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Computing A-Homotopy Groups of Graphs Using Coverings and Lifting Properties

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Computing A-Homotopy Groups of Graphs Using Coverings and Lifting Properties

by

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Abstract

In classical homotopy theory, two spaces are homotopy equivalent if one space can be continuously deformed into the other. This theory, however, does not respect the discrete nature of graphs. For this reason, a discrete homotopy theory that recognizes the difference between the vertices and edges of a graph was invented, called A-homotopy theory. In classical homotopy theory, covering spaces and lifting properties are often used to compute the fundamental group of a space. In this thesis, we develop the lifting properties for A-homotopy theory. Using a covering graph and these lifting properties, we compute the fundamental group of the cycle C_5 and use this computation to show that C_5 is not contractible in this theory, even though the cycles C_3 and C_4 are contractible.

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To my father who taught me to love learning.

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List of Figures and Illustrations

List of Symbols, Abbreviations and Nomenclature

$\mathbb{Z}$	Set of integers
$\mathbb{N}$	Set of natural numbers starting at 1
$V(G)$	Vertex set of a graph G
$E(G)$	Edge set of a graph G
$\{v, w\}$	Edge incident to the vertices v and w
C_k	Cycle on k vertices
$\mathbf{1}_G$	Identity map from G to G
c_{v_0}	Constant map from G to G
$G_1 \Box G_2$	Cartesian product of G_1 and G_2
I_n	Path of length n
I_m^n	n-fold Cartesian product of I_m
δI_m^n	Boundary of I_m^n
$G[S]$	Induced subgraph of G on the vertex set S
$f \simeq_A g$	$f : G_1 \to G_2$ is A-homotopic to $g : G_1 \to G_2$ by a homotopy $H : I_n \Box G_1 \to G_2$
$*$	Graph with a single vertex $*$ and no edges
$A_1(G, v_0)$	Fundamental group of G
$A_n(G, v_0)$	n^{th} A-homotopy group of G
I_∞	Infinite path
$m_0(f, \varepsilon i)$	Integer that $f : I_\infty^n \to G$ stabilizes at on the i^{th}-axis in the ε direction
$f \cdot g$	Concatenation of f and g
$f \cdot_i g$	Concatenation of f and g on the i^{th}-axis
$\alpha_{\varepsilon i}^n$	Face map in the εi direction
β_i^n	Degeneracy map on the i^{th}-axis
$C_n(G)$	Set of stable graph homomorphisms from I_∞^n to G
$f \sim g$	$f : I_\infty^n \to G$ is A-homotopic to $g : I_\infty^n \to G$ by a homotopy $H : I_\infty^{n+1} \to G$
$B_n(G, v_0)$	Set of stable graph homomorphisms from I_∞^n to G based at v_0
p_{v_0}	Constant map from I_∞ to G
$\bar{f}$	Inverse of a graph homomorphism $f : I_\infty \to G$
$\widetilde{f}$	Lift of a graph homomorphism f
$N[x]$	Closed neighborhood of a vertex x
f_*	Induced map of the graph homomorphism f

Chapter 1

Introduction

In algebraic topology, we consider two *topological spaces*, or just *spaces* for short, to be the same if one can be continuously deformed into the other. For example, the shape of a coffee mug can be continuously deformed into the shape of a doughnut by gradually shifting the cup part of the mug onto the handle to eventually form the doughnut. Thus to a topologist, these two shapes represent the same space.

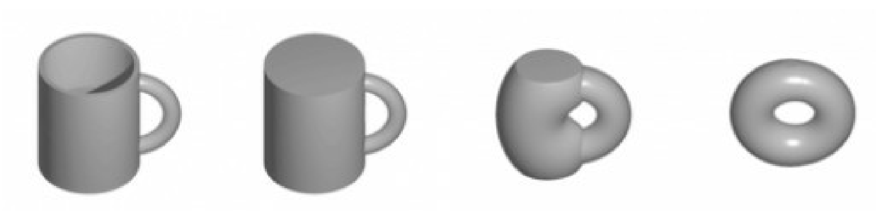

Figure 1.1: Deformation of coffee mug to doughnut [12]

A *graph* consists of a set of vertices and a set of edges where each edge is an unordered pair of vertices. In figures, the vertices of a graph are represented as points and the edges as line segments between the vertices of the unordered pair. The vertices often represent a set of objects or ideas, while the edges represent a relationship between these objects or ideas. When considering a graph as a space, that is, as a subset of $\mathbb{R}^2$ or $\mathbb{R}^3$, any continuous deformation would ignore the inherent discrete structure of the graph, not distinguishing between the vertices and edges. All connected graphs can be continuously deformed into a *bouquet of loops*, that is, a single vertex with some number of edges having both endpoints at that vertex. For example in the Figure 1.2, we can continuously shorten the red edges in the graph on the left until we have the graph on the right, a bouquet of five loops. This can be done for every connected graph by continuously contracting the edges of a spanning tree of the original graph [9, Proposition 1A.2]. Thinking of all graphs as being equivalent to

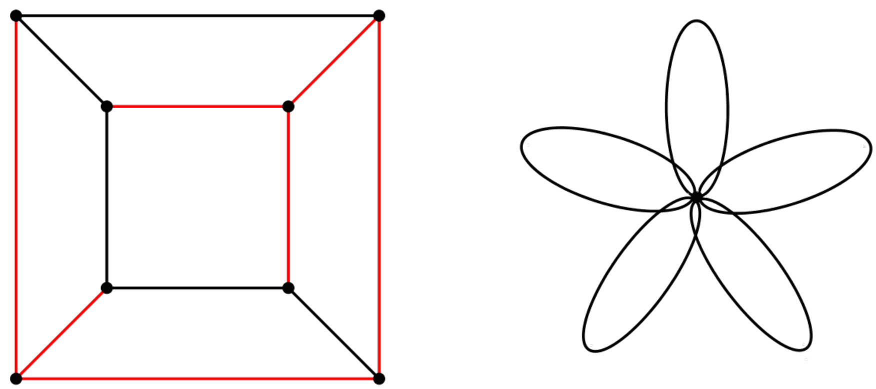

Figure 1.2: Bouquet of loops

bouquets of loops, however, is not particularly useful because this equivalence is too coarse, placing graphs in the same equivalence class that should be kept distinct. For this reason, discrete homotopy theories were developed that would respect the structure of graphs, i.e., the vertices and edges, and give us more relevant information about graphs by applying ideas from algebraic topology in a combinatorial way [6, 8].

In this thesis, we focus on A-homotopy theory, a theory first developed by Atkin (see [5]). In a space, the properties that are preserved by continuous deformation are called *invariants*. For example, the hole created by the handle of a coffee mug which is deformed into the hole of a doughnut is an invariant of the space, because no continuous deformation can remove the hole. Another type of invariant of a space is given by examining the loops of a space based at a distinguished point of the space. A *loop* of a space X with a distinguished point x_0 is a continuous map $f : [0, 1] \to X$ with $f(0) = f(1) = x_0$. The set of equivalence classes of these loops is known as the *fundamental group of the space*. In a graph, these loops do not distinguish between the vertices and edges of the graph, and thus cannot record the combinatorial information of the graph. Hence, the classical fundamental group cannot find very informative or useful invariants for graphs.

One of the most basic graphs, a *cycle*, is a set of vertices connected by edges in a closed chain [11]. The loops mentioned earlier find cycle subgraphs of a graph, which is useful, but we need a way of keeping track of how many vertices the cycle contains. For example, the loops would detect a 3-cycle, that is, a cycle on three vertices, but each vertex of a 3-cycle is connected to the other two, so it should not be viewed as a 'hole' in the graph. In A-homotopy theory, we look for areas where there are fewer edges connecting the vertices of the graph. Since graphs are often used to represent real world networks and systems, these areas with fewer edges can either point to missing information in the network or areas where the network could be made more efficient by adding connections. To find some of these areas, we examine the A-homotopy theory fundamental group of the graph, that is, the set of equivalence classes of paths (a sequence of edges and a sequence of vertices) mapped into the graph with both endpoints of the path mapped to the distinguished vertex of the graph.

In classical homotopy theory, all cycles can be continuously deformed into the circle. In A-homotopy theory, however, the 3-cycle C_3 and 4-cycle C_4 are contractible, that is, they are considered to be the same as a single vertex, and all cycles on five or more vertices are not contractible (see Propositions 3.6 and 3.7). In [4, Proposition 5.12], Barcelo, Kramer,

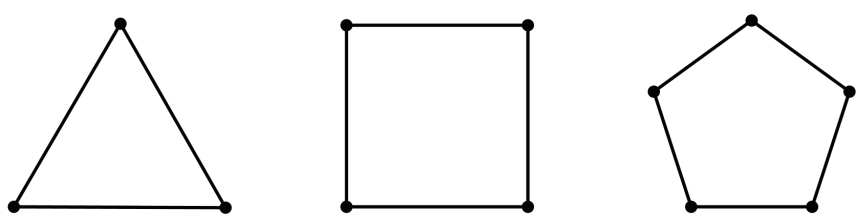

Figure 1.3: Cycles C_3, C_4, and C_5

Laubenbacher, and Weaver show this by proving that attaching 2-cells to the 3-cycles and 4-cycles of graphs, and using classical homotopy theory on the spaces created, is equivalent to using A-homotopy theory on the original graphs.

In this thesis, we explore the question of why the 3-cycle and 4-cycle are contractible in A-homotopy theory, but the cycles on five or more vertices are not contractible. In classical homotopy theory, the circle is one of the first spaces for which we compute the fundamental group. Since the cycles C_k, for $k \geq 5$, are not contractible, they are the best candidates for graphs that have a behavior analogous to the behavior of the circle as a topological space. For this reason, we prove that, similar to the fundamental group of the circle, there is an isomorphism between the A-homotopy theory fundamental group of C_5 and the integers $\mathbb{Z}$, using combinatorial methods within A-homotopy theory. This computation implies that C_5 is not contractible in a direct way, and the proof fails for C_3 and C_4 in a way that lends insight into our question.

The methods used in the computation of the A-homotopy fundamental group of $\mathcal{C}_5$ are inspired by the methods used in the computation of the fundamental group of the circle in classical homotopy theory found in [9], namely, covering spaces and lifting properties. While an analogous definition of covering spaces can be found in the literature for graphs (see [10]), no such analogous theory of lifting properties exists for graphs. Thus we develop these lifting properties in Chapter 6 of this thesis. Since covering spaces and lifting properties are one of the frequently-used methods to compute the homotopy groups of spaces in classical homotopy theory, these analogous lifting properties are a significant contribution to A-homotopy theory. While developing the *Homotopy Lifting Property* (6.11), we found that it does not hold for graphs containing 3-cycles or 4-cycles. Since the *Homotopy Lifting Property* (6.11) is used in the computation of the A-homotopy fundamental group of C_5 (Theorem 7.8), this same method cannot be used to compute the A-homotopy fundamental group of C_3 or C_4. In fact, an entirely different method is used to compute the A-homotopy fundamental groups of contractible graphs (Theorem 7.1), which is included in Chapter 7. The fact that the *Homotopy Lifting Property* (6.11) does not hold for C_3 or C_4, and that the A-homotopy fundamental groups of C_3 and C_4 must be computed in a different way than the A-homotopy fundamental groups of C_k, for $k \geq 5$, helps us better understand why the cycles C_3 and C_4 have such interesting behavior in A-homotopy theory.

In Chapter 2, we introduce the basic definitions of graphs and graph homomorphisms that are used in A-homotopy theory. Each definition is followed by an example and figure. We also establish basic notation that is used throughout this thesis.

In Chapter 3, we provide an introduction to A-homotopy theory, summarizing the main definitions found in the literature. More specifically, we provide the basic definitions of A-homotopy theory along with examples and figures depicting those examples. We also give the precise definitions of the A-homotopy fundamental group and the n^{th} A-homotopy group. In this chapter, we also include proofs that the A-homotopy relation is an equivalence relation and that the cycles C_3 and C_4 are A-contractible.

In Chapter 4, we provide an alternate definition for A-homotopy theory first defined in [3], which establishes an equivalence relation on the set of graph homomorphisms from infinite paths into a graph G. These graph homomorphisms must be *active* (p. 29) for only a finite region of the infinite path. This alternate definition is essential, because the original homotopy relation, included in Chapter 3, only compares graph homomorphisms with the same domain, but the A-homotopy fundamental group of a graph G must compare graph homomorphisms from paths of any length into G. With this new A-homotopy relation, we define a new set $B_1(G, v_0)/\sim$, which is isomorphic to the A-homotopy fundamental group defined in Chapter 3. We use the set $B_1(G, v_0)/\sim$ as the A-homotopy fundamental group of G in all of the remaining chapters.

In Chapter 5, we show that the set $B_1(G, v_0)/\sim$ is a group. While this result is stated in the literature, a full proof does not appear. This is likely because the proof is long and highly technical. For this reason, we include the proof here, and this constitutes part of the original work of this thesis.

In Chapter 6, we provide the definition of a covering graph along with examples and develop the lifting properties for A-homotopy theory. These properties include the *Path*

Lifting Property (6.10), the *Homotopy Lifting Property* (6.11), and the *Lifting Criterion* (6.18). These theorems are the main results of this thesis. We also include examples that illustrate why the *Homotopy Lifting Property* (6.11) does not hold for graphs containing 3-cycles or 4-cycles.

In Chapter 7, we conclude this thesis by showing that the A-homotopy fundamental groups of all A-contractible graphs is zero and by using a covering graph and the lifting properties to show that the A-homotopy fundamental group of the cycle C_5 is isomorphic to $\mathbb{Z}$. This implies that the cycle C_5 is not contractible, even though C_3 and C_4 are A-contractible. Indeed, the cycle C_k is not A-contractible for any $k \geq 5$.

Chapter 2

Graphs and Graph Homomorphisms

Before introducing A-homotopy theory, we need to consider some basic definitions and lemmas that are the building blocks of this discrete homotopy theory. Since graphs are the main objects that we consider, we start with a more rigorous definition of a graph.

Definition 2.1. A *graph* G consists of a set of vertices, $V(G)$, and a set of edges, $E(G)$, where each edge in $E(G)$ is an unorder pair of distinct vertices. Let $\{v, w\}$ denote an edge between the vertices v and w.

This definition ensures that the graphs we consider are *simple*, that is, the graphs do not have more than one edge connecting the same two vertices or edges with both endpoints at the same vertex. If $\{v, w\} \in E(G)$, then we say that the vertices v and w are *adjacent* and the edge $\{v, w\}$ is *incident* to the vertices v and w. Some graphs we consider have one selected vertex called a *distinguished vertex*, even when not explicitly stated. We denote a graph G with distinguished vertex v by (G, v). In figures, this distinguished vertex will generally be colored green.

In the following definitions, we use the notation $G_1, G_2, \ldots$ for simple graphs with distinguished vertices, $v_1, v_2, \ldots$ respectively. Throughout this thesis, we use the graphs (S, x) and (T, a) depicted in Figure 2.1 as examples. Here, S is a 3-cycle with particular labels. We reserve the notation C_3 for the unlabeled graph with the same shape.

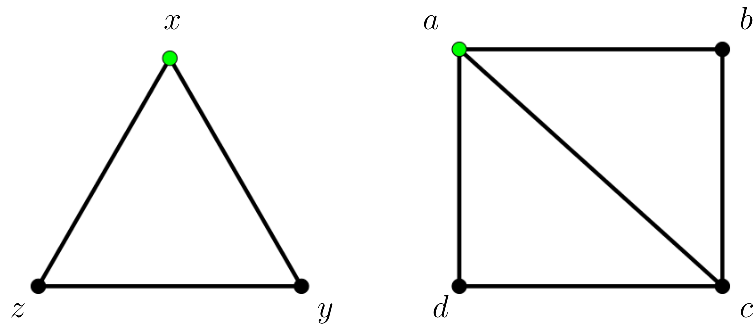

Figure 2.1: Graphs S and T

In classical homotopy theory, we examine continuous maps from topological spaces to topological spaces. In A-homotopy theory, we need a discrete mapping that respects the structure of the graphs.

Definition 2.2. [3, Definition 2.1(2)] A *graph homomorphism* $f : G_1 \to G_2$ is a map of sets$V(G_1) \to V(G_2)$ such that, if $\{u, v\} \in E(G_1)$, then either $f(u) = f(v)$ or $\{f(u), f(v)\} \in E(G_2)$, that is, adjacent vertices in G_1 are mapped to the same vertex of G_2 or adjacent vertices of G_2.

This definition is slightly altered from the standard graph theory definition of a graph homomorphism [7, p. 3]. In this version, adjacent vertices can always be mapped to the same vertex.

Example 2.3. Let the identity map $\mathbf{1}_G : G \to G$ be defined by $\mathbf{1}_G(v) = v$ for all $v \in V(G)$. This map $\mathbf{1}_G$ is a graph homomorphism, because if $\{v, w\} \in E(G)$, then $\{f(v), f(w)\} = \{v, w\} \in E(G)$.

Example 2.4. Given $v_0 \in V(G)$, let the constant map $c_{v_0} : G \to G$ be defined by $c_{v_0}(x) = v_0$ for all $x \in V(G)$. This map c_{v_0} is a graph homomorphism, because if $\{u, w\} \in E(G)$, then $c_{v_0}(u) = v_0 = c_{v_0}(w)$.

Example 2.5. Let the vertex set maps $f : S \to T$ and $g : T \to S$ be defined by

$$\begin{aligned} f(x) &= a & \qquad & g(a) = x \\ f(y) &= d & \text{and} \qquad & g(b) = y \\ f(z) &= c & & g(c) = z \\ & & & g(d) = y. \end{aligned}$$

It is routine to verify that both of these set maps f and g are graph homomorphisms.

Definition 2.6. A graph G' is a *subgraph* of the graph G if $V(G') \subseteq V(G)$ and $E(G') \subseteq E(G)$, where each unordered pair of $E(G')$ only contains vertices of $V(G')$.

Definition 2.7. [3, Definition 2.1(3)] Let G'_1 be a subgraph of G_1 and G'_2 be a subgraph of G_2. A *relative graph homomorphism* $f : (G_1, G'_1) \to (G_2, G'_2)$ is a graph homomorphism $f : G_1 \to G_2$ which restricts to a graph homomorphism $f|_{G'_1} : G'_1 \to G'_2$.

We use relative graph homomorphisms to ensure that the distinguished vertex of the first graph is mapped to the distinguished vertex of the second graph.

Definition 2.8. [4, Definition 5.1(4)] A *based graph homomorphism* $f : (G_1, v_1) \to (G_2, v_2)$ is a relative graph homomorphism $f : G_1 \to G_2$ that maps the distinguished vertex v_1 to the distinguished vertex v_2.

Example 2.9. Consider the graph homomorphisms f and g from Example 2.5. Since $f(x) = a$ and $g(a) = x$, both f and g are based graph homomorphisms.

We assume that all graph homomorphisms are based, unless otherwise specified. Next, we show that the composition of two based graph homomorphisms is also a based graph homomorphism.

Lemma 2.10 (**Composition Lemma**). *If $f : (G_1, v_1) \to (G_2, v_2)$ and $g : (G_2, v_2) \to (G_3, v_3)$ are graph homomorphisms, then the composition $g \circ f : (G_1, v_1) \to (G_3, v_3)$ is a graph homomorphism.*

Proof. Let G_1, G_2, and G_3 be simple graphs and $f : G_1 \to G_2$ and $g : G_2 \to G_3$ be graph homomorphisms. Suppose $\{u, w\} \in E(G_1)$. Since f is a graph homomorphism, either $\{f(u), f(w)\} \in E(G_2)$ or $f(u) = f(w)$.

- **Case 1:** Suppose $\{f(u), f(w)\} \in E(G_2)$, that is, the vertex $f(u)$ is adjacent to $f(w)$. Since g is a graph homomorphism, either $\{g(f(u)), g(f(w))\} \in E(G_3)$ or $g(f(u)) = g(f(w))$. Thus $\{(g \circ f)(u), (g \circ f)(w)\} \in E(G_3)$ or $(g \circ f)(u) = (g \circ f)(w)$.

- **Case 2:** Suppose $f(u) = f(w)$, that is, $f(u)$ and $f(w)$ are the same vertex. Then $g(f(u)) = g(f(w))$. Thus $(g \circ f)(u) = (g \circ f)(w)$.

Thus $g \circ f$ is a graph homomorphism. Also, $(g \circ f)(v_1) = g(f(v_1)) = g(v_2) = v_3$. Thus $g \circ f$ is a based graph homomorphism. □

In classical homotopy theory, we frequently use the product of two spaces. In A-homotopy theory, we use a discrete version of this product that produces a graph.

Definition 2.11. The *Cartesian product* of the graphs G_1 and G_2, denoted $G_1 \Box G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$. There is an edge between the vertices (u_1, u_2) and (w_1, w_2) if either $u_1 = w_1$ and $\{u_2, w_2\} \in E(G_2)$ or $u_2 = w_2$ and $\{u_1, w_1\} \in E(G_1)$.

By default, the distinguished vertex of the Cartesian product of two graphs G_1 and G_2 is (v_1, v_2), the 2-tuple with the distinguished vertices of each separate graph.

Example 2.12. The Cartesian product of the graphs S and T is illustrated in Figure 2.2. If you move the vertices of the graphs S and T into a straight line, you can see that S is repeated horizontally and T is repeated vertically in the Cartesian product. The edges of the copies of T are shown in red. The distinguished vertex of $S \Box T$ is (x, a) and shown in green.

In classical homotopy theory, we continuously deform maps over the unit interval, and when forming the fundamental group, we map the unit interval into the space. In A-

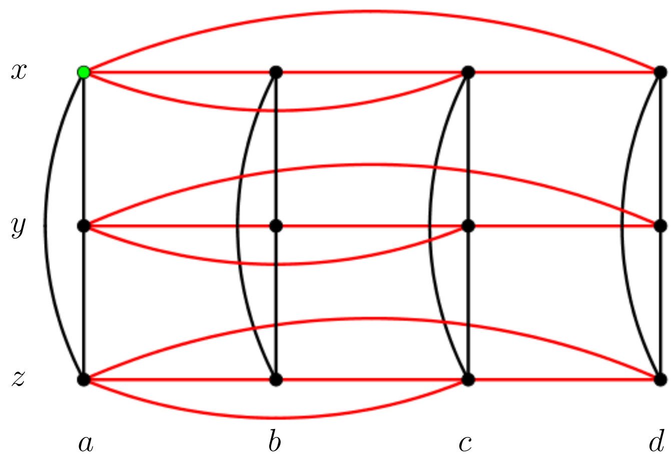

Figure 2.2: The Cartesian product of S and T

homotopy theory, in order to better distinguish between vertices and edges in the graphs that we examine, we replace the unit interval with graphs known as paths.

Definition 2.13. [4, Definition 5.1(3)] Let I_n be a graph with $n+1$ vertices labeled $0, 1, \ldots, n$ and n edges $\{i-1, i\}$ for $1 \leq i \leq n$. This graph is referred to as a *path of length* n.

The distinguished vertex of a path of length n is the vertex 0, unless otherwise stated.

Example 2.14. The path of length five, I_5, is illustrated in Figure 2.3.

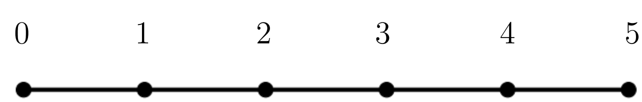

Figure 2.3: The graph I_5

We also use the path of infinite length, denoted by I_∞, with vertices labeled by the integers. This graph becomes important in Chapter 4. We now proceed to an introduction to A-homotopy theory.

Chapter 3

A-Homotopy Theory

In classical homotopy theory, two maps $f, g : A \to B$ are homotopic if we can take the product of the space A with the unit interval and continuously deform the map f into the map g over time from 0 to 1 [9, p. 3]. In A-homotopy theory, as mentioned in the previous chapter, we use the Cartesian product of a graph with a path I_n to deform one graph homomorphism into another graph homomorphism in a combinatorial way that keeps track the vertices and edges of the graph.

Definition 3.1. [4, Definition 5.2(1)] Let $f,\ g : (G_1, v_1) \to (G_2, v_2)$ be graph homomorphisms. If there exists an integer $n \in \mathbb{N}$ and a graph homomorphism $H : G_1 \square I_n \to G_2$ such that

- $H(v, 0) = f(v)$ for all $v \in V(G_1)$,
- $H(v, n) = g(v)$ for all $v \in V(G_1)$, and
- $H(v_1, i) = v_2$ for all $0 \leq i \leq n$,

then f and g are *A-homotopic*, denoted $f \simeq_A g$. The graph homomorphism H is called a *graph homotopy* from f to g.

Since $H(v_1, 0) = v_2$ by definiton, the graph homotopy H is a based graph homomorphism.

Example 3.2. Recall the graphs S and T from Figure 2.1. Let $f, g : (S, x) \to (T, a)$ be the graph homomorphisms defined by

$$\begin{aligned} f(x) &= a, & \qquad & g(x) = a, \\ f(y) &= d, & \text{and} \qquad & g(y) = b, \\ f(z) &= c, & & g(z) = c. \end{aligned}$$

Figure 3.1 depicts the graph homomorphisms f and g. The image under f of each vertex in S is shown in red, while the image under g of each vertex in S is shown in blue.

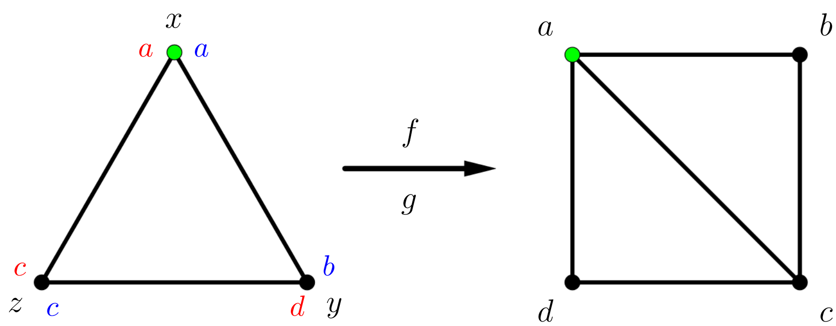

Figure 3.1: Graph homomorphisms from S to T

Define a map $H : S \square I_2 \to T$ by

$$\begin{aligned} H(x,0) &= a, & \quad H(x,1) &= a, & \quad H(x,2) &= a, \\ H(y,0) &= d, & \quad H(y,1) &= a, & \quad H(y,2) &= b, \\ H(z,0) &= c, & \quad H(z,1) &= c, & \quad H(z,2) &= c. \end{aligned}$$

Figure 3.2 depicts this map H with the image of each vertex shown in red. Then H is a graph homomorphism with $H(v, 0) = f(v)$ and $H(v, 2) = g(v)$ for all $v \in V(S)$, and $H(x, i) = a$ for all $0 \le i \le 2$. Thus H is a graph homotopy from f to g. However, H is not unique. It is only one of several possible graph homotopies.

We now show that this relation between graph homomorphisms is an equivalence relation on $\mathrm{Hom}((G_1, v_1),\ (G_2, v_2))$, the set of graph homomorphism from (G_1, v_1) to (G_2, v_2). We

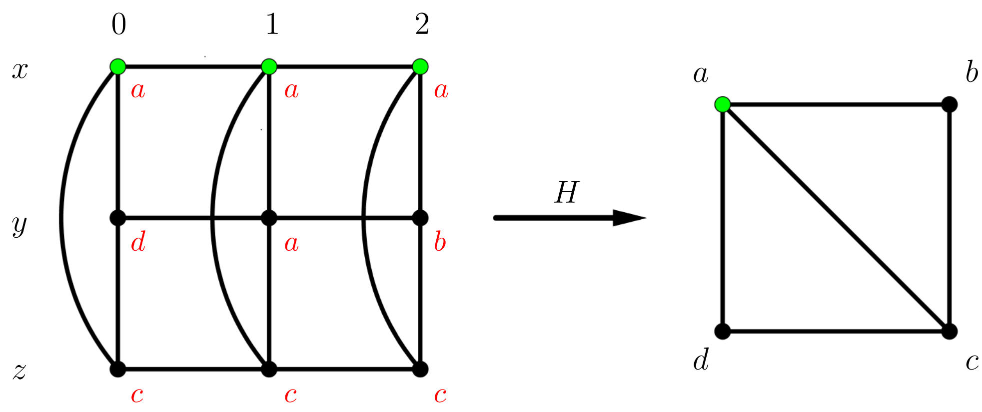

Figure 3.2: A graph homotopy H from f to g

often abbreviate this set as $\mathrm{Hom}_*(G_1, G_2)$.

Proposition 3.3. The A-homotopy relation $\simeq_A$ is an equivalence relation on $\mathrm{Hom}_*(G_1, G_2)$.

Proof. To show that $\simeq_A$ is reflexive, symmetric, and transitive, we define maps and show that each map is well-defined, is a graph homomorphism, and is a graph homotopy.

- $\simeq_A$ is reflexive.

  Let $f \in \mathrm{Hom}_*(G_1, G_2)$. To show that $f \simeq_A f$, define $H : G_1 \square I_1 \to G_2$ by

  $$H(v, i) \quad = \quad f(v) \quad \text{for all} \quad v \in V(G_1) \quad \text{and} \quad i \in \{0, 1\}.$$

  The map H is well-defined, since f is well-defined. We now show that H is a graph homomorphism. Suppose that $\{(u, j),\ (w, k)\} \in E(G_1 \square I_1)$. By the definition of the Cartesian product, either $u = w$ and $\{j, k\} \in E(I_1)$, or $\{u, w\} \in E(G_1)$ and $j = k$.

  - If $u = w$ and $\{j, k\} \in E(I_1)$, then $\{(u, j),\ (w, k)\} = \{(u, j),\ (u, k)\}$. Since $H(v, i) = f(v)$ for all $i \in \{0, 1\}$, it follows that $H(u, j) = f(u)$ and $H(u, k) = f(u)$. Hence, $H(u, j) = H(u, k) = H(w, k)$.
  - Otherwise, $\{u, w\} \in E(G_1)$ and $j = k$. Hence, $\{(u, j),\ (w, k)\} = \{(u, j),\ (w, j)\}$. Since $j \in \{0, 1\}$, it follows that $H(u, j) = f(u)$ and $H(w, j) = f(w)$. Moreover, $\{u, w\} \in E(G_1)$ and f is a graph homomorphism, so $f(u) = f(w)$ or

$\{f(u), f(w)\} \in E(G_2)$. Thus $H(u,j) = H(w,j) = H(w,k)$ or $\{H(u,j), H(w,k)\} \in E(G_2)$.

Therefore, in both cases $H(u,j) = H(w,k)$ or $\{H(u,j), H(w,k)\} \in E(G_2)$ for each edge $\{(u,j),(w,k)\} \in E(G_1 \Box I_1)$, so H is a graph homomorphism. By the definition of H and since $f(v_1) = v_2$, it follows that $H(v,0) = f(v)$ and $H(v,1) = f(v)$ for all $v \in V(G_1)$, and $H(v_1, i) = v_2$ for all $i \in \{0,1\}$. Thus H is a graph homotopy from f to f, so $f \simeq_A f$.

- $\simeq_A$ is symmetric.

Let $f, g \in \mathrm{Hom}_*(G_1, G_2)$, and suppose $f \simeq_A g$. Then there exists an $n \in \mathbb{N}$ and a graph homomorphism $H_1 : G_1 \Box I_n \to G_2$ such that

$$\begin{aligned} H_1(v,0) &= f(v) \quad \text{for all} \quad v \in V(G_1), \\ H_1(v,n) &= g(v) \quad \text{for all} \quad v \in V(G_1), \\ H_1(v_1,i) &= v_2 \quad \text{for all} \quad i \in \{0,\dots,n\}. \end{aligned}$$

To show that $g \simeq_A f$, define $H_2 : G_1 \Box I_n \to G_2$ by

$$H_2(v,i) = H_1(v, n-i) \quad \text{for all} \quad v \in V(G_1) \text{ and } i \in \{0,\dots,n\}.$$

The map H_2 is well-defined, since H_1 is well-defined. We now show that H_2 is a graph homomorphism. Suppose $\{(u,j),(w,k)\} \in E(G_1 \Box I_n)$. By definition of the Cartesian product, either $u = w$ and $\{j,k\} \in E(I_n)$, or $\{u,w\} \in E(G_1)$ and $j = k$.

  - If $u = w$ and $\{j,k\} \in E(I_n)$, then $|j-k| = 1$. Thus, without loss of generality, we may assume that $k = j+1$, and hence,

$$H_2(u,j) = H_1(u, n-j) \quad \text{and} \quad H_2(w,k) = H_2(u,j+1) = H_1(u, n-j-1).$$

Since $\{(u, n-j), (u, n-j-1)\} \in E(G_1 \square I_n)$ for $0 \leq j < n$ and H_1 is a graph homomorphism, it follows that $H_1(u, n-j) = H_1(u, n-j-1)$ or $\{H_1(u, n-j), H_1(u, n-j-1)\} \in E(G_1)$. Thus $H_2(u, j) = H_2(w, k)$ or $\{H_2(u, j), H_2(w, k)\} \in E(G_2)$.

– Otherwise, $\{u, w\} \in E(G_1)$ and $j = k$, and hence,

$$H_2(u, j) = H_1(u, n-j) \quad \text{and} \quad H_2(w, k) = H_2(w, j) = H_1(w, n-j).$$

Since $\{(u, n-j), (w, n-j)\} \in E(G_1 \square I_n)$ for $0 \leq j \leq n$ and H_1 is a graph homomorphism, it follows that $H_1(u, n-j) = H_1(w, n-j)$ and $\{H_1(u, n-j), H_1(w, n-j)\} \in E(G_2)$. Thus $H_2(u, j) = H_2(w, k)$ or $\{H_2(u, j), H_2(w, k)\} \in E(G_2)$.

Therefore, in both cases $H_2(u, j) = H_2(w, k)$ or $\{H_2(u, j), H_2(w, k)\} \in E(G_2)$ for each edge $\{(u, j), (w, k)\} \in E(G_1 \square I_n)$, so H_2 is a graph homomorphism. By definition of H_1 and H_2,

$$\begin{aligned} H_2(v, 0) &= H_1(v, n-0) = H_1(v, n) = g(v) \quad \text{for all} \quad v \in V(G_1) \\ H_2(v, n) &= H_1(v, n-n) = H_1(v, 0) = f(v) \quad \text{for all} \quad v \in V(G_1), \\ H_2(v_1, i) &= H_1(v_1, n-i) = v_2 \quad \text{for all} \quad i \in \{0, \ldots, n\}. \end{aligned}$$

Thus H_2 is a graph homotopy from g to f, and hence, $g \simeq_A f$.

- $\simeq_A$ is transitive.

Let $f, g, h \in \mathrm{Hom}_*(G_1, G_2)$, and suppose $f \simeq_A g$ and $g \simeq_A h$. Then there exists an

$n \in \mathbb{N}$ and a graph homomorphism $H_1 : G_1 \Box I_n \to G_2$ such that

$$\begin{aligned} H_1(v,0) &= f(v) \quad \text{for all} \quad v \in V(G_1), \\ H_1(v,n) &= g(v) \quad \text{for all} \quad v \in V(G_1), \\ H_1(v_1,i) &= v_2 \quad \text{for all} \quad i \in \{0,\dots,n\}. \end{aligned}$$

Similarly, there exists an $m \in \mathbb{N}$ and a graph homomorphism $H_2 : G_1 \Box I_m \to G_2$ such that

$$\begin{aligned} H_2(v,0) &= g(v) \quad \text{for all} \quad v \in V(G_1), \\ H_2(v,m) &= h(v) \quad \text{for all} \quad v \in V(G_1), \\ H_2(v_1,i) &= v_2 \quad \text{for all} \quad i \in \{0,\dots,m\}. \end{aligned}$$

To show that $f \simeq_A h$, define $H_3 : G_1 \Box I_{n+m} \to G_2$ by

$$H_3(v,i) = \begin{cases} H_1(v,i) & \text{for} \quad 0 \le i \le n, \\ H_2(v,i-n) & \text{for} \quad n \le i \le n+m, \end{cases}$$

for all $v \in V(G_1)$. The map H_3 is well-defined, since $H_1(v,n) = H_2(v,0)$ for all $v \in V(G_1)$. We now show that H_3 is a graph homomorphism. Suppose $\{(u,j),(w,k)\} \in E(G_1 \Box I_{n+m})$. By definition of Cartesian product, either $u = w$ and $\{j,k\} \in E(I_{n+m})$, or $\{u,w\} \in E(G_1)$ and $j = k$.

– If $u = w$ and $\{j,k\} \in E(I_{n+m})$, then $|j-k| = 1$. Thus, without loss of generality, we may assume that $k = j+1$, and hence,

$$H_3(u,j) = \begin{cases} H_1(u,j) & \text{for} \quad 0 \le j \le n, \\ H_2(u,j-n) & \text{for} \quad n \le j \le n+m, \end{cases}$$

and

$$H_3(w,k) = H_3(u,j+1) = \begin{cases} H_1(u,j+1) & \text{for} \quad 0 \leq j \leq n, \\ H_2(u,j+1-n) & \text{for} \quad n \leq j \leq n+m. \end{cases}$$

Since $\{(u,j),(u,j+1)\} \in E(G_1 \Box I_n)$ for $0 \leq j < n$ and H_1 is a graph homomorphism, $H_1(u,j) = H_1(u,j+1)$ or $\{H_1(u,j), H_1(u,j+1)\} \in E(G_2)$. Thus $H_3(u,j) = H_3(w,k)$ or $\{H_3(u,j), H_3(w,k)\} \in E(G_2)$ for $0 \leq j < n$. Similarly, since $\{(u,j-n),(u,j+1-n)\} \in G_1 \Box I_m$ for $n \leq j < n+m$ and H_2 is a graph homomorphism, $H_2(u,j-n) = H_2(u,j+1-n)$ or $\{H_2(u,j-n), H_2(u,j+1-n)\} \in E(G_2)$. Therefore, $H_3(u,j) = H_3(w,k)$ or $\{H_3(u,j), H_3(w,k)\} \in E(G_2)$ for $n \leq j < n+m$.

– Otherwise, $\{u,w\} \in E(G_1)$ and $j = k$, and hence,

$$H_3(u,j) = \begin{cases} H_1(u,j) & \text{for} \quad 0 \leq j \leq n, \\ H_2(u,j-n) & \text{for} \quad n \leq j \leq n+m, \end{cases}$$

and

$$H_3(w,k) = H_3(w,j) = \begin{cases} H_1(w,j) & \text{for} \quad 0 \leq j \leq n, \\ H_2(w,j-n) & \text{for} \quad n \leq j \leq n+m. \end{cases}$$

Since $\{(u,j),(w,j)\} \in E(G_1 \Box I_n)$ for $0 \leq j < n$ and H_1 is a graph homomorphism, $H_1(u,j) = H_1(w,j)$ or $\{H_1(u,j), H_1(w,j)\} \in E(G_2)$. Thus $H_3(u,j) = H_3(w,k)$ or $\{H_3(u,j), H_3(w,k)\} \in E(G_2)$ for $0 \leq j < n$. Since $\{(u,j-n),(w,j-n)\} \in E(G_1 \Box I_m)$ for $n \leq j < n+m$ and H_2 is a graph homomorphism, $H_2(u,j-n) = H_2(w,j-n)$ or $\{H_2(u,j-n), H_2(w,j-n)\} \in E(G_2)$. Thus $H_3(u,j) = H_3(w,k)$ or $\{H_3(u,j), H_3(w,k)\} \in E(G_2)$ for $n \leq j < n+m$.

Therefore, in both cases $H_3(u,j) = H_3(w,k)$ or $\{H_3(u,j), H_3(w,k)\} \in E(G_2)$ for each edge $\{(u,j),(w,k)\} \in E(G_1 \Box I_{n+m})$, so H_3 is a graph homomorphism. By the definitions of H_1, H_2, and H_3,

$$\begin{aligned} H_3(v,0) &= H_1(v,0) = f(v) \quad \text{for all} \quad v \in V(G_1), \\ H_3(v,n+m) &= H_2(v,n+m-n) = H_2(v,m) = h(v) \quad \text{for all} \quad v \in V(G_1), \\ H_3(v_1,i) &= v_2 \quad \text{for all} \quad i \in \{0,\ldots,n+m\}. \end{aligned}$$

Thus H_3 is a graph homotopy from f to h, so $f \simeq_A h$.

