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Systems of Polynomial Equations

by

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Abstract

In this thesis we study possible relations between the solutions of related systems of polynomial equations. In particular, we have considered conjugate systems of polynomial equations and transpose systems of binary homogeneous polynomial equations.

In case of conjugate systems of polynomial equations, we compared the number of solutions by using the structure theorem for a finite dimensional commutative associative algebras with identity.

In case of transpose systems of binary homogeneous polynomial equations, we have proved topological (in terms of the Zariski topology) properties of the set of all matrices with rank less than or equal to a certain number such that both a system and its transpose system represent the same number of projective points.

As a by-product of this analysis we have proved that, for a given partition $(m_1, \ldots, m_s)$ of r, the set of binary forms f of degree r in the variables X_0, X_1 over the field of complex numbers $\mathbb{C}$ such that f has the form $l_1^{m_1} \ldots l_s^{m_s}$ for some linear forms $l_1, \ldots, l_s$, is a Zariski irreducible closed set with dimension $s + 1$. Furthermore, we have proved that the corresponding prime ideal of this closed set is the radical of a coefficient ideal of a covariant (cf. 2.5 for the definition), for two part partitions.

We have illustrated these in detail for binary cubic, binary quartic and binary quintic forms.

Dedicated to my late daughter Mary Thayalini Thangarajah (1993 - 1997).

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Chapter 1

Introduction: A Brief Overview

The study of polynomial equations is one of the important branches of Mathematics. It dates back to 1600 BC, initially with no sign of algebraic formulations such as in Babylonian tablets and ancient Greek geometrical constructions. Our objective in this thesis is to explore connections between the solutions of related systems of polynomial equations. In particular, we have studied two versions: These are

- Conjugate systems of polynomial equations, and
- Transpose systems of binary homogeneous polynomial equations.

A partial solution to the first version involves finite dimensional commutative algebra and a partial solution to the second version involves algebraic geometry.

In Chapter 2, we have introduced basic concepts which are needed for this thesis, namely: algebraic geometry, and invariant theory.

In Chapter 3, we have stated the main problem, and have considered two different versions of it. A solution to the basic case of the main problem involves elementary linear algebra. My supervisor, Prof. H. K. Farahat, explained to me his approach to conjugate systems using the structure theorem for finite dimensional associative commutative algebras over an algebraically closed field. In an attempt to solve the second version, we have studied the set of all square matrices of rank less than or equal to l such that both a system and its transpose system represent same number of projective points. The case of all matrices with rank less than or equal to 1 corresponds to the study of binary forms.

Invariant theory was developed in the nineteenth century by Boole [Boole,1841], Cayley [Cayley 1889], Clebsch [Clebsch 1872], Gordan [Gordan 1885], Hilbert [Hilbert 1886], Sylvester [Sylvester 1879] and others. It has been studied intermittently ever since. In recent times, newly developed techniques have been applied with great success to some of its outstanding problems. This has moved invariant theory, once again, to the forefront of mathematical research (cf. [Kung, Rota 1984], [Mumford 1994]).

As a part of this thesis we study a problem concerning factors of binary forms of degree r over the complex field $\mathbb{C}$. Hilbert had shown that $\mathcal{I}(r) = Rad\langle\mathcal{H}\rangle$, and Gordan had proved that $\mathcal{I}(r-1,1) = Rad\langle\mathcal{P}\rangle$ for $r \neq 4, 6, 8, 12$ (cf.4.5 for definitions). But for each $0 < m < r$, we have found a covariant such that the radical of the coefficient ideal of this covariant is $\mathcal{I}(r-m, m)$. This is presented in Theorem 4.23.

Further, in Chapter 4, we have explored the use of Gröbner bases, and have presented results for binary cubic, binary quartic and binary quintic forms. Some of the cases for sextic forms are covered by general results. But the full problem for sextic forms is presently not completely solved. This is a good place to start future research.

In Chapter 5, we have presented our results of the investigation of transpose systems of binary homogeneous polynomial equations. In this case we have found that the set of all $(r+1)\times(r+1)$ matrices of rank less than or equal to 1 such that both the system and its transpose system represent k projective points

- together with 0, is an affine closed set when $k = 1$,

- is an intersection of an affine closed set and an affine open set, when $2 \leq k \leq r$,
- is a dense subset, when $k = r + 1$.

Further we have found that the set of all $(r+1) \times (r+1)$ matrices of rank less than or equal to l such that both a system and its transpose system have only the trivial solution is a dense subset of the set of all $(r+1) \times (r+1)$ matrices of rank less than or equal to l, for $2 \leq l \leq r+1$.

In Appendix A, we have discussed a recurrence formula for positioning monomials with respect to lexicographic order. In other Appendix sections , we have attached a list of polynomials from Gröbner bases which are needed for the proofs.

Thus, in brief, almost everything in Chapter 3, Chapter 4, and Chapter 5 is new and the results are original. The main novelty of Chapter 4 lies in the theorem for a covariant generator for the two part partition ideal (cf.Theorem 4.23). The results which do not indicate any reference are my own. In particular, the proofs given in terms of Gröbner bases are my own.

We conclude with some observations and notations in this thesis:

It is to be noted that the results thought to be most significant are labeled as theorems or occasionally lemmas.

References have generally been given in the following forms: ([Gordan 1885] p.35). Here [Gordan 1885] refers to the entry in the bibiliography under Gordan and the given year, and p.35 refers to the page number where a proof can be found.

<u>Notations</u>: We will follow the following notations for $f \in \mathbb{C}(X_0, X_1)$:

1. $\dfrac{\partial f}{\partial X_i} = \partial_i f$, for $i = 0, 1$,

2. $\dfrac{\partial^2 f}{\partial X_i^2} = \partial_i^2 f$, for $i = 0, 1$,

3. $\dfrac{\partial^2 f}{\partial X_0 \partial X_1} = \partial_0 \partial_1 f$,

4. $\mathbb{K}_{(r,s)}$ denotes the set of all $r \times s$ matrices over a field $\mathbb{K}$.

Chapter 2

Preliminaries

2.1 Algebraic Geometry

2.1.1 Affine Space

Let V be an n-dimensional vector space over the field of complex numbers $\mathbb{C}$. Then the set of all $\mathbb{C}$-valued functions on V, $\mathbb{C}^V$, with pointwise operations, forms a $\mathbb{C}$-algebra. Now $\mathbb{C}^V$ contains all the constant functions and the $\mathbb{C}$-linear functions. Therefore, the space of all linear functions $V^* = Hom_{\mathbb{C}}(V, \mathbb{C})$, is a subset of $\mathbb{C}^V$. The subalgebra of $\mathbb{C}^V$ generated by V^* is denoted by $\mathbb{C}[V]$. This subalgebra $\mathbb{C}[V]$ is clearly generated by any basis of V^*. Thus $\mathbb{C}[V] = \mathbb{C}[X_1, \ldots, X_n] =$ the subalgebra generated by any choice of co-ordinate functions $X_1, \ldots, X_n$ on V, the so-called *coordinate ring* of V. We call the elements of $\mathbb{C}[X_1, \ldots, X_n]$ *polynomial functions* on V. A polynomial function $h \in \mathbb{C}[X_1, \ldots, X_n]$ is *homogeneous* of degree m if $h(ax) = a^m h(x)$ for $a \in \mathbb{C}$, $x \in V$.

Viewed with its ring of polynomial functions, V is called an *affine n-space* over the field of complex numbers $\mathbb{C}$.

Given a subset G of $\mathbb{C}[X_1, \ldots, X_n]$, we define a corresponding subset of V called the *zero set* of G, namely:

$$\mathbf{V}(G) = \{x \in V \,|\, g(x) = 0 \text{ for all } g \in G\}.$$

From the definition of the zero set $\mathbf{V}(G)$, it is clear that G may be replaced by the ideal that it generates in $\mathbb{C}[X_1, \ldots, X_n]$ without changing $\mathbf{V}(G)$. If $S = \mathbf{V}(G)$ is a zero set, then a zero subset T of S is a set of the form $T = \mathbf{V}(J)$, for some J a subset of $\mathbb{C}[X_1, \ldots, X_n]$, that happens to be contained in S. The ***Zariski topology*** on S is the topology whose closed sets are the zero subsets of S. We shall call these closed sets affine closed sets to distinguish them from projective objects we shall define later. Topological notions in this thesis will always be relative to the Zariski topology.

There is a sort of inverse to the construction of a zero set : Given any set $Q \subset V$ we define

$$\mathbf{I}(Q) = \{g \in \mathbb{C}[X_1, \ldots, X_n] \mid g(x) = 0 \text{ for all } x \in Q\}.$$

It is clear that $\mathbf{I}(Q)$ is an ideal, which we shall call the ***vanishing ideal*** of Q. A polynomial function on Q is by definition the restriction to Q of a polynomial function on V. Identifying two polynomial functions if they agree at all the points of Q, we get the coordinate ring, $\mathbb{C}[Q]$ of Q (so called because it is the $\mathbb{C}$-algebra of functions on Q generated by the coordinate functions). Clearly we have $\mathbb{C}[Q] \cong \mathbb{C}[X_1, \ldots, X_n]/\mathbf{I}(Q)$. The correspondence between zero sets and vanishing ideals is given by Hilbert's Nullstellensatz [1893].

Theorem 2.1 *(Nullstellensatz)*

If $I \subset \mathbb{C}[X_1, \ldots, X_n]$ is an ideal , then

$$\mathbf{I}(\mathbf{V}(I)) = \mathit{Rad}(I),$$

where

$$Rad(I) = \{f \in \mathbb{C}[X_1, \ldots, X_n] \mid f^m \in I \text{ for some positive integer } m\}.$$

Thus, the correspondences $I \mapsto \mathbf{V}(I)$ *and* $Q \mapsto \mathbf{I}(Q)$ *induce a bijection between the collection of zero subsets of* V *and radical ideals of* $\mathbb{C}[X_1, \ldots, X_n]$.

The intersection of all closed subsets of X containing a given subset $M \subset X$ is closed. It is called the closure of M and is denoted by $\bar{M}$. A subset M is called dense in X if $\bar{M} = X$. This means that M is not contained properly in any closed subset $Y \subset X, Y \neq X$.

Let W be an m-dimensional vector space. A mapping $\phi : V \to W$ is called a *polynomial mapping* if, with respect to some basis of W, the coordinates of $\phi(x), x \in V$, are polynomial functions on V.

Let

$$\alpha : V \to W$$

be a polynomial mapping. Then the map

$$\alpha^* : \mathbb{C}[W] \to \mathbb{C}[V]$$

defined by

$$\alpha^*(f) = f\alpha$$

is a ring homomorphism which is the identity on the constant functions $\mathbb{C} \subset \mathbb{C}[W]$. (See [Shafarevich 1974] p.19).

A non-empty subset Y of a topological space X is *irreducible* if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper subsets, each one of which is closed in Y. The empty set is not considered to be irreducible.

It can be proved from the definition that a topological space X is irreducible if and only if every non-empty open subset of X is dense.

The following is an equivalent condition for irreducibility in the Zariski topology:

An affine closed subset S of V is irreducible if and only if $\mathbf{I}(S)$ is a prime ideal of $\mathbb{C}[V]$ (see [Shafarevich 1974] p. 23).

2.1.2 Projective Space

Projective space over the field $\mathbb{C}$, written $\mathbb{P}^n$, is the set of all one-dimensional subspaces of $\mathbb{C}_{1,(n+1)}$, the vector space of $1 \times n+1$ row matrices over $\mathbb{C}$. Sometimes, we will want to refer to the projective space of all one dimensional subspaces of a vector space V over the field $\mathbb{C}$; in this case we will denote it by $\mathbb{P}(V)$.

A point in $\mathbb{P}^n$ is usually written as a homogeneous vector $[z_0, \ldots, z_n]$ by which we mean the one dimensional subspace spanned by $(z_0, \ldots, z_n) \in \mathbb{C}_{1,(n+1)}$. Likewise for any non-zero vector $v \in V$ we denote by $[v]$ the corresponding point in $\mathbb{P}(V)$. A polynomial $f \in \mathbb{C}[X_0, \ldots, X_n]$, where $X_0, \ldots, X_n$ are co-ordinate functions on $\mathbb{C}_{1,(n+1)}$ does not define a function on $\mathbb{P}^n$. On the other hand if f happens to be homogeneous of degree d then since

$$f(\lambda X_0, \ldots, \lambda X_n) = \lambda^d f(X_0, \ldots, X_n),$$

it does make sense to talk about the zero set of the polynomial f as a subset of $\mathbb{P}^n$.

A subset $X \subset \mathbb{P}^n$ is called *projective closed* if it consists of all points at which finitely many homogeneous polynomials with coefficients in $\mathbb{C}$ vanish simultaneously. In this case $\mathbf{I}(X)$ has the property that if a polynomial is contained in it, then so are all its homogeneous components. Ideals having this property are called homogeneous ideals.

2.1.3 Products

Definition 2.2 *1. A subset A of $\mathbb{P}^n \times \mathbb{P}^m$ is projective closed if and only if it is a zero set of a system of polynomial functions*

$$G_i(U_0, \ldots, U_n; V_0, \ldots, V_m), (i = 1, \ldots, t)$$

homogeneous in each set of co-ordinate functions U_j on $\mathbb{P}^n$ and V_j on $\mathbb{P}^m$ separately.

2. The closed subsets of $\mathbb{P}^n \times \mathbb{C}_{1,m}$ are the zero sets of systems of polynomial functions

$$g_i(U_0, \ldots, U_n; Y_1, \ldots, Y_m), (i = 1, \ldots, t)$$

homogeneous in the coordinate functions $U_0, \ldots, U_n$ on $\mathbb{P}^n$, where Y_j are coordinate functions on $\mathbb{C}_{1,m}$.

3. The closed sets in $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_l}$ are the zero sets of systems of polynomial functions, homogeneous in each of the l groups of coordinate functions.

2.1.4 Dimension

Definition 2.3 *Let X be a topological space , $Y \subset X$ a closed irreducible subset. If $X \neq \emptyset$, the dimension $dim(X)$ of X is the supremum of the lengths n of all chains*

$$X_0 \subset X_1 \subset \ldots \subset X_n, (X_{i+1} \neq X_i)$$

of non-empty closed irreducible subsets X_i of X. If $Y \neq \emptyset$, then the codimension $codim_X(Y)$ of Y in X is defined as the supremum of the lengths of all chains

$$Y = X_0 \subset X_1 \subset \ldots \subset X_n, (X_{i+1} \neq X_i).$$

The empty topological space is assigned dimension -1, and the empty subset of X is assigned codimension ∞.

2.2 Binary forms and Action of $GL(2,\mathbb{C})$

Let X_0, X_1 be algebraically independent indeterminates over $\mathbb{C}$. Then the ring of polynomials in X_0, X_1 over $\mathbb{C}$, $\mathbb{C}[X_0, X_1]$, is a commutative associative graded algebra over $\mathbb{C}$ graded by degree. That is,

$$\mathbb{C}[X_0,X_1] = \mathbb{C} \dot{+} \mathbb{C}[X_0,X_1]_1 \dot{+} \mathbb{C}[X_0,X_1]_2 \dot{+} \ldots,$$

where $\mathbb{C}[X_0, X_1]_r$ is the set of all homogeneous polynomials in X_0, X_1 over $\mathbb{C}$ of degree r, the so called *complex binary forms* in X_0, X_1 of degree r.

The set of all homogeneous polynomials in X_0, X_1 over $\mathbb{C}$ of degree r, $\mathbb{C}[X_0, X_1]_r$ is a vector space over $\mathbb{C}$ of dimension $r+1$. The set of monomials in X_0, X_1 of degree r, $\{X_0^r, X_0^{r-1}X_1, \ldots, X_1^r\}$, is the standard ordered monomial basis for $\mathbb{C}[X_0, X_1]_r$.

The group of all 2×2 invertible complex matrices, $GL(2,\mathbb{C})$, acts on $\mathbb{C}[X_0, X_1]_1$ as follows:

For $g \in GL(2,\mathbb{C})$

$$\left.\begin{array}{rcl} gX_0 & = & g_{11}X_0 + g_{21}X_1 \\ gX_1 & = & g_{12}X_0 + g_{22}X_1 \end{array}\right\}.$$

That is, g acts on $\mathbb{C}[X_0, X_1]_1$ as the linear transformation whose matrix relative to the basis $\{X_0, X_1\}$ is g. The group $GL(2,\mathbb{C})$ acts on all of $\mathbb{C}[X_0, X_1]$ by degree preserving algebra automorphisms. Hence $GL(2,\mathbb{C})$ acts on each $\mathbb{C}[X_0, X_1]_r$ by linear automorphisms. The r^{th} *induced matrix* $g^{[r]}$ is the matrix of the linear automorphism defined by g on $\mathbb{C}[X_0, X_1]_r$, with respect to the standard ordered monomial basis.

Example 2.4

$$g^{[2]} = \begin{pmatrix} g_{11}^2 & g_{11}g_{12} & g_{12}^2 \\ 2g_{11}g_{21} & g_{11}g_{22} + g_{21}g_{12} & 2g_{12}g_{22} \\ g_{21}^2 & g_{21}g_{22} & g_{22}^2 \end{pmatrix}.$$

Coordinate ring of $\mathbb{C}[X_0, X_1]_r$

Recall that the ring of polynomial functions from $\mathbb{C}[X_0, X_1]_r$ to $\mathbb{C}$ is generated by any set of coordinate functions (i.e. a basis of the dual) of the vector space $\mathbb{C}[X_0, X_1]_r$. Thus if $A_0, A_1, \ldots, A_r$ are such coordinate functions then $\mathbb{C}[A_0, A_1, \ldots, A_r]$ is the ring of polynomial functions on $\mathbb{C}[X_0, X_1]_r$, the so-called coordinate ring of $\mathbb{C}[X_0, X_1]_r$. A polynomial function is homogeneous of degree k if it is a $\mathbb{C}$−linear combination of monomials in $A_0, A_1, \ldots, A_r$ of degree k.

Polynomial mappings

Recall also that a polynomial mapping from $\mathbb{C}[X_0, X_1]_r$ to $\mathbb{C}[X_0, X_1]_m$ is given in terms of coordinate functions by $m+1$ polynomial functions on $\mathbb{C}[X_0, X_1]_r$. Equivalently, g is a polynomial mapping iff the composition $l \circ g$ is a polynomial function on $\mathbb{C}[X_0, X_1]_r$ for every linear function l from $\mathbb{C}[X_0, X_1]_m$ to $\mathbb{C}$.

Covariants

Definition 2.5 *1. A polynomial mapping C from $\mathbb{C}[X_0, X_1]_r$ to $\mathbb{C}[X_0, X_1]_m$ is called a* covariant *of weight w if*

(a) C is homogeneous of degree k(say), and

(b) for all $g \in GL(2,\mathbb{C})$ and for all $f \in \mathbb{C}[X_0, X_1]_r$ we have $gC(f) = (\det g)^w\, C(gf)$.

When $m=0$, C *is called an invariant.*

2. *A polynomial mapping* C *from* $\mathbb{C}[X_0,X_1]_r \oplus \mathbb{C}[X_0,X_1]_s$ *to* $\mathbb{C}[X_0,X_1]_m$ *is called a joint covariant of weight* w *if*

   (a) C *is homogeneous of degree* k *(say), and*

   (b) *for all* $g \in GL(2,\mathbb{C})$, *for all* $f \in \mathbb{C}[X_0,X_1]_r$ *and for all* $h \in \mathbb{C}[X_0,X_1]_s$ *we have* $gC(f,h) = (\det g)^w C(gf, gh)$.

   When $m=0$, C *is called a joint invariant.*

3. *The coefficient ideal of a covariant* C *is the ideal of the coordinate ring of* $\mathbb{C}[X_0,X_1]_r$, *generated by the compositions* $l \circ C$, *for every coordinate function* l *from* $\mathbb{C}[X_0,X_1]_m$ *to* $\mathbb{C}$.

The simplest example of a covariant is the identity mapping $\mathfrak{I}$ from $\mathbb{C}[X_0,X_1]_r$ to itself. It has weight 0.

The discriminant

A particularly important invariant from $\mathbb{C}[X_0,X_1]_r$ to $\mathbb{C}$ is the discriminant.

Definition 2.6 *1. Let*

$$f = \sum_{i=0}^{r} a_i {X_0}^{r-i} {X_1}^{i}, (r \geq 1),$$

$$g = \sum_{i=0}^{m} b_i {X_0}^{m-i} {X_1}^{i}, (m \geq 1).$$

Then the resultant $Res(f,g)$ *of* f *and* g, *is the determinant of the following*

$(r+m)\times(r+m)$ matrix,

$$\begin{pmatrix}
a_0 & & & & b_0 & & & \\
a_1 & a_0 & & & b_1 & b_0 & \ddots & \\
a_2 & a_1 & \ddots & & b_2 & b_1 & \ddots & \\
\vdots & & \ddots & a_0 & \vdots & & \vdots & b_0 \\
 & \vdots & & a_1 & b_m & & & b_1 \\
a_r & & & \vdots & & b_m & & \vdots \\
 & a_r & & & & & & \\
 & & \ddots & & & & \ddots & \\
 & & & a_r & & & & b_m
\end{pmatrix},$$

where the empty spaces are filled by zeros.

2. *The discriminant is the polynomial function $\mathcal{D}$ from $\mathbb{C}[X_0,X_1]_r$ to $\mathbb{C}$ defined by*

$$\mathcal{D}(f) = Res(\partial_0 f, \partial_1 f), \text{ for } f \in \mathbb{C}[X_0,X_1]_r.$$

Properties of discriminant:

1. ([Bôcher 1964] p. 259) The discriminant is an invariant of weight $r(r-1)$.

2. ([Bôcher 1964] p. 237) A necessary and sufficient condition that the binary form f has a multiple linear factor is that the discriminant of f vanishes.

3. ([Bôcher 1964] p. 259) The discriminant of a binary form is an irreducible polynomial function.

The Hessian

The *Hessian* is the polynomial mapping $\mathcal{H}$ from $\mathbb{C}[X_0, X_1]_r$ to $\mathbb{C}[X_0, X_1]_{2r-4}$ defined by

$$\mathcal{H}(f) = \begin{vmatrix} \partial_0^2(f) & \partial_0\partial_1(f) \\ \partial_0\partial_1(f) & \partial_1^2(f) \end{vmatrix}, f \in \mathbb{C}[X_0, X_1]_r.$$

It is a covariant of weight 2.

The Jacobian

The *Jacobian* is the polynomial mapping $\mathcal{J}$ from $\mathbb{C}[X_0, X_1]_r$ to $\mathbb{C}[X_0, X_1]_{3r-6}$ defined by

$$\mathcal{J}(f) = \begin{vmatrix} \partial_0(f) & \partial_1(f) \\ \partial_0(\mathcal{H}(f)) & \partial_1(\mathcal{H}(f)) \end{vmatrix}, f \in \mathbb{C}[X_0, X_1]_r.$$

It is a covariant of weight 3.

This use of the word "Jacobian" is not to be confused with the usual terminology in calculus.

The transvectants

The Hessian and the Jacobian are special cases of a general type of covariant called *transvectant*. To define transvectants, we will briefly explain the symbolic representation of binary forms, which originated with Clebsh.

We shall represent a binary form

$$f = \sum_{i=0}^{r} \binom{r}{i} a_i {X_0}^{r-i} {X_1}^{i}, (r \geq 1),$$

symbolically as

$$f = (\alpha_0 X_0 + \alpha_1 X_1)^r = (\alpha_0' X_0 + \alpha_1' X_1)^r = \dots$$

where the symbols appearing here are subject to the formal relations:

$a_k = \alpha_0^{r-k}\alpha_1^k = {\alpha_0'}^{r-k}{\alpha_1'}^k = \dots$ for $k = 0, \dots, r$.

