+(2q_2-2q_2)z\\
&+\left(q_1+q_1+(2q_{-1}-\overline{e}_{-1})\frac{1}{R_1^2}\right)+\left(2q_2R_1^2+2(3q_{-2}-\overline{e}_{-2})\frac{1}{R_1^2}\right)z^{-1}\\
&+\left(3q_3R_1^4-q_{-1}-2q_{-1}+3(4q_{-3}-\overline{e}_{-3})\frac{1}{R_1^2}\right)z^{-2}\\
&+\left(-2q_{-2}+4(5q_{-4}-\overline{e}_{-4})\frac{1}{R_1^2}-6q_{-2}\right)z^{-3}
\end{aligned}$$

$$+\left(-3q_{-3}+30q_{-5}\frac{1}{R_1^2}-12q_{-3}\right)z^{-4}+\left(-4q_{-4}+42q_{-6}\frac{1}{R_1^2}-20q_{-4}\right)z^{-5}$$
$$+\left(-5q_{-5}-30q_{-5}\right)z^{-6}+\left(-6q_{-6}-42q_{-6}\right)z^{-7}.$$

A4.4 Derivation of Equation (3.23)

The displacement jump condition along the Γ_1 (2.14) is

$$\begin{aligned}
&(m_1-n_1)(k_1+1)\frac{R_1}{z}\varphi_{12}(z)+(m_1+n_1)(k_1+1)\frac{z}{R_1}\overline{\varphi_{12}}\left(\frac{R_1^2}{z}\right) \qquad (2.14)\\
= \;& -4\mu_1\left(\varphi_2'(z)+\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)-z\varphi_2''(z)-\frac{z^2}{R_1^2}\psi_2'(z)\right)\\
&+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(R_1\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)+\frac{R_1}{z}\overline{\psi_2}\left(\frac{R_1^2}{z}\right)\right)\\
&+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)\frac{R_1}{z}\varphi_2(z)\\
&+(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)\frac{z}{R_1}\overline{\varphi_2}\left(\frac{R_1^2}{z}\right)\\
&+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(R_1\varphi_2'(z)+\frac{z}{R_1}\psi_2(z)\right).
\end{aligned}$$

Let's derive the each items required to be substituted:

$$\varphi_{12}(z)=q_3z^3+q_2z^2+q_1z^1+q_0+q_{-1}z^{-1}+q_{-2}z^{-2}+q_{-3}z^{-3}+q_{-4}z^{-4}+q_{-5}z^{-5}+q_{-6}z^{-6}, \qquad (A4.31)$$

$$\begin{aligned}
\frac{R_1}{z}\varphi_{12}(z) \;=\;& R_1q_3z^2+R_1q_2z+R_1q_1+R_1q_0z^{-1}+R_1q_{-1}z^{-2} \qquad (A4.32)\\
&+R_1q_{-2}z^{-3}+R_1q_{-3}z^{-4}+R_1q_{-4}z^{-5}+R_1q_{-5}z^{-6}+R_1q_{-6}z^{-7},
\end{aligned}$$

$$\frac{z}{R_1}\overline{\varphi_{12}}\left(\frac{R_1^2}{z}\right) = \frac{z}{R_1}\left(\begin{array}{c} q_3\left(R_1^2z^{-1}\right)^3 + q_2\left(R_1^2z^{-1}\right)^2 + q_1\left(R_1^2z^{-1}\right)^1 \\ +q_0 + q_{-1}\left(R_1^2z^{-1}\right)^{-1} + q_{-2}\left(R_1^2z^{-1}\right)^{-2} \\ +q_{-3}\left(R_1^2z^{-1}\right)^{-3} + q_{-4}\left(R_1^2z^{-1}\right)^{-4} \\ +q_{-5}\left(R_1^2z^{-1}\right)^{-5} + q_{-6}\left(R_1^2z^{-1}\right)^{-6} \end{array}\right) \tag{A4.33}$$

$$= q_3R_1^5z^{-2} + q_2R_1^3z^{-1} + q_1R_1 + \frac{1}{R_1}q_0z + q_{-1}\frac{1}{R_1^3}z^2 + q_{-2}\frac{1}{R_1^5}z^3 + q_{-3}\frac{1}{R_1^7}z^4 + q_{-4}\frac{1}{R_1^9}z^5 + q_{-5}\frac{1}{R_1^{11}}z^6 + q_{-6}\frac{1}{R_1^{13}}z^7.$$

The first bracket $\left(\varphi_2'(z) + \overline{\varphi_2'}\left(\frac{R_1^2}{z}\right) - z\varphi_2''(z) - \frac{z^2}{R_1^2}\psi_2'(z)\right)$ in the R.H.S. of (2.14) is same as L.H.S. of (2.13), namely (A4.29).

$$R_1\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right) = R_1\left(\begin{array}{c} A - \frac{1}{R_1^4}c_1z^2 - \frac{2}{R_1^6}c_2z^3 - \frac{3}{R_1^8}c_3z^4 - \frac{4}{R_1^{10}}c_4z^5 \\ -\frac{5}{R_1^{12}}c_5z^6 - \frac{6}{R_1^{14}}c_6z^7 - \frac{7}{R_1^{16}}c_7z^8 - \frac{8}{R_1^{18}}c_8z^9 \end{array}\right) \tag{A4.34}$$

$$= R_1A - \frac{1}{R_1^3}c_1z^2 - \frac{2}{R_1^5}c_2z^3 - \frac{3}{R_1^7}c_3z^4 - \frac{4}{R_1^9}c_4z^5 - \frac{5}{R_1^{11}}c_5z^6 - \frac{6}{R_1^{13}}c_6z^7 - \frac{7}{R_1^{15}}c_7z^8 - \frac{8}{R_1^{17}}c_8z^9,$$

$$\frac{R_1}{z}\overline{\psi_2}\left(\frac{R_1^2}{z}\right) = \frac{R_1}{z}\left(\begin{array}{c} B\left(\frac{R_1^2}{z}\right) + d_1\left(\frac{R_1^2}{z}\right)^{-1} + d_2\left(\frac{R_1^2}{z}\right)^{-2} \\ +d_3\left(\frac{R_1^2}{z}\right)^{-3} + d_4\left(\frac{R_1^2}{z}\right)^{-4} + d_5\left(\frac{R_1^2}{z}\right)^{-5} \end{array}\right) \tag{A4.35}$$

$$= BR_1^3z^{-2} + d_1\frac{1}{R_1} + d_2\frac{1}{R^3}z + d_3\frac{1}{R_1^5}z^2 + d_4\frac{1}{R_1^7}z^3 + d_5\frac{1}{R_1^9}z^4,$$

$$\frac{R_1}{z}\varphi_2(z) = AR_1 + R_1c_1z^{-2} + R_1c_2z^{-3} + R_1c_3z^{-4} + R_1c_4z^{-5}, \tag{A4.36}$$

$$\frac{z}{R_1}\overline{\varphi_2}\left(\frac{R_1^2}{z}\right) = AR_1 + c_1\frac{1}{R_1^3}z^2 + c_2\frac{1}{R_1^5}z^3 + c_3\frac{1}{R_1^7}z^4 + c_4\frac{1}{R_1^9}z^5, \tag{A4.37}$$

$$R_1\varphi_2'(z) = R_1\left(A - c_1z^{-2} - 2c_2z^{-3} - 3c_3z^{-4} - 4c_4z^{-5}\right) \tag{A4.38}$$

$$= AR_1 - R_1c_1z^{-2} - 2R_1c_2z^{-3} - 3R_1c_3z^{-4} - 4R_1c_4z^{-5},$$

$$\begin{aligned} \frac{z}{R_1}\psi_2(z) &= B\frac{z}{R_1}z + d_1\frac{z}{R_1}z^{-1} + d_2\frac{z}{R_1}z^{-2} \\ &\quad + d_3\frac{z}{R_1}z^{-3} + d_4\frac{z}{R_1}z^{-4} + d_5\frac{z}{R_1}z^{-5} \\ &= B\frac{1}{R_1}z^2 + d_1\frac{1}{R_1} + d_2\frac{1}{R_1}z^{-1} + d_3\frac{1}{R_1}z^{-2} + d_4\frac{1}{R_1}z^{-3} + d_5\frac{1}{R_1}z^{-4}. \end{aligned} \tag{A4.39}$$

By substituting (A4.31) and (A4.33) into (2.14), the L.H.S. of (2.14) takes the following form

$$\begin{aligned} &(m_1 - n_1)(k_1+1)\frac{R_1}{z}\varphi_1(z) + (m_1+n_1)(k_1+1)\frac{z}{R_1}\overline{\varphi_1}\left(\frac{R_1^2}{z}\right) \\ &= (m_1-n_1)(k_1+1)\begin{pmatrix} R_1q_3z^2 + R_1q_2z + R_1q_1 + R_1q_0z^{-1} \\ +R_1q_{-1}z^{-2} + R_1q_{-2}z^{-3} + R_1q_{-3}z^{-4} \\ +R_1q_{-4}z^{-5} + R_1q_{-5}z^{-6} + R_1q_{-6}z^{-7} \end{pmatrix} \\ &\quad + (m_1+n_1)(k_1+1)\begin{pmatrix} q_3R_1^5z^{-2} + q_2R_1^3z^{-1} + q_1R_1 + \frac{1}{R_1}q_0z \\ +q_{-1}\frac{1}{R_1^3}z^2 + q_{-2}\frac{1}{R_1^5}z^3 + q_{-3}\frac{1}{R_1^7}z^4 \\ +q_{-4}\frac{1}{R_1^9}z^5 + q_{-5}\frac{1}{R_1^{11}}z^6 + q_{-6}\frac{1}{R_1^{13}}z^7 \end{pmatrix} \\ &= \begin{pmatrix} (m_1-n_1)(k_1+1)R_1q_3z^2 + (m_1-n_1)(k_1+1)R_1q_2z \\ +(m_1-n_1)(k_1+1)R_1q_1 + (m_1-n_1)(k_1+1)R_1q_0z^{-1} \\ +(m_1-n_1)(k_1+1)R_1q_{-1}z^{-2} \\ +(m_1-n_1)(k_1+1)R_1q_{-2}z^{-3} + (m_1-n_1)(k_1+1)R_1q_{-3}z^{-4} \\ +(m_1-n_1)(k_1+1)R_1q_{-4}z^{-5} + (m_1-n_1)(k_1+1)R_1q_{-5}z^{-6} \\ +(m_1-n_1)(k_1+1)R_1q_{-6}z^{-7} \end{pmatrix} \end{aligned} \tag{A4.40}$$

$$
\begin{aligned}
&+\left(\begin{array}{c}
(m_1+n_1)(k_1+1)\,q_3R_1^5z^{-2}+(m_1+n_1)(k_1+1)\,q_2R_1^3z^{-1}\\
+(m_1+n_1)(k_1+1)\,q_1R_1+(m_1+n_1)(k_1+1)\,\tfrac{1}{R_1}q_0z\\
+(m_1+n_1)(k_1+1)\,q_{-1}\tfrac{1}{R_1^3}z^2+(m_1+n_1)(k_1+1)\,q_{-2}\tfrac{1}{R_1^5}z^3\\
+(m_1+n_1)(k_1+1)\,q_{-3}\tfrac{1}{R_1^7}z^4+(m_1+n_1)(k_1+1)\,q_{-4}\tfrac{1}{R_1^9}z^5\\
+(m_1+n_1)(k_1+1)\,q_{-5}\tfrac{1}{R_1^{11}}z^6+(m_1+n_1)(k_1+1)\,q_{-6}\tfrac{1}{R_1^{13}}z^7
\end{array}\right)\\
=\;&(m_1+n_1)(k_1+1)\,q_{-6}\frac{1}{R_1^{13}}z^7+(m_1+n_1)(k_1+1)\,q_{-5}\frac{1}{R_1^{11}}z^6\\
&+(m_1+n_1)(k_1+1)\,q_{-4}\frac{1}{R_1^9}z^5+(m_1+n_1)(k_1+1)\,q_{-3}\frac{1}{R_1^7}z^4\\
&+(m_1+n_1)(k_1+1)\,q_{-2}\frac{1}{R_1^5}z^3\\
&+\left((m_1-n_1)(k_1+1)\,R_1q_3+(m_1+n_1)(k_1+1)\,q_{-1}\frac{1}{R_1^3}\right)z^2\\
&+\left((m_1-n_1)(k_1+1)\,R_1q_2+(m_1+n_1)(k_1+1)\,\frac{1}{R_1}q_0\right)z\\
&+2m_1(k_1+1)\,R_1q_1\\
&+\left((m_1-n_1)(k_1+1)\,R_1q_0+(m_1+n_1)(k_1+1)\,R_1^3q_2\right)z^{-1}\\
&+\left[(m_1-n_1)(k_1+1)\,R_1q_{-1}+(m_1+n_1)(k_1+1)\,R_1^5q_3\right]z^{-2}\\
&+(m_1-n_1)(k_1+1)\,R_1q_{-2}z^{-3}+(m_1-n_1)(k_1+1)\,R_1q_{-3}z^{-4}\\
&+(m_1-n_1)(k_1+1)\,R_1q_{-4}z^{-5}+(m_1-n_1)(k_1+1)\,R_1q_{-5}z^{-6}\\
&+(m_1-n_1)(k_1+1)\,R_1q_{-6}z^{-7}.
\end{aligned}
$$

Also, the R.H.S. of (2.14) is re-written as

$$
\begin{aligned}
&-4\mu_1\left(\varphi_2'(z)+\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)-z\varphi_2''(z)-\frac{z^2}{R_1^2}\psi_2'(z)\right)\\
&+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(R_1\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)+\frac{R_1}{z}\overline{\psi_2}\left(\frac{R_1^2}{z}\right)\right)\\
&+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)\frac{R_1}{z}\varphi_2(z)+(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)\frac{z}{R_1}\overline{\varphi_2}\left(\frac{R_1^2}{z}\right)
\end{aligned}
\qquad \text{(A4.41)}
$$

$$+\left(m_1+n_1\right)\left(1-\frac{\mu_1}{\mu_2}\right)\left(R_1\varphi_2'(z)+\frac{z}{R_1}\psi_2(z)\right)$$

$$= \quad -4\mu_1\left(\begin{array}{c} -\frac{8}{R_1^{18}}c_8z^9-\frac{7}{R_1^{16}}c_7z^8-\frac{6}{R_1^{14}}c_6z^7-\frac{5}{R_1^{12}}c_5z^6-\frac{4}{R_1^{10}}c_4z^5 \\ -\frac{3}{R_1^{8}}c_3z^4-\frac{2}{R_1^{6}}c_2z^3+\left(-\frac{1}{R_1^4}c_1z^2-\frac{1}{R_1^2}Bz^2\right) \\ +\left(2A+\frac{1}{R_1^2}d_1\right)+\frac{2}{R_1^2}d_2z^{-1}+\left(-3c_1z^{-2}+\frac{3}{R_1^2}d_3z^{-2}\right) \\ +\left(-8c_2z^{-3}+\frac{4}{R_1^2}d_4z^{-3}\right)+\left(-15c_3z^{-4}+\frac{5}{R_1^2}d_5z^{-4}\right) \\ +\left(-24c_4z^{-5}+\frac{6}{R_1^2}d_6z^{-5}\right)+\left(-35c_5z^{-6}+\frac{7}{R_1^2}d_7z^{-6}\right) \\ +\left(-48c_6z^{-7}+\frac{8}{R_1^2}d_8z^{-7}\right)-63c_7z^{-8}-80c_8z^{-9} \end{array}\right)$$

$$+\left(m_1-n_1\right)\left(1-\frac{\mu_1}{\mu_2}\right)\left(\begin{array}{c} \left(\begin{array}{c} R_1A-\frac{1}{R_1^3}c_1z^2-\frac{2}{R_1^5}c_2z^3 \\ -\frac{3}{R_1^7}c_3z^4-\frac{4}{R_1^9}c_4z^5 \end{array}\right) \\ +\left(\begin{array}{c} BR_1^3z^{-2}+d_1\frac{1}{R_1}+d_2\frac{1}{R^3}z \\ +d_3\frac{1}{R_1^5}z^2+d_4\frac{1}{R_1^7}z^3+d_5\frac{1}{R_1^9}z^4 \end{array}\right) \end{array}\right)$$

$$+\left(m_1-n_1\right)\left(1+\frac{\mu_1}{\mu_2}k_2\right)\left(AR_1+R_1c_1z^{-2}+R_1c_2z^{-3}+R_1c_3z^{-4}+R_1c_4z^{-5}\right)$$

$$+\left(m_1+n_1\right)\left(1+\frac{\mu_1}{\mu_2}k_2\right)\left(AR_1+c_1\frac{1}{R_1^3}z^2+c_2\frac{1}{R_1^5}z^3+c_3\frac{1}{R_1^7}z^4+c_4\frac{1}{R_1^9}z^5\right)$$

$$+\left(m_1+n_1\right)\left(1-\frac{\mu_1}{\mu_2}\right)\left(\begin{array}{c} \left(\begin{array}{c} AR_1-R_1c_1z^{-2}-2R_1c_2z^{-3} \\ -3R_1c_3z^{-4}-4R_1c_4z^{-5} \end{array}\right) \\ +\left(\begin{array}{c} B\frac{1}{R_1}z^2+d_1\frac{1}{R_1}+d_2\frac{1}{R_1}z^{-1} \\ +d_3\frac{1}{R_1}z^{-2}+d_4\frac{1}{R_1}z^{-3}+d_5\frac{1}{R_1}z^{-4} \end{array}\right) \end{array}\right)$$