Therefore, $\simeq_A$ is an equivalence relation on $\mathrm{Hom}_*(G_1, G_2)$. □

Just as in classical homotopy theory we seek to know when two spaces are homotopy equivalent, in A-homotopy theory we seek to know when two graphs are A-homotopy equivalent. The next definition is drawn directly from [9, p. 3], except with 'graph homomorphism' in the place of 'continuous map' and 'A-homotopic' in the place of 'homotopic'.

Definition 3.4. [4, Definition 5.2(2)] The graph homomorphism $f : G_1 \to G_2$ is an *A-homotopy equivalence* if there exists a graph homomorphism $g : G_2 \to G_1$ such that $f \circ g \simeq_A \mathbf{1}_{G_2}$ and $g \circ f \simeq_A \mathbf{1}_{G_1}$. In this case, the graphs G_1 and G_2 are *A-homotopy equivalent*.

We introduce one more definition in order to give two simple and relevant examples of A-homotopy equivalence. This definition is a slight modification of the definition of contractible found in [9, p. 4].

Definition 3.5. A graph G is *A-contractible* if G is A-homotopy equivalent to the graph with a single vertex, called $*$, and no edge. For convenience, we will abuse the notation slightly and refer to this graph as $*$.

As mentioned in the introduction, the results of [4] imply that the cycles C_3 and C_4 are A-contractible graphs. We prove this directly using the previous definitions and combinatorial methods.

Proposition 3.6. [5, p. 47] The cycle C_3 is A-contractible.

Proof. We use the labeled 3-cycle, S, in this proof. First, we must define our graph homomorphisms $f : S \to *$ and $g : * \to S$. Notice that there is only one possible choice. Namely, $f : S \to *$ must be defined by $f(x) = f(y) = f(z) = *$, since the graph $*$ has only one vertex. Similarly, $g : * \to S$ must be defined by $g(*) = x$, since x is the distinguished vertex of S.

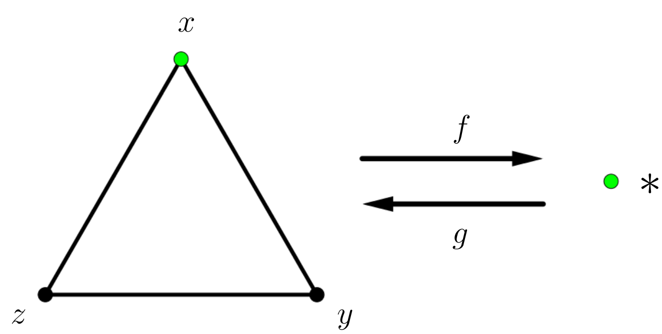

Figure 3.3: Graph homomorphisms f and g

Then $f \circ g$ is defined by $(f \circ g)(*) = f(g(*)) = f(x) = *$, and thus $f \circ g = \mathbf{1}_*$. Also,

$$(g \circ f)(x) = g(f(x)) = g(*) = x,$$
$$(g \circ f)(y) = g(f(y)) = g(*) = x,$$
$$(g \circ f)(z) = g(f(z)) = g(*) = x.$$

Thus the composition $g \circ f$ is equal to $c_x : S \to S$, the constant graph homomorphism mapping every vertex to x. We must now show that $c_x \simeq_A \mathbf{1}_S$. Define $H : S \square I_1 \to S$ by

$$H(x, 0) = x, \qquad H(x, 1) = x,$$
$$H(y, 0) = x, \qquad H(y, 1) = y,$$
$$H(z, 0) = x, \qquad H(z, 1) = z.$$

The image under H of each vertex in $S \square I_1$ is shown in red in Figure 3.4. For H to be a graph homomorphism, it must be the case that for all $\{u, w\} \in E(S \square I_1)$, either

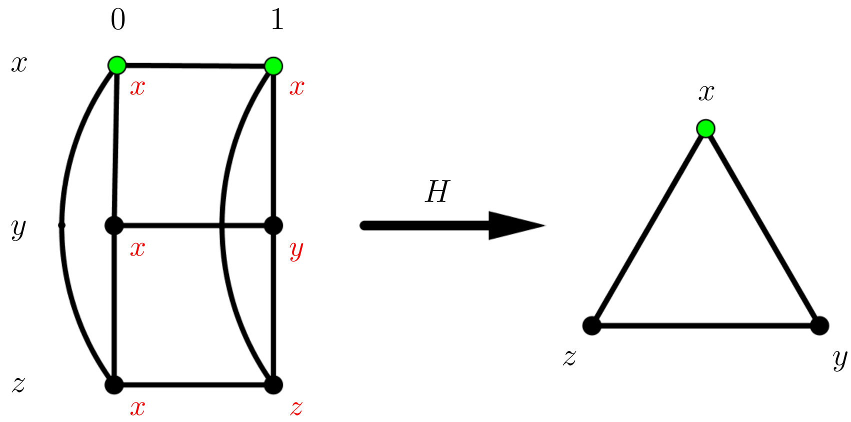

Figure 3.4: Homotopy from c_x to $\mathbf{1}_S$

$H(u) = H(w)$ or $\{H(u), H(w)\} \in E(S)$. Since every vertex of S is adjacent to every other vertex of S, the map H is a graph homomorphism. By construction of H, $H(v,0) = c_x(v)$ and $H(v,1) = \mathbf{1}_S(v)$ for all $v \in V(S)$, and $H(x,i) = x$ for all $i \in \{0,1\}$. Hence, H is a graph homotopy from c_x to $\mathbf{1}_S$, and $g \circ f \simeq_A \mathbf{1}_S$. Thus the graph S is A-contractible. $\square$

Proposition 3.7. [5, p.46] The cycle C_4 is A-contractible.

Proof. Let R be a labeled 4-cycle obtained from the graph T by deleting the edge $\{a, c\}$. There is again only one choice for the graph homomorphisms f and g. Namely, $f : R \to *$ is defined by $f(a) = f(b) = f(c) = f(d) = *$ and $g : * \to R$ is definedn by $g(*) = a$.

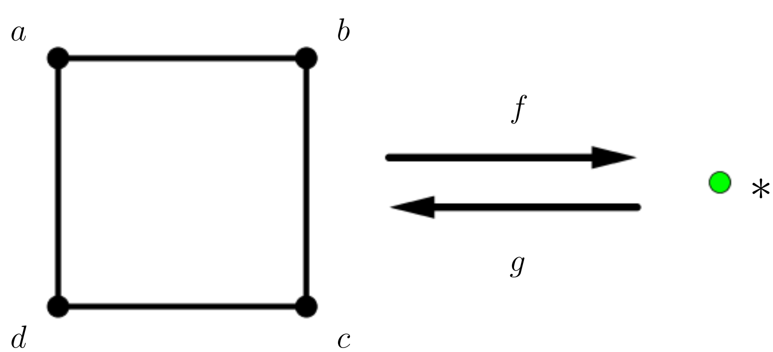

Figure 3.5: Graph homomorphisms f and g

Then $f \circ g$ is defined by $(f \circ g)(*) = f(g(*)) = f(a) = *$, and thus $f \circ g = \mathbf{1}_*$. Also,

$$(g \circ f)(a) = g(f(a)) = g(*) = a,$$
$$(g \circ f)(b) = g(f(b)) = g(*) = a,$$
$$(g \circ f)(c) = g(f(c)) = g(*) = a,$$
$$(g \circ f)(d) = g(f(d)) = g(*) = a.$$

Thus $g \circ f$ is equal to $c_a : R \to R$, the constant graph homomorphism mapping every vertex to a. We must now show that $c_a \simeq_A \mathbf{1}_R$. Define $H : R \square I_2 \to R$ by

$$\begin{aligned} &H(a,0) = a, \quad &&H(a,1) = a, \quad &&H(a,2) = a,\\ &H(b,0) = a, &&H(b,1) = a, &&H(b,2) = b,\\ &H(c,0) = a, &&H(c,1) = d, &&H(c,2) = c,\\ &H(d,0) = a, &&H(d,1) = d, &&H(d,2) = d. \end{aligned}$$

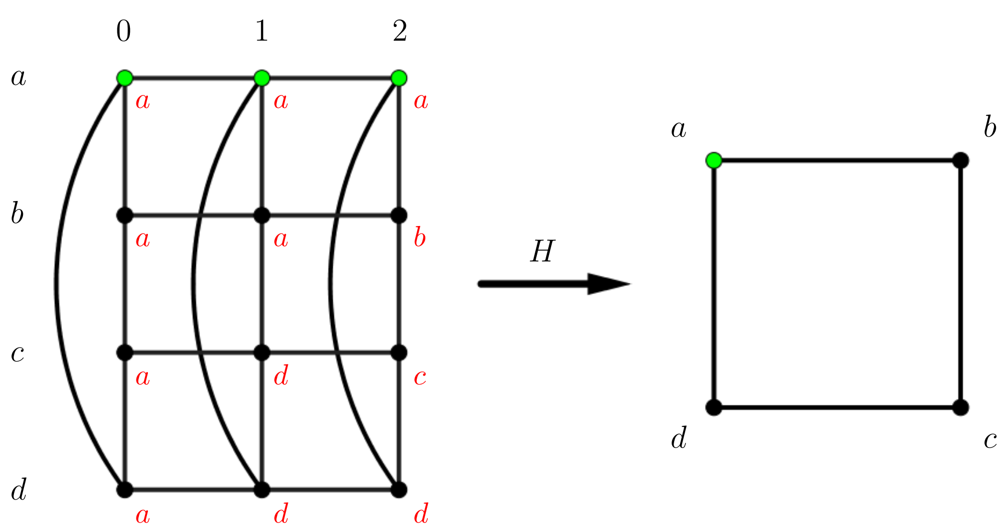

Figure 3.6: Graph homotopy from c_a to $\mathbf{1}_{C_4}$

The image under H of each vertex in $R \square I_2$ is shown in red in Figure 3.6. It is routine to verify that H is a graph homomorphism. By construction of H, $H(v, 0) = c_a(v)$ and $H(v, 2) = \mathbf{1}_R(v)$ for all $v \in V(R)$, and $H(a, i) = a$ for all $i \in \{0, 1, 2\}$. Hence, H is a graph homotopy from c_a to $\mathbf{1}_R$, and $g \circ f \simeq_A \mathbf{1}_R$. Thus the graph R is A-contractible. $\square$

Therefore, the cycles C_3 and C_4 are A-contractible. As mentioned in the introduction, the results in [4, Proposition 5.12] imply that the cycle C_5 is not A-contractible. To prove this in a more direct way, we need to examine the A-homotopy invariants of the cycle. For example, we show that the A-homotopy theory fundamental group of an A-contractible graph is equal to zero (Theorem 7.1). Thus, if the fundamental group of a graph is not equal to zero, then the graph cannot be A-contractible. In a later chapter, we show that the A-homotopy theory fundamental group of C_5 is isomorphic to the group $\mathbb{Z}$, using classical homotopy inspired methods in a combinatorical way (see Theorem 7.8). This allows us to explore the question of why the cycles C_3 and C_4 are A-contractible and the cycles C_k, for $k \geq 5$, are not A-contractible. Then we need a more rigorous definition of the fundamental group of a graph in A-homotopy theory.

Definition 3.8. [4, Definition 5.5] The *fundamental group* of the graph (G, v_0), denoted $A_1(G, v_0)$, is the set of homotopy classes of relative graph homomorphisms $f : (I_m, \{0, m\}) \to (G, v_0)$ from I_m, $m \geq 0$, to G that map the vertices 0 and m to the distinguished vertex v_0, using Definition 3.1 of A-homotopic.

Remark 3.9. The fundamental group is a set, but we show that it has group structure with the operation of concatenation in Chapter 5.

A graph homomorphism $f : I_n \to G$ is also referred to as a *path* in the graph G. This is not the standard definition of a path found in graph theory, but it does reflect the classical homotopy terminology.

Example 3.10. Figure 3.7 depicts the graph homomorphism $f : I_6 \to S$, which wraps around the 3-cycle S twice in a clockwise direction. The image under f of each vertex in I_6 is labeled in red.

While this thesis only deals with the fundamental group of graphs in A-homotopy theory, we include the general definition of the A-homotopy groups for the sake of completeness.

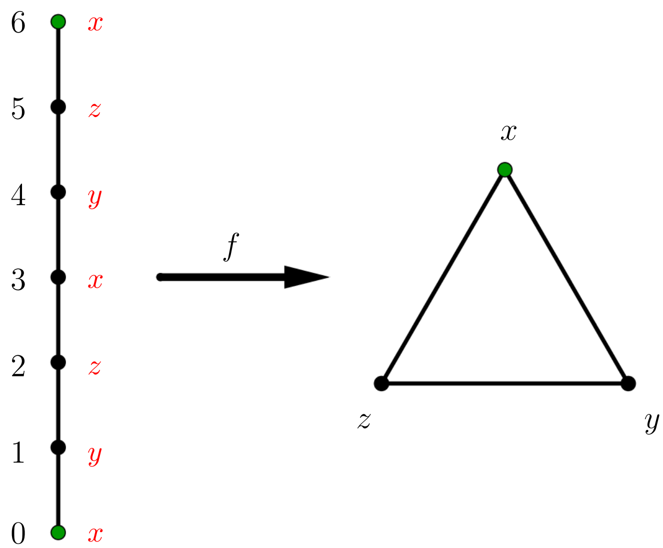

Figure 3.7: A graph homomorphism from I_6 to S

Before doing this, we require two additional definitions. First, we need a higher dimensional graph to map into a graph.

Definition 3.11. [4, Definition 5.3(1)] The graph $I_m^n = I_m \square \cdots \square I_m$ is the *n-fold Cartesian product of* I_m for some integers $n, m \geq 0$ with distinguished vertex $\mathbf{0} = (0, \ldots, 0)$.

Example 3.12. Figure 3.8 illustrates the 2-fold Cartesian product of I_2 and the 3-fold Cartesian products of I_1 and I_2, without labels.

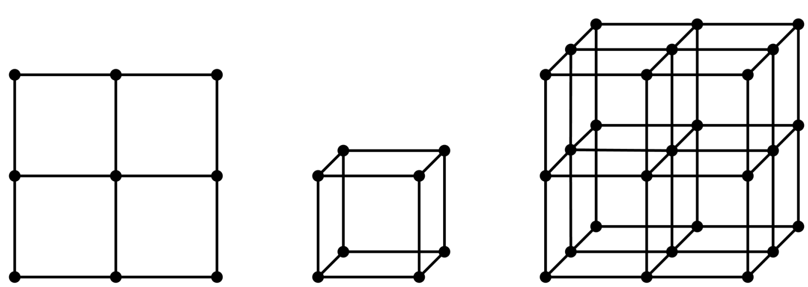

Figure 3.8: The graphs I_2^2 and I_1^3 and I_2^3

Remark 3.13. In topology, the space $[0,1]^n$ is the n-dimensional cube in $\mathbb{R}^n$. Similarly, the graph I_m^n resembles an n-dimensional cube graph with sides of length m, as seen in Figure 3.8.

Definition 3.14. Let $G = (V, E)$ be a graph and $V' \subseteq V$. The *induced subgraph* $G[V']$ is the graph with vertex set V' and edge set $E' = \{\{v, w\} \in E \mid v, w \in V'\}$, that is, all edges with vertices of V' as both endpoints.

Definition 3.15. [4, Definition 5.3(2)] The *boundary* of I_m^n, denoted δI_m^n, is the subgraph of I_m^n induced by the vertices with at least one coordinate equal to 0 or m.

Example 3.16. Figure 3.9 illustrates the unlabeled boundaries of the 2-fold Cartesian product of I_2 and the 3-fold Cartesian products of I_1 and I_2.

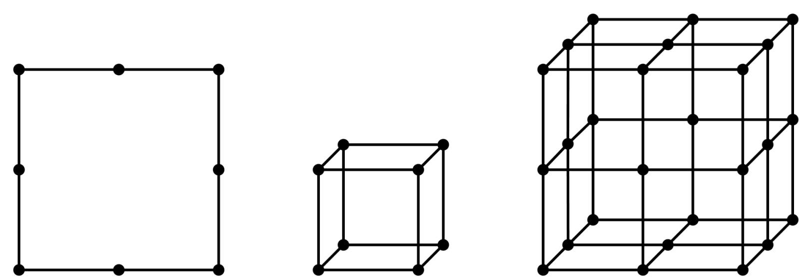

Figure 3.9: The graphs δI_2^2 and δI_1^3 and δI_2^3

Now we can define the A-homotopy groups for every dimension.

Definition 3.17. [4, Definition 5.5] The n^{th} *A-homotopy group* $A_n(G, v_0)$, for $n \geq 1$, is the set of homotopy classes of relative graph homomorphisms $f : (I_m^n, \delta I_m^n) \to (G, v_0)$ from I_m^n, $m \geq 0$, to G which map the vertices of δI_m^n to the distinguished vertex v_0, using Definition 3.1 of A-homotopic.

Remark 3.18. We do not provide the group structure for A-homotopy groups in general, because we are only interested in the fundamental groups of graphs in this thesis.

While graph homotopies are only defined to compare graph homomorphisms with the same domain and codomain, the fundamental group of a graph G must compare graph homomorphisms from paths of different lengths into G. For this reason, the authors of [3] defined an alternate set and equivalence relation to use with A-homotopy groups of graphs. These are presented in the next chapter.

Chapter 4

Alternate Definitions for A-Homotopy Theory

The A-homotopy theory fundamental group of a graph G, from Definition 3.8, is the set of equivalence classes of the graph homomorphisms from paths I_n into G, where n ranges over all nonnegative integers. Thus we must compare graph homomorphisms starting at paths of different lengths, but graph homotopies are only defined to compare graph homomorphisms that have the same domain and codomain. For this reason, the authors of [3] give an alternate definition of A-homotopy groups that compare graph homomorphisms starting at products of infinite paths I_∞ that are what we term *active* for finite regions. The definitions for this alternate theory are given here. While these definitions are notationally heavy, each is followed by an example and figure to illustrate the idea. The n-fold Cartesian product I_∞^n, labeled by $\mathbb{Z}^n$, features frequently in these definitions.

Definition 4.1. [3, Defintion 3.1] A graph homomorphism $f : I_\infty^n \to G$ *stabilizes in direction* εi with $1 \leq i \leq n$ and $\varepsilon \in \{-1, +1\}$, if there exists a least integer $m_0(f, \varepsilon i)$ such that either:

- if $\varepsilon = +1$, then for all $m \geq m_0(f, +i)$,

$$f(a_1, \cdots, a_{i-1},\ m,\ a_{i+1}, \cdots, a_n) \ = \ f(a_1, \cdots, a_{i-1},\ m_0(f, +i),\ a_{i+1}, \cdots, a_n),$$

- if $\varepsilon = -1$, then for all $m \leq m_0(f, -i)$,

$$f(a_1, \cdots, a_{i-1},\ m,\ a_{i+1}, \cdots, a_n) \ = \ f(a_1, \cdots, a_{i-1},\ m_0(f, -i),\ a_{i+1}, \cdots, a_n).$$

Remark 4.2. The graph homomorphisms $f : I_\infty^n \to G$ are not based graph homomorphisms.

The integer $m_0(f, \varepsilon i)$ gives us the point at which the graph homomorphism f stabilizes on the i^{th}-axis in the ε direction of that axis. In figures, the graphs I_∞ and I_∞^2 are depicted with the 1^{st}-axis vertical and the 2^{nd}-axis horizontal. No n-cubes of higher dimension are depicted.

Example 4.3. Figure 4.1 depicts a graph homomorphism $f : I_\infty \to S$ with the image of each vertex under f shown in red. Since $f(i) = x$ for all $i \leq -1$, the integer $m_0(f, -1) = -1$,

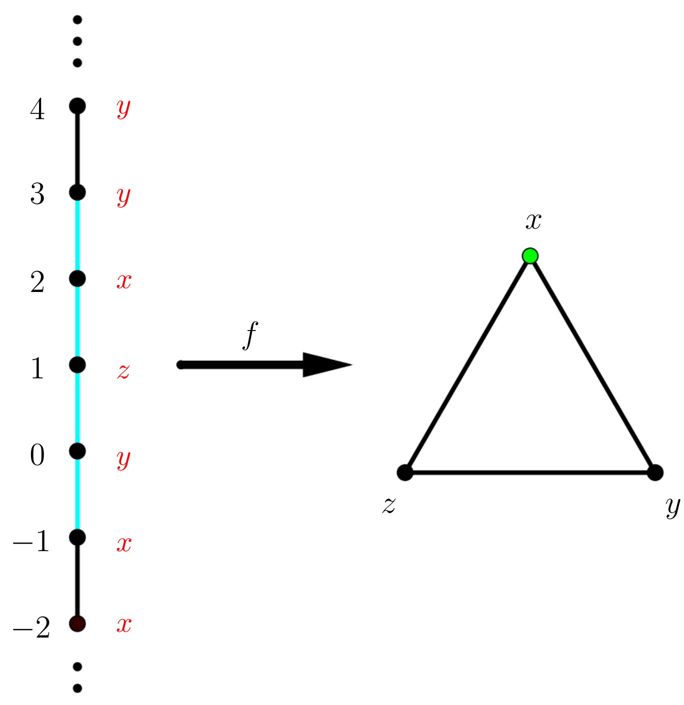

Figure 4.1: A stable graph homomorphism f from I_∞ to S

that is, f stabilizes on the 1^{st}-axis in the negative direction at -1. Similarly, since $f(i) = y$ for all $i \geq 3$, the integer $m_0(f, +1) = 3$, that is, f stabilizes on the 1^{st}-axis in the positive direction at 3.

When a graph homomorphism stabilizes in every direction, there is a finite region of the n-dimensional lattice with "relevant information". For instance, the information stored by the graph homomorphism in Example 4.3 could be presented in a graph homomorphism from I_4 to S, since $m_0(f,+1) - m_0(f,-1) = 3 - (-1) = 4$. We call the region of I_∞^n, induced by the vertex set $\prod_{i\in[n]}[m_0(f,-i), m_0(f,+i)]$, the *active region* for each graph homomorphism $f : I_\infty^n \to G$. In Figure 4.1, the edges of the active region of the graph homomorphism f are shown in light blue. For each path $f : I_\infty \to G$, we say that f *starts* at $f(m_0(f,-1))$ and f *ends* at $f(m_0(f,+1))$ when these integers exist. In Example 4.3, $f : I_\infty \to S$ starts at the vertex x and ends at the vertex y.

Definition 4.4. [3, Defintion 3.1] Let $C_n(G)$ be the set of graph homomorphisms from the infinite n-cube I_∞^n to the graph G that stabilize in each direction εi for $1 \le i \le n$ and $\varepsilon \in \{-1,+1\}$. These graph homomorphisms are referred to as *stable graph homomorphisms*.

The set $C_0(G)$ consists of the graph homomorphisms from the graph $*$, with a single vertex $*$ and no edges, to the graph G.

Definition 4.5. A graph G is *connected* if for each $v, w \in V(G)$, there exists a stable graph homomorphism $f \in C_1(G)$ such that $f(m_0(f,-1)) = v$ and $f(m_0(f,+1) = w$.

While this is not the standard definition of a connected graph found in [11], it is equivalent. In order to better understand and discuss the graph homomorphisms of $C_n(G)$, we need the following tools.

Definition 4.6. [3, Definition 3.1] The *face map* $\alpha_{\varepsilon i}^n : C_n(G) \to C_{n-1}(G)$, with $1 \le i \le n$ and $\varepsilon \in \{-1,+1\}$, is defined by $f \mapsto \alpha_{\varepsilon i}^n(f)$, where

$$\alpha_{\varepsilon i}^n(f)(a_1,\ldots,a_{n-1}) = f(a_1,\ldots,a_{i-1}, m_0(f,\varepsilon i), a_i,\ldots,a_{n-1}).$$

We refer to the map $\alpha_{\varepsilon i}^n(f)$ as the *face of f* in the εi direction.

For each graph homomorphism $f \in C_n(G)$, the face $\alpha^n_{\varepsilon i}(f) : I^{n-1}_\infty \to G$ is a restriction of f to $m_0(f, \varepsilon i)$ on the i^{th}-axis, that is, the face of f in the εi direction. Thus, since f is a stable graph homomorphism, each face $\alpha^n_{\varepsilon i}(f)$ is a stable graph homomorphism.

Example 4.7. Consider the graph homomorphism $f \in C_1(S)$ in Example 4.3. The face $\alpha^1_{-1}(f) : * \to S$ is $\alpha^1_{-1}(f)(*) = f(m_0(f,-1)) = f(-1) = x$, that is, the face of f on 1^{st}-axis in the negative direction, is x. Similarly, the face $\alpha^1_{+1}(f) : * \to S$ is $\alpha^1_{+1}(f)(*) = f(m_0(f,+1)) = f(3) = y$, that is, the face of f on the 1^{st}-axis in the positive direction, is y.

Example 4.8. Figure 4.2 depicts a graph homomorphism $H : I^2_\infty \to T$ with the image of each vertex in I^2_∞ under H shown in red. This map H stabilizes in every direction with $m_0(H,-1) = -1$, $m_0(H,+1) = 2$, $m_0(H,-2) = 0$, and $m_0(H,+2) = 2$. Thus $H \in C_2(T)$.

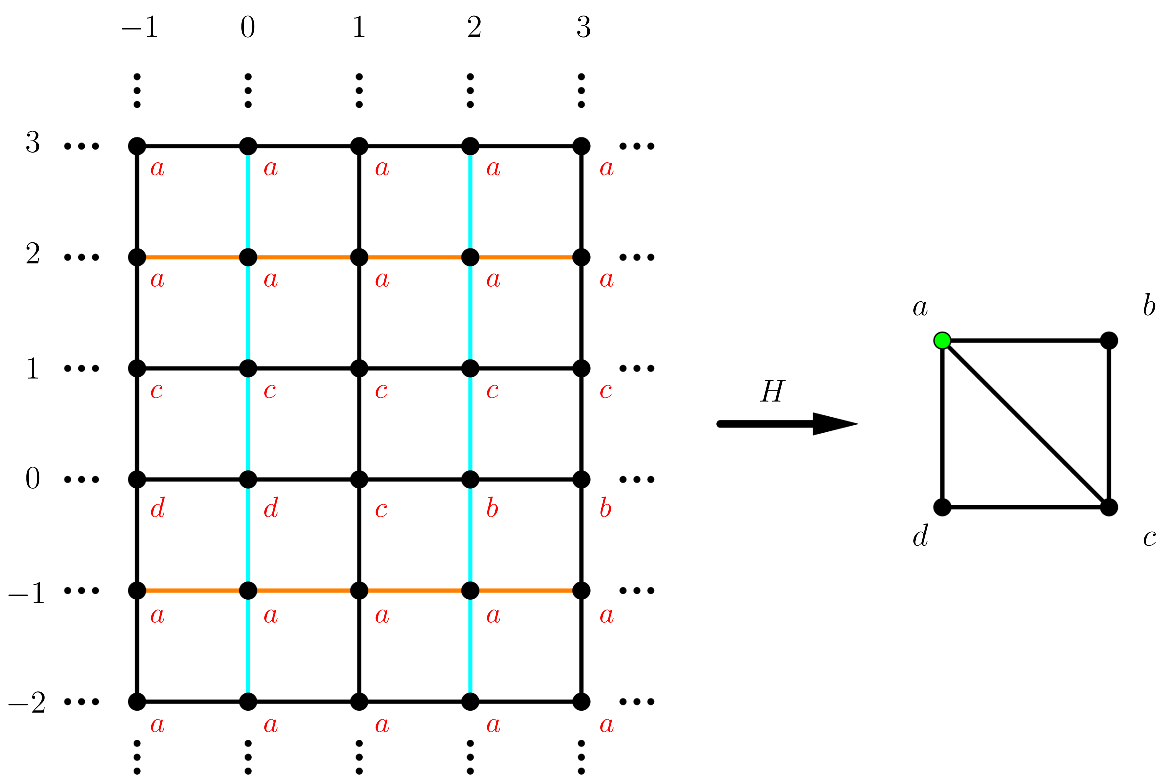

Figure 4.2: A stable graph homomorphism H from I^2_∞ to T

The face $\alpha^2_{-1}(H)$ is $\alpha^2_{-1}(H)(i) = H(m_0(H,-1), i) = H(-1, i) = a$ for all $i \in \mathbb{Z}$. Thus $\alpha^2_{-1}(H) : I_\infty \to T$ is constantly equal to a and is shown in orange as the bottom face of the lattice. Similarly, $\alpha^2_{+1}(H)$ is $\alpha^2_{+1}(H)(i) = H(m_0(H, +1), i) = H(2, i) = a$ for all $i \in \mathbb{Z}$ and is shown in orange as the top face of the lattice. The face $\alpha^2_{-2}(H)$, that is, the face of H on the 2^{nd}-axis in the negative direction, is $\alpha^2_{-2}(H)(i) = H(i, m_0(H, -2)) = H(i, 0)$ for all $i \in \mathbb{Z}$ and is shown in light blue as the left face of the lattice. The face $\alpha^2_{+2}(H)$, that is, the face of H on the 2^{nd}-axis in the positive direction, is $\alpha^2_{+2}(H)(i) = H(m_0(h, +2), i) = H(i, 2)$ for all $i \in \mathbb{Z}$ and is shown in light blue as the right face of the lattice.

Definition 4.9. [3, Definition 3.1] The *degeneracy maps* $\beta^n_i : C_{n-1}(G) \to C_n(G)$ with $1 \leq i \leq n$ is defined by $f \mapsto \beta^n_i(f)$, where

$$\beta^n_i(f)(a_1, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n).$$

Example 4.10. Consider the graph homomorphism $f \in C_1(S)$ from Example 4.3. Figure 4.3 illustrates that the graph homomorphism $\beta^2_1(f) : I^2_\infty \to S$ is $\beta^2_1(f)(i, j) = f(j)$ for all $i, j \in \mathbb{Z}$. The image of each vertex under $\beta^2_1(f)$ is shown in red. This map repeats f along

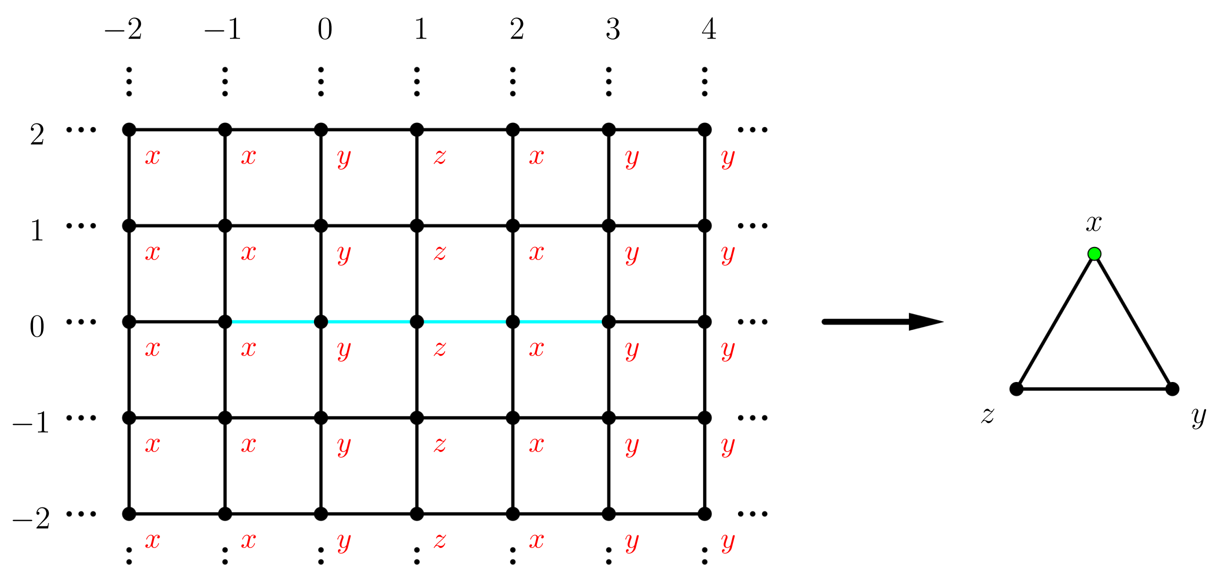

Figure 4.3: The graph homomorphism β^2_1 from I^2_∞ to G

the 1^{st}-axis. The edges of the active region of the lattice are shown in light blue.

Example 4.11. Consider the graph homomorphism $f \in C_1(S)$ from Example 4.3. Figure 4.4 illustrates the graph homomorphism $\beta_2^2(f) : I_\infty^2 \to S$ defined by $\beta_1^2(f)(i,j) = f(i)$ for all $i, j \in \mathbb{Z}$. The image of each vertex under $\beta_2^2(f)$ is shown in red. This map repeats f along

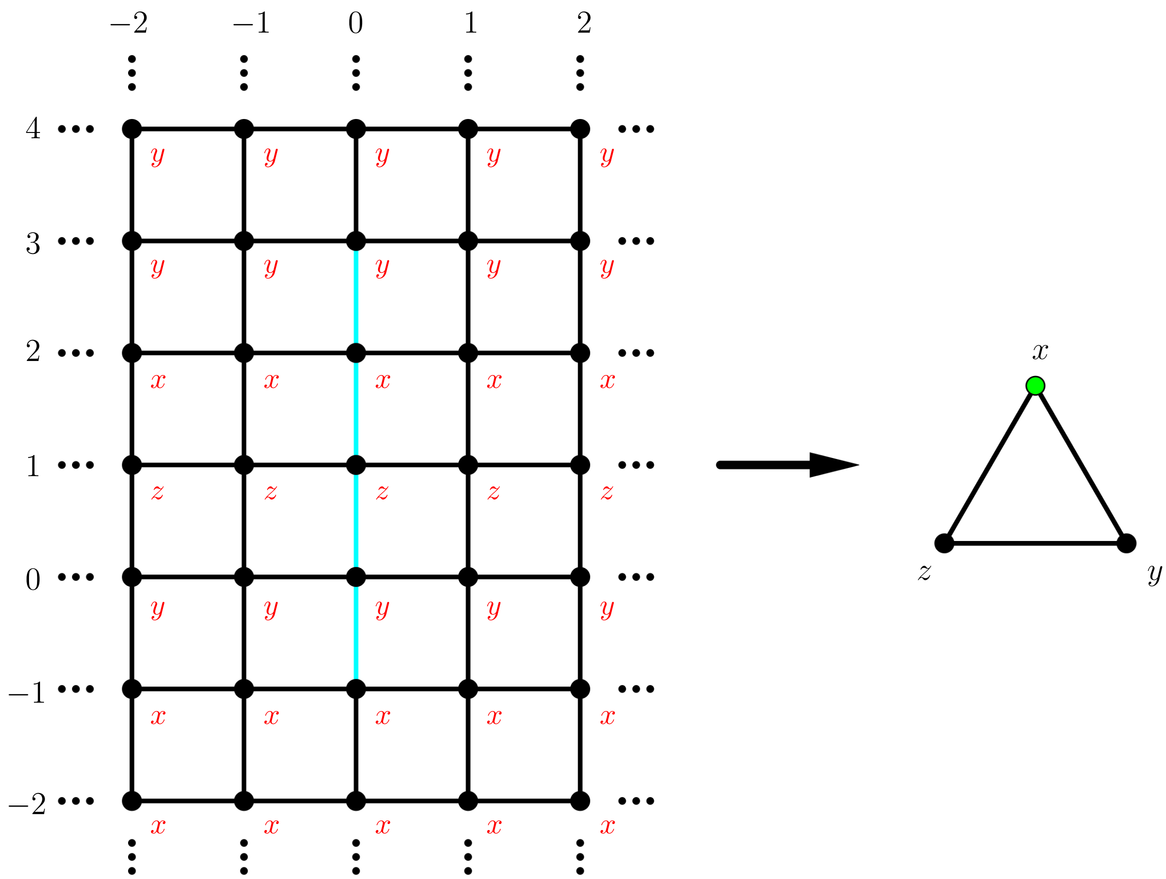

Figure 4.4: The graph homomorphism $\beta_2^2(f)$ from I_∞^2 to S

the 2^{nd}-axis. Again, the edges of the active region of the lattice are shown in light blue. Note that this graph homomorphism is just a rotation of Figure 4.3 by $-\pi/2$ radians.

In general, these degeneracy maps β_i^n repeat the graph homomorphisms $f : I_\infty^{n-1} \to G$ along the i^{th}-axis with $1 \le i \le n$, giving us a graph homomorphism from I_∞^n to G. For our purpose, we need only map between the sets $C_0(G)$ and $C_1(G)$, and between the sets $C_1(G)$

and $C_2(G)$ for each graph G.

$$
\begin{tikzcd}
C_2(G) \arrow[d, bend right=80, "\alpha^2_{+2}"'] \arrow[d, bend right=60, "\alpha^2_{-2}"'] \arrow[d, bend right=40, "\alpha^2_{+1}"'] \arrow[d, bend right=20, "\alpha^2_{-1}"'] \\
C_1(G) \arrow[u, bend right=20, "\beta^2_1"'] \arrow[u, bend right=40, "\beta^2_2"'] \arrow[d, bend right=40, "\alpha^1_{+1}"'] \arrow[d, bend right=20, "\alpha^1_{-1}"'] \\
C_0(G) \arrow[u, bend right=20, "\beta^1_1"']
\end{tikzcd}
$$

Using these face and degeneracy maps, we can give a definition for a graph homotopy between two graph homomorphisms of $C_n(G)$.

Definition 4.12. [3, Definition 3.2] Let $f, g \in C_n(G)$. The graph homomorphisms f and g are *A-homotopic*, denoted $f \sim g$, if there exists a graph homomorphism $H \in C_{n+1}(G)$ such that for all $1 \leq i \leq n$ and $\varepsilon \in \{-1, +1\}$:

(a) $\alpha^n_{\varepsilon i}(f) \;=\; \alpha^n_{\varepsilon i}(g)$,

(b) $\alpha^{n+1}_{\varepsilon i}(H) \;=\; \beta^n_n \alpha^n_{\varepsilon i}(f) \;=\; \beta^n_n \alpha^n_{\varepsilon i}(g)$,

(c) $\alpha^{n+1}_{-(n+1)}(H) \;=\; f$ and $\alpha^{n+1}_{+(n+1)}(h) \;=\; g$.

The graph homomorphism $H : I^{n+1}_\infty \to G$ is referred to as a *graph homotopy* from f to g.

By part (a), in order for the graph homomorphisms $f, g \in C_n(G)$ to be homotopic, they must stabilize to the same graph homomorphism of $C_{n-1}(G)$ in each εi direction for $1 \leq i \leq n$ and $\varepsilon \in \{-1, +1\}$, that is, they must have the same faces. By part (b), the graph homomorphism H must stabilize in each εi direction for $1 \leq i \leq n$ and $\varepsilon \in \{-1, +1\}$ to the faces of f and g repeated along the n^{th}-axis. By part (c), the graph homomorphism H must stabilize to f in the negative direction of the $(n+1)^{st}$-axis and stabilize to g in the positive direction of the $(n+1)^{st}$-axis.

Example 4.13. Recall the graph homomorphism $H \in C_2(T)$ depicted in Figure 4.2. Let the graph homomorphisms $f, g \in C_1(T)$ be defined by

$$f(i) = \begin{cases} a & \text{for} \quad i \geq 2, \\ c & \text{for} \quad i = 1, \\ d & \text{for} \quad i = 0, \\ a & \text{for} \quad i \leq -1, \end{cases} \qquad \text{and} \qquad g(i) = \begin{cases} a & \text{for} \quad i \geq 2, \\ c & \text{for} \quad i = 1, \\ b & \text{for} \quad i = 0, \\ a & \text{for} \quad i \leq -1. \end{cases}$$

We show that H is a graph homotopy from f to g by verifying conditions (a)-(c) of Definition 4.12.