Definition 2.7 *The* k^{th} *transvectant is the polynomial mapping* $(\,,\,)^{(k)}$ *from* $\mathbb{C}[X_0, X_1]_r \oplus \mathbb{C}[X_0, X_1]_s$ *to* $\mathbb{C}[X_0, X_1]_{r+s-2k}$ *defined by*

$$(f,h)^{(k)} = (\alpha_0\beta_1 - \alpha_1\beta_0)^k(\alpha_0 X_0 + \alpha_1 X_1)^{r-k}\,(\beta_0 X_0 + \beta_1 X_1)^{s-k},$$

where $f = (\alpha_0 X_0 + \alpha_1 X_1)^r \in \mathbb{C}[X_0, X_1]_r$ *and* $h = (\beta_0 X_0 + \beta_1 X_1)^s \in \mathbb{C}[X_0, X_1]_s$. *It is a joint covariant. In this, the right hand side is converted, using the above relations, to an expression involving* X_0, X_1 *and the coefficients of* f, h.

Example 2.8 *Let* $f = (\alpha_0 X_0 + \alpha_1 X_1)^r = (\beta_0 X_0 + \beta_1 X_1)^r$. *Then*

$$\begin{aligned}
(f,f)^{(2)} &= (\alpha_0\beta_1 - \alpha_1\beta_0)^2(\alpha_0 X_0 + \alpha_1 X_1)^{r-2}\,(\beta_0 X_0 + \beta_1 X_1)^{r-2} \\
&= (\alpha_0^2\beta_1^2 - 2\alpha_0\beta_1\alpha_1\beta_0 + \alpha_1^2\beta_0^2)(\alpha_0 X_0 + \alpha_1 X_1)^{r-2}\,(\beta_0 X_0 + \beta_1 X_1)^{r-2} \\
&= \alpha_0^2(\alpha_0 X_0 + \alpha_1 X_1)^{r-2}\,\beta_1^2(\beta_0 X_0 + \beta_1 X_1)^{r-2} \\
&\quad -2\alpha_0\beta_1(\alpha_0 X_0 + \alpha_1 X_1)^{r-2}\,\alpha_1\beta_0(\beta_0 X_0 + \beta_1 X_1)^{r-2} \\
&\quad +\alpha_1^2(\alpha_0 X_0 + \alpha_1 X_1)^{r-2}\,\beta_0^2(\beta_0 X_0 + \beta_1 X_1)^{r-2} \\
&= \frac{1}{r^2(r-1)^2}\left\{\partial_0^2(f)\,\partial_1^2(f) - 2\partial_0\partial_1(f)\,\partial_0\partial_1(f) + \partial_1^2(f)\,\partial_0^2(f)\right\} \\
&= \frac{2}{r^2(r-1)^2}\mathcal{H}(f)
\end{aligned}$$

Some examples of transvectants used in this dissertation are: For $f \in \mathbb{C}[X_0.X_1]_r$,

$$\begin{aligned}\frac{1}{r^2(r-1)^2}\mathcal{H}(f) &= \frac{1}{2}(f,f)^{(2)}\\ \frac{1}{r^3(r-1)^2(r-2)}\mathcal{J}(f) &= \frac{1}{2}(f,\mathcal{H}(f))^{(1)}\\ \mathcal{P}(f) &= (f,f)^{(4)}\end{aligned}$$

As the next theorem shows, it is possible to express the Hessian and the Jacobian in terms of only one of the partial derivatives ∂_0, ∂_1, mainly because of Euler's Theorem on homogeneous functions([Bôcher 1964] p. 237).

Theorem 2.9 *Let f be a binary form of degree r. Then*

$$X_0^2\mathcal{H}(f) = r(r-1)f\partial_1^2 f - (r-1)^2(\partial_1 f)^2,$$

and

$$X_1^2\mathcal{H}(f) = r(r-1)f\partial_0^2 f - (r-1)^2(\partial_0 f)^2.$$

Proof: ([Farahat])

Let f have degree r. Then $\partial_0 f, \partial_1 f$ are binary forms of degree $r-1$. The Hessian of f is

$$\mathcal{H}(f) = \begin{vmatrix} \partial_0^2 f & \partial_0\partial_1 f \\ \partial_0\partial_1 f & \partial_1^2 f \end{vmatrix}.$$

Multiply the first row by X_0 , then multiply the second row by X_1 and add to the first row. We get

$$\begin{aligned}
X_0\mathcal{H}(f) &= \begin{vmatrix} X_0\partial_0^2 f + X_1\partial_0\partial_1 f & X_0\partial_0\partial_1 f + X_1\partial_1^2 f \\ \partial_0\partial_1 f & \partial_1^2 f \end{vmatrix} \\
&= \begin{vmatrix} (X_0\partial_0 + X_1\partial_1)\partial_0 f & (X_0\partial_0 + X_1\partial_1)\partial_1 f \\ \partial_0\partial_1 f & \partial_1^2 f \end{vmatrix}
\end{aligned}$$

By Euler's formula, we have

$$(X_0\partial_0 + X_1\partial_1)\partial_0 f = (r-1)\partial_0 f, \text{ and}$$

$$(X_0\partial_0 + X_1\partial_1)\partial_1 f = (r-1)\partial_1 f.$$

Therefore,

$$X_0\mathcal{H}(f) = \begin{vmatrix} (r-1)\partial_0 f & (r-1)\partial_1 f \\ \partial_0\partial_1 f & \partial_1^2 f \end{vmatrix}.$$

Now multiply the first column by X_0 and then multiply the second column by X_1 and add to the first column, we get

$$\begin{aligned}
X_0^2\mathcal{H}(f) &= \begin{vmatrix} (r-1)X_0\,\partial_0 f + (r-1)\,X_1\partial_1 f & (r-1)\partial_1 f \\ X_0\,\partial_0\partial_1 f + X_1\,\partial_1^2 f & \partial_1^2 f \end{vmatrix} \\
&= \begin{vmatrix} (r-1)(X_0\partial_0 + X_1\partial_1) f & (r-1)\partial_1 f \\ (X_0\partial_0 + X_1\partial_1)\partial_1 f & \partial_1^2 f \end{vmatrix}
\end{aligned}$$

By Euler's formula, we have

$$(X_0\partial_0 + X_1\partial_1)\partial_1 f = (r-1)\partial_1 f$$

and

$$(X_0\partial_0 + X_1\partial_1) f = rf.$$

Thus,

$$X_0^2\mathcal{H}(f) = \begin{vmatrix} r(r-1)f & (r-1)\partial_1 f \\ (r-1)\partial_1 f & \partial_1^2 f \end{vmatrix}.$$

Hence,

$$X_0^2\mathcal{H}(f) = r(r-1)f\partial_1^2 f - (r-1)^2(\partial_1 f)^2.$$

In a similar manner we have,

$$X_1^2\mathcal{H}(f) = r(r-1)f\partial_0^2 f - (r-1)^2(\partial_0 f)^2.$$

□

Theorem 2.10 *Let f be a binary form of degree $r > 2$. Then*

$$\begin{aligned} X_0^3\mathcal{J}(f) &= -3r(r-1)(r-2)\, f\partial_1 f\, \partial_1^2 f \\ &\quad + r^2(r-1)\, f^2\, \partial_1^3 f + (2r-4)(r-1)^2\,(\partial_1 f)^3, \ \textit{and} \\ X_1^3\mathcal{J}(f) &= -3r(r-1)(r-2) f\, \partial_0 f\, \partial_0^2 f \\ &\quad + r^2(r-1)\, f^2\, \partial_0^3 f + (2r-4)(r-1)^2\,(\partial_0 f)^3. \end{aligned}$$

Proof: Let f have degree r. Then the Hessian $\mathcal{H}(f)$ of f is a binary form of degree $2r-4$ in the variables X_0 and X_1. The Jacobian of f is,

$$\mathcal{J}(f) = \begin{vmatrix} \partial_0 f & \partial_1 f \\ \partial_0\mathcal{H}(f) & \partial_1\mathcal{H}(f) \end{vmatrix}.$$

Multiply the first column by X_0, and then multiply the second column by X_1 and add to the first column, we get

$$\begin{aligned} X_0\mathcal{J}(f) &= \begin{vmatrix} X_0\partial_0 f + X_1\partial_1 f & \partial_1 f \\ X_0\partial_0\mathcal{H}(f) + X_1\partial_1\mathcal{H}(f) & \partial_1\mathcal{H}(f) \end{vmatrix} \\ &= \begin{vmatrix} (X_0\,\partial_0 + X_1\,\partial_1)\,f & \partial_1 f \\ (X_0\,\partial_0 + X_1\,\partial_1)\,\mathcal{H}(f) & \partial_1(\mathcal{H}(f)) \end{vmatrix} \\ &= \begin{vmatrix} rf & \partial_1 f \\ (2r-4)\mathcal{H}(f) & \partial_1\mathcal{H}(f) \end{vmatrix} \text{(by Euler's formula)} \end{aligned}$$

Hence,

$$X_0\mathcal{J}(f) = rf\,\partial_1(\mathcal{H}(f)) - (2r-4)\,\mathcal{H}(f)\,\partial_1 f.$$

By Theorem 2.9,

$$\mathcal{H}(f) = X_0^{-2}\left\{r(r-1)f\partial_1^2 f - (r-1)^2(\partial_1 f)^2\right\}.$$

Hence

$$X_0^3\mathcal{J}(f) \;=\; rf\left\{(r-1)(2-r)\partial_1 f\partial_1^2 f + r(r-1)f\partial_1^3 f\right\}$$

$$
\begin{aligned}
& -(2r-4)\partial_1 f\left\{r(r-1)f\partial_1^2 f-(r-1)^2(\partial_1 f)^2\right\} \\
= \; & -3r(r-1)(r-2)f\partial_1 f\partial_1^2 f+r^2(r-1)f^2\partial_1^3 f \\
& +2(r-2)(r-1)^2(\partial_1 f)^3.
\end{aligned}
$$

The second identity can be obtained by similar means. □

Remark 2.11 *Defining f_i by*

$$\partial_0^i f=\frac{r!}{(r-i)!}f_i$$

we have

$$X_1^2\mathcal{H}(f)=r^2(r-1)^2f_0f_2-r^2(r-1)^2f_1^2.$$

$$X_1^3\mathcal{J}(f)=-r^3(r-1)^2(r-2)\left\{3f_0f_1f_2-f_0^2f_3-2f_1^3\right\}.$$

When $r>2$, we have

$$\frac{\mathcal{H}(f)}{r^2(r-1)^2} = X_1^{-2}\left\{f_0f_2-f_1^2\right\}, \tag{2.1}$$

$$\frac{\mathcal{J}(f)}{(-1)r^3(r-1)^2(r-2)} = X_1^{-3}\left\{3f_0f_1f_2-f_0^2f_3-2f_1^3\right\}. \tag{2.2}$$

Similarly defining $\tilde{f}_i$ by ,

$$\partial_1^i f=\frac{r!}{(r-i)!}\tilde{f}_i$$

we have when $r>2$

$$\frac{\mathcal{H}(f)}{r^2(r-1)^2} = X_0^{-2}\left\{\tilde{f}_0\tilde{f}_2-\tilde{f}_1^2\right\}, \text{ and} \tag{2.3}$$

$$\frac{\mathfrak{J}(f)}{(-1)r^3(r-1)^2(r-2)} = X_0^{-3}\left\{3\tilde{f}_0\tilde{f}_1\tilde{f}_2 - \tilde{f}_0^2\tilde{f}_3 - 2\tilde{f}_1^3\right\}. \tag{2.4}$$

Chapter 3

Problem Statement and Some Special Cases

In this chapter, we first introduce the main problem. The basic case of the main problem follows easily from linear algebra. Then we explore two versions of the main problem. The solution to version 1 was obtained by Prof. H. K. Farahat in 1995 and discussed in a seminar in 1997. Finally at the end of this chapter we state version 2 of the main problem.

Let n, r be positive integers, and let $X_1, \ldots, X_n$ be commuting indeterminates over a field $\mathbb{K}$. Then any monomial in $X_1, \ldots, X_n$ can be written as $X_1^{\alpha_1} \ldots X_n^{\alpha_n}$, and the degree of the monomial $X_1^{\alpha_1} \ldots X_n^{\alpha_n}$ is the sum $\alpha_1 + \ldots + \alpha_n$. We shall order the monomials of degree r by using lexicographic order, which is defined below.

Definition 3.1 *Lexicographic order is a relation $\succeq$ defined on the set of monomials in $X_1, \ldots, X_n$ satisfying $X_1^{\alpha_1} \ldots X_n^{\alpha_n} \succeq X_1^{\beta_1} \ldots X_n^{\beta_n}$ if and only if $\alpha_1 > \beta_1$, or $\alpha_1 = \beta_1$ and $\alpha_2 > \beta_2$, etc.*

Definition 3.2 *Define $N(n,r)$ to be the number of monomials in $X_1, \ldots, X_n$ of degree r. (See [Cameron 1994] pages 32-33.) For all $n > 0$, $r \geq 0$,*

$$N(n,r) = \binom{n+r-1}{r}.$$

Definition 3.3 *Let*

$$X = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}$$

where $X_1, \ldots, X_n$ are variables. Then for $r \geq 1$, define $X^{[r]}$ to be the column matrix whose entries are the monomials $X_{i_1} \ldots X_{i_r}$, where $1 \leq i_1 \leq \ldots \leq i_r \leq n$, listed in lexicographic order.

That is,

$$X^{[r]} = \begin{pmatrix} X_1^r \\ X_1^{(r-1)} X_2 \\ \vdots \\ X_n^r \end{pmatrix}_{(N(n,r)\times 1)} .$$

Note that $X^{[1]} = X$.

For example when n=2,

$$X^{[3]} = \begin{pmatrix} X_1^3 \\ X_1^2 X_2 \\ X_1 X_2^2 \\ X_2^3 \end{pmatrix} .$$

We have found a recurrence formula for positioning a monomial of degree r in $X^{[r]}$, which is attached in Appendix A.

Next we shall state the main problem.

Problem Statement: (Transpose system of polynomial equations)

Let $r \geq 1$, $s \geq 1$, and let C be a $N(n,r) \times N(n,s)$ matrix over $\mathbb{K}$.

Consider the following systems of polynomial equations,

$$X^{[r]} = CX^{[s]}, \tag{3.1}$$

$$X^{[s]} = C^T X^{[r]}, \tag{3.2}$$

where C^T is the transpose of the matrix C. Our aim is to find any relations that may exist between the solutions of the systems of equations 3.1 and 3.2.

The basic case $r = s = 1$ is covered by the following:

Theorem 3.4 *(Basic case) If $C \in \mathbb{K}_{n,n}$ then the solution space of the system of linear equations*

$$X = CX, \tag{3.3}$$

and of the system of linear equations

$$X = C^T X, \tag{3.4}$$

have the same dimension.

Proof: The matrix equation $X = CX$, is equivalent to $(I - C)X = 0$. This is a system of homogeneous linear equations, whose solution set is a vector space with dimension equal to $n - rank(I - C)$.

The matrix equation $X = C^T X$, is equivalent to $(I - C^T)X = 0$. This is also a system of homogeneous linear equations, whose solution set is a vector space with dimension equal to $n - rank(I - C^T)$. Since $\text{rank}(I - C) = \text{rank }(I - C)^T = \text{rank}(I - C^T)$, the solution space of the system (3.3) and the solution space of the system (3.4) have the same dimension. □

3.1 Conjugate Systems of Quadratic Equations

Let $\mathbb{K}$ be an algebraically closed field, and let $n \geq 1$.

Now we shall state the problem of conjugate systems of polynomial equations.

Problem Statement:(Conjugate Systems of Quadratic Equations)

Suppose that we have a family of scalars (meaning elements of $\mathbb{K}$) c_{ijk} for $1 \leq i, j, k \leq n$, with the property that $c_{ijk} = c_{jik}$ for all $i, j, k = 1, \ldots n$.

Consider the following system of quadratic equations in n variables $X_1, \ldots, X_n$,

$$X_i X_j = \sum_{k=1}^{n} c_{ijk} X_k, \text{ for all } i, j = 1, \ldots n \tag{3.5}$$

and its conjugate system of quadratic equations in n variables $X_1, \ldots, X_n$,

$$X_k = \sum_{i,j=1}^{n} c_{ijk} X_i X_j, \text{ for all } k = 1, \ldots n \tag{3.6}$$

Find any relations that may exist between the solutions of the systems of equations 3.5 and 3.6.

It turns out that the structure theory of finite dimensional commutative algebras is useful in this connection.

Definition 3.5 *Let V be n-dimensional vector space over $\mathbb{K}$. Then there exist $v_1, \ldots, v_n \in$*

V such that,

$$V = \mathbb{K}v_1 \dot{+} \ldots \dot{+} \mathbb{K}v_n, \text{ (internal direct sum).}$$

Also suppose that $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$ are the corresponding co-ordinate functions in the dual space of V. These are linear functions

$$\mathfrak{X}_i : V \to \mathbb{K}$$

such that

$$\mathfrak{X}_i(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Define a bilinear multiplication $\star$ on V by

$$v_i \star v_j = v_j \star v_i = \sum_{k=1}^{n} c_{ijk} v_k.$$

The vector space V together with the multiplication $\star$ defined above, is a finite dimensional commutative algebra over $\mathbb{K}$. We denote this (possibly non-associative) algebra by $\mathbf{V}_c$.

Next we shall show that the idempotents in the algebra $\mathbf{V}_c$ correspond to the solutions of the system of quadratic equations 3.6. This follows from the following lemma.

Lemma 3.6 *The following are equivalent for $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$:*

1. *$\alpha_1 v_1 + \ldots + \alpha_n v_n$ is an idempotent in $\mathbf{V}_c$.*
2. *$\sum_{i,j=1}^{n} \alpha_i \alpha_j c_{ijk} = \alpha_k$, for all $k = 1, \ldots, n$.*

Proof: Let $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$. Then

$$\begin{aligned}(\alpha_1 v_1 + \ldots + \alpha_n v_n)^2 &= \sum_{i,j=1}^{n} \alpha_i \alpha_j \, v_i \star v_j \\ &= \sum_{k=1}^{n} \left(\sum_{i,j=1}^{n} \alpha_i \alpha_j c_{ijk} \right) v_k\end{aligned}$$

Hence $\alpha_1 v_1 + \ldots + \alpha_n v_n$ is an idempotent in $\mathbf{V}_c$, iff

$$\alpha_k - \sum_{i,j=1}^{n} \alpha_i \alpha_j c_{ijk} = 0, \forall k = 1, \ldots n.$$

$\square$

Next we shall show that the algebra homomorphisms from $\mathbf{V}_c$ to the field $\mathbb{K}$ correspond to the solutions of the system of quadratic equations 3.5. This follows from the following lemma.

Lemma 3.7 *The following are equivalent for $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$:*

1. *The $\mathbb{K}$-linear function*

$$\alpha_1 \mathfrak{X}_1 + \ldots + \alpha_n \mathfrak{X}_n : \mathbf{V}_c \to \mathbb{K}$$

is an algebra homomorphism.

2. $\sum_{k=1}^{n} c_{ijk} \alpha_k = \alpha_i \alpha_j$, *for all* $i, j = 1, \ldots, n$.

Proof: The $\mathbb{K}$-linear function

$$h = \alpha_1 \mathfrak{X}_1 + \ldots + \alpha_n \mathfrak{X}_n : \mathbf{V}_c \to \mathbb{K}$$

is an algebra homomorphism iff

$$h(v_i)h(v_j) = h(v_i \star v_j), \text{ for all } i, j = 1, \dots, n.$$

The result follows from the following:

$$h(v_i)h(v_j) = \alpha_i \alpha_j,$$

$$h(v_i \star v_j) = h(\sum_{k=1}^{n} c_{ijk} v_k) = \sum_{k=1}^{n} c_{ijk} h(v_k) = \sum_{k=1}^{n} c_{ijk} \alpha_k.$$

□

Next we shall state the main theorem in this chapter.

Theorem 3.8 *([Farahat])*

Consider the following conjugate systems of polynomial equations,

$$X_i X_j = \sum_{k=1}^{n} c_{ijk} X_k, \text{ for all } i, j = 1, \dots n,$$

$$X_k = \sum_{i,j=1}^{n} c_{ijk} X_i X_j, \text{ for all } k = 1, \dots n.$$

where all c_{ijk} are in the algebraically closed field $\mathbb{K}$.

Suppose that scalars c_{ijk} satisfy both of the following statements

1. *There exist* $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ *such that for all* $j = 1, \ldots, n$, *and all* $k \neq j$,

$$\sum_{i=1}^{n} \alpha_i c_{ijk} = 0,$$

and

$$\sum_{i=1}^{n} \alpha_i c_{ijj} = 1.$$

2. $\sum_{k=1}^{n} c_{ijk} c_{klp} = \sum_{k=1}^{n} c_{jlk} c_{ikp}$, *and* $c_{ijk} = c_{jik}$, *for all* $1 \leq i, j, l, p \leq n$.

Then the system of quadratic equations 3.5 has $m + 1$ *solutions if and only if the system of quadratic equations 3.6 has* 2^m *solutions in* $\mathbb{K}$.

In order to give a proof of this theorem we shall establish the following two lemmas, providing conditions on the constants c_{ijk}, equivalent to $\mathbf{V}_c$ being associative with identity element.

Lemma 3.9 1. *The following are equivalent:*

(a) $\mathbf{V}_c$ *has an identity element.*

(b) *There exist* $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ *such that for all* $j = 1, \ldots, n$ *and for all* $k \neq j$,

$$\sum_{i=1}^{n} \alpha_i \, c_{ijk} = 0,$$

and

$$\sum_{i=1}^{n} \alpha_i \, c_{ijj} = 1.$$

2. *The following are equivalent:*

(a) $\mathbf{V}_c$ *is associative*

(b) The c_{ijk} satisfy the following quadratic conditions,

$$\sum_{k=1}^{n} c_{ijk}c_{klp} = \sum_{k=1}^{n} c_{jlk}c_{ikp}, \text{ for all } 1 \leq i,j,l,p \leq n.$$

Proof:

1. $\mathbf{V}_c$ has an identity element iff there exist $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ such that

$$\sum_{j=1}^{n} \alpha_j v_j \star v_i = v_i = v_i \star \sum_{j=1}^{n} \alpha_j v_j, \ \forall i = 1, \ldots, n.$$

Since $\star$ is commutative, only one of these will suffice. That is,

$$\sum_{j=1}^{n} \alpha_j v_j \ \star \ v_i = v_i, \forall i = 1, \ldots, n.$$

By the definition of the multiplication, we have

$$\sum_{j=1}^{n} \alpha_j \left(\sum_{k=1}^{n} c_{ijk} v_k \right) = v_i, \forall i = 1, \ldots, n.$$

That is,

$$\sum_{k=1}^{n} \sum_{j=1}^{n} (\alpha_j c_{ijk})\, v_k = v_i, \forall i = 1, \ldots, n.$$

Since $v_1, \ldots, v_n$ are linearly independent, for all $i = 1, \ldots, n$,

$$\sum_{j=1}^{n} \alpha_j c_{ijk} = 0, \ \text{for all } k \neq i,$$

and

$$\sum_{j=1}^{n} \alpha_j c_{iji} = 1.$$

Hence the result.