$$
\begin{aligned}
= & \left(\begin{array}{c}
4\mu_1 \frac{8}{R_1^{18}} c_8 z^9 + 4\mu_1 \frac{7}{R_1^{16}} c_7 z^8 + 4\mu_1 \frac{6}{R_1^{14}} c_6 z^7 + 4\mu_1 \frac{5}{R_1^{12}} c_5 z^6 + 4\mu_1 \frac{4}{R_1^{10}} c_4 z^5 \\
+4\mu_1 \frac{3}{R_1^8} c_3 z^4 + 4\mu_1 \frac{2}{R_1^6} c_2 z^3 + 4\mu_1 \left(\frac{1}{R_1^4} c_1 z^2 + \frac{1}{R_1^2} B z^2 \right) \\
-4\mu_1 \left(2A + \frac{1}{R_1^2} d_1 \right) - 4\mu_1 \frac{2}{R_1^2} d_2 z^{-1} - 4\mu_1 \left(-3c_1 z^{-2} + \frac{3}{R_1^2} d_3 z^{-2} \right) \\
-4\mu_1 \left(-8c_2 z^{-3} + \frac{4}{R_1^2} d_4 z^{-3} \right) - 4\mu_1 \left(-15c_3 z^{-4} + \frac{5}{R_1^2} d_5 z^{-4} \right) \\
-4\mu_1 \left(-24c_4 z^{-5} + \frac{6}{R_1^2} d_6 z^{-5} \right) - 4\mu_1 \left(-35c_5 z^{-6} + \frac{7}{R_1^2} d_7 z^{-6} \right) \\
-4\mu_1 \left(-48c_6 z^{-7} + \frac{8}{R_1^2} d_8 z^{-7} \right) + 4\mu_1 63 c_7 z^{-8} + 4\mu_1 80 c_8 z^{-9}
\end{array} \right) \\
& + \left(\begin{array}{c}
(m_1 - n_1)\left(1 - \frac{\mu_1}{\mu_2}\right) B R_1^3 z^{-2} + (m_1 - n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\left(R_1 A + d_1 \frac{1}{R_1}\right) \\
+(m_1 - n_1)\left(1 - \frac{\mu_1}{\mu_2}\right) d_2 \frac{1}{R^3} z + (m_1 - n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\left(d_3 \frac{1}{R_1^5} z^2 - \frac{1}{R_1^3} c_1 z^2\right) \\
+(m_1 - n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\left(d_4 \frac{1}{R_1^7} z^3 - \frac{2}{R_1^5} c_2 z^3\right) \\
+(m_1 - n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\left(d_5 \frac{1}{R_1^9} z^4 - \frac{3}{R_1^7} c_3 z^4\right) - (m_1 - n_1)\left(1 - \frac{\mu_1}{\mu_2}\right) \frac{4}{R_1^9} c_4 z^5
\end{array} \right) \\
& + \left(\begin{array}{c}
(m_1 - n_1)\left(1 + \frac{\mu_1}{\mu_2} k_2\right) A R_1 + (m_1 - n_1)\left(1 + \frac{\mu_1}{\mu_2} k_2\right) R_1 c_1 z^{-2} \\
+(m_1 - n_1)\left(1 + \frac{\mu_1}{\mu_2} k_2\right) R_1 c_2 z^{-3} + (m_1 - n_1)\left(1 + \frac{\mu_1}{\mu_2} k_2\right) R_1 c_3 z^{-4} \\
+(m_1 - n_1)\left(1 + \frac{\mu_1}{\mu_2} k_2\right) R_1 c_4 z^{-5}
\end{array} \right) \\
& + \left(\begin{array}{c}
(m_1 + n_1)\left(1 + \frac{\mu_1}{\mu_2} k_2\right) A R_1 + (m_1 + n_1)\left(1 + \frac{\mu_1}{\mu_2} k_2\right) c_1 \frac{1}{R_1^3} z^2 \\
+(m_1 + n_1)\left(1 + \frac{\mu_1}{\mu_2} k_2\right) c_2 \frac{1}{R_1^5} z^3 + (m_1 + n_1)\left(1 + \frac{\mu_1}{\mu_2} k_2\right) c_3 \frac{1}{R_1^7} z^4 \\
+(m_1 + n_1)\left(1 + \frac{\mu_1}{\mu_2} k_2\right) c_4 \frac{1}{R_1^9} z^5
\end{array} \right) \\
& + \left(\begin{array}{c}
(m_1 + n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\left(A R_1 + d_1 \frac{1}{R_1}\right) + (m_1 + n_1)\left(1 - \frac{\mu_1}{\mu_2}\right) d_2 \frac{1}{R_1} z^{-1} \\
+(m_1 + n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\left(d_3 \frac{1}{R_1} z^{-2} - R_1 c_1 z^{-2}\right) \\
+(m_1 + n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\left(d_4 \frac{1}{R_1} z^{-3} - 2R_1 c_2 z^{-3}\right) \\
+(m_1 + n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\left(d_5 \frac{1}{R_1} z^{-4} - 3R_1 c_3 z^{-4}\right) \\
-(m_1 + n_1)\left(1 - \frac{\mu_1}{\mu_2}\right) 4R_1 c_4 z^{-5} + (m_1 + n_1)\left(1 - \frac{\mu_1}{\mu_2}\right) B \frac{1}{R_1} z^2
\end{array} \right)
\end{aligned}
$$

$$
= \quad 4\mu_1 \frac{4}{R_1^{10}} c_4 z^5 - (m_1 - n_1)\left(1 - \frac{\mu_1}{\mu_2}\right) \frac{4}{R_1^9} c_4 z^5
$$

$$
\begin{aligned}
&+(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)c_4\frac{1}{R_1^9}z^5+4\mu_1\frac{3}{R_1^8}c_3z^4\\
&+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_5\frac{1}{R_1^9}z^4-\frac{3}{R_1^7}c_3z^4\right)+(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)c_3\frac{1}{R_1^7}z^4\\
&+4\mu_1\frac{2}{R_1^6}c_2z^3+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_4\frac{1}{R_1^7}z^3-\frac{2}{R_1^5}c_2z^3\right)\\
&+(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)c_2\frac{1}{R_1^5}z^3+4\mu_1\left(\frac{1}{R_1^4}c_1z^2+\frac{1}{R_1^2}Bz^2\right)\\
&+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_3\frac{1}{R_1^5}z^2-\frac{1}{R_1^3}c_1z^2\right)\\
&+(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)c_1\frac{1}{R_1^3}z^2+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)B\frac{1}{R_1}z^2\\
&+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)d_2\frac{1}{R^3}z-4\mu_1\left(2A+\frac{1}{R_1^2}d_1\right)\\
&+(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)AR_1+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(AR_1+d_1\frac{1}{R_1}\right)\\
&+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(AR_1+d_1\frac{1}{R_1}\right)+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)AR_1\\
&-4\mu_1\frac{2}{R_1^2}d_2z^{-1}+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)d_2\frac{1}{R_1}z^{-1}\\
&-4\mu_1\left(-3c_1z^{-2}+\frac{3}{R_1^2}d_3z^{-2}\right)+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)BR_1^3z^{-2}\\
&+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)R_1c_1z^{-2}+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_3\frac{1}{R_1}z^{-2}-R_1c_1z^{-2}\right)\\
&-4\mu_1\left(-8c_2z^{-3}+\frac{4}{R_1^2}d_4z^{-3}\right)+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)R_1c_2z^{-3}\\
&+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_4\frac{1}{R_1}z^{-3}-2R_1c_2z^{-3}\right)\\
&-4\mu_1\left(-15c_3z^{-4}+\frac{5}{R_1^2}d_5z^{-4}\right)+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)R_1c_3z^{-4}\\
&+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_5\frac{1}{R_1}z^{-4}-3R_1c_3z^{-4}\right)\\
&-4\mu_1\left(-24c_4z^{-5}+\frac{6}{R_1^2}d_6z^{-5}\right)+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)R_1c_4z^{-5}\\
&-(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)4R_1c_4z^{-5}.
\end{aligned}
$$

Thus, (2.14) is solely expressed in terms of the coefficients in the following form

$$
\begin{aligned}
&(m_1+n_1)(k_1+1)\,q_{-6}\frac{1}{R_1^{13}}z^7+(m_1+n_1)(k_1+1)\,q_{-5}\frac{1}{R_1^{11}}z^6 \qquad (3.23)\\
&+(m_1+n_1)(k_1+1)\,q_{-4}\frac{1}{R_1^{9}}z^5+(m_1+n_1)(k_1+1)\,q_{-3}\frac{1}{R_1^{7}}z^4\\
&+(m_1+n_1)(k_1+1)\,q_{-2}\frac{1}{R_1^{5}}z^3\\
&+\left((m_1-n_1)(k_1+1)\,R_1q_3+(m_1+n_1)(k_1+1)\,q_{-1}\frac{1}{R_1^{3}}\right)z^2\\
&+\left((m_1-n_1)(k_1+1)\,R_1q_2+(m_1+n_1)(k_1+1)\,\frac{1}{R_1}q_0\right)z+2m_1(k_1+1)\,R_1q_1\\
&+\left((m_1-n_1)(k_1+1)\,R_1q_0+(m_1+n_1)(k_1+1)\,R_1^3q_2\right)z^{-1}\\
&+\left((m_1-n_1)(k_1+1)\,R_1q_{-1}+(m_1+n_1)(k_1+1)\,R_1^5q_3\right)z^{-2}\\
&+(m_1-n_1)(k_1+1)\,R_1q_{-2}z^{-3}+(m_1-n_1)(k_1+1)\,R_1q_{-3}z^{-4}\\
&+(m_1-n_1)(k_1+1)\,R_1q_{-4}z^{-5}+(m_1-n_1)(k_1+1)\,R_1q_{-5}z^{-6}\\
&+(m_1-n_1)(k_1+1)\,R_1q_{-6}z^{-7}\\
=\;&\left(4\mu_1\frac{4}{R_1^{10}}c_4-(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\frac{4}{R_1^{9}}c_4+(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)c_4\frac{1}{R_1^{9}}\right)z^5\\
&+\begin{pmatrix} 4\mu_1\frac{3}{R_1^8}c_3+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_5\frac{1}{R_1^9}-\frac{3}{R_1^7}c_3\right)\\ +(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)c_3\frac{1}{R_1^7}\end{pmatrix}z^4\\
&+\begin{pmatrix} 4\mu_1\frac{2}{R_1^6}c_2+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_4\frac{1}{R_1^7}-\frac{2}{R_1^5}c_2\right)\\ +(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)c_2\frac{1}{R_1^5}\end{pmatrix}z^3\\
&+\begin{pmatrix} 4\mu_1\left(\frac{1}{R_1^4}c_1+\frac{1}{R_1^2}B\right)+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_3\frac{1}{R_1^5}-\frac{1}{R_1^3}c_1\right)\\ +(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)c_1\frac{1}{R_1^3}\;+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)B\frac{1}{R_1}\end{pmatrix}z^2\\
&+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)d_2\frac{1}{R^3}z\\
&-4\mu_1\left(2A+\frac{1}{R_1^2}d_1\right)+(m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)AR_1
\end{aligned}
$$

$$
\begin{aligned}
&+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(AR_1+d_1\frac{1}{R_1}\right)\\
&+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(AR_1+d_1\frac{1}{R_1}\right)+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)AR_1\\
&+\left(-4\mu_1\frac{2}{R_1^2}d_2+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)d_2\frac{1}{R_1}\right)z^{-1}\\
&+\left(\begin{array}{c}-4\mu_1\left(-3c_1+\frac{3}{R_1^2}d_3\right)+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)BR_1^3\\
+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)R_1c_1\\
+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_3\frac{1}{R_1}-R_1c_1\right)\end{array}\right)z^{-2}\\
&+\left(\begin{array}{c}-4\mu_1\left(-8c_2+\frac{4}{R_1^2}d_4\right)+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)R_1c_2\\
+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_4\frac{1}{R_1}-2R_1c_2\right)\end{array}\right)z^{-3}\\
&+\left(\begin{array}{c}-4\mu_1\left(-15c_3+\frac{5}{R_1^2}d_5\right)+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)R_1c_3\\
+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left(d_5\frac{1}{R_1}-3R_1c_3\right)\end{array}\right)z^{-4}\\
&+\left(\begin{array}{c}-4\mu_1\left(-24c_4+\frac{6}{R_1^2}d_6\right)+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)R_1c_4\\
-(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)4R_1c_4\end{array}\right)z^{-5}.
\end{aligned}
$$

APPENDIX 5
38 Algebraic Equations

(3.3) - (3.21) and (3.24) - (3.42) can be re-written as follows:

Traction continuation along the Γ_0

1) z^5

$$24a_6 + \frac{4}{R_0^2}b_4 + \frac{4}{R_0^2}e_4 - 24p_6 + \frac{12}{R_0^2}p_4 - \frac{4}{R_0^{10}}p_{-4} = 0,$$

2) z^4

$$15a_5 + \frac{3}{R_0^2}b_3 + \frac{3}{R_0^2}e_3 - 15p_5 + \frac{6}{R_0^2}p_3 - \frac{3}{R_0^8}p_{-3} = 0,$$

3) z^3

$$8a_4 + \frac{2}{R_0^2}b_2 + \frac{2}{R_0^2}e_2 - 8p_4 + \frac{2}{R_0^2}p_2 - \frac{2}{R_0^6}p_{-2} = 0,$$

4) z^2

$$3a_3 + \frac{1}{R_0^2}b_1 + \frac{1}{R_0^2}e_1 - 3p_3 - \frac{1}{R_0^4}p_{-1} = 0,$$

5) z^0

$$-2a_1 - \frac{1}{R_0^2}e_{-1} + 2p_1 + \frac{2}{R_0^2}p_{-1} = 0,$$

6) z^{-1}

$$-2R_0^2a_2 - \frac{2}{R_0^2}e_{-2} + 2R_0^2p_2 + \frac{6}{R_0^2}p_{-2} = 0,$$

7) z^{-2}

$$-3R_0^4a_3 - \frac{3}{R_0^2}e_{-3} + 3R_0^4p_3 - 3p_{-1} + \frac{12}{R_0^2}p_{-3} = 0,$$

8)z^{-3}

$$-4R_0^6a_4 - \frac{4}{R_0^2}e_{-4} + 4R_0^6p_4 - 8p_{-2} + \frac{20}{R_0^2}p_{-4} = 0,$$

9) z^{-4}

$$-5R_0^8a_5 + 5R_0^8p_5 - 15p_{-3} = 0,$$

Displacement jump along the Γ_0

10) z^5

$$\begin{aligned} 0 &= (m_0 - n_0)(\kappa_1 + 1)R_0p_6 + (m_0 + n_0)(\kappa_1 + 1)\frac{1}{R_0^9}p_{-4} \\ &+ \left(96\mu_1 - (m_0 - n_0)(1 + \kappa_0\frac{\mu_1}{\mu_0})R_0 - 6(m_0 + n_0)(1 - \frac{\mu_1}{\mu_0})R_0\right) a_6 \\ &+ \left(-(m_0 + n_0)(1 - \frac{\mu_1}{\mu_0})\frac{1}{R_0} + \mu_1\frac{16}{R_0^2}\right) b_4, \end{aligned}$$

11) z^4

$$\begin{aligned} 0 &= (m_0 - n_0)(\kappa_1 + 1)R_0p_5 + (m_0 + n_0)(\kappa_1 + 1)\frac{1}{R_0^7}p_{-3} \\ &+ \left(60\mu_1 - (m_0 - n_0)(1 + \kappa_0\frac{\mu_1}{\mu_0})R_0 - 5(m_0 + n_0)(1 - \frac{\mu_1}{\mu_0})R_0\right) a_5 \\ &+ \left(-(m_0 + n_0)(1 - \frac{\mu_1}{\mu_0})\frac{1}{R_0} + \mu_1\frac{12}{R_0^2}\right) b_3, \end{aligned}$$

12) z^3

$$
\begin{aligned}
0 \;=\; & (m_0-n_0)(\kappa_1+1)R_0p_4+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0^5}p_{-2}\\
&+\left(32\mu_1-(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0-4(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0\right)a_4\\
&+\left(-(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}+\mu_1\frac{8}{R_0^2}\right)b_2,
\end{aligned}
$$

13) z^2

$$
\begin{aligned}
&\frac{2\mu_1(m_0+n_0)(\varepsilon_2-i\varepsilon_3)}{R_0}\\
=\; & (m_0-n_0)(\kappa_1+1)R_0p_3+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0^3}p_{-1}\\
&+\left(12\mu_1-(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0-3(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0\right)a_3\\
&+\left(-(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}+\mu_1\frac{4}{R_0^2}\right)b_1,
\end{aligned}
$$

14) z

$$
\begin{aligned}
0 \;=\; & (m_0-n_0)(\kappa_1+1)R_0p_2+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0}p_0\\
&+\left(-(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0-2(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0\right)a_2\\
&-(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})\frac{1}{R_0}a_0-(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_0,
\end{aligned}
$$