(a) Since f and g both stabilize to the vertex a in the negative direction of the 1^{st}-axis and the positive direction of the 1^{st}-axis, $\alpha^1_{-1}(f) = \alpha^1_{-1}(g)$ and $\alpha^1_{+1}(f) = \alpha^1_{+1}(g)$.

(b) Let $p_a : I_\infty \to T$ denote the graph homomorphism which is constantly equal to a. Hence, H stabilizes to p_a in the negative direction of the 1^{st}-axis and positive direction of the 1^{st}-axis. Since the degeneracy map $\beta^1_1 : C_0(T) \to C_1(T)$ repeats a graph homomorphism along the 1^{st}-axis, it follows that $p_a = \beta^1_1\alpha^1_{-1}(f) = \beta^1_1\alpha^1_{-1}(g)$ and $p_a = \beta^1_1\alpha^1_{+1}(f) = \beta^1_1\alpha^1_{+1}(g)$. Thus $\alpha^2_{-1}(H) = \beta^1_1\alpha^1_{-1}(f) = \beta^1_1\alpha^1_{-1}(g)$ and $\alpha^2_{+1}(H) = \beta^1_1\alpha^1_{+1}(f) = \beta^1_1\alpha^1_{+1}(g)$.

(c) Since H stabilizes to f in the negative direction of the 2^{nd}-axis and stabilizes to g in the positive direction of the 2^{nd}-axis, $\alpha^2_{-2}(H) = f$ and $\alpha^2_{+2}(H) = g$.

Thus H is a graph homotopy from f to g, and hence, $f \sim g$.

Now that we have a way to compare graph homomorphisms from paths of different lengths to a graph G, we need an operation that combines the graph homomorphisms of $C_n(G)$.

Definition 4.14. Let f and g be graph homomorphisms of $C_1(G)$ with $\alpha^1_{-1}(f) = \alpha^1_{+1}(g)$.

The *concatenation of f and g*, denoted $f \cdot g$, is defined by

$$(f \cdot g)(a) = \begin{cases} f(a + m_0(f, -1)) & \text{for} \quad a \geq 0, \\ g(a + m_0(g, +1)) & \text{for} \quad a \leq 0. \end{cases}$$

More generally, if f and g are graph homomorphisms of $C_n(G)$ with $\alpha^n_{-i}(f) = \alpha^n_{+i}(g)$, then the *concatenation of f and g on the i^{th}-axis*, denoted $f \cdot_i g$, is defined by

$$(f \cdot_i g)(a_1, \cdots, a_i, \cdots, a_n) = \begin{cases} f(a_1, \ldots, a_{i-1},\ a_i + m_0(f, -i),\ a_{i+1}, \ldots, a_n) & \text{for} \quad a_i \geq 0, \\ g(a_1, \ldots, a_{i-1},\ a_i + m_0(g, +i),\ a_{i+1}, \ldots, a_n) & \text{for} \quad a_i \leq 0. \end{cases}$$

This operation essentially shifts the first graph homomorphism f to stabilize in the negative direction on the i^{th}-axis at zero and shifts the second graph homomorphism g to stabilize in the positive direction on the i^{th}-axis at zero. For this reason, the face of f in the negative direction on the i^{th}-axis must be the same as the face of g in the positive direction on the i^{th}-axis.

Proposition 4.15. If $f, g \in C_1(G)$ with $\alpha^n_{-1}(f) = \alpha^n_{+1}(g)$, then the concatenation $f \cdot g$ is a graph homomorphism of $C_1(G)$.

Proof. Let $f, g \in C_1(G)$ with $\alpha^1_{-1}(f) = \alpha^1_{+1}(g)$. By the definition of concatenation,

$$(f \cdot g)(i) = \begin{cases} f(i + m_0(f, -1)) & \text{for} \quad i \geq 0, \\ g(i + m_0(g, +1)) & \text{for} \quad i \leq 0. \end{cases}$$

Since $\alpha^1_{-1}(f) = \alpha^1_{+1}(g)$, it follows that $f(m_0(f, -1)) = g(m_0(g, +1))$. Thus $f \cdot g$ is well-defined. In order for $f \cdot g$ to be a graph homomorphism, each pair of adjacent vertices in I_∞ must be mapped to the same vertex or adjacent vertices in G. By definition of the graph I_∞, there is an edge $\{j, j+1\} \in E(I_\infty)$ for each $j \in \mathbb{Z}$.

- If $j \geq 0$, then

$$(f \cdot g)(j) = f(j + m_0(f, -1)) \quad \text{and} \quad (f \cdot g)(j+1) = f(j + 1 + m_0(f, -1)).$$

Since $\{j + m_0(f, -1), j + 1 + m_0(f, -1)\} \in E(I_\infty)$ and f is a graph homomorphism, $f(j+m_0(f,-1)) = f(j+1+m_0(f,-1))$ or $\{f(j+m_0(f,-1)), f(j+1+m_0(f,-1))\} \in E(I_\infty)$. Thus $(f \cdot g)(j) = (f \cdot g)(j+1)$ or $\{(f \cdot g)(j), (f \cdot g)(j+1)\} \in E(I_\infty)$ for all $j \geq 0$.

- Otherwise $j < 0$, and it follows that

$$(f \cdot g)(j) = g(j + m_0(g, +1)) \quad \text{and} \quad (f \cdot g)(j+1) = g(j + 1 + m_0(g, +1)).$$

Since $\{j + m_0(f, -1), j + 1 + m_0(f, -1)\} \in E(I_\infty)$ and g is a graph homomorphism, $g(j+m_0(g,+1)) = g(j+1+m_0(g,+1))$ or $\{g(j+m_0(g,+1)), g(j+1+m_0(g,+1))\} \in E(I_\infty)$. Thus $(f \cdot g)(j) = (f \cdot g)(j+1)$ or $\{(f \cdot g)(j), (f \cdot g)(j+1)\} \in E(I_\infty)$ for all $j < 0$.

Therefore, $(f \cdot g)(j) = (f \cdot g)(j+1)$ or $\{(f \cdot g)(j), (f \cdot g)(j+1)\} \in E(I_\infty)$ for all $j \in \mathbb{Z}$, and thus the concatenation $f \cdot g$ is a graph homomorphism. $\square$

Lemma 4.16. *For each $f, g \in C_1(G)$ with $\alpha^1_{-1}(f) = \alpha^1_{+1}(g)$, the concatenation $f \cdot g \in C_1(G)$ stabilizes in the positive direction at $m_0(f \cdot g, +1) = m_0(f, +1) - m_0(f, -1)$ and in the negative direction at $m_0(f \cdot g, -1) = m_0(g, -1) - m_0(g, +1)$.*

Proof. Let $f, g \in C_1(G)$ be such that $\alpha^1_{-1}(f) = \alpha^1_{+1}(g)$. Then by Proposition 4.15, the concatenation $f \cdot g$ is a graph homomorphism. For $i \geq 0$, $(f \cdot g)(i) = f(i + m_0(f, -1))$. Since $m_0(f, +1) - m_0(f, -1) \geq 0$,

$$\begin{aligned} (f \cdot g)(m_0(f, +1) - m_0(f, -1)) &= f(m_0(f, +1) - m_0(f, -1) + m_0(f, -1)) \\ &= f(m_0(f, +1)). \end{aligned}$$

By Definition 4.1, $m_0(f,+1)$ is the least integer such that $f(m) = f(m_0(f,+1))$ for all $m \geq m_0(f,+1)$, so it follows that $m_0(f,+1) - m_0(f,-1)$ is the least integer such that $(f \cdot g)(i) = f(m_0(f,+1))$ for all $i \geq m_0(f,+1) - m_0(f,-1)$. Therefore, $m_0(f \cdot g,+1) = m_0(f,+1) - m_0(f,-1)$. For $i \leq 0$, $(f \cdot g)(i) = g(i + m_0(g,+1))$. Since $m_0(g,-1) - m_0(g,+1) \leq 0$,

$$\begin{aligned}(f \cdot g)(m_0(g,-1) - m_0(g,+1)) &= g(m_0(g,-1) - m_0(g,+1) + m_0(g,+1)) \\ &= g(m_0(g,-1)).\end{aligned}$$

By Definition 4.1, $m_0(g,-1)$ is the greatest integer such that $g(m) = g(m_0(g,-1))$ for all $m \leq m_0(f,+1)$, so it follows that $m_0(g,-1) - m_0(g,+1)$ is the greatest integer such that $(f \cdot g)(i) = g(m_0(g,-1))$ for all $i \leq m_0(g,-1) - m_0(g,+1)$. Therefore, $m_0(f \cdot g,-1) = m_0(g,-1) - m_0(g,+1)$. □

We now continue with an example of the concatenation of two graph homomorphisms.

Example 4.17. Figure 4.5 depicts the stable graph homomorphism $f : I_\infty \to S$ that starts at x, wraps around S in a clockwise direction, and stops at y, and the stable graph homomorphism $g : I_\infty \to S$ that starts at y, wraps around S in a counterclockwise direction, and stops at x.

Since f stabilizes to x in the the negative direction and g stabilizes to x in the positive direction, $\alpha^1_{-1}(f) = \alpha^1_{+1}(g)$. Since g stabilizes to y in the the negative direction and f stabilizes to y in the positive direction, $\alpha^1_{-1}(g) = \alpha^1_{+1}(f)$. Thus we can take the concatenations $f \cdot g$ and $g \cdot f$, which are illustrated in Figure 4.6.

Since $m_0(f,-1) = -1$, $m_0(f,+1) = 3$, $m_0(g,-1) = -2$, and $m_0(g,+1) = 2$, the

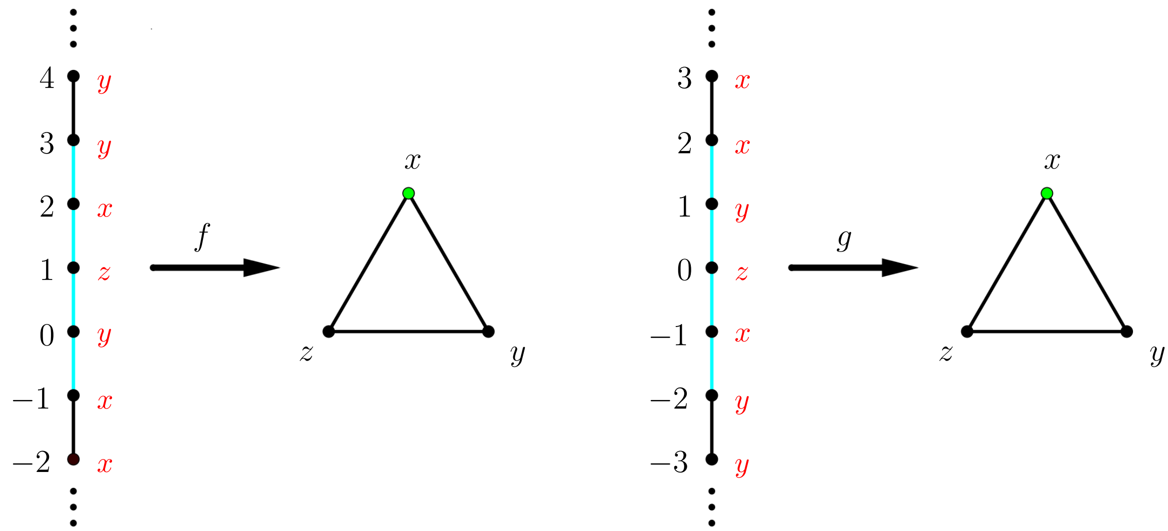

Figure 4.5: Stable graph homomorphisms from I_∞ to S

concatenations $f \cdot g$ and $g \cdot f$ stabilize at the following integers:

$$m_0(f \cdot g, -1) = m_0(g, -1) - m_0(g, +1) = -2 - 2 = -4,$$

$$m_0(f \cdot g, +1) = m_0(f, +1) - m_0(f, -1) = 3 - (-1) = 4,$$

$$m_0(g \cdot f, -1) = m_0(f, -1) - m_0(f, +1) = -1 - 3 = -4,$$

$$m_0(g \cdot f, +1) = m_0(g, +1) - m_0(g, -1) = 2 - (-2) = 4.$$

This definition of concatenation is only one of many variations that are all A-homotopic to each other. We use the version defined here because we know exactly where the concatenation of two graph homomorphisms stabilizes.

Proposition 4.18. The homotopy relation $\sim$ is an equivalence relation on $C_1(G)$.

Proof. To show that $\sim$ is reflexive, symmetric, and transitive, we define maps, and show that each map is well-defined, a stable graph homomorphism, and a graph homotopy.

- $\sim$ is reflexive.

  Let $f \in C_1(G)$. Define $H = \beta_2^2(f)$, that is, f repeated along the 2^{nd}-axis. By definition of H and since $f \in C_1(G)$, the map $H : I_\infty^2 \to G$ is well-defined and a stable graph

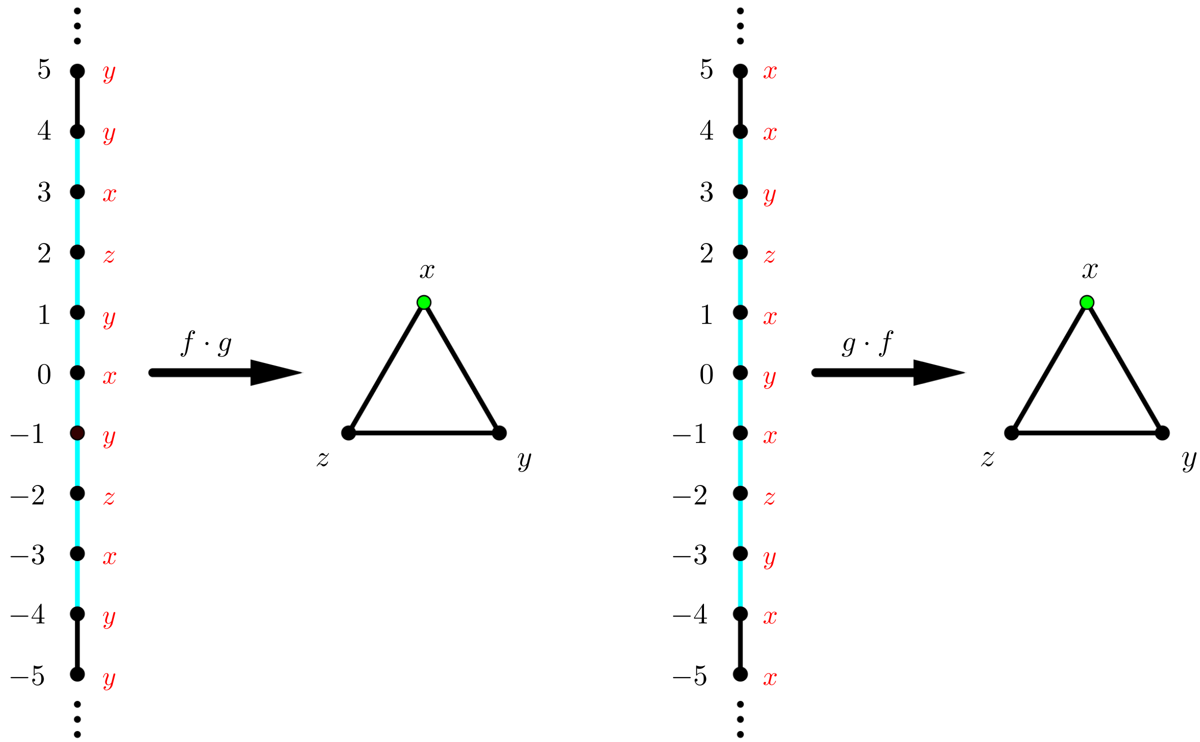

Figure 4.6: The concatenations $f \cdot g$ and $g \cdot f$

homomorphism in $C_2(G)$. To show that H is a graph homotopy from f to f, we must verify conditions (a)-(c) of Definition 4.12.

(a) Since $f(m_0(f,+1)) = f(m_0(f,+1))$ and $f(m_0(f,-1)) = f(m_0(f,-1))$ trivially, it follows that $\alpha^1_{+1}(f) = \alpha^1_{+1}(f)$ and $\alpha^1_{-1}(f) = \alpha^1_{-1}(f)$.

(b) By definition of H, $H(i,j) = \beta^2_2(f)(i,j) = f(i)$ for all $i, j \in \mathbb{Z}$. Thus the graph homomorphism H stabilizes on the 1^{st}-axis at $m_0(H,+1) = m_0(f,+1)$ and $m_0(H,-1) = m_0(f,-1)$. The face $\alpha^2_{+1}(H) : I_\infty \to G$ is given by

$$\begin{aligned}
\alpha^2_{+1}(H)(i) &= H(m_0(H,+1), i) \\
&= H(m_0(f,+1), i) \\
&= \beta^2_2(f)(m_0(f,+1), i) \\
&= f(m_0(f,+1))
\end{aligned}$$

for all $i \in \mathbb{Z}$, that is, taking the top face of H is the same as taking the top face of f and repeating it along the 1^{st}-axis. Thus $\alpha^2_{+1}(H) = \beta^1_1\alpha^1_{+1}(f) = \beta^1_1\alpha^1_{+1}(f)$. Similarly, the face $\alpha^2_{-1}(H) : I_\infty \to G$ is given by

$$\begin{aligned}
\alpha^2_{-1}(H)(i) &= H(m_0(H,-1),i) \\
&= H(m_0(f,-1),i) \\
&= \beta^2_2(f)(m_0(f,-1),i) \\
&= f(m_0(f,-1))
\end{aligned}$$

for all $i \in \mathbb{Z}$, that is, taking the bottom face of H is the same as taking the bottom face of f and repeating it along the 1^{st}-axis. Thus $\alpha^2_{-1}(H) = \beta^1_1\alpha^1_{-1}(f) = \beta^1_1\alpha^1_{-1}(f)$.

(c) Since $H(i,j) = f(i)$ for all $i,j \in \mathbb{Z}$, the graph homomorphism H stabilizes on the 2^{nd}-axis at $m_0(H,+2) = m_0(H,-2) = 0$. The face $\alpha^2_{-2}(H)$ is given by

$$\begin{aligned}
\alpha^2_{-2}(H)(i) &= H(i,m_0(H,-2)) \\
&= H(i,0) \\
&= \beta^2_2(f)(i,0) \\
&= f(i)
\end{aligned}$$

for all $i \in \mathbb{Z}$. Thus $\alpha^2_{-2}(H) = f$. Similarly, the face $\alpha^2_{+2}(H) = f$.

Hence, H is a graph homotopy from f to f, so $f \sim f$. Thus the relation $\sim$ is reflexive.

- $\sim$ is symmetric.

Let $f, g \in C_1(G)$, and suppose $f \sim g$. Then there exists a graph homomorphism $H_1 \in C_2(G)$ such that

(1) $\alpha^1_{+1}(f) = \alpha^1_{+1}(g)$, and $\alpha^1_{-1}(f) = \alpha^1_{-1}(g)$

(2) $\alpha^2_{+1}(H_1) = \beta^1_1\alpha^1_{+1}(f) = \beta^1_1\alpha^1_{+1}(g)$ and $\alpha^2_{-1}(H_1) = \beta^1_1\alpha^1_{-1}(f) = \beta^1_1\alpha^1_{-1}(g)$,

(3) $\alpha^2_{-2}(H_1) = f$ and $\alpha^2_{+2}(H_1) = g$.

Define the map $H_2 : I^2_\infty \to G$ by $H_2(i,j) = H_1(i,-j)$ for all $i,j \in \mathbb{Z}$. Since $H_1 \in C_2(G)$, the map H_2 is well-defined. Since

$$\begin{aligned} H_2(i,j) &= H_1(i,-j), \\ H_2(i+1,j) &= H_1(i+1,-j), \\ H_2(i,j+1) &= H_1(i,-j-1), \end{aligned}$$

and H_1 is a graph homomorphism, the map H_2 is a graph homomorphism. To show that H_2 is a graph homotopy from g to f, we must verify conditions (a)-(c) of Definition 4.12.

(a) Trivially by condition (1), $\alpha^1_{+1}(g) = \alpha^1_{+1}(f)$ and $\alpha^1_{-1}(g) = \alpha^1_{-1}(f)$.

(b) By condition (2), $\alpha^2_{-1}(H_1) = \beta^1_1\alpha^1_{-1}(f) = \beta^1_1\alpha^1_{-1}(g)$ and $\alpha^2_{+1}(H_1) = \beta^1_1\alpha^1_{+1}(f) = \beta^1_1\alpha^1_{+1}(g)$. Since

$$\begin{aligned} m_0(H_1,-1) &\leq m_0(f,-1), \\ m_0(H_1,-1) &\leq m_0(g,-1), \\ m_0(H_1,+1) &\geq m_0(f,+1), \\ m_0(H_1,+1) &\geq m_0(g,+1), \end{aligned}$$

the faces $\alpha^2_{-1}(H_1)$ and $\alpha^2_{+1}(H_1)$ are given by

$$\alpha^2_{-1}(H_1)(i) = H_1(m_0(H_1,-1),i) = f(m_0(f,-1)) = g(m_0(g,-1))$$

and

$$\alpha^2_{+1}(H_1)(i) = H_1(m_0(H_1,+1),i) = f(m_0(f,+1)) = g(m_0(g,+1)),$$

respectively, for all $i \in \mathbb{Z}$. Since $H_2(m_0(H_1,-1),i) = H_1(m_0(H_1,-1),-i)$ and $H_2(m_0(H_1,+1),i) = H_1(m_0(H_1,+1),-i)$ for all $i \in \mathbb{Z}$, it follows that H_2 stabilizes on the 1^{st}-axis in the negative direction at $m_0(H_2,-1) = m_0(H_1,-1)$ and in the positive direction at $m_0(H_2,+1) = m_0(H_1,+1)$. The faces $\alpha^2_{-1}(H_2)$ and $\alpha^2_{+1}(H_2)$ are given by

$$\begin{aligned}
\alpha^2_{-1}(H_2)(i) &= H_2(m_0(H_2,-1),i) \\
&= H_1(m_0(H_2,-1),-i) \\
&= H_1(m_0(H_1,-1),-i)
\end{aligned}$$

and

$$\begin{aligned}
\alpha^2_{+1}(H_2)(i) &= H_2(m_0(H_2,+1),i) \\
&= H_1(m_0(H_2,+1),-i) \\
&= H_1(m_0(H_1,+1),-i)
\end{aligned}$$

for all $i \in \mathbb{Z}$. This implies that $\alpha^2_{-1}(H_2)(i) = f(m_0(f,-1)) = g(m_0(g,-1))$ and $\alpha^2_{+1}(H_2)(i) = f(m_0(f,+1)) = g(m_0(g,+1))$ for all $i \in \mathbb{Z}$. Therefore, $\alpha^2_{-1}(H_2) = \beta^1_1\alpha^1_{-1}(g) = \beta^1_1\alpha^1_{-1}(f)$ and $\alpha^2_{+1}(H_2) = \beta^1_1\alpha^1_{+1}(g) = \beta^1_1\alpha^1_{+1}(f)$.

(c) By condition (3),

$$H_1(i, m_0(H_1,-2)) = f(i) \quad \text{and} \quad H_1(i, m_0(H_1,+2)) = g(i)$$

for all $i \in \mathbb{Z}$. Since $H_2(i,j) = H_1(i,-j)$ for all $i,j \in \mathbb{Z}$, it follows that H_2 stabilizes on the 2^{nd}-axis in the negative direction at $m_0(H_2, -2) = -m_0(H_1, +2)$ and in the positive direction at $m_0(H_2, +2) = -m_0(H_1, -2)$. The face $\alpha^2_{-2}(H_2)$ is given by

$$\begin{aligned} \alpha^2_{-2}(H_2)(i) &= H_2(i, m_0(H_2, -2)) \\ &= H_2(i, -m_0(H_1, +2)) \\ &= H_1(i, m_0(H_1, +2)) \\ &= g(i) \end{aligned}$$

for all $i \in \mathbb{Z}$. Similarly, the face $\alpha^2_{+2}(H_2)$ is given by

$$\begin{aligned} \alpha^2_{+2}(H_2)(i) &= H_2(i, m_0(H_2, +2)) \\ &= H_2(i, -m_0(H_1, -2)) \\ &= H_1(i, m_0(H_1, -2)) \\ &= f(i) \end{aligned}$$

for all $i \in \mathbb{Z}$. Thus it follows that $\alpha^2_{-2}(H_2) = g$ and $\alpha^2_{+2}(H_2) = f$.

Therefore, $g \sim f$, so the relation $\sim$ is symmetric.

- $\sim$ is transitive.

Let $f, g, h \in C_1(G)$, and suppose $f \sim g$ and $g \sim h$. Then there exists a graph homomorphism $H_1 \in C_2(G)$ such that

(1) $\alpha^1_{-1}(f) = \alpha^1_{-1}(g)$ and $\alpha^1_{+1}(f) = \alpha^1_{+1}(g)$,

(2) $\alpha^2_{-1}(H_1) = \beta^1_1\alpha^1_{-1}(f) = \beta^1_1\alpha^1_{-1}(g)$ and $\alpha^2_{+1}(H_1) = \beta^1_1\alpha^1_{+1}(f) = \beta^1_1\alpha^1_{+1}(g)$,

(3) $\alpha^2_{-2}(H_1) = f$ and $\alpha^2_{+2}(H_1) = g$,

and there exists a graph homomorphism $H_2 \in C_2(G)$ such that

(4) $\alpha^1_{-1}(g) = \alpha^1_{-1}(h)$ and $\alpha^1_{+1}(g) = \alpha^1_{+1}(h)$,

(5) $\alpha^2_{-1}(H_2) = \beta^1_1\alpha^1_{-1}(g) = \beta^1_1\alpha^1_{-1}(h)$ and $\alpha^2_{+1}(H_2) = \beta^1_1\alpha^1_{+1}(g) = \beta^1_1\alpha^1_{+1}(h)$,

(5) $\alpha^2_{-2}(H_2) = g$ and $\alpha^2_{+2}(H_2) = h$.

Define $H_3 : I^2_\infty \to G$ by $H_3 = H_2 \cdot_2 H_1$, namely,

$$H_3(i,j) = \begin{cases} H_2(i, j + m_0(H_2, -2)) & \text{for} \quad j \geq 0, \\ H_1(i, j + m_0(H_1, +2)) & \text{for} \quad j \leq 0. \end{cases}$$

Since H_1 and H_2 are graph homomorphism and $\alpha^2_{+2}(H_1) = g = \alpha^2_{-2}(H_2)$, the concatenation H_3 is a graph homomorphism. To show that H_3 is a graph homotopy from f to h, we must verify conditions (a)-(c) of Definition 4.12.

(a) By conditions (1) and (4), $\alpha^1_{-1}(f) = \alpha^1_{-1}(g) = \alpha^1_{-1}(h)$ and $\alpha^1_{+1}(f) = \alpha^1_{+1}(g) = \alpha^1_{+1}(h)$.

(b) By conditions (2) and (5),

$$\begin{aligned} H_1(m_0(H_1, -1), j) &= f(m_0(f, -1)) = g(m_0(g, -1)), \\ H_1(m_0(H_1, +1), j) &= f(m_0(f, +1)) = g(m_0(g, +1)), \\ H_2(m_0(H_2, -1), j) &= g(m_0(g, -1)) = h(m_0(h, -1)), \\ H_2(m_0(H_2, +1), j) &= g(m_0(g, +1)) = h(m_0(h, +1)) \end{aligned}$$

for all $j \in \mathbb{Z}$. Since H_3 is the concatenation of H_1 and H_2 on the 2^{nd}-axis, it follows that $m_0(H_3, -1) = \min\{m_0(H_1, -1),\ m_0(H_2, -1)\}$ and $m_0(H_3, +1) =$

$\max\{m_0(H_1, +1),\ m_0(H_2, +1)\}$. Thus the face $\alpha^2_{-1}(H_3)$ is given by

$$\begin{aligned}
\alpha^2_{-1}(H_3)(j) &= H_3(m_0(H_3, -1), j) \\
&= \begin{cases} H_2(m_0(H_3, -1),\ j + m_0(H_2, -2)) & \text{for} \quad j \geq 0, \\ H_1(m_0(H_3, -1),\ j + m_0(H_1, +2)) & \text{for} \quad j \leq 0, \end{cases} \\
&= \begin{cases} H_2(m_0(H_2, -1),\ j + m_0(H_2, -2)) & \text{for} \quad j \geq 0, \\ H_1(m_0(H_1, -1),\ j + m_0(H_1, +2)) & \text{for} \quad j \leq 0, \end{cases}
\end{aligned}$$

and the face $\alpha^2_{+1}(H_3)$ is given by

$$\begin{aligned}
\alpha^2_{+1}(H_3)(j) &= H_3(m_0(H_3, +1), j) \\
&= \begin{cases} H_2(m_0(H_3, +1),\ j + m_0(H_2, -2)) & \text{for} \quad j \geq 0, \\ H_1(m_0(H_3, +1),\ j + m_0(H_1, +2)) & \text{for} \quad j \leq 0, \end{cases} \\
&= \begin{cases} H_2(m_0(H_2, +1),\ j + m_0(H_2, -2)) & \text{for} \quad j \geq 0, \\ H_1(m_0(H_1, +1),\ j + m_0(H_1, +2)) & \text{for} \quad j \leq 0. \end{cases}
\end{aligned}$$

Since by parts (1) and (4) $f(m_0(f, -1)) = g(m_0(g, -1)) = h(m_0(h, -1))$ and $f(m_0(f, +1)) = g(m_0(g, +1)) = h(m_0(h, +1))$, it follows that $\alpha^2_{-1}(H_3)(j) = f(m_0(f, -1)) = h(m_0(h, -1))$ and $\alpha^2_{+1}(H_3)(j) = f(m_0(f, +1)) = h(m_0(h, +1))$ for all $j \in \mathbb{Z}$. Thus $\alpha^2_{-1}(H_3) = \beta^1_1\alpha^1_{-1}(f) = \beta^1_1\alpha^1_{-1}(h)$ and $\alpha^2_{+1}(H_3) = \beta^1_1\alpha^1_{+1}(f) = \beta^1_1\alpha^1_{+1}(h)$.

(c) By condition (3) and (6),

$$H_1(i, m_0(H_1, -2)) = f(i) \quad \text{and} \quad H_2(i, m_0(H_2, +2)) = h(i)$$

for all $i \in \mathbb{Z}$. Since H_3 is a concatenation of H_1 and H_2 on the 2^{nd}-axis, it follows that $m_0(H_3, -2) = m_0(H_1, -2) - m_0(H_1, +2)$ and $m_0(H_3, +2) = m_0(H_2, +2) -$

$m_0(H_2, -2)$. Thus the face $\alpha^2_{-2}(H_3)$ is given by

$$\begin{aligned}
\alpha^2_{-2}(H_3)(i) &= H_3(i, m_0(H_3, -2)) \\
&= H_3(i, m_0(H_1, -2) - m_0(H_1, +2)) \\
&= H_1(i, m_0(H_1, -2) - m_0(H_1, +2) + m_0(H_1, +2)) \\
&= H_1(i, m_0(H_1, -2)) \\
&= f(i)
\end{aligned}$$

for all $i \in \mathbb{Z}$, since $m_0(H_1, -2) - m_0(H_1, +2) \leq 0$. Similarly, the face $\alpha^2_{+2}(H_3)$ is given by

$$\begin{aligned}
\alpha^2_{+2}(H_3)(i) &= H_3(i, m_0(H_3, +2)) \\
&= H_3(i, m_0(H_2, +2) - m_0(H_2, -2)) \\
&= H_2(i, m_0(H_2, +2) - m_0(H_2, -2) + m_0(H_2, -2)) \\
&= H_2(i, m_0(H_2, +2)) \\
&= h(i)
\end{aligned}$$

for all $i \in \mathbb{Z}$, since $m_0(H_2, +2) - m_0(H_2, -2) \geq 0$. Thus $\alpha^2_{-2}(H_3) = f$ and $\alpha^2_{+2}(H_3) = h$.

Hence, H_3 is a graph homotopy from f to h, so $f \sim h$. Therefore, the relation $\sim$ is transitive.

Thus $\sim$ is an equivalence relation on $C_1(G)$. $\square$

Definition 4.19. [3, Definition 3.4] Let $v_0 \in G$ be a distinguished vertex of the graph G. The set $B_n(G, v_0) \subseteq C_n(G)$ is the subset of all graph homomorphisms from I^n_∞ to G that are equal to v_0 outside of a finite region of I^n_∞ for $n \geq 0$.

Theorem 4.20. [3, Proposition 3.5] The n^{th} A-homotopy group of the graph G with distinguished vertex v_0 is

$$A_n(G, v_0) \cong (B_n(G, v_0)/\sim).$$

From now on, we refer to the set $B_1(G, v_0)/\sim$ as the fundamental group of G. In the next chapter, we show that $B_1(G, v_0)/\sim$ is a group under the operation of concatenation.

Chapter 5

The Group $B_1(G, v_0)/\sim$

Since $B_1(G, v_0) \subseteq C_1(G)$ and $\sim$ is an equivalence relation on $C_1(G)$, the relation $\sim$ is a equivalence relation on $B_1(G, v_0)$. Thus the set $B_1(G, v_0)/\sim$ is well-defined. We now need to show that the set $B_1(G, v_0)/\sim$ is a group with the operation of concatenation.

Remark 5.1. This result is stated in the existing literature, but the full proof is not, since it is similar to the proof that the discrete fundamental group of a simplicial complex is a group, which is including in the literature.

To do this, we need a series of lemmas. The first is called the *Padding Lemma* (5.2). When a path $f : I_\infty \to G$ maps a sequence of consecutive vertices to the same vertex in G, this section is called *padding*. The *Padding Lemma* (5.2) states that a path with padding is homotopic to that same path with the padding removed.

Lemma 5.2 (Padding Lemma). *Let $f \in C_1(G)$. Define $f', f'' \in C_1(G)$ by*

$$f'(i) = \begin{cases} f(i-n) & \text{for} \quad i \geq b+n, \\ f(b) & \text{for} \quad b \leq i \leq b+n, \\ f(i) & i \text{ for} \quad \leq b, \end{cases}$$

and

$$f''(i) = \begin{cases} f(i) & \textit{for} \quad i \geq b, \\ f(b) & \textit{for} \quad b-n \leq i \leq b, \\ f(i+n) & \textit{for} \quad i \leq b-n, \end{cases}$$

for some $n \in \mathbb{N}$ *and some* $b \in \mathbb{Z}$ *such that* $m_0(f,-1) \leq b \leq m_0(f,+1)$. *Then* $f \sim f' \sim f''$.

Proof. Let $f \in C_1(G)$, and $f' \in C_1(G)$ be defined as in the statement of the lemma. To show that $f \sim f'$, we define a map $H' : I_\infty^2 \to G$, show that H' is a stable graph homomorphism, and show that H' is a graph homotopy from f to f'. Define $H' : I_\infty^2 \to G$ by

$$H'(i,j) = \begin{cases} f(i) & \text{for} \quad j \leq 0 \\ f(i-j) & \text{for} \quad 0 \leq j \leq n, \ i \geq b+j, \\ f(b) & \text{for} \quad 0 \leq j \leq n, \ b \leq i \leq b+j, \\ f(i) & \text{for} \quad 0 \leq j \leq n, \ i \leq b \\ f'(i) & j \geq n. \end{cases}$$

We now show that H' is a graph homomorphism. By the definitions of I_∞ and Cartesian product, there are edges $\{(i,j),(i+1,j)\}, \{(i,j),(i,j+1)\} \in E(I_\infty^2)$ for all $i,j \in \mathbb{Z}$. Thus the map H' is a graph homomorphism if $H'(i,j) = H'(i+1,j)$ or $\{H'(i,j), H'(i+1,j)\} \in E(G)$, and $H'(i,j) = H'(i,j+1)$ or $\{H'(i,j), H'(i,j+1)\} \in E(G)$. Since f and f' are graph homomorphisms, and H' is constantly equal to f for $j \leq 0$ and constantly equal to f' for $j \geq n$, it is sufficient to examine H' for $0 \leq j < n$. The restriction $H'|_{I_\infty \square \{j\}} : I_\infty \to G$ is defined by

$$H'|_{I_\infty \square \{j\}}(i) = \begin{cases} f(i-j) & \text{for} \quad i \geq b+j, \\ f(b) & \text{for} \quad b \leq i \leq b+j, \\ f(i) & \text{for} \quad i \leq b, \end{cases}$$

for each $0 \leq j < n$. Since $H'|_{I_\infty \square \{j\}}$ is a graph homomorphism and $H'|_{I_\infty \square \{j\}}(i) = H'(i,j)$ and

$H'|_{I_\infty \square \{j\}}(i+1) = H'(i+1, j)$, it follows that $H'(i,j) = H'(i+1,j)$ or $\{H'(i,j), H'(i+1,j)\} \in E(G)$. Thus it suffices to show that $H'(i,j) = H'(i,j+1)$ or $\{H'(i,j), H'(i,j+1)\} \in E(G)$. Let $0 \leq j < n$.

- For $i \geq b + j$,

  $$H'(i,j) = f(i-j) \quad \text{and} \quad H'(i,j+1) = f(i-j-1).$$

  Since $\{i-j, i-j-1\} \in E(I_\infty$ for all $j \in \mathbb{Z}$ and f is a graph homomorphism, $f(i-j) = f(i-j-1)$ or $\{f(i-j),\ f(i-j-1)\} \in E(G)$. Thus $H'(i,j) = H'(i,j+1)$ or $\{H'(i,j),\ H'(i,j+1)\} \in E(G)$.

- For $b \leq i \leq b+j$,

  $$H'(i,j) = f(b) \quad \text{and} \quad H'(i,j+1) = f(b).$$

  Thus $H'(i,j) = f(b) = H'(i,j+1)$.

- For $i \leq b$,

  $$H'(i,j) = f(i) \quad \text{and} \quad H'(i,j+1) = f(i).$$

  Thus $H'(i,j) = f(i) = H'(i,j+1)$.

Therefore, the map H' is a graph homomorphism. To show that H' is a graph homotopy from f to f', we must verify conditions (a)-(c) from Definition 4.12.

(a) Since $f'(m_0(f,+1)+n) = f(m_0(f,+1)+n-n) = f(m_0(f,+1))$, it follows that $m_0(f',+1) = m_0(f,+1)+n$. Since $b \leq m_0(f,+1)$ implies that $b+n \leq m_0(f,+1)+n$, the face $\alpha^1_{+1}(f)$ is given by

$$\begin{aligned}
\alpha^1_{+1}(f)(*) &= f(m_0(f,+1)) \\
&= f(m_0(f,+1)+n-n) \\
&= f'(m_0(f,+1)+n) \\
&= f'(m_0(f',+1)) \\
&= \alpha^1_{+1}(f')(*).
\end{aligned}$$

Thus $\alpha^1_{+1}(f) = \alpha^1_{+1}(f')$. Similarly, since $f'(m_0(f,-1)) = f(m_0(f,-1))$, it follows that $m_0(f',-1) = m_0(f,-1)$. Since $m_0(f,-1) \leq b$, the face $\alpha^1_{-1}(f)$ is given by

$$\begin{aligned}
\alpha^1_{-1}(f)(*) &= f(m_0(f,-1)) \\
&= f'(m_0(f,-1)) \\
&= f'(m_0(f',-1)) \\
&= \alpha^1_{-1}(f')(*).
\end{aligned}$$

Thus $\alpha^1_{-1}(f) = \alpha^1_{-1}(f')$.