2. Let $a = \sum_{i=1}^{n} \alpha_i v_i$, $b = \sum_{j=1}^{n} \beta_j v_j$, $c = \sum_{k=1}^{n} \gamma_k v_k$ be any elements of $\mathbf{V}_c$. Then

$$(a \star b) \star c = \left(\sum_{i,j,l=1}^{n} \alpha_i \beta_j c_{ijl} v_l \right) \star \left(\sum_{k=1}^{n} \gamma_k v_k \right) = \sum_{i,j,l,k,p=1}^{n} \alpha_i \beta_j \gamma_k c_{ijl} c_{lkp} v_p,$$

$$a \star (b \star c) = \left(\sum_{i=1}^{n} \alpha_i v_i \right) \star \left(\sum_{j,k,l=1}^{n} \beta_j \gamma_k c_{jkl} v_l \right) = \sum_{i,j,l,k,p=1}^{n} \beta_j \gamma_k \alpha_i c_{jkl} c_{ilp} v_p.$$

The condition for associativity of $\mathbf{V}_c$ follows from this by comparison of the coefficients of $\beta_j \gamma_k \alpha_i$. □

Now we are ready to give a proof of Theorem 3.8.

<u>Proof of Theorem 3.8:</u>

The conditions of the theorem ensure that $\mathbf{V}_c$ is a finite dimensional associative commutative algebra over $\mathbb{K}$ with an identity. The structure of such algebras is well known, and can be found for example in [Hungerford 1974] on page 153. That is, $\mathbf{V}_c/Rad(\mathbf{V}_c)$ is isomorphic to a direct sum of a finite number of copies of $\mathbb{K}$, where $Rad(\mathbf{V}_c)$ is the set of all nilpotent elements in $\mathbf{V}_c$:

$$\mathbf{V}_c/Rad(\mathbf{V}_c) \;\cong\; \underbrace{\mathbb{K} \oplus \ldots \oplus \mathbb{K}}_{m}.$$

Now an element $\alpha = (\alpha_1, \ldots, \alpha_m)$ in $\mathbb{K} \oplus \ldots \oplus \mathbb{K}$ is an idempotent iff $\alpha_i^2 = \alpha_i$, for all $i = 1, \ldots, m$. Since a field has only 2 idempotents, $\mathbb{K} \oplus \ldots \oplus \mathbb{K}$ has exactly

2^m idempotents. Therefore $\mathbf{V}_c/Rad(\mathbf{V}_c)$ has exactly 2^m idempotents. But every idempotent in $\mathbf{V}_c/Rad(\mathbf{V}_c)$, can be lifted uniquely to an idempotent in $\mathbf{V}_c$ (see lifting idempotents in [Eisenbud 1995] p. 189). Hence, we have that $\mathbf{V}_c$ has exactly 2^m idempotents. Note that $e_1 = (1, 0 \ldots, 0), \ldots, e_m = (0, 0, \ldots, 1)$ are primitive non-zero orthogonal idempotents in $\mathbb{K} \oplus \ldots \oplus \mathbb{K}$, and every idempotent is a sum of a subset of them.

Suppose that g is a $\mathbb{K}$–algebra homomorphism from $\mathbb{K} \oplus \ldots \oplus \mathbb{K}$ to $\mathbb{K}$. Then $g(\alpha_1, \ldots, \alpha_m) = \sum_{i=1}^{m} \alpha_i \, g(e_i)$, for $\alpha_1, \ldots, \alpha_m \in \mathbb{K}$, where $g(e_i)^2 = g(e_i)$ for all $i = 1, \ldots, m$, and $g(e_i)g(e_j) = 0$ for all $1 \leq i < j \leq m$. Therefore, for each $i = 1, \ldots, m$, $g(e_i)$ is either 0 or 1 and $g(e_i)g(e_j) = 0$ for all $1 \leq i < j \leq m$. Hence, there are $m+1$ $\mathbb{K}$–algebra homomorphisms from $\mathbf{V}_c$ to field $\mathbb{K}$, namely 0 and the m projections. $\square$

We shall illustrate Theorem 3.8 with the following examples.

Example 3.10 *Consider the following system of polynomial equations,*

$$\begin{pmatrix} X_1^2 \\ X_1 X_2 \\ X_2 X_1 \\ X_2^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \tag{3.7}$$

First we look at the algebra $A = \mathbb{K}v_1 \dot{+} \mathbb{K}v_2$. The multiplication table for the basis of A is as follows:

	v_1	v_2
v_1	v_1	v_2
v_2	v_2	v_2

It is easy to verify that

1. *A is associative with identity element v_1, and primitive idempotents $v_2, v_1 - v_2$.*

2. *$Rad(A)$ is zero.*

3. *$A = \mathbb{K}v_1 \dot{+} \mathbb{K}(v_1 - v_2)$ (direct sum of fields isomorphic to $\mathbb{K}$).*

The above mentioned system 3.7 has 3 *solutions, namely* $(0,0), (1,0)$, *and* $(1,1)$. *There are exactly* 3 *algebra homomorphisms from A to $\mathbb{K}$, namely:*

1. *trivial homomorphism*

2. $-\mathfrak{X}_2$

3. $\mathfrak{X}_1 + \mathfrak{X}_2$,

where for each $i = 1, 2$,

$$\mathfrak{X}_i : A \to \mathbb{K}$$

is defined by

$$\mathfrak{X}_i(v_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Now consider the following system of polynomial equations,

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} X_1^2 \\ X_1X_2 \\ X_2X_1 \\ X_2^2 \end{pmatrix} \tag{3.8}$$

This system is conjugate to the system 3.7 and it has 4 *solutions, namely* $(0,0),(1,0),(0,1)$ *and* $(1,-1)$. *There are four idempotents in* A, *namely:* 0, v_1, v_2, $v_1 - v_2$.

3.2 Transpose Systems of Binary Homogeneous Polynomial Equations

First we shall state the problem of transpose systems of binary homogeneous polynomial equations:

Problem Statement:

Let $r \geq 1, A \in \mathbb{C}_{r+1,r+1}$, and $X = \begin{pmatrix} X_0 \\ X_1 \end{pmatrix}$.

Then find any relations that may exist between the solutions of the transpose systems of binary homogeneous polynomial equations

$$AX^{[r]} = 0, \tag{3.9}$$

and

$$A^T X^{[r]} = 0. \tag{3.10}$$

As we shall see in Chapter 5, this problem is connected with rather basic concepts of algebraic geometry. For this purpose we shall consider the vector space $\mathbb{C}_{r+1,r+1}$ of all $(r+1)\times(r+1)$ matrices over $\mathbb{C}$. This is a complex vector space of dimension $(r+1)^2$, and its co-ordinate ring is generated by any dual basis of this vector space. For fixed l, the set $\mathbb{C}^{(l)}_{r+1,r+1}$ of all $(r+1)\times(r+1)$ matrices of rank less than or equal to l is a Zariski closed subset. It consists of those matrices with all $(l+1)\times(l+1)$ minors equal to zero. Formally:

Definition 3.11

$$\mathbb{C}^{(l)}_{r+1,r+1} = \mathbf{V}(\ \textit{all}\ (l+1)\times(l+1)\ \textit{minors}).$$

In fact it was proved in [Bruns, Vetter 1988] on p. 5 that for $0 \le l \le (r+1)$, $\mathbb{C}^{(l)}_{r+1,r+1}$ is an irreducible closed subset of $\mathbb{C}_{r+1,r+1}$ with dimension $l(2r+2-l)$.

The ideal of the co-ordinate ring generated by minors of a given size is called a *determinantal ideal*. It is in fact prime but this is a non-trivial statement. The subject of determinantal ideals is fairly extensive.(See [Bruns, Vetter 1988] on page 14.)

Thus we have the following ascending chain of irreducible Zariski closed subsets of $\mathbb{C}_{r+1,r+1}$,

$$\{0\} = \mathbb{C}^{(0)}_{r+1,r+1} \subset \dots \subset \mathbb{C}^{(r+1)}_{r+1,r+1} = \mathbb{C}_{r+1,r+1}.$$

Definition 3.12 *For $C \in \mathbb{C}_{r+1,r+1}$, define $\mathcal{P}(C)$ to be the set of all projective points $[X]=[X_0,X_1]$ in the one dimensional projective space $\mathbb{P}^1$ such that $CX^{[r]}=0$. That* *is,*

$$\mathcal{P}(C) = \{[X] = [X_0, X_1] \in \mathbb{P}^1 \,|CX^{[r]} = 0\}.$$

When $C = 0$, $\mathcal{P}(C) = \mathbb{P}^1$ is infinite. Otherwise it has at most r points. Hence the following definition makes sense.

Definition 3.13 *For $k \geq 0$,*

$$\mathcal{E}^{(l)}(k) = \{C \in \mathbb{C}^{(l)}_{r+1,r+1} | \#\mathcal{P}(C) = \#\mathcal{P}(C^T) = k\}.$$

We are interested in the properties of the sets $\mathcal{E}^{(l)}(k)$.

We know that $\mathbb{C}^{(0)}_{r+1,r+1} = \{0\}$ and therefore $\mathcal{E}^{(0)}(k) = \emptyset$ for all $k \geq 0$.

It is obvious that if $C \in \mathbb{C}^{(1)}_{r+1,r+1} \setminus \{0\}$ then the system $CX^{[r]} = 0$ is equivalent to a single binary homogeneous polynomial equation. Thus the projective points in the set $\mathcal{P}(C)$ are same as the projective points represented by the corresponding binary form. Therefore it is necessary to get further information about binary forms. This is the subject of the next chapter.

Chapter 4

Binary Forms

In this chapter we want to explore the geometrical nature of the set of all binary forms having a certain factorization. In Section 4.1, we have proved that the set of all binary forms having certain factorizations are affine irreducible closed sets.

In Section 4.2, we determine the dimension of these closed sets.

In Section 4.3, we present our findings regarding the following question:

Let $(m_1, \ldots, m_s)$ be a partition of r. Can one find covariants whose vanishing for a binary form f is a necessary and sufficient condition that f has the form $l_1^{m_1} \ldots l_s^{m_s}$ for some linear forms $l_1, \ldots, l_s$?

Our investigation is by no means complete. But for degrees 2, 3, 4 and 5 it is complete. We present the results in the Subsections 4.3.2 , 4.3.3 and 4.3.4.

4.1 The Affine Closed Sets $\mathcal{F}(m_1, \ldots, m_s)$

Let $(m_1, \ldots, m_s)$ be a partition of r, that is :

$$m_1 + \ldots + m_s = r,\ m_1 \geq m_2 \geq \ldots \geq m_s > 0.$$

We consider the mapping :

$$\begin{array}{rcl} \mathbb{C}[X_0, X_1]_1 \oplus \ldots \oplus \mathbb{C}[X_0, X_1]_1 & \rightarrow & \mathbb{C}[X_0, X_1]_r \\ (l_1, \ldots, l_s) & \mapsto & l_1^{m_1} \ldots l_s^{m_s}. \end{array}$$

The domain and destination are vector spaces and this mapping is a polynomial mapping. It turns out that its image, i.e. the set of binary forms of degree r with factorization multiplicities $m_1,\ldots,m_s$, is an irreducible closed subset of $\mathbb{C}[X_0,X_1]_r$.

Explicitly, writing $l_i = l_{i0}X_0 + l_{i1}X_1$ we have

$$l_1^{m_1}\ldots l_s^{m_s}. = \prod_{i=1}^{s}(l_{i0}X_0 + l_{i1}X_1)^{m_i}.$$

Now by expanding the right hand side, using the binomial theorem we have,

$$\begin{aligned}
\prod_{i=1}^{s}(l_{i0}X_0 + l_{i1}X_1)^{m_i} &= \prod_{i=1}^{s}\sum_{q_i=0}^{m_i}\binom{m_i}{q_i} l_{i0}^{m_i-q_i} l_{i1}^{q_i} X_0^{m_i-q_i} X_1^{q_i} \\
&= \sum_{j=0}^{r}\sum_{q_1+\ldots+q_s=j}\prod_{i=1}^{s}\left(\binom{m_i}{q_i} l_{i0}^{m_i-q_i} l_{i1}^{q_i}\right) X_0^{r-j}X_1^{j} \\
&= \sum_{j=0}^{r} c_j X_0^{r-j}X_1^{j} \text{ (say).} \qquad (4.1)
\end{aligned}$$

It is important to note from this that $c_0,\ldots,c_r$ are polynomial functions of the coordinates of $l_1,\ldots,l_s$, and that each c_j is separately homogeneous of degree m_i in l_{i0} and l_{i1}.

We are interested in the set of all such binary forms for a fixed choice of partition $(m_1,\ldots,m_s)$. To this end let $\mathcal{F}(m_1,\ldots,m_s)$ denote the set of binary forms of degree r corresponding to all choices $l_1,\ldots,l_s \in \mathbb{C}[X_0,X_1]_1$. Formally:

Definition 4.1

$$\mathcal{F}(m_1,\ldots,m_s) = \{f \in \mathbb{C}[X_0,X_1]_r \,|\, f = l_1^{m_1}\ldots l_s^{m_s},\ \textit{for some } l_1,\ldots,l_s \in \mathbb{C}[X_0,X_1]_1\}.$$

Theorem 4.2 *([Farahat])*

For any partition $(m_1,\ldots,m_s)$ of r, $\mathcal{F}(m_1,\ldots,m_s)$ is a closed subset of $\mathbb{C}[X_0,X_1]_r$.

Proof:

We are going to show that $\mathcal{F}(m_1,\ldots,m_s)$ is an affine closed subset of $\mathbb{C}[X_0,X_1]_r$, by exhibiting a closed subset Q of the product

$$\underbrace{\mathbb{P}(\mathbb{C}[X_0,X_1]_1)\times\ldots\times\mathbb{P}(\mathbb{C}[X_0,X_1]_1)}_{s\text{ copies}}\times\mathbb{C}[X_0,X_1]_r$$

whose projection on $\mathbb{C}[X_0,X_1]_r$ is $\mathcal{F}(m_1,\ldots,m_s)$. In fact

$$Q = \left\{\left([l_1],\ldots,[l_s],\sum_{j=0}^{r} a_j X_0^{r-j}X_1^j\right)\in\mathbb{P}(\mathbb{C}[X_0,X_1]_1)\times\ldots\times\mathbb{P}(\mathbb{C}[X_0,X_1]_1)\times\mathbb{C}[X_0,X_1]_r \;\middle|\; a_ic_j-a_jc_i=0\ \forall\, 0\le i<j\le r \text{ where the } c_j \text{ are defined by 4.1}\right\}.$$

Recalling the definition of closed sets in a product, and the above remark concerning the function c_i, it is evident that Q is a closed subset of $\mathbb{P}(\mathbb{C}[X_0,X_1]_1)\times\ldots\times\mathbb{P}(\mathbb{C}[X_0,X_1]_1)\times\mathbb{C}[X_0,X_1]_r$. Since $\mathbb{P}(\mathbb{C}[X_0,X_1]_1)\times\ldots\times\mathbb{P}(\mathbb{C}[X_0,X_1]_1)$ is a projective closed set, it follows from (Theorem 3 [Shafarevich 1974] p. 45) that the projection onto $\mathbb{C}[X_0,X_1]_r$ carries the closed subset Q to a closed subset of $\mathbb{C}[X_0,X_1]_r$. It only remains to show that the image of Q under the projection, is exactly $\mathcal{F}(m_1,\ldots,m_s)$.

If $a=\sum_{j=0}^{r} a_jX_0^{r-j}X_1^j$ is an element of the image of Q, then Q contains an element $([l_1],\ldots,[l_s],a)$, and the corresponding $c=\sum_{j=0}^{r} c_jX_0^{r-j}X_1^j=l_1^{m_1}\ldots l_s^{m_s}$ is non-zero, because each l_i is non-zero. The conditions $a_ic_j - a_jc_i = 0$ for all $0 \leq i < j \leq r$, now imply that a is scalar multiple of c. Hence a belongs to $\mathcal{F}(m_1, \ldots, m_s)$.

On the other hand, it is clear that every non-zero element of $\mathcal{F}(m_1, \ldots, m_s)$ belongs to the image of Q. The zero element of $\mathcal{F}(m_1, \ldots, m_s)$ is obviously also in the image. □

It turns out that each of these closed sets is irreducible:

Theorem 4.3 *For any partition $(m_1, \ldots, m_s)$ of r, $\mathcal{F}(m_1, \ldots, m_s)$ is irreducible.*

Proof: Now $\mathcal{F}(m_1, \ldots, m_s)$ is the image of the polynomial mapping

$$\Gamma : (l_1, \ldots, l_s) \in \mathbb{C}[X_0, X_1]_1 \oplus \ldots \oplus \mathbb{C}[X_0, X_1]_1 \to l_1^{m_1} \ldots l_s^{m_s} \in \mathbb{C}[X_0, X_1]_r.$$

The domain, being a vector space, is irreducible. The image is closed by the above theorem. The polynomial mapping Γ induces a ring homomorphism $\tilde{\Gamma}$ from the coordinate ring $\mathbb{C}\left[\mathbb{C}[X_0, X_1]_r\right]$ to the coordinate ring $\mathbb{C}\left[\mathbb{C}[X_0, X_1]_1 \oplus \ldots \oplus \mathbb{C}[X_0, X_1]_1\right]$ with kernel $\mathbf{I}(\mathcal{F}(m_1, \ldots, m_s))$. Hence $\mathbb{C}\left[\mathbb{C}[X_0, X_1]_r\right]/\mathbf{I}(\mathcal{F}(m_1, \ldots, m_s))$ is isomorphic to a subring of the coordinate ring $\mathbb{C}\left[\mathbb{C}[X_0, X_1]_1 \oplus \ldots \oplus \mathbb{C}[X_0, X_1]_1\right]$. Since the coordinate ring $\mathbb{C}\left[\mathbb{C}[X_0, X_1]_1 \oplus \ldots \oplus \mathbb{C}[X_0, X_1]_1\right]$ is an integral domain, every subring of the coordinate ring $\mathbb{C}\left[\mathbb{C}[X_0, X_1]_1 \oplus \ldots \oplus \mathbb{C}[X_0, X_1]_1\right]$ is an integral domain. Therefore $\mathbb{C}\left[\mathbb{C}[X_0, X_1]_r\right]/\mathbf{I}(\mathcal{F}(m_1, \ldots, m_s))$ is an integral domain. Hence $\mathbf{I}(\mathcal{F}(m_1, \ldots, m_s))$ is a prime ideal. Hence the result. □

Now we turn to the problem of the dimensions of these closed sets:

4.2 Dimensions of the closed sets of the binary forms

The Theorem of Dimension of Fibers (see [Shafarevich 1974] p.60) applied to the polynomial mapping in the proof of Theorem 4.3 provides an upper bound for the dimension of $\mathcal{F}(m_1, \ldots, m_s)$. Namely, the dimension must be less than or equal to $2s$. It turns out that the dimension of $\mathcal{F}(m_1, \ldots, m_s)$ is in fact $s+1$. In order to give a proof of this result, we shall define the following operation.

Let $r > 1$, $s > 1$, and let $(m_1, \ldots, m_s)$ be a partition of r with s parts. Then adding any two entries in the sequence $m_1, \ldots, m_s$ produces another partition $(m_1', \ldots, m_{s-1}')$ of r with $s-1$ parts. We shall call this a merging operation. The source of this definition is [Farahat].

Evidently all the partitions of r can be formed by recursively doing merging operations starting with the partition $(1, \ldots, 1)$ of r. For given any partition $(m_1, \ldots, m_s)$ of r, $(m_1, 1, \ldots, 1)$ can be formed from $(1, \ldots, 1)$ by successively doing $m_1 - 1$ merging operations on the first two entries. Then $(m_1, m_2, 1, \ldots, 1)$ could be formed from $(m_1, 1, \ldots, 1)$ by successively doing $m_2 - 1$ merging operations on the second and third entries. Repeating similar merging operations, after $(m_1 - 1) + \ldots + (m_s - 1)$ merging operations produces the partition $(m_1, \ldots, m_s)$.

We group the closed sets $\mathcal{F}(m_1, \ldots, m_s)$ according to the number of parts in the partition.

We have listed these closed sets in fig. 4.1 for the case $r = 6$.

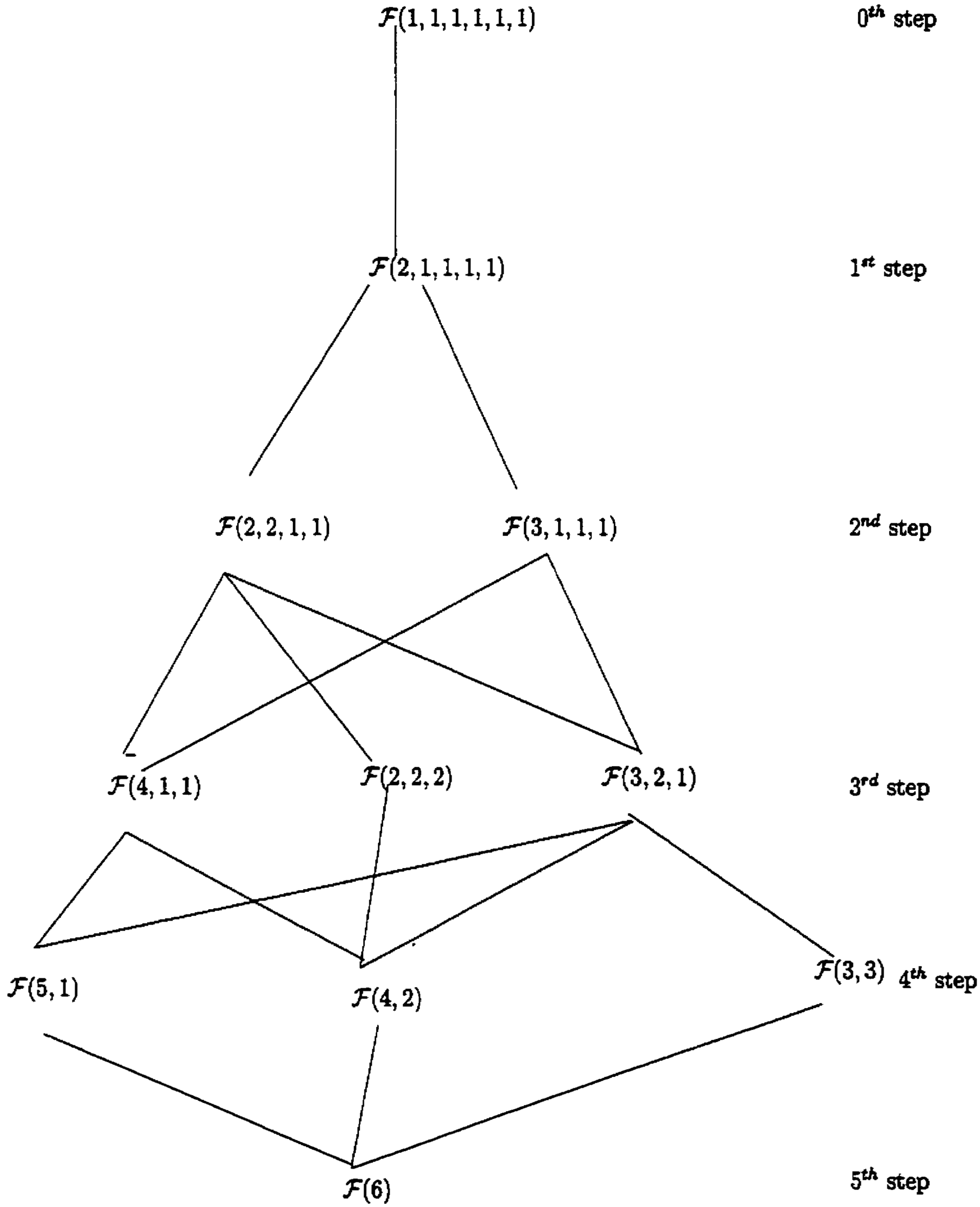

Figure 4.1: The affine closed sets for $r = 6$

Theorem 4.4 *For any partition* $(m_1, \ldots, m_s)$ *of* r,

$$dim(\mathcal{F}(m_1, \ldots, m_s)) = s + 1.$$

Proof:

Let s be a number between 1 and r. Assuming that $(m_1', \ldots, m_{s-1}')$ is a partition obtained by merging the partition $(m_1, \ldots, m_s)$, we shall show that $\mathcal{F}(m_1', \ldots, m_{s-1}')$ is a proper subset of $\mathcal{F}(m_1, \ldots, m_s)$.