15) z^0

$$
\begin{aligned}
&4m_0R_0\mu_1\varepsilon_1\\
=\; & 2m_0(\kappa_1+1)R_0p_1\\
&+\begin{pmatrix} -8\mu_1-(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0-(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0\\ -(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0-(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0 \end{pmatrix}a_1,
\end{aligned}
$$

16) z^{-1}

$$
\begin{aligned}
0 \;=\;& (m_0+n_0)(\kappa_1+1)R_0^3p_2+(m_0-n_0)(\kappa_1+1)R_0p_0 \\
&+\left(-8\mu_1R_0^2-2(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^3-(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^3\right)a_2 \\
&-(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_0-(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0b_0,
\end{aligned}
$$

17) z^{-2}

$$
\begin{aligned}
& 2\mu_1(m_0-n_0)(\varepsilon_2+i\varepsilon_3)R_0^3 \\
=\;& (m_0+n_0)(\kappa_1+1)R_0^5p_3+(m_0-n_0)(\kappa_1+1)R_0p_{-1} \\
&+\left(-12\mu_1R_0^4-3(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^5-(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^5\right)a_3 \\
&-(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^3b_1,
\end{aligned}
$$

18) z^{-3}

$$
\begin{aligned}
0 \;=\;& (m_0+n_0)(\kappa_1+1)R_0^7p_4+(m_0-n_0)(\kappa_1+1)R_0p_{-2} \\
&+\left(-16\mu_1R_0^6-4(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^7-(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^7\right)a_4 \\
&-(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^5b_2,
\end{aligned}
$$

19) z^{-4}

$$
\begin{aligned}
0 \;=\;& (m_0+n_0)(\kappa_1+1)R_0^9p_5+(m_0-n_0)(\kappa_1+1)R_0p_{-3} \\
&+\left(-20\mu_1R_0^8-5(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^9-(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^9\right)a_5 \\
&-(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^7b_3,
\end{aligned}
$$

Traction continuation along the Γ_1

20) z^5

$$0 = \frac{4}{R_1^{10}}c_4 + 4\frac{1}{R_1^2}e_4 - 4\frac{1}{R_1^{10}}q_{-4},$$

21) z^4

$$0 = \frac{3}{R_1^8}c_3 + 3\frac{1}{R_1^2}e_3 + 6\frac{1}{R_1^2}q_3 - 3\frac{1}{R_1^8}q_{-3},$$

22) z^3

$$0 = \frac{2}{R_1^6}c_2 + 2\frac{1}{R_1^2}e_2 + 2\frac{1}{R_1^2}q_2 - 2\frac{1}{R_1^6}q_{-2},$$

23) z^2

$$-\frac{1}{R_1^2}B = \frac{1}{R_1^4}c_1 + \frac{1}{R_1^2}e_1 - 3q_3 - \frac{1}{R_1^4}q_{-1},$$

24) z^0

$$2A = -\frac{1}{R_1^2}d_1 - \frac{1}{R_1^2}e_{-1} + 2q_1 + 2\frac{1}{R_1^2}q_{-1},$$

25) z^{-1}

$$0 = -\frac{2}{R_1^2}d_2 - 2\frac{1}{R_1^2}e_{-2} + 2R_1^2q_2 + 6\frac{1}{R_1^2}q_{-2},$$

26) z^{-2}

$$0 = 3c_1 - \frac{3}{R_1^2}d_3 - 3\frac{1}{R_1^2}e_{-3} + 3R_1^4q_3 - 3q_{-1} + 12\frac{1}{R_1^2}q_{-3},$$

27) z^{-3}

$$0 = 8c_2 - \frac{4}{R_1^2}d_4 - 4\frac{1}{R_1^2}e_{-4} - 8q_{-2} + 20\frac{1}{R_1^2}q_{-4},$$

28) z^{-4}

$$0 = 15c_3 - \frac{5}{R_1^2}d_5 - 15q_{-3} + 30\frac{1}{R_1^2}q_{-5},$$

Displacement jump along the Γ_1

29) z^5

$$\begin{aligned} 0 &= \left(\frac{16}{R_1^{10}}\mu_1 - (m_1 - n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\frac{4}{R_1^9} + (m_1 + n_1)\left(1 + \frac{\mu_1}{\mu_2}k_2\right)\frac{1}{R_1^9}\right)c_4 \\ &\quad - (m_1 + n_1)(k_1 + 1)\frac{1}{R_1^9}q_{-4}, \end{aligned}$$

30) z^4

$$\begin{aligned} 0 &= \left(\frac{12}{R_1^8}\mu_1 - (m_1 - n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\frac{3}{R_1^7} + (m_1 + n_1)\left(1 + \frac{\mu_1}{\mu_2}k_2\right)\frac{1}{R_1^7}\right)c_3 \\ &\quad + (m_1 - n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^9}d_5 - (m_1 + n_1)(k_1 + 1)\frac{1}{R_1^7}q_{-3}, \end{aligned}$$

31) z^3

$$\begin{aligned} 0 &= \left(\frac{8}{R_1^6}\mu_1 - (m_1 - n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\frac{2}{R_1^5} + (m_1 + n_1)\left(1 + \frac{\mu_1}{\mu_2}k_2\right)\frac{1}{R_1^5}\right)c_2 \\ &\quad + (m_1 - n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^7}d_4 - (m_1 + n_1)(k_1 + 1)\frac{1}{R_1^5}q_{-2}, \end{aligned}$$

32) z^2

$$\left(-(m_1 + n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\frac{1}{R_1} - 4\mu_1\frac{1}{R_1^2}\right)B$$

$$= \left(\frac{4}{R_1^4}\mu_1 - (m_1 - n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^3} + (m_1 + n_1)\left(1 + \frac{\mu_1}{\mu_2}k_2\right)\frac{1}{R_1^3}\right)c_1$$
$$+ (m_1 - n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^5}d_3 - (m_1 - n_1)(k_1 + 1)R_1 q_3 - (m_1 + n_1)(k_1 + 1)q_{-1}\frac{1}{R_1^3},$$

33) z^1

$$0 = (m_1 - n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\frac{1}{R^3}d_2 - (m_1 - n_1)(k_1 + 1)R_1 q_2 - (m_1 + n_1)(k_1 + 1)\frac{1}{R_1}q_0,$$

34) z^0

$$\begin{aligned} & -\left(-8\mu_1 A + 2m_1\left(1 + \frac{\mu_1}{\mu_2}k_2\right)AR_1 + 2m_1\left(1 - \frac{\mu_1}{\mu_2}\right)AR_1\right) \\ = & \left(-\frac{4}{R_1^2}\mu_1 + (m_1 + n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\frac{1}{R_1} + (m_1 - n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\frac{1}{R_1}\right)d_1 \\ & -2m_1(k_1 + 1)R_1 q_1, \end{aligned}$$

35) z^{-1}

$$\begin{aligned} 0 = & \left(-\frac{8}{R_1^2}\mu_1 + (m_1 + n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\frac{1}{R_1}\right)d_2 \\ & -(m_1 + n_1)(k_1 + 1)R_1^3 q_2 - (m_1 - n_1)(k_1 + 1)R_1 q_0, \end{aligned}$$

36) z^{-2}

$$\begin{aligned} & -(m_1 - n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)BR_1^3 \\ = & \left(12\mu_1 + (m_1 - n_1)\left(1 + \frac{\mu_1}{\mu_2}k_2\right)R_1 - (m_1 + n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)R_1\right)c_1 \\ & + \left(-\frac{12}{R_1^2}\mu_1 + (m_1 + n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\frac{1}{R_1}\right)d_3 \\ & -(m_1 + n_1)(k_1 + 1)R_1^5 q_3 - (m_1 - n_1)(k_1 + 1)R_1 q_{-1}, \end{aligned}$$

37) z^{-3}

$$
\begin{aligned}
0 \;=\;& \left(+32\mu_1 + (m_1 - n_1)\left(1 + \frac{\mu_1}{\mu_2}k_2\right)R_1 - 2R_1(m_1 + n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\right)c_2 \\
&+\left(-\frac{16}{R_1^2}\mu_1 + (m_1 + n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\frac{1}{R_1}\right)d_4 - (m_1 - n_1)(k_1 + 1)R_1 q_{-2},
\end{aligned}
$$

38) z^{-4}

$$
\begin{aligned}
0 \;=\;& \left(+60\mu_1 + (m_1 - n_1)\left(1 + \frac{\mu_1}{\mu_2}k_2\right)R_1 - 3R_1(m_1 + n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\right)c_3 \\
&+\left(-\frac{20}{R_1^2}\mu_1 + (m_1 + n_1)\left(1 - \frac{\mu_1}{\mu_2}\right)\frac{1}{R_1}\right)d_5 - (m_1 - n_1)(k_1 + 1)R_1 q_{-3}.
\end{aligned}
$$

Now p_k and q_k are substituted into the above equations.

$p_{-4..6}$ are :

$$
\begin{aligned}
p_6 &= (g_7 f_{-1} + g_6 f_0 + g_5 f_1 + g_4 f_2 + g_3 f_3 + g_2 f_4), \\
p_5 &= (g_7 f_{-2} + g_6 f_{-1} + g_5 f_0 + g_4 f_1 + g_3 f_2 + g_2 f_3 + g_1 f_4), \\
p_4 &= \left(g_7 f_{-3} + g_6 f_{-2} + g_5 f_{-1} + g_4 f_0 + g_3 f_1 + g_2 f_2 + g_1 f_3 + g_0 f_4 + \frac{1}{2}e_4\right), \\
p_3 &= \left(g_6 f_{-3} + g_5 f_{-2} + g_4 f_{-1} + g_3 f_0 + g_2 f_1 + g_1 f_2 + g_0 f_3 + \frac{1}{2}e_3\right), \\
p_2 &= \left(g_5 f_{-3} + g_4 f_{-2} + g_3 f_{-1} + g_2 f_0 + g_1 f_1 + g_0 f_2 + \frac{1}{2}e_2\right), \\
p_1 &= \left(g_4 f_{-3} + g_3 f_{-2} + g_2 f_{-1} + g_1 f_0 + g_0 f_1 + \frac{1}{2}e_1\right), \\
p_0 &= \left(g_3 f_{-3} + g_2 f_{-2} + g_1 f_{-1} + g_0 f_0 + \frac{1}{2}e_0\right), \\
p_{-1} &= \left(g_2 f_{-3} + g_1 f_{-2} + g_0 f_{-1} + \frac{1}{2}e_{-1}\right), \\
p_{-2} &= \left(g_1 f_{-3} + g_0 f_{-2} + \frac{1}{2}e_{-2}\right), \\
p_{-3} &= \left(g_0 f_{-3} + \frac{1}{2}e_{-3}\right),
\end{aligned}
$$

$$p_{-4} = \left(\frac{1}{2}e_{-4}\right).$$

$q_{-6..3}$ are :

$$\begin{aligned}
q_3 &= \left(h_{-1}f_4 + \frac{1}{2}e_3\right), \\
q_2 &= \left(h_{-2}f_4 + h_{-1}f_3 + \frac{1}{2}e_2\right), \\
q_1 &= \left(h_{-3}f_4 + h_{-2}f_3 + h_{-1}f_2 + \frac{1}{2}e_1\right), \\
q_0 &= \left(h_{-4}f_4 + h_{-3}f_3 + h_{-2}f_2 + h_{-1}f_1 + \frac{1}{2}e_0\right), \\
q_{-1} &= \left(h_{-5}f_4 + h_{-4}f_3 + h_{-3}f_2 + h_{-2}f_1 + h_{-1}f_0 + \frac{1}{2}e_{-1}\right), \\
q_{-2} &= \left(h_{-6}f_4 + h_{-5}f_3 + h_{-4}f_2 + h_{-3}f_1 + h_{-2}f_0 + h_{-1}f_{-1} + \frac{1}{2}e_{-2}\right), \\
q_{-3} &= \left(h_{-7}f_4 + h_{-6}f_3 + h_{-5}f_2 + h_{-4}f_1 + h_{-3}f_0 + h_{-2}f_{-1} + h_{-1}f_{-2} + \frac{1}{2}e_{-3}\right), \\
q_{-4} &= \left(\begin{array}{c} h_{-8}f_4 + h_{-7}f_3 + h_{-6}f_2 + h_{-5}f_1 + h_{-4}f_0 + h_{-3}f_{-1} + h_{-2}f_{-2} \\ +h_{-1}f_{-3} + \frac{1}{2}e_{-4} \end{array}\right), \\
q_{-5} &= \left(\begin{array}{c} h_{-8}f_3 + h_{-7}f_2 + h_{-6}f_1 + h_{-5}f_0 + h_{-4}f_{-1} + h_{-3}f_{-2} + h_{-2}f_{-3} \\ +h_{-1}f_{-4} + \frac{1}{2}e_{-5} \end{array}\right), \\
q_{-6} &= \left(h_{-8}f_2 + h_{-7}f_1 + h_{-6}f_0 + h_{-5}f_{-1} + h_{-4}f_{-2} + h_{-3}f_{-3} + h_{-2}f_{-4} + \frac{1}{2}e_{-6}\right).
\end{aligned}$$

Consequently, the first 9 equations from the traction continuation condition along Γ_0 , namely, (3.43) thru (3.51) are;

1) z^5

$$0 = 24a_6 + \frac{4}{R_0^2}b_4 - \frac{4}{R_0^{10}}\frac{1}{2}e_{-4} + \frac{10}{R_0^2}e_4 + \frac{12}{R_0^2}g_7f_{-3} + \frac{12}{R_0^2}g_6f_{-2}$$

$$
\begin{aligned}
&+\left(\frac{12}{R_0^2}g_5-24g_7\right)f_{-1}+\left(\frac{12}{R_0^2}g_4-24g_6\right)f_0+\left(\frac{12}{R_0^2}g_3-24g_5\right)f_1\\
&+\left(\frac{12}{R_0^2}g_2-24g_4\right)f_2+\left(\frac{12}{R_0^2}g_1-24g_3\right)f_3+\left(\frac{12}{R_0^2}g_0-24g_2\right)f_4,
\end{aligned}
$$

2) z^4

$$
\begin{aligned}
0 \;=\;& 15a_5+\frac{3}{R_0^2}b_3-\frac{3}{R_0^8}\frac{1}{2}e_{-3}+\frac{6}{R_0^2}e_3+\left(\frac{6}{R_0^2}g_6-\frac{3}{R_0^8}g_0\right)f_{-3}\\
&+\left(\frac{6}{R_0^2}g_5-15g_7\right)f_{-2}+\left(\frac{6}{R_0^2}g_4-15g_6\right)f_{-1}+\left(\frac{6}{R_0^2}g_3-15g_5\right)f_0\\
&+\left(\frac{6}{R_0^2}g_2-15g_4\right)f_1+\left(\frac{6}{R_0^2}g_1-15g_3\right)f_2+\left(\frac{6}{R_0^2}g_0-15g_2\right)f_3-15g_1f_4,
\end{aligned}
$$

3) z^3

$$
\begin{aligned}
0 \;=\;& 8a_4+\frac{2}{R_0^2}b_2-\frac{1}{R_0^6}e_{-2}+\frac{3}{R_0^2}e_2-4e_4+\left(\frac{2}{R_0^2}g_5-8g_7-\frac{2}{R_0^6}g_1\right)f_{-3}\\
&+\left(\frac{2}{R_0^2}g_4-8g_6-\frac{2}{R_0^6}g_0\right)f_{-2}+\left(\frac{2}{R_0^2}g_3-8g_5\right)f_{-1}+\left(\frac{2}{R_0^2}g_2-8g_4\right)f_0\\
&+\left(\frac{2}{R_0^2}g_1-8g_3\right)f_1+\left(\frac{2}{R_0^2}g_0-8g_2\right)f_2-8g_1f_3-8g_0f_4,
\end{aligned}
$$

4) z^2

$$
\begin{aligned}
0 \;=\;& 3a_3+\frac{1}{R_0^2}b_1-\frac{1}{2}\frac{1}{R_0^4}e_{-1}+\frac{1}{R_0^2}e_1-3\frac{1}{2}e_3+\left(-3g_6-\frac{1}{R_0^4}g_2\right)f_{-3}\\
&+\left(-3g_5-\frac{1}{R_0^4}g_1\right)f_{-2}+\left(-3g_4-\frac{1}{R_0^4}g_0\right)f_{-1}\\
&-3g_3f_0-3g_2f_1-3g_1f_2-3g_0f_3,
\end{aligned}
$$

5) z^0

$$
0 \;=\; -2a_1+e_1+\left(2g_4+\frac{2}{R_0^2}g_2\right)f_{-3}
$$

$$+\left(2g_3+\frac{2}{R_0^2}g_1\right)f_{-2}+\left(2g_2+\frac{2}{R_0^2}g_0\right)f_{-1}+2g_1f_0+2g_0f_1,$$

6) z^{-1}

$$\begin{aligned}0 &= -2R_0^2a_2+\frac{1}{R_0^2}e_{-2}+R_0^2e_2+\left(2R_0^2g_5+\frac{6}{R_0^2}g_1\right)f_{-3}\\ &\quad+\left(2R_0^2g_4+\frac{6}{R_0^2}g_0\right)f_{-2}+2R_0^2g_3f_{-1}+2R_0^2g_2f_0+2R_0^2g_1f_1+2R_0^2g_0f_2,\end{aligned}$$