(b) For all $i \leq b$, $H'(i,j) = f(i)$ when $j \leq n$ and $H'(i,j) = f'(i)$ when $j \geq n$. Since $f'(i) = f(i)$ for all $i \leq b$, it follows that H' stabilizes on the 1^{st}-axis in the negative direction at $m_0(H',-1) = m_0(f',-1) = m_0(f,-1)$. Since $m_0(H',-1) = m_0(f,-1) \leq b$, the face

$\alpha^2_{-1}(H')$ is given by

$$\begin{aligned}
\alpha^2_{-1}(H')(j) &= H'(m_0(H',-1),j) \\
&= \begin{cases} f(m_0(H',-1)) & \text{for} \quad j \le 0, \\ f(m_0(H',-1)) & \text{for} \quad 0 \le j \le n, \\ f'(m_0(H',-1)) & \text{for} \quad j \ge n, \end{cases} \\
&= \begin{cases} f(m_0(f,-1)) & \text{for} \quad j \le 0, \\ f(m_0(f,-1)) & \text{for} \quad 0 \le j \le n, \\ f'(m_0(f',-1)) & \text{for} \quad j \ge n. \end{cases}
\end{aligned}$$

Thus $\alpha^2_{-1}(H')(j) = \alpha^1_{-1}(f)(*) = \alpha^1_{-1}(f')(*)$ for all $j \in \mathbb{Z}$. Therefore, $\alpha^2_{-1}(H') = \beta^1_1\alpha^1_{-1}(f) = \beta^1_1\alpha^1_{-1}(f')$. Also, H' stabilizes on the 1^{st}-axis in the positive direction at $m_0(H',+1) = m_0(f',+1) = m_0(f,+1) + n$. Since $b \le m_0(f,+1)$ implies that $b + j \le m_0(f,+1) + j \le m_0(f,+1) + n = m_0(H',+1)$, the face $\alpha^2_{+1}(H')$ is given by

$$\begin{aligned}
\alpha^2_{+1}(H')(j) &= H'(m_0(H',+1),j) \\
&= \begin{cases} f(m_0(H',+1)) & \text{for} \quad j \le 0, \\ f(m_0(H',+1) - j) & \text{for} \quad 0 \le j \le n, \\ f'(m_0(H',+1)) & \text{for} \quad j \ge n, \end{cases} \\
&= \begin{cases} f(m_0(f,+1) + n) & \text{for} \quad j \le 0, \\ f(m_0(f,+1) + n - j) & \text{for} \quad 0 \le j \le n, \\ f'(m_0(f',+1)) & \text{for} \quad j \ge n. \end{cases}
\end{aligned}$$

Thus $\alpha^2_{+1}(H')(j) = \alpha^1_{+1}(f)(*) = \alpha^1_{+1}(f')(*)$ for all $j \in \mathbb{Z}$. Therefore, $\alpha^2_{+1}(H') = \beta^1_1\alpha^1_{+1}(f) = \beta^1_1\alpha^1_{+1}(f')$.

(c) By construction of H', $m_0(H', -2) = 0$ and $m_0(H', +2) = n$. Hence, the faces $\alpha^2_{-2}(H')$ and $\alpha^2_{+2}(H')$ are given by

$$\alpha^2_{-2}(H')(i) = H'(i,0) = f(i) \quad \text{and} \quad \alpha^2_{+2}(H')(i) = H'(i,n) = f'(i),$$

for all $i \in \mathbb{Z}$. Thus $\alpha^2_{-2}(H') = f$ and $\alpha^2_{+2}(H') = f'$.

Therefore, H' is a homotopy from f to f', so $f \sim f'$. The proof that $f \sim f''$ proceeds in the same way using the homotopy $H'' \in C_2(G)$ defined by

$$H''(i,j) = \begin{cases} f(i) & \text{for} \quad j \leq 0, \\ f(i) & \text{for} \quad 0 \leq j \leq n,\ i \geq b, \\ f(b) & \text{for} \quad 0 \leq j \leq n,\ b - j \leq i \leq b, \\ f(i+j) & \text{for} \quad 0 \leq j \leq n,\ i \leq b - j, \\ f''(i) & \text{for} \quad j \geq n. \end{cases}$$

□

We combine the two cases of the previous lemma into the following convenient statement.

Lemma 5.3 (**General Padding Lemma**). *Let $f \in C_1(G)$. Define $f' \in C_1(G)$ by*

$$f'(i) = \begin{cases} f(i-m) & \textit{for} \quad i \geq b + m, \\ f(b) & \textit{for} \quad b - n \leq i \leq b + m, \\ f(i+n) & \textit{for} \quad i \leq b - n, \end{cases}$$

for some $n, m \in \mathbb{N}$ and $b \in \mathbb{Z}$ with $m_0(f, -1) < b < m_0(f, +1)$. Then $f \sim f'$.

Proof. Let $f \in C_1(G)$ and $f' \in C_1(G)$ be defined as in the statement of the lemma. By the

Padding Lemma (5.2), $f \sim g$, where

$$g(i) = \begin{cases} f(i-m) & \text{for} \quad i \geq b+m, \\ f(b) & \text{for} \quad b \leq i \leq b+m, \\ f(i) & \text{for} \quad i \leq b. \end{cases}$$

Also by the *Padding Lemma* (5.2), $g \sim h$, where

$$\begin{aligned} h(i) &= \begin{cases} g(i) & \text{for} \quad i \geq b, \\ g(b) & \text{for} \quad b-n \leq i \leq b, \\ g(i+n) & \text{for} \quad i \leq b-n, \end{cases} \\ &= \begin{cases} f(i-m) & \text{for} \quad i \geq b+m, \\ f(b) & \text{for} \quad b \leq i \leq b+m, \\ f(b) & \text{for} \quad b-n \leq i \leq b, \\ f(i+n) & \text{for} \quad i \leq b-n, \end{cases} \\ &= \begin{cases} f(i-m) & \text{for} \quad i \geq b+m, \\ f(b) & \text{for} \quad b-n \leq i \leq b+m, \\ f(i+n) & \text{for} \quad i \leq b-n. \end{cases} \end{aligned}$$

Thus $f \sim g \sim h = f'$. Since $\sim$ is an equivalence relation, $f \sim f'$. $\square$

Lastly, we need the *Shifting Lemma* (5.4). This lemma states that a path is homotopic to that same path shifted down to start at an earlier vertex and to that same path shifted up to start at a later vertex.

Lemma 5.4 (Shifting Lemma). *Let $f \in C_1(G)$ and $n \in \mathbb{N}$. Define $f_n \in C_1(G)$ by $f_n(i) = f(i-n)$ (f shifted by n). Then $f \sim f_n$. Similarly, if $f_{-n} \in C_1(G)$ is defined by*

$f_{-n}(i) = f(i+n)$ (f shifted down by n), then $f \sim f_{-n}$.

Proof. Let $f \in C_1(G)$, and suppose $f_n \in C_1(G)$ is defined by $f_n(i) = f(i-n)$ for some $n \in \mathbb{N}$. To show $f \sim f_n$, we define a map $H_n : I_\infty^2 \to G$, show that H_n is a graph homomorphism, and show that H_n is a graph homotopy from f to f_n. Define $H_n : I_\infty^2 \to G$ by

$$H_n(i,j) = \begin{cases} f(i) & \text{for} \quad j \leq 0, \\ f(i-j) & \text{for} \quad 0 \leq j \leq n, \\ f(i-n) & \text{for} \quad j \geq n. \end{cases}$$

Since $f(i) = f(i-j)$ for $j = 0$ and $f(i-j) = f(i-n)$ for $j = n$, the map H_n is well-defined. We now show that H_n is a graph homomorphism. By the definitions of I_∞^2 and the Cartesian product, there are edges $\{(i,j),(i+1,j)\}, \{(i,j),(i,j+1)\} \in E(I_\infty)$ for all $i,j \in \mathbb{Z}$. Thus the map H_n is a graph homomorphism if $H_n(i,j) = H_n(i+1,j)$ or $\{H_n(i,j), H_n(i+1,j)\} \in E(G)$, and $H_n(i,j) = H_n(i,j+1)$ or $\{H_n(i,j), H_n(i,j+1)\} \in E(G)$ for all $i,j \in \mathbb{Z}$. Since H_n is constantly equal to f for $j \leq 0$ and constantly equal to f_n for $j \geq n$, it suffices to examine H_n for $0 \leq j < n$. Let $0 \leq j < n$.

- For all $i \in \mathbb{Z}$,

  $$H_n(i,j) = f(i-j) \quad \text{and} \quad H_n(i+1,j) = f(i+1-j).$$

  Since $\{i-j, i+1-j\} \in E(I_\infty)$ for all $i,j \in \mathbb{Z}$ and f is a graph homomorphism, $f(i-j) = f(i+1-j)$ or $\{f(i-j),\ f(i+1-j)\} \in E(G)$. Hence, $H_n(i,j) = H_n(i+1,j)$ or $\{H_n(i,j),\ H_n(i+1,j)\} \in E(G)$.

- Similarly, for all $i \in \mathbb{Z}$,

  $$H_n(i,j) = f(i-j) \quad \text{and} \quad H_n(i,j+1) = f(i-j-1)).$$

Since $\{i-j, i-j-1\} \in E(I_\infty)$ for all $i, j \in \mathbb{Z}$ and f is a graph homomorphism, $f(i-j) = f(i-j-1)$ or $\{f(i-j),\ f(i-j-1)\} \in E(G)$. Hence, $H_n(i,j) = H_n(i, j+1)$ or $\{H_n(i,j),\ H_n(i,j+1)\} \in E(G)$.

Thus H_n is a graph homomorphism. We now show that H_n is a graph homotopy from f to f_n by verifying conditions (a)-(c) of Definition 4.12.

(a) Since $m_0(f_n, +1) = m_0(f, +1) + n$ and $m_0(f_n, -1) = m_0(f, -1) + n$, the face $\alpha^1_{+1}(f)$ is given by

$$\begin{aligned}
\alpha^1_{+1}(f)(*) &= f(m_0(f, +1)) \\
&= f(m_0(f, +1) + n - n) \\
&= f_n(m_0(f, +1) + n) \\
&= f_n(m_0(f_n, +1)) \\
&= \alpha^1_{+1}(f_n)(*),
\end{aligned}$$

and the face $\alpha^1_{-1}(f)$ is given by

$$\begin{aligned}
\alpha^1_{-1}(f)(*) &= f(m_0(f, -1)) \\
&= f(m_0(f, -1) + n - n) \\
&= f_n(m_0(f, -1) + n) \\
&= f_n(m_0(f_n, -1)) \\
&= \alpha^1_{-1}(f_n)(*).
\end{aligned}$$

Thus $\alpha^1_{+1}(f) = \alpha^1_{+1}(f_n)$ and $\alpha^1_{-1}(f) = \alpha^1_{-1}(f_n)$.

(b) Let $(H_n)_j : I_\infty \to G$ be defined by $(H_n)_j(i) = H_n(i,j)$ for all $i, j \in \mathbb{Z}$. Then $(H_n)_j(i) = f(i-j)$ for $0 \le i \le n$, which implies that $m_0((H_n)_j, +1) = m_0(f, +1) + j$ and $m_0((H_n)_j, -1) = m_0(f, -1) + j$. Since H_n is constantly equal to f for $j \le 0$ and H_n is constantly equal to f_n for $j \geq n$, it follows that $m_0(H_n, +1) = \max\{m_0((H_n)_j, +1) \mid 0 \leq j \leq n\} = \max\{m_0(f, +1)+j \mid 0 \leq j \leq n\} = m_0(f, +1)+n$. Similarly, $m_0(H_n, -1) = \min\{m_0((H_n)_j, -1) \mid 0 \leq j \leq n\} = \min\{m_0(f, -1) + j \mid 0 \leq j \leq n\} = m_0(f, -1)$. Hence, the face $\alpha^2_{+1}(H_n)$ is given by

$$\begin{aligned}
\alpha^2_{+1}(H_n)(j) &= H_n(m_0(H_n, +1), j) \\
&= \begin{cases} f(m_0(H_n, +1)) & \text{for} \quad j \leq 0, \\ f(m_0(H_n, +1) - j) & \text{for} \quad 0 \leq j \leq n, \\ f(m_0(H_n, +1) - n) & \text{for} \quad j \geq n, \end{cases} \\
&= \begin{cases} f(m_0(f, +1) + n) & \text{for} \quad j \leq 0, \\ f(m_0(f, +1) + n - j) & \text{for} \quad 0 \leq j \leq n, \\ f(m_0(f, +1) + n - n) & \text{for} \quad j \geq n, \end{cases} \\
&= \begin{cases} f(m_0(f, +1) + n) & \text{for} \quad j \leq 0, \\ f(m_0(f, +1) + n - j) & \text{for} \quad 0 \leq j \leq n, \\ f(m_0(f, +1)) & \text{for} \quad j \geq n. \end{cases}
\end{aligned}$$

Thus $\alpha^2_{+1}(H_n)(j) = \alpha^1_{+1}(f)(*) = \alpha^1_{+1}(f_n)(*)$ for all $j \in \mathbb{Z}$. Therefore, $\alpha^2_{+1}(H_n) =$

$\beta_1^1\alpha_{+1}^1(f) = \beta_1^1\alpha_{+1}^1(f_n)$. Similarly, the face $\alpha_{-1}^2(H_n)$ is given by

$$\begin{aligned}
\alpha_{-1}^2(H_n)(j) &= H_n(m_0(H_n,-1),j) \\
&= \begin{cases} f(m_0(H_n,-1)) & \text{for} \quad j \leq 0, \\ f(m_0(H_n,-1)-j) & \text{for} \quad 0 \leq j \leq n, \\ f(m_0(H_n,-1)-n) & \text{for} \quad j \geq n, \end{cases} \\
&= \begin{cases} f(m_0(f,-1)) & \text{for} \quad j \leq 0, \\ f(m_0(f,-1)-j) & \text{for} \quad 0 \leq j \leq n, \\ f(m_0(f,-1)-n) & \text{for} \quad j \geq n. \end{cases}
\end{aligned}$$

Thus $\alpha_{-1}^2(H_n)(j) = \alpha_{-1}^1(f)(*) = \alpha_{-1}^1(f_n)(*)$ for all $j \in \mathbb{Z}$. Therefore, $\alpha_{-1}^2(H_n) = \beta_1^1\alpha_{-1}^1(f) = \beta_1^1\alpha_{-1}^1(f_n)$.

(c) By construction, H_n stabilizes on the 2^{nd}-axis at the integers $m_0(H_n,-2) = 0$ and $m_0(H_n,+2) = n$. Thus the faces $\alpha_{-2}^2(H_n)$ and $\alpha_{+2}^2(H_n)$ are given by

$$\alpha_{-2}^2(H_n)(i) = H_n(i,0) = f(i) \quad \text{and} \quad \alpha_{+2}^2(H_n)(i) = H_n(i,n) = f(i-n) = f_n(i)$$

for all $i \in \mathbb{Z}$. Thus $\alpha_{-2}^2(H_n) = f$ and $\alpha_{+2}^2(H_n) = f_n$.

Thus H_n is a graph homotopy from f to f_n, so $f \sim f_n$ for all $n \in \mathbb{N}$. The proof of $f \sim f_{-n}$ proceeds in the same way using the graph homotopy $H_{-n} \in C_2(G)$ defined by

$$H_{-n}(i,j) = \begin{cases} f(i) & \text{for} \quad j \leq 0, \\ f(i+j) & \text{for} \quad 0 \leq j \leq n, \\ f(i+n) & \text{for} \quad j \geq n, \end{cases}$$

for all $i \in \mathbb{Z}$. $\square$

With the *General Padding Lemma* (5.3) and the *Shifting Lemma* (5.4), we can now proceed to the proof that the set $B_1(G, v_0)/\sim$ with the operation of concatenation has group structure. We prove this in five propositions:

- Concatenation is well-defined on the equivalence classes of $B_1(G, v_0)/\sim$.
- The set $B_1(G, v_0)$ is closed with respect to concatenation.
- The set $B_1(G, v_0)/\sim$ has an identity element.
- Every element of the set $B_1(G, v_0)/\sim$ has an inverse in the set.
- Concatenation on the set $B_1(G, v_0)/\sim$ is associative.

Proposition 5.5 (Well-Defined). Concatenation is well-defined on the equivalence classes of $B_1(G, v_0)/\sim$.

Proof. Let $f_1, g_1, f_2, g_2 \in B_1(G, v_0)$ be such that $f_1 \sim g_1$ and $f_2 \sim g_2$. Then there exists a graph homotopy $H_1 \in C_2(G)$ such that

(1) $\alpha^1_{-1}(f_1) = \alpha^1_{-1}(g_1)$ and $\alpha^1_{+1}(f_1) = \alpha^1_{+1}(g_1)$,

(2) $\alpha^2_{-1}(H_1) = \beta^1_1\alpha^1_{-1}(f_1) = \beta^1_1\alpha^1_{-1}(g_1)$ and $\alpha^2_{-1}(H_1) = \beta^1_1\alpha^1_{+1}(f_1) = \beta^1_1\alpha^1_{+1}(g_1)$,

(3) $\alpha^2_{-2}(H_1) = f_1$ and $\alpha^2_{+2}(H_1) = g_1$,

and there exists a graph homotopy $H_2 \in C_2(G)$ such that

(4) $\alpha^1_{-1}(f_2) = \alpha^1_{-1}(g_2)$ and $\alpha^1_{+1}(f_2) = \alpha^1_{+1}(g_2)$,

(5) $\alpha^2_{-1}(H_2) = \beta^1_1\alpha^1_{-1}(f_2) = \beta^1_1\alpha^1_{-1}(g_2)$ and $\alpha^2_{+1}(H_2) = \beta^1_1\alpha^1_{+1}(f_2) = \beta^1_1\alpha^1_{+1}(g_2)$,

(6) $\alpha^2_{-2}(H_2) = f_2$ and $\alpha^2_{+2}(H_2) = g_2$.

These graph homotopies are illustrated in Figure 5.1 with only the active regions of the lattice I^2_∞ shown as light blue rectangles. The graph homomorphisms f_1 and f_2 are shown

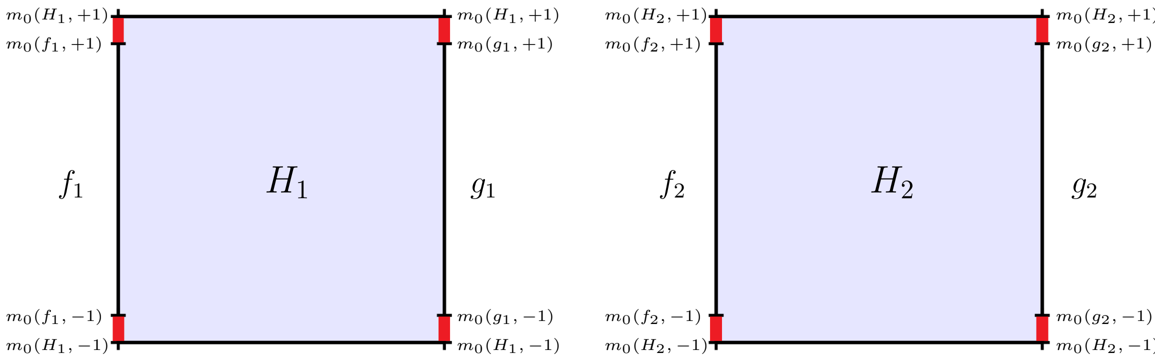

Figure 5.1: The homotopies H_1 and H_2

on the left sides, and the graph homomorphisms g_1 and g_2 are shown on the right sides of the graph homotopies. By parts (3) and (6), for all $i \in \mathbb{Z}$,

$$H_1(i, m_0(H_1, -2)) = f_1(i) \quad \text{and} \quad H_1(i, m_0(H_1, +2)) = g_1(i),$$

and

$$H_2(i, m_0(H_1, -2)) = f_2(i) \quad \text{and} \quad H_2(i, m_0(H_2, +2)) = g_2(i).$$

This implies that

$$\begin{aligned} m_0(H_1, +1) \geq m_0(f_1, +1) \quad &\text{and} \quad m_0(H_1, +1) \geq m_0(g_1, +1), \\ m_0(H_1, -1) \leq m_0(f_1, -1) \quad &\text{and} \quad m_0(H_1, -1) \leq m_0(g_1, -1), \\ m_0(H_2, +1) \geq m_0(f_2, +1) \quad &\text{and} \quad m_0(H_2, +1) \geq m_0(g_2, +1), \\ m_0(H_2, -1) \leq m_0(f_2, -1) \quad &\text{and} \quad m_0(H_2, -1) \leq m_0(g_2, -1). \end{aligned}$$

Because of these inequalities, there is potentially some padding between each of the following pairs of the vertices:

$$
\begin{array}{ccc}
(m_0(H_1,+1), m_0(H_1,-2)) & \text{and} & (m_0(f_1,+1), m_0(H_1,-2)), \\
(m_0(H_1,-1), m_0(H_1,-2)) & \text{and} & (m_0(f_1,-1), m_0(H_1,-2)), \\
(m_0(H_1,+1), m_0(H_1,+2)) & \text{and} & (m_0(g_1,+1), m_0(H_1,+2)), \\
(m_0(H_1,-1), m_0(H_1,+2)) & \text{and} & (m_0(g_1,-1), m_0(H_1,+2)), \\
(m_0(H_2,+1), m_0(H_2,-2)) & \text{and} & (m_0(f_2,+1), m_0(H_2,-2)), \\
(m_0(H_2,-1), m_0(H_2,-2)) & \text{and} & (m_0(f_2,-1), m_0(H_2,-2)), \\
(m_0(H_2,+1), m_0(H_2,+2)) & \text{and} & (m_0(g_2,+1), m_0(H_2,+2)), \\
(m_0(H_2,-1), m_0(H_2,+2)) & \text{and} & (m_0(g_2,-1), m_0(H_2,+2)).
\end{array}
$$

These sections of potential padding are depicted as thick red lines in Figure 5.1.

The concatenations $f_1 \cdot f_2$ and $g_1 \cdot g_2$ are defined by

$$
(f_1 \cdot f_2)(i) = \begin{cases} f_1(i + m_0(f_1,-1)) & \text{for} \quad i \geq 0, \\ f_2(i + m_0(f_2,+1)) & \text{for} \quad i \leq 0. \end{cases}
$$

and

$$
(g_1 \cdot g_2)(i) = \begin{cases} g_1(i + m_0(g_1,-1)) & \text{for} \quad i \geq 0, \\ g_2(i + m_0(g_2,+1)) & \text{for} \quad i \leq 0. \end{cases}
$$

Since $f_1, g_1, f_2, g_2 \in B_1(G, v_0)$, it follows that $\alpha^1_{-1}(f_1) = \alpha^1_{+1}(f_2)$ and $\alpha^1_{-1}(g_1) = \alpha^1_{+1}(g_2)$, which implies that the concatenations $f_1 \cdot f_2$ and $g_1 \cdot g_2$ are well-defined and graph homomorphisms. In order to show that $f_1 \cdot f_2 \sim g_1 \cdot g_2$, we need to define a graph homotopy from $f_1 \cdot f_2$ to $g_1 \cdot g_2$. Consider the concatenation of the two graph homotopies H_1 and H_2 on the

1^{st}-axis defined by

$$(H_1 \cdot_1 H_2)(i,j) = \begin{cases} H_1(i + m_0(H_1, -1),\ j) & \text{for} \quad i \geq 0, \\ H_2(i + m_0(H_2, +1),\ j) & \text{for} \quad i \leq 0. \end{cases}$$

This concatentation is depicted Figure 5.2. Since $f_1, g_1, f_2, g_2 \in B_1(G, v_0)$, by parts (2)

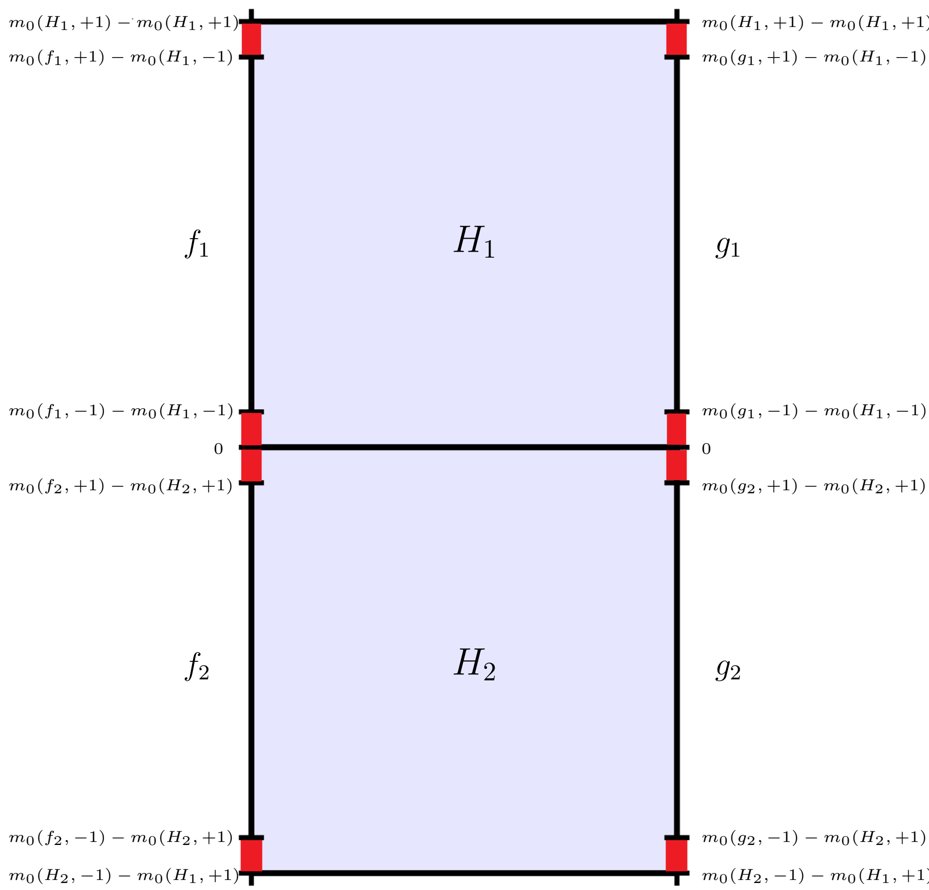

Figure 5.2: The concatenation of H_1 and H_2

and (4), $H_1(m_0(H_1, -1), j) = v_0$ and $H_2(m_0(H_2, +1), j) = v_0$ for all $j \in \mathbb{Z}$. Therefore, $\alpha^2_{-1}(H_1) = \alpha^2_{+1}(H_2)$, which implies that $H_1 \cdot_1 H_2$ is well-defined and a graph homomorphism. However, this concatenation is not necessarily a graph homotopy from $f_1 \cdot f_2$ to $g_1 \cdot g_2$, as we would hope, but $H_1 \cdot_1 H_2$ is still useful. Let us examine the faces $\alpha^2_{-2}(H_1 \cdot_1 H_2)$ and

$\alpha^2_{-2}(H_1 \cdot_1 H_2)$ to show that this is the case.

Since H_1 stabilizes on the 2^{nd}-axis in the negative direction at $m_0(H_1,-2)$, H_2 stabilizes on the 2^{nd}-axis in the negative direction at $m_0(H_2,-2)$, and $H_1 \cdot_1 H_2$ is the concatenation of H_1 and H_2 on the 1^{st}-axis, it follows that $H_1 \cdot_1 H_2$ stabilizes on the 2^{nd}-axis in the negative direction at $m_0(H_1 \cdot_1 H_2,-2) = \min\{m_0(H_1,-2),\ m_0(H_2,-2)\}$. Thus by parts (3) and (6), the face $\alpha^2_{-2}(H_1 \cdot_1 H_2)$ is given by

$$\begin{aligned}
\alpha^2_{-2}(H_1 \cdot_1 H_2)(i) &= (H_1 \cdot_1 H_2)(i, m_0(H_1 \cdot_1 H_2,-2)) \\
&= \begin{cases} H_1(i+m_0(H_1,-1), m_0(H_1 \cdot_1 H_2,-2)) & \text{for} \quad i \geq 0, \\ H_2(i+m_0(H_2,+1), m_0(H_1 \cdot_1 H_2,-2)) & \text{for} \quad i \leq 0, \end{cases} \\
&= \begin{cases} H_1(i+m_0(H_1,-1), m_0(H_1,-2)) & \text{for} \quad i \geq 0, \\ H_2(i+m_0(H_2,+1), m_0(H_2,-2)) & \text{for} \quad i \leq 0, \end{cases} \\
&= \begin{cases} f_1(i+m_0(H_1,-1)) & \text{for} \quad i \geq 0, \\ f_2(i+m_0(H_2,+1)) & \text{for} \quad i \leq 0. \end{cases}
\end{aligned}$$

Thus $\alpha^2_{-2}(H_1 \cdot_1 H_2)$ is constantly equal to v_0 from the vertex $m_0(f_2,+1)-m_0(H_2,+1)$ to the vertex $m_0(f_1,-1)-m_0(H_1,-1)$. Therefore, $\alpha^2_{-2}(H_1 \cdot_1 H_2) \neq f_1 \cdot f_2$ if

$$m_0(f_2,+1) \neq m_0(H_2,+1) \quad \text{or} \quad m_0(f_1,-1) \neq m_0(H_1,-1),$$

because there is padding between f_1 and f_2. However, by the *General Padding Lemma* (5.3), $\alpha^2_{-2}(H_1 \cdot_1 H_2) \sim f_1 \cdot f_2$. Also, since H_1 stabilizes on the 2^{nd}-axis in the positive direction at $m_0(H_1,+2)$, H_2 stabilizes on the 2^{nd}-axis in the positive direction at $m_0(H_2,+2)$, and $H_1 \cdot_1 H_2$ is the concatenation of H_1 and H_2 on the 1^{st}-axis, it follows that $H_1 \cdot_1 H_2$ stabilizes on the 2^{nd}-axis in the positive direction at $m_0(H_1 \cdot_1 H_2,+2) = \max\{m_0(H_1,+2),\ m_0(H_2,+2)\}$.

Thus by part (3) and (6), the face $\alpha^2_{+2}(H_1 \cdot_1 H_2)$ is given by

$$
\begin{aligned}
\alpha^2_{+2}(H_1 \cdot_1 H_2)(i) &= (H_1 \cdot_1 H_2)(i, m_0(H_1 \cdot_1 H_2, +2)) \\
&= \begin{cases} H_1(i + m_0(H_1, -1), m_0(H_1 \cdot_1 H_2, +2)) & \text{for} \quad i \geq 0, \\ H_2(i + m_0(H_2, +1), m_0(H_1 \cdot_1 H_2, +2)) & \text{for} \quad i \leq 0, \end{cases} \\
&= \begin{cases} H_1(i + m_0(H_1, -1), m_0(H_1, +2)) & \text{for} \quad i \geq 0, \\ H_2(i + m_0(H_2, +1), m_0(H_2, +2)) & \text{for} \quad i \leq 0, \end{cases} \\
&= \begin{cases} g_1(i + m_0(H_1, -1)) & \text{for} \quad i \geq 0, \\ g_2(i + m_0(H_2, +1)) & \text{for} \quad i \leq 0. \end{cases}
\end{aligned}
$$

Thus $\alpha^2_{+2}(H_1 \cdot_1 H_2)$ is constantly equal to v_0 from the vertex $m_0(g_2, +1) - m_0(H_2, +1)$ to the vertex $m_0(g_1, -1) - m_0(H_1, -1)$. Therefore, $\alpha^2_{+2}(H_1 \cdot_1 H_2) \neq g_1 \cdot g_2$ if

$$m_0(g_2, +1) \neq m_0(H_2, +1) \quad \text{and} \quad m_0(g_1, -1) \neq m_0(H_1, -1),$$

because there is padding between g_1 and g_2. However, by the *General Padding Lemma* (5.3), $\alpha^2_{+2}(H_1 \cdot_1 H_2) \sim g_1 \cdot g_2$. Thus $H_1 \cdot_1 H_2$ may not be a homotopy from $f_1 \cdot f_2$ to $g_1 \cdot g_2$, but if the concatenation $H_1 \cdot_1 H_2$ is a homotopy from $\alpha^2_{-2}(H_1 \cdot_1 H_2)$ to $\alpha^2_{+2}(H_1 \cdot_1 H_2)$, then $f_1 \cdot f_2 \sim g_1 \cdot g_2$. We now show that $H_1 \cdot_1 H_2$ is a graph homotopy from $\alpha^2_{-2}(H_1 \cdot_1 H_2)$ to $\alpha^2_{+2}(H_1 \cdot_1 H_2)$ by verifying conditions (a)-(c) found in Definition 4.12.

(a) By the definition of concatentation, and since $\alpha^2_{-2}(H_1 \cdot_1 H_2) \sim f_1 \cdot f_2$,

$$\alpha^1_{+1}(\alpha^2_{-2}(H_1 \cdot_1 H_2)) = \alpha^1_{+1}(f_1 \cdot f_2) = \alpha^1_{+1}(f_1)$$

and

$$\alpha^1_{-1}(\alpha^2_{-2}(H_1 \cdot_1 H_2)) = \alpha^1_{-1}(f_1 \cdot f_2) = \alpha^1_{-1}(f_2).$$

By the definition of concatenation, and since $\alpha^2_{+2}(H_1 \cdot_1 H_2) \sim g_1 \cdot g_2$,

$$\alpha^1_{+1}(\alpha^2_{+2}(H_1 \cdot_1 H_2)) \;=\; \alpha^1_{+1}(g_1 \cdot g_2) \;=\; \alpha^1_{+1}(g_1)$$

and

$$\alpha^1_{-1}(\alpha^2_{+2}(H_1 \cdot_1 H_2)) \;=\; \alpha^1_{-1}(g_1 \cdot g_2) \;=\; \alpha^1_{-1}(g_2).$$

By part (1), $\alpha^1_{+1}(f_1) = \alpha^1_{+1}(g_1)$, and by part (4), $\alpha^1_{-1}(f_2) = \alpha^1_{-1}(g_2)$. Therefore, by the previous statements,

$$\alpha^1_{+1}(\alpha^2_{-2}(H_1 \cdot_1 H_2)) \;=\; \alpha^1_{+1}(\alpha^2_{+2}(H_1 \cdot_1 H_2))$$

and

$$\alpha^1_{-1}(\alpha^2_{-2}(H_1 \cdot_1 H_2)) \;=\; \alpha^1_{-1}(\alpha^2_{+2}(H_1 \cdot_1 H_2)).$$

(b) By the definition of concatenation, $\alpha^2_{+1}(H_1 \cdot_1 H_2) = \alpha^2_{+1}(H_1)$ and $\alpha^2_{-1}(H_1 \cdot_1 H_2) = \alpha^2_{-1}(H_2)$. Recall that H_1 is a graph homotopy from f_1 to g_1, and H_2 is a graph homotopy from f_2 to g_2. Thus by part (2), $\alpha^2_{+1}(H_1) = \beta^1_1\alpha^1_{+1}(f_1)$ and $\alpha^2_{+1}(H_1) = \beta^1_1\alpha^1_{+1}(g_1)$, and by part (5), $\alpha^2_{-1}(H_2) = \beta^1_1\alpha^1_{-1}(f_2)$ and $\alpha^2_{-1}(H_2) = \beta^1_1\alpha^1_{-1}(g_2)$. By definition of concatenation,

$$\begin{aligned}
\beta^1_1\alpha^1_{+1}(f_1) &= \beta^1_1\alpha^1_{+1}(f_1 \cdot f_2),\\
\beta^1_1\alpha^1_{+1}(g_1) &= \beta^1_1\alpha^1_{+1}(g_1 \cdot g_2),\\
\beta^1_1\alpha^1_{-1}(f_2) &= \beta^1_1\alpha^1_{-1}(f_1 \cdot f_2),\\
\beta^1_1\alpha^1_{-1}(g_2) &= \beta^1_1\alpha^1_{-1}(g_1 \cdot g_2).
\end{aligned}$$

Since $\alpha^2_{-2}(H_1 \cdot_1 H_2) \sim f_1 \cdot f_2$, it follows that $\beta^1_1\alpha^1_{+1}(f_1 \cdot f_2) = \beta^1_1\alpha^1_{+1}(\alpha^2_{-2}(H_1 \cdot_1 H_2))$ and $\beta^1_1\alpha^1_{+1}(g_1 \cdot g_2) = \beta^1_1\alpha^1_{+1}(\alpha^2_{+2}(H_1 \cdot_1 H_2))$. Similarly, $\alpha^2_{+2}(H_1 \cdot_1 H_2) \sim g_1 \cdot g_2$ implies that $\beta^1_1\alpha^1_{-1}(f_1 \cdot f_2) = \beta^1_1\alpha^1_{-1}(\alpha^2_{-2}(H_1 \cdot_1 H_2))$ and $\beta^1_1\alpha^1_{-1}(g_1 \cdot g_2) = \beta^1_1\alpha^1_{-1}(\alpha^2_{+2}(H_1 \cdot_1 H_2))$.

Therefore,

$$\begin{aligned}\alpha^2_{+1}(H_1 \cdot_1 H_2) = \alpha^2_{+1}(H_1) &= \beta^1_1\alpha^1_{+1}(\alpha^2_{-2}(H_1 \cdot_1 H_2)) \\ &= \beta^1_1\alpha^1_{+1}(\alpha^2_{+2}(H_1 \cdot_1 H_2))\end{aligned}$$

and

$$\begin{aligned}\alpha^2_{-1}(H_1 \cdot_1 H_2) = \alpha^2_{-1}(H_2) &= \beta^1_1\alpha^1_{-1}(\alpha^2_{-2}(H_1 \cdot_1 H_2)) \\ &= \beta^1_1\alpha^1_{-1}(\alpha^2_{+2}(H_1 \cdot_1 H_2)).\end{aligned}$$

(c) Trivially, $\alpha^2_{-2}(H_1 \cdot_1 H_2) = \alpha^2_{-2}(H_1 \cdot_1 H_2)$ and $\alpha^2_{+2}(H_1 \cdot_1 H_2) = \alpha^2_{+2}(H_1 \cdot_1 H_2)$.

Thus $H_1 \cdot_1 H_2$ is a homotopy from $\alpha^2_{-2}(H_1 \cdot_1 H_2)$ to $\alpha^2_{+2}(H_1 \cdot_1 H_2)$, so $f_1 \cdot f_2 \sim \alpha^2_{-2}(H_1 \cdot_1 H_2) \sim \alpha^2_{+2}(H_1 \cdot_1 H_2) \sim g_1 \cdot g_2$. Hence, concatenation is well-defined on the set $B_1(G, v_0)/\sim$, that is, if $[f_1] = [g_1]$ and $[f_2] = [g_2]$, then $[f_1 \cdot f_2] = [g_1 \cdot g_2]$. □

Thus for each pair of elements $[f], [g] \in B_1(G, v_0)/\sim$, the concatenation of $[f]$ and $[g]$ is defined by $[f] \cdot [g] = [f \cdot g]$. We now continue by showing that the set $B_1(G, v_0)$ is closed under concatenation.

Proposition 5.6 (Closure). The set $B_1(G, v_0)$ is closed under concatenation.

Proof. Let $f, g \in B_1(G, v_0)$. Then $f(m_0(f, -1)) = g(m_0(g, +1)) = v_0$, and $\alpha^1_{-1}(f) = \alpha^1_{+1}(g)$. Thus the concatenation $f \cdot g$ is well-defined and defined by

$$(f \cdot g)(i) = \begin{cases} f(i + m_0(f, -1)) & \text{for} \quad i \geq 0, \\ g(i + m_0(g, +1)) & \text{for} \quad i \leq 0. \end{cases}$$

By Lemma 4.16, $m_0(f \cdot g, +1) = m_0(f, +1) - m_0(f, -1)$ and $m_0(f \cdot g, -1) = m_0(g, -1) -$

$m_0(g,+1)$. Thus the faces $\alpha^1_{+1}(f \cdot g)$ and $\alpha^1_{-1}(f \cdot g)$ are given by

$$\begin{aligned}
\alpha^1_{+1}(f \cdot g)(*) &= f \cdot g(m_0(f,+1) - m_0(f,-1)) \\
&= f(m_0(f,+1) - m_0(f,-1) + m_0(f,-1)) \\
&= f(m_0(f,+1)),
\end{aligned}$$

and

$$\begin{aligned}
\alpha^1_{-1}(f \cdot g)(*) &= f \cdot g(m_0(g,-1) - m_0(g,+1)) \\
&= g(m_0(g,-1) - m_0(g,+1) + m_0(g,+1)) \\
&= g(m_0(g,-1)).
\end{aligned}$$

Since $f, g \in B_1(G, v_0)$, it follows that f and g stabilize to v_0 in both directions. Thus $f \cdot g \in B_1(G, v_0)$ and $B_1(G, v_0)$ is closed with respect to concatenation. $\square$

Definition 5.7. Let the *constant path* $p_{v_0} : I_\infty \to G$ be defined by $p_{v_0}(i) = v_0$ for all $i \in \mathbb{Z}$.