We choose $l_1, \ldots, l_s \in \mathbb{C}[X_0, X_1]_1$ all mutually distinct, meaning l_i is not a scalar multiple of l_j for all $1 \leq i < j \leq s$. Now $l_1^{m_1} \ldots l_s^{m_s}$ belongs to $\mathcal{F}(m_1, \ldots, m_s)$, and not in $\mathcal{F}(m_1', \ldots, m_{s-1}')$. Thus

$$\mathcal{F}(m_1', \ldots, m_{s-1}') \subset \mathcal{F}(m_1, \ldots, m_s).$$

Since these closed sets are irreducible, the codimension of $\mathcal{F}(m_1', \ldots, m_{s-1}')$ in $\mathcal{F}(m_1, \ldots, m_s)$, is at least 1 (See [Shafarevich 1974] Theorem 1 on page 54). That is,

$$dim(\mathcal{F}(m_1, \ldots, m_s)) - dim(\mathcal{F}(m_1', \ldots, m_{s-1}')) \geq 1.$$

Since there are $r-1$ different steps between $\mathcal{F}(1, \ldots, 1)$ and $\mathcal{F}(r)$, the codimension of $\mathcal{F}(r)$ in $\mathcal{F}(1, \ldots, 1)$ is at least $r-1$. Thus

$$dim(\mathcal{F}(1, \ldots, 1)) - dim(\mathcal{F}(r)) \geq r - 1.$$

Since $dim(\mathcal{F}(1, \ldots, 1)) = r + 1$, $dim(\mathcal{F}(r))$ is less than or equal to 2.

Now if, as we shall prove, the dimension of $\mathcal{F}(r)$ is 2 then it follows that the codimension of $\mathcal{F}(m_1', \ldots, m_{s-1}')$ in $\mathcal{F}(m_1, \ldots, m_s)$ is in fact 1. Since there are $r-s$ different steps between $\mathcal{F}(1, \ldots, 1)$ and $\mathcal{F}(m_1, \ldots, m_s)$, the codimension of $\mathcal{F}(m_1, \ldots, m_s)$ in $\mathcal{F}(1, \ldots, 1)$ is $r-s$. Therefore the dimension of $\mathcal{F}(m_1, \ldots, m_s)$ is $r+1-(r-s) = s+1$. Thus it only remains to show that the dimension of $\mathcal{F}(r)$ is 2.

In order to show that the dimension of $\mathcal{F}(r)$ is 2, we shall show that

$$dim(\mathcal{F}(r)) \geq 2$$

by recalling the polynomial mapping θ from $\mathbb{C}[X_0, X_1]_1$ to $\mathbb{C}[X_0, X_1]_r$ whose image is $\mathcal{F}(r)$. In fact

$$\theta(\alpha X_0 + \beta X_1) = (\alpha X_0 + \beta X_1)^r = \sum_{j=1}^{r} \binom{r}{j} \alpha^{r-j} \beta^j X_0^{r-j} X_1^j.$$

Since $\mathbb{C}[X_0, X_1]_1$ and $\mathcal{F}(r)$ are irreducible, and the fiber $\theta^{-1}0 = \{0\}$ is a singleton set with dimension zero, it follows from the Theorem of the dimension of fibers (see [Shafarevich 1974] page 60) that

$$0 = dim(\theta^{-1}(0,0)) \geq dim(\mathbb{C}[X_0, X_1]_1) - dim(\mathcal{F}(r)) = 2 - dim(\mathcal{F}(r)).$$

That is,

$$dim(\mathcal{F}(r)) \geq 2.$$

Hence the dimension of $\mathcal{F}(r)$ is 2. $\square$

4.3 The Ideals $\mathcal{I}(m_1, \ldots, m_s)$

Finally there is the problem of the ideals corresponding to these closed sets :

Definition 4.5 *Let $m_1, \ldots, m_s$ be a partition of r. Then $\mathcal{I}(m_1, \ldots, m_s)$ is the ideal of all polynomial functions on $\mathbb{C}[X_0, X_1]_r$ which vanish on $\mathcal{F}(m_1, \ldots, m_s)$.*

We have listed these ideals in fig. 4.2 for the case $r = 6$.

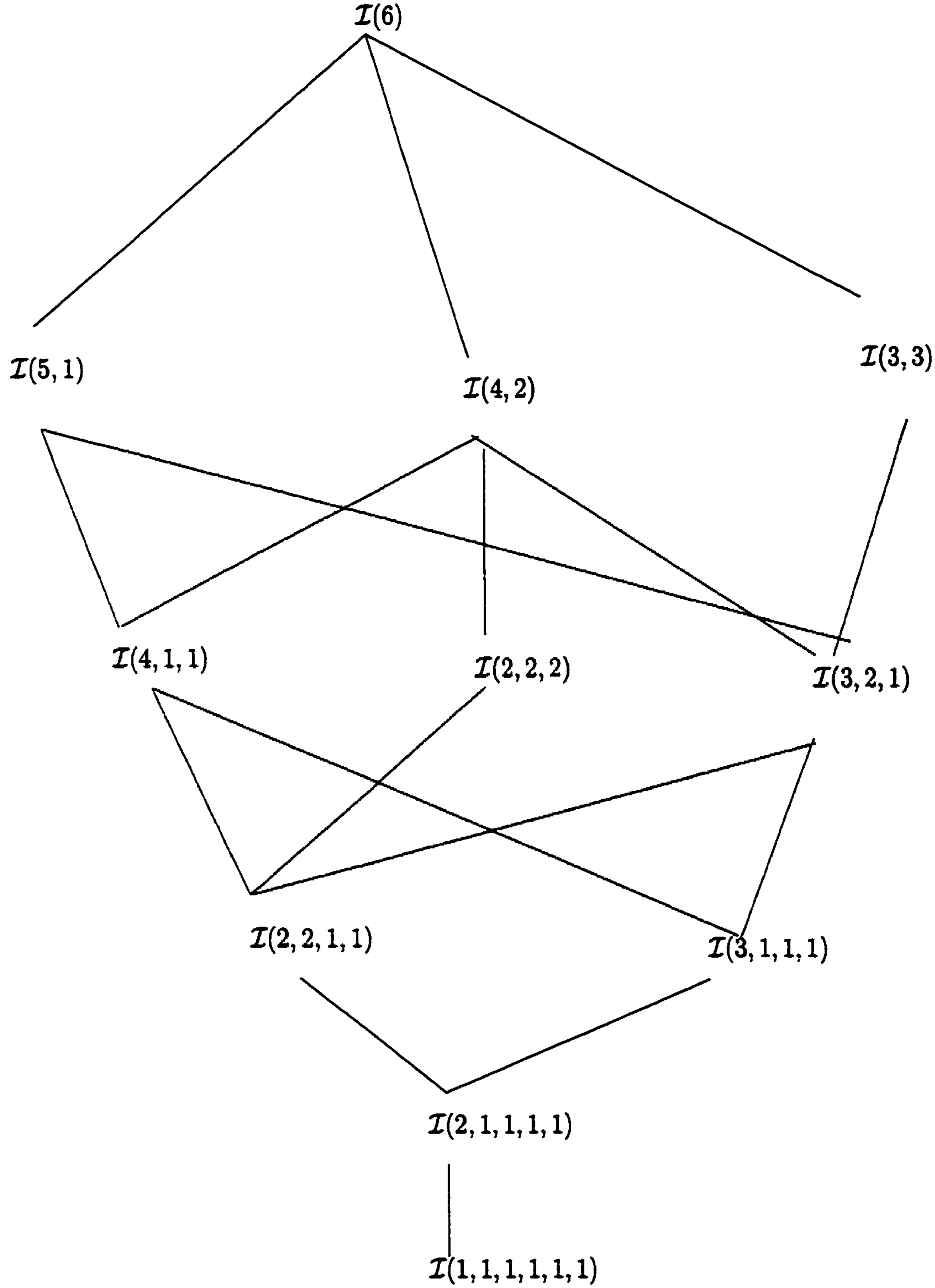

Figure 4.2: The ideals for $r = 6$

Since each of the closed sets $\mathcal{F}(m_1, \ldots, m_s)$ is irreducible, the ideals $\mathcal{I}(m_1, \ldots, m_s)$ corresponding to these closed sets are prime.

These ideals could be described by finding polynomial ideals whose radical is $\mathcal{I}(m_1, \ldots, m_s)$. Instead of just looking for a set of generators for ideals whose radical is $\mathcal{I}(m_1, \ldots, m_s)$, we are interested in the following problem:

Find covariants whose vanishing for a binary form f is a necessary and sufficient condition that f has the form $l_1^{m_1} \ldots l_s^{m_s}$ for some linear forms $l_1, \ldots, l_s$.

Since every binary form is a product of linear forms (Reference [Bôcher 1964] page 188), $\mathcal{F}(1, \ldots, 1) = \mathbb{C}[X_0, X_1]_r$, and hence $\mathcal{I}(1, \ldots, 1) = \{0\}$.

Recall the following facts about the discriminant:

1. ([Bôcher 1964] p. 237) A necessary and sufficient condition that the binary form $f(X_0, X_1)$ have a multiple linear factor is that discriminant of f, i.e the resultant of the two binary forms $\frac{\partial f}{\partial X_0}$ and $\frac{\partial f}{\partial X_1}$, vanishes.

2. ([Bôcher 1964] p. 259) The discriminant of a binary form is an irreducible polynomial function.

These facts prove the following:

Lemma 4.6 $\mathcal{I}(2, 1, \ldots, 1)$ *is the principal prime ideal generated by the discriminant.*

Thus we have

$$\mathcal{I}(2, 1, \ldots, 1) = \langle \textit{discriminant} \rangle.$$

The following theorems of Hilbert provide solutions to the above problem for some partitions.

Theorem 4.7 *([Hilbert 1893]) Let $f(X_0, X_1)$ have degree r. Then f has a linear factor of multiplicity $> \frac{r}{2}$ if and only if every invariant vanishes for f.*

From Theorem 4.7 we have when r is odd, $\mathcal{I}(\lceil \frac{r}{2} \rceil, 1, \ldots, 1)$ is the radical of the ideal generated by all the invariants . When r is even, $\mathcal{I}(\frac{r}{2}+1, 1, \ldots, 1)$ is the radical of the ideal generated by all the invariants.

These ideals are the radicals of coefficient ideals of covariants (invariants).

Definition 4.8 *Let $r = \mu\nu$ and let f be a binary form of degree r in the variables X_0 and X_1. We think of f as a polynomial in one variable X, i.e.*

$$f = \sum_{i=0}^{r} \binom{r}{i} a_i X^{r-i}.$$

Then define f_i by [1]

$$\frac{\partial^i(f)}{\partial X^i} = \frac{r!}{(r-i)!} f_i$$

The polynomial mapping C_ν from $\mathbb{C}[X_0, X_1]_r$ to $\mathbb{C}[X_0, X_1]_{(r-2)(\nu+1)}$ defined by

$$C_\nu(f) = f_0^{\nu - \frac{1}{\mu} + 1} \Delta^{\nu+1} f_0^{\frac{1}{\mu}},$$

where $\Delta = r f_1 \dfrac{\partial}{\partial f_0} + \ldots + f_r \dfrac{\partial}{\partial f_{r-1}}$, is a covariant of weight $\nu + 1$.

Theorem 4.9 *([Hilbert 1886]) The following are equivalent for a binary form f of degree $r = \mu\nu$ in the variables X_0 and X_1 over the field $\mathbb{C}$:*

1. *There exists a binary form g of degree ν such that $f = g^\mu$.*

[1] *here f_i is same as the f_i defined in the Preliminaries section 2.2 with the assumption that $X_0 = X$ and $X_1 = 1$.*

2. The covariant C_ν (defined above) vanishes for f.

From Theorem 4.9, we have $\mathcal{I}(m,\ldots,m)$ is the radical of the coefficient ideal of the covariant C_ν on $\mathbb{C}[X_0,X_1]_r$, where $r=\mu\nu$.

Some special cases of Theorem 4.9 are:

1. $\mathcal{I}(r)$ is the radical of the coefficient ideal of the Hessian.

2. When r is even, $\mathcal{I}(\frac{r}{2},\frac{r}{2})$ is the radical of the coefficient ideal of the Jacobian.

4.3.1 The ideal $\mathcal{I}(r-m,m)$

In this section, we prove our main theorem about a covariant generator for the two part partition ideal $\mathcal{I}(r-m,m)$. We explore this at the end of this section, after establishing some necessary technical lemmas. First we shall need the following definition:

Definition 4.10 *Let $\mathbb{K}$ be field with char($\mathbb{K}$) $=0$. A $\mathbb{K}$-derivation δ of an associative algebra A over $\mathbb{K}$ is a $\mathbb{K}$-linear map from A into itself satisfying the following condition:*

$$\delta(ab)=a\delta(b)+b\delta(a),\ \textit{for all } a,b\in A.$$

The kernel of δ is a subfield called the field of constants of the derivation δ.

The formal partial derivative ∂_0 is a $\mathbb{C}(X_1)$-derivation on the field of rational fractions $\mathbb{C}(X_0,X_1)$, and the kernel of ∂_0 is the field of fractions $\mathbb{C}(X_1)$.

The first result we require is

Lemma 4.11 *([Farahat]) Let $r>2$, $0<m<r$, and let f be a binary form of degree r in the variables X_0 and X_1 over the field of complex numbers $\mathbb{C}$. Then the following conditions are equivalent:*

1. *f has the form $l_1^{r-m}l_2^m$ for some linear forms l_1 and l_2.*
2. *There exist linear forms l_1 and l_2 such that f satisfies the following differential equations*
$$\frac{\partial_0(f)}{f}=(r-m)\frac{\partial_0(l_1)}{l_1}+m\frac{\partial_0(l_2)}{l_2},$$
$$\partial_0\left(\frac{\partial_0(f)}{f}\right)=-(r-m)\left(\frac{\partial_0(l_1)}{l_1}\right)^2-m\left(\frac{\partial_0(l_2)}{l_2}\right)^2,$$
$$\partial_0^2\left(\frac{\partial_0(f)}{f}\right)=2(r-m)\left(\frac{\partial_0(l_1)}{l_1}\right)^3+2m\left(\frac{\partial_0(l_2)}{l_2}\right)^3.$$
3. *There exist linear forms l_1 and l_2 such that f satisfies the following differential equations*
$$\frac{\partial_1(f)}{f}=(r-m)\frac{\partial_1(l_1)}{l_1}+m\frac{\partial_1(l_2)}{l_2},$$
$$\partial_1\left(\frac{\partial_1(f)}{f}\right)=-(r-m)\left(\frac{\partial_1(l_1)}{l_1}\right)^2-m\left(\frac{\partial_1(l_2)}{l_2}\right)^2,$$
$$\partial_1^2\left(\frac{\partial_1(f)}{f}\right)=2(r-m)\left(\frac{\partial_1(l_1)}{l_1}\right)^3+2m\left(\frac{\partial_1(l_2)}{l_2}\right)^3.$$

Proof: To prove (1) $\Rightarrow$ (2), assume that $f=l_1^{r-m}l_2^m$, for some linear forms l_1 and l_2. By logarithmic differentiation with respect to X_0, we get

$$\frac{\partial_0(f)}{f}=(r-m)\frac{\partial_0(l_1)}{l_1}+m\frac{\partial_0(l_2)}{l_2}.$$

Now by repeated differentiation with respect to X_0, we get

$$\partial_0\left(\frac{\partial_0(f)}{f}\right) = -(r-m)\left(\frac{\partial_0(l_1)}{l_1}\right)^2 - m\left(\frac{\partial_0(l_2)}{l_2}\right)^2,$$

$$\partial_0^2\left(\frac{\partial_0(f)}{f}\right) = 2(r-m)\left(\frac{\partial_0(l_1)}{l_1}\right)^3 + 2m\left(\frac{\partial_0(l_2)}{l_2}\right)^3.$$

To prove (2) $\Rightarrow$ (1), assume that there exist linear forms l_1 and l_2 such that f satisfies the following differential equation

$$\frac{\partial_0(f)}{f} \;=\; (r-m)\frac{\partial_0(l_1)}{l_1} + m\frac{\partial_0(l_2)}{l_2}. \tag{4.2}$$

We shall show that f has the form $l_1^{r-m}l_2^m$ by first showing that the partial derivative with respect to X_0 of $\frac{l_1^{r-m}l_2^m}{f}$ is zero. By the quotient rule we have

$$\partial_0\left(\frac{l_1^{r-m}l_2^m}{f}\right) = \frac{f((r-m)l_1^{(r-m-1)}l_2^m\partial_0 l_1 + m\,l_1^{r-m}l_2^{m-1}\partial_0 l_2) - l_1^{(r-m)}l_2^m\,\partial_0(f)}{f^2}.$$

Factoring out $\frac{l_1^{r-m}l_2^m}{f}$, we get

$$\partial_0\left(\frac{l_1^{r-m}l_2^m}{f}\right) = \frac{l_1^{r-m}l_2^m}{f}\left((r-m)\frac{\partial_0(l_1)}{l_1} + m\frac{\partial_0(l_2)}{l_2} - \frac{\partial_0(f)}{f}\right).$$

By the equation 4.2, we have

$$\partial_0\left(\frac{l_1^{r-m}l_2^m}{f}\right) = 0.$$

Hence,

$$\frac{l_1^{r-m}l_2^m}{f} = g,$$

where g is an element in the field $\mathbb{C}(X_1)$.

We shall show that g is a constant in $\mathbb{C}$. Suppose that

$$g = \frac{q}{p}$$

where p, q belong to $\mathbb{C}[X_1]$ and have no common factors.

Then

$$q\,f = p l_1^{r-m} l_2^m.$$

By comparing the degrees, we get the degree of p is same as the degree of q.

If this degree is zero, then p and q are in $\mathbb{C}$ and g is constant. Hence the result.

Otherwise p and q are not in $\mathbb{C}$. We know that every polynomial in $\mathbb{C}[X_1]$, of positive degree, factors completely in $\mathbb{C}[X_1]$ into polynomials of degree 1. We suppose that for $k \geq 1$,

$$p = \prod_{i=1}^{k}(\gamma_i X_1 + \eta_i),$$

and

$$q = \prod_{i=1}^{k}(\tilde{\gamma}_i X_1 + \tilde{\eta}_i),$$

where $\gamma_i, \eta_i, \tilde{\gamma}_i, \tilde{\eta}_i \in \mathbb{C}$, for all $i = 1, \ldots, l$.

Let $1 \leq i \leq k$. The irreducible factor of q, $(\tilde{\gamma}_i X_1 + \tilde{\eta}_i)$ divides $q\,f$ in $\mathbb{C}[X_0, X_1]$. Therefore $(\tilde{\gamma}_i X_1 + \tilde{\eta}_i)$ divides $l_1^{(r-m)} l_2^m$ in $\mathbb{C}[X_0, X_1]$. Hence $(\tilde{\gamma}_i X_1 + \tilde{\eta}_i)$ divides l_1 or l_2 in $\mathbb{C}[X_0, X_1]$. If $(\tilde{\gamma}_i X_1 + \tilde{\eta}_i)$ divides the linear form l_1, $\tilde{\eta}_i = 0$. Hence $\tilde{\eta}_i = 0$ for all

$1 \leq i \leq k$. Thus

$$q = \tilde{\gamma}_1 \ldots \tilde{\gamma}_k X_1^k.$$

By a similar argument we can show that

$$p = \gamma_1 \ldots \gamma_k X_1^k.$$

Since $k \geq 1$, p and q have at least one common factor X_1. This contradicts the fact that p and q have no common factors.

Therefore, p and q must be constants. Hence f has the form $l_1^{r-m} l_2^m$.

By a similar argument we can prove that (1) is equivalent to (3). □

It is proven in [Sturmfels 1998] on page 31 that we need a system of k homogeneous polynomial equations in order to eliminate k variables. That is the reason why we include three differential equations in the 2^{nd} statement of Lemma 4.11, to eliminate the variables $\frac{\partial_0(l_1)}{l_1}$ and $\frac{\partial_0(l_2)}{l_2}$ from the non-homogeneous polynomial equation 4.2, even though in the proof that (2) implies (1) we only needed the 1^{st} differential equation in the 2^{nd} statement of Lemma 4.11.

Thus we need to eliminate p, q, s in the following system of equations:

$$\begin{aligned} (r-m)a + mb &= p \\ (r-m)a^2 + mb^2 &= -q \\ (r-m)a^3 + mb^3 &= s. \end{aligned}$$

In terms of elimination theory, we have the following problem:

Let A, B, P, Q, S be algebraically independent indeterminates over a field F, let

$r > 2$, $0 < m < r$, and let I be the ideal in the ring $F[A,B,P,Q,S]$ generated by the polynomials

$$(r-m)A + mB - P, (r-m)A^2 + mB^2 + Q, (r-m)A^3 + mB^3 - S.$$

Then compute the intersection $I \cap F[P,Q,S]$ of the ideal I and the polynomial ring $F[P,Q,S]$.

Next we shall explain briefly how to do elimination by using a Gröbner basis.

Definition 4.12 *Let J be a polynomial ideal of $F[X_1,\ldots,X_n]$ other than $\{0\}$.*

1. *We denote by $LT(J)$, the set of leading terms of elements of J. Thus*

$$LT(J) = \{LT(h) | h \in J\},$$

where leading term $LT(h)$ of h is the term having the monomial which is ranked highest under lexicographic order of all monomials which have nonzero coefficients in h.

2. *We denote by $\langle LT(J)\rangle$ the ideal generated by the elements of $LT(J)$.*

3. *A finite sequence $(g_1,\ldots,g_t)$ of elements of the ideal J forms a Gröbner basis for J if*

$$\langle LT(g_1),\ldots,LT(g_t)\rangle = \langle LT(J)\rangle.$$

In fact a Gröbner basis is a basis for the ideal J.

4. *Let $J = \langle f_1,\ldots,f_s\rangle$, where $f_1,\ldots,f_s$ are in the polynomial ring $F[X_1,\ldots,X_n]$ in the algebraically independent indeterminates $X_1,\ldots,X_n$. Then the l^{th} elim-*

ination ideal J_l *is the ideal of* $\mathbb{C}[X_{l+1},\dots,X_n]$ *is defined by*

$$J_l = I \cap F[X_{l+1},\dots,X_n].$$

The following theorem provides a basis for J_k.

Theorem 4.13 *([Cox, Little, O'Shea 1996] p.113)(Elimination Theorem)*

Let F *be a field with* $char(F) = 0$. *If* $J = \langle f_1,\dots,f_s\rangle \subset F[X_1,\dots,X_n]$ *is an ideal and* $G = (g_1,\dots,g_t)$ *is a Gröbner basis for* J *for lexicographic order with* $X_1 > \dots > X_n$, *then for each* k *between* 1 *and* $n-1$, *the set*

$$G \cap F[X_{k+1},\dots,X_n]$$

is a Gröbner basis for the elimination ideal J_k.

A related question is answered by the extension theorem: given a point $(a_2,\dots,a_n) \in \mathbf{V}(J_1)$, when can we find a value a_1 such that $(a_1,\dots,a_n) \in \mathbf{V}(J)$?

Theorem 4.14 *([Cox, Little, O'Shea 1996] p.115)(Extension Theorem)*

Let F *be an algebraically closed field with* $char(F) = 0$. *Given*

$$J = \langle f_1,\dots,f_s\rangle \subset F[X_1,\dots,X_n],$$

we get the elimination ideal $J_1 = J \cap F[X_2,\dots,X_n]$. *For each* $1 \le i \le s$, *write* f_i *in the form*

$$f_i = \bar{f}_i(X_2,\dots,X_n)X_1^{N_i} + \text{ terms in which } X_1 \text{ has degree } < N_i,$$

where $N_i \geq 0$ and $\bar{f}_i \in F[X_2,\ldots,X_n]$ is non-zero. Now let $(a_2,\ldots,a_n) \in \mathbf{V}(J_1)$. If $\bar{f}_i(a_2,\ldots,a_n) \neq 0$ for at least one $1 \leq i \leq s$, then there exists $a_1 \in F$ such that $(a_1,\ldots,a_n) \in \mathbf{V}(J)$.