7) z^{-2}

$$\begin{aligned}0 &= -3R_0^4a_3+\frac{3}{R_0^2}e_{-3}-\frac{3}{2}e_{-1}+3R_0^4\frac{1}{2}e_3\\ &\quad+\left(3R_0^4g_6-3g_2+\frac{12}{R_0^2}g_0\right)f_{-3}+\left(3R_0^4g_5-3g_1\right)f_{-2}\\ &\quad+\left(3R_0^4g_4-3g_0\right)f_{-1}+3R_0^4g_3f_0+3R_0^4g_2f_1+3R_0^4g_1f_2+3R_0^4g_0f_3,\end{aligned}$$

8)z^{-3}

$$\begin{aligned}0 &= -4R_0^6a_4+\frac{6}{R_0^2}e_{-4}-4e_{-2}+2R_0^6e_4\\ &\quad+\left(4R_0^6g_7-8g_1\right)f_{-3}+\left(4R_0^6g_6-8g_0\right)f_{-2}+4R_0^6g_5f_{-1}\\ &\quad+4R_0^6g_4f_0+4R_0^6g_3f_1+4R_0^6g_2f_2+4R_0^6g_1f_3+4R_0^6g_0f_4,\end{aligned}$$

9) z^{-4}

$$\begin{aligned}0 &= -5R_0^8a_5-15\frac{1}{2}e_{-3}-15g_0f_{-3}+5R_0^8g_7f_{-2}+5R_0^8g_6f_{-1}\\ &\quad+5R_0^8g_5f_0+5R_0^8g_4f_1+5R_0^8g_3f_2+5R_0^8g_2f_3+5R_0^8g_1f_4\ .\end{aligned}$$

In addition, non-dimentional parameter $M_0=\frac{m_0+n_0}{2\mu_1}R_0$ is deployed into displacement jump equations along the Γ_0. Thus, another 10 equations, such as (3.52) thru

(3.61) are;

10) z^5

$$\begin{aligned}
0 &= \left(96\mu_1 - (m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0 - 6(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0\right)a_6 \\
&+\left(-(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}+\mu_1\frac{16}{R_0^2}\right)b_4 \\
&+(m_0+n_0)(\kappa_1+1)\frac{1}{2R_0^9}e_{-4} \\
&+(m_0-n_0)(\kappa_1+1)R_0g_7f_{-1}+(m_0-n_0)(\kappa_1+1)R_0g_6f_0 \\
&+(m_0-n_0)(\kappa_1+1)R_0g_5f_1+(m_0-n_0)(\kappa_1+1)R_0g_4f_2 \\
&+(m_0-n_0)(\kappa_1+1)R_0g_3f_3+(m_0-n_0)(\kappa_1+1)R_0g_2f_4,
\end{aligned}$$

$$\begin{aligned}
0 &= \left(48\frac{1}{M_0}-\frac{(m_0-n_0)}{(m_0+n_0)}(1+\kappa_0\frac{\mu_1}{\mu_0})-6(1-\frac{\mu_1}{\mu_0})\right)a_6 \\
&+\left(-(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0^2}+\frac{8}{M_0}\frac{1}{R_0^2}\right)b_4+(\kappa_1+1)\frac{1}{2R_0^{10}}e_{-4} \\
&+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_7f_{-1}+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_6f_0 \\
&+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_5f_1+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_4f_2 \\
&+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_3f_3+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_2f_4,
\end{aligned}$$

11) z^4

$$\begin{aligned}
0 &= \left(60\mu_1 - (m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0 - 5(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0\right)a_5 \\
&+\left(-(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}+\mu_1\frac{12}{R_0^2}\right)b_3 \\
&+(m_0+n_0)(\kappa_1+1)\frac{1}{2R_0^7}e_{-3}+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0^7}g_0f_{-3} \\
&+(m_0-n_0)(\kappa_1+1)R_0g_7f_{-2}+(m_0-n_0)(\kappa_1+1)R_0g_6f_{-1}
\end{aligned}$$

$$+(m_0-n_0)(\kappa_1+1)R_0g_5f_0+(m_0-n_0)(\kappa_1+1)R_0g_4f_1$$
$$+(m_0-n_0)(\kappa_1+1)R_0g_3f_2+(m_0-n_0)(\kappa_1+1)R_0g_2f_3+(m_0-n_0)(\kappa_1+1)R_0g_1f_4,$$

$$\begin{aligned}0 \;=\; & \left(30\frac{1}{M_0}-\frac{(m_0-n_0)}{(m_0+n_0)}(1+\kappa_0\frac{\mu_1}{\mu_0})-5(1-\frac{\mu_1}{\mu_0})\right)a_5 \\ & +\left(-(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0^2}+\frac{1}{M_0}\frac{6}{R_0^2}\right)b_3+(\kappa_1+1)\frac{1}{2R_0^8}e_{-3} \\ & +(\kappa_1+1)\frac{1}{R_0^8}g_0f_{-3}+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_7f_{-2}+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_6f_{-1} \\ & +\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_5f_0+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_4f_1+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_3f_2 \\ & +\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_2f_3+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_1f_4,\end{aligned}$$

12) z^3

$$\begin{aligned}0 \;=\; & \left(32\mu_1-(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0-4(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0\right)a_4 \\ & +\left(-(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}+\mu_1\frac{8}{R_0^2}\right)b_2 \\ & +(m_0+n_0)(\kappa_1+1)\frac{1}{R_0^5}\frac{1}{2}e_{-2}+(m_0-n_0)(\kappa_1+1)R_0\frac{1}{2}e_4 \\ & +\left((m_0-n_0)(\kappa_1+1)R_0g_7f_{-3}+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0^5}g_1f_{-3}\right) \\ & +\left((m_0-n_0)(\kappa_1+1)R_0g_6f_{-2}+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0^5}g_0f_{-2}\right) \\ & +(m_0-n_0)(\kappa_1+1)R_0g_5f_{-1}+(m_0-n_0)(\kappa_1+1)R_0g_4f_0 \\ & +(m_0-n_0)(\kappa_1+1)R_0g_3f_1+(m_0-n_0)(\kappa_1+1)R_0g_2f_2 \\ & +(m_0-n_0)(\kappa_1+1)R_0g_1f_3+(m_0-n_0)(\kappa_1+1)R_0g_0f_4,\end{aligned}$$

$$0 \;=\; \left(16\frac{1}{M_0}-\frac{(m_0-n_0)}{(m_0+n_0)}(1+\kappa_0\frac{\mu_1}{\mu_0})-4(1-\frac{\mu_1}{\mu_0})\right)a_4$$

$$+\left(-(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0^2}+\frac{1}{M_0}\frac{4}{R_0^2}\right)b_2+(\kappa_1+1)\frac{1}{R_0^6}\frac{1}{2}e_{-2}+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)\frac{1}{2}e_4$$
$$+\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_7+(\kappa_1+1)\frac{1}{R_0^6}g_1\right)f_{-3}$$
$$+\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_6+(\kappa_1+1)\frac{1}{R_0^6}g_0\right)f_{-2}+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_5f_{-1}$$
$$+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_4f_0+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_3f_1$$
$$+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_2f_2+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_1f_3+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_0f_4,$$

13) z^2 $\left[\frac{2\mu_1(m_0+n_0)(\varepsilon_2-i\varepsilon_3)}{R_0}=0\right]$

$$\begin{aligned}0 \;=\; & \left(12\mu_1-(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0-3(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0\right)a_3\\
&+\left(-(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}+\mu_1\frac{4}{R_0^2}\right)b_1+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0^3}\frac{1}{2}e_{-1}\\
&+(m_0-n_0)(\kappa_1+1)R_0\frac{1}{2}e_3\\
&+\left((m_0-n_0)(\kappa_1+1)R_0g_6f_{-3}+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0^3}g_2f_{-3}\right)\\
&+\left((m_0-n_0)(\kappa_1+1)R_0g_5f_{-2}+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0^3}g_1f_{-2}\right)\\
&+\left((m_0-n_0)(\kappa_1+1)R_0g_4f_{-1}+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0^3}g_0f_{-1}\right)\\
&+(m_0-n_0)(\kappa_1+1)R_0g_3f_0+(m_0-n_0)(\kappa_1+1)R_0g_2f_1\\
&+(m_0-n_0)(\kappa_1+1)R_0g_1f_2+(m_0-n_0)(\kappa_1+1)R_0g_0f_3,\end{aligned}$$

$$\begin{aligned}0 \;=\; & \left(6\frac{1}{M_0}-\frac{(m_0-n_0)}{(m_0+n_0)}(1+\kappa_0\frac{\mu_1}{\mu_0})-3(1-\frac{\mu_1}{\mu_0})\right)a_3\\
&+\left(-(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0^2}+\frac{1}{M_0}\frac{2}{R_0^2}\right)b_1+(\kappa_1+1)\frac{1}{R_0^4}\frac{1}{2}e_{-1}\\
&+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)\frac{1}{2}e_3+\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_6+(\kappa_1+1)\frac{1}{R_0^4}g_2\right)f_{-3}\end{aligned}$$

$$+\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_5+(\kappa_1+1)\frac{1}{R_0^4}g_1\right)f_{-2}$$
$$+\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_4+(\kappa_1+1)\frac{1}{R_0^4}g_0\right)f_{-1}$$
$$+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_3f_0+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_2f_1$$
$$+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_1f_2+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_0f_3,$$

14) z

$$\begin{aligned}
0 \;=\; & +\left(-(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0-2(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0\right)a_2 \\
& -(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})\frac{1}{R_0}a_0-(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_0 \\
& +(m_0+n_0)(\kappa_1+1)\frac{1}{R_0}\frac{1}{2}e_0+(m_0-n_0)(\kappa_1+1)R_0\frac{1}{2}e_2 \\
& +\left((m_0-n_0)(\kappa_1+1)R_0g_5f_{-3}+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0}g_3f_{-3}\right) \\
& +\left((m_0-n_0)(\kappa_1+1)R_0g_4f_{-2}+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0}g_2f_{-2}\right) \\
& +\left((m_0-n_0)(\kappa_1+1)R_0g_3f_{-1}+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0}g_1f_{-1}\right) \\
& +\left((m_0-n_0)(\kappa_1+1)R_0g_2f_0+(m_0+n_0)(\kappa_1+1)\frac{1}{R_0}g_0f_0\right) \\
& +(m_0-n_0)(\kappa_1+1)R_0g_1f_1+(m_0-n_0)(\kappa_1+1)R_0g_0f_2,
\end{aligned}$$

$$\begin{aligned}
0 \;=\; & \left(-\frac{(m_0-n_0)}{(m_0+n_0)}(1+\kappa_0\frac{\mu_1}{\mu_0})R_0-2(1-\frac{\mu_1}{\mu_0})R_0\right)a_2-(1+\kappa_0\frac{\mu_1}{\mu_0})\frac{1}{R_0}a_0 \\
& -(1-\frac{\mu_1}{\mu_0})\frac{1}{R_0}b_0+(\kappa_1+1)\frac{1}{R_0}\frac{1}{2}e_0+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)R_0\frac{1}{2}e_2 \\
& +\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)R_0g_5+(\kappa_1+1)\frac{1}{R_0}g_3\right)f_{-3} \\
& +\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)R_0g_4+(\kappa_1+1)\frac{1}{R_0}g_2\right)f_{-2}
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)R_0g_3+(\kappa_1+1)\frac{1}{R_0}g_1\right)f_{-1}\\
&+\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)R_0g_2+(\kappa_1+1)\frac{1}{R_0}g_0\right)f_0\\
&+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)R_0g_1f_1+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)R_0g_0f_2,
\end{aligned}
$$

15) z^0 $\quad (4m_0R_0\mu_1\varepsilon_1=0)$

$$
\begin{aligned}
0 \;=\;& \begin{pmatrix} -8\mu_1-(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0-(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0 \\ -(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0-(m_0+n_0)(1-\frac{\mu_1}{\mu_0})R_0 \end{pmatrix} a_1\\
&+2m_0(\kappa_1+1)R_0\frac{1}{2}e_1+2m_0(\kappa_1+1)R_0g_4f_{-3}+2m_0(\kappa_1+1)R_0g_3f_{-2}\\
&+2m_0(\kappa_1+1)R_0g_2f_{-1}+2m_0(\kappa_1+1)R_0g_1f_0+2m_0(\kappa_1+1)R_0g_0f_1,
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\;& \begin{pmatrix} -4\frac{1}{M_0}-\frac{(m_0-n_0)}{(m_0+n_0)}(1+\kappa_0\frac{\mu_1}{\mu_0})-\frac{(m_0-n_0)}{(m_0+n_0)}(1-\frac{\mu_1}{\mu_0}) \\ -(1+\kappa_0\frac{\mu_1}{\mu_0})-(1-\frac{\mu_1}{\mu_0}) \end{pmatrix} a_1\\
&+\frac{m_0(\kappa_1+1)}{(m_0+n_0)}e_1+\frac{2m_0(\kappa_1+1)}{(m_0+n_0)}g_4f_{-3}+\frac{2m_0(\kappa_1+1)}{(m_0+n_0)}g_3f_{-2}\\
&+\frac{2m_0(\kappa_1+1)}{(m_0+n_0)}g_2f_{-1}+\frac{2m_0(\kappa_1+1)}{(m_0+n_0)}g_1f_0+\frac{2m_0(\kappa_1+1)}{(m_0+n_0)}g_0f_1,
\end{aligned}
$$

16) z^{-1}

$$
\begin{aligned}
0 \;=\;& -(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0a_0+\begin{pmatrix} -8\mu_1R_0^2-2(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^3 \\ -(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^3 \end{pmatrix} a_2\\
&-(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0b_0+(m_0-n_0)(\kappa_1+1)R_0\frac{1}{2}e_0\\
&+(m_0+n_0)(\kappa_1+1)R_0^3\frac{1}{2}e_2+\begin{pmatrix} (m_0-n_0)(\kappa_1+1)R_0g_3f_{-3} \\ +(m_0+n_0)(\kappa_1+1)R_0^3g_5f_{-3} \end{pmatrix}\\
&+\left((m_0-n_0)(\kappa_1+1)R_0g_2f_{-2}+(m_0+n_0)(\kappa_1+1)R_0^3g_4f_{-2}\right)
\end{aligned}
$$

$$+\left((m_0-n_0)(\kappa_1+1)R_0g_1f_{-1}+(m_0+n_0)(\kappa_1+1)R_0^3g_3f_{-1}\right)$$
$$+\left((m_0-n_0)(\kappa_1+1)R_0g_0f_0+(m_0+n_0)(\kappa_1+1)R_0^3g_2f_0\right)$$
$$+(m_0+n_0)(\kappa_1+1)R_0^3g_1f_1+(m_0+n_0)(\kappa_1+1)R_0^3g_0f_2,$$

$$\begin{aligned}
0 \;=\; & -\frac{(m_0-n_0)}{(m_0+n_0)}(1+\kappa_0\frac{\mu_1}{\mu_0})a_0+\begin{pmatrix} -4\frac{1}{M_0}R_0^2-\frac{(m_0-n_0)}{(m_0+n_0)}2(1-\frac{\mu_1}{\mu_0})R_0^2 \\ -(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^2 \end{pmatrix}a_2 \\
& -\frac{(m_0-n_0)}{(m_0+n_0)}(1-\frac{\mu_1}{\mu_0})b_0+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)\frac{1}{2}e_0+(\kappa_1+1)R_0^2\frac{1}{2}e_2 \\
& +\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_3+(\kappa_1+1)R_0^2g_5\right)f_{-3} \\
& +\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_2+(\kappa_1+1)R_0^2g_4\right)f_{-2} \\
& +\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_1+(\kappa_1+1)R_0^2g_3\right)f_{-1} \\
& +\left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_0+(\kappa_1+1)R_0^2g_2\right)f_0+(\kappa_1+1)R_0^2g_1f_1+(\kappa_1+1)R_0^2g_0f_2,
\end{aligned}$$