Proposition 5.8 (Identity). The equivalence class of the constant path $p_{v_0} : I_\infty \to G$ is the identity element of $B_1(G, v_0)/\sim$.

Proof. Let $f \in B_1(G, v_0)$. Consider the concatenation $p_{v_0} \cdot f : I_\infty \to G$. Since $m_0(p_{v_0}, -1) =$

0 and $f \in B_1(G, v_0)$,

$$
\begin{aligned}
(p_{v_0} \cdot f)(i) &= \begin{cases} p_{v_0}(i + m_0(p_{v_0}, -1)) & \text{for} \quad i \geq 0, \\ f(i + m_0(f, +1)) & \text{for} \quad i \leq 0, \end{cases} \\
&= \begin{cases} p_{v_0}(i) & \text{for} \quad i \geq 0, \\ f(i + m_0(f, +1)) & \text{for} \quad i \leq 0, \end{cases} \\
&= \begin{cases} v_0 & \text{for} \quad i \geq 0, \\ f(i + m_0(f, +1)) & \text{for} \quad i \leq 0, \end{cases} \\
&= f(i + m_0(f, +1)).
\end{aligned}
$$

Thus $p_{v_0} \cdot f = f_{-m_0(f,+1)}$, the graph homomorphism f shifted by $-m_0(f, +1)$. Therefore, $f \sim p_{v_0} \cdot f$ by the *Shifting Lemma* (5.4). Now consider the concatenation $f \cdot p_{v_0} : I_\infty \to G$. Since $m_0(p_{v_0}, +1) = 0$ and $f \in B_1(G, v_0)$,

$$
\begin{aligned}
(f \cdot p_{v_0})(i) &= \begin{cases} f(i + m_0(f, -1)) & \text{for} \quad i \geq 0, \\ p_{v_0}(i + m_0(p_{v_0}, +1)) & \text{for} \quad i \leq 0, \end{cases} \\
&= \begin{cases} f(i + m_0(f, -1)) & \text{for} \quad i \geq 0, \\ p_{v_0}(i) & \text{for} \quad i \leq 0, \end{cases} \\
&= \begin{cases} f(i + m_0(f, -1)) & \text{for} \quad i \geq 0, \\ v_0 & \text{for} \quad i \leq 0, \end{cases} \\
&= f(i + m_0(f, -1)).
\end{aligned}
$$

Thus $f \cdot p_{v_0} = f_{-m_0(f,-1)}$, the graph homomorphism f shifted by $-m_0(f, -1)$. Hence, $f \cdot p_{v_0} \sim f$ by the *Shifting Lemma* (5.4). Thus the equivalence class of p_{v_0} is the identity element of $B_1(G, v_0)/\sim$. $\square$

Definition 5.9. For each $f \in C_1(G)$, let $\bar{f} \in C_1(G)$ be defined by $\bar{f}(i) = f(-i)$ for all $i \in \mathbb{Z}$.

Proposition 5.10 (Inverses). For each $[f] \in B_1(G, v_0)/\sim$, the equivalence class $[\bar{f}] \in B_1(G, v_0)/\sim$ is the inverse of $[f]$.

Proof. Let $f \in B_1(G, v_0)$. Then $\bar{f} \in B_1(G, v_0)$. By definition, $\bar{f}$ stabilizes in the positive direction at the integer $m_0(\bar{f}, +1) = -m_0(f, -1)$. Thus the concatenation $f \cdot \bar{f}$ is given by

$$
\begin{aligned}
(f \cdot \bar{f})(i) &= \begin{cases} f(i + m_0(f, -1)) & \text{for} \quad i \geq 0, \\ \bar{f}(i + m_0(\bar{f}, +1)) & \text{for} \quad i \leq 0, \end{cases} \\
&= \begin{cases} f(i + m_0(f, -1)) & \text{for} \quad i \geq 0, \\ \bar{f}(i - m_0(f, -1)) & \text{for} \quad i \leq 0, \end{cases} \\
&= \begin{cases} f(i + m_0(f, -1)) & \text{for} \quad i \geq 0, \\ f(-i + m_0(f, -1)) & \text{for} \quad i \leq 0. \end{cases}
\end{aligned}
$$

To show that $f \cdot \bar{f} \sim p_{v_0}$, we define a map $H_1 : I_\infty^2 \to G$ and show that H_1 is well-defined, is a stable graph homomorphism, and is a graph homotopy from $f \cdot \bar{f}$ to p_{v_0}. Define $H_1 : I_\infty^2 \to G$ by

$$
H_1(i, j) = \begin{cases} (f \cdot \bar{f})(i) & \text{for} \quad j \leq 0, \\ (f \cdot \bar{f})(i + j) & \text{for} \quad 0 \leq j \leq m_0(f, +1) - m_0(f, -1), \ i \geq 0, \\ (f \cdot \bar{f})(i - j) & \text{for} \quad 0 \leq j \leq m_0(f, +1) - m_0(f, -1), \ i \leq 0, \\ p_{v_0}(i) & \text{for} \quad j \geq m_0(f, +1) - m_0(f, -1). \end{cases}
$$

By definition of concatenation, $(f \cdot \bar{f})(i + j) = f(i + j + m_0(f, -1))$ for $i + j \geq 0$, and $(f \cdot \bar{f})(i - j) = \bar{f}(i - j + m_0(\bar{f}, +1))$ for $i - j \leq 0$. Since $m_0(\bar{f}, +1) = -m_0(f, -1)$, it follows that $\bar{f}(i-j+m_0(\bar{f},+1)) = f(-i+j+m_0(f,-1))$ by definition of $\bar{f}$. Thus

$$H_1(i,j) = \begin{cases} (f\cdot\bar{f})(i) & \text{for} \quad j \leq 0, \\ f(i+j+m_0(f,-1)) & \text{for} \quad 0 \leq j \leq m_0(f,+1) - m_0(f,-1),\ i \geq 0, \\ f(-i+j+m_0(f,-1)) & \text{for} \quad 0 \leq j \leq m_0(f,+1) - m_0(f,-1),\ i \leq 0, \\ v_0 & \text{for} \quad j \geq m_0(f,+1) - m_0(f,-1). \end{cases}$$

First, we show that H_1 is well-defined where it is doubly defined: when $0 \leq j \leq m_0(f,+1) - m_0(f,-1)$ and $i = 0$; when $j = 0$ and $i \leq 0$; when $j = 0$ and $i \geq 0$; when $j = m_0(f,+1) - m_0(f,-1)$ and $i \leq 0$; and when $j = m_0(f,+1) - m_0(f,-1)$ and $i \geq 0$.

- When $0 \leq j \leq m_0(f,+1) - m_0(f,-1)$ and $i = 0$, $f(i+j+m_0(f,-1)) = f(j + m_0(f,-1)) = f(-i+j+m_0(f,-1))$.

- Suppose $j = 0$. For $i \leq 0$, $H_1(i,j) = (f\cdot\bar{f})(i) = \bar{f}(i+m_0(\bar{f},+1)) = f(-i + m_0(f,-1)) = f(-i+j+m_0(f,-1))$, and for $i \geq 0$, $H_1(i,j) = (f\cdot\bar{f})(i) = f(i + m_0(f,-1)) = f(i+j+m_0(f,-1))$.

- Suppose $j = m_0(f,+1) - m_0(f,-1)$. For $i \leq 0$,

$$\begin{aligned} H_1(i,j) &= f(-i+j+m_0(f,-1)) \\ &= f(-i+m_0(f,+1) - m_0(f,-1) + m_0(f,-1)) \\ &= f(-i+m_0(f,+1)) \\ &= v_0, \end{aligned}$$

and for all $i \geq 0$,

$$\begin{aligned} H_1(i,j) &= f(i+j+m_0(f,-1)) \\ &= f(i+m_0(f,+1)-m_0(f,-1)+m_0(f,-1)) \\ &= f(i+m_0(f,+1)) \\ &= v_0. \end{aligned}$$

Thus H_1 is well-defined. We now show that H_1 is a graph homomorphism. Since there are edges $\{(i,j),(i+1,j)\},\{(i,j),(i,j+1)\} \in E(I_\infty^2)$ for all $i,j \in \mathbb{Z}$, the map H_1 is a graph homomorphism if either $H_1(i,j) = H_1(i+1,j)$ or $\{H_1(i,j), H_1(i+1,j)\} \in E(G)$, and either $H_1(i,j) = H_1(i,j+1)$ or $\{H_1(i,j), H_1(i,j+1)\} \in E(G)$ for all $i,j \in \mathbb{Z}$. Since $f \cdot \bar{f}$ and p_{v_0} are graph homomorphisms, and since H_1 is constantly equal to $f \cdot \bar{f}$ for $j \leq 0$ and constantly equal to p_{v_0} for $j \geq m_0(f,+1) - m_0(f,-1)$, we only need to examine H_1 for $0 \leq j < m_0(f,+1) - m_0(f,-1)$. Let $0 \leq j < m_0(f,+1) - m_0(f,-1)$.

- First, consider $H_1(i,j)$ and $H_1(i+1,j)$.

  For $i \geq 0$,

  $$H_1(i,j) = f(i+j+m_0(f,-1)) \quad \text{and} \quad H_1(i+1,j) = f(i+1+j+m_0(f,-1)).$$

  Since f is a graph homomorphism, $f(i+j+m_0(f,-1)) = f(i+1+j+m_0(f,-1))$ or $\{f(i+j+m_0(f,-1)),\ f(i+1+j+m_0(f,-1))\} \in E(G)$.

  For $i < 0$,

  $$H_1(i,j) = f(-i+j+m_0(f,-1)) \quad \text{and} \quad H_1(i+1,j) = f(-i-1+j+m_0(f,-1)).$$

  Since f is a graph homomorphism, either $f(-i+j+m_0(f,-1)) = f(-i-1+j+m_0(f,-1))$ or $\{f(-i+j+m_0(f,-1)),\ f(-i-1+j+m_0(f,-1))\} \in E(G)$. Thus

$H_1(i,j) = H_1(i+1,j)$ or $\{H_1(i,j),\ H_1(i+1,j)\} \in E(G)$ for all $i \in \mathbb{Z}$.

- Next, consider $H_1(i,j)$ and $H_1(i,j+1)$.

  For $i \geq 0$,

$$H_1(i,j) = f(i+j+m_0(f,-1)) \quad \text{and} \quad H_1(i,j+1) = f(i+j+1+m_0(f,-1)).$$

  Since f is a graph homomorphism, $f(i+j+m_0(f,-1)) = f(i+j+1+m_0(f,-1))$ or $\{f(i+j+m_0(f,-1)),\ f(i+j+1+m_0(f,-1))\} \in E(G)$.

  For $i < 0$,

$$H_1(i,j) = f(-i+j+m_0(f,-1)) \quad \text{and} \quad H_1(i,j+1) = f(-i+j+1+m_0(f,-1)).$$

  Since f is a graph homomorphism, $f(-i+j+m_0(f,-1)) = f(-i+j+1+m_0(f,-1))$ or $\{f(-i+j+m_0(f,-1)),\ f(-i+j+1+m_0(f,-1))\} \in E(G)$. Thus $H_1(i,j) = H_1(i,j+1)$ or $\{H_1(i,j),\ H_1(i,j+1)\} \in E(G)$ for all $i \in \mathbb{Z}$.

Thus H_1 is a graph homomorphism. We now show that H_1 is a graph homotopy from $f \cdot \bar{f}$ to p_{v_0} by verifying conditions (a)-(c) found in Definition 4.12.

(a) Since $f \cdot \bar{f},\ p_{v_0} \in B_1(G, v_0)$, both graph homomorphisms stabilize to the vertex v_0 in the positive and negative directions. Thus $\alpha^1_{-1}(f \cdot \bar{f}) = \alpha^1_{-1}(p_{v_0})$ and $\alpha^1_{+1}(f \cdot \bar{f}) = \alpha^1_{+1}(p_{v_0})$.

(b) Let $(H_1)_j : I_\infty \to G$ be defined by $(H_1)_j(i) = H_1(i,j)$ for each $i, j \in \mathbb{Z}$. Since H_1 is constantly equal to $f \cdot \bar{f}$ for $j \leq 0$ and constantly equal to p_{v_0} for $j \geq m_0(f,+1) - m_0(f,-1)$, it follows that H_1 stabilizes on the 1^{st}-axis in the positive direction at $m_0(H_1,+1) = \max\{m_0((H_1)_j,+1) \mid 0 \leq j \leq m_0(f,+1) - m_0(f,-1)\}$. For $0 \leq j \leq m_0(f,+1) - m_0(f,-1)$ and $i \geq 0$, $(H_1)_j(i) = f(i+j+m_0(f,-1))$. Since $(H_1)_j(m_0(f,+1) - m_0(f,-1) - j) = f(m_0(f,+1) - m_0(f,-1) - j + j + m_0(f,-1)) = f(m_0(f,+1))$ and f stabilizes in the positive direction at $m_0(f,+1)$, it follows that

$(H_1)_j$ stabilizes in the positive direction at $m_0((H_1)_j, +1) = m_0(f, +1) - m_0(f, -1) - j$. Therefore, $m_0(H_1, +1) = \max\{m_0(f, +1) - m_0(f, -1) - j \mid 0 \leq j \leq m_0(f, +1) - m_0(f, -1)\} = m_0(f, +1) - m_0(f, -1)$. For clarity, let $M = m_0(f, +1) - m_0(f, -1)$. Since $(f \cdot \bar{f})(i) = f(i + m_0(f, +1))$ for all $i \geq 0$ and since $m_0(f, +1) - m_0(f, -1) \geq 0$, the face $\alpha^2_{+1}(H_1)$ is given by

$$
\begin{aligned}
\alpha^2_{+1}(H_1)(j) &= H_1(m_0(H_1, +1), j) \\
&= H_1(m_0(f, +1) - m_0(f, -1), j) \\
&= \begin{cases} (f \cdot \bar{f})(m_0(f, +1) - m_0(f, -1)) & \text{for } j \leq 0, \\ f(m_0(f, +1) - m_0(f, -1) + j + m_0(f, -1)) & \text{for } 0 \leq j \leq M, \\ v_0 & \text{for } j \geq M, \end{cases} \\
&= \begin{cases} f(m_0(f, +1) - m_0(f, -1) + m_0(f, -1)) & \text{for } j \leq 0, \\ f(m_0(f, +1) + j) & \text{for } 0 \leq j \leq M, \\ v_0 & \text{for } j \geq M, \end{cases} \\
&= \begin{cases} f(m_0(f, +1)) & \text{for } j \leq 0, \\ f(m_0(f, +1) + j) & \text{for } 0 \leq j \leq M, \\ v_0 & \text{for } j \geq M, \end{cases} \\
&= \begin{cases} v_0 & \text{for } j \leq 0, \\ v_0 & \text{for } 0 \leq j \leq M, \\ v_0 & \text{for } j \geq M. \end{cases}
\end{aligned}
$$

Since $f \cdot \bar{f}$ and p_{v_0} stabilize to v_0 in the positive direction, $\alpha^2_{+1}(H_1)(j) = \alpha^1_{+1}(f \cdot \bar{f})(*) = \alpha^1_{+1}(p_{v_0})(*)$ for all $j \in \mathbb{Z}$. Therefore, $\alpha^2_{+1}(H_1) = \beta^1_1 \alpha^1_{+1}(f \cdot \bar{f}) = \beta^1_1 \alpha^1_{+1}(p_{v_0})$.

Similarly, the graph homomorphism H_1 stabilizes on the 1^{st}-axis in the negative direction at $m_0(H_1, -1) = \min\{m_0((H_1)_j, -1) \mid 0 \leq j \leq m_0(f, +1) - m_0(f, -1)\}$. For

$0 \leq j \leq m_0(f,+1) - m_0(f,-1)$ and $i \leq 0$, $(H_1)_j(i) = f(-i + j + m_0(f,-1))$. Since $(H_1)_j(-m_0(f,+1)+m_0(f,-1)+j) = f(m_0(f,+1)-m_0(f,-1)-j+j+m_0(f,-1)) = f(m_0(f,+1)) = \bar{f}(-m_0(f,+1)) = \bar{f}(m_0(\bar{f},-1))$ and $\bar{f}$ stabilizes in the negative direction at $m_0(\bar{f},-1)$, it follows that $(H_1)_j$ stabilizes in the negative direction at $m_0((H_1)_j,-1) = -m_0(f,+1)+m_0(f,-1)+j$. Thus $m_0(H_1,+1) = \min\{-m_0(f,+1)+m_0(f,-1)+j \mid 0 \leq j \leq m_0(f,+1) - m_0(f,-1)\} = -m_0(f,+1) + m_0(f,-1)$. Since $(f \cdot \bar{f})(i) = \bar{f}(i + m_0(\bar{f},+1)) = f(-i - m_0(\bar{f},+1)) = f(-i + m_0(f,-1))$ for all $i \leq 0$ and since $-m_0(f,+1) + m_0(f,-1) \leq 0$, the face $\alpha^2_{-1}(H_1)$ is given by

$$
\begin{aligned}
\alpha^2_{-1}(H_1)(j) &= H_1(m_0(H_1,-1),j) \\
&= H_1(m_0(f,-1) - m_0(f,+1), j) \\
&= \begin{cases} (f \cdot \bar{f})(m_0(f,-1) - m_0(f,+1)) & \text{for } \ j \leq 0, \\ f(-m_0(f,-1) + m_0(f,+1) + j + m_0(f,-1)) & \text{for } \ 0 \leq j \leq M, \\ v_0 & \text{for } \ j \geq M, \end{cases} \\
&= \begin{cases} f(-m_0(f,-1) + m_0(f,+1) + m_0(f,-1)) & \text{for } \ j \leq 0, \\ f(m_0(f,+1) + j) & \text{for } \ 0 \leq j \leq M, \\ v_0 & \text{for } \ j \geq M, \end{cases} \\
&= \begin{cases} f(m_0(f,+1)) & \text{for } \ j \leq 0, \\ f(m_0(f,+1) + j) & \text{for } \ 0 \leq j \leq M, \\ v_0 & \text{for } \ j \geq M, \end{cases} \\
&= \begin{cases} v_0 & \text{for } \ j \leq 0, \\ v_0 & \text{for } \ 0 \leq j \leq M, \\ v_0 & \text{for } \ j \geq M. \end{cases}
\end{aligned}
$$

Since $f \cdot \bar{f}$ and p_{v_0} stabilize to v_0 in both directions, $\alpha^2_{-1}(H_1)(j) = \alpha^1_{-1}(f \cdot \bar{f})(*) =$

$\alpha^1_{-1}(p_{v_0})(*)$ for all $j \in \mathbb{Z}$. Therefore, $\alpha^2_{-1}(H_1) = \beta^1_1\alpha^1_{-1}(f \cdot \bar{f}) = \beta^1_1\alpha^1_{-1}(p_{v_0})$.

(c) By construction, H_1 stabilizes on the 2^{nd}-axis at $m_0(H_1, -2) = 0$ and $m_0(H_1, +2) = m_0(f, +1) - m_0(f, -1)$. Thus the faces $\alpha^2_{-2}(H_1)$ and $\alpha^2_{+2}(H_1)$ are given by

$$\alpha^2_{-2}(H_1)(i) = H_1(i, 0) = f \cdot \bar{f}(i)$$

and

$$\alpha^2_{+2}(H_1)(i) = H_1(i,\ m_0(f, +1) - m_0(f, -1)) = p_{v_0}(i),$$

respectively, for all $i \in \mathbb{Z}$. Hence, $\alpha^2_{-2}(H_1) = f \cdot \bar{f}$ and $\alpha^2_{+2}(H_1) = p_{v_0}$.

Thus H_1 is a graph homotopy from $f \cdot \bar{f}$ to p_{v_0}, and hence, $f \cdot \bar{f} \sim p_{v_0}$. We show that $\bar{f} \cdot f \sim p_{v_0}$ by proceeding in the same way using the homotopy $H_2 \in C_2(G)$ defined by

$$H_2(i,j) = \begin{cases} (\bar{f} \cdot f)(i) & \text{for} \quad j \leq 0, \\ (\bar{f} \cdot f)(i+j) & \text{for} \quad 0 \leq j \leq m_0(f,+1) - m_0(f,-1),\ i \geq 0, \\ (\bar{f} \cdot f)(i-j) & \text{for} \quad 0 \leq j \leq m_0(f,+1) - m_0(f,-1),\ i \leq 0, \\ p_{v_0}(i) & \text{for} \quad j \geq m_0(f,+1) - m_0(f,-1). \end{cases}$$

□

Now the only thing left to show is that concatenation on the set $B_1(G, v_0)/\sim$ is associative.

Proposition 5.11 (Associativity). Concatenation on the set $B_1(G, v_0)/\sim$ is associative.

Proof. Let $f, g, h \in B_1(G, v_0)$. The concatenations $f \cdot g$ and $g \cdot h$ are defined by

$$(f \cdot g)(i) = \begin{cases} f(i + m_0(f, -1)) & \text{for} \quad i \geq 0, \\ g(i + m_0(g, +1)) & \text{for} \quad i \leq 0, \end{cases}$$

and

$$(g \cdot h)(i) = \begin{cases} g(i + m_0(g, -1)) & \text{for} \quad i \geq 0, \\ h(i + m_0(h, +1)) & \text{for} \quad i \leq 0. \end{cases}$$

By Proposition 5.6, $f \cdot g, g \cdot h \in B_1(G, v_0)$. Recall that by Lemma 4.16,

$$\begin{aligned} m_0(f \cdot g, -1) &= m_0(g, -1) - m_0(g, +1), \\ m_0(f \cdot g, +1) &= m_0(f, +1) - m_0(f, -1), \\ m_0(g \cdot h, -1) &= m_0(h, -1) - m_0(h, +1), \\ m_0(g \cdot h, +1) &= m_0(g, +1) - m_0(g, -1). \end{aligned}$$

Thus the concatenation $(f \cdot g) \cdot h : I_\infty \to G$ is defined by

$$
\begin{aligned}
&((f \cdot g) \cdot h)(i) \\
&= \begin{cases} (f \cdot g)(i + m_0(f \cdot g, -1)) & \text{for} \quad i \geq 0, \\ h(i + m_0(h, +1)) & \text{for} \quad i \leq 0, \end{cases} \\
&= \begin{cases} f(i + m_0(f \cdot g, -1) + m_0(f, -1)) & \text{for} \quad i + m_0(f \cdot g, -1) \geq 0, \\ g(i + m_0(f \cdot g, -1) + m_0(g, +1)) & \text{for} \quad m_0(f \cdot g, -1) \leq i + m_0(f \cdot g, -1) \leq 0, \\ h(i + m_0(h, +1)) & \text{for} \quad i \leq 0, \end{cases} \\
&= \begin{cases} f(i + m_0(f \cdot g, -1) + m_0(f, -1)) & \text{for} \quad i \geq -m_0(f \cdot g, -1), \\ g(i + m_0(f \cdot g, -1) + m_0(g, +1)) & \text{for} \quad 0 \leq i \leq -m_0(f \cdot g, -1), \\ h(i + m_0(h, +1)) & \text{for} \quad i \leq 0, \end{cases} \\
&= \begin{cases} f(i + m_0(g, -1) - m_0(g, +1) + m_0(f, -1)) & \text{for} \quad i \geq -m_0(g, -1) + m_0(g, +1), \\ g(i + m_0(g, -1) - m_0(g, +1) + m_0(g, +1)) & \text{for} \quad 0 \leq i \leq -m_0(g, -1) + m_0(g, +1), \\ h(i + m_0(h, +1)) & \text{for} \quad i \leq 0, \end{cases} \\
&= \begin{cases} f(i + m_0(g, -1) - m_0(g, +1) + m_0(f, -1)) & \text{for} \quad i \geq m_0(g, -1) - m_0(g, +1), \\ g(i + m_0(g, -1)) & \text{for} \quad 0 \leq i \leq m_0(g, -1) - m_0(g, +1), \\ h(i + m_0(h, +1)) & \text{for} \quad i \leq 0. \end{cases}
\end{aligned}
$$

Similarly, the concatenation $f \cdot (g \cdot h) : I_\infty \to G$ is defined by

$$
\begin{aligned}
&(f \cdot (g \cdot h))(i) \\
&= \begin{cases} f(i + m_0(f, -1)) & \text{for} \quad i \geq 0, \\ (g \cdot h)(i + m_0(g \cdot h, +1)) & \text{for} \quad i \leq 0, \end{cases} \\
&= \begin{cases} f(i + m_0(f, -1)) & \text{for} \quad i \geq 0, \\ g(i + m_0(g \cdot h, +1) + m_0(g, -1)) & \text{for} \quad 0 \leq i + m_0(g \cdot h, +1) \leq m_0(g \cdot h, +1), \\ h(i + m_0(g \cdot h, +1) + m_0(h, +1)) & \text{for} \quad i + m_0(g \cdot h, +1) \leq 0, \end{cases} \\
&= \begin{cases} f(i + m_0(f, -1)) & \text{for} \quad i \geq 0, \\ g(i + m_0(g \cdot h, +1) + m_0(g, -1)) & \text{for} \quad -m_0(g \cdot h, +1) \leq i \leq 0, \\ h(i + m_0(g \cdot h, +1) + m_0(h, +1)) & \text{for} \quad i \leq -m_0(g \cdot h, +1), \end{cases} \\
&= \begin{cases} f(i + m_0(f, -1)) & \text{for} \quad i \geq 0, \\ g(i + m_0(g, +1) - m_0(g, -1) + m_0(g, -1)) & \text{for} \quad -m_0(g, +1) + m_0(g, -1) \leq i \leq 0, \\ h(i + m_0(g, +1) - m_0(g, -1) + m_0(h, +1)) & \text{for} \quad i \leq -m_0(g, +1) + m_0(g, -1), \end{cases} \\
&= \begin{cases} f(i + m_0(f, -1)) & \text{for} \quad i \geq 0, \\ g(i + m_0(g, +1)) & \text{for} \quad -m_0(g, +1) + m_0(g, -1) \leq i \leq 0, \\ h(i + m_0(g, +1) - m_0(g, -1) + m_0(h, +1)) & \text{for} \quad i \leq -m_0(g, +1) + m_0(g, -1). \end{cases}
\end{aligned}
$$

However, for all $i \in \mathbb{Z}$,

$$((f \cdot g) \cdot h)(i + m_0(g, +1) - m_0(g, -1)) = (f \cdot (g \cdot h))(i).$$

Thus $f \cdot (g \cdot h) = (f \cdot g) \cdot h_{-m_0(g,+1)+m_0(g,-1)}$, that is, $f \cdot (g \cdot h)$ is equal to the graph homomorphism $(f \cdot g) \cdot h$ shifted down by $m_0(g, +1) - m_0(g, -1)$. Therefore, by the *Shifting Lemma* (5.4), $f \cdot (g \cdot h) \sim (f \cdot g) \cdot h$, and concatenation on the set $B_1(G, v)/\sim$ is associative. $\square$

Since concatenation is well-defined on $B_1(G, v_0)/\sim$, and the set $B_1(G, v_0)/\sim$ is closed under concatenation, has an identity, inverses, and concatenation is associative on the set $B_1(G, v_0)/\sim$, we can conclude the following.

Theorem 5.12. The set of equivalence classes $B_1(G, v_0)/\sim$ is a group with the operation of concatenation.

Now that we have shown $B_1(G, v_0)/\sim$ is a group with the operation of concatenation, in the next chapter we move to the main results of this thesis, the development of the theory of graph coverings and lifting properties.

Chapter 6

Covering Graphs and Lifting Properties

In topology, a covering space is a continuous map $p : \widetilde{X} \to X$ that preserves the local structure of the space. When considering a graph as a space, these covering spaces again fail to recognize the structure of the graph, namely, the vertices and edges. Thus there are covering graphs, that is, graph homomorphisms $p : \widetilde{G} \to G$ that preserve the local structures of the graphs. In particular, the graph $\widetilde{G}$ should 'look like' the graph G locally with the map p formalizing this structure. In topology, given a covering space $p : \widetilde{X} \to X$ and a continuous map $f : Y \to X$, there are also *lifts* $\widetilde{f} : Y \to \widetilde{X}$ which factor f through the space $\widetilde{X}$. There are lifting properties in topology that determine when a lift does or does not exist. While an analogous term and properties do not exist in the current literature for A-homotopy theory, we define a discrete version of lifts and develop the corresponding lifting properties in this chapter. The next three definitions give us a more precise idea of what covering graphs are.

Definition 6.1. Let G be a graph and be $v \in V(G)$. The *closed neighborhood* of v, denoted $N[v]$, is the set of vertices adjacent to v as well as v itself, more precisely,

$$N[v] = \{a \in V(G) \mid \{a, v\} \in E(G) \text{ or } a = v\}.$$

Definition 6.2. [10] The graph homomorphism $p : G_1 \to G_2$ is a *local isomorphism* if p is onto and for each vertex $v \in V(G_2)$ and each vertex $w \in p^{-1}(v)$, the induced mapping $p|_{N[w]} : N[w] \to N[v]$ is bijective.

Remark 6.3. While the restriction $p|_{N[w]} : N[w] \to N[v]$ given in the previous definition is a bijection between the vertex sets $N[w]$ and $N[v]$, it is not necessarily a bijection between the edges of the induced subgraphs $G_1(N[w])$ and $G_2(N[v])$ (see Definition 3.14).

Example 6.4. Let $\mathcal{C}_k$ be a k-cycle on $k \geq 3$ and vertices labeled $[0], [1], \ldots, [k-1]$. Figure 6.1 depicts a local isomorphism $p : \mathcal{C}_6 \to \mathcal{C}_3$ defined by $p([i]) = [i \mod 3]$ for $i \in \{0, \ldots, 6\}$. The

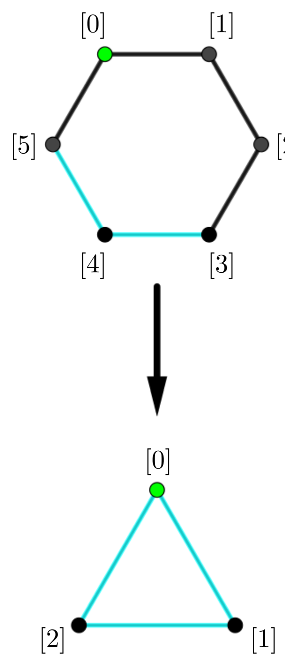

Figure 6.1: The local isomorphism $p : \mathcal{C}_6 \to \mathcal{C}_3$

edges of the induced subgraphs $\mathcal{C}_6(N[[4]])$ and $\mathcal{C}_3(N[[1]])$ are shown in light blue. While there is an edge $\{[0], [2]\}$ in $\mathcal{C}_3$, there is no edge $\{[3], [5]\}$ in $\mathcal{C}_6$). Thus the restriction $p|_{N[[4]]} : N[[4]] \to N[[1]]$ is a bijective on the vertices but not the edges of the induced subgraphs $\mathcal{C}_6(N[[4]])$ and $\mathcal{C}_3(N[[1]])$.

We define a different subgraph with the property p restricted to this subgraph is bijective on both vertices and edges. For $x \in V(G_1)$, let N_x denote the subgraph of G_1 with vertex set $V(N_x) = N[x]$ and edge set $E(N_x) = \{\{x, v\} \mid v \in N[x], v \neq x\}$. If $p : G_1 \to G_2$ is a local isomorphism, then p induces a graph homomorphism from the subgraph N_x to the subgraph $N_{p(x)}$ for each $x \in V(G_1)$, that is, there is a graph homomorphism

$$p|_{N_x} : N_x \to N_{p(x)},$$

that is bijective on the vertices and edges of the subgraphs. This implies the following lemma.

Lemma 6.5. *Let $p : G_1 \to G_2$ be a local isomorphism and $x \in V(G_1)$. Then the graph homomorphism $p|_{N_x}$ is invertible, and its inverse $(p|_{N_x})^{-1} : N_{p(x)} \to N_x$ is a graph homomorphism.*

These restrictions of local isomorphisms are useful when we discuss *lifting properties.*

Definition 6.6. [10] Let G and $\widetilde{G}$ be graphs, and let $p : \widetilde{G} \to G$ be a graph homomorphism. The pair $(\widetilde{G}, p)$ is a *covering graph* of G if p is a local isomorphism.

We now give some examples of covering graphs and how they differ from covering spaces.

Example 6.7. Let $\mathcal{C}_k$ be a cycle with $k \geq 3$ and vertices labeled $[0], [1], \ldots, [k-1]$. If the graph homomorphism $p_k : I_\infty \to \mathcal{C}_k$ is defined by $p_k(i) = [i \mod k]$, then the pair (I_∞, p_k) forms a covering graph of the cycle $\mathcal{C}_k$.

As mentioned previously, in classical homotopy theory, all cycles are homotopy equivalent as topological space to the circle. Example 6.7 is analogous to covering the circle with the real line $\mathbb{R}$ by mapping it onto the circle as a helix. This is illustrated in Figure 6.2.

Example 6.8. If the graph homomorphism $p : \mathcal{C}_{2k} \to \mathcal{C}_k$ is defined by $p([i]) = [i \mod k]$ for all $i \in \{0, \ldots, 2k-1\}$, then the pair $(\mathcal{C}_{2k}, p)$ forms a covering graph of the cycle $\mathcal{C}_k$.

The local isomorphism p depicted in Figure 6.1 is an example of a covering graph of $\mathcal{C}_k$ by $\mathcal{C}_{2k}$ with $k = 3$. Example 6.8 is analogous to mapping the topological circle onto another circle so that the first wraps around the second twice. We now continue to the definition of a

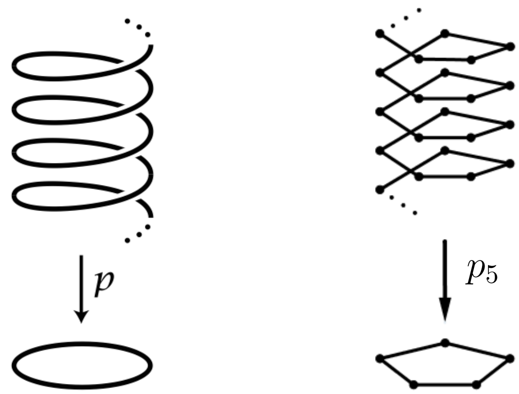

Figure 6.2: The maps $p : \mathbb{R} \to S^1$ and $p_5 : I_\infty \to \mathcal{C}_5$

lift and lifting properties, material that is not found in the existing literature for A-homotopy theory. The following definition is taken from [9, p. 5] but with 'graph homomorphism' in place of 'continuous map'.

Definition 6.9. Let G be a graph, and let $(\widetilde{G}, p)$ be a covering graph of G. Given a graph homomorphism $f : K \to G$, a *lift* of f is a graph homomorphism $\widetilde{f} : K \to \widetilde{G}$ such that $p \circ \widetilde{f} = f$.

Theorem 6.10 (Path Lifting Property). Let $(\widetilde{G}, p)$ be a covering graph of G. For each $f \in C_1(G)$ with $f(m_0(f, -1)) = v_0 \in V(G)$ and each vertex $\widetilde{v_0} \in p^{-1}(v_0)$, there exists a unique lift $\widetilde{f}$ of f starting at the vertex $\widetilde{v_0}$.

$$\begin{tikzcd} & \widetilde{G} \arrow[d, "p"] \\ I_\infty \arrow[r, "f"] \arrow[ur, dashed, "\exists! \widetilde{f}"] & G \end{tikzcd}$$

Proof. Let $f \in C_1(G)$ with $f(m_0(f, -1)) = v_0 \in V(G)$, and suppose $\widetilde{v_0} \in p^{-1}(v_0)$. Define the map $\widetilde{f} : I_\infty \to \widetilde{G}$ by $\widetilde{f}(i) = \widetilde{v_0}$ for all $i \le m_0(f,-1)$ and recursively by

$$\widetilde{f}(i) = (p|_{N_{\widetilde{f}(i-1)}})^{-1}(f(i)) \quad \text{for} \quad i > m_0(f,-1).$$

We must show that the map $\widetilde{f}$ is well-defined, is a graph homomorphism, is a lift of f, and is unique. Since $\widetilde{f}$ is defined to be constant for $i \le m_0(f,-1)$ and defined recursively for $i > m_0(f,-1)$, in the following proofs of the four properties we address the case for $i \le m_0(f,-1)$ separately and use induction to prove the properties for $i \ge m_0(f,-1)$.

(1) $\widetilde{f}$ is well-defined.

- By definition, $\widetilde{f}(i) = \widetilde{v_0}$ for all $i \le m_0(f,-1)$. Thus $\widetilde{f}(i)$ is well-defined for $i \le m_0(f,-1)$.
- For $i \ge m_0(f,-1)$, we show that the correspondence $i \mapsto \widetilde{f}(i)$ is well-defined by induction on i.

  Base Case: By definition of $\widetilde{f}$, $\widetilde{f}(m_0(f,-1)) = \widetilde{v_0}$ and

$$\begin{aligned}\widetilde{f}(m_0(f,-1)+1) &= (p|_{N_{\widetilde{f}(m_0(f,-1))}})^{-1}(f(m_0(f,-1)+1)) \\ &= (p|_{N_{\widetilde{v_0}}})^{-1}(f(m_0(f,-1)+1)).\end{aligned}$$

  By Lemma 6.5, the inverse $(p|_{N_{\widetilde{v_0}}})^{-1} : N_{p(\widetilde{v_0})} \to N_{\widetilde{v_0}}$ exists. Since $\widetilde{v_0} \in p^{-1}(v_0)$, the domain of $(p|_{N_{\widetilde{v_0}}})^{-1}$ is equal to N_{v_0}. Moreover, $f(m_0(f,-1)+1) \in N[v_0]$, since f is a graph homomorphism. Thus $f(m_0(f,-1)+1)$ is in the domain of $(p|_{N_{\widetilde{v_0}}})^{-1}$, and hence, $(p|_{N_{\widetilde{v_0}}})^{-1}(f(m_0(f,-1)+1))$ is well-defined.

  Inductive Hypothesis: Suppose $\widetilde{f}(i)$ is well-defined for some $i > m_0(f,-1)$.

  By definition, $\widetilde{f}(i+1) = (p|_{N_{\widetilde{f}(i)}})^{-1}(f(i+1))$. In order for $\widetilde{f}(i+1)$ to be well-defined, we must verify that $f(i+1)$ is in the domain of $(p|_{N_{\widetilde{f}(i)}})^{-1}$. By the inductive hypothesis, $\widetilde{f}(i) = (p|_{N_{\widetilde{f}(i-1)}})^{-1}(f(i))$ is well-defined. Since p is a graph homomorphism,

$$\begin{aligned} p|_{N_{\widetilde{f}(i-1)}}(\widetilde{f}(i)) &= p|_{N_{\widetilde{f}(i-1)}}((p|_{N_{\widetilde{f}(i-1)}})^{-1}(f(i))) \\ &= f(i). \end{aligned}$$

By Lemma 6.5, the inverse $(p|_{N_{\widetilde{f}(i)}})^{-1} : N_{p(\widetilde{f}(i))} \to N_{\widetilde{f}(i)}$ exists. Since $p(\widetilde{f}(i)) = f(i)$, the domain of $(p|_{N_{\widetilde{f}(i)}})^{-1}$ is equal to $N_{f(i)}$. Moreover, since f is a graph homomorphism, it follows that $f(i+1) \in N[f(i)]$, so $f(i+1)$ is in the domain of $(p|_{N_{\widetilde{f}(i)}})^{-1}$. Therefore, $(p|_{N_{\widetilde{f}(i)}})^{-1}(f(i+1))$ is well-defined.

Thus by induction, the map $\widetilde{f}$ is well-defined for $i \geq m_0(f,-1)$.