Remark 4.15 *The ideal I_{l+1} is the first elimination ideal of I_l. This allows us to use the extension theorem multiple times when eliminating more than one variable.*

See [Cox, Little, O'Shea 1996] for further details.

Lemma 4.16 *Let A, B, P, Q, S be algebraically independent indeterminates over a field F, let $r > 2$, $0 < m < r$, and let I be the ideal in the ring $F[A,B,P,Q,S]$, generated by the polynomials*

$$(r-m)A + mB - P, (r-m)A^2 + mB^2 + Q, (r-m)A^3 + mB^3 - S.$$

Then the intersection $I \cap F[P,Q,S]$ is a principal ideal in $F[P,Q,S]$ generated by the polynomial G, where

$$\begin{aligned} G &= 3rQP^4 + (4mr - 4m^2)SP^3 + (3m^2 + 3r^2 - 3mr)Q^2P^2 \\ &\quad + P^6 + (mr^3 - m^2r^2)S^2 + (-4mr^2 + 4m^2r + r^3)Q^3 \\ &\quad + (-6m^2r + 6mr^2)QSP. \end{aligned} \tag{4.3}$$

Furthermore, we have the following

1. *If there exist $p,q,s \in F$ such that the zero set $\mathbf{V}(I)(\subset F_{1,5})$ of I contains a point whose last three coordinates are p,q,s then G vanishes for $P=p, Q=q, S=s$ in F.*

2. *If F is algebraically closed and G vanishes for some $P = p, Q = q, S = s$ in F, then $\mathbf{V}(I)$ contains a point whose last three coordinates are p, q, s.*

<u>Proof</u>:

A Gröbner basis for the ideal I with respect to lexicographic order, computed using Maple is

$$\begin{aligned}
G &= 3rQP^4 + (4mr - 4m^2)SP^3 + (3m^2 + 3r^2 - 3mr)Q^2P^2 + P^6 + \\
&\quad (mr^3 - m^2r^2)S^2 + (-4mr^2 + 4m^2r + r^3)Q^3 + (-6m^2r + 6mr^2)QSP, \\
G2 &= (-mr^2 + 2m^2r)Q^3B + (-mr^3 + 2m^2r^2)S^2B - QP^5 + rSP^4 - 2rQ^2P^3 + \\
&\quad (-4mr + 2r^2 + 4m^2)QSP^2 + (-3m^2 + 2mr - r^2)PQ^3 + (3r^2m - 4rm^2)S^2P \\
&\quad + (5rm^2 - 5r^2m + r^3)SQ^2, \\
G3 &= (-mr^2 + 2m^2r)SBP + (mr^2 - 2m^2r)Q^2B + P^5 + 2rQP^3 + \\
&\quad (3mr - 4m^2)SP^2 + (3m^2 - 2mr + r^2)Q^2P + (mr^2 - m^2r)QS, \\
G4 &= (2m^2r - mr^2)SB + (2m^2 - mr)QBP + (-4m^2 + 3mr)PS + \\
&\quad (-4mr + r^2 + 4m^2)Q^2 + (-rm + 2r)QP^2 + P^4, \\
G5 &= (-r^2 + rm)S + (2rm - r^2)QB + (m - 2r)PQ + (-r + 2m)BP^2 - P^3, \\
G6 &= rmB^2 + (r - m)Q - 2mBP + P^2, \\
G7 &= (-r + m)A - mB + P.
\end{aligned}$$

By the Elimination Theorem 4.13, we obtain

$$I \cap F(B, P, Q, S) = I_1 = \langle G, G2, G3, G4, G5, G6 \rangle,$$

$$I \cap F(P,Q,S) = I_2 = \langle G \rangle.$$

Hence if there exist $p,q,s \in F$ such that $\mathbf{V}(I)$ contains a point whose last three coordinates are p,q,s, then G vanishes for $P=p, Q=q, S=s$.

To prove the converse, assume that G vanishes for $P=p, Q=q, S=s$, then $(p,q,s) \in \mathbf{V}(I_2)$. The idea is to extend (p,q,s) one coordinate at a time: first to (b,p,q,s), then to (a,b,p,q,s). Since the field F is algebraically closed, we can use the Extension Theorem 4.14 at each step. The crucial observation is that I_2 is the first elimination ideal of I_1. The coefficient of B^2 in $G6$ is rm, which is non zero. Therefore by the Extension Theorem 4.14, there exists $b \in F$ such that $(b,p,q,s) \in \mathbf{V}(I_1)$.

The next step is to go from I_1 to I. Since $G7 \in I$ and the coefficient $m-r$ of A in $G7$ is non zero, there exists $a \in F$ such that $(a,b,p,q,s) \in \mathbf{V}(I)$. Hence the result. □

Remark 4.17 *The above proof may strike the reader as lacking in conviction due to reliance on machine calculations. However it is also possible to find the polynomial* G, *from the following equations*

$$\begin{aligned}(r-m)a+mb &= Pc\\(r-m)a^2+mb^2 &= -Qc^2\\(r-m)a^3+mb^3 &= Sc^3,\end{aligned}$$

by eliminating one variable at a time, by hand.

Lemma 4.18 *Let $r > 2$, $0 < m < r$ and f be a binary form of degree r in the variables X_0 and X_1 over the complex field $\mathbb{C}$. With the following substitution*

$$P = \frac{\partial_0(f)}{f}, Q = \partial_0\left(\frac{\partial_0(f)}{f}\right), S = \left(\partial_0^2\left(\frac{\partial_0(f)}{f}\right)\right)/2,$$

G (stated in 4.3) becomes $\frac{1}{4f^6}g(f)$, where

$$\begin{aligned}
g(f) \;=\; & \{(16\,m^2\,r + 4\,r^3 - 16\,m\,r^2)\,(\partial_0^2 f)^3\,f^3 + (r^3\,m - m^2\,r^2)\,(\partial_0^3\,f)^2\,f^4 \\
& +(12\,r - 12\,r^3\,m + 12\,r^3 + 12\,m^2\,r^2 + 12\,m\,r^2 - 24\,r^2 - 12\,m^2\,r)\,(\partial_0^2 f)(\partial_0 f)^4\,f \\
& +(-12\,m^2\,r - 12\,m\,r - 9\,m^2\,r^2 - 12\,r^3 + 12\,m\,r^2 + 12\,m^2 + 9\,r^3\,m + 12\,r^2) \\
& (\partial_0^2 f)^2\,(\partial_0 f)^2\,f^2 \\
& +(8\,m\,r - 4\,m^2\,r^2 + 4\,r^3\,m + 12\,m^2\,r - 12\,m\,r^2 - 8\,m^2)\,(\partial_0^3 f)\,(\partial_0\,f)^3 f^2 \\
& +(6\,m^2\,r^2 + 12\,m\,r^2 - 6\,r^3\,m - 12\,m^2\,r)\,(\partial_0^2 f)\,(\partial_0 f)\,(\partial_0^3 f)\,f^3 \\
& +(4\,m\,r - 4\,r^3 + 8\,m^2\,r - 4\,m^2\,r^2 + 4\,r^3\,m + 12\,r^2 - 12\,r - 4\,m^2 + 4 - 8\,m\,r^2) \\
& (\partial_0 f)^6\}\,. \qquad (4.4)
\end{aligned}$$

This is a straightforward tedious calculation, which was done by Maple. The work sheet is attached in Appendix C. □

On the other hand, with the derivatives with respect to X_1, we have the following result.

Lemma 4.19 *Let $0 < m < r$ and f be a binary form of degree r in the variables X_0*

and X_1 over the complex field $\mathbb{C}$. With the following substitution

$$P = \frac{\partial_1(f)}{f}, Q = \partial_1\left(\frac{\partial_1(f)}{f}\right), S = \left(\partial_1^2\left(\frac{\partial_1(f)}{f}\right)\right)/2$$

G(stated in 4.3) becomes $\frac{1}{4f^6}\bar{g}(f) \in \mathbb{C}(X_0, X_1)$, where

$$\begin{aligned}
\bar{g}(f) \;=\; & \{(16\,m^2\,r + 4\,r^3 - 16\,m\,r^2)\,(\partial_1^2 f)^3\,f^3 + (r^3\,m - m^2\,r^2)\,(\partial_1^3\,f)^2\,f^4 \\
& +(12\,r - 12\,r^3\,m + 12\,r^3 + 12\,m^2\,r^2 + 12\,m\,r^2 - 24\,r^2 - 12\,m^2\,r)\,(\partial_1^2 f)(\partial_1 f)^4\,f \\
& +(-12\,m^2\,r - 12\,m\,r - 9\,m^2\,r^2 - 12\,r^3 + 12\,m\,r^2 + 12\,m^2 + 9\,r^3\,m + 12\,r^2) \\
& (\partial_1^2 f)^2\,(\partial_1 f)^2\,f^2 \\
& +(8\,m\,r - 4\,m^2\,r^2 + 4\,r^3\,m + 12\,m^2\,r - 12\,m\,r^2 - 8\,m^2)\,(\partial_1^3 f)\,(\partial_1\,f)^3\,f^2 \\
& +(6\,m^2\,r^2 + 12\,m\,r^2 - 6\,r^3\,m - 12\,m^2\,r)\,(\partial_1^2 f)\,(\partial_1 f)\,(\partial_1^3 f)\,f^3 \\
& +(4\,m\,r - 4\,r^3 + 8\,m^2\,r - 4\,m^2\,r^2 + 4\,r^3\,m + 12\,r^2 - 12\,r - 4\,m^2 + 4 - 8\,m\,r^2) \\
& (\partial_1 f)^6\}. \qquad (4.5)
\end{aligned}$$

Lemma 4.20 *The following are equivalent for a binary form f of degree $r\,(>2)$ in the variables X_0, X_1 over the complex field $\mathbb{C}$.*

1. *g stated in 4.4, vanishes for f.*
2. *(2^{nd} statement of Lemma 4.11) There exist linear forms l_1 and l_2 such that f satisfies the following differential equations*

$$\frac{\partial_0(f)}{f} = (r-m)\frac{\partial_0(l_1)}{l_1} + m\frac{\partial_0(l_2)}{l_2},$$

$$\partial_0\left(\frac{\partial_0(f)}{f}\right) = -(r-m)\left(\frac{\partial_0(l_1)}{l_1}\right)^2 - m\left(\frac{\partial_0(l_2)}{l_2}\right)^2,$$

$$\partial_0^2\left(\frac{\partial_0(f)}{f}\right) = 2(r-m)\left(\frac{\partial_0(l_1)}{l_1}\right)^3 + 2m\left(\frac{\partial_0(l_2)}{l_2}\right)^3.$$

Proof:

(2) $\Rightarrow$ (1) : Assume that statement (2) is true. Since f is non zero, this implication follows from Lemma 4.16.

(1) $\Rightarrow$ (2) : Assume that statement (1) is true. Suppose F is an algebraic closure of the field $\mathbb{C}(X_0, X_1)$. Since ∂_0 is a $\mathbb{C}$-derivation on $\mathbb{C}(X_0, X_1)$ and F is an algebraic extension field of the field $\mathbb{C}(X_0, X_1)$, there exists a $\mathbb{C}$-derivation extension Ω on F such that

$$\Omega|\mathbb{C}(X_0, X_1) = \partial_0$$

(Reference [Jacobson 1964] pages 168-170). From Lemma 4.16, there exist $a, b \in F$ such that

$$\frac{\partial_0(f)}{f} = (r-m)a + mb, \tag{4.6}$$

$$\partial_0\left(\frac{\partial_0(f)}{f}\right) = -(r-m)a^2 - mb^2, \tag{4.7}$$

$$\partial_0^2\left(\frac{\partial_0(f)}{f}\right) = 2(r-m)a^3 + 2mb^3. \tag{4.8}$$

We shall show that $\Omega(a) = -a^2, \Omega(b) = -b^2$. By applying Ω to the differential equation 4.6, and then comparing with differential equation 4.7, we obtain

$$(r-m)(\Omega(a) + a^2) + m(\Omega(b) + b^2) = 0. \tag{4.9}$$

Also by applying Ω to the differential equation 4.7, and then comparing with the differential equation 4.8, we obtain

$$(r-m)a(\Omega(a)+a^2)+mb(\Omega(b)+b^2) \;=\; 0. \tag{4.10}$$

From the equations 4.9 and 4.10, we have the following system of homogeneous linear equations,

$$\begin{pmatrix} (r-m) & m \\ (r-m)a & mb \end{pmatrix}\begin{pmatrix} \Omega(a)+a^2 \\ \Omega(b)+b^2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The determinant of the coefficient matrix is $(r-m)\,m\,(b-a)$. We know that $(r-m)m \neq 0$. If $b \neq a$, then the coefficient matrix is invertible, so $\Omega(a) = -a^2$, and $\Omega(b) = -b^2$. On the other hand, if $a = b$ then by the equation 4.9,

$$r(\Omega(a)+a^2) \;=\; 0.$$

Since $r \neq 0$, $\Omega(a) = -a^2$ and $\Omega(b) = -b^2$.

In order to find linear forms l_1 and l_2 in $\mathbb{C}[X_0, X_1]$ satisfying the differential equations in the statement of Lemma 4.11, we will consider three cases.

<u>**Case 1:**</u>

If a and b are zero, from equation 4.6 we have $\dfrac{\partial_0(f)}{f} = 0$. Thus $\partial_0(f) = 0$. We choose $l_1 = X_1$ and $l_2 = X_1$, hence the result.

<u>**Case 2:**</u>

Suppose $b = 0$ and $a \neq 0$, then from equation 4.6 we have

$$\frac{\partial_0(f)}{f} = (r-m)a.$$

Then a is in the field $\mathbb{C}(X_0, X_1)$ and

$$\partial_0\left(\frac{1}{a}\right) = -a^{-2}\,\partial_0 a = (-a^{-2})\,(-a^2) = 1.$$

We shall show that $\frac{1}{a}$ is a linear form in the variables X_0 and X_1 over the field of complex numbers $\mathbb{C}$. Since $\partial_0\left(\frac{1}{a} - X_0\right) = 0$, $\frac{1}{a} - X_0 \in ker\partial_0 = \mathbb{C}(X_1)$. Thus

$$\frac{1}{a} - X_0 = h, \text{ for some } h \text{ in the field } \mathbb{C}(X_1).$$

We shall show that $h \in \mathbb{C}X_1$.

From equation 4.6, we have

$$\frac{\partial_0(f)}{f} \;=\; (r-m)\frac{1}{X_0+h}.$$

Which implies,

$$\partial_0(f)(X_0+h) = (r-m)f. \tag{4.11}$$

We know that X_0 and X_1 are algebraically independent over $\mathbb{C}$, therefore X_0 is transcendental over the field $\mathbb{C}(X_1)$.

Since $X_0 + h$ is a polynomial of degree 1 in X_0 over the field $\mathbb{C}(X_1)$, $X_0 + h$ is an irreducible polynomial in the polynomial ring $\mathbb{C}(X_1)[X_0]$. Consider f as a polynomial in the polynomial ring $\mathbb{C}(X_1)[X_0]$. The crucial observation is that every irreducible linear factor of f in the polynomial ring $\mathbb{C}[X_1][X_0]$, is also irreducible linear factor as a polynomial in the polynomial ring $\mathbb{C}(X_1)[X_0]$.

Since the irreducible polynomial X_0+h divides the polynomial $(X_0+h)\,\partial_0 f$, it follows from equation 4.11 that X_0+h divides f in the polynomial ring $\mathbb{C}(X_1)[X_0]$. Hence X_0+h divides some irreducible linear factor of f in the polynomial ring $\mathbb{C}(X_1)[X_0]$, (say) $\alpha X_0+\beta X_1$, where $\alpha,\beta\in\mathbb{C}$. Therefore, $\alpha(X_0+h)=(\alpha X_0+\beta X_1)$. Notice that if $\alpha=0$ then $\beta=0$. This contradicts the fact that f is a binary form. Therefore $\alpha\neq 0$. Hence X_0+h is a linear form in $\mathbb{C}[X_0,X_1]$. Thus $\frac{1}{a}$ is a linear form in $\mathbb{C}(X_0,X_1)$.

In this case we choose $l_1=\frac{1}{a}$ and $l_2=X_1$, hence the result.

<u>Case 3:</u>

Suppose a and b are non zero. Since Ω is a derivation, we have

$$0=\Omega(1)=\Omega\left(a\frac{1}{a}\right)=a^{-1}\,\Omega(a)+a\,\Omega\left(\frac{1}{a}\right)=a^{-1}(-a^2)+a\,\Omega\left(\frac{1}{a}\right).$$

Hence, $\Omega\left(\frac{1}{a}\right)=1$. Therefore

$$\Omega\left(\frac{1}{a}-X_0\right)=0.$$

Similarly we have, $\Omega\left(\frac{1}{b}-X_0\right)=0$. Hence,

$$\frac{1}{a}-X_0=h,\ \frac{1}{b}-X_0=j,\ \text{ for some } h,j \text{ in the field } ker\,\Omega.$$

For convenience we will denote $ker\,\Omega$ as L. Since Ω is the extension of ∂_0, L is algebraic over $\mathbb{C}(X_1)$. If X_0 is algebraic over L, then X_0 is algebraic over the field $\mathbb{C}(X_1)$. This contradicts the fact that X_0 and X_1 are algebraically independent. Therefore, X_0 is transcendental over the field L.

The figure fig. 4.3 shows the various field extensions involved in this discussion.

Now we have,

$$\frac{\partial_0(f)}{f} = (r-m)\frac{1}{X_0+h} + m\frac{1}{X_0+j}.$$

Therefore,

$$\partial_0(f)(X_0+h)(X_0+j) = (rX_0+(r-m)j+mh)f. \tag{4.12}$$

The binary form f is in the polynomial ring $\mathbb{C}[X_o,X_1]$, and has a factorization

$$f = \prod_{i=1}^{r}(\alpha_i X_0 + \beta_i X_1),$$

where $\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_r \in \mathbb{C}$.

For each $1 \leq i \leq r$, $(\alpha_i X_0 + \beta_i X_1)$ is an irreducible in the polynomial ring $\mathbb{C}[X_o,X_1]$.

Consider f as a polynomial in the polynomial ring $L[X_0]$. We claim that each irreducible factor $(\alpha_i X_0 + \beta_i X_1), i = 1,\ldots,r$ of f in the polynomial ring $\mathbb{C}[X_o,X_1]$, is also irreducible as a polynomial in the polynomial ring $L[X_0]$. Assume that

$$(\alpha_i X_0 + \beta_i X_1) = (a_0 + a_1X_0 + \ldots + a_kX_0^k)(b_0 + b_1X_0 + \ldots + b_sX_0^s),$$

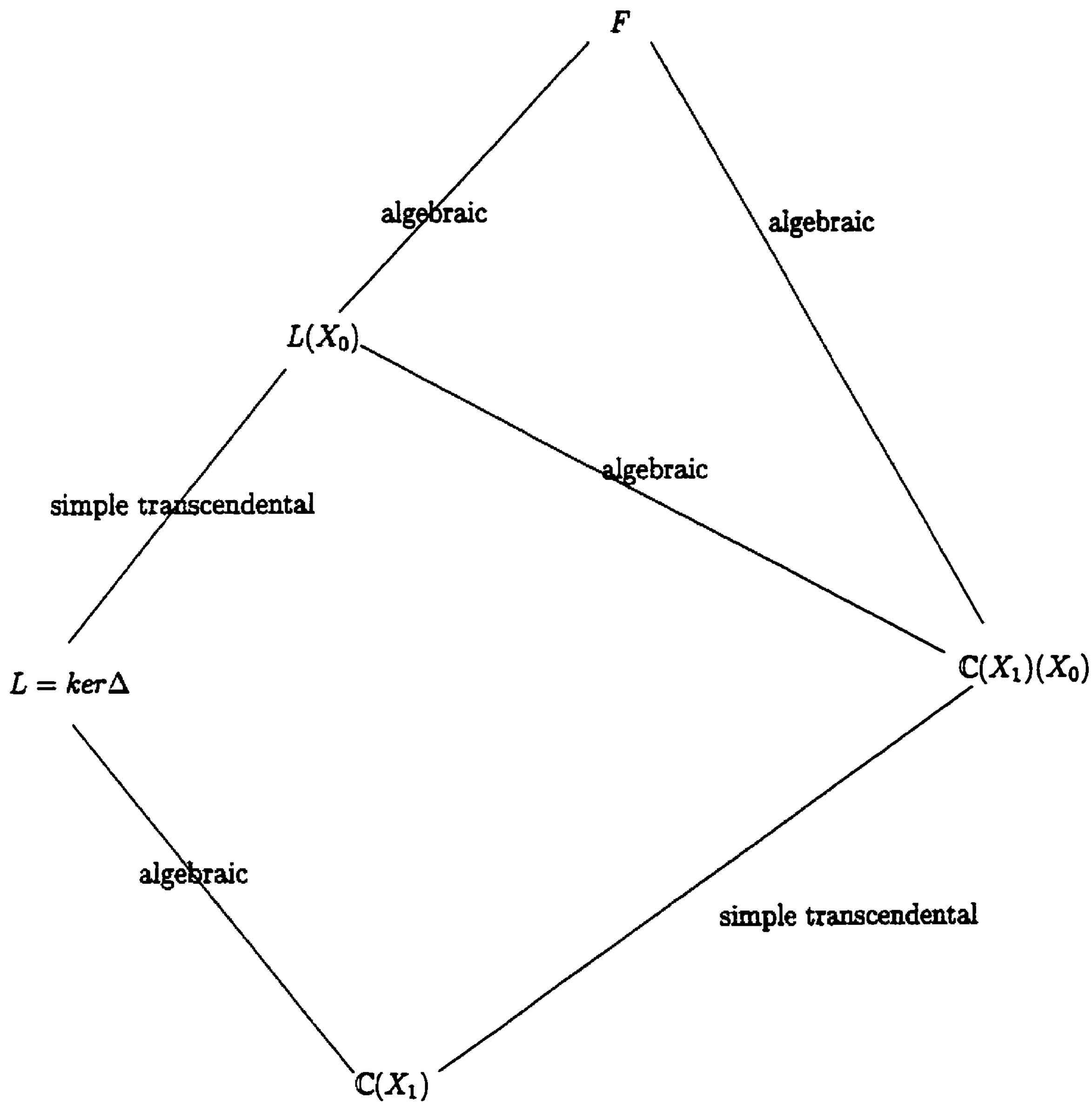

Figure 4.3: Field extensions

where $a_0,\ldots,a_k,b_0,\ldots,b_s \in L$. Then by equating the leading coefficients, $k+s=1$. Without loss of generality we may assume, $k=0$ and $s=1$. Thus

$$(\alpha_i X_0 + \beta_i X_1) = a_0(b_0 + b_1 X_0),$$

where $a_0, b_0, b_1 \in L$ and $\alpha_i, \beta_i \in \mathbb{C}$. Hence $(\alpha_i X_0 + \beta_i X_1)$ is irreducible in the polynomial ring $L[X_0]$, where $\alpha_i, \beta_i \in \mathbb{C}$.

Since $L[X_0]$ is a unique factorization domain, f has the factorization

$$f = \prod_{i=1}^{r}(\alpha_i X_0 + \beta_i X_1),$$

where $\alpha_1,\ldots,\alpha_r,\beta_1,\ldots,\beta_r \in \mathbb{C}$, in $L[X_0]$.