17) z^{-2} $\qquad (2\mu_1(m_0-n_0)(\varepsilon_2+i\varepsilon_3)R_0^3=0)$

$$\begin{aligned}
0 \;=\; & \left(-12\mu_1R_0^4-3(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^5-(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^5\right)a_3 \\
& -(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^3b_1+(m_0-n_0)(\kappa_1+1)R_0\frac{1}{2}e_{-1} \\
& +(m_0+n_0)(\kappa_1+1)R_0^5\frac{1}{2}e_3+\begin{pmatrix} (m_0-n_0)(\kappa_1+1)R_0g_2f_{-3} \\ +(m_0+n_0)(\kappa_1+1)R_0^5g_6f_{-3} \end{pmatrix} \\
& +\left((m_0-n_0)(\kappa_1+1)R_0g_1f_{-2}+(m_0+n_0)(\kappa_1+1)R_0^5g_5f_{-2}\right) \\
& +\left((m_0-n_0)(\kappa_1+1)R_0g_0f_{-1}+(m_0+n_0)(\kappa_1+1)R_0^5g_4f_{-1}\right) \\
& +(m_0+n_0)(\kappa_1+1)R_0^5g_3f_0+(m_0+n_0)(\kappa_1+1)R_0^5g_2f_1 \\
& +(m_0+n_0)(\kappa_1+1)R_0^5g_1f_2+(m_0+n_0)(\kappa_1+1)R_0^5g_0f_3,
\end{aligned}$$

$$
\begin{aligned}
0 \;=\;& \left(-\frac{6}{M_0}R_0^4 - \frac{(m_0-n_0)}{(m_0+n_0)}3(1-\frac{\mu_1}{\mu_0})R_0^4 - (1+\kappa_0\frac{\mu_1}{\mu_0})R_0^4\right) a_3 \\
& -\frac{(m_0-n_0)}{(m_0+n_0)}(1-\frac{\mu_1}{\mu_0})R_0^2 b_1 + \frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)\frac{1}{2}e_{-1} \\
& +(\kappa_1+1)R_0^4\frac{1}{2}e_3 + \left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_2 + (\kappa_1+1)R_0^4 g_6\right) f_{-3} \\
& + \left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_1 + (\kappa_1+1)R_0^4 g_5\right) f_{-2} \\
& + \left(\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_0 + (\kappa_1+1)R_0^4 g_4\right) f_{-1} + (\kappa_1+1)R_0^4 g_3 f_0 \\
& +(\kappa_1+1)R_0^4 g_2 f_1 + (\kappa_1+1)R_0^4 g_1 f_2 + (\kappa_1+1)R_0^4 g_0 f_3,
\end{aligned}
$$

18) z^{-3}

$$
\begin{aligned}
0 \;=\;& \left(-16\mu_1 R_0^6 - 4(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^7 - (m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^7\right) a_4 \\
& -(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^5 b_2 + (m_0-n_0)(\kappa_1+1)R_0\frac{1}{2}e_{-2} \\
& +(m_0+n_0)(\kappa_1+1)R_0^7\frac{1}{2}e_4 + \left(\begin{array}{c} (m_0+n_0)(\kappa_1+1)R_0^7 g_7 f_{-3} \\ +(m_0-n_0)(\kappa_1+1)R_0 g_1 f_{-3} \end{array}\right) \\
& + \left((m_0+n_0)(\kappa_1+1)R_0^7 g_6 f_{-2} + (m_0-n_0)(\kappa_1+1)R_0 g_0 f_{-2}\right) \\
& +(m_0+n_0)(\kappa_1+1)R_0^7 g_5 f_{-1} \\
& +(m_0+n_0)(\kappa_1+1)R_0^7 g_4 f_0 + (m_0+n_0)(\kappa_1+1)R_0^7 g_3 f_1 \\
& +(m_0+n_0)(\kappa_1+1)R_0^7 g_2 f_2 + (m_0+n_0)(\kappa_1+1)R_0^7 g_1 f_3 \\
& +(m_0+n_0)(\kappa_1+1)R_0^7 g_0 f_4,
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\;& \left(-8\frac{1}{M_0}R_0^6 - \frac{(m_0-n_0)}{(m_0+n_0)}4(1-\frac{\mu_1}{\mu_0})R_0^6 - (1+\kappa_0\frac{\mu_1}{\mu_0})R_0^6\right) a_4 \\
& -\frac{(m_0-n_0)}{(m_0+n_0)}(1-\frac{\mu_1}{\mu_0})R_0^4 b_2 + \frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)\frac{1}{2}e_{-2}
\end{aligned}
$$

$$+(\kappa_1+1)R_0^6\frac{1}{2}e_4+\left((\kappa_1+1)R_0^6g_7+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_1\right)f_{-3}$$
$$+\left((\kappa_1+1)R_0^6g_6+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_0\right)f_{-2}+(\kappa_1+1)R_0^6g_5f_{-1}$$
$$+(\kappa_1+1)R_0^6g_4f_0+(\kappa_1+1)R_0^6g_3f_1+(\kappa_1+1)R_0^6g_2f_2$$
$$+(\kappa_1+1)R_0^6g_1f_3+(\kappa_1+1)R_0^6g_0f_4,$$

19) z^{-4} :

$$\begin{aligned}
0 \;=\; & \left(-20\mu_1R_0^8-5(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^9-(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^9\right)a_5\\
&-(m_0-n_0)(1-\frac{\mu_1}{\mu_0})R_0^7b_3+(m_0-n_0)(\kappa_1+1)R_0\frac{1}{2}e_{-3}\\
&+(m_0-n_0)(\kappa_1+1)R_0g_0f_{-3}+(m_0+n_0)(\kappa_1+1)R_0^9g_7f_{-2}\\
&+(m_0+n_0)(\kappa_1+1)R_0^9g_6f_{-1}+(m_0+n_0)(\kappa_1+1)R_0^9g_5f_0\\
&+(m_0+n_0)(\kappa_1+1)R_0^9g_4f_1+(m_0+n_0)(\kappa_1+1)R_0^9g_3f_2\\
&+(m_0+n_0)(\kappa_1+1)R_0^9g_2f_3+(m_0+n_0)(\kappa_1+1)R_0^9g_1f_4,
\end{aligned}$$

$$\begin{aligned}
0 \;=\; & \left(-10\frac{1}{M_0}R_0^8-\frac{(m_0-n_0)}{(m_0+n_0)}5(1-\frac{\mu_1}{\mu_0})R_0^8-(1+\kappa_0\frac{\mu_1}{\mu_0})R_0^8\right)a_5\\
&-\frac{(m_0-n_0)}{(m_0+n_0)}(1-\frac{\mu_1}{\mu_0})R_0^6b_3+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)\frac{1}{2}e_{-3}\\
&+\frac{(m_0-n_0)}{(m_0+n_0)}(\kappa_1+1)g_0f_{-3}+(\kappa_1+1)R_0^8g_7f_{-2}+(\kappa_1+1)R_0^8g_6f_{-1}\\
&+(\kappa_1+1)R_0^8g_5f_0+(\kappa_1+1)R_0^8g_4f_1+(\kappa_1+1)R_0^8g_3f_2\\
&+(\kappa_1+1)R_0^8g_2f_3+(\kappa_1+1)R_0^8g_1f_4.
\end{aligned}$$

Other 9 equations, (3.62) thru (3.70) from the traction continuation condition along Γ_1 are;

20) z^5

$$0 = +\frac{4}{R_1^{10}}c_4 + 4\frac{1}{R_1^2}e_4 - 4\frac{1}{R_1^{10}}\begin{pmatrix} h_{-8}f_4 + h_{-7}f_3 + h_{-6}f_2 + h_{-5}f_1 + h_{-4}f_0 \\ +h_{-3}f_{-1} + h_{-2}f_{-2} + h_{-1}f_{-3} + \frac{1}{2}e_{-4} \end{pmatrix},$$

$$\begin{aligned} 0 = {} & +\frac{4}{R_1^{10}}c_4 + 4\frac{1}{R_1^2}e_4 - \frac{2}{R_1^{10}}e_{-4} - \frac{4}{R_1^{10}}h_{-1}f_{-3} - \frac{4}{R_1^{10}}h_{-2}f_{-2} \\ & -\frac{4}{R_1^{10}}h_{-3}f_{-1} - \frac{4}{R_1^{10}}h_{-4}f_0 - \frac{4}{R_1^{10}}h_{-5}f_1 - \frac{4}{R_1^{10}}h_{-6}f_2 \\ & -\frac{4}{R_1^{10}}h_{-7}f_3 - \frac{4}{R_1^{10}}h_{-8}f_4, \end{aligned}$$

21) z^4

$$0 = +\frac{3}{R_1^8}c_3 + 3\frac{1}{R_1^2}e_3 + 6\frac{1}{R_1^2}\left(h_{-1}f_4 + \frac{1}{2}e_3\right) - 3\frac{1}{R_1^8}\begin{pmatrix} h_{-7}f_4 + h_{-6}f_3 + h_{-5}f_2 + h_{-4}f_1 + h_{-3}f_0 \\ +h_{-2}f_{-1} + h_{-1}f_{-2} + \frac{1}{2}e_{-3} \end{pmatrix},$$

$$\begin{aligned} 0 = {} & +\frac{3}{R_1^8}c_3 + \frac{6}{R_1^2}e_3 + \left(\frac{6}{R_1^2}h_{-1} - \frac{3}{R_1^8}h_{-7}\right)f_4 \\ & -\frac{3}{R_1^8}h_{-6}f_3 - \frac{3}{R_1^8}h_{-5}f_2 - \frac{3}{R_1^8}h_{-4}f_1 - \frac{3}{R_1^8}h_{-3}f_0 \\ & -\frac{3}{R_1^8}h_{-2}f_{-1} - \frac{3}{R_1^8}h_{-1}f_{-2} - \frac{1}{R_1^8}\frac{3}{2}e_{-3}, \end{aligned}$$

22) z^3

$$0 = +\frac{2}{R_1^6}c_2 + 2\frac{1}{R_1^2}e_2 + 2\frac{1}{R_1^2}\left(h_{-2}f_4 + h_{-1}f_3 + \frac{1}{2}e_2\right)$$

$$-2\frac{1}{R_1^6}\begin{pmatrix} h_{-6}f_4+h_{-5}f_3+h_{-4}f_2+h_{-3}f_1 \\ +h_{-2}f_0+h_{-1}f_{-1}+\frac{1}{2}e_{-2} \end{pmatrix},$$

$$\begin{aligned} 0 &= \frac{2}{R_1^6}c_2+\frac{3}{R_1^2}e_2-\frac{1}{R_1^6}e_{-2}+\left(\frac{2}{R_1^2}h_{-2}-\frac{2}{R_1^6}h_{-6}\right)f_4 \\ &+\left(\frac{2}{R_1^2}h_{-1}-\frac{2}{R_1^6}h_{-5}\right)f_3-\frac{2}{R_1^6}h_{-4}f_2-\frac{2}{R_1^6}h_{-3}f_1-\frac{2}{R_1^6}h_{-2}f_0-\frac{2}{R_1^6}h_{-1}f_{-1}, \end{aligned}$$

23) z^2

$$\begin{aligned} -\frac{1}{R_1^2}B &= +\frac{1}{R_1^4}c_1+\frac{1}{R_1^2}e_1-3\left(h_{-1}f_4+\frac{1}{2}e_3\right) \\ &-\frac{1}{R_1^4}\left(h_{-5}f_4+h_{-4}f_3+h_{-3}f_2+h_{-2}f_1+h_{-1}f_0+\frac{1}{2}e_{-1}\right), \end{aligned}$$

$$\begin{aligned} -\frac{1}{R_1^2}B &= +\frac{1}{R_1^4}c_1-\frac{3}{2}e_3+\frac{1}{R_1^2}e_1-\frac{1}{R_1^4}\frac{1}{2}e_{-1}+\left(-3h_{-1}-\frac{1}{R_1^4}h_{-5}\right)f_4 \\ &-\frac{1}{R_1^4}h_{-4}f_3-\frac{1}{R_1^4}h_{-3}f_2-\frac{1}{R_1^4}h_{-2}f_1-\frac{1}{R_1^4}h_{-1}f_0, \end{aligned}$$

24) z^0

$$\begin{aligned} 2A &= -\frac{1}{R_1^2}d_1-\frac{1}{R_1^2}e_{-1}+2\left(h_{-3}f_4+h_{-2}f_3+h_{-1}f_2+\frac{1}{2}e_1\right) \\ &+2\frac{1}{R_1^2}\left(h_{-5}f_4+h_{-4}f_3+h_{-3}f_2+h_{-2}f_1+h_{-1}f_0+\frac{1}{2}e_{-1}\right), \end{aligned}$$

$$\begin{aligned} 2A &= -\frac{1}{R_1^2}d_1+e_1+\left(2h_{-3}+\frac{2}{R_1^2}h_{-5}\right)f_4+\left(2h_{-2}+\frac{2}{R_1^2}h_{-4}\right)f_3 \\ &+\left(2h_{-1}+\frac{2}{R_1^2}h_{-3}\right)f_2+\frac{2}{R_1^2}h_{-2}f_1+\frac{2}{R_1^2}h_{-1}f_0, \end{aligned}$$

25) z^{-1}

$$
\begin{aligned}
0 &= -\frac{2}{R_1^2}d_2 - 2\frac{1}{R_1^2}e_{-2} + 2R_1^2\left(h_{-2}f_4 + h_{-1}f_3 + \frac{1}{2}e_2\right) \\
&\quad +6\frac{1}{R_1^2}\left(h_{-6}f_4 + h_{-5}f_3 + h_{-4}f_2 + h_{-3}f_1 + h_{-2}f_0 + h_{-1}f_{-1} + \frac{1}{2}e_{-2}\right),
\end{aligned}
$$

$$
\begin{aligned}
0 &= -\frac{2}{R_1^2}d_2 + R_1^2 e_2 + \frac{1}{R_1^2}e_{-2} + \left(2R_1^2 h_{-2} + \frac{6}{R_1^2}h_{-6}\right)f_4 \\
&\quad + \left(2R_1^2 h_{-1} + \frac{6}{R_1^2}h_{-5}\right)f_3 + \frac{6}{R_1^2}h_{-4}f_2 + \frac{6}{R_1^2}h_{-3}f_1 + \frac{6}{R_1^2}h_{-2}f_0 + \frac{6}{R_1^2}h_{-1}f_{-1},
\end{aligned}
$$

26) z^{-2}

$$
\begin{aligned}
0 &= +3c_1 - \frac{3}{R_1^2}d_3 - 3\frac{1}{R_1^2}e_{-3} + 3R_1^4\left(h_{-1}f_4 + \frac{1}{2}e_3\right) \\
&\quad -3\left(h_{-5}f_4 + h_{-4}f_3 + h_{-3}f_2 + h_{-2}f_1 + h_{-1}f_0 + \frac{1}{2}e_{-1}\right) \\
&\quad +12\frac{1}{R_1^2}\left(h_{-7}f_4 + h_{-6}f_3 + h_{-5}f_2 + h_{-4}f_1 + h_{-3}f_0 + h_{-2}f_{-1} + h_{-1}f_{-2} + \frac{1}{2}e_{-3}\right),
\end{aligned}
$$

$$
\begin{aligned}
0 &= +3c_1 - \frac{3}{R_1^2}d_3 + R_1^4\frac{3}{2}e_3 - \frac{3}{2}e_{-1} + \frac{3}{R_1^2}e_{-3} \\
&\quad + \left(\frac{12}{R_1^2}h_{-7} + 3R_1^4 h_{-1} - 3h_{-5}\right)f_4 + \left(\frac{12}{R_1^2}h_{-6} - 3h_{-4}\right)f_3 + \left(\frac{12}{R_1^2}h_{-5} - 3h_{-3}\right)f_2 \\
&\quad + \left(\frac{12}{R_1^2}h_{-4} - 3h_{-2}\right)f_1 + \left(\frac{12}{R_1^2}h_{-3} - 3h_{-1}\right)f_0 + \frac{12}{R_1^2}h_{-2}f_{-1} + \frac{12}{R_1^2}h_{-1}f_{-2},
\end{aligned}
$$

27) z^{-3}

$$
\begin{aligned}
0 &= +8c_2 - \frac{4}{R_1^2}d_4 - 4\frac{1}{R_1^2}e_{-4} \\
&\quad -8\left(h_{-6}f_4 + h_{-5}f_3 + h_{-4}f_2 + h_{-3}f_1 + h_{-2}f_0 + h_{-1}f_{-1} + \frac{1}{2}e_{-2}\right)
\end{aligned}
$$

$$+20\frac{1}{R_1^2}\begin{pmatrix} h_{-8}f_4+h_{-7}f_3+h_{-6}f_2+h_{-5}f_1+h_{-4}f_0+h_{-3}f_{-1} \\ +h_{-2}f_{-2}+h_{-1}f_{-3}+\frac{1}{2}e_{-4} \end{pmatrix},$$

$$\begin{aligned} 0 \;=\; & +8c_2-\frac{4}{R_1^2}d_4-4e_{-2}+\frac{6}{R_1^2}e_{-4}+\left(\frac{20}{R_1^2}h_{-8}-8h_{-6}\right)f_4 \\ & +\left(\frac{20}{R_1^2}h_{-7}-8h_{-5}\right)f_3+\left(\frac{20}{R_1^2}h_{-6}-8h_{-4}\right)f_2+\left(\frac{20}{R_1^2}h_{-5}-8h_{-3}\right)f_1 \\ & +\left(\frac{20}{R_1^2}h_{-4}-8h_{-2}\right)f_0+\left(\frac{20}{R_1^2}h_{-3}-8h_{-1}\right)f_{-1}+\frac{20}{R_1^2}h_{-2}f_{-2}+\frac{20}{R_1^2}h_{-1}f_{-3}, \end{aligned}$$