Hence, $\widetilde{f}$ is well-defined.

(2) $\widetilde{f}$ is a graph homomorphism.

There is an edge $\{i, i+1\} \in E(I_\infty)$ for all $i \in \mathbb{Z}$. Thus to show that $\widetilde{f}$ is a graph homomorhism, we must show that either $\widetilde{f}(i) = \widetilde{f}(i+1)$ or $\{\widetilde{f}(i), \widetilde{f}(i+1)\} \in E(G)$ for all $i \in \mathbb{Z}$.

- By definition, $\widetilde{f}(i) = \widetilde{v_0}$ for all $i \leq m_0(f,-1)$. Thus $\widetilde{f}(i) = \widetilde{v_0} = \widetilde{f}(i+1)$ for all $i < m_0(f,-1)$.
- For $i \geq m_0(f,-1)$, we show that either $\widetilde{f}(i) = \widetilde{f}(i+1)$ or $\{\widetilde{f}(i), \widetilde{f}(i+1)\} \in E(G)$ by induction on i.

  Base Case: By definition, $\widetilde{f}(m_0(f,-1)) = \widetilde{v_0} \in p^{-1}(v_0)$, and by part (1),

$$\tilde{f}(m_0(f,-1)+1) = (p|_{N_{\widetilde{v_0}}})^{-1}(f(m_0(f,-1)+1).$$

  Thus, since $f(m_0(f,-1)) = v_0$, it follows that $\widetilde{f}(m_0(f,-1)) = (p|_{N_{\widetilde{v_0}}})^{-1}(v_0) = (p|_{N_{\widetilde{v_0}}})^{-1}(f(m_0(f,-1))$. Moreover, since f is a graph homomorphism, either $f(m_0(f,-1)) = f(m_0(f,-1)+1)$ or $\{f(m_0(f,-1)), f(m_0(f,-1)+1)\} \in E(G)$.

Therefore, it follows that $(p|_{N_{\widetilde{v_0}}})^{-1}(f(m_0(f,-1))) = (p|_{N_{\widetilde{v_0}}})^{-1}(f(m_0(f,-1)+1))$ or $\{(p|_{N_{\widetilde{v_0}}})^{-1}(f(m_0(f,-1))), (p|_{N_{\widetilde{v_0}}})^{-1}(f(m_0(f,-1)+1))\} \in E(\widetilde{G})$, since $(p|_{N_{\widetilde{v_0}}})^{-1}$ is a graph homomorphism. Thus either

$$\widetilde{f}(m_0(f,-1)) = \widetilde{f}(m_0(f,-1)+1)$$

or

$$\{\widetilde{f}(m_0(f,-1)),\ \widetilde{f}(m_0(f,-1)+1)\} \in E(G).$$

Inductive Hypothesis: Suppose that for some $i > m_0(f,-1)$, $\widetilde{f}(i-1) = \widetilde{f}(i)$ or $\{\widetilde{f}(i-1),\ \widetilde{f}(i)\} \in E(G)$.

By definition, $\widetilde{f}(i) = (p|_{N_{\widetilde{f}(i-1)}})^{-1}(f(i))$ and $\widetilde{f}(i+1) = (p|_{N_{\widetilde{f}(i)}})^{-1}(f(i+1))$. By Lemma 6.5, the inverses $(p|_{N_{\widetilde{f}(i-1)}})^{-1} : N_{p(\widetilde{f}(i-1))} \to N_{\widetilde{f}(i-1)}$ and $(p|_{N_{\widetilde{f}(i)}})^{-1} : N_{p(\widetilde{f}(i))} \to N_{\widetilde{f}(i)}$ exist. By the inductive hypothesis, either $\widetilde{f}(i-1) = \widetilde{f}(i)$ or $\{\widetilde{f}(i-1),\ \widetilde{f}(i)\} \in E(G)$, so $\widetilde{f}(i) \in N[\widetilde{f}(i-1)] \cap N[\widetilde{f}(i)]$. Since both $p|_{N_{\widetilde{f}(i-1)}}$ and $p|_{N_{\widetilde{f}(i)}}$ are bijective, $p(\widetilde{f}(i)) \in N[p(\widetilde{f}(i-1))] \cap N[p(\widetilde{f}(i))]$. By part (1), $p(\widetilde{f}(i)) = f(i)$. Thus, we can write

$$(p|_{N_{\widetilde{f}(i-1)}})^{-1}(f(i)) = (p|_{N_{\widetilde{f}(i)}})^{-1}(f(i)).$$

Since f is a graph homomorphism, $f(i) = f(i+1)$ or $\{f(i),\ f(i+1)\} \in E(G)$. Thus $(p|_{N_{\widetilde{f}(i)}})^{-1}(f(i)) = (p|_{N_{\widetilde{f}(i)}})^{-1}(f(i+1))$ or $\{(p|_{N_{\widetilde{f}(i)}})^{-1}(f(i)), (p|_{N_{\widetilde{f}(i)}})^{-1}(f(i+1))\} \in E(\widetilde{G})$, since $(p|_{N_{\widetilde{f}(i)}})^{-1}$ is a graph homomorphism. Hence, $\widetilde{f}(i) = \widetilde{f}(i+1)$ or $\{\widetilde{f}(i), \widetilde{f}(i+1)\} \in E(\widetilde{G})$ for all $i > m_0(f,-1)$ by induction on i.

Thus $\widetilde{f}$ is a graph homomorphism.

(3) $\widetilde{f}$ is a lift of f.

- For all $i \leq m_0(f,-1)$, the composition $p \circ \widetilde{f}$ is defined by $p(\widetilde{f}(i)) = p(\widetilde{v_0}) = v_0$.

Thus $p(\widetilde{f}(i)) = f(i)$ for all $i \le m_0(f, -1)$.

- For all $i > m_0(f, -1)$,

$$\begin{aligned} p(\widetilde{f}(i)) &= p((p|_{N_{\widetilde{f}(i-1)}})^{-1}(f(i))) \\ &= f(i). \end{aligned}$$

  Thus $p(\widetilde{f}(i)) = f(i)$ for all $i > m_0(f, -1)$.

Therefore, $p \circ \widetilde{f} = f$, and hence, the graph homomorphism $\widetilde{f}$ is a lift of f.

(4) $\widetilde{f}$ is unique for each choice of $\widetilde{v_0} \in p^{-1}(v_0)$.

Let $\widetilde{g} : I_\infty \to \widetilde{G}$ be a graph homomorphism such that $\widetilde{g}(m_0(\widetilde{g}, -1)) = \widetilde{v_0}$ and $p \circ \widetilde{g} = f$.

- Since $f(i) = v_0$ for all $i \le m_0(f, -1)$ and $p \circ \widetilde{g} = f$, it follows that $p(\widetilde{g}(i)) = v_0$ for all $i \le m_0(f, -1)$. By Lemma 6.5, $p|_{N_{\widetilde{v_0}}})^{-1} : N_{v_0} \to N_{\widetilde{v_0}}$ is a graph homomorphism, since $\widetilde{v_0} \in p^{-1}(v_0)$. Thus $(p|_{N_{\widetilde{v_0}}})^{-1}(p(\widetilde{g}(i))) = (p|_{N_{\widetilde{v_0}}})^{-1}(v_0)$ for all $i \le m_0(f, -1)$. This implies that $\widetilde{g}(i) = \widetilde{v_0}$ for all $i \le m_0(f, -1)$. By definition $\widetilde{f}(i) = \widetilde{v_0}$ for all $i \le m_0(f, -1)$, so $\widetilde{g}(i) = \widetilde{f}(i)$ for all $i \le m_0(f, -1)$.

- We now show that $\widetilde{g}(i) = \widetilde{f}(i)$ for all $i > m_0(f, -1)$ by induction on i.

  Base Case: By the previous case, $\widetilde{f}(m_0(f, -1)) = \widetilde{v_0} = \widetilde{g}(m_0(f, -1))$.

  Inductive Hypothesis: Suppose $\widetilde{g}(i) = \widetilde{f}(i)$ for some $i \ge m_0(f, -1)$.

  Since $\widetilde{f}$ and $\widetilde{g}$ are graph homomorphisms,

$$\widetilde{f}(i) = \widetilde{f}(i+1) \quad \text{or} \quad \{\widetilde{f}(i),\ \widetilde{f}(i+1)\} \in E(G)$$

  and

$$\widetilde{g}(i) = \widetilde{g}(i+1) \quad \text{or} \quad \{\widetilde{g}(i),\ \widetilde{g}(i+1)\} \in E(G).$$

  By the inductive hypothesis, $\widetilde{g}(i) = \widetilde{f}(i)$. Hence, $\widetilde{f}(i+1), \widetilde{g}(i+1) \in N[\widetilde{f}(i)]$. Since $p \circ \widetilde{g} = f = p \circ \widetilde{f}$, it follows that $p|_{N_{\widetilde{f}(i)}}(\widetilde{g}(i+1)) = p|_{N_{\widetilde{f}(i)}}(\widetilde{f}(i+1))$.

Thus $(p|_{N_{\widetilde{f}(i)}})^{-1}(p|_{N_{\widetilde{f}(i)}}(\widetilde{g}(i+1))) = (p_{N_{\widetilde{f}(i)}})^{-1}(p|_{N_{\widetilde{f}(i)}}(\widetilde{f}(i+1)))$, since $(p|_{N_{\widetilde{f}(i)}})^{-1} : N_{f(i)} \to N_{\widetilde{f}(i)}$ is a graph homomorphism. Therefore, $\widetilde{f}(i+1) = \widetilde{g}(i+1)$ for all $i \geq m_0(f,-1)$.

Thus by induction, $\widetilde{g}(i+1) = \widetilde{f}(i+1)$ for all $i \geq m_0(f,-1)$.

Hence, $\widetilde{g} = \widetilde{f}$, so the lift $\widetilde{f}$ of f is unique.

$\square$

We now use the *Path Lifting Property* (Theorem 6.10) to prove the *Homotopy Lifting Property* (Theorem 6.11). In the introduction, we discussed the question of why the 3-cycle and 4-cycle are A-contractible, but the cycles on five or more vertices are not. This question is answered in Chapter 7, where we use the *Homotopy Lifting Property* (Theorem 6.11) to show that C_5 is not A-contractible. The fact that homotopy lifting does not hold for C_3 or C_4 is significant.

Theorem 6.11 (**Homotopy Lifting Property**)**.** Let G be a graph containing no 3-cycles or 4-cycles and $(\widetilde{G}, p)$ be a covering graph of G. Given a homotopy $H : K \square I_n \to G$ from f to g and a lift $\widetilde{f} : K \to \widetilde{G}$ of f, there exists a unique homotopy $\widetilde{H} : K \square I_n \to \widetilde{G}$ that lifts H.

The statement of this theorem can be summarized by the following diagram.

$$\begin{tikzcd} & \widetilde{G} \arrow[d, "p"] \\ K \arrow[ur, dashed, bend left=10, "\widetilde{g}"] \arrow[ur, "\widetilde{f}"'] \arrow[ur, phantom, "\sim"] \arrow[r, shift left, "g"] \arrow[r, shift right, "f"'] \arrow[r, phantom, "\sim"] & G \end{tikzcd}$$

Here $\widetilde{G}$ is a cover of G by the graph homomorphism p, the $\sim$ between f and g represents the graph homotopy H from f to g, and the $\sim$ between $\widetilde{f}$ and $\widetilde{g}$ represents a lift $\widetilde{H}$ of H, a graph homotopy between a lift $\widetilde{f}$ of f and a lift $\widetilde{g}$ of g. Thus if a lift $\widetilde{H}$ of H exists, then a lift $\widetilde{g}$ of g exists as well. Now we proceed to the proof.

Proof. Let G, $(\widetilde{G}, p)$, H and $\widetilde{f}$ be as in the statement of the theorem. The strategy of this proof is to build the lift $\widetilde{H}$ inductively. For each $y \in V(K)$, we produce a lift of H restricted to $N_y \Box I_n$. First, we use the lift of f to construct a lift of $H|_{N_{(y,0)}}$. Then we proceed by induction to define a lift of $H|_{N_{(y,i+1)}}$ for each $0 \leq i < n$, which agrees with the previous lift of $H|_{N_{(y,i)}}$ on $V(N_{(y,i-1)}) \cap v(N_{(y,i)})$. This produces a lift of $H|_{N_{(y,0)} \cup \cdots \cup N_{(y,n-1)} \cup N_{(y,n)}}$, which we can then complete to a lift of $H|_{N_y \Box I_n}$. Once we have constructed a lift $\widetilde{H}|_{N_y \Box I_n}$, we use it to build the lift $\widetilde{H}$ by appealing to the uniqueness of the *Path Lifting Property* (Theorem 6.10).

Now we proceed to the construction of $\widetilde{H}_{N_y \Box I_n}$. Let $y \in V(K)$. Since H is a graph homotopy from f to g, it follows that $H|_{N_y \Box \{0\}} = f|_{N_y}$. Define $\widetilde{H}|_{N_y \Box \{0\}} = \widetilde{f}|_{N_y}$. Since p is a covering map, the restriction $p|_{N_{\widetilde{H}(y,0)}} : N_{\widetilde{H}(y,0)} \to N_{p(\widetilde{H}(y,0))}$ is a bijection on the vertices and edges of these subgraphs. By definition of $H|_{N_y \Box \{0\}}$, it follows that $p(\widetilde{H}(y,0)) = f(y) = H(y,0)$. Thus the inverse $(p|_{N_{\widetilde{H}(y,0)}})^{-1} : N_{H(y,0)} \to N_{\widetilde{H}(y,0)}$ exists by Lemma 6.5. Since H is a graph homomorphism, there is an inclusion of sets $H(N[y,0]) \subseteq N[H(y,0)]$ and, in particular, $H(y,1) \in N[H(y,0)]$. That is, $H(y,1)$ is in the domain of the inverse $(p|_{N_{\widetilde{H}(y,0)}})^{-1}$. Define $\widetilde{H}(y,1) = (p|_{N_{\widetilde{H}(y,0)}})^{-1}(H(y,1))$. Since $\widetilde{f}$ is a lift of f and $H|_{N_y \Box \{0\}} = f|_{N_y}$, it follows that $\widetilde{f}|_{N_y} = (p|_{N_{\widetilde{H}(y,0)}})^{-1} \circ H|_{N_y \Box \{0\}}$. Thus we have defined $\widetilde{H}|_{N_{(y,0)}}$, and it is a graph homomorphism because it is the composition of graph homomorphisms.

For the inductive step, assume that $H|_{N_{(y,0)} \cup \cdots \cup N_{(y,i)}}$ has a lift $\widetilde{H}|_{N_{(y,0)} \cup \cdots \cup N_{(y,i)}}$ for some $0 \leq i < n$. Figure 6.3 illustrates the graph $N_{(y,0)} \cup \cdots \cup N_{(y,i)}$, in the case that the vertex y has three adjacent vertices. The subgraph $N_{(y,i)}$ is shown in light blue, and the dashed edges shown in red are not included in the graph $N_{(y,0)} \cup \cdots \cup N_{(y,i)}$. Since $(y, i+1) \in N[y,i]$, it follows that $\widetilde{H}(y, i+1)$ is defined.

Since p is a covering map, the restriction $p|_{N_{\widetilde{H}(y,i+1)}} : N_{\widetilde{H}(y,i+1)} \to N_{H(y,i+1)}$ is a bijection on the vertices and edges of these subgraphs. Thus by Lemma 6.5, the inverse $(p|_{N_{\widetilde{H}(y,i+1)}})^{-1}$:

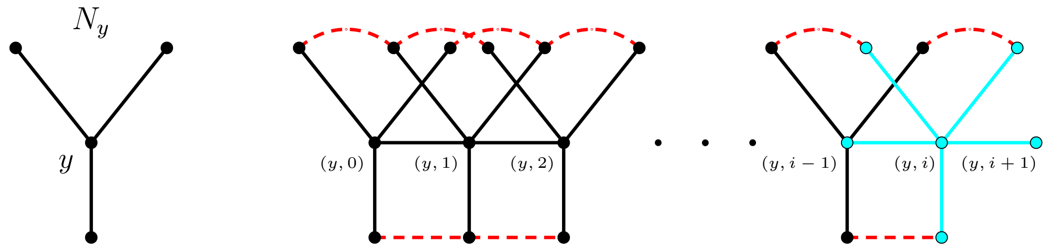

Figure 6.3: The union of neighborhoods $N_{(y,0)} \cup \cdots \cup N_{(y,i-1)} \cup N_{(y,i)}$

$N_{H(y,i+1)} \to N_{\widetilde{H}(y,i+1)}$ exists and is a graph homomorphism. Define

$$\widetilde{H}|_{N_{(y,i+1)}} = (p|_{N_{\widetilde{H}(y,i+1)}})^{-1} \circ H|_{N_{(y,i+1)}}.$$

Since H is a graph homomorphism, there is an inclusion $H(N[(y,i+1)]) \subseteq N[H(y,i+1)]$. Thus $\widetilde{H}|_{N_{(y,i+1)}}$ is well-defined. Since $\widetilde{H}|_{N_{(y,i+1)}}$ is the composition of graph homomorphisms, it follows that $\widetilde{H}|_{N_{(y,i+1)}}$ is a graph homomorphism. This is illustrated by the following diagram.

$$\begin{tikzcd}
 & & N_{\widetilde{H}(y,i+1)} \\
N_{(y,i+1)} \arrow[r, "H|_{N_{(y,i+1)}}"'] \arrow[urr, "\widetilde{H}|_{N_{(y,i+1)}}"] & H(N_{(y,i+1)}) \arrow[r, hook] & N_{H(y,i+1)} \arrow[u, "(p|_{N_{\widetilde{H}(y,i+1)}})^{-1}"']
\end{tikzcd}$$

Thus after a finite number of steps, the lift $\widetilde{H}|_{N_{(y,0)} \cup \cdots \cup N_{(y,n-1)} \cup N_{(y,n)}}$ is defined.

Suppose $x \in N[y]$. Then $\{(x,i),\ (x,i+1)\} \in E(N_y \square I_n)$ for all $0 \le i < n$. Hence, in order for $\widetilde{H}|_{N_{(y,0)} \cup \cdots \cup N_{(y,n-1)} \cup N_{(y,n)}}$ to be extended to a graph homomorphism with domain $N_y \square I_n$, we must show that $\widetilde{H}(x,i) = \widetilde{H}(x,i+1)$ or $\{\widetilde{H}(x,i), \widetilde{H}(x,i+1)\} \in E(\widetilde{G})$ for all

$0 \leq i < n$. By definition of $\widetilde{H}|_{N_{(y,0)} \cup \cdots \cup N_{(y,n-1)} \cup N_{(y,n)}}$,

$$\widetilde{H}(x,i) = (p|_{N_{\widetilde{H}(y,i)}})^{-1} \circ H|_{N_{(y,i)}}(x,i) = (p|_{N_{\widetilde{H}(y,i)}})^{-1} \circ H(x,i)$$

and

$$\widetilde{H}(x,i+1) = (p|_{N_{\widetilde{H}(y,i+1)}})^{-1} \circ H|_{N_{(y,i+1)}}(x,i+1) = (p|_{N_{\widetilde{H}(y,i+1)}})^{-1} \circ H(x,i+1).$$

That is, $\widetilde{H}(x,i)$ is constructed using the graph homomorphism $(p|_{N_{\widetilde{H}(y,i)}})^{-1}$, and $\widetilde{H}(x,i+1)$ is constructed using the graph homomorphism $(p|_{N_{\widetilde{H}(y,i+1)}})^{-1}$. In order to show that $\widetilde{H}(x,i) = \widetilde{H}(x,i+1)$ or $\{\widetilde{H}(x,i), \widetilde{H}(x,i+1)\} \in E(\widetilde{G})$ for all $0 \leq i < n$, we will examine the 4-cycle of $N_y \square I_n$ shown in light blue in Figure 6.4.

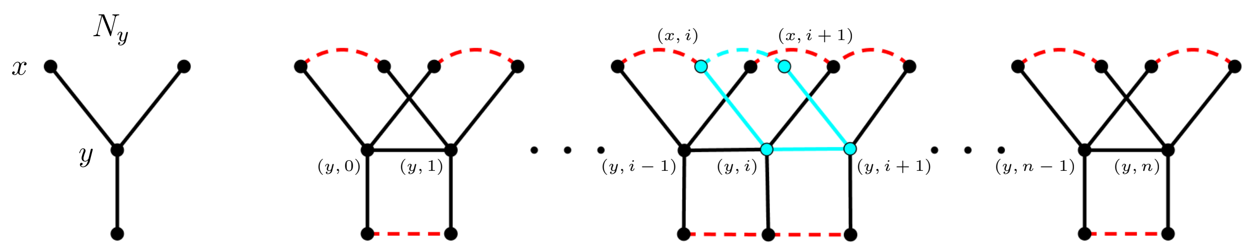

Figure 6.4: The union of neighborhoods $N_{(y,0)} \cup \cdots \cup N_{(y,n-1)} \cup N_{(y,n)}$

We denote this 4-cycle subgraph by $C_{x,i}$. Since H is a graph homomorphism and G contains no 3-cycles or 4-cycles, we have the following nine cases of how H maps $C_{x,i}$ to G, illustrated in Figure 6.5. The label '=' means that H maps the pair of vertices to the same vertex in G. The label a means that H maps the pair of vertices to adjacent vertices in G. In cases (8) and (9), the pair of vertices being mapped to the same vertex are circled in red.

For cases (1)-(8), there is an inclusion of sets $H(C_{x,i}) \subseteq N[H(y,i)]$, and $H(x,i) = H(x,i+1)$ or $\{H(x,i), H(x,i+1)\} \in E(G)$ for all $0 \leq i < n$. Thus the subgraph $C_{x,i}$ is mapped by H into the domain of the inverse $(p|_{N_{\widetilde{H}(y,i)}})^{-1} : N_{H(y,i)} \to N_{\widetilde{H}(y,i)}$. Since

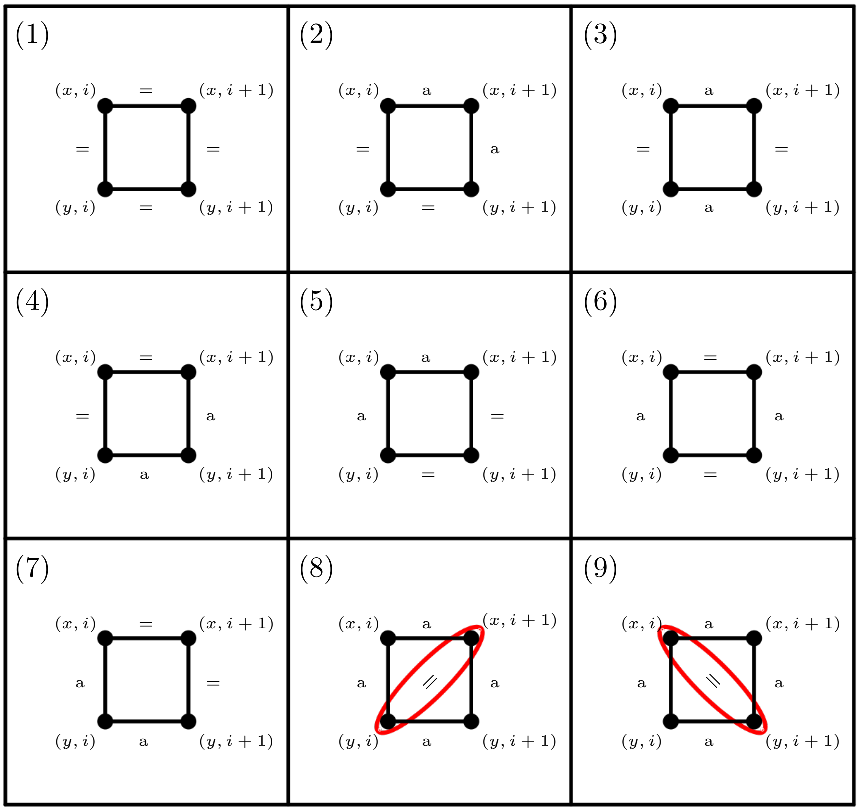

Figure 6.5: The cases of how H maps $C_{x,i}$ to G

$\widetilde{H}|_{C_{x,i}} = (p|_{N_{\widetilde{H}(y,i)}})^{-1} \circ H|_{C_{x,i}}$, it follows that

$$\widetilde{H}(x,i) = \widetilde{H}(x,i+1) \quad \text{or} \quad \{\widetilde{H}(x,i),\ \widetilde{H}(x,i+1)\} \in E(\widetilde{G}).$$

For case (9), there is an inclusion of sets $H(C_{x,i}) \subseteq N[H(y,i+1)]$, and $H(x,i) = H(x,i+1)$ or $\{H(x,i), H(x,i+1)\} \in E(G)$ for all $0 \leq i < n$. Thus the subgraph $C_{x,i}$ is mapped by H into the domain of the inverse $(p|_{N_{\widetilde{H}(y,i+1)}})^{-1} : N_{H(y,i+1)} \to N_{\widetilde{H}(y,i+1)}$. Since $\widetilde{H}|_{C_{x,i}} = (p|_{N_{\widetilde{H}(y,i+1)}})^{-1} \circ H|_{C_{x,i}}$, it follows that

$$\widetilde{H}(x,i) = \widetilde{H}(x,i+1) \quad \text{or} \quad \{\widetilde{H}(x,i),\ \widetilde{H}(x,i+1)\} \in E(\widetilde{G}).$$

Thus we can extend the graph homomorphism $\widetilde{H}|_{N_{(y,0)} \cup \cdots \cup N_{(y,n-1)} \cup N_{(y,n)}}$ to $\widetilde{H}|_{N_y \square I_n}$.

The restriction $H|_{\{y\}\Box I_n}$ is a graph homomorphism from I_n to G and can be written as $H_y : I_n \to G$. By the *Uniqueness of Path Lifting* (6.10), the lift $\widetilde{H}_y : I_n \to \widetilde{G}$ is unique with $\widetilde{H}_y(0) = \widetilde{H}(y,0) = \widetilde{f}(y)$. Since each graph homomorphism $H_x : I_n \to G$ must have a unique lift $\widetilde{H}_x : I_n \to \widetilde{G}$ for all $x \in N[y]$ with $\widetilde{H}_x(0) = \widetilde{H}(x,0) = \widetilde{f}(x)$, the lift $\widetilde{H}|_{N_y\Box I_n}$ must be unique for each $y \in V(K)$. Since $\widetilde{H}_x$ is unique for each $x \in V(K)$ and is a restriction of the graph homomorphism $\widetilde{H}|_{N_y\Box I_n}$ for each $y \in V(K)$ such that $x \in N[y]$, the graph homomorphisms $\widetilde{H}|_{N(y)\Box I_n}$ must form a unique lift $\widetilde{H}$ of the homotopy H. $\Box$

Here, we provide two examples of homotopies into $\mathcal{C}_3$ and $\mathcal{C}_4$ that do not have lifts.

Example 6.12. Let $f : I_3 \to \mathcal{C}_3$ be the graph homomorphism that starts at $[0]$ and wraps around $\mathcal{C}_3$ once in a clockwise direction and is defined by

$$f(0) = [0],\ f(1) = [1],\ f(2) = [2],\ \text{ and }\ f(3) = [0].$$

Let $g : I_3 \to \mathcal{C}_3$ be the graph homomorphism that stays constantly at $[0]$ and is defined by $g(i) = [0]$ for all $i \in \{0, 1, 2, 3\}$. Recall that (I_∞, p_3) is a covering graph of $\mathcal{C}_3$, where p_3 is defined by $p_3(i) = [i \mod 3]$ for all $i \in \mathbb{Z}$. Figure 6.6 depicts, on the left, a graph homotopy $H : I_3\Box I_1 \to \mathcal{C}_3$ from f to g.

A lift $\widetilde{H} : I_3\Box I_1 \to I_\infty$ of H is depicted in Figure 6.6, on the right. However, this map $\widetilde{H}$ is not a graph homomorphism. The edges shown in red are incident to vertices that are not mapped to the same vertex or adjacent vertices of I_∞.

By the *Path Lifting Property* (6.10), since the restriction $H|_{I_\infty\Box\{j\}}$ is a path for each $j \in \{0, 1\}$, there is a unique lift $\widetilde{H}|_{I_\infty\Box\{j\}}$ starting at $0 \in V(I_\infty)$ for each $j \in \{0, 1\}$. Thus $\widetilde{H}$ is the only possible lift of H given the lift $\widetilde{f}$ of f starting at $0 \in V(I_\infty)$.

Example 6.13. Let $f : I_4 \to \mathcal{C}_4$ be the graph homomorphism that starts at $[0]$ and wraps around $\mathcal{C}_4$ once in a clockwise direction and is defined by

$$f(0) = [0],\ f(1) = [1],\ f(2) = [2],\ f(3) = [3],\ \text{ and }\ f(4) = [0].$$

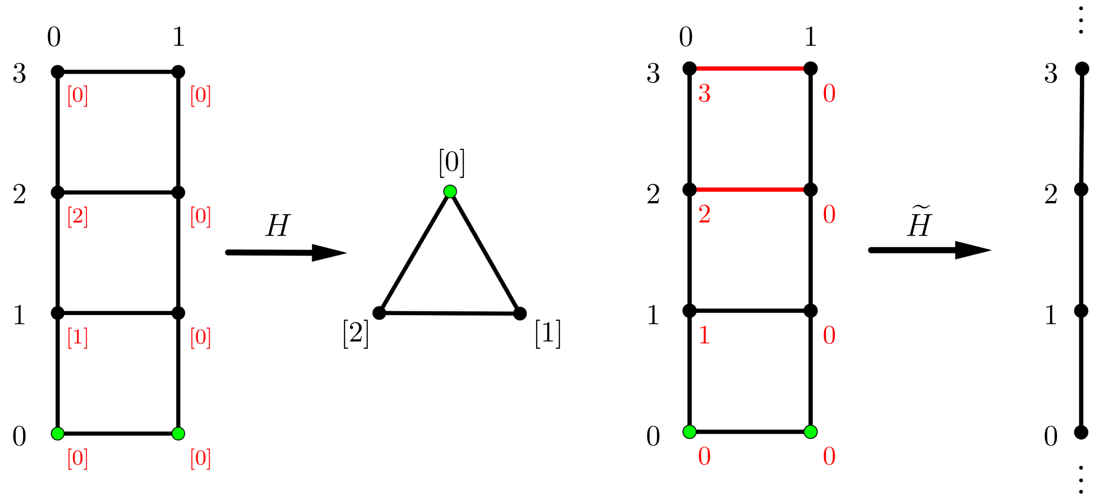

Figure 6.6: A homotopy $H : I_3 \square I_1 \to \mathcal{C}_3$ and the lift $\widetilde{H} : I_3 \square I_1 \to I_\infty$

Let $g : I_4 \to \mathcal{C}_4$ be the graph homomorphism that stays constantly at [0] and is defined by $g(i) = [0]$ for all $i \in \{0, 1, 2, 3, 4\}$. Recall that (I_∞, p_4) is a covering graph of $\mathcal{C}_4$, where p_4 is defined by $p_4(i) = [i \mod 4]$. Figure 6.7 depicts, on the left, a graph homotopy $H : I_4 \square I_2 \to \mathcal{C}_4$ from f to g.

The lift $\widetilde{H} : I_4 \square I_2 \to I_\infty$ of H is depicted in Figure 6.7, on the right. Again, this map $\widetilde{H}$ is not a graph homomorphism. The edges shown in red are incident to vertices that are not mapped to the same vertex or adjacent vertices of I_∞.

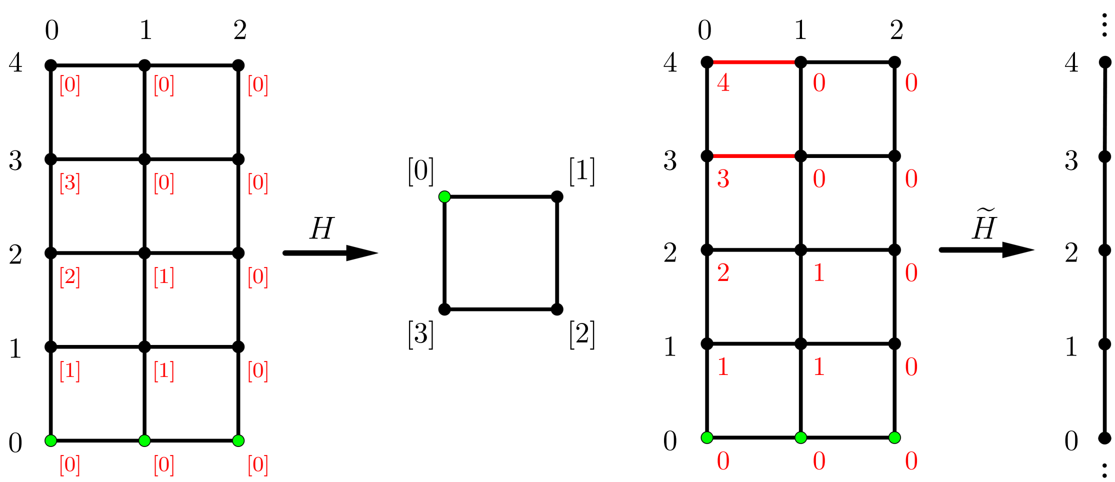

Figure 6.7: A homotopy $H : I_4 \square I_2 \to \mathcal{C}_4$ and the lift $\widetilde{H} : I_4 \square I_2 \to I_\infty$

By the *Path Lifting Property* (6.10), since the restriction $H|_{I_\infty \square \{j\}}$ is a path for each

$j \in \{0,1,2\}$, there is a unique lift $\widetilde{H}|_{I_\infty \Box \{j\}}$ starting at $0 \in V(I_\infty)$ for each $j \in \{0,1,2\}$. Thus $\widetilde{H}$ is the only possible lift of H given the lift $\widetilde{f}$ of f starting at $0 \in V(I_\infty)$.

We now use the *Path Lifting Property* (6.10) and the *Homotopy Lifting Property* (6.11) to prove the general *Lifting Criterion* (6.18), but first we need the following definition.

Definition 6.14. Let $f : (K, y_0) \to (G, x_0)$ be a graph homomorphism. The induced map $f_* : B_1(K, y_0)/\sim \to B_1(G, x_0)/\sim$ is defined by $f_*([\gamma]) = [f \circ \gamma]$, where $[\gamma]$ is an equivalence class of $B_1(K, y_0)/\sim$.

Lemma 6.15. *If $f : (K, y_0) \to (G, x_0)$ is a graph homomorphism, then the induced map $f_* : B_1(K, y_0)/\sim \to B_1(G, x_0)/\sim$ is well-defined.*

Proof. Let $f : (K, y_0) \to (G, x_0)$ be a graph homomorphism, and let the induced map $f_* : B_1(K, y_0)/\sim \to B_1(G, x_0)/\sim$ be defined by $f_*([\gamma]) = [f \circ \gamma]$, where $[\gamma] \in B_1(K, y_0)/\sim$. Suppose $\gamma_1, \gamma_2 \in B_1(K, y_0)$ such that $\gamma_1 \sim \gamma_2$. Thus there exist a graph homomorphism $H_1 \in C_2(K)$ such that

(1) $\alpha^1_{-1}(\gamma_1) = \alpha^1_{-1}(\gamma_2)$ and $\alpha^1_{+1}(\gamma_1) = \alpha^1_{+1}(\gamma_2)$,

(2) $\alpha^2_{-1}(H_1) = \beta^1_1\alpha^1_{-1}(\gamma_1) = \beta^1_1\alpha^1_{-1}(\gamma_2)$ and $\alpha^2_{+1}(H_1) = \beta^1_1\alpha^1_{+1}(\gamma_1) = \beta^1_1\alpha^1_{+1}(\gamma_2)$,

(3) $\alpha^2_{-2}(H_1) = \gamma_1$ and $\alpha^2_{+2}(H_1) = \gamma_2$.

We need to show that $f_*([\gamma_1]) = f_*([\gamma_2])$, that is, $f \circ \gamma_1 \sim f \circ \gamma_2$. Thus we must define a map $H_2 : I^2_\infty \to G$ and show that H_2 is well-defined, a graph homomorphism, and is a graph homotopy from $f \circ \gamma_1$ to $f \circ \gamma_2$. Define $H_2 : I^2_\infty \to G$ by $H_2 = f \circ H_1$. Since H_2 is a composition of the graph homomorphisms $H_1 : I^2_\infty \to K$ and $f : K \to G$, it follows that H_2 is a graph homomorphism. We now show that H_2 is a graph homotopy by verifying conditions (a)-(c) of Definition 4.12.

(a) By part (1), $\gamma_1(m_0(\gamma_1, -1)) = \gamma_2(m_0(\gamma_2, -1))$ and $\gamma_1(m_0(\gamma_1, +1)) = \gamma_2(m_0(\gamma_2, +1))$. Since f is a graph homomorphism, it follows that

$$f(\gamma_1(m_0(\gamma_1, -1))) = f(\gamma_2(m_0(\gamma_2, -1)))$$

and

$$f(\gamma_1(m_0(\gamma_1, +1))) = f(\gamma_2(m_0(\gamma_2, +1))).$$

Therefore, $\alpha^1_{-1}(f \circ \gamma_1) = \alpha^1_{-1}(f \circ \gamma_2)$ and $\alpha^1_{+1}(f \circ \gamma_1) = \alpha^1_{+1}(f \circ \gamma_2)$.

(b) By part (2), it follows that $H_1(m_0(H_1, -1), j) = \gamma_1(m_0(\gamma_1, -1)) = \gamma_2(m_0(\gamma_2, -1))$ and $H_1(m_0(H_1, +1), j) = \gamma_1(m_0(\gamma_1, +1)) = \gamma_2(m_0(\gamma_2, +1))$ for all $j \in \mathbb{Z}$. Since f is a graph homomorphism, it follows that $f(H_1(m_0(H_1, -1), j)) = f(\gamma_1(m_0(\gamma_1, -1))) = f(\gamma_2(m_0(\gamma_2, -1)))$ and $f(H_1(m_0(H_1, +1), j)) = f(\gamma_1(m_0(\gamma_1, +1))) = f(\gamma_2(m_0(\gamma_2, +1)))$ for all $j \in \mathbb{Z}$. Thus $\alpha^2_{-1}(H_2) = \beta^1_1 \alpha^1_{-1}(f \circ \gamma_1) = \beta^1_1 \alpha^1_{-1}(f \circ \gamma_2)$ and $\alpha^2_{+1}(H_1) = \beta^1_1 \alpha^1_{+1}(f \circ \gamma_1) = \beta^1_1 \alpha^1_{+1}(f \circ \gamma_2)$, since $H_2 = f \circ H_1$.

(c) By part (3), $H_1(i, m_0(H_1, -2)) = \gamma_1(i)$ and $H_1(i, m_0(H_1, +2)) = \gamma_2(i)$ for all $i \in \mathbb{Z}$. Since f is a graph homomorphism, it follows that $f(H_1(i, m_0(H_1, -2))) = f(\gamma_1(i))$ and $f(H_1(i, m_0(H_1, +2))) = f(\gamma_2(i))$ for all $i \in \mathbb{Z}$. Therefore, $\alpha^2_{-2}(H_2) = f \circ \gamma_1$ and $\alpha^2_{+2}(H_2) = f \circ \gamma_2$, since $H_2 = f \circ H_1$.