Consider the polynomials $\partial_0(f), (X_0+h), (X_0+j), (rX_0+(r-m)j+mh), f$ in the variable X_0 over the field L.

Since $X_0+h, X_0+j, rX_0+(r-m)j+mh$ are polynomials of degree 1 in X_0 in the polynomial ring $L[X_0]$, they are irreducible polynomials over the field L. Since the irreducible polynomial X_0+h divides $\partial_0(f)(X_0+h)(X_0+j)$ (the left hand side of equation 4.12) over the field L, X_0+h divides $(rX_0+(r-m)j+mh)f$ (the right hand side of equation 4.12) in the polynomial ring $L[X_0]$. That is, X_0+h divides $rX_0+(r-m)j+mh$ or f in the polynomial ring $L[X_0]$.

Suppose that X_0+h divides $rX_0+(r-m)j+mh$ in the polynomial ring $L[X_0]$. Then

$$(\gamma_0+\ldots+\gamma_k X_0^k)(X_0+h) = (rX_0+(r-m)j+mh),$$

where $\gamma_0,\ldots\gamma_k$ are in the field L. Then by equating the coefficients of the leading term, we have $k = 0$. Therefore, $\gamma_0 = r$ and $j = h$. Which implies

$$\frac{\partial_0(f)}{f} = ra.$$

The result follows from case 2.

Suppose that $X_0 + h$ divides f in the polynomial ring $L[X_0]$. Hence $X_0 + h$ divides some irreducible linear factor of f, say $(\alpha_{i_0} X_0 + \beta_{i_0} X_1)$. Then

$$(\alpha_{i_0} X_0 + \beta_{i_0} X_1) = \alpha_{i_0}(X_0 + h).$$

Notice that if $\alpha_{i_0} = 0$ then $\beta_{i_0} = 0$; this cannot happen because of the fact that f is a binary form. Therefore $\alpha_{i_0} \neq 0$. Hence $X_0 + h$ is a linear form in $\mathbb{C}[X_0, X_1]$.

The proof of $X_0 + j$ is a linear form in $\mathbb{C}[X_0, X_1]$ can be done by the same argument.

Thus in this case we choose $l_1 = \dfrac{1}{a}$ and $l_2 = \dfrac{1}{b}$, hence the result. □

Repeating the same arguments for the partial derivatives with respect to X_1, we get the following result.

Lemma 4.21 *The following are equivalent for a binary form f of degree $r\,(> 2)$ in the variables X_0, X_1.*

1. *$\tilde{g}$ stated in 4.5, vanishes for f.*

2. *(3^{rd} statement of Lemma 4.11)There exist linear forms l_1 and l_2 such that f satisfies the following differential equations;*

$$\frac{\partial_1(f)}{f} = (r-m)\frac{\partial_1(l_1)}{l_1} + m\frac{\partial_1(l_2)}{l_2},$$

$$\partial_1\left(\frac{\partial_1(f)}{f}\right) = -(r-m)\left(\frac{\partial_1(l_1)}{l_1}\right)^2 - m\left(\frac{\partial_1(l_2)}{l_2}\right)^2,$$
$$\partial_1^2\left(\frac{\partial_1(f)}{f}\right) = 2(r-m)\left(\frac{\partial_1(l_1)}{l_1}\right)^3 + 2m\left(\frac{\partial_1(l_2)}{l_2}\right)^3.$$

Combining Lemma 4.11, Lemma 4.20 and Lemma 4.21, we have the following result.

Lemma 4.22 *The following are equivalent for a binary form f of degree $r\,(>2)$ in the variables X_0, X_1 over the complex field $\mathbb{C}$.*

1. *f has the form $l_1^{r-m}l_2^m$ for some linear forms l_1 and l_2.*
2. *g stated in 4.4 vanishes for f.*
3. *$\bar{g}$ stated in 4.5 vanishes for f.*

Now we are ready to state the main theorem.

Theorem 4.23 *Let $r > 2$, $0 < m < r$. Then the prime ideal $\mathcal{I}(r-m,m)$ is the radical of the coefficients ideal of the following covariant*

$$4\,(r-2m)^2\,(r-1)\left\{\frac{\mathcal{H}}{r^2(r-1)^2}\right\}^3 + m\,(r-m)\,(r-2)^2\left\{\frac{\mathfrak{J}}{(-1)r^3(r-1)^2(r-2)}\right\}^2,$$

where $\mathcal{H}$ denotes the Hessian covariant and $\mathfrak{J}$ denotes the Jacobian covariant.

Proof: Let $r > 2$, $0 < m < r$ and f be a binary form of degree r namely,

$$f = \sum_{i=0}^{r}\binom{r}{i}a_iX_0^{r-i}X_1^i.$$

Using the definitions of the closed set $\mathcal{F}(r-m,m)$, its corresponding prime ideal $\mathcal{I}(r-m,m)$, and Lemma 4.22 we have that $\mathcal{I}(r-m,m)$ is the radical of the coefficient ideal of the polynomial mapping $X_1^{-6}g$ (g stated in 4.4) from $\mathbb{C}[X_0,X_1]_r$ to $\mathbb{C}[X_0,X_1]_{6r-12}$. Also $\mathcal{I}(r-m,m)$ is the radical of the coefficient ideal of the polynomial mapping $X_0^{-6}\tilde{g}$ ($\tilde{g}$ stated in 4.5) from $\mathbb{C}[X_0,X_1]_r$ to $\mathbb{C}[X_0,X_1]_{6r-12}$.

We shall show that $\dfrac{X_1^{-6}g}{r^4(r-1)^2}$ and $\dfrac{X_0^{-6}\tilde{g}}{r^4(r-1)^2}$ are the same covariants namely,

$$4(r-2m)^2(r-1)\left\{\frac{\mathcal{H}}{r^2(r-1)^2}\right\}^3+m(r-m)(r-2)^2\left\{\frac{\mathcal{J}}{(-1)r^3(r-1)^2(r-2)}\right\}^2,$$

where $\mathcal{H}$ denotes the Hessian covariant and $\mathcal{J}$ denotes the Jacobian covariant. For that we shall substitute

$$\partial_0^i f=\frac{r!}{(r-i)!}f_i$$

in $g(f)$ (g stated in 4.4) we get

$$\begin{aligned}
g(f) \;=\;& 4r(r-2m)^2\left(\frac{r!}{(r-2)!}\right)^3 f_0^3f_2^3+mr^2(r-m)\left(\frac{r!}{(r-3)!}\right)^2 f_0^4f_3^2\\
&-12r(m-1)(r-1)(r-m-1)\left(\frac{r!}{(r-1)!}\right)^4\left(\frac{r!}{(r-2)!}\right)f_0f_1^4f_2\\
&+\left(-12\,m^2\,r-12\,m\,r-9\,m^2\,r^2-12\,r^3+12\,m\,r^2+12\,m^2+9\,r^3\,m+12\,r^2\right)\\
&\left(\frac{r!}{(r-1)!}\right)^2\left(\frac{r!}{(r-2)!}\right)^2 f_0^2f_1^2f_2^2\\
&+4m(r-1)(r-2)(r-m)\left(\frac{r!}{(r-1)!}\right)^3\left(\frac{r!}{(r-3)!}\right)f_0^2f_1^3f_3\\
&-6mr(r-2)(r-m)\left(\frac{r!}{(r-1)!}\right)\left(\frac{r!}{(r-2)!}\right)\left(\frac{r!}{(r-3)!}\right)f_0^3f_1f_2f_3\\
&+4(m-1)(r-1)^2(r-m-1)\left(\frac{r!}{(r-1)!}\right)^6 f_1^6.
\end{aligned}$$

Thus,

$$\begin{aligned}
g(f) &= 4r^4(r-2m)^2(r-1)^3f_0^3f_2^3 + mr^4(r-m)(r-1)^2(r-2)^2f_0^4f_3^2 \\
&\quad -12r^6(m-1)(r-1)^2(r-m-1)f_0f_1^4f_2 \\
&\quad +r^4(r-1)^2(-12\,m^2\,r-12\,m\,r-9\,m^2\,r^2-12\,r^3+12\,m\,r^2+12\,m^2+9\,r^3\,m+12\,r^2) \\
&\quad f_0^2f_1^2f_2^2 \\
&\quad +4mr^4(r-1)^2(r-2)^2(r-m)f_0^2f_1^3f_3 - 6mr^4(r-2)^2(r-m)(r-1)^2f_0^3f_1f_2f_3 \\
&\quad +4mr^6(m-1)(r-1)^2(r-m-1)f_1^6.
\end{aligned}$$

By taking out common factors we get,

$$\begin{aligned}
g(f) &= r^4(r-1)^2\Big\{4(r-2m)^2(r-1)f_0^3f_2^3 + m(r-m)(r-2)^2f_0^4f_3^2 \\
&\quad -12r^2(m-1)(r-m-1)f_0f_1^4f_2 \\
&\quad +(-12\,m^2\,r-12\,m\,r-9\,m^2\,r^2-12\,r^3+12\,m\,r^2+12\,m^2+9\,r^3\,m+12\,r^2)f_0^2f_1^2f_2^2 \\
&\quad +4m(r-2)^2(r-m)f_0^2f_1^3f_3 - 6m(r-m)(r-2)^2f_0^3f_1f_2f_3 \\
&\quad +4r^2(m-1)(r-m-1)f_1^6)\Big\}.
\end{aligned}$$

From equations 2.1 and 2.2,

$$\begin{aligned}
\frac{\mathcal{H}(f)}{r^2(r-1)^2} &= X_1^{-2}(f_0f_2-f_1^2) \\
\frac{\mathcal{J}(f)}{(-1)r^3(r-1)^2(r-2)} &= X_1^{-3}(-f_0^2f_3+3f_0f_1f_2-2f_1^3)
\end{aligned}$$

are covariants having respective weights 2 and 3. Hence the powers $\mathcal{H}^3$ and $\mathcal{J}^2$ are covariants having the same weight 6, and so is any linear combination. Therefore, if $r>2$ then $4\,(r-2m)^2\,(r-1)\left\{\frac{\mathcal{H}}{r^2(r-1)^2}\right\}^3+m\,(r-m)\,(r-2)^2\left\{\frac{\mathcal{J}}{(-1)r^3(r-1)^2(r-2)}\right\}^2$, from $\mathbb{C}[X_0,X_1]_r$ to $\mathbb{C}[X_0,X_1]_{6r-12}$ is a covariant of weight 6. It can be easily verified from the table below that this covariant is in fact $\dfrac{X_1^{-6}g}{r^4(r-1)^2}$.

We shall calculate coefficients of the monomials,

$f_0^3f_2^3, f_0^4f_3^2, f_0f_1^4f_2, f_0^2f_1^2f_2^2, f_0^2f_1^3f_3, f_0^3f_1f_2f_3, f_1^6$, occurring in

$$4\,(r-2m)^2\,(r-1)\left\{\frac{\mathcal{H}}{r^2(r-1)^2}\right\}^3+m\,(r-m)\,(r-2)^2\left\{\frac{\mathcal{J}}{(-1)r^3(r-1)^2(r-2)}\right\}^2.$$

	$X_1^6\left\{\frac{\mathcal{H}(f)}{r^2(r-1)^2}\right\}^3$	$X_1^6\left\{\frac{\mathcal{J}(f)}{r^3(r-1)^2(r-2)}\right\}^2$	$X_1^6\left\{4\,(r-2m)^2\,(r-1)\left\{\frac{\mathcal{H}(f)}{r^2(r-1)^2}\right\}^3 +m\,(r-m)\,(r-2)^2\left\{\frac{\mathcal{J}(f)}{(-1)r^3(r-1)^2(r-2)}\right\}^2\right\}$
$f_0^3f_2^3$	1	0	$4(r-2m)^2(r-1)$
$f_0^4f_3^2$	0	1	$m(r-m)(r-2)^2$
$f_0f_1^4f_2$	3	-12	$12(r-2m)^2(r-1)-12m(r-m)(r-2)^2$
$f_0^2f_1^2f_2^2$	-3	9	$-12(r-2m)^2(r-1)+9m(r-m)(r-2)^2$
$f_0^2f_1^3f_3$	0	4	$4m(r-m)(r-2)^2$
$f_0^3f_1f_2f_3$	0	-6	$-6m(r-m)(r-2)^2$
f_1^6	-1	4	$-4(r-2m)^2(r-1)+4m(r-m)(r-2)^2$

Table 4.1: Calculation of the coefficients of the monomials

Hence

$$X_1^{-6}\frac{g}{r^4(r-1)^2} = 4\,(r-2m)^2\,(r-1)\left\{\frac{\mathcal{H}}{r^2(r-1)^2}\right\}^3 + m\,(r-m)\,(r-2)^2\left\{\frac{\mathcal{J}}{(-1)r^3(r-1)^2(r-2)}\right\}^2$$

By doing similar calculations with the substitution

$$\partial_1^i f = \frac{r!}{(r-i)!}\bar{f}_i$$

in $\dfrac{\bar{g}}{r^4(r-1)^2}$ ($\bar{g}$ stated in 4.5) we have

$$X_0^{-6}\frac{\bar{g}}{r^4(r-1)^2} = 4\,(r-2m)^2\,(r-1)\left\{\frac{\mathcal{H}}{r^2(r-1)^2}\right\}^3 + m\,(r-m)\,(r-2)^2\left\{\frac{\mathcal{J}}{(-1)r^3(r-1)^2(r-2)}\right\}^2$$

where

$$\begin{aligned}\frac{\mathcal{H}(f)}{r^2(r-1)^2} &= X_0^{-2}(\bar{f}_0\bar{f}_2 - \bar{f}_1^2)\\ \frac{\mathcal{J}(f)}{(-1)r^3(r-1)^2(r-2)} &= X_0^{-3}(-\bar{f}_0^2\bar{f}_3 + 3\bar{f}_0\bar{f}_1\bar{f}_2 - 2\bar{f}_1^3).\end{aligned}$$

Thus $\mathcal{I}(r-m,m)$ is the radical of the coefficient ideal of the covariant

$$4\,(r-2m)^2\,(r-1)\left\{\frac{\mathcal{H}}{r^2(r-1)^2}\right\}^3 + m\,(r-m)\,(r-2)^2\left\{\frac{\mathcal{J}}{(-1)r^3(r-1)^2(r-2)}\right\}^2.$$

Hence the result. □

Remark 4.24 *1. As a consequence of Theorem 4.23 we have the following known* *results 4.9:*

- *When $m = 0$ we have the following equivalent statements for a binary form of degree r:*

  (a) f has the form l_1^r for some linear form l_1 over $\mathbb{C}$.

  (b) The Hessian $\mathcal{H}$ vanishes for f.

- *When r is even and $m = \frac{r}{2}$ we have the following equivalent statements for a binary form of degree r:*

  (a) f has the form $(l_1 l_2)^{\frac{r}{2}}$ for some linear forms l_1 and l_2 over $\mathbb{C}$.

  (b) The Jacobian $\mathfrak{J}$ vanishes for f.

2. *The following theorem, originally due to Clebsch, was proved in [Gordan 1885] by Gordan :*

   Theorem 4.25 *the following statements are equivalent for a binary form f of degree r, where $r \neq 4, 6, 8, 12$.*

   (a) f has the form $l_1^{r-1} l_2$ for some linear forms l_1 and l_2 over $\mathbb{C}$.

   (b) The fourth transvectant $\mathcal{P}$ vanishes for f.

   Thus $\mathcal{I}(r-1, 1) =$ Radical of the coefficient ideal of $\mathcal{P}$
   $=$ Radical of the coefficient ideal of $4\left\{\frac{\mathcal{H}}{r^2(r-1)^2}\right\}^3 + \left\{\frac{\mathfrak{J}}{(-1)r^3(r-1)^2(r-2)}\right\}^2$, for $r \neq 4, 6, 8, 12$.

4.3.2 Binary Quadratic and Cubic Forms

When $r = 2$, we have complete description of these ideals.

Every binary quadratic form is a product of linear forms. Hence $\mathcal{I}(1,1) = \{0\}$. On the other hand $\mathcal{I}(2) = \langle A_0 A_2 - A_1^2 \rangle$, where A_0, A_1, A_2 are the coordinate functions on $\mathbb{C}[X_0, X_1]_2$ given by

$$A_i(a_0X_0^2 + 2a_1X_0X_1 + a_2X_1^2) = a_i, i = 0, 1, 2.$$

Consider a binary cubic form

$$f = a_0X_0^3 + 3a_1X_0^2X_1 + 3a_2X_0X_1^2 + a_3X_1^3.$$

The following facts about covariants of binary cubic forms can be found in [Schur 1968] on page 77.

1. The discriminant $\mathcal{D}(f)$ of f, apart from a numerical factor, is

$$a_0^2a_3^2 - 6a_0a_1a_2a_3 + 4a_0a_2^3 - 3a_1^2a_2^2 + 4a_1^3a_3,$$

2. The Hessian $\mathcal{H}(f)$ of f, apart from a numerical factor, is

$$(a_0a_2 - a_1^2)X_0^2 + (a_0a_3 - a_1a_2)X_0X_1 + (a_1a_3 - a_2^2)X_1^2,$$

3. The Jacobian $\mathfrak{J}(f)$ of f, apart from a numerical factor, is

$$(a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3)X_0^3 + \dots.$$

4. The following is essentially the only relation between them

$$\mathcal{J}(f)^2 + \mathcal{H}(f)^3 = 2^4\, 3^6\, f^2\, \mathcal{D}(f).$$

Note the following consequence of previous results:

- Since every form is a product of linear forms (see [Bôcher 1964] page 188), $\mathcal{I}(1,1,1) = \{0\}$.
- (By Theorem 4.23) $\mathcal{I}(2,1)$ is the radical of the coefficient ideal of the covariant $\mathcal{H}^3 + \mathcal{J}^2$.
- (By Lemma 4.6) $\mathcal{I}(2,1)$ is the ideal generated by the invariant discriminant $\mathcal{D}$.
- (By Theorem 4.9) $\mathcal{I}(3)$ is the radical of the coefficient ideal of the covariant Hessian $\mathcal{H}$.

Remark 4.26 *The discriminant of a cubic form f is proportional to the discriminant of the Hessian of f.*

As a summary we have,

$$\mathcal{I}(3) = \mathit{Rad}.\text{ of the coefficient ideal of } \mathit{Hessian}$$

$$|$$

$$\mathcal{I}(2,1) = \langle \mathit{disc} \rangle = \mathit{Rad}.\text{ of the coefficient ideal of } \mathcal{H}^3 + \mathcal{J}^2$$

$$|$$

$$\mathcal{I}(1,1,1) = \{0\}.$$

4.3.3 Binary Quartic Form

For a binary quartic form

$$f = \sum_{i=0}^{4} \binom{4}{i} a_i {X_0}^{4-i} {X_1}^{i},$$

the following facts can be found in [Schur 1968] on page 80.

1. The following are algebraically independent invariants from $\mathbb{C}[X_0, X_1]_4$ and they generate all invariants from $\mathbb{C}[X_0, X_1]_4$:

   (a)

$$\begin{aligned} \mathcal{P}(f) &= (f, f)^{(4)} \\ &= a_0 a_4 - 4 a_1 a_3 + 3 a_2^2. \end{aligned}$$

   (b)

$$\begin{aligned} \mathcal{Q}(f) &= \begin{vmatrix} a_0 & a_1 & a_2 \\ a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \end{vmatrix} \text{ (Hankel determinant)} \\ &= a_0 a_2 a_4 + 2 a_1 a_2 a_3 - a_0 a_3^2 - a_1^2 a_4 - a_2^3. \end{aligned}$$

2. The Hessian $\mathcal{H}(f)$ of f, apart from a numerical factor, is

$$(a_0 a_2 - a_1^2)\, X_0^4 + \ldots$$

3. The Jacobian $\mathcal{J}(f)$ of f, apart from a numerical factor, is

$$(a_0^2a_3 - 3a_0a_1a_2 + 2a_1^3)X_0^6 + \dots$$

4. There is a relation between all of the above,

$$9\left(\mathcal{J}(f)\right)^2 + 16\left(\mathcal{H}(f)\right)^3 = 2^{10}3^4 f^2\left(\mathcal{P}(f)\,f - 2^4 3^2 \mathcal{H}(f)\,\mathcal{Q}(f)\right).$$

5. (p. 52) The discriminant of f is given by the formula

$$\mathcal{D}(f) = 256(\mathcal{P}(f)^3 - 27\mathcal{Q}(f)^2).$$

Note the following consequences of previous results:

- (By Theorem 4.9) $\mathcal{I}(4)$ is the radical of the coefficient ideal of the Hessian $\mathcal{H}$.
- (By Theorem 4.7) $\mathcal{I}(3,1)$ is the radical of the ideal generated by $\mathcal{P}, \mathcal{Q}$. Also (by Theorem 4.23), $\mathcal{I}(3,1)$ is the radical of the coefficient ideal of $16\,\mathcal{H}^3 + 9\,\mathcal{J}^2$.
- (By Theorem 4.9) $\mathcal{I}(2,2)$ is the radical of the coefficient ideal of the Jacobian $\mathcal{J}$.
- (By Lemma 4.6) $\mathcal{I}(2,1,1)$ is generated by the discriminant $\mathcal{D}$.
- $\mathcal{I}(1,1,1,1) = \{0\}$.

We are going to give a variety of other proofs of some of these special cases.

Lemma 4.27 *A necessary and sufficient condition that a binary quartic form f belong to $\mathcal{F}(3,1)$ is that the invariants $\mathcal{P}$ and Ω vanish for f.*

Proof: We have $f \in \mathcal{F}(3,1)$ iff f has a linear factor of multiplicity $> 2 = \frac{4}{2}$. Therefore the lemma follows immediately from Theorem 4.7. □

It is interesting to compare the above with a computational proof using elimination theory. As a matter of fact, in the proof of Theorem 4.2 there is a way of constructing polynomials whose vanishing gives a necessary and sufficient condition for a binary form f of degree r in the variables X_0, X_1 to represent k projective points. Many of our later discussions and calculations are based on this method.

The method is as follows:

Let

$$f = \sum_{j=0}^{r} \binom{r}{j} a_j X_0^{r-j} X_1^j,$$

and let $(m_1, \ldots, m_s)$ be a partition of r. Then

$f \in \mathcal{F}(m_1, \ldots, m_s)$ iff there exist $\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_s \in \mathbb{C}$ such that

$$f = \prod_{i=1}^{s} (\alpha_i X_0 + \beta_i X_1)^{m_i}.$$

This is equivalent to the following system of equations:

$$a_j = \sum_{q_1+\cdots+q_s=j} \prod_{i=1}^{s} \left(\binom{m_i}{q_i} \alpha_i^{m_i-q_i} \beta_i^{q_i} \right). \tag{4.13}$$

The idea is to eliminate $\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_s$ from the above equations 4.13. This can be done by using Gröbner basis techniques. But, it would take too long to do by hand. The use of computer algebra system made it possible for $r \leq 5$.

Now we shall use this method to prove Lemma 4.27.