28) z^{-4}

$$\begin{aligned} 0 \;=\; & +15c_3-\frac{5}{R_1^2}d_5-15\begin{pmatrix} h_{-7}f_4+h_{-6}f_3+h_{-5}f_2+h_{-4}f_1 \\ +h_{-3}f_0+h_{-2}f_{-1}+h_{-1}f_{-2}+\frac{1}{2}e_{-3} \end{pmatrix} \\ & +30\frac{1}{R_1^2}\begin{pmatrix} h_{-8}f_3+h_{-7}f_2+h_{-6}f_1+h_{-5}f_0+h_{-4}f_{-1} \\ +h_{-3}f_{-2}+h_{-2}f_{-3}+h_{-1}f_{-4}+\frac{1}{2}e_{-5} \end{pmatrix}, \end{aligned}$$

$$\begin{aligned} 0 \;=\; & 15c_3-\frac{5}{R_1^2}d_5-\frac{15}{2}e_{-3}+\frac{15}{R_1^2}e_{-5}-15h_{-7}f_4 \\ & +\left(\frac{30}{R_1^2}h_{-8}-15h_{-6}\right)f_3+\left(\frac{30}{R_1^2}h_{-7}-15h_{-5}\right)f_2 \\ & +\left(\frac{30}{R_1^2}h_{-6}-15h_{-4}\right)f_1+\left(\frac{30}{R_1^2}h_{-5}-15h_{-3}\right)f_0 \\ & +\left(\frac{30}{R_1^2}h_{-4}-15h_{-2}\right)f_{-1}+\left(\frac{30}{R_1^2}h_{-3}-15h_{-1}\right)f_{-2}+\frac{30}{R_1^2}h_{-2}f_{-3}+\frac{30}{R_1^2}h_{-1}f_{-4}. \end{aligned}$$

Again, non-dimentional parameter $M_1=\frac{m_1+n_1}{2\mu_1}R_1$ is deployed into displacement jump equations along the Γ_1. Finally, the other 10 equations, (3.71) thru (3.80) are;

29) z^5

$$0 = \left(+\frac{16}{R_1^{10}}\mu_1 - (m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\frac{4}{R_1^9} + (m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)\frac{1}{R_1^9}\right)c_4 - (m_1+n_1)(k_1+1)\frac{1}{R_1^9}\left(\begin{array}{c} h_{-8}f_4 + h_{-7}f_3 + h_{-6}f_2 + h_{-5}f_1 \\ +h_{-4}f_0 + h_{-3}f_{-1} + h_{-2}f_{-2} + h_{-1}f_{-3} + \frac{1}{2}e_{-4} \end{array}\right),$$

$$\begin{aligned} 0 = {} & \left(\frac{1}{M_1}\frac{8}{R_1^9} - \frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)\frac{4}{R_1^9} + \left(1+\frac{\mu_1}{\mu_2}k_2\right)\frac{1}{R_1^9}\right)c_4 \\ & -(k_1+1)\frac{1}{R_1^9}h_{-8}f_4 - (k_1+1)\frac{1}{R_1^9}h_{-7}f_3 - (k_1+1)\frac{1}{R_1^9}h_{-6}f_2 \\ & -(k_1+1)\frac{1}{R_1^9}h_{-5}f_1 - (k_1+1)\frac{1}{R_1^9}h_{-4}f_0 - (k_1+1)\frac{1}{R_1^9}h_{-3}f_{-1} \\ & -(k_1+1)\frac{1}{R_1^9}h_{-2}f_{-2} - (k_1+1)\frac{1}{R_1^9}h_{-1}f_{-3} - (k_1+1)\frac{1}{R_1^9}\frac{1}{2}e_{-4}, \end{aligned}$$

30) z^4

$$\begin{aligned} 0 = {} & \left(+\frac{12}{R_1^8}\mu_1 - (m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\frac{3}{R_1^7} + (m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)\frac{1}{R_1^7}\right)c_3 \\ & +(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^9}d_5 \\ & -\frac{(m_1+n_1)(k_1+1)}{R_1^7}\left(\begin{array}{c} h_{-7}f_4 + h_{-6}f_3 + h_{-5}f_2 + h_{-4}f_1 + h_{-3}f_0 \\ +h_{-2}f_{-1} + h_{-1}f_{-2} + \frac{1}{2}e_{-3} \end{array}\right), \end{aligned}$$

$$\begin{aligned} 0 = {} & \left(\frac{1}{M_1}\frac{6}{R_1^7} - \frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)\frac{3}{R_1^7} + \left(1+\frac{\mu_1}{\mu_2}k_2\right)\frac{1}{R_1^7}\right)c_3 \\ & +\frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^9}d_5 - \frac{(k_1+1)}{R_1^7}\frac{1}{2}e_{-3} - \frac{(k_1+1)}{R_1^7}h_{-7}f_4 \\ & -\frac{(k_1+1)}{R_1^7}h_{-6}f_3 - \frac{(k_1+1)}{R_1^7}h_{-5}f_2 - \frac{(k_1+1)}{R_1^7}h_{-4}f_1 - \frac{(k_1+1)}{R_1^7}h_{-3}f_0 \\ & -\frac{(k_1+1)}{R_1^7}h_{-2}f_{-1} - \frac{(k_1+1)}{R_1^7}h_{-1}f_{-2}, \end{aligned}$$

31) z^3

$$
\begin{aligned}
0 \;=\;& \left(+\frac{8}{R_1^6}\mu_1 - (m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\frac{2}{R_1^5} + (m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)\frac{1}{R_1^5}\right)c_2 \\
&+ (m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^7}d_4 \\
&- (m_1+n_1)(k_1+1)\frac{1}{R_1^5}\left(\begin{array}{c} h_{-6}f_4 + h_{-5}f_3 + h_{-4}f_2 + h_{-3}f_1 \\ +h_{-2}f_0 + h_{-1}f_{-1} + \frac{1}{2}e_{-2} \end{array}\right),
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\;& \left(\frac{1}{M_1}\frac{4}{R_1^5} - \frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)\frac{2}{R_1^5} + \left(1+\frac{\mu_1}{\mu_2}k_2\right)\frac{1}{R_1^5}\right)c_2 \\
&+\frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^7}d_4 - \frac{(k_1+1)}{R_1^5}\frac{1}{2}e_{-2} - \frac{(k_1+1)}{R_1^5}h_{-6}f_4 - \frac{(k_1+1)}{R_1^5}h_{-5}f_3 \\
&-\frac{(k_1+1)}{R_1^5}h_{-4}f_2 - \frac{(k_1+1)}{R_1^5}h_{-3}f_1 - \frac{(k_1+1)}{R_1^5}h_{-2}f_0 - \frac{(k_1+1)}{R_1^5}h_{-1}f_{-1},
\end{aligned}
$$

32) z^2

$$
\begin{aligned}
&\left(-(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1} - 4\mu_1\frac{1}{R_1^2}\right)B \\
=\;& \left(+\frac{4}{R_1^4}\mu_1 - (m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^3} + (m_1+n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)\frac{1}{R_1^3}\right)c_1 \\
&+ (m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^5}d_3 - (m_1-n_1)(k_1+1)R_1\left(h_{-1}f_4 + \frac{1}{2}e_3\right) \\
&- (m_1+n_1)(k_1+1)\frac{1}{R_1^3}\left(h_{-5}f_4 + h_{-4}f_3 + h_{-3}f_2 + h_{-2}f_1 + h_{-1}f_0 + \frac{1}{2}e_{-1}\right),
\end{aligned}
$$

$$
\begin{aligned}
&\left(-\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1} - \frac{1}{M_1}\frac{2}{R_1}\right)B \\
=\;& \left(\frac{1}{M_1}\frac{2}{R_1^3} - \frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^3} + \left(1+\frac{\mu_1}{\mu_2}k_2\right)\frac{1}{R_1^3}\right)c_1 \\
&+\frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^5}d_3 - \frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)R_1\frac{1}{2}e_3
\end{aligned}
$$

$$-\frac{(k_1+1)}{R_1^3}\frac{1}{2}e_{-1}+\left(-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)R_1h_{-1}-\frac{(k_1+1)}{R_1^3}h_{-5}\right)f_4$$
$$-\frac{(k_1+1)}{R_1^3}h_{-4}f_3-\frac{(k_1+1)}{R_1^3}h_{-3}f_2-\frac{(k_1+1)}{R_1^3}h_{-2}f_1-\frac{(k_1+1)}{R_1^3}h_{-1}f_0,$$

33) z^1

$$\begin{aligned}0 &= +(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R^3}d_2-(m_1-n_1)(k_1+1)R_1\left(h_{-2}f_4+h_{-1}f_3+\frac{1}{2}e_2\right)\\ &\quad -(m_1+n_1)(k_1+1)\frac{1}{R_1}\left(h_{-4}f_4+h_{-3}f_3+h_{-2}f_2+h_{-1}f_1+\frac{1}{2}e_0\right),\end{aligned}$$

$$\begin{aligned}0 &= \frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R^3}d_2-\frac{(m_1-n_1)}{2(m_1+n_1)}(k_1+1)R_1e_2\\ &\quad -(k_1+1)\frac{1}{R_1}\frac{1}{2}e_0+\left(-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)R_1h_{-2}-(k_1+1)\frac{1}{R_1}h_{-4}\right)f_4\\ &\quad +\left(-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)R_1h_{-1}-(k_1+1)\frac{1}{R_1}h_{-3}\right)f_3\\ &\quad -(k_1+1)\frac{1}{R_1}h_{-2}f_2-(k_1+1)\frac{1}{R_1}h_{-1}f_1,\end{aligned}$$

34) z^0

$$\begin{aligned}&-\left(-8\mu_1A+2m_1\left(1+\frac{\mu_1}{\mu_2}k_2\right)AR_1+2m_1\left(1-\frac{\mu_1}{\mu_2}\right)AR_1\right)\\ &= \left(-\frac{4}{R_1^2}\mu_1+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1}+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1}\right)d_1\\ &\quad -2m_1(k_1+1)R_1\left(h_{-3}f_4+h_{-2}f_3+h_{-1}f_2+\frac{1}{2}e_1\right),\end{aligned}$$

$$\begin{aligned}&\frac{4A}{M_1}-\frac{2m_1A}{(m_1+n_1)}\left(1+\frac{\mu_1}{\mu_2}k_2\right)-\frac{2m_1A}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)\\ &= \left(-\frac{2}{R_1^2}\frac{1}{M_1}+\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^2}+\frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^2}\right)d_1\end{aligned}$$

$$-\frac{m_1(k_1+1)}{(m_1+n_1)}e_1-\frac{2m_1(k_1+1)}{(m_1+n_1)}h_{-3}f_4-\frac{2m_1(k_1+1)}{(m_1+n_1)}h_{-2}f_3-\frac{2m_1(k_1+1)}{(m_1+n_1)}h_{-1}f_2,$$

35) z^{-1}

$$\begin{aligned}0 &= \left(-\frac{8}{R_1^2}\mu_1+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1}\right)d_2\\ &\quad -(m_1+n_1)(k_1+1)R_1^3\left(h_{-2}f_4+h_{-1}f_3+\frac{1}{2}e_2\right)\\ &\quad -(m_1-n_1)(k_1+1)R_1\left(h_{-4}f_4+h_{-3}f_3+h_{-2}f_2+h_{-1}f_1+\frac{1}{2}e_0\right),\end{aligned}$$

$$\begin{aligned}0 &= \left(-\frac{4}{R_1}\frac{1}{M_1}+\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1}\right)d_2-(k_1+1)R_1^3\frac{1}{2}e_2\\ &\quad -\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)R_1\frac{1}{2}e_0+\left(-(k_1+1)R_1^3h_{-2}-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)R_1h_{-4}\right)f_4\\ &\quad +\left(-(k_1+1)R_1^3h_{-1}-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)R_1h_{-3}\right)f_3\\ &\quad -\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)R_1h_{-2}f_2-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)R_1h_{-1}f_1,\end{aligned}$$

36) z^{-2}

$$\begin{aligned}&-(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)BR_1^3\\ &= +\left[12\mu_1+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)R_1-(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)R_1\right]c_1\\ &\quad +\left[-\frac{12}{R_1^2}\mu_1+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1}\right]d_3-(m_1+n_1)(k_1+1)R_1^5\left(h_{-1}f_4+\frac{1}{2}e_3\right)\\ &\quad -(m_1-n_1)(k_1+1)R_1\left(h_{-5}f_4+h_{-4}f_3+h_{-3}f_2+h_{-2}f_1+h_{-1}f_0+\frac{1}{2}e_{-1}\right),\end{aligned}$$

$$-\frac{(m_1-n_1)}{(m_1+n_1)}\left(1-\frac{\mu_1}{\mu_2}\right)BR_1^2$$

$$
\begin{aligned}
&= \left(\frac{6}{M_1}+\frac{(m_1-n_1)}{(m_1+n_1)}\left(1+\frac{\mu_1}{\mu_2}k_2\right)-\left(1-\frac{\mu_1}{\mu_2}\right)\right)c_1 \\
&\quad+\left(-\frac{6}{R_1^2}\frac{1}{M_1}+\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^2}\right)d_3-(k_1+1)R_1^4\frac{1}{2}e_3-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)\frac{1}{2}e_{-1} \\
&\quad+\left(-(k_1+1)R_1^4h_{-1}-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-5}\right)f_4-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-4}f_3 \\
&\quad-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-3}f_2-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-2}f_1-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-1}f_0,
\end{aligned}
$$

37) z^{-3}

$$
\begin{aligned}
0 &= \left[+32\mu_1+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)R_1-2R_1(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\right]c_2 \\
&\quad+\left[-\frac{16}{R_1^2}\mu_1+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1}\right]d_4 \\
&\quad-(m_1-n_1)(k_1+1)R_1\begin{pmatrix} h_{-6}f_4+h_{-5}f_3+h_{-4}f_2+h_{-3}f_1 \\ +h_{-2}f_0+h_{-1}f_{-1}+\frac{1}{2}e_{-2}\end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
0 &= \left(\frac{16}{M_1}+\frac{(m_1-n_1)}{(m_1+n_1)}\left(1+\frac{\mu_1}{\mu_2}k_2\right)-2\left(1-\frac{\mu_1}{\mu_2}\right)\right)c_2 \\
&\quad+\left(-\frac{1}{R_1^2}\frac{8}{M_1}+\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^2}\right)d_4-\frac{(m_1-n_1)(k_1+1)}{2(m_1+n_1)}e_{-2} \\
&\quad-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-6}f_4-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-5}f_3 \\
&\quad-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-4}f_2-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-3}f_1 \\
&\quad-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-2}f_0-\frac{(m_1-n_1)}{(m_1+n_1)}(k_1+1)h_{-1}f_{-1};
\end{aligned}
$$

38) z^{-4}

$$
\begin{aligned}
0 &= \left(+60\mu_1+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}k_2\right)R_1-3R_1(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\right)c_3 \\
&\quad+\left(-\frac{20}{R_1^2}\mu_1+(m_1+n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\frac{1}{R_1}\right)d_5-(m_1-n_1)(k_1+1)R_1q_{-3},
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\; & \left(\frac{30}{M_1} + \frac{(m_1 - n_1)}{(m_1 + n_1)}\left(1 + \frac{\mu_1}{\mu_2}k_2\right) - 3\left(1 - \frac{\mu_1}{\mu_2}\right)\right)c_3 \\
& + \left(-\frac{1}{R_1^2}\frac{10}{M_1} + \left(1 - \frac{\mu_1}{\mu_2}\right)\frac{1}{R_1^2}\right)d_5 - \frac{(m_1 - n_1)(k_1 + 1)}{(m_1 + n_1)}h_{-7}f_4 \\
& - \frac{(m_1 - n_1)(k_1 + 1)}{(m_1 + n_1)}h_{-6}f_3 - \frac{(m_1 - n_1)(k_1 + 1)}{(m_1 + n_1)}h_{-5}f_2 - \frac{(m_1 - n_1)(k_1 + 1)}{(m_1 + n_1)}h_{-4}f_1 \\
& - \frac{(m_1 - n_1)(k_1 + 1)}{(m_1 + n_1)}h_{-3}f_0 - \frac{(m_1 - n_1)(k_1 + 1)}{(m_1 + n_1)}h_{-2}f_{-1} \\
& - \frac{(m_1 - n_1)(k_1 + 1)}{(m_1 + n_1)}h_{-1}f_{-2} - \frac{(m_1 - n_1)(k_1 + 1)}{2(m_1 + n_1)}e_{-3}.
\end{aligned}
$$

APPENDIX 6
Stress Fields Around Crack Tips

A6.1 Stress Field

Stress fields are expressed in terms of stress potentials as follows

$$\begin{aligned}
\sigma_{xx} &= \mathrm{Re}\left[\varphi_1'(z) - \varphi_1'(\bar{z}) + 2\overline{\varphi_1'(z)} + (\bar{z}-z)\,\overline{\varphi_1''(z)} + X'(\bar{z})\right], \qquad (4.7)\\
\sigma_{yy} &= \mathrm{Re}\left[\varphi_1'(z) + \varphi_1'(\bar{z}) + (z-\bar{z})\,\overline{\varphi_1''(z)} - X'(\bar{z})\right],\\
\sigma_{xy} &= \mathrm{Im}\left[\varphi_1'(z) - \varphi_1'(\bar{z}) + 2\overline{\varphi_1'(z)} + (\bar{z}-z)\,\overline{\varphi_1''(z)} + X'(\bar{z})\right].
\end{aligned}$$