Thus H_2 is a graph homotopy from $f \circ \gamma_1$ to $f \circ \gamma_2$, so $f \circ \gamma_1 \sim f \circ \gamma_2$. Therefore, f_* is well-defined. □

Lemma 6.16. *If $f : (K, y_0) \to (G, x_0)$ is a graph homomorphism, then the induced map $f_* : B_1(K, y_0)/\sim \to B_1(G, x_0)/\sim$ is a group homomorphism.*

Proof. Let $f : (K, y_0) \to (G, x_0)$ be a graph homomorphism, and let the induced map $f_* : B_1(K, y_0)/\sim \to B_1(G, x_0)/\sim$ be defined by $f_*([\gamma]) = [f \circ \gamma]$, where $[\gamma] \in B_1(K, y_0)/\sim$. Suppose $\gamma_1, \gamma_2 \in B_1(K, y_0)$. Since $B_1(K, y_0)$ is closed with respect to concatenation, it follows that $\gamma_1 \cdot \gamma_2 \in B_1(K, y_0)$. We need to show that $f_*([\gamma_1 \cdot \gamma_2]) = f_*([\gamma_1]) \cdot f_*([\gamma_2])$, that is, $f \circ (\gamma_1 \cdot \gamma_2) = (f \circ \gamma_1) \cdot (f \circ \gamma_2)$. The concatenation $(f \circ \gamma_1) \cdot (f \circ \gamma_2)$ is defined by

$$\begin{aligned}
((f \circ \gamma_1) \cdot (f \circ \gamma_2))(i) &= \begin{cases} (f \circ \gamma_1)(i + m_0(f \circ \gamma_1, -1)) & \text{for } i \geq 0, \\ (f \circ \gamma_2)(i + m_0(f \circ \gamma_2, +1)) & \text{for } i \leq 0, \end{cases} \\
&= \begin{cases} (f(\gamma_1(i + m_0(f \circ \gamma_1, -1))) & \text{for } i \geq 0, \\ (f(\gamma_2(i + m_0(f \circ \gamma_2, +1))) & \text{for } i \leq 0. \end{cases}
\end{aligned}$$

Similarly, the concatenation $\gamma_1 \cdot \gamma_2$ is defined by

$$(\gamma_1 \cdot \gamma_2)(i) = \begin{cases} \gamma_1(i + m_0(\gamma_1, -1)) & \text{for } i \geq 0, \\ \gamma_2(i + m_0(\gamma_2, +1)) & \text{for } i \leq 0. \end{cases}$$

Thus the composition $f \circ (\gamma_1 \cdot \gamma_2)$ is defined by

$$f((\gamma_1 \cdot \gamma_2)(i)) = \begin{cases} f(\gamma_1(i + m_0(\gamma_1, -1))) & \text{for } i \geq 0, \\ f(\gamma_2(i + m_0(\gamma_2, +1))) & \text{for } i \leq 0. \end{cases}$$

Since f might possibly map vertices to x_0 after γ_1 stabilizes at $m_0(\gamma_1, -1)$, it follows that $m_0(f \circ \gamma_1, -1) \geq m_0(\gamma_1, -1)$. Thus $m_0(f \circ \gamma_1, -1) - m_0(\gamma_1, -1) \geq 0$, which implies that

$$\begin{aligned}
&f((\gamma_1 \cdot \gamma_2)(m_0(f \circ \gamma_1, -1) - m_0(\gamma_1, -1))) \\
&\quad = f(\gamma_1(m_0(f \circ \gamma_1, -1) - m_0(\gamma_1, -1) + m_0(\gamma_1, -1))) \\
&\quad = f(\gamma_1(m_0(f \circ \gamma_1, -1))).
\end{aligned}$$

Since $m_0(f \circ \gamma_1, -1)$ is the greatest integer such that $(f \circ \gamma_1)(m) = f \circ \gamma_1(m_0(f \circ \gamma_1, -1))$ for all $m \leq m_0(f \circ \gamma_1, -1)$, it follows that $f \circ (\gamma_1 \cdot \gamma_2)$ maps all vertices between 0 and $m_0(f \circ \gamma_1, -1) - m_0(\gamma_1, -1)$ to v_0.

Since f might possibly map vertices to x_0 before the end of γ_2 at $m_0(\gamma_2, +1)$, it follows that $m_0(f \circ \gamma_1, +1) \leq m_0(\gamma_1, +1)$. Thus $m_0(f \circ \gamma_1, +1) - m_0(\gamma_1, +1) \leq 0$, which implies that

$$\begin{aligned}
&f((\gamma_1 \cdot \gamma_2)(m_0(f \circ \gamma_2, +1) - m_0(\gamma_2, +1))) \\
&\qquad = f(\gamma_2(m_0(f \circ \gamma_2, +1) - m_0(\gamma_2, +1) + m_0(\gamma_2, +1))) \\
&\qquad = f(\gamma_2(m_0(f \circ \gamma_2, +1))).
\end{aligned}$$

Since $m_0(f \circ \gamma_2, +1)$ is the least integer such that $(f \circ \gamma_2)(m) = f \circ \gamma_2(m_0(f \circ \gamma_2, +1))$ for all $m \geq m_0(f \circ \gamma_2, +1)$, it follows that $f \circ (\gamma_1 \cdot \gamma_2)$ maps all vertices between $m_0(f \circ \gamma_2, +1) - m_0(\gamma_2, +1)$ and 0 to v_0.

Thus there is potentially padding in $f \circ (\gamma_1 \cdot \gamma_2)$ from the vertex $m_0(f \circ \gamma_2, +1) - m_0(\gamma_2, +1))$ to the vertex $m_0(f \circ \gamma_1, -1) - m_0(\gamma_1, -1))$. Therefore, $f \circ (\gamma_1 \cdot \gamma_2) \sim (f \circ \gamma_1) \cdot (f \circ \gamma_2)$ by the *General Padding Lemma* (5.3), and it follows that f_* is a group homomorphism. $\square$

Lemma 6.17. *Let $(\widetilde{G}, p)$ with $p : (\widetilde{G}, \widetilde{x}_0) \to (G, x_0)$ be a covering graph of G and $f : (K, y_0) \to (G, x_0)$ be a graph homomorphism. Given a lift $\widetilde{f} : (K, y_0) \to (\widetilde{G}, \widetilde{x}_0)$ of f, $p_* \circ \widetilde{f}_* = f_*$.*

Proof. Let $(\widetilde{G}, p)$ with $p : (\widetilde{G}, \widetilde{x}_0) \to (G, x_0)$ be a covering graph of G, let $f : (K, y_0) \to (G, x_0)$ be a graph homomorphism, and let $\widetilde{f} : (K, y_0) \to (\widetilde{G}, \widetilde{x}_0)$ be a lift of f. For all $[\gamma] \in B_1(K, y_0)/\sim$,

$$\begin{aligned}
(p_* \circ \widetilde{f}_*)([\gamma]) &= p_*(\widetilde{f}_*([\gamma])) \\
&= p_*([\widetilde{f} \circ \gamma]) \\
&= [p \circ (\widetilde{f} \circ \gamma)] \\
&= [(p \circ \widetilde{f}) \circ \gamma] \\
&= [f \circ \gamma] \\
&= f_*([\gamma]).
\end{aligned}$$

Therefore, $p_* \circ \widetilde{f}_* = f_*$. $\square$

Theorem 6.18 (**Lifting Criterion**). Let G be a connected graph, let $(\widetilde{G}, p)$ be a covering graph of G, and let $f : (K, y_0) \to (G, x_0)$ be a stable graph homomorphism. If G contains neither 3-cycles nor 4-cycles, then there is a lift $\widetilde{f} : (K, y_0) \to (\widetilde{G}, \widetilde{x}_0)$ of f if and only if $f_*(B_1(K, y_0)/\sim) \subseteq p_*(B_1(\widetilde{G}, \widetilde{x}_0)/\sim)$.

$$\begin{tikzcd} & (\widetilde{G}, \widetilde{x}_0) \arrow[d, "p"] \\ (K, y_0) \arrow[ur, dashed, "\exists! \tilde{f}"] \arrow[r, "f"] & (G, x_0) \end{tikzcd}$$

Proof. Let G be a connected graph with no 3-cycles or 4-cycles, $(\widetilde{G}, p)$ be a covering graph of G, and $f : (K, y_0) \to (G, x_0)$ be a stable graph homomorphism.

- Suppose a lift $\tilde{f} : (K, y_0) \to (\widetilde{G}, \widetilde{x}_0)$ of f exists. Then $p \circ \widetilde{f} = f$, which implies that $p_* \circ \widetilde{f}_* = f_*$ by Lemma 6.17. Let $[\gamma] \in B_1(K, y_0)/\sim$. Thus $f_*([\gamma]) = (p_* \circ \widetilde{f}_*)([\gamma]) = p_*(\widetilde{f}_*([\gamma])) \in p_*(B_1(\widetilde{G}, \widetilde{x}_0)/\sim)$, since $\widetilde{f}_*([\gamma]) = [\widetilde{f} \circ \gamma] \in B_1(\widetilde{G}, \widetilde{x}_0)/\sim$. Therefore, $f_*(B_1(K, y_0)/\sim) \subseteq p_*(B_1(\widetilde{G}, \widetilde{x}_0)/\sim)$.

- Conversely, suppose $f_*(B_1(K, y_0)/\sim) \subseteq p_*(B_1(\widetilde{G}, \widetilde{x}_0)/\sim)$. Let $y \in V(K)$. Since G is connected, there is a stable graph homomorphism $\gamma_y : I_\infty \to K$ with $\gamma_y(m_0(\gamma_y, -1)) = y_0$ and $\gamma_y(m_0(\gamma_y, +1)) = y$. Thus $f \circ \gamma_y : I_\infty \to G$ is a stable graph homomorphism with $f(\gamma_y(m_0(\gamma_y, -1))) = x_0$ and $f(\gamma_y(m_0(\gamma_y, +1))) = f(y) \in V(G)$. Hence, by the *Path Lifting Property* (6.10), there is a unique lift $\widetilde{f\gamma_y} : I_\infty \to \widetilde{G}$ with $\widetilde{f\gamma_y}(m_0(\gamma_y, -1)) = \widetilde{x}_0 \in p^{-1}(x_0)$. Define $\widetilde{f} : K \to \widetilde{G}$ by $\widetilde{f}(y) = \widetilde{f\gamma_y}(m_0(\gamma_y, +1)) \in p^{-1}(f(y))$.

$$\begin{tikzcd}
 & & (\widetilde{G},\widetilde{x}_0) \arrow[d, "p"] \\
I_\infty \arrow[r, "\gamma_y"'] \arrow[urr, bend left, "\widetilde{f\gamma_y}"] & (K,y_0) \arrow[r, "f"'] \arrow[ur, dashed, "\exists\widetilde{f}"] & (G,x_0)
\end{tikzcd}$$

(1) The map $\tilde{f}$ is well-defined.

We must show that $\widetilde{f}(y)$ does not depend on the choice of γ_y. Suppose $\gamma'_y : I_\infty \to K$ is another stable graph homomorphism with $\gamma'_y(m_0(\gamma'_y,-1)) = y_0$ and $\gamma'_y(m_0(\gamma'_y,+1)) = y$. Then $f \circ \gamma'_y : I_\infty \to G$ is a stable graph homomorphism with $f(\gamma'_y(m_0(\gamma'_y,-1))) = x_0$ and $f(\gamma'_y(m_0(\gamma'_y,+1))) = f(y)$. Recall from Definition 5.9 that $\overline{\gamma_y} : I_\infty \to K$ is defined by $\overline{\gamma_y}(i) = \gamma_y(i-1)$ for all $i \in \mathbb{Z}$. Therefore, the concatenation $\overline{\gamma_y} \cdot \gamma'_y : I_\infty \to K$ is defined by

$$\begin{aligned}
\overline{\gamma_y} \cdot \gamma'_y(i) &= \begin{cases} \overline{\gamma_y}(i + m_0(\overline{\gamma_y},-1)) & \text{for } i \geq 0, \\ \gamma'_y(i + m_0(\gamma'_y,+1)) & \text{for } i \leq 0, \end{cases} \\
&= \begin{cases} \overline{\gamma_y}(i - m_0(\gamma_y,+1)) & \text{for } i \geq 0, \\ \gamma'_y(i + m_0(\gamma'_y,+1)) & \text{for } i \leq 0, \end{cases} \\
&= \begin{cases} \gamma_y(-i + m_0(\gamma_y,+1)) & \text{for } i \geq 0, \\ \gamma'_y(i + m_0(\gamma'_y,+1)) & \text{for } i \leq 0. \end{cases}
\end{aligned}$$

Since $\overline{\gamma_y}(m_0(\overline{\gamma_y},-1)) = \gamma_y(-m_0(\overline{\gamma_y},-1)) = \gamma_y(m_0(\gamma_y,+1)) = y = \gamma'_y(\gamma'_y,+1))$, it follows that $\alpha^1_{-1}(\overline{\gamma_y}) = \alpha^1_{+1}(\gamma'_y)$. Thus by Proposition 4.15, $\overline{\gamma_y} \cdot \gamma'_y$ is a graph homomorphism. By Lemma 4.16, $\overline{\gamma_y} \cdot \gamma'_y$ stabilizes in the negative direction at $m_0(\overline{\gamma_y} \cdot \gamma'_y,-1) = m_0(\gamma'_y,-1) - m_0(\gamma'_y,+1)$ and in the positive direction at $m_0(\overline{\gamma_y} \cdot$

$\gamma'_y, +1) = m_0(\overline{\gamma_y}, +1) - m_0(\overline{\gamma_y}, -1) = -m_0(\gamma_y, -1) + m_0(\gamma_y, +1)$. Therefore,

$$\begin{aligned}
\overline{\gamma_y} \cdot \gamma'_y(m_0(\overline{\gamma_y} \cdot \gamma'_y, -1)) &= \overline{\gamma_y} \cdot \gamma'_y(m_0(\gamma'_y, -1) - m_0(\gamma'_y, +1)) \\
&= \gamma'_y(m_0(\gamma'_y, -1) - m_0(\gamma'_y, +1) + m_0(\gamma'_y, +1)) \\
&= \gamma'_y(m_0(\gamma'_y, -1)) \\
&= y_0
\end{aligned}$$

and

$$\begin{aligned}
\overline{\gamma_y} \cdot \gamma'_y(m_0(\overline{\gamma_y} \cdot \gamma'_y, +1)) &= \overline{\gamma_y} \cdot \gamma'_y(-m_0(\gamma_y, -1) + m_0(\gamma_y, +1)) \\
&= \gamma_y(m_0(\gamma_y, -1) - m_0(\gamma_y, +1) + m_0(\gamma_y, +1)) \\
&= \gamma_y(m_0(\gamma_y, -1)) \\
&= y_0.
\end{aligned}$$

Thus $[\overline{\gamma_y} \cdot \gamma'_y] \in B_1(K, y_0)/\sim$, namely, a 'loop' in the graph K based at the distinguished vertex y_0. Since f_* is a group homomorphism by Lemma 6.16, $f_*([\overline{\gamma_y} \cdot \gamma'_y]) = [f(\overline{\gamma_y} \cdot \gamma'_y)] = [\overline{f\gamma_y} \cdot f\gamma'_y]$. Therefore, $[\overline{f\gamma_y} \cdot f\gamma'_y] \in f_*(B_1(K, y_0)/\sim) \subseteq p_*(B_1(\widetilde{G}, \widetilde{x}_0)/\sim)$. Thus there exists an equivalence class $[g] \in B_1(\widetilde{G}, \widetilde{x}_0)/\sim$ such that $p_*([g]) = [\overline{f\gamma_y} \cdot f\gamma'_y]$. Hence, $[pg] = [\overline{f\gamma_y} \cdot f\gamma'_y]$, which implies that $pg \sim \overline{f\gamma_y} \cdot f\gamma'_y$. Therefore, it follows that there exists a graph homotopy $H : I^2_\infty \to G$ from pg to $\overline{f\gamma_y} \cdot f\gamma'_y$. The graph homomorphism $g : I_\infty \to \widetilde{G}$ is a lift of pg. By the *Path Lifting Property* (6.10), there is a unique lift $\widetilde{\overline{f\gamma_y} \cdot f\gamma'_y} : I_\infty \to \widetilde{G}$ of $\overline{f\gamma_y} \cdot f\gamma'_y$ with $\widetilde{\overline{f\gamma_y} \cdot f\gamma'_y}(m_0(\widetilde{\overline{f\gamma_y} \cdot f\gamma'_y}, -1)) = \widetilde{x}_0$. Since G contains neither 3-cycles nor 4-cycles, the *Homotopy Lifting Property* (6.11) holds. Thus there exists a lifted homotopy $\widetilde{H} : I^2_\infty \to \widetilde{G}$ from g to $\widetilde{\overline{f\gamma_y} \cdot f\gamma'_y}$. Since $[g] \in B_1(\widetilde{G}, \widetilde{x}_0)/\sim$, it follows that $\widetilde{\overline{f\gamma_y} \cdot f\gamma'_y}(m_0(\overline{f\gamma_y} \cdot f\gamma'_y, -1)) = \widetilde{\overline{f\gamma_y} \cdot f\gamma'_y}(m_0(\overline{f\gamma_y} \cdot f\gamma'_y, +1)) = \widetilde{x}_0$ as well. By definition of concatenation, $\overline{f\gamma_y} \cdot f\gamma'_y : I_\infty \to G$ is first defined by $f\gamma'_y$ followed by

$\overline{f\gamma_y}$. By definition of inverses, $\overline{f\gamma_y}$ is defined by $f\gamma_y$ in reverse. Therefore, by the uniqueness of the *Path Lifting Property* (6.10), the first part of $\widetilde{\overline{f\gamma_y} \cdot f\gamma_y'}$ is the lift $\widetilde{f\gamma_y'}$ of $f\gamma_y'$ followed by the lift $\widetilde{f\gamma_y}$ of $f\gamma_y$ in reverse with the common vertex $\widetilde{f\gamma_y'}(m_0(\gamma_y', +1)) = \widetilde{f\gamma_y}(m_0(\gamma_y, +1))$. Thus $\widetilde{f}(y)$ is not dependent on the choice of path γ_y starting at y_0 and ending at y. Therefore, $\widetilde{f}$ is well-defined.

(2) $\widetilde{f}$ is a graph homomorphism.

Suppose $x \in N[y]$, the closed neighborhood of y. The map $\widetilde{f}$ is a graph homomorphism if either $\widetilde{f}(y) = \widetilde{f}(x)$ or $\{\widetilde{f}(y), \widetilde{f}(x)\} \in E(\widetilde{G})$. Define $\beta : I_\infty \to G$ by

$$\beta(i) = \begin{cases} \gamma_y(i) & \text{for} \quad i \leq m_0(\gamma_y, +1), \\ x & \text{for} \quad i > m_0(\gamma_y, +1). \end{cases}$$

Since γ_y is a stable graph homomorphism and $x \in N[y]$, the map β is a stable graph homomorphism with $m_0(\beta, +1) = m_0(\gamma_y, +1) + 1$. Therefore, $\widetilde{f}(x) = \widetilde{f\beta}(m_0(\beta, +1))$. Since $\beta(m_0(\beta, +1) - 1) = \beta(m_0(\gamma_y, +1)) = \gamma_y(m_0(\gamma_y, +1)) = y$ and f is a graph homomorphism, $f(\beta(m_0(\beta, +1) - 1)) = f(\gamma_y(m_0(\gamma_y, +1)))$. Thus $\widetilde{f\gamma_y}(m_0(\gamma_y, +1)) = \widetilde{f\beta}(m_0(\beta, +1) - 1)$, which implies that $\widetilde{f\gamma_y}(m_0(\gamma_y, +1)) = \widetilde{f\beta}(m_0(\beta, +1))$ or $\{\widetilde{f\gamma_y}(m_0(\gamma_y, +1)), \widetilde{f\beta}(m_0(\beta, +1))\} \in E(\widetilde{G})$. Therefore, $\widetilde{f}(y) = \widetilde{f}(x)$ or $\{\widetilde{f}(y), \widetilde{f}(x)\} \in E(\widetilde{G})$, and hence, $\widetilde{f}$ is a graph homomorphism.

(3) The graph homomorphism $\widetilde{f}$ is a lift of f, that is, $p \circ \widetilde{f} = f$.

Since $\widetilde{f\gamma_y} : I_\infty \to \widetilde{G}$ is a lift of $f\gamma_y$ and $p \circ \widetilde{f\gamma_y} = f\gamma_y$, it follows that

$$\begin{aligned} p \circ \widetilde{f}(y) &= p(\widetilde{f\gamma_y}(m_0(\gamma_y, +1))) \\ &= f(\gamma_y(m_0(\gamma_y, +1))) \\ &= f(y) \end{aligned}$$

for all $y \in V(K)$. Thus $p \circ \widetilde{f} = f$, and the graph homomorphism $\widetilde{f} : K \to \widetilde{G}$ is lift of f.

□

In the next chapter, we use these lifting properties to show that the fundamental group of the cycle C_5 is isomorphic to $\mathbb{Z}$ in a combinatorial way, concluding our question of why the cycles C_3 and C_4 are A-contractible, while cycles on five or more vertices are not.

Chapter 7

Fundamental Group

In this final chapter, we answer the question of why the cycles C_3 and C_4 are A-contractible and the cycles C_k with $k \geq 5$ are not contractible. In topology, the lifting properties are used to prove that the fundamental group of the circle is isomorphic to $\mathbb{Z}$. We use the analogous lifting properties defined in Chapter 6 in a similar way to show that $(B_1(\mathcal{C}_5, [0])/\sim) \cong \mathbb{Z}$ in this chapter. This method cannot be used for C_3 and C_4, however, because the *Homotopy Lifting Property* only holds for graphs containing neither 3-cycles or 4-cycles. Before we proceed to the computation of the fundamental group of C_5, we first address the fundamental group of all A-contractible graphs, including C_3 and C_4.

Theorem 7.1. If a graph G is A-contractible, then the fundamental group of G based at v_0 is $(B_1(G, v_0)/\sim) = 0$.

Proof. Let G be an A-contractible graph. Recall from Definition 4.14, this implies that there exists graph homomorphisms $f : G \to *$ defined by $f(x) = *$ for all $x \in V(G)$ and $g : * \to G$ defined by $g(*) = v_0$ such that $f \circ g \simeq_A \mathbf{1}_*$ and $g \circ f \simeq_A \mathbf{1}_G$. The composition $f \circ g$ is defined by $f(g(*)) = f(v_0) = *$, and the composition $g \circ f$ is defined by

$$g(f(x)) = g(*) = v_0 \quad \text{for all} \quad x \in V(G).$$

Therefore, $f \circ g$ is equal to the identity $\mathbf{1}_*$ (see Example 2.3) and $g \circ f$ is equal to the constant map $c_{v_0} : G \to G$ (see Example 2.4) that maps every vertex to v_0. Since $g \circ f \simeq_A \mathbf{1}_G$, it follows that $\mathbf{1}_G \simeq_A c_{v_0}$. Thus there exists an integer $n \in \mathbb{N}$ and a graph homomorphism $H : G \square I_n \to G$ such that

- $H(x, 0) = \mathbf{1}_G(x)$ for all $x \in V(G)$,
- $H(x, n) = c_{v_0}(x)$ for all $x \in V(G)$,
- $H(v_0, j) = v_0$ for all $0 \leq j \leq n$.

Define $H_\infty : G \square I_\infty \to G$ by

$$H_\infty(x, j) = \begin{cases} H(x, 0) & \text{for} \quad j \leq 0, \\ H(x, j) & \text{for} \quad 0 \leq j \leq n, \\ H(x, n) & \text{for} \quad j \geq n, \end{cases}$$

for all $x \in V(G)$. Since H is a graph homomorphism, H_∞ is a graph homomorphism. The fundamental group of G is isomorphic to zero if every element of $B_1(G, v_0)$ is homotopic to the constant path $p_{v_0} : I_\infty \to G$ that maps every vertex to v_0 (see Definition 5.7). Let $\gamma \in B_1(G, v_0)$. We use the graph homomorphism H_∞ and γ itself to build a homotopy from γ to p_{v_0}. Define a map $\gamma \square \mathbf{1}_I : I_\infty \square I_\infty \to G \square I_\infty$ by

$$(\gamma \square \mathbf{1}_I)(i, j) \;=\; (\gamma(i), \mathbf{1}_I(j)) \;=\; (\gamma(i), j) \quad \text{for all} \quad i, j \in \mathbb{Z}.$$

We must now show that $\gamma \square \mathbf{1}_I$ is a graph homomorphism. By the definitions of I_∞ and the Cartesian product, there are edges $\{(i, j), (i + 1, j)\}, \{(i, j), (i, j + 1)\} \in E(I_\infty \square I_\infty)$ for all $i, j \in \mathbb{Z}$. Thus $\gamma \square \mathbf{1}_I$ is a graph homomorphism if either $(\gamma \square \mathbf{1}_I)(i, j) = (\gamma \square \mathbf{1}_I)(i + 1, j)$ or $\{(\gamma \square \mathbf{1}_I)(i, j), (\gamma \square \mathbf{1}_I)(i + 1, j)\} \in E(G \square I_\infty)$, and either $(\gamma \square \mathbf{1}_I)(i, j) = (\gamma \square \mathbf{1}_I)(i, j + 1)$ or $\{(\gamma \square \mathbf{1}_I)(i, j), (\gamma \square \mathbf{1}_I)(i, j + 1)\} \in E(G \square I_\infty)$.

- First consider $(\gamma\Box\mathbf{1}_I)(i,j)$ and $(\gamma\Box\mathbf{1}_I)(i+1,j)$.

  By definition of $\gamma\Box\mathbf{1}_I$,

  $$(\gamma\Box\mathbf{1}_I)(i,j) = (\gamma(i),j) \quad \text{and} \quad (\gamma\Box\mathbf{1}_I)(i+1,j) = (\gamma(i+1),j).$$

  Since $\{i,i+1\} \in E(I_\infty)$ for all $i \in \mathbb{Z}$ and γ is a graph homomorphism, it follows that either $\gamma(i) = \gamma(i+1)$ or $\{\gamma(i),\gamma(i+1)\} \in E(G)$. Thus $(\gamma\Box\mathbf{1}_I)(i,j) = (\gamma\Box\mathbf{1}_I)(i+1,j)$ or $\{(\gamma\Box\mathbf{1}_I)(i,j),(\gamma\Box\mathbf{1}_I)(i+1,j)\} \in E(G\Box I_\infty)$.

- Next consider $(\gamma\Box\mathbf{1}_I)(i,j)$ and $(\gamma\Box\mathbf{1}_I)(i,j+1)$.

  By definition of $\gamma\Box\mathbf{1}_I$,

  $$(\gamma\Box\mathbf{1}_I)(i,j) = (\gamma(i),j) \quad \text{and} \quad (\gamma\Box\mathbf{1}_I)(i,j+1) = (\gamma(i),j+1).$$

  By definition of I_∞, the edge $\{j,j+1\} \in E(I_\infty)$ for all $j \in \mathbb{Z}$. Therefore, $\{(\gamma(i),j),$ $(\gamma(i),j+1)\} \in E(G\Box I_\infty)$ by definition of the Cartesian product. Thus

  $$\{(\gamma\Box\mathbf{1}_I)(i,j),(\gamma\Box\mathbf{1}_I)(i,j+1)\} \in E(G\Box I_\infty).$$

Therefore, $\gamma\Box\mathbf{1_I}$ is a graph homomorphism. Define a map $H_1 = H_\infty \circ (\gamma\Box\mathbf{1}_I)$. Since $\gamma\Box\mathbf{1}_I : I_\infty\Box I_\infty \to G\Box I_\infty$ and $H_\infty : G\Box I_\infty \to G$, it follows that $H_1 : I_\infty\Box I_\infty \to G$. The map H_1 is

$$\begin{aligned} H_1(i,j) &= (H_\infty \circ (\gamma\Box\mathbf{1}_I))(i,j) \\ &= H_\infty(\gamma(i),j) \\ &= \begin{cases} H(\gamma(i),0) & \text{for} \quad j \leq 0, \\ H(\gamma(i),j) & \text{for} \quad 0 \leq j \leq n, \\ H(\gamma(i),n) & \text{for} \quad j \geq n, \end{cases} \end{aligned}$$

for all $i \in \mathbb{Z}$. Since H_∞ and $\gamma \Box \mathbf{1}_I$ are graph homomorphisms, the composition $H_\infty \circ (\gamma \Box \mathbf{1}_I)$ is a graph homomorphism by Lemma 2.10. We must now show that H_1 is a homotopy from γ to p_{v_0} by verifying conditions (a)-(c) of Definition 4.12.

(a) Since the path p_{v_0} is constantly equal to v_0, $m_0(p_{v_0}, +1) = 0 = m_0(p_{v_0}, -1)$, and since $\gamma \in B_1(G, v_0)$, γ must start and end at v_0. Thus

$$\alpha^1_{+1}(\gamma)(*) = \gamma(m_0(\gamma, +1)) = v_0 \quad \text{and} \quad \alpha^1_{+1}(p_{v_0})(*) = p_{v_0}(0) = v_0$$

and

$$\alpha^1_{-1}(\gamma)(*) = \gamma(m_0(\gamma, -1)) = v_0 \quad \text{and} \quad \alpha^1_{-1}(p_{v_0})(*) = p_{v_0}(0) = v_0,$$

Therefore, $\alpha^1_{+1}(\gamma) = \alpha^1_{+1}(p_{v_0})$ and $\alpha^1_{-1}(\gamma) = \alpha^1_{-1}(p_{v_0})$.

(b) Since $H_1(i,j) = H(\gamma(i), 0)$ for $j \leq 0$, $H_1(i,j) = H(\gamma(i), j)$ for $0 \leq j \leq n$, and $H_1(i,j) = H(\gamma(i), n)$ for $j \geq n$, it follows that H_1 stabilizes on the 1^{st}-axis when γ stabilizes. Thus $m_0(H_1, +1) = m_0(\gamma, +1)$ and $m_0(H_1, -1) = m_0(\gamma, -1)$. Therefore, the face $\alpha^2_{+1}(H_1)$ is given by

$$\begin{aligned}
\alpha^2_{+1}(H_1)(j) &= H_1(m_0(\gamma, +1), j) \\
&= \begin{cases} H(\gamma(m_0(\gamma, +1)), 0) & \text{for} \quad j \leq 0, \\ H(\gamma(m_0(\gamma, +1)), j) & \text{for} \quad 0 \leq j \leq n, \\ H(\gamma(m_0(\gamma, +1)), n) & \text{for} \quad j \geq n, \end{cases} \\
&= \begin{cases} H(v_0, 0) & \text{for} \quad j \leq 0, \\ H(v_0, j) & \text{for} \quad 0 \leq j \leq n, \\ H(v_0, n) & \text{for} \quad j \geq n. \end{cases}
\end{aligned}$$

Since $H(v_0, i) = v_0$ for all $0 \leq i \leq n$, it follows that $\alpha^2_{+1}(H_1)(j) = v_0 = \alpha^1_{+1}(\gamma)(*) = \alpha^1_{+1}(p_{v_0})(*)$ for all $j \in \mathbb{Z}$. Thus $\alpha^2_{+1}(H_1) = \beta^1_1 \alpha^1_{+1}(\gamma) = \beta^1_1 \alpha^1_{+1}(p_{v_0})$. Similarly, the face

$\alpha^2_{-1}(H_1)$ is given by

$$\begin{aligned}
\alpha^2_{-1}(H_1)(j) &= H_1(m_0(\gamma,-1),j) \\
&= \begin{cases} H(\gamma(m_0(\gamma,-1)),0) & \text{for} \quad j \le 0, \\ H(\gamma(m_0(\gamma,-1)),j) & \text{for} \quad 0 \le j \le n, \\ H(\gamma(m_0(\gamma,-1)),n) & \text{for} \quad j \ge n, \end{cases} \\
&= \begin{cases} H(v_0,0) & \text{for} \quad j \le 0, \\ H(v_0,j) & \text{for} \quad 0 \le j \le n, \\ H(v_0,n) & \text{for} \quad j \ge n. \end{cases}
\end{aligned}$$

Since $H(v_0,i) = v_0$ for all $0 \le i \le n$, it follows that $\alpha^2_{-1}(H_1)(j) = v_0 = \alpha^1_{-1}(\gamma)(*) = \alpha^1_{-1}(p_{v_0})(*)$ for all $j \in \mathbb{Z}$. Thus $\alpha^2_{-1}(H_1) = \beta^1_1\alpha^1_{-1}(\gamma) = \beta^1_1\alpha^1_{-1}(p_{v_0})$.

(c) Since $H_1(i,j) = H(\gamma(i),0)$ for $j \le 0$ and $H_1(i,j) = H(\gamma(i),n)$ for $j \ge n$, it follows that H_1 stabilizes on the 2^{nd}-axis at $m_0(H_1,-2) = 0$ and $m_0(H_1,+2) = n$. Thus the face $\alpha^2_{-2}(H_1)$ is

$$\alpha^2_{-2}(H_1)(i) = H_1(i,0) = H(\gamma(i),0) = \mathbf{1}_G(\gamma(i)) = \gamma(i) \quad \text{for all} \quad i \in \mathbb{Z}.$$

Therefore, $\alpha^2_{-2}(H_1) = \gamma$. Similarly, the face $\alpha^2_{+2}(H_1)$ is

$$\alpha^2_{+2}(H_1)(i) = H_1(i,n) = H(\gamma(i),n) = c_{v_0}(\gamma(i)) = v_0 \quad \text{for all} \quad i \in \mathbb{Z}.$$

Hence, $\alpha^2_{+2}(H_1) = p_{v_0}$.

Thus H_1 is a graph homotopy from γ to p_{v_0}, and it follows that $\gamma \sim p_{v_0}$. Therefore, $(B_1(G,v_0)/\sim) = 0$, since γ is an arbitrary element of $B_1(G,v_0)$. $\square$

The next step is to find the fundamental group of the cycle C_5, which is not A-contractible.

We need a few tools in order to do this.

Definition 7.2. Let $\mathcal{C}_5$ be a 5-cycle with vertices labeled [0], [1], [2], [3], and [4], and let $p_5 : I_\infty \to \mathcal{C}_5$ be the graph homomorphism defined by $p_5(i) = [i \mod 5]$ for all $i \in \mathbb{Z}$.

Note that p_5 does not stabilize in either direction. Let $[i-1, i+1]$ denote the subgraph of I_∞ with vertex set $V([i-1,i+1]) = \{i-1, i, i+1\}$ and edge set $E([i-1,i+1]) = \{\{i-1,i\},\{i,i+1\}\}$ for all $i \in \mathbb{Z}$. The relative graph homomorphism $p_5|_{N[i]} = p_5|_{[i-1,i+1]}$ is bijective for all $i \in \mathbb{Z}$. Thus p_5 is a local isomorphism, and the pair (I_∞, p_5) forms a covering graph of $\mathcal{C}_5$.

If $\alpha : I_\infty \to \mathcal{C}_5$ is a stable graph homomorphism with $\alpha(m_0(\alpha,-1)) = [0]$, then by the *Path Lifting Property* (Theorem 6.10) there is a unique graph homomorphism $\widetilde{\alpha} : I_\infty \to I_\infty$ with $\widetilde{\alpha}(m_0(\widetilde{\alpha},-1)) = \widetilde{x}$ for each $\widetilde{x} \in p_5^{-1}([0])$ such that the diagram

$$\begin{tikzcd} & (I_\infty, \widetilde{x}) \arrow[d, "p_5"] \\ I_\infty \arrow[ur, "\widetilde{\alpha}"] \arrow[r, "\alpha"] & (\mathcal{C}_5, [0]) \end{tikzcd}$$

commutes, that is, $p_5 \circ \widetilde{\alpha} = \alpha$.

Lemma 7.3 (Path Lift). *Let $\alpha \in B_1(\mathcal{C}_5, x_0)$, and let the pair (I_∞, p_5) be as in Definition 7.2. Suppose $\widetilde{x_0} \in p_5^{-1}(x_0)$. Then a lift $\widetilde{\alpha} : I_\infty \to I_\infty$ of α is defined for all $i \leq m_0(\alpha,-1)$ by $\widetilde{\alpha}(i) = \widetilde{x_0}$ and for all $i > m_0(\alpha,-1)$ recursively by*

$$\widetilde{\alpha}(i) = \begin{cases} \widetilde{\alpha}(i-1)+1 & \textit{if } \ \alpha(i) = \alpha(i-1) + [1], \\ \widetilde{\alpha}(i-1) & \textit{if } \ \alpha(i) = \alpha(i-1), \\ \widetilde{\alpha}(i-1)-1 & \textit{if } \alpha(i) = \alpha(i-1) - [1]. \end{cases}$$

Proof. Let $\alpha \in B_1(\mathcal{C}_5, x_0)$ and the pair (I_∞, p_5) be as defined previously. Suppose $\widetilde{x_0} \in p_5^{-1}(x_0)$. By the *Path Lifting Property* (6.10), there is a unique lift $\widetilde{\alpha} : I_\infty \to I_\infty$ defined by $\widetilde{\alpha}(i) = \widetilde{x_0}$ for all $i \leq m_0(\alpha, -1)$, and recursively by $\widetilde{\alpha}(i) = (p_5|_{N_{\widetilde{\alpha}(i-1)}})^{-1}(\alpha(i))$ for all $i > m_0(\alpha, -1)$, so we only need to compute $(p_5|_{N_{\widetilde{\alpha}(i-1)}})^{-1}(\alpha(i))$ for $i > m_0(\alpha, -1)$. By definition of I_∞, it follows that the subgraph $N_{\widetilde{\alpha}(i-1)} = [\widetilde{\alpha}(i-1)-1, \widetilde{\alpha}(i-1)+1]$. Therefore, $p_5|_{N_{\widetilde{\alpha}(i-1)}}$ is a graph homomorphism from the subgraph with vertex set $\{\widetilde{\alpha}(i-1)-1, \widetilde{\alpha}(i-1), \widetilde{\alpha}(i-1)+1\}$ to the subgraph with vertex set $\{p_5(\widetilde{\alpha}(i-1)-1), p_5(\widetilde{\alpha}(i-1)), p_5(\widetilde{\alpha}(i-1)+1)\}$. By definition of p_5 and since $p_5 \circ \widetilde{\alpha} = \alpha$,

$$\begin{aligned}
p_5(\widetilde{\alpha}(i-1)-1) &= [(\widetilde{\alpha}(i-1)-1) \mod 5] \\
&= [\widetilde{\alpha}(i-1) \mod 5] - [1] \\
&= p_5(\widetilde{\alpha}(i-1)) - [1] \\
&= \alpha(i-1) - [1],
\end{aligned}$$

and

$$p_5(\widetilde{\alpha}(i-1)) = \alpha(i-1),$$

and

$$\begin{aligned}
p_5(\widetilde{\alpha}(i-1)+1) &= [(\widetilde{\alpha}(i-1)+1) \mod 5] \\
&= [\widetilde{\alpha}(i-1) \mod 5] + [1] \\
&= p_5(\widetilde{\alpha}(i-1) + [1] \\
&= \alpha(i-1) + [1].
\end{aligned}$$

Thus $(p_5|_{N_{\widetilde{\alpha}(i-1)}})^{-1}(\alpha(i-1) - [1]) = \widetilde{\alpha}(i-1) - 1$, $(p_5|_{N_{\widetilde{\alpha}(i-1)}})^{-1}(\alpha(i-1)) = \widetilde{\alpha}(i-1)$ $(p_5|_{N_{\widetilde{\alpha}(i-1)}})^{-1}(\alpha(i-1) + [1]) = \widetilde{\alpha}(i-1) + 1$. Therefore, $\widetilde{\alpha}$ is defined by $\widetilde{\alpha}(i) = \widetilde{x_0}$ for all

$i \leq m_0(\alpha, -1)$ and recursively by

$$\widetilde{\alpha}(i) = \begin{cases} \widetilde{\alpha}(i-1)+1 & \text{if } \ \alpha(i) = \alpha(i-1)+[1], \\ \widetilde{\alpha}(i-1) & \text{if } \ \alpha(i) = \alpha(i-1), \\ \widetilde{\alpha}(i-1)-1 & \text{if } \alpha(i) = \alpha(i-1)-[1], \end{cases}$$

for all $i > m_0(\alpha, -1)$. □

We also need to propose representatives for the equivalence classes of $B_1(\mathcal{C}_5, [0])/\sim$.