Let $f = pX_0^4 + 4qX_0^3X_1 + 6rX_0^2X_1^2 + 4sX_0X_1^3 + tX_1^4$ be a binary form which has degree 4. Then $f \in \mathcal{F}(3,1)$ if and only if there exist $a, b, c, d \in \mathbb{C}$ such that

$$f = (aX_0 + bX_1)^3(cX_0 + dX_1).$$

This is equivalent to the following system of equations:

$$\begin{aligned} ca^3 &= p, \\ 3\,cba^2 + da^3 &= 4q, \\ 3\,ab^2c + 3\,a^2bd &= 6r, \\ 3\,ab^2d + cb^3 &= 4s, \\ db^3 &= t. \end{aligned}$$

Let $A, B, C, D, P, Q, R, S, T$ be coordinate functions on $\mathbb{C}[X_0, X_1]_1 \oplus \mathbb{C}[X_0, X_1]_1 \oplus \mathbb{C}[X_0, X_1]_4$ such that

$P(0,0,f) = p, Q(0,0,f) = q, R(0,0,f) = r, S(0,0,f) = s, T(0,0,f) = t,$
$A(aX_0 + bX_1, 0, 0) = a, B(aX_0 + bX_1, 0, 0) = b, C(0, cX_0 + dX_1, 0) = c,$
$D(0, cX_0 + dX_1, 0) = d.$

Let I be the ideal of $\mathbb{C}[A, B, C, D, P, Q, R, S, T]$ generated by

$$\{CA^3 - P, 3\,CBA^2 + DA^3 - 4Q, 3\,AB^2C + 3\,A^2BD - 6R, 3\,AB^2D + CB^3 - 4S, DB^3 - T\}.$$

There are 37 polynomials in the Gröbner basis for I with respect to lexicographic order. Only the polynomials which are needed for this proof are attached in Appendix B.3. The interested reader may contact the author for the complete and extensive Maple output.

By the Elimination Theorem 4.13, we obtain

$$\begin{aligned} I \cap \mathbb{C}[P,Q,R,S,T] = I_4 &= \langle \mathsf{h}_1, \mathsf{h}_2, \mathsf{h}_3 \rangle, \\ I \cap \mathbb{C}[D,P,Q,R,S,T] = I_3 &= \langle \mathsf{h}_1, \mathsf{h}_2, \mathsf{h}_3 \rangle, \\ I \cap \mathbb{C}[C,D,P,Q,R,S,T] = I_2 &= \langle \mathsf{h}_1, \dots, \mathsf{h}_{23} \rangle, \\ I \cap \mathbb{C}[B,C,D,P,Q,R,S,T] = I_1 &= \langle \mathsf{h}_1, \dots, \mathsf{h}_{37} \rangle, \end{aligned}$$

Hence if there exist $p,q,r,s,t \in \mathbb{C}$ such that $\mathbf{V}(I)$ contains a point whose last coordinate is f, then $\mathsf{h}_1, \mathsf{h}_2, \mathsf{h}_3$ vanish for f.

Assume that $\mathsf{h}_1, \mathsf{h}_2, \mathsf{h}_3$ vanish for f. Then there exist $p,q,r,s,t \in \mathbb{C}$ such that $f \in \mathbf{V}(I_4)$. The idea is to extend (f) one coordinate at a time: first to (d,f), then to (c,d,f) then to (b,c,d,f) and then to (a,b,c,d,f). We will use the Extension Theorem 4.14 at each step.

Since I_4 is the first elimination ideal of I_3 and $I_3 = I_4$, it follows that for all $d \in \mathbb{C}$, $(d,f) \in \mathbf{V}(I_3)$. We choose d to be non-zero.

The extension step fails only when the leading coefficients vanish simultaneously.

From the Gröbner basis for I we have, $\mathsf{h}_{20}, \dots, \mathsf{h}_{23}$, are in the ideal I_2 and

- the coefficient of C^4 in h_{23} is t,
- the coefficient of C^2 in h_{21} is $(3ps - 2rq)$,

- the coefficient of C^2 in h_{22} is $(9pr - 8q^2)$,
- the coefficient of C^2 in h_{20} is $(4sq - 3r^2)$,

Suppose firstly that at least one of these coefficients $t, (3ps-2rq), (9pr-8q^2), (4qs-3r^2)$ is non-zero, by the Extension Theorem 4.14 there exists $c \in \mathbb{C}$ such that $(c, d, f) \in \mathbf{V}(I_2)$.

Since I_2 is the first elimination ideal of I_1, the next step is to go from I_2 to I_1. Since $\mathsf{h}_{24} \in I_1$ and the coefficient of B^3 in h_{24} is d, which is non-zero, it follows from the Extension Theorem 4.14 that there exists $b \in \mathbb{C}$ such that $(b, c, d, f) \in \mathbf{V}(I_1)$.

Since I_1 is the first elimination ideal of I, the next step is to go from I_1 to I. Since $\mathsf{h}_{60} \in I$ and the coefficient of A^3 in h_{60} is d, which is non-zero, it follows from the Extension Theorem 4.14 that there exists $a \in \mathbb{C}$ such that $(aX_0 + bX_1, cX_0 + dX_1, f) \in \mathbf{V}(I)$. Thus $f \in \mathcal{F}(3, 1)$.

If on the other hand, all the coefficients $t, (3ps - 2rq), (9pr - 8q^2), (4qs - 3r^2)$ are zero, then

$$\begin{aligned} f &= pX_0^4 + 4qX_0^3X_1 + 6rX_0^2X_1^2 + 4sX_0X_1^3 \\ &= X_0(pX_0^3 + 4qX_0^2X_1 + 6rX_0X_1^2 + 4sX_1^3) \end{aligned}$$

with the Hessian of the binary cubic form $(pX_0^3 + 4qX_0^2X_1 + 6rX_0X_1^2 + 4sX_1^3)$ is

$$(9pr - 8q^2)X_0^2 + (3ps - 2rq)X_0X_1 + (4qs - 3r^2)X_1^2.$$

By Hilbert's Theorem 4.9, this binary cubic form is the cube of a linear factor, meaning there exist $a, b \in \mathbb{C}$ such that

$$(pX_0^3 + 4qX_0^2X_1 + 6rX_0X_1^2 + 4sX_1^3) = (aX_0 + bX_1)^3.$$

This implies

$$f = X_0(aX_0 + bX_1)^3.$$

Thus $f \in \mathcal{F}(3,1)$.

Thus $\mathcal{I}(3,1) = Rad\langle \mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \rangle$.

By the following relations,

1. $\mathfrak{h}_2 = \mathcal{Q}$,

2. $\mathcal{P} = R\,\mathfrak{h}_2 - \mathfrak{h}_3$,

3. $\mathfrak{h}_1 = -S^2\,\mathcal{P} + T\,\mathcal{Q}$,

we have,

$$\langle \mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \rangle = \langle \mathcal{P}, \mathcal{Q} \rangle.$$

Hence the result. □

Next we shall give two different proofs to show that $\mathcal{I}(2,2)$ = radical of the coefficient ideal of the Jacobian.

Theorem 4.28 *The following are equivalent for a binary quartic form f,*

1. *$f = q^2$, for some binary quadratic form q.*

2. *The Hessian of f is a scalar multiple of f.*

3. The Jacobian of f is zero.

Proof: (Method 1)

First we shall show that the statements (1) and (2) are equivalent.

Assume that $f = q^2$, for some binary quadratic form q. Then $\partial_1 f = 2q\partial_1 q$, and $\partial_1^2 f = 2q\partial_1^2 q + 2(\partial_1 q)^2$. Consider,

$$\begin{aligned} X_0^2\mathcal{H}(f) &= 12f\partial_1^2 f - 9(\partial_1 f)^2 \text{ (by Theorem 2.9)} \\ &= 12q^2(2q\partial_1^2 q + 2(\partial_1 q)^2) - 9(2q\partial_1 q)^2 \\ &= 24q^3\partial_1^2 q + 24q^2(\partial_1 q)^2 - 36q^2(\partial_1 q)^2 \\ &= 4q^2(6q\partial_1^2 q - 3(\partial_1 q)^2). \end{aligned}$$

Since q is a binary quadratic form, say $q = aX_0^2 + 2bX_0X_1 + cX_1^2$, for $a, b, c, \in \mathbb{C}$, $\partial_1 q = 2bX_0 + 2cX_1$ and $\partial_1^2 q = 2c$. Hence,

$$\begin{aligned} X_0^2\mathcal{H}(f) &= 4q^2(6(aX_0^2 + 2bX_0X_1 + cX_1^2)2c - 3(2bX_0 + 2cX_1)^2) \\ &= 4q^2(12acX_0^2 + 24bcX_0X_1 + 12c^2X_1^2 - 12b^2X_0^2 - 24bcX_0X_1 - 12c^2X_1^2) \\ &= 48f(ac - b^2)X_0^2. \end{aligned}$$

Thus, $\mathcal{H}(f) = 48f(ac - b^2)$. Therefore $\mathcal{H}(f)$ is is a scalar multiple to f.

Conversely assume that $\mathcal{H}(f)$ is a scalar multiple of f. Then f divides $\mathcal{H}(f)$. Since

$$X_0^2\mathcal{H}(f) = 12f\partial_1^2 f - 9(\partial_1 f)^2,$$

f divides $(\partial_1 f)^2$. Hence every linear factor of f divides $(\partial_1 f)^2$. Linear factors of f are irreducible and $\mathbb{C}[X_0, X_1]$ is a unique factorization domain. Therefore every linear factor of f divides $\partial_1 f$. In similar manner by using the formula

$$X_1^2\mathcal{H}(f) = 12f\partial_0^2 f - 9(\partial_0 f)^2$$

we have every linear factor of f divides $\partial_0 f$.

Let

$$f = (\alpha_1 X_0 - \beta_1 X_1)(\alpha_2 X_0 - \beta_2 X_1)(\alpha_3 X_0 - \beta_3 X_1)(\alpha_4 X_0 - \beta_4 X_1).$$

Let

$$l_j = (\alpha_j X_0 - \beta_j X_1), j = 1, 2, 3, 4.$$

Then

$$\partial_0(f) = \alpha_1 l_2 l_3 l_4 + \alpha_2 l_1 l_3 l_4 + \alpha_3 l_1 l_2 l_4 + \alpha_4 l_1 l_2 l_3,$$

and

$$\partial_1(f) = -\beta_1 l_2 l_3 l_4 - \beta_2 l_1 l_3 l_4 - \beta_3 l_1 l_2 l_4 - \beta_4 l_1 l_2 l_3.$$

Now l_1 divides both $\partial_0 f$ and $\partial_1 f$. Therefore, l_1 divides both $\alpha_1 l_2 l_3 l_4$, and $-\beta_1 l_2 l_3 l_4$. We know that either $\alpha_1 \neq 0$ or $\beta_1 \neq 0$, and l_1 is an irreducible polynomial. Therefore, l_1 is a scalar multiple of l_j for some $j \in \{2, 3, 4\}$. Hence l_1 has a multiplicity > 1. Similarly, we can show that all the linear factors f must have multiplicity > 1. Thus all the linear factors f must have multiplicity 2 or 4. Therefore in either case $f = q^2$, for some binary quadratic form q.

Now we shall show that the statements (2) and (3) are equivalent.

From the definition of the Jacobian of f , it easily follows that if $\mathcal{H}(f)$ is a scalar multiple of f then the Jacobian of f is zero.

Conversely, assume that the Jacobian of f is zero. Thus,

$$\begin{aligned} 0 &= \begin{vmatrix} \partial_0 f & \partial_1 f \\ \partial_0\mathcal{H}(f) & \partial_1\mathcal{H}(f) \end{vmatrix} \\ &= \partial_0 f\, \partial_1\mathcal{H}(f) - \partial_1 f\, \partial_0\mathcal{H}(f). \end{aligned}$$

Since f is a binary form, therefore either $\partial_0 f$ or $\partial_1 f$ is non-zero. Without loss of generality we may assume that $\partial_0 f$ is non-zero. Then

$$\partial_1\mathcal{H}(f) = \frac{\partial_0\mathcal{H}(f)}{\partial_0 f}\partial_1 f, \tag{4.14}$$

$$\partial_0\mathcal{H}(f) = \frac{\partial_0\mathcal{H}(f)}{\partial_0 f}\partial_0 f. \tag{4.15}$$

Notice that

$$\frac{\partial_0\mathcal{H}(f)}{\partial_0 f}$$

is a rational function in the field $\mathbb{C}(X_0, X_1)$, we shall denote it by C.

From Euler's formula for homogeneous functions we have,

$$4\mathcal{H}(f) = X_0\partial_0\mathcal{H}(f) + X_1\partial_1\mathcal{H}(f).$$

Then

$$4\mathcal{H}(f) = X_0 C\partial_0 f + X_1 C\partial_1 f.$$

Since f is a binary quartic form, it follows from Euler's formula that

$$4\mathcal{H}(f) = 4Cf,$$

that is

$$\mathcal{H}(f) = Cf.$$

Now we shall show that the rational function C is in fact a constant.

For $i = 0, 1$, By partially differentiating with respect to X_i we get

$$\partial_i \mathcal{H}(f) = f\partial_i C + C\partial_i f.$$

Since

$$C\partial_i f = \partial_i \mathcal{H}(f),$$

$$f\partial_i C = 0.$$

Since f is non-zero, $\partial_i C = 0$, for all $i = 0, 1$. Thus $C \in ker\, \partial_0 \cap ker\, \partial_1 = \mathbb{C}$. That is, C is a constant. □

We shall give another proof by using elimination theory:

<u>Proof</u>: (Method 2)

(1) ⇔ (3)

Let $f = pX_0^4 + 4qX_0^3X_1 + 6rX_0^2X_1^2 + 4sX_0X_1^3 + tX_1^4$, and $g = aX_0^2 + 2bX_0X_1 + cX_1^2$.

Then the condition $f = g^2$ is equivalent to the following system of equations:

$$p \;=\; a^2,$$

$$\begin{aligned} 4q &= 4ab, \\ 6r &= 4b^2 + 2ac, \\ 4s &= 4bc, \\ t &= c^2. \end{aligned}$$

Let A, B, C, P, Q, R, S, T be coordinate functions on $\mathbb{C}[X_0, X_1]_2 \oplus \mathbb{C}[X_0, X_1]_4$ such that

$P(0, f) = p, Q(0, f) = q, R(0, f) = r, S(0, f) = s, T(0, f) = t, A(g, 0) = a,$ $B(g, 0) = b, C(g, 0) = c.$

Let I be the ideal in $\mathbb{C}[A, B, C, P, Q, R, S, T]$, generated by

$$\{A^2 - P, AB - Q, 3R - 2B^2 - AC, S - BC, T - C^2\}.$$

Note that f is a square of a binary quadratic form iff the zero set $\mathbf{V}(I)(\subset \mathbb{C}[X_0, X_1]_2 \oplus \mathbb{C}[X_0, X_1]_4)$ of I contains a point whose last coordinate is f.

There are 20 polynomials in the Gröbner basis for I with respect to lexicographic order. Only the polynomials which are needed for this proof are attached in Appendix B.2. The interested reader may contact the author for the complete and extensive Maple output.

By the Elimination Theorem 4.13, we obtain

$$\begin{aligned} I \cap \mathbb{C}[P, Q, R, S, T] = I_3 &= \langle \mathbf{g}_1, \ldots, \mathbf{g}_7 \rangle, \\ I \cap \mathbb{C}[C, P, Q, R, S, T] = I_2 &= \langle \mathbf{g}_1, \ldots, \mathbf{g}_8 \rangle, \\ I \cap \mathbb{C}[B, C, P, Q, R, S, T] = I_1 &= \langle \mathbf{g}_1, \ldots, \mathbf{g}_{20} \rangle. \end{aligned}$$

Assume that there exist $f \in \mathbb{C}[X_0, X_1]_4$ such that $\mathbf{V}(I)$ contains a point whose last coordinate is f. Then $g_1, \ldots, g_7$ vanish for f.

Conversely, assume that $g_1, \ldots, g_7$ vanish for f.

Then $f \in \mathbf{V}(I_3)$. Since $g_8 \in I_2$ and the coefficient of C^2 in g_8 is 1, by the Extension Theorem 4.14, there exists $c \in \mathbb{C}$ such that $(c, f) \in \mathbf{V}(I_2)$.

Since $g_{20} \in I_1$ and the coefficient of B^3 in g_{20} is 2, it follows from the Extension Theorem 4.14, there exists $b \in \mathbb{C}$ such that $(2b, c, f) \in \mathbf{V}(I_1)$.

Since $g_{27} \in I$ and the coefficient of A^2 in g_{27} is 1, it follows from the Extension Theorem 4.14, there exists $a \in \mathbb{C}$ such that $(g, f) \in \mathbf{V}(I)$. Hence

$$f = (aX_0^2 + 2bX_0X_1 + cX_1^2)^2.$$

The above argument shows that $\mathcal{I}(2,2) = \langle g_1, \ldots, g_7 \rangle$.

The Jacobian of f is in fact

$$-1152\,(g_7(f)X_0^6 - g_6(f)X_0^5X_1 - 5\,g_5(f)X_0^4X_1^2 - 10\,g_4(f)X_0^3X_1^3$$
$$-5\,g_3(f)X_0^2X_1^4 - g_2(f)X_0X_1^5 - g_1(f)X_1^6).$$

To prove (1) $\Rightarrow$ (2), let

$$f = pX_0^4 + 4qX_0^3X_1 + 6rX_0^2X_1^2 + 4sX_0X_1^3 + tX_1^4.$$

Then

$$\begin{aligned}\mathcal{H}(f) &= 144((pr - q^2)X_0^4 + (2ps - 2qr)X_0^3X_1 + (pt + 2qs - 3r^2)X_0^2X_1^2 \\ &\quad + (2qt - 2rs)X_0X_1^3 + (rt - s^2)X_1^4).\end{aligned}$$

Assume that $\mathcal{H}(f)$ is a scalar multiple of f . Then the rank of the matrix

$$\begin{pmatrix} pr-q^2 & 2ps-2qr & pt+2qs-3^2 & 2qt-2rs & rt-s^2 \\ p & 4q & 6r & 4s & t \end{pmatrix}$$

is 1. Therefore all the 2×2 minors of this matrix are zero. There are 10 minors. The minors and the connection between the polynomials $g_1,\ldots,g_7$ for $P(f)=p, Q(f)=q, R(f)=r, S(f)=s, T(f)=t$ are listed below.

$$0=(pr-q^2)4q-(2ps-2qr)p=6pqr-4q^3-2p^2s=2g_7(f)$$

$$0=(pr-q^2)6r-(pt+2qs-3r^2)p=9pr^2-6q^2r-p^2t-2pqs=-g_6(f)$$

$$0=(pr-q^2)4s-(2qt-2rs)p=6prs-4q^2s-2pqt=-2g_5(f)$$

$$0=(pr-q^2)t-(rt-s^2)p=-q^2t+s^2p=g_4(f)$$

$$0=(2ps-2qr)6r-(pt+2qs-3r^2)4q=12psr-4pqt-8q^2s=-4g_5(f)$$

$$0=(2ps-2qr)4s-(2qt-2rs)4q=8ps^2-8q^2t=8g_4(f)$$

$$0=(2ps-2qr)t-(rt-s^2)4q=2pst-6qrt+4s^2q=2g_3(f)$$

$$0=(pt+2qs-3r^2)4s-(2qt-2rs)6r=4pts+8qs^2-12qrt=4g_3(f)$$

$$0=(pt+2qs-3r^2)t-(rt-s^2)6r=pt^2+2qst-9r^2t+6s^2r=g_2(f)$$

$$0=(2qt-2rs)t-(rt-s^2)4s=2qt^2-6rst+4s^3=2g_1(f)$$

□

As a summary we have listed the ideals for binary quartic forms in the following Fig. 4.4.

$$\begin{tikzcd}[column sep=tiny]
 & \mathcal{I}(4) = \text{Rad. of the coef. ideal of } \mathcal{H} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathcal{I}(3,1) = \text{Rad. of the ideal generated by } \mathcal{P}, \mathcal{Q} \arrow[dr, dash] & & \mathcal{I}(2,2) = \text{Rad. of the coef. ideal of } \mathcal{J} \arrow[dl, dash] \\
 & \mathcal{I}(2,1,1) = \text{ideal generated by } \mathcal{D} \arrow[d, dash] & \\
 & \mathcal{I}(1,1,1,1) = \{0\} &
\end{tikzcd}$$

Figure 4.4: The ideals for binary quartic forms

4.3.4 Binary Quintic Form

Let

$$f = pX_0^5 + 5qX_0^4X_1 + 10rX_0^3X_1^2 + 10sX_0^2X_1^3 + 5tX_0X_1^4 + uX_1^5$$

be a binary quintic form.

We have the following special cases of previous general results:

- (By Theorem 4.9) $\mathcal{I}(5)$ is the radical of the coefficient ideal of the covariant Hessian $\mathcal{H}$.
- (By Theorem 4.25) $\mathcal{I}(4,1)$ is the radical of the coefficient ideal of the fourth transvectant $\mathcal{P}$. Also (by Theorem 4.23) $\mathcal{I}(4,1)$ is the radical of the coefficient ideal of the covariant $9\,\mathcal{H}^3 + 4\,\mathfrak{J}^2$.
- (By Theorem 4.23) $\mathcal{I}(3,2)$ is the radical of the coefficient ideal of the covariant $\mathcal{H}^3 + 6\,\mathfrak{J}^2$.
- (By Theorem 4.7) $\mathcal{I}(3,1,1)$ is the radical of the ideal generated by all the invariants of binary quintic forms.
- (By Lemma 4.6) $\mathcal{I}(2,1,1,1)$ is generated by the invariant discriminant $\mathcal{D}$.
- $\mathcal{I}(1,1,1,1,1) = \{0\}$.

We proceed to provide and compare alternative proofs of some of these cases. First we shall illustrate the use of elimination for the case where f has the form $l_1^4 l_2$ for some linear forms l_1 and l_2 over $\mathbb{C}$.

Lemma 4.29 *The following are equivalent for a binary quintic form f.*

1. *f has the form $l_1^4 l_2$ for some linear forms l_1 and l_2 over $\mathbb{C}$, i.e. f belongs to $\mathcal{F}(4,1)$.*

2. *$\mathrm{i}_1, \ldots, \mathrm{i}_6$ vanish for f (listed in Appendix B.4).*

Proof:

Let

$$f = pX_0^5 + 5qX_0^4X_1 + 10rX_0^3X_1^2 + 10sX_0^2X_1^3 + 5tX_0X_1^4 + uX_1^5$$

be a binary quintic form. Then f has the form $l_1^4 l_2$ for some linear forms l_1 and l_2 over $\mathbb{C}$ if and only if there exist $a, b, c, d \in \mathbb{C}$ such that

$$f = (aX_0 + bX_1)^4(cX_0 + dX_1).$$

This is equivalent to the following system of equations:

$$\begin{aligned} ca^4 &= p, \\ 4\,cba^3 + da^4 &= 5q, \\ 6\,a^2b^2c + 4\,a^3bd &= 10r, \\ 6\,a^2b^2d + 4acb^3 &= 10s, \\ 4ab^3d + b^4c &= 5t, \\ db^4 &= u. \end{aligned}$$

Let $A, B, C, D, P, Q, R, S, T, U$ be coordinate functions on $\mathbb{C}[X_0, X_1]_1 \oplus \mathbb{C}[X_0, X_1]_1 \oplus \mathbb{C}[X_0, X_1]_5$ such that

$P(0,0,f) = p, Q(0,0,f) = q, R(0,0,f) = r, S(0,0,f) = s, T(0,0,f) = t,$
$U(0,0,f) = u, A(aX_0 + bX_1, 0, 0) = a, B(aX_0 + bX_1, 0, 0) = b,$
$C(0, cX_0 + dX_1, 0) = c, D(0, cX_0 + dX_1, 0) = d.$

Let I be the ideal in $\mathbb{C}[A, B, C, D, P, Q, R, S, T, U]$ generated by

$\{CA^4 - P, 4\,CBA^3 + DA^4 - 5Q, 6\,A^2B^2C + 4\,A^3BD - 10R, 6\,A^2B^2D + 4ACB^3 - 10S, 4AB^3D + B^4C - 5T, DB^4 - U\},$

Note that f has the form $l_1^4 l_2$ for some linear forms l_1 and l_2 over $\mathbb{C}$ iff $\mathbf{V}(I)(\subset \mathbb{C}[X_0, X_1]_1 \oplus \mathbb{C}[X_0, X_1]_1 \oplus \mathbb{C}[X_0, X_1]_5)$ contains a point whose last co-ordinate is f. There are 88 polynomials in the Gröbner basis for I with respect to lexicographic order. Only the polynomials which are needed for this proof are attached in Appendix B.4. The interested reader may contact the author for the complete and extensive Maple output.