Let's derive $\varphi_1'(z)$ first. From (2.32)

$$\begin{aligned}
\frac{d}{dz}\varphi_1(z) &= \frac{d}{dz}\left[\frac{1}{2}(z-a)^{-\frac{1}{2}}(z-b)^{-\frac{1}{2}}Y(z) + \frac{1}{2}X(z)\right] \qquad (A6.1)\\
&= -\frac{1}{4}(z-a)^{-\frac{3}{2}}(z-b)^{-\frac{1}{2}}Y(z) - \frac{1}{4}(z-a)^{-\frac{1}{2}}(z-b)^{-\frac{3}{2}}Y(z)\\
&\quad +\frac{1}{2}(z-a)^{-\frac{1}{2}}(z-b)^{-\frac{1}{2}}Y'(z) + \frac{1}{2}X'(z)\\
&= -\frac{1}{4}\frac{(z-b)+(z-a)}{(z-a)^{\frac{3}{2}}(z-b)^{\frac{3}{2}}}Y(z) + \frac{1}{2}(z-a)^{-\frac{1}{2}}(z-b)^{-\frac{1}{2}}Y'(z) + \frac{1}{2}X'(z)\\
&= \frac{1}{4}\frac{a+b-2z}{(z-a)^{\frac{3}{2}}(z-b)^{\frac{3}{2}}}Y(z) + \frac{1}{2(z-a)^{\frac{1}{2}}(z-b)^{\frac{1}{2}}}Y'(z) + \frac{1}{2}X'(z).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\varphi_1''(z) \qquad (A6.2)\\
&= \frac{d}{dz}\left[\begin{array}{l} \frac{1}{4}(a+b-2z)(z-a)^{-\frac{3}{2}}(z-b)^{-\frac{3}{2}}Y(z)\\ +\frac{1}{2}(z-a)^{-\frac{1}{2}}(z-b)^{-\frac{1}{2}}Y'(z) + \frac{1}{2}X'(z)\end{array}\right]\\
&= \frac{1}{4}(-2)(z-a)^{-\frac{3}{2}}(z-b)^{-\frac{3}{2}}Y(z) - \frac{3}{8}(a+b-2z)(z-a)^{-\frac{5}{2}}(z-b)^{-\frac{3}{2}}Y(z)\\
&\quad -\frac{3}{8}(a+b-2z)(z-a)^{-\frac{3}{2}}(z-b)^{-\frac{5}{2}}Y(z) + \frac{1}{4}(a+b-2z)(z-a)^{-\frac{3}{2}}(z-b)^{-\frac{3}{2}}Y'(z)
\end{aligned}$$

$$-\frac{1}{4}(z-a)^{-\frac{3}{2}}(z-b)^{-\frac{1}{2}}Y'(z)-\frac{1}{4}(z-a)^{-\frac{1}{2}}(z-b)^{-\frac{3}{2}}Y'(z)$$
$$+\frac{1}{2}(z-a)^{-\frac{1}{2}}(z-b)^{-\frac{1}{2}}Y''(z)+\frac{1}{2}X''(z).$$

The 2nd and 3rd terms in (A6.2) turn out

$$\begin{aligned}
& -\frac{3}{8}(a+b-2z)\left[(z-a)^{-\frac{5}{2}}(z-b)^{-\frac{3}{2}}+(z-a)^{-\frac{3}{2}}(z-b)^{-\frac{5}{2}}\right]Y(z) \\
= & -\frac{3}{8}(a+b-2z)\frac{(z-b)+(z-a)}{(z-a)^{\frac{5}{2}}(z-b)^{\frac{5}{2}}}Y(z) \\
= & \frac{3}{8}\frac{(a+b-2z)^2}{(z-a)^{\frac{5}{2}}(z-b)^{\frac{5}{2}}}Y(z).
\end{aligned} \quad \text{(A6.3)}$$

The 4th, 5th and 6th terms are also re-written as

$$\frac{(a+b-2z)-(z-b)-(z-a)}{4(z-a)^{\frac{3}{2}}(z-b)^{\frac{3}{2}}}Y'(z)=\frac{(a+b-2z)}{2(z-a)^{\frac{3}{2}}(z-b)^{\frac{3}{2}}}Y'(z). \quad \text{(A6.4)}$$

Thus,

$$\begin{aligned}
\varphi_1''(z) = & -\frac{1}{2(z-a)^{\frac{3}{2}}(z-b)^{\frac{3}{2}}}Y(z)+\frac{3(a+b-2z)^2}{8(z-a)^{\frac{5}{2}}(z-b)^{\frac{5}{2}}}Y(z) \\
& +\frac{(a+b-2z)}{2(z-a)^{\frac{3}{2}}(z-b)^{\frac{3}{2}}}Y'(z)+\frac{1}{2(z-a)^{\frac{1}{2}}(z-b)^{\frac{1}{2}}}Y''(z)+\frac{1}{2}X''(z).
\end{aligned} \quad \text{(A6.5)}$$

Since our aim is to determine the values of stress potentials near crack tips, $Y(z)$, $Y'(z)$ and $Y''(z)$ need to be determined near crack tip a and b. First let's determine the stress potential $Y(z)$ in the neighbourhood of crack tip a as follows

$$\begin{aligned}
Y(z) = & \ Y(a)+Y'(a)(z-a)+\frac{1}{2!}Y''(a)(z-a)^2+\ldots \\
& +\frac{1}{n!}Y^{(n)}(a)(z-a)^n+\ldots \\
= & \ Y(a)+Y'(a)r_1e^{i\theta_1}+\frac{1}{2!}Y''(a)r_1^2e^{i2\theta_1}+\ldots+\frac{1}{n!}Y^{(n)}(a)r_1^ne^{in\theta_1}+\ldots,
\end{aligned} \quad \text{(A6.6)}$$

where $(z-a) = r_1 e^{i\theta_1}$ $(0 \le \theta_1 \le 2\pi)$. Thus,

$$
\begin{aligned}
Y(z) &= Y(a) + Y'(a) r_1 e^{i\theta_1} + \dots , && \text{(A6.7)}\\
Y'(z) &= Y'(a) + Y''(a) r_1 e^{i\theta_1} \dots , && \text{(A6.8)}\\
Y''(z) &= Y''(a) + Y'''(a) r_1 e^{i\theta_1} \dots . && \text{(A6.9)}
\end{aligned}
$$

Substituting (A6.7) and (A6.8) into (A6.1) yields

$$
\begin{aligned}
\varphi_1'(z) &= \left[\frac{1}{4} \frac{(a+b-2z)}{(z-a)^{\frac{3}{2}}(z-b)^{\frac{3}{2}}} Y(z) + \frac{1}{2(z-a)^{\frac{1}{2}}(z-b)^{\frac{1}{2}}} Y'(z) + \frac{1}{2} X'(z) \right]_{around\ z=a} \quad \text{(A6.10)}\\
&= \frac{1}{4} \frac{-(a-b)\left[Y(a) + Y'(a) r_1 e^{i\theta_1}\right]}{(r_1 e^{i\theta_1})^{\frac{3}{2}} (a-b)^{\frac{3}{2}}} + \frac{1}{2} \frac{\left[Y'(a) + Y''(a) r_1 e^{i\theta_1}\right]}{(r_1 e^{i\theta_1})^{\frac{1}{2}} (a-b)^{\frac{1}{2}}} + \frac{1}{2} X'(a)\\
&= \frac{1}{4} \frac{-Y(a)}{(r_1 e^{i\theta_1})^{\frac{3}{2}} (a-b)^{\frac{1}{2}}} + \frac{1}{4} \frac{-Y'(a)}{(r_1 e^{i\theta_1})^{\frac{1}{2}} (a-b)^{\frac{1}{2}}} + \frac{1}{2} \frac{Y'(a)}{(r_1 e^{i\theta_1})^{\frac{1}{2}} (a-b)^{\frac{1}{2}}}\\
&\quad + \frac{1}{2} \frac{Y''(a) (r_1 e^{i\theta_1})^{\frac{1}{2}}}{(a-b)^{\frac{1}{2}}} + \frac{1}{2} X'(a).
\end{aligned}
$$

Taking only leading-order items yields

$$
\begin{aligned}
\varphi_1'(z) &= \frac{1}{4} \frac{-Y'(a)}{(r_1 e^{i\theta_1})^{\frac{1}{2}} (a-b)^{\frac{1}{2}}} + \frac{1}{2} \frac{Y'(a)}{(r_1 e^{i\theta_1})^{\frac{1}{2}} (a-b)^{\frac{1}{2}}} \quad \text{(A6.11)}\\
&= \frac{1}{4} \frac{Y'(a)}{\sqrt{r_1} e^{i\frac{\theta_1}{2}} \sqrt{a-b}}.
\end{aligned}
$$

Note that $b > a$ so that $\sqrt{a-b} = i\sqrt{b-a}$ and $i = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$. Therefore,

$$
\varphi_1'(z) = \frac{Y'(a)}{4\sqrt{r_1} e^{i\frac{\theta_1}{2}} e^{i\frac{\pi}{2}} \sqrt{b-a}} = \frac{Y'(a)}{4\sqrt{r_1}\sqrt{b-a}} e^{-i\left(\frac{\theta_1+\pi}{2}\right)}. \quad \text{(A6.12)}
$$

In addition,

$$
\varphi_1'(\bar{z}) = \frac{Y'(a)}{4\sqrt{r_1} e^{-i\frac{\theta_1}{2}} e^{i\frac{\pi}{2}} \sqrt{b-a}} = \frac{Y'(a)}{4\sqrt{r_1}\sqrt{b-a}} e^{i\left(\frac{\theta_1-\pi}{2}\right)}. \quad \text{(A6.13)}
$$

The exponential terms can be expressed with trigonometric functions in the following form

$$
\begin{aligned}
e^{-i\left(\frac{\theta_1+\pi}{2}\right)} &= \cos\left(\frac{\theta_1+\pi}{2}\right) - i\sin\left(\frac{\theta_1+\pi}{2}\right) = -\sin\frac{\theta_1}{2} - i\cos\frac{\theta_1}{2}, \\
e^{i\left(\frac{\theta_1-\pi}{2}\right)} &= \cos\left(\frac{\theta_1-\pi}{2}\right) + i\sin\left(\frac{\theta_1-\pi}{2}\right) = \sin\frac{\theta_1}{2} - i\cos\frac{\theta_1}{2}.
\end{aligned}
$$

Substituting the above equations into (A6.12) and (A6.13) leads to

$$
\begin{aligned}
\varphi_1'(z) &= \frac{Y'(a)}{4\sqrt{r_1}\sqrt{b-a}}\left(-\sin\frac{\theta_1}{2} - i\cos\frac{\theta_1}{2}\right), \\
\varphi_1'(\bar{z}) &= \frac{Y'(a)}{4\sqrt{r_1}\sqrt{b-a}}\left(\sin\frac{\theta_1}{2} - i\cos\frac{\theta_1}{2}\right).
\end{aligned}
\tag{A6.14}
$$

Note that $\operatorname{Re}(\varphi_1'(z)) = -\operatorname{Re}(\varphi_1'(\bar{z}))$, which means $\varphi_1'(z)$ has multi-values at the crack face. Therefore, we need to define a new $\varphi_1'(\bar{z})$ value leading single valueness at the crack face. Now, let's consider a branch cut which gives us single valueness and makes $\varphi_1'(z)$ single-valued function at the crack face. The branches of $e^{-i\frac{\theta_1}{2}}$ are $e^{-i\frac{1}{2}(\theta_1+2n\pi)}$ and $e^{-i\frac{1}{2}(\theta_1-2n\pi)}$. Let's take $e^{-i\frac{1}{2}(\theta_1-2\pi)}$ for $\varphi_1'(\bar{z})$. From (6.13), $\varphi_1'(\bar{z})$ can be expressed as follows

$$
\begin{aligned}
\varphi_1'(\bar{z}) &= \frac{Y'(a)}{4\sqrt{r_1}\sqrt{b-a}e^{-i\frac{1}{4}(\theta_1-2\pi)}e^{i\frac{\pi}{2}}} = \frac{Y'(a)}{4\sqrt{r_1}\sqrt{b-a}}e^{i\left(\frac{\theta_1}{2}-\frac{3\pi}{2}\right)} \\
&= \frac{Y'(a)}{4\sqrt{r_1}\sqrt{b-a}}\left[\cos\left(\frac{\theta_1}{2}-\frac{3\pi}{2}\right) + i\sin\left(\frac{\theta_1}{2}-\frac{3\pi}{2}\right)\right] \\
&= \frac{Y'(a)}{4\sqrt{r_1}\sqrt{b-a}}\left[-\sin\frac{\theta_1}{2} + i\cos\frac{\theta_1}{2}\right].
\end{aligned}
\tag{A6.15}
$$

Thus, at the branch, $\varphi_1'(z)$ has a single value at the upper/lower $(+/-)$ crack face such as

$$\varphi_1'(z) = \frac{Y'(a)}{4\sqrt{r_1}\sqrt{b-a}}\left(-\sin\frac{\theta_1}{2} - i\cos\frac{\theta_1}{2}\right), \qquad \text{(A6.16)}$$
$$\varphi_1'(\bar{z}) = \frac{Y'(a)}{4\sqrt{r_1}\sqrt{b-a}}\left[-\sin\frac{\theta_1}{2} + i\cos\frac{\theta_1}{2}\right].$$

Next, $\varphi_1''(z)$ is derived. In view of (4.7), (A6.5) is rewritten as

$$(z-\bar{z})\,\varphi_1''(z) = (z-\bar{z})\left[\begin{array}{c} \frac{(a+b-2z)}{2(z-a)^{\frac{3}{2}}(z-b)^{\frac{3}{2}}}Y'(z) - \frac{1}{2(z-a)^{\frac{3}{2}}(z-b)^{\frac{3}{2}}}Y(z) \\ +\frac{3(a+b-2z)^2}{8(z-a)^{\frac{5}{2}}(z-b)^{\frac{5}{2}}}Y(z) + \frac{1}{2(z-a)^{\frac{1}{2}}(z-b)^{\frac{1}{2}}}Y''(z) + \frac{1}{2}X''(z) \end{array}\right]_{around\ z=a} \qquad \text{(A6.17)}$$

At $z \approx a$, from (A6.7), (A6.8) and (A6.9) are

$$(z-a) = r_1 e^{i\theta_1},$$
$$Y(z) = Y(a) + Y'(a) r_1 e^{i\theta_1},$$
$$Y'(z) = Y'(a) \text{ and } Y''(z) = Y''(a).$$

In addition,

$$(z-\bar{z}) = r_1(\cos\theta_1 + i\sin\theta_1) - r_1(\cos\theta_1 - i\sin\theta_1) = 2r_1 i\sin\theta_1.$$

Substituting above equations into (A6.17) yields

$$(z-\bar{z})\,\varphi_1''(z) = 2r_1 i\sin\theta_1\left[\begin{array}{c} \frac{(-a+b)Y'(a)}{2(r_1e^{i\theta_1})^{\frac{3}{2}}(a-b)^{\frac{3}{2}}} - \frac{Y(a)+Y'(a)r_1e^{i\theta_1}}{2(r_1e^{i\theta_1})^{\frac{3}{2}}(a-b)^{\frac{3}{2}}} \\ +\frac{3(-a+b)^2\left[Y(a)+Y'(a)r_1e^{i\theta_1}\right]}{8(r_1e^{i\theta_1})^{\frac{5}{2}}(a-b)^{\frac{5}{2}}} \end{array}\right] \qquad \text{(A6.18)}$$
$$+2r_1 i\sin\theta_1\frac{Y''(a)}{2(r_1e^{i\theta_1})^{\frac{1}{2}}(a-b)^{\frac{1}{2}}} + 2r_1 i\sin\theta_1\frac{1}{2}X''(z).$$

Note that $Y(a) = 0$ by definition. By taking leading order singularity only, we get

$$(z-\bar{z})\,\varphi_1''(z) = -\frac{Y'(a)\sin\theta_1}{4\sqrt{r_1}\sqrt{b-a}e^{i\frac{3}{2}\theta_1}} = -\frac{Y'(a)\sin\theta_1}{4\sqrt{r_1}\sqrt{b-a}}e^{-i\frac{3}{2}\theta_1}. \tag{A6.19}$$

Now, let's check single valueness of $(z-\bar{z})\,\varphi_1''(z)$ at a branch cut. We choose $\theta_1 + 2\pi$ first.

$$\begin{aligned}
e^{-i\left(3\pi+\frac{3\theta_1}{2}\right)} &= \cos\left(3\pi+\frac{3\theta_1}{2}\right) - i\sin\left(3\pi+\frac{3\theta_1}{2}\right) = -\cos\frac{3\theta_1}{2} + i\sin\frac{3\theta_1}{2},\\
e^{i\left(3\pi+\frac{3\theta_1}{2}\right)} &= \cos\left(3\pi+\frac{3\theta_1}{2}\right) + i\sin\left(3\pi+\frac{3\theta_1}{2}\right) = -\cos\frac{3\theta_1}{2} - i\sin\frac{3\theta_1}{2}.
\end{aligned}$$

Note that $\mathrm{Re}\,(\varphi_1''(z)) = \mathrm{Re}\,(\varphi_1''(\bar{z}))$ and $\mathrm{Im}\,(\varphi_1''(z)) = -\,\mathrm{Im}\,(\varphi_1''(\bar{z}))$ and it has single value at the crack face. Thus,

$$(z-\bar{z})\,\varphi_1''(z) = -\frac{Y'(a)\sin\theta_1}{4\sqrt{r_1}\sqrt{b-a}}\left(-\cos\frac{3\theta_1}{2} + i\sin\frac{3\theta_1}{2}\right). \tag{A6.20}$$