Definition 7.4. Let the map $\gamma_n : I_\infty \to \mathcal{C}_5$ be defined for each $n \geq 0$ by

$$\gamma_n(i) = \begin{cases} [0] & \text{for } \ i \leq 0, \\ [i \mod 5] & \text{for } \ 0 \leq i \leq 5n, \\ [0] & \text{for } \ i \geq 5n, \end{cases}$$

and for each $n \leq 0$ by

$$\gamma_n(i) = \begin{cases} [0] & \text{for } \ i \leq 0, \\ [(-i) \mod 5] & \text{for } \ 0 \leq i \leq -5n, \\ [0] & \text{for } \ i \geq -5n. \end{cases}$$

When $n = 0$, γ_n is the constant map at $[0]$. For $n > 0$, the graph homomorphism γ_n starts at $[0]$ and wraps around $\mathcal{C}_5$ in a clockwise direction n times. Similarly, for $n < 0$, the graph homomorphism γ_n starts at $[0]$ and wraps around $\mathcal{C}_5$ in a counterclockwise direction

n times. Given these γ_n representatives, we need lifts $\widetilde{\gamma}_n$. If $n \geq 0$, then

$$\begin{aligned}
\gamma_n(i) &= [i \mod 5] \\
&= [(i-1+1) \mod 5] \\
&= [(i-1) \mod 5] + [1] \\
&= \gamma_n(i-1) + [1]
\end{aligned}$$

for all $0 < i \leq 5n$, and $\gamma_n(i) = [0]$ otherwise. Similarly, if $n \leq 0$, then

$$\begin{aligned}
\gamma_n(i) &= [(-i) \mod 5] \\
&= [(-i+1-1) \mod 5] \\
&= [(-i+1) \mod 5] - [1] \\
&= \gamma_n(i-1) - [1]
\end{aligned}$$

for all $0 < i \leq -5n$, and $\gamma_n(i) = [0]$ otherwise. Thus by Lemma 7.3, the lift of γ_n starting at 0 is $\widetilde{\gamma}_n : I_\infty \to I_\infty$ defined by

$$\widetilde{\gamma}_n(i) = \begin{cases} 0 & \text{for} \quad i \leq 0, \\ i & \text{for} \quad 0 \leq i \leq 5n, \\ 5n & \text{for} \quad i \geq 5n, \end{cases} \qquad \text{if } n \geq 0$$

and

$$\widetilde{\gamma}_n(i) = \begin{cases} 0 & \text{for} \quad i \leq 0, \\ -i & \text{for} \quad 0 \leq i \leq -5n, \\ 5n & \text{for} \quad i \geq -5n, \end{cases} \qquad \text{if } n \leq 0.$$

We also need to know how the representatives γ_n relate to each other. We do this by the following lemma.

Lemma 7.5. *Let $\gamma_n, \gamma_{-n} \in B_1(\mathcal{C}_5, [0])$ be as defined in Definition 7.4 for $n \in \mathbb{Z}$. Then $\gamma_{-n} \sim \overline{\gamma_n}$, the inverse of graph homomorphism γ_n.*

Proof. Suppose $n \geq 0$. By Definition 7.4 and Definition 5.9,

$$
\begin{aligned}
\overline{\gamma_n}(i) &= \gamma_n(-i) \\
&= \begin{cases} [0] & \text{for} \quad -i \leq 0, \\ [(-i) \mod 5] & \text{for} \quad 0 \leq -i \leq 5n, \\ [0] & \text{for} \quad -i \geq 5n, \end{cases} \\
&= \begin{cases} [0] & \text{for} \quad i \leq -5n, \\ [(-i) \mod 5] & \text{for} \quad -5n \leq i \leq 0, \\ [0] & \text{for} \quad i \geq 0. \end{cases}
\end{aligned}
$$

By the construction of γ_n,

$$
\begin{aligned}
\gamma_{-n}(i+5n) &= \begin{cases} [0] & \text{for} \quad i+5n \leq 0, \\ [(-i-5n) \mod 5] & \text{for} \quad 0 \leq i+5n \leq 5n, \\ [0] & \text{for} \quad i+5n \geq 5n, \end{cases} \\
&= \begin{cases} [0] & \text{for} \quad i \leq -5n, \\ [(-i) \mod 5] & \text{for} \quad -5n \leq i \leq 0, \\ [0] & \text{for} \quad i \geq 0, \end{cases} \\
&= \overline{\gamma_n}(i),
\end{aligned}
$$

for all $i \in \mathbb{Z}$. Therefore, the inverse $\overline{\gamma_n}$ is γ_{-n} shifted down by $5n$. Thus it follows by the *Shifting Lemma* (5.4) that $\gamma_{-n} \sim \overline{\gamma_n}$ for $n \geq 0$. Suppose $n \leq 0$. By Definition 5.9 and

Definition 7.4,

$$\begin{aligned}
\overline{\gamma_n}(i) &= \gamma_n(-i) \\
&= \begin{cases} [0] & \text{for} \quad -i \leq 0, \\ [i \mod 5] & \text{for} \quad 0 \leq -i \leq -5n, \\ [0] & \text{for} \quad -i \geq -5n, \end{cases} \\
&= \begin{cases} [0] & \text{for} \quad i \leq 5n, \\ [i \mod 5] & \text{for} \quad 5n \leq i \leq 0, \\ [0] & \text{for} \quad i \geq 0. \end{cases}
\end{aligned}$$

By construction of γ_n,

$$\begin{aligned}
\gamma_{-n}(i-5n) &= \begin{cases} [0] & \text{for} \quad i-5n \leq 0, \\ [(i-5n) \mod 5] & \text{for} \quad 0 \leq i-5n \leq -5n, \\ [0] & \text{for} \quad i-5n \geq -5n, \end{cases} \\
&= \begin{cases} [0] & \text{for} \quad i \leq 5n, \\ [i \mod 5] & \text{for} \quad 5n \leq i \leq 0, \\ [0] & \text{for} \quad i \geq 0, \end{cases} \\
&= \overline{\gamma_n}(i),
\end{aligned}$$

for all $i \in \mathbb{Z}$. Therefore, the inverse $\overline{\gamma_n}$ is γ_{-n} shifted down by $-5n$. Thus it follows by the *Shifting Lemma* (5.4) that $\gamma_{-n} \sim \overline{\gamma_n}$ for $n \leq 0$. Therefore, $\gamma_{-n} \sim \overline{\gamma_n}$ for all $n \in \mathbb{Z}$. $\square$

We need one last lemma before proceeding to the proof that $(B_1(\mathcal{C}_5, [0])/\sim) \cong \mathbb{Z}$.

Definition 7.6. Let $\widetilde{f} : I_\infty \to I_\infty$ be a stable graph homomorphism. For $i \in \mathbb{Z}$, the value $\widetilde{f}(i)$ is *increasing* if $\widetilde{f}(i) < \widetilde{f}(i+1)$ and is *decreasing* if $\widetilde{f}(i) > \widetilde{f}(i+1)$ and is *constant* if

$\widetilde{f}(i) = \widetilde{f}(i+1)$.

Lemma 7.7. *If $\widetilde{f} : I_\infty \to I_\infty$ is a stable graph homomorphism with $\widetilde{f}(m_0(\widetilde{f}, -1)) = 0$ and $\widetilde{f}(m_0(\widetilde{f}, +1)) = 5n$, then $[\widetilde{f}] = [\widetilde{\gamma_n}]$, where $\widetilde{\gamma_n}$ is a lift of $\gamma_n : I_\infty \to \mathcal{C}_5$.*

Proof. Let $\widetilde{f} : I_\infty \to I_\infty$ be a stable graph homomorphism with $\widetilde{f}(m_0(\widetilde{f}, -1)) = 0$ and $\widetilde{f}(m_0(\widetilde{f}, +1)) = 5n$ with $n \in \mathbb{Z}$. Although the path $\widetilde{f}$ starts at 0 and ends at $5n$, $\widetilde{f}$ may increase, decrease, or remain constant from the vertex $m_0(\widetilde{f}, -1)$ to the vertex $m_0(\widetilde{f}, +1)$. In contrast, for $n \geq 0$, $\widetilde{\gamma_n}$ increases constantly from starting at 0 to ending at $5n$, and for $n \leq 0$, $\widetilde{\gamma_n}$ decreases constantly from starting at 0 to ending at $5n$. We show that $\widetilde{f}$ is homotopic to $\widetilde{\gamma_n}$ by first showing that $\widetilde{f}$ is homotopic to a path $\widetilde{f}'$ that has no negative increasing values and no positive decreasing values. Since $\widetilde{f}'$ starts at 0 as well, if $n \geq 0$, no negative increasing values implies that $\widetilde{f}'$ has no negative values at all, and no positive decreasing values implies that $\widetilde{f}'$ is constant or increasing from 0 to $5n$. If $n \leq 0$, no positive decreasing values implies that $\widetilde{f}'$ has no positive values at all, and no negative increasing values implies that $\widetilde{f}'$ is constant or decreasing from 0 to $5n$. Then we use the *General Padding Lemma* (5.3) to show that this path $\widetilde{f}'$ is homotopic to $\widetilde{\gamma_n}$.

Define $H : I_\infty \square I_\infty \to I_\infty$ for all $j \leq 0$ by $H(i,j) = \widetilde{f}(i)$, and recursively for all $j > 0$ by

$$H(i,j) = \begin{cases} H(i,j-1) - 1 & \text{if } \ 0 \leq H(i+1,j-1) < H(i,j-1), \\ H(i,j-1) & \text{if } \ 0 \leq H(i,j-1) \leq H(i+1,j-1), \\ H(i,j-1) + 1 & \text{if } \ H(i,j-1) < H(i+1,j-1) \leq 0, \\ H(i,j-1) & \text{if } \ H(i+1,j-1) \leq H(i,j-1) \leq 0. \end{cases}$$

First, we must confirm that these are all of the cases. Define $H_j : I_\infty \to I_\infty$ by $H_j(i) = H(i,j)$ of all $i, j \in \mathbb{Z}$. The first case is if $H_{j-1}(i)$ is a positive decreasing value. The second case is if $H_{j-1}(i)$ is a non-negative increasing or constant value. The third case is if $H_{j-1}(i)$ is a negative increasing value. The fourth case is if $H_{j-1}(i)$ is a non-positive decreasing or constant value. These are all possible cases. Note that the second and fourth cases overlap when $H_{j-1}(i) = 0$ and is a constant value. The map H is well-defined, however, since $H(i,j) = H(i,j-1)$ in both cases. We now need to show that H is a graph homomorphism. By the definitions of I_∞ and the Cartesian product, there are edges $\{(i,j),(i+1,j)\}, \{(i,j),(i,j+1)\} \in E(I_\infty \square I_\infty)$. Thus H is a graph homomorphism if either $H(i,j) = H(i+1,j)$ or $\{H(i,j), H(i+1,j)\} \in E(I_\infty)$, and either $H(i,j) = H(i,j+1)$ or $\{H(i,j), H(i,j+1)\} \in E(I_\infty)$. Since $H(i,j) = \widetilde{f}(i)$ for all $j \leq 0$ and $\widetilde{f}$ is a graph homomorphism, we only need to examine H for $j \geq 0$. Let $j \geq 0$.

- First consider $H(i,j)$ and $H(i+1,j)$.

  Since H is defined recursively for $j > 0$, we show that either $H(i,j) = H(i+1,j)$ or $\{H(i,j), H(i+1,j)\} \in E(I_\infty)$ by induction on j.

  Base case: For $j = 0$, $H(i,j) = H(i,0) = \widetilde{f}(i)$ and $H(i+1,j) = H(i+1,0) = \widetilde{f}(i+1)$. Since $\{i, i+1\} \in E(I_\infty)$ and $\widetilde{f}$ is a graph homomorphism, either $\widetilde{f}(i) = \widetilde{f}(i+1)$ or $\{\widetilde{f}(i), \widetilde{f}(i+1)\} \in E(I_\infty)$. Thus $H(i,0) = H(i+1,0)$ or $\{H(i,0), H(i+1,0)\} \in E(I_\infty)$.

  Inductive Hypothesis: Assume $H(i,j-1) = H(i+1,j-1)$ or $\{H(i,j-1), H(i+1,j-1)\} \in E(I_\infty)$ for some $j > 0$.

  We examine the four cases for how $H(i,j)$ is defined, and for each of these cases, the four cases for how $H(i+1,j)$ is defined.

  1. Suppose $0 \leq H(i+1,j-1) < H(i,j-1)$. By definition of H, $H(i,j) = H(i,j-1) - 1$ in this case. By the inductive hypothesis, since $H(i+1,j-1) < H(i,j-1)$, it follows that $H(i+1,j-1) = H(i,j-1) - 1 = H(i,j)$. We now examine the four cases for how $H(i+1,j)$ is defined in this case.

     (a) Suppose $0 \leq H(i+2,j-1) < H(i+1,j-1)$. By definition of H, $H(i+1,j) = H(i+1,j-1) - 1$. Thus $H(i+1,j) = H(i,j) - 1$, so $\{H(i,j), H(i+1,j)\} \in E(I_\infty)$.

     (b) Suppose $0 \leq H(i+1,j-1) \leq H(i+2,j-1)$. By definition of H, $H(i+1,j) = H(i+1,j-1)$. Thus $H(i+1,j) = H(i,j)$.

(c) Suppose $H(i+1, j-1) < H(i+2, j-1) \leq 0$. Since $0 \leq H(i+1, j-1)$, this is a contradiction.

(d) Suppose $H(i+2, j-1) \leq H(i+1, j-1) = 0$. By definition of H, $H(i+1, j) = H(i+1, j-1)$, so $H(i+1, j) = H(i, j)$.

2. Suppose $0 \leq H(i, j-1) \leq H(i+1, j-1)$. By definition of H, $H(i, j) = H(i, j-1)$ in this case. By the inductive hypothesis, since $H(i, j-1) \leq H(i+1, j-1)$, it follows that $H(i+1, j-1) = H(i, j-1)$ or $H(i+1, j-1) = H(i, j-1)+1$. Thus $H(i+1, j-1) = H(i, j)$ or $H(i+1, j-1) = H(i, j)+1$. We now examine the four cases for how $H(i+1, j)$ is defined in this case.

   (a) Suppose $0 \leq H(i+2, j-1) < H(i+1, j-1)$. By definition of H, $H(i+1, j) = H(i+1, j-1) - 1$. Thus

$$H(i+1, j) = H(i, j) - 1 \quad \text{or} \quad H(i+1, j) = H(i, j) + 1 - 1 = H(i, j),$$

   which implies that $H(i+1, j) = H(i, j)$ or $\{H(i, j), H(i+1, j)\} \in E(I_\infty)$.

   (b) Suppose $0 \leq H(i+1, j-1) \leq H(i+2, j-1)$. By definition of H, $H(i+1, j) = H(i+1, j-1)$. Thus

$$H(i+1, j) = H(i, j) \quad \text{or} \quad H(i+1, j) = H(i, j) + 1,$$

   which implies that $H(i+1, j) = H(i, j)$ or $\{H(i, j), H(i+1, j)\} \in E(I_\infty)$.

   (c) Suppose $H(i+1, j-1) < H(i+2, j-1) \leq 0$. Since $0 \leq H(i+1, j-1)$, this is a contradiction.

   (d) Suppose $H(i+2, j-1) \leq H(i+1, j-1) = 0$. By definition of H, $H(i+1, j) = H(i+1, j-1)$. Thus

$$H(i+1, j) = H(i, j) \quad \text{or} \quad H(i+1, j) = H(i, j) + 1.$$

Therefore, $H(i,j) = H(i+1,j)$ or $\{H(i,j), H(i+1,j)\} \in E(I_\infty)$.

3. Suppose $H(i,j-1) < H(i+1,j-1) \leq 0$. By definition of H, $H(i,j) = H(i,j-1)+1$ in this case. By the inductive hypothesis, since $H(i,j-1) < H(i+1,j-1)$, it follows that $H(i+1,j-1) = H(i,j-1)+1 = H(i,j)$. We now examine the four cases for how $H(i+1,j)$ is defined in this case.

   (a) Suppose $0 \leq H(i+2,j-1) < H(i+1,j-1)$. Since $H(i+1,j-1) \leq 0$, this is a contradiction.

   (b) Suppose $0 = H(i+1,j-1) \leq H(i+2,j-1)$. By definition of H, $H(i+1,j) = H(i+1,j-1)$. Thus $H(i+1,j) = H(i,j)$.

   (c) Suppose $H(i+1,j-1) < H(i+2,j-1) \leq 0$. By definition of H, $H(i+1,j) = H(i+1,j-1)+1$. Thus $H(i+1,j) = H(i,j)+1$, which implies that $\{H(i,j), H(i+1,j)\} \in E(I_\infty)$.

   (d) Suppose $H(i+2,j-1) \leq H(i+1,j-1) \leq 0$. By definition of H, $H(i+1,j) = H(i+1,j-1)$. Thus $H(i+1,j) = H(i,j)$.

4. Suppose $H(i+1,j-1) \leq H(i,j-1) \leq 0$. By definition of H, $H(i,j) = H(i,j-1)$ in this case. By the inductive hypothesis, since $H(i+1,j-1) \leq H(i,j-1)$, it follows that $H(i+1,j-1) = H(i,j-1)$ or $H(i+1,j-1) = H(i,j-1)-1$. Thus $H(i+1,j-1) = H(i,j)$ or $H(i+1,j-1) = H(i,j)-1$. We now examine the four cases for how $H(i+1,j)$ is defined in this case.

   (a) Suppose $0 \leq H(i+2,j-1) < H(i+1,j-1)$. Since $H(i+1,j-1) \leq 0$, then this is a contradiction.

   (b) Suppose $0 = H(i+1,j-1) \leq H(i+2,j-1)$. By definition of H, $H(i+1,j) = H(i+1,j-1)$. Since $H(i+1,j-1) \leq H(i,j-1) \leq 0$, it follows that $H(i+1,j-1) = H(i,j-1) = 0$. Therefore, $H(i+1,j) = 0 = H(i,j)$.

   (c) Suppose $H(i+1,j-1) < H(i+2,j-1) \leq 0$. By definition of H, $H(i+1,j) =$

$H(i+1, j-1)+1$. Thus

$$H(i+1, j) = H(i, j)+1 \quad \text{or} \quad H(i+1, j) = H(i, j)-1+1 = H(i, j),$$

which implies that $H(i, j) = H(i+1, j)$ or $\{H(i, j), H(i+1, j)\} \in E(I_\infty)$.

(d) Suppose $H(i+2, j-1) \leq H(i+1, j-1) \leq 0$. By definition of H, $H(i+1, j) = H(i+1, j-1)$. Thus

$$H(i+1, j) = H(i, j) \quad \text{or} \quad H(i+1, j) = H(i, j)-1,$$

which implies that $H(i, j) = H(i+1, j)$ or $\{H(i, j), H(i+1, j)\} \in E(I_\infty)$.

Therefore, $H(i, j) = H(i+1, j)$ or $\{H(i, j), H(i+1, j)\} \in E(I_\infty)$ for all $i, j \in \mathbb{Z}$.

- Next consider $H(i, j)$ and $H(i, j+1)$.

  For each $i \in \mathbb{Z}$ and $j \geq 0$, we show that either $H(i, j) = H(i, j+1)$ or $\{H(i, j), H(i, j+1)\} \in E(I_\infty)$ directly by examining the four possible cases which define $H(i, j+1)$.

  1. Suppose $0 \leq H(i+1, j) < H(i, j)$. By definition of H, $H(i, j+1) = H(i, j)-1$. Thus $\{H(i, j), H(i, j+1)\} \in E(I_\infty)$.
  2. Suppose $0 \leq H(i, j) \leq H(i+1, j)$. By definition of H, $H(i, j+1) = H(i, j)$.
  3. Suppose $H(i, j) < H(i+1, j) \leq 0$. By definition of H, $H(i, j+1) = H(i, j)+1$. Thus $\{H(i, j), H(i, j+1)\} \in E(I_\infty)$.
  4. Suppose $H(i+1, j) \leq H(i, j) \leq 0$. By definition of H, $H(i, j+1) = H(i, j)$.

  Therefore, $H(i, j) = H(i, j+1)$ or $\{H(i, j), H(i, j+1)\} \in E(I_\infty)$ for all $i, j \in \mathbb{Z}$.

Thus H is a graph homomorphism.

We now show that H is stable. Recall that $H_j : I_\infty \to I_\infty$ is defined by $H_j(i) = H(i, j)$ for all $i, j \in \mathbb{Z}$. Since H is a graph homomorphism, the restriction H_j is a graph homomorphism.

Since $\widetilde{f}$ is a stable graph homomorphism, the difference between $m_0(\widetilde{f},-1)$ and $m_0(\widetilde{f},+1)$ is finite. Thus there are a finite number of $m \in \mathbb{Z}$ with $m_0(\widetilde{f},-1) \leq m \leq m_0(\widetilde{f},+1)$.

(1) Suppose $H_j(m) = 0$. By definition of H, either $0 = H_j(m) \leq H_j(m+1)$ and $H_{j+1}(m) = H_j(m)$, or $H_j(m+1) \leq H_j(m) = 0$ and $H_{j+1}(m) = H_j(m)$. Thus if $H_j(m) = 0$, then $H_{j+1}(m) = 0$. This also implies that if $H_0(m) = \widetilde{f}(m) > 0$, then $H_j(m) \geq 0$ for all $j \geq 0$, and if $H_0(m) = \widetilde{f}(m) < 0$, then $H_j(m) \leq 0$ for all $j \geq 0$.

(2) Suppose $H_j(m) > 0$. By definition of H, either $0 \leq H_j(m+1) < H_j(m)$ and $H_{j+1}(m) = H_j(m)-1$, or $0 \leq H_j(m) \leq H_j(m+1)$ and $H_{j+1}(m) = H_j(m)$. Thus $H_j(m)$ is constant or decreasing as j increases.

(3) Suppose $H_j(m) < 0$. By definition of H, either $H_j(m) < H_j(m+1) \leq 0$ and $H_{j+1}(m) = H_j(m)+1$, or $H_j(m+1) \leq H_j(m) \leq 0$ and $H_{j+1}(m) = H_j(m)$. Thus $H_j(m)$ is constant or increasing as j increases.

Observe that if there exists $j \geq 0$ such that $H_{j+1}(i) = H_j(i)$ for all $i \in \mathbb{Z}$, then H stabilizes in the positive direction on the 2^{nd}-axis, that is, the integer $m_0(H,+2)$ exists. For each $j \geq 0$, H does not stabilize at j in the positive direction in the 2^{nd}-axis if and only if there exists some $m \in \mathbb{Z}$ with $m_0(\widetilde{f},-1) \leq m \leq m_0(\widetilde{f},+1)$ such that $H_j(m) \neq H_{j+1}(m)$. We now count how many times it is possible for $H_j(m) \neq H_{j+1}(m)$ for $j \geq 0$ and $m_0(\widetilde{f},-1) \leq m \leq m_0(\widetilde{f},+1)$. There are at most $m_0(\widetilde{f},+1) - m_0(\widetilde{f},-1)$ choices for $m \in \mathbb{Z}$ with $m_0(\widetilde{f},-1) \leq m \leq m_0(\widetilde{f},+1)$. By parts (1)-(3), for each such m, there are at most $|\widetilde{f}(m)|$ times that $H_j(m) \neq H_{j+1}(m)$. This implies that H is not stable in the positive direction on the 2^{nd}-axis at a maximum of $j = \sum_m |\widetilde{f}(m)| < \infty$. Therefore, the integer $m_0(H,+2)$ exists.

We now show that H is a graph homotopy from $\widetilde{f}$ to $\alpha^2_{+2}(H)$ by verifying conditions (a)-(c) of Definition 4.12.

(a) We use induction on j to show that $H_j(m_0(H_j,-1)) = 0$ for all $j \geq 0$.

Basis Case: By construction of H, $H_0 = \widetilde{f}$. Since $\widetilde{f}(m_0(\widetilde{f},-1)) = 0$, it follows that $H_0(m_0(H_0,-1)) = 0$.

Induction Hypothesis: Suppose $H_j(m_0(H_j, -1)) = 0$ for some $j \geq 0$. Then $0 = H_j(m_0(H_j, -1)) \leq H_j(m_0(H_j, -1) + 1)$, or $H_j(m_0(H_j, -1) + 1) \leq H_j(m_0(H_j, -1)) = 0$, which implies that $H_{j+1}(m_0(H_j, -1)) = H_j(m_0(H_j, -1)) = 0$ by definition of H. Thus by induction, $H_j(m_0(H_j, -1)) = 0$ for all $j \geq 0$. Therefore, $\alpha^1_{-1}(\alpha^2_{+2}(H))(*) = \alpha^2_{+2}(H)(m_0(H_{m_0(H,+2)}, -1)) = 0$, which implies that $\alpha^1_{-1}(\widetilde{f}) = \alpha^1_{-1}(\alpha^2_{+2}(H))$.

We now use induction on j to show that $H_j(m_0(H_j, +1)) = 5n$ for all $j \geq 0$.

Basic Case: By construction of H, $H_0 = \widetilde{f}$. Since $\widetilde{f}(m_0(\widetilde{f}, +1)) = 5n$, it follows that $H_0(m_0(H_0, +1)) = 5n$.

Induction Hypothesis: Suppose $H_j(m_0(H_j, +1)) = 5n$ for some $j \geq 0$. Then it follows that $H_j(m_0(H_j, +1) + 1) = 5n$. Therefore, $H_{j+1}(m_0(H_j, +1)) = H_j(m_0(H_j, +1)) = 5n$ by definition of H. Thus by induction, $H_j(m_0(H_j, +1)) = 5n$ for all $j \geq 0$. Therefore, $\alpha^1_{+1}(\alpha^2_{+2}(H))(*) = \alpha^2_{+2}(H)(m_0(H_{m_0(H,+2)}, +1)) = 5n$, which implies that $\alpha^1_{+1}(\widetilde{f}) = \alpha^1_{+1}(\alpha^2_{+2}(H))$.

(b) This condition is a consequence of the inductive arguments in part (a). By part (a), $H(m_0(H, -1), j) = 0 = \alpha^1_{-1}(\widetilde{f})(*) = \alpha^1_{-1}(\alpha^2_{+2}(H))(*)$ for all $j \in \mathbb{Z}$, and similarly, $H(m_0(H, +1), j) = 5n = \alpha^1_{+1}(\widetilde{f})(*) = \alpha^1_{+1}(\alpha^2_{+2}(H))(*)$ for all $j \in \mathbb{Z}$. Therefore, $\alpha^2_{-1}(H) = \beta^1_1\alpha^1_{-1}(\widetilde{f}) = \beta^1_1\alpha^1_{-1}(\alpha^2_{+2}(H))$ and $\alpha^2_{+1}(H) = \beta^1_1\alpha^1_{+1}(\widetilde{f}) = \beta^1_1\alpha^1_{+1}(\alpha^2_{+2}(H))$.

(c) By construction of H, $\alpha^2_{-2}(H) = \widetilde{f}$. Trivially, $\alpha^2_{+2}(H) = \alpha^2_{+2}(H)$.

Thus H is a homotopy from $\widetilde{f}$ to $\alpha^2_{+2}(H)$, so $\widetilde{f} \sim \alpha^2_{+2}(H)$. By definition of H, the face $\alpha^2_{+2}(H)$ has no positive decreasing value and no negative increasing values. Since $\alpha^1_{-1}(\alpha^2_{+2}(H))(*) = 0$ and $\alpha^1_{+1}(\alpha^2_{+2}(h))(*) = 5n$, it follows that $\alpha^2_{+2}(H)$ must be increasing or constant from 0 to $5n$. Thus by the *General Padding Lemma* (5.3), $\alpha^2_{+2}(H) \sim \widetilde{\gamma_n}$. Therefore, $\widetilde{f} \sim \widetilde{\gamma_n}$ for all $n \in \mathbb{Z}$. □

We conclude this chapter by computing the fundamental group of the 5-cycle.

Theorem 7.8. The fundamental group of $\mathcal{C}_5$ is $(B_1(\mathcal{C}_5, [0])/\sim, \cdot) \cong (\mathbb{Z}, +)$.

Proof. Define $\varphi : \mathbb{Z} \to B_1(\mathcal{C}_5, [0])/\sim$ by $n \mapsto [\gamma_n]$, the homotopy class of the stable graph homomorphism $\gamma_n : I_\infty \to \mathcal{C}_5$ defined in Definition 7.4. We now show that this map φ is an isomorphism.

- **Group Homomorphism:** We show that $\varphi(n+m) = \varphi(n) \cdot \varphi(m)$ for all $n, m \in \mathbb{Z}$.
  - **Case 1:** Suppose $n, m \geq 0$. The concatenation $\gamma_n \cdot \gamma_m$ is defined by

$$
\begin{aligned}
(\gamma_n \cdot \gamma_m)(i) &= \begin{cases} \gamma_n(i + m_0(\gamma_n, -1)) & \text{for} \quad i \geq 0, \\ \gamma_m(i + m_0(\gamma_m, +1)) & \text{for} \quad i \leq 0, \end{cases} \\
&= \begin{cases} \gamma_n(i+0) & \text{for} \quad i \geq 0, \\ \gamma_m(i+5m) & \text{for} \quad i \leq 0, \end{cases} \\
&= \begin{cases} [0] & i \geq 5n \\ [i \mod 5] & \text{for} \quad 0 \leq i \leq 5n, \\ [(i+5m) \mod 5] & \text{for} \quad -5m \leq i \leq 0, \\ [0] & \text{for} \quad i \leq -5m, \end{cases} \\
&= \begin{cases} [0] & \text{for} \quad i \geq 5n, \\ [i \mod 5] & \text{for} \quad -5m \leq i \leq 5n, \\ [0] & \text{for} \quad i \leq -5m. \end{cases}
\end{aligned}
$$

Thus $(\gamma_n \cdot \gamma_m)(i - 5m) = \gamma_{n+m}(i)$, and $\gamma_n \cdot \gamma_m \sim \gamma_{n+m}$ by the *Shifting Lemma* (5.4).

– **Case 2:** Suppose $n, m < 0$. The concatenation $\gamma_n \cdot \gamma_m$ is defined by

$$
\begin{aligned}
(\gamma_n \cdot \gamma_m)(i) &= \begin{cases} \gamma_n(i + m_0(\gamma_n, -1)) & \text{for} \quad i \geq 0, \\ \gamma_m(i + m_0(\gamma_m, +1)) & \text{for} \quad i \leq 0, \end{cases} \\
&= \begin{cases} \gamma_n(i + 0) & \text{for} \quad i \geq 0, \\ \gamma_m(i - 5m) & \text{for} \quad i \leq 0, \end{cases} \\
&= \begin{cases} [0] & \text{for} \quad i \geq -5n, \\ [(-i) \mod 5] & \text{for} \quad 0 \leq i \leq -5n, \\ [(-i + 5m) \mod 5] & \text{for} \quad 5m \leq i \leq 0, \\ [0] & \text{for} \quad i \leq 5m, \end{cases} \\
&= \begin{cases} [0] & \text{for} \quad i \geq -5n, \\ [(-i) \mod 5] & \text{for} \quad 5m \leq i \leq -5n, \\ [0] & \text{for} \quad i \leq 5m. \end{cases}
\end{aligned}
$$

Thus $\gamma_n \cdot \gamma_m(i + 5m) = \gamma_{n+m}(i)$, and $\gamma_n \cdot \gamma_m \sim \gamma_{n+m}$ by the *Shifting Lemma* (5.4).

– **Case 3:** Suppose $n \geq 0, m < 0$. By Lemma 7.5, $\gamma_n \sim \overline{\gamma_{-n}}$ and $\gamma_m \sim \overline{\gamma_{-m}}$. By Case 1, if $n + m \geq 0$, then $\gamma_n = \gamma_{n+m-m} \sim \gamma_{n+m} \cdot \gamma_{-m}$. By Case 2, if $n + m < 0$, then $\gamma_m = \gamma_{-n+n+m} \sim \gamma_{-n} \cdot \gamma_{n+m}$. Thus

$$\gamma_n \cdot \gamma_m \sim \gamma_n \cdot \overline{\gamma_{-m}} \sim \gamma_{n+m} \cdot \gamma_{-m} \cdot \overline{\gamma_{-m}} \sim \gamma_{n+m} \qquad \text{if} \quad n + m \geq 0,$$

and

$$\gamma_n \cdot \gamma_m \sim \overline{\gamma_{-n}} \cdot \gamma_m \sim \overline{\gamma_{-n}} \cdot \gamma_{-n} \cdot \gamma_{n+m} \sim \gamma_{n+m} \qquad \text{if} \quad n + m < 0.$$

– **Case 4:** Suppose that $n < 0, m \geq 0$. Again by Lemma 7.5, $\gamma_n \sim \overline{\gamma_{-n}}$ and $\gamma_m \sim \overline{\gamma_{-m}}$. By Case 2, if $n + m < 0$, then $\gamma_n = \gamma_{n+m-m} \sim \gamma_{n+m} \cdot \gamma_{-m}$. By Case

1, if $n+m \geq 0$, then $\gamma_m = \gamma_{-n+n+m} \sim \gamma_{-n} \cdot \gamma_{n+m}$. Thus

$$\gamma_n \cdot \gamma_m \sim \gamma_n \cdot \overline{\gamma_{-m}} \sim \gamma_{n+m} \cdot \gamma_{-m} \cdot \overline{\gamma_{-m}} \sim \gamma_{n+m} \qquad \text{if} \quad n+m < 0,$$

and

$$\gamma_n \cdot \gamma_m \sim \overline{\gamma_{-n}} \cdot \gamma_m \sim \overline{\gamma_{-n}} \cdot \gamma_{-n} \cdot \gamma_{n+m} \sim \gamma_{n+m} \qquad \text{if} \quad n+m \geq 0.$$

Therefore, $\varphi(n+m) = [\gamma_{n+m}] = [\gamma_n \cdot \gamma_m] = [\gamma_n] \cdot [\gamma_m] = \varphi(n) \cdot \varphi(m)$ for all $n, m \in \mathbb{Z}$.

- **Surjective:** We show that if $[f] \in B_1(\mathcal{C}_5, [0])/\sim$, then there exists $n \in \mathbb{Z}$ such that $\varphi(n) = [f]$.

  Let $[f] \in B_1(\mathcal{C}_5, [0])/\sim$. Then f is a stable graph homomorphism with $f(m_0(f, -1)) = f(m_0(f, +1)) = [0]$. Hence, there exists a unique lift $\widetilde{f} : I_\infty \to I_\infty$ with $\widetilde{f}(m_0(f, -1)) = 0$ and $f = p \circ \widetilde{f}$. Since $f(m_0(f, +1)) = [0]$, it follows that $p(\widetilde{f}(m_0(f, +1))) = [0]$, so $\widetilde{f}(m_0(f, +1)) \mod 5 = 0$. Thus there exists $n \in \mathbb{Z}$ such that $\widetilde{f}(m_0(f, +1)) = 5n$. Hence, by the Lemma 7.7, we have that $\widetilde{f} \sim \widetilde{\gamma_n}$, which implies that there exists a graph homotopy $H : I_\infty^2 \to I_\infty$ from $\widetilde{f}$ to $\widetilde{\gamma_n}$. Since H and p_5 are graph homomorphisms, the composition $p_5 \circ H : I_\infty^2 \to \mathcal{C}_5$ is a graph homomorphism. We now show that $p_5 \circ H$ is a graph homotopy from f to γ_n by verifying conditions (a)-(c) of Definition 4.12

  (a) By the definitions of $\widetilde{f}$ and $\widetilde{\gamma_n}$,

  $$\widetilde{f}(m_0(\widetilde{f}, -1)) = \widetilde{\gamma_n}(m_0(\widetilde{\gamma_n}, -1)) = 0$$

  and

  $$\widetilde{f}(m_0(\tilde{f}, +1)) = \widetilde{\gamma_n}(m_0(\widetilde{\gamma_n}, +1)) = 5n.$$

  Since p_5 is a graph homomorphism, $p_5(\widetilde{f}(m_0(\widetilde{f}, -1))) = p_5(\widetilde{\gamma_n}(m_0(\widetilde{\gamma_n}, -1))) = [0]$ and $p_5(\widetilde{f}(m_0(\widetilde{f}, +1))) = p_5(\widetilde{\gamma_n}(m_0(\widetilde{\gamma_n}, +1))) = [0]$. Therefore, $\alpha^1_{-1}(f) = \alpha^1_{-1}(\gamma_n)$ and $\alpha^1_{+1}(f) = \alpha^1_{+1}(\gamma_n)$.

(b) Since H is a graph homotopy from $\widetilde{f}$ to $\widetilde{\gamma_n}$, $\alpha^1_{-1}(H)(j) = H(m_0(H,-1),j) = 0$ and $\alpha^1_{+1}(H)(j) = H(m_0(H,+1),j) = 5n$ for all $j \in \mathbb{Z}$. Thus $(p_5 \circ H)(m_0(H,-1),j) = [0] = p_5 \circ \widetilde{f}(m_0(\widetilde{f},-1)) = (p_5 \circ \widetilde{\gamma_n})(m_0(\gamma_n,-1))$ and $(p_5 \circ H)(m_0(H,+1),j) = [0] = (p_5 \circ \widetilde{f})(m_0(\widetilde{f},+1)) = p_5 \circ \widetilde{\gamma_n}(m_0(\gamma_n,+1))$ for all $j \in \mathbb{Z}$. Therefore, $\alpha^2_{-1}(p_5 \circ H) = \beta^1_1 \alpha^1_{-1}(f) = \beta^1_1 \alpha^1_{-1}(\gamma_n)$ and $\alpha^2_{+1}(p_5 \circ H) = \beta^1_1 \alpha^1_{+1}(f) = \beta^1_1 \alpha^1_{+1}(\gamma_n)$.

(c) Since $H(i, m_0(H,-2)) = \widetilde{f}(i)$ and $H(i, m_0(H,+2)) = \widetilde{\gamma_n}(i)$ for all $i \in \mathbb{Z}$, it follows that $p_5 \circ H(i, m_0(H,-2)) = p_5 \circ \widetilde{f}(i)$ and $p_5 \circ H(i, m_0(H,+2)) = p_5 \circ \widetilde{\gamma_n}(i)$ for all $i \in \mathbb{Z}$. Thus $\alpha^2_{-2}(p_5 \circ H) = f$ and $\alpha^2_{+2}(p_5 \circ H) = \gamma_n$.

Therefore, $p_5 \circ H$ is a homotopy from f to γ_n, so it follows that $[f] = [\gamma_n]$. Hence, $\varphi(n) = [f]$.

- **Injective:** We show that if $\varphi(n) = \varphi(m)$, then $n = m$.

  Let $\varphi(n) = \varphi(m)$. Then $[\gamma_n] = [\gamma_m]$, which implies that $\gamma_n \sim \gamma_m$. Therefore, there exists a graph homotopy $H : I^2_\infty \to \mathcal{C}_5$ from γ_n to γ_m. By the *Homotopy Lifting Property* (6.11), there is a graph homotopy $\widetilde{H} : I^2_\infty \to I_\infty$ from $\widetilde{\gamma_n}$ to $\widetilde{\gamma_m}$. Thus $\widetilde{\gamma_n} \sim \widetilde{\gamma_m}$, and it follows that $\alpha^1_{+1}(\widetilde{\gamma_n}) = \alpha^1_{+1}(\widetilde{\gamma_m})$. Therefore, $\widetilde{\gamma_n}(m_0(\gamma_n,+1)) = \widetilde{\gamma_m}(m_0(\gamma_m,+1))$. Hence it follows that $5n = 5m$, which implies that $n = m$.

Thus φ is an isomorphism, and $(B_1(\mathcal{C}_5,[0])/\sim) \cong \mathbb{Z}$. $\square$

Since $(B_1(\mathcal{C}_5,[0])/\sim) \cong \mathbb{Z}$, it follows by Theorem 7.1 that C_5 is not A-contractible. The proof of Theorem 7.8 can also be slightly altered to show that $(B_1(\mathcal{C}_k,[0])/\sim) \cong \mathbb{Z}$ for any $k \geq 5$, and thus that the cycle C_k is not A-contractible for $k \geq 5$. This proof cannot be used for the cycles C_3 and C_4, however, because the *Homotopy Lifting Property* (6.11) does not hold for graphs containing 3-cycles or 4-cycles.

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Appendix A

The image in Figure 1.1 is used in accordance with Imgur's user policy, which we have included below.

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