By the Elimination Theorem 4.13, we obtain

$$\begin{aligned}
I \cap \mathbb{C}[P, Q, R, S, T, U] = I_4 &= \langle \mathrm{i}_1, \ldots, \mathrm{i}_6 \rangle, \\
I \cap \mathbb{C}[D, P, Q, R, S, T] = I_3 &= \langle \mathrm{i}_1, \ldots, \mathrm{i}_6 \rangle, \\
I \cap \mathbb{C}[C, D, P, Q, R, S, T] = I_2 &= \langle \mathrm{i}_1, \ldots, \mathrm{i}_{33} \rangle, \\
I \cap \mathbb{C}[B, C, D, P, Q, R, S, T] = I_1 &= \langle \mathrm{i}_1, \ldots, \mathrm{i}_{49} \rangle.
\end{aligned}$$

Hence if f has the form $l_1^4 l_2$ for some linear forms l_1 and l_2 over $\mathbb{C}$ then $\mathrm{i}_1, \ldots, \mathrm{i}_6$ vanish for f.

To prove the converse, assume that $\mathrm{i}_1, \ldots, \mathrm{i}_6$ vanish for f. Then $f \in \mathbf{V}(I_4)$. The idea is to extend f one co-ordinate at a time: first to (d, f), to (c, d, f) then (b, c, d, f) and then to (a, b, c, d, f). We will use the Extension Theorem 4.14 at each step.

Notice that $I_3 = I_4$. Therefore, for all $d \in \mathbb{C}$, $(d, f) \in \mathbf{V}(I_3)$. We choose d to be non-zero. Since I_3 is the first elimination ideal of I_2, the next step is to go from I_3 to I_2. The extension step fails only when the leading coefficients vanish simultaneously.

Notice that $\mathrm{i}_{33}, \mathrm{i}_{32}, \mathrm{i}_{31}, \mathrm{i}_{30}, \mathrm{i}_{29}, \mathrm{i}_{28} \in I_2$, and

- the coefficient of C^5 in i_{33} is u,
- the coefficient of C^2 in i_{32} is $(16pr - 15q^2)$,
- the coefficient of C^2 in i_{31} is $(6ps - 5rq)$,
- the coefficient of C^2 in i_{30} is $(9qs - 8r^2)$,
- the coefficient of C^2 in i_{29} is $(3qt - 2rs)$,
- the coefficient of C^2 in i_{28} is $(4rt - 3s^2)$.

Assume firstly that at least one of these coefficients is non-zero. Then by the Extension Theorem 4.14, there exists $c \in \mathbb{C}$ such that $(c, d, f) \in \mathbf{V}(I_2)$.

Since I_2 is the first elimination ideal of I_1, the next step is to go from I_2 to I_1. Since $\mathrm{i}_{34} \in I_1$ and the coefficient of B^4 in i_{34} is d, which is non-zero, it follows from the Extension Theorem 4.14 that there exists $b \in \mathbb{C}$ such that $(b, c, d, f) \in \mathbf{V}(I_1)$.

Since I_1 is the first elimination ideal of I, the next step is to go from I_1 to I. Since $\mathrm{i}_{87} \in I$ and the coefficient of A^4 in i_{87} is d, which is non-zero, it follows from the Extension Theorem 4.14 that there exists $a \in \mathbb{C}$ such that $(aX_0 + bX_1, cX_0 + dX_1, f) \in \mathbf{V}(I)$. Hence the result.

If on the other hand, all of the coefficients in the above list are zero, then

$$f = pX_0^5 + 5qX_0^4X_1 + 10rX_0^3X_1^2 + 10sX_0^2X_1^3 + 5tX_0X_1^4$$

$$= X_0(pX_0^4 + 5qX_0^3X_1 + 10rX_0^2X_1^2 + 10sX_0X_1^3 + 5tX_1^4)$$

and

$$\mathrm{i}_5(f) = -4sq + pt + 3r^2.$$

Thus $(pX_0^4 + 5qX_0^3X_1 + 10rX_0^2X_1^2 + 10sX_0X_1^3 + 5tX_1^4)$ is a binary quartic form, and the coefficients of the Hessian of this binary quartic form are (apart from a numerical factor)

$$(16pr - 15q^2), (6ps - 5rq), (8pt - 5qs) + 2(9qs - 8r^2), (3qt - 5rs), (4rt - 3s^2).$$

All of these polynomials are appearing in the coefficients list except $(8pt - 5qs)$. But

$$(8pt - 5qs) = 8(-4sq + pt + 3r^2) + 3(9qs - 8r^2).$$

Hence by Theorem 4.9, there exist $a, b \in \mathbb{C}$ such that

$$(pX_0^4 + 5qX_0^3X_1 + 10rX_0^2X_1^2 + 10sX_0X_1^3 + 5tX_1^4) = (aX_0 + bX_1)^4.$$

Therefore,

$$f = X_0\,(aX_0 + bX_1)^4.$$

Hence the result. ∎

From the relations listed below we have that the ideal $\mathcal{I}(4,1)$ is the radical of the ideal generated by the coefficients of the fourth transvectant $\mathcal{P}$ of binary quintic forms.

$$\begin{aligned}
\mathcal{P}(f) &= \mathrm{i}_5(f)\,X_0^2 + \mathrm{i}_4(f)\,X_0X_1 + \mathrm{i}_2(f)\,X_1^2 && (4.16)\\
\mathrm{i}_1(f) &= (US-T^2)(f)\,\mathrm{i}_2(f) - T(f)U(f)\,\mathrm{i}_4(f) + U^2(f)\,\mathrm{i}_5(f), && (4.17)\\
\mathrm{i}_3(f) &= \frac{1}{3}\{T(f)\,\mathrm{i}_4(f) + U(f)\,\mathrm{i}_5(f) + 4S(f)\,\mathrm{i}_2(f)\}, && (4.18)\\
\mathrm{i}_6(f) &= -\frac{1}{3}\{4R(f)\,\mathrm{i}_5(f) - Q(f)\,\mathrm{i}_4(f) + P(f)\,\mathrm{i}_2(f)\}. && (4.19)
\end{aligned}$$

Now we shall look for a covariant such that the radical of the coefficient ideal of this covariant is $\mathcal{I}(3,2)$.

Lemma 4.30 *The following are equivalent for a binary quintic form f.*

1. *f has the form $l_1^3 l_2^2$ for some linear forms l_1 and l_2 over $\mathbb{C}$, i.e. f belongs to $\mathcal{F}(3,2)$.*
2. *$\mathrm{j}_1, \ldots, \mathrm{j}_{60}$ vanish for f (listed in Appendix B.5).*

Proof:

Let

$$f = pX_0^5 + 5qX_0^4X_1 + 10rX_0^3X_1^2 + 10sX_0^2X_1^3 + 5tX_0X_1^4 + uX_1^5$$

be a binary quintic form. Then f has the form $l_1^3 l_2^2$ for some linear forms l_1 and l_2 over $\mathbb{C}$ if and only if there exist $a, b, c, d \in \mathbb{C}$ such that

$$f = (aX_0 + bX_1)^3(cX_0 + dX_1)^2.$$

This is equivalent to the following system of equations:

$$\begin{aligned}
a^3c^2 &= p, \\
(2a^3cd + 3a^2bc^2) &= 5q, \\
(a^3d^2 + 6a^2bcd + 3ab^2c^2) &= 10r, \\
(3a^2bd^2 + b^3c^2 + 6ab^2cd) &= 10s, \\
(3ab^2d^2 + 2b^3cd) &= 5t, \\
b^3d^2 &= u.
\end{aligned}$$

Let $A, B, C, D, P, Q, R, S, T, U$ be co-ordinate functions on $\mathbb{C}[X_0, X_1]_1 \oplus \mathbb{C}[X_0, X_1]_1 \oplus \mathbb{C}[X_0, X_1]_5$ such that

$P(0,0,f) = p, Q(0,0,f) = q, R(0,0,f) = r, S(0,0,f) = s, T(0,0,f) = t,$ $U(0,0,f) = u, A(aX_0 + bX_1, 0, 0) = a, B(aX_0 + bX_1, 0, 0) = b,$ $C(0, cX_0 + dX_1, 0) = c, D(0, cX_0 + dX_1, 0) = d.$

Let I be the ideal in $\mathbb{C}[A, B, C, D, P, Q, R, S, T, U]$ generated by

$\{A^3C^2 - P, (A^3D^2 + 6A^2BCD + 3AB^2C^2) - 10R, (2A^3CD + 3A^2BC^2) - 5Q,$ $B^3D^2 - U, (3AB^2D^2 + 2B^3CD) - 5T, (3A^2BD^2 + B^3C^2 + 6AB^2CD) - 10S\},$

Note that f has the form $l_1^3 l_2^2$ for some linear forms l_1 and l_2 over $\mathbb{C}$ iff $\mathbf{V}(I)(\subset \mathbb{C}[X_0, X_1]_1 \oplus \mathbb{C}[X_0, X_1]_1 \oplus \mathbb{C}[X_0, X_1]_5)$ contains a point whose last co-ordinate is f.

There are 189 polynomials in the Gröbner basis for I with respect to lexicographic order. Only the polynomials which are needed for this proof are attached in Appendix B.5. The interested reader may contact the author for the complete and extensive Maple output.

By the Elimination Theorem 4.13, we obtain

$$\begin{aligned} I \cap \mathbb{C}[P,Q,R,S,T,U] = I_4 &= \langle \mathrm{j}_1, \ldots, \mathrm{j}_{60} \rangle, \\ I \cap \mathbb{C}[D,P,Q,R,S,T] = I_3 &= \langle \mathrm{j}_1, \ldots, \mathrm{j}_{60} \rangle, \\ I \cap \mathbb{C}[C,D,P,Q,R,S,T] = I_2 &= \langle \mathrm{j}_1, \ldots, \mathrm{j}_{111} \rangle, \\ I \cap \mathbb{C}[B,C,D,P,Q,R,S,T] = I_1 &= \langle \mathrm{j}_1, \ldots, \mathrm{j}_{143} \rangle. \end{aligned}$$

Hence if f has the form $l_1^3 l_2^2$ for some linear forms l_1 and l_2 over $\mathbb{C}$, $\mathrm{j}_1, \ldots, \mathrm{j}_{60}$ vanish for f.

To prove the converse, assume that $\mathrm{j}_1, \ldots, \mathrm{j}_{60}$ vanish for f. Then $f \in \mathbf{V}(I_4)$. The idea is to extend f one co-ordinate at a time: first to (d, f), to (c, d, f) then (b, c, d, f) and then to (a, b, c, d, f). We will use the Extension Theorem 4.14 at each step.

Notice that $I_3 = I_4$. Therefore, for all $d \in \mathbb{C}$, $(d, f) \in \mathbf{V}(I_3)$. We choose d to be non-zero. Since I_3 is the first elimination ideal of I_2, the next step is to go from I_3 to I_2. The extension step fails only when the leading coefficients vanish simultaneously. Notice that $\mathrm{j}_{109}, \mathrm{j}_{108}, \mathrm{j}_{105}, \mathrm{j}_{96} \in I_3$ and the coefficient of C^2

- in j_{109} is $(6pr - 5q^2)$,
- in j_{108} is $(9ps - 5rq)$,
- in j_{105} is $2(3qs - 2r^2)$,
- in j_{96} is u^2.

Assume firstly that at least one of these coefficients is non-zero. By the Extension Theorem 4.14, there exists $c \in \mathbb{C}$ such that $(c, d, f) \in \mathbf{V}(I_2)$.

Since I_2 is the first elimination ideal of I_1, the next step is to go from I_2 to I_1. Since $\mathrm{j}_{112} \in I_1$ and the coefficient of B^3 in j_{112} is d^2, which is non-zero, it follows from the Extension Theorem 4.14 that there exists $b \in \mathbb{C}$ such that $(b, c, d, f) \in \mathbf{V}(I_1)$.

Since I_1 is the first elimination ideal of I, the next step is to go from I_1 to I. Since $\mathrm{j}_{187} \in I$ and the coefficient of A^3 in j_{187} is d^3, which is non-zero, it follows from the Extension Theorem 4.14 that there exists $a \in \mathbb{C}$ such that $(aX_0 + bX_1, cX_0 + dX_1, f) \in \mathbf{V}(I)$. Thus f has the form $l_1^3 l_2^2$ for some linear forms l_1 and l_2 over $\mathbb{C}$.

If on the other hand all of the coefficients in the above list are zero, then substituting $u = 0$ in j_1 implies $t = 0$. Therefore,

$$\begin{aligned} f &= pX_0^5 + 5qX_0^4X_1 + 10rX_0^3X_1^2 + 10sX_0^2X_1^3 \\ &= X_0^2(pX_0^3 + 5qX_0^2X_1 + 10rX_0X_1^2 + 10sX_1^3) \end{aligned}$$

Now $(pX_0^3 + 5qX_0^2X_1 + 10rX_0X_1^2 + 10sX_1^3)$ is binary cubic form, and the Hessian of this cubic form is

$$(6pr - 5q^2)X_0^2 + (9ps - 5rq)X_0X_1 + (3qs - 2r^2)X_1^2.$$

Since the Hessian of this cubic form is zero, this binary cubic form is a cube of a linear form. This implies f has the form $l_1^3 l_2^2$ for some linear forms l_1 and l_2 over $\mathbb{C}$. Hence the result. □

It turns out that the ideal $\mathcal{I}(3,2)$ is the radical of the ideal generated by the coefficients of the covariant

$$4(\mathfrak{I}, \mathfrak{J})^{(1)} + \mathcal{H}^2,$$

where $(\mathcal{J},\mathcal{J})^{(1)}$ is the covariant from $\mathbb{C}[X_0,X_1]_5$ to $\mathbb{C}[X_0,X_1]_{12}$ defined by

$$(\mathcal{J},\mathcal{J})^{(1)}(f) = (f,\mathcal{J}(f))^{(1)}.$$

Notice that from the calculations of the above covariant (Maple work sheet attached in Appendix D)

$$\begin{aligned}
4(f,\mathcal{J}(f))^{(1)} + \mathcal{H}^2(f) \;=\;& 80000\left\{\mathrm{j}_{59}(f)X_0^{12} + 3\mathrm{j}_{58}(f)X_0^{11}X_1\right.\\
&+\left(\frac{-51}{2}\mathrm{j}_{55}(f) + \frac{33}{2}\mathrm{j}_{57}(f)\right)X_0^{10}X_1^2 + \left(10\mathrm{j}_{54}(f) - 65\mathrm{j}_{52}(f)\right)X_0^9X_1^3\\
&+\left(-72\mathrm{j}_{45}(f) + 18\mathrm{j}_{51}(f) + 39\mathrm{j}_{50}(f)\right)X_0^8X_1^4\\
&-\left(\frac{264}{4}\mathrm{j}_{42}(f) + \frac{990}{32}\mathrm{j}_{41}(f) + 66\mathrm{j}_{49}(f)\right)X_0^7X_1^5\\
&-\left(\frac{220}{12}\mathrm{j}_{33}(f) + \frac{19}{3}\mathrm{j}_{48}(f) + \frac{608}{6}\mathrm{j}_{40}(f) - \frac{512}{3}\mathrm{j}_{34}(f)\right)X_0^6X_1^6\\
&-\left(\frac{114}{9}\mathrm{j}_{39}(f) - \frac{468}{9}\mathrm{j}_{27}(f) + \frac{264}{9}\mathrm{j}_{32}(f)\right)X_0^5X_1^7\\
&+\left(120\mathrm{j}_{26} - 90\mathrm{j}_{31}(f) + \frac{225}{3}\mathrm{j}_{24}(f)\right)\\
&+\left(\frac{10}{3}\mathrm{j}_{25}(f) + \frac{155}{3}\mathrm{j}_{23}(f)\right)X_0^3X_1^9\\
&\left(\frac{-51}{2}\mathrm{j}_{6}(f) + \frac{33}{2}\mathrm{j}_{22}(f)\right)X_0^2X_1^{10}\\
&\left.+3\,\mathrm{j}_{21}(f)X_0X_1^{11} + \mathrm{j}_2(f)X_1^{12}\right\}.
\end{aligned}$$

The ideal generated by the polynomials appearing in the above covariant (i.e. $\mathrm{j}_{59}, \mathrm{j}_{58}, \mathrm{j}_{57}, \mathrm{j}_{55}, \mathrm{j}_{54}, \mathrm{j}_{52}, \mathrm{j}_{51}, \mathrm{j}_{50}, \mathrm{j}_{49}, \mathrm{j}_{48}, \mathrm{j}_{45}, \mathrm{j}_{42}, \mathrm{j}_{41}, \mathrm{j}_{40},$
$\mathrm{j}_{39}, \mathrm{j}_{34}, \mathrm{j}_{33}, \mathrm{j}_{32}, \mathrm{j}_{31}, \mathrm{j}_{27}, \mathrm{j}_{26}, \mathrm{j}_{25}, \mathrm{j}_{24}, \mathrm{j}_{23}, \mathrm{j}_{22}, \mathrm{j}_{21}, \mathrm{j}_{6}, \mathrm{j}_{2}$) is in fact also generated by the poly-

nomials $\mathrm{j}_1, \ldots, \mathrm{j}_{60}$ (work sheet is attached in Appendix B.7).

Hence we have the following:

The following are equivalent for a binary quintic form f.

1. f has the form $l_1^3 l_2^2$ for some linear forms l_1, l_2 over $\mathbb{C}$.

2. The covariant $4(\mathfrak{J}, \mathfrak{J})^{(1)} + \mathcal{H}^2$ vanishes for f.

Next we shall look for a covariant generator for an ideal whose radical is $\mathcal{I}(2,2,1)$.

Lemma 4.31 *The following are equivalent for a binary quintic form f.*

1. *f has the form $q^2 l$ for some quadratic form q over $\mathbb{C}$ and linear form l over $\mathbb{C}$, i.e. f belongs to $\mathcal{F}(2,2,1)$.*

2. *$\mathrm{k}_1, \ldots, \mathrm{k}_{25}$ vanish for f (listed in Appendix B.6).*

Proof: Let

$$f = pX_0^5 + 5qX_0^4X_1 + 10rX_0^3X_1^2 + 10sX_0^2X_1^3 + 5tX_0X_1^4 + uX_1^5$$

be a binary quintic form. Then f has the form $q^2 l$ for some quadratic, linear forms q, l over $\mathbb{C}$ if and only if there exist $a, b, c, d, e \in \mathbb{C}$ such that

$$f = (aX_0^2 + 2bX_0X_1 + cX_1^2)^2(dX_0 + eX_1).$$

This is equivalent to the following system of equations:

$$a^2 d = p,$$

$$\begin{aligned}(a^2e + 4abd) &= 5q,\\ (2acd + 4b^2d + 4abe) &= 10r,\\ (4b^2e + 4bcd + 2ace) &= 10s,\\ (4bce + c^2d) &= 5t,\\ c^2e &= u.\end{aligned}$$

Let $A, B, C, D, P, Q, R, S, T, U$ be co-ordinate functions on $\mathbb{C}[X_0, X_1]_2 \oplus \mathbb{C}[X_0, X_1]_1 \oplus \mathbb{C}[X_0, X_1]_5$ such that

$P(0,0,f) = p, Q(0,0,f) = q, R(0,0,f) = r, S(0,0,f) = s, T(0,0,f) = t,$ $U(0,0,f) = u, A(aX_0^2 + 2bX_0X_1 + cX_1^2, 0, 0) = a, B(aX_0^2 + 2bX_0X_1 + cX_1^2, 0, 0) = b,$ $C(aX_0^2 + 2bX_0X_1 + cX_1^2, 0, 0) = c, D(0, dX_0 + eX_1, 0) = d, E(0, dX_0 + eX_1, 0) = e.$

Let I be the ideal in $\mathbb{C}[A, B, C, D, E, P, Q, R, S, T, U]$ generated by

$\{(2ACD + 4B^2D + 4ABE) - 10R, (A^2E + 4ABD) - 5Q, (4BCE + C^2D) - 5T,$ $A^2D - P, (4B^2E + 4BCD + 2ACE) - 10S, C^2E - U\},$

and note that f has the form q^2l for some quadratic form q over $\mathbb{C}$ and linear form l over $\mathbb{C}$ iff $\mathbf{V}(I)(\subset \mathbb{C}[X_0, X_1]_2 \oplus \mathbb{C}[X_0, X_1]_1 \oplus \mathbb{C}[X_0, X_1]_5)$ contains a point whose last co-ordinate is f.

The Sun microsystem computer took approximately 3 days to compute a Gröbner basis. There are 588 polynomials in the Gröbner basis for I with respect to lexicographic order. Only the polynomials which are needed for this proof are listed in Appendix B.6. The interested reader may contact the author for the complete and extensive Maple output.

By the Elimination Theorem 4.13, we obtain

$$I \cap \mathbb{C}[P, Q, R, S, T, U] = I_5 \;=\; \langle \mathsf{k}_1, \ldots, \mathsf{k}_{25} \rangle.$$

Hence if f has the form $q^2 l$ for some quadratic form q over $\mathbb{C}$ and linear form l over $\mathbb{C}$, then $\mathsf{k}_1, \ldots, \mathsf{k}_{25}$ vanish for f.

To prove the converse, assume that $\mathsf{k}_1, \ldots, \mathsf{k}_{25}$ vanish for f. Then $f \in \mathbf{V}(I_4)$.

Notice that $I_4 = I_5$. Therefore, for all $e \in \mathbb{C}$, $(e, f) \in \mathbf{V}(I_3)$. We choose e to be non-zero. Since I_4 is the first elimination ideal of I_3, the next step is to go from I_4 to I_3. The extension step fails only when the leading coefficients vanish simultaneously.

Notice that $\mathsf{k}_{218}, \mathsf{k}_{224}, \mathsf{k}_{226}, \mathsf{k}_{227}, \mathsf{k}_{230}, \mathsf{k}_{232}, \mathsf{k}_{233}, \mathsf{k}_{234} \in I_3$ and the coefficient of D^3

- in k_{218} is $(3\,t^2\,q - 6\,r\,t\,s + 3\,s^3)$,
- in k_{224} is $(24t^2 p + 30qst - 120r^2 t + 60rs^2)$,
- in k_{226} is $(36pst - 90qrt + 45qs^2)$,
- in k_{227} is $(4ps^2 - 5q^2 t)$,
- in k_{230} is $(12ptq - 24psr + 15sq^2)$,
- in k_{232} is $(8p^2 t + 10pqs - 40r^2 p + 25rq^2)$,
- in k_{233} is $(16p^2 S + 25q^3 - 40pqr)$

and the coefficient of D^5 in k_{234} is u.

Assume firstly that at least one of these coefficients is non-zero. By the Extension Theorem 4.14, there exists $d \in \mathbb{C}$ such that $(d, e, f) \in \mathbf{V}(I_3)$.

Since I_3 is the first elimination ideal of I_2, the next step is to go from I_3 to I_2. Since $\mathsf{k}_{235} \in I_2$ and the coefficient of C^2 in k_{235} is equal to e which is non-zero, it follows from the Extension Theorem 4.14 that there exists $c \in \mathbb{C}$ such that $(c, d, e, f) \in \mathbf{V}(I_2)$.

Since I_2 is the first elimination ideal of I_1, the next step is to go from I_2 to I_1. Since $\mathsf{k}_{550} \in I_1$ and the coefficient of B^3 in k_{550} is $4e^2$, which is non-zero, it follows from the Extension Theorem 4.14 that there exists $b \in \mathbb{C}$ such that $(2b, c, d, e, f) \in \mathbf{V}(I_1)$.

Since I_1 is the first elimination ideal of I, the next