Let's substitute (A6.16) and (A6.20) into $(4.7)_2$ to define stress fields.

$$\begin{aligned}
\sigma_{yy} &= \mathrm{Re}\left[\varphi_1'(z) + \varphi_1'(\bar{z}) + (z-\bar{z})\,\overline{\varphi_1''(z)} - X'(\bar{z})\right] \qquad \text{(A6.21)}\\
&= -\frac{Y'(a)}{2\sqrt{r_1}\sqrt{b-a}}\sin\frac{\theta_1}{2} + \frac{Y'(a)}{4\sqrt{r_1}\sqrt{b-a}}\sin\theta_1\cos\frac{3\theta_1}{2} - \mathrm{Re}\left[X'(\bar{z})\right]\\
&= \frac{Y'(a)}{\sqrt{r_1}\sqrt{b-a}}\left[-\frac{1}{2}\sin\frac{\theta_1}{2} + \frac{1}{4}\sin\theta_1\cos\frac{3\theta_1}{2}\right] + O(r_1).
\end{aligned}$$

Note that

$$\begin{aligned}
&-\frac{1}{2}\sin\frac{\theta_1}{2} + \frac{1}{4}\cos\frac{3\theta_1}{2}\sin\theta_1\\
=\;& -\frac{1}{2}\sin\frac{\theta_1}{2} + \frac{1}{8}\left[\sin\left(\frac{3\theta_1}{2}+\theta_1\right) - \sin\left(\frac{3\theta_1}{2}-\theta_1\right)\right]\\
=\;& -\frac{1}{2}\sin\frac{\theta_1}{2} + \frac{1}{8}\left(\sin\frac{5\theta_1}{2} - \sin\frac{\theta_1}{2}\right)
\end{aligned}$$

$$= \frac{1}{8}\sin\frac{5\theta_1}{2} - \frac{5}{8}\sin\frac{\theta_1}{2}.$$

Thus,

$$\sigma_{yy} = \frac{Y'(a)}{\sqrt{r_1}\sqrt{b-a}}\left(-\frac{5}{8}\sin\frac{\theta_1}{2} + \frac{1}{8}\sin\frac{5\theta_1}{2}\right) + O(r_1). \tag{A6.22}$$

In view of (4.7), adding σ_{yy} to σ_{xx} leads to the expression

$$\sigma_{yy} + \sigma_{xx} = 2\,\mathrm{Re}\left[\varphi_1'(z) + \overline{\varphi_1'(z)}\right]. \tag{A6.23}$$

Thus,

$$\begin{aligned}
\sigma_{xx} &= -\frac{Y'(a)}{\sqrt{r_1}\sqrt{b-a}}\sin\frac{\theta_1}{2} - \frac{Y'(a)}{\sqrt{r_1}\sqrt{b-a}}\left(-\frac{5}{8}\sin\frac{\theta_1}{2} + \frac{1}{8}\sin\frac{5\theta_1}{2}\right) + O(r_1) \\
&= \frac{Y'(a)}{\sqrt{r_1}\sqrt{b-a}}\left[-\frac{3}{8}\sin\frac{\theta_1}{2} - \frac{1}{8}\sin\frac{5\theta_1}{2}\right] + O(r_1).
\end{aligned} \tag{A6.24}$$

Also, σ_{xy} is given by

$$\begin{aligned}
\sigma_{xy} &= \mathrm{Im}\left[\varphi_1'(z) - \varphi_1'(\bar{z}) + 2\overline{\varphi_1'(z)} + (\bar{z} - z)\,\overline{\varphi_1''(z)} + X'(\bar{z})\right] \qquad \text{(A6.25)} \\
&= \frac{Y'(a)}{4\sqrt{r_1}\sqrt{b-a}}\left[-\cos\frac{\theta_1}{2} - \cos\frac{\theta_1}{2} + 2\cos\frac{\theta_1}{2} - \sin\theta_1\sin\frac{3\theta_1}{2}\right] + O(r_1) \\
&= \frac{Y'(a)}{\sqrt{r_1}\sqrt{b-a}}\left[-\frac{1}{8}\cos\frac{\theta_1}{2} + \frac{1}{8}\cos\frac{5\theta_1}{2}\right] + O(r_1).
\end{aligned}$$

Consequently, the stress fields σ_{yy}, σ_{xx} and σ_{xy} near crack tip a have been determined as follows

$$\begin{aligned}
\sigma_{yy} &= \frac{Y'(a)}{\sqrt{r_1}\sqrt{b-a}}\left(-\frac{5}{8}\sin\frac{\theta_1}{2} + \frac{1}{8}\sin\frac{5\theta_1}{2}\right) + O(r_1), \qquad \text{(A6.26)} \\
\sigma_{xx} &= \frac{Y'(a)}{\sqrt{r_1}\sqrt{b-a}}\left(-\frac{3}{8}\sin\frac{\theta_1}{2} - \frac{1}{8}\sin\frac{5\theta_1}{2}\right) + O(r_1),
\end{aligned}$$

$$\sigma_{xy} = \frac{Y'(a)}{\sqrt{r_1}\sqrt{b-a}}\left(-\frac{1}{8}\cos\frac{\theta_1}{2}+\frac{1}{8}\cos\frac{5\theta_1}{2}\right)+O\left(r_1\right).$$

In the similar way of (A6.6), the stress potential $Y(z)$ near crack tip b can also be determined in the following form

$$\begin{aligned} Y(z) &= Y(b)+Y'(b)\left(z-b\right)+\frac{1}{2!}Y''(b)\left(z-b\right)^2+... \\ &\quad +\frac{1}{n!}Y^{(n)}(b)\left(z-b\right)^n+... \\ &= Y(b)+Y'(b)r_2e^{i\theta_2}+\frac{1}{2!}Y''(b)r_2^2e^{i2\theta_2}+...+\frac{1}{n!}Y^{(n)}(b)r_2^ne^{in\theta_2}+...\,, \end{aligned} \tag{A6.27}$$

where $(z-b)=r_2e^{i\theta_2}$ $(-\pi\leq\theta_2\leq\pi)$. And

$$Y(z) = Y(b)+Y'(b)r_2e^{i\theta_2}, \tag{A6.28}$$

$$Y'(z) = Y'(b)+Y''(b)r_2e^{i\theta_2}, \tag{A6.29}$$

$$Y''(z) = Y''(b)+Y'''(b)r_2e^{i\theta_2}. \tag{A6.30}$$

Thus,

$$\begin{aligned} \varphi_1'(z) &= \left[\frac{1}{4}\frac{(a+b-2z)}{(z-a)^{\frac{3}{2}}(z-b)^{\frac{3}{2}}}Y(z)+\frac{1}{2(z-a)^{\frac{1}{2}}(z-b)^{\frac{1}{2}}}Y'(z)+\frac{1}{2}X'(z)\right]_{around\ z=b} \\ &= \frac{1}{4}\frac{-(b-a)}{(b-a)^{\frac{3}{2}}(r_2e^{i\theta_2})^{\frac{3}{2}}}\left[Y(b)+Y'(b)r_2e^{i\theta_2}\right] \\ &\quad +\frac{1}{2(b-a)^{\frac{1}{2}}(r_2e^{i\theta_2})^{\frac{1}{2}}}\left[Y'(b)+Y''(b)r_2e^{i\theta_2}\right]+\frac{1}{2}X'(z). \end{aligned} \tag{A6.31}$$

Taking only leading-order items yields

$$\varphi_1'(z)=\frac{1}{4}\frac{Y'(b)}{\sqrt{r_2e^{i\frac{\theta_2}{2}}}\sqrt{b-a}}=\frac{1}{4}\frac{Y'(b)}{\sqrt{r_2}\sqrt{b-a}}\left(\cos\frac{\theta_2}{2}-i\sin\frac{\theta_2}{2}\right). \tag{A6.32}$$

In addition,

$$\varphi_1'(\bar{z}) = \frac{1}{4}\frac{Y'(b)}{\sqrt{r_2}e^{-i\frac{\theta_2}{2}}\sqrt{b-a}} = \frac{1}{4}\frac{Y'(b)}{\sqrt{r_2}\sqrt{b-a}}\left(\cos\frac{\theta_2}{2} + i\sin\frac{\theta_2}{2}\right). \tag{A6.33}$$

Thus, the $\varphi_1'(z)$ has single value at the crack face since $\operatorname{Re}[\varphi_1'(z)] = \operatorname{Re}[\varphi_1'(\bar{z})]$ and $\operatorname{Im}[\varphi_1'(z)] = -\operatorname{Im}[\varphi_1'(\bar{z})]$.

Similar to (A6.17) and (A6.18),

$$\begin{aligned}
(z-\bar{z})\,\varphi_1''(z) &= 2r_2 i\sin\theta_2 \begin{bmatrix} \frac{(a+b-2z)}{2(z-a)^{\frac{3}{2}}(z-b)^{\frac{3}{2}}}\left[Y'(b) + Y''(b)r_2e^{i\theta_2}\right] \\ -\frac{1}{2(z-a)^{\frac{3}{2}}(z-b)^{\frac{3}{2}}}\left[Y(b) + Y'(b)r_2e^{i\theta_2}\right] \\ +\frac{3(a+b-2z)^2}{8(z-a)^{\frac{5}{2}}(z-b)^{\frac{5}{2}}}\left[Y(b) + Y'(b)r_2e^{i\theta_2}\right] \\ +\frac{1}{2(z-a)^{\frac{1}{2}}(z-b)^{\frac{1}{2}}}\left[Y''(b)\right] + \frac{1}{2}X''(z) \end{bmatrix}_{around\ z=b} \\
&= \frac{-2i\sin\theta_2}{2(b-a)^{\frac{1}{2}}(r_2)^{\frac{1}{2}}(e^{i\theta_2})^{\frac{3}{2}}}Y'(b) + \frac{-2\,(r_2)^{\frac{1}{2}}\,i\sin\theta_2}{2(b-a)^{\frac{1}{2}}(e^{i\theta_2})^{\frac{1}{2}}}Y''(b) \\
&\quad -\frac{2(r_2)^{\frac{1}{2}}i\sin\theta_2}{2(b-a)^{\frac{3}{2}}(e^{i\theta_2})^{\frac{1}{2}}}Y'(b) + \frac{3*2i\sin\theta_2}{8(b-a)^{\frac{1}{2}}(r_2)^{\frac{1}{2}}(e^{i\theta_2})^{\frac{3}{2}}}Y'(b) \\
&\quad +\frac{2(r_2)^{\frac{1}{2}}i\sin\theta_2}{2(b-a)^{\frac{1}{2}}(e^{i\theta_2})^{\frac{1}{2}}}Y''(b) + \frac{1}{2}X''(z)2r_2 i\sin\theta_2.
\end{aligned} \tag{A6.34}$$

Note that $Y(b) = 0$ by definition. By taking leading order singularity only, we get

$$(z-\bar{z})\,\varphi_1''(z) = -\frac{Y'(b)e^{i\frac{\pi}{2}}\sin\theta_2}{4\sqrt{r_2}\sqrt{b-a}\,e^{i\frac{3}{2}\theta_2}} = -\frac{Y'(b)\sin\theta_2}{4\sqrt{r_2}\sqrt{b-a}}e^{i\left(\frac{\pi}{2}-\frac{3}{2}\theta_2\right)}. \tag{A6.35}$$

By taking $\theta_2 + 2\pi$, the single valueness of $(z-\bar{z})\,\varphi_1''(z)$ at a branch cut is obtained in the following expression

$$\begin{aligned}
\frac{\pi}{2} - \frac{3}{2}\theta_2 &= \frac{\pi}{2} - \frac{3}{2}(\theta_2 + 2\pi) = -\frac{\pi}{2} - \frac{3}{2}\theta_2, \\
e^{-i\left(-\frac{\pi}{2}-\frac{3}{2}\theta_2\right)} &= \cos\left(-\frac{\pi}{2} - \frac{3}{2}\theta_2\right) - i\sin\left(-\frac{\pi}{2} - \frac{3}{2}\theta_2\right) = -\sin\frac{3\theta_2}{2} + i\cos\frac{3\theta_2}{2},
\end{aligned}$$

$$e^{i\left(-\frac{\pi}{2}-\frac{3}{2}\theta_2\right)} = \cos\left(-\frac{\pi}{2}-\frac{3}{2}\theta_2\right) + i\sin\left(-\frac{\pi}{2}-\frac{3}{2}\theta_2\right) = -\sin\frac{3\theta_2}{2} - i\cos\frac{3\theta_2}{2}.$$

Thus,

$$(z-\bar{z})\,\varphi_1''(z) = -\frac{Y'(b)\sin\theta_2}{4\sqrt{r_2}\sqrt{b-a}}\left(-\sin\frac{3\theta_2}{2} - i\cos\frac{3\theta_2}{2}\right). \tag{A6.36}$$

By substituting (A6.32), (A6.33) and (A6.36) into (4.7), we get stress fields around crack tip b as follows

$$\begin{aligned}
\sigma_{yy} &= \frac{Y'(b)}{\sqrt{r_2}\sqrt{b-a}}\left(\frac{5}{8}\cos\frac{\theta_2}{2} - \frac{1}{8}\cos\frac{5\theta_2}{2}\right) + O\left(r_2\right), && \text{(A6.37)}\\
\sigma_{xx} &= \frac{Y'(b)}{\sqrt{r_2}\sqrt{b-a}}\left(\frac{3}{8}\cos\frac{\theta_2}{2} + \frac{1}{8}\cos\frac{5\theta_2}{2}\right) + O\left(r_2\right),\\
\sigma_{xy} &= \frac{Y'(b)}{\sqrt{r_2}\sqrt{b-a}}\left(-\frac{1}{8}\sin\frac{\theta_2}{2} + \frac{1}{8}\sin\frac{5\theta_2}{2}\right) + O\left(r_2\right).
\end{aligned}$$

The $Y(z)$, (2.31) determined in Chapter 2 yields

$$\begin{aligned}
Y(z) &= \sum_{k=-\infty}^{\infty} f_k z^k = ...f_{-2}z^{-2} + f_{-1}z^{-1} + f_0 + f_1 z^1 + f_2 z^2 ... , && \text{(A6.38)}\\
Y'(z) &= \sum_{k=-\infty}^{\infty} k f_k z^{k-1} = ... - 2f_{-2}z^{-3} - f_{-1}z^{-2} + 0 + f_1 + 2f_2 z ... .
\end{aligned}$$

Near the crack tip a and b,

$$Y'(a) = \sum_{k=-\infty}^{\infty} k f_k a^{k-1}, Y'(b) = \sum_{k=-\infty}^{\infty} k f_k b^{k-1}. \tag{A6.39}$$

Finally, with (A6.39) and $b-a=2l$ substituted into (6.26), we have stress fields at crack tips a in the following forms

$$\begin{aligned}
\sigma_{yy} &= \frac{1}{\sqrt{r_1}\sqrt{2l}}\left(-\frac{5}{8}\sin\frac{\theta_1}{2}+\frac{1}{8}\sin\frac{5\theta_1}{2}\right)\left[\sum_{k=-\infty}^{\infty} kf_k a^{k-1}\right]+O\left(r_1\right), \qquad (4.8)\\
\sigma_{xx} &= \frac{1}{\sqrt{r_1}\sqrt{2l}}\left(-\frac{3}{8}\sin\frac{\theta_1}{2}-\frac{1}{8}\sin\frac{5\theta_1}{2}\right)\left[\sum_{k=-\infty}^{\infty} kf_k a^{k-1}\right]+O\left(r_1\right),\\
\sigma_{xy} &= \frac{1}{\sqrt{r_1}\sqrt{2l}}\left(-\frac{1}{8}\cos\frac{\theta_1}{2}+\frac{1}{8}\cos\frac{5\theta_1}{2}\right)\left[\sum_{k=-\infty}^{\infty} kf_k a^{k-1}\right]+O\left(r_1\right),
\end{aligned}$$

where $(z-a)=r_1e^{i\theta_1}$ $(0\leq\theta_1\leq 2\pi)$.

Also, the stresses, (A6.37) near crack tip b are determined as

$$\begin{aligned}
\sigma_{yy} &= \frac{1}{\sqrt{r_2}\sqrt{2l}}\left(\frac{5}{8}\cos\frac{\theta_2}{2}-\frac{1}{8}\cos\frac{5\theta_2}{2}\right)\left[\sum_{k=-\infty}^{\infty} kf_k b^{k-1}\right]+O\left(r_2\right), \qquad (4.9)\\
\sigma_{xx} &= \frac{1}{\sqrt{r_2}\sqrt{2l}}\left(\frac{3}{8}\cos\frac{\theta_2}{2}+\frac{1}{8}\cos\frac{5\theta_2}{2}\right)\left[\sum_{k=-\infty}^{\infty} kf_k b^{k-1}\right]+O\left(r_2\right),\\
\sigma_{xy} &= \frac{1}{\sqrt{r_2}\sqrt{2l}}\left(-\frac{1}{8}\sin\frac{\theta_2}{2}+\frac{1}{8}\sin\frac{5\theta_2}{2}\right)\left[\sum_{k=-\infty}^{\infty} kf_k b^{k-1}\right]+O\left(r_2\right),
\end{aligned}$$

where $(z-b)=r_2e^{i\theta_2}$ $(-\pi\leq\theta_2\leq\pi)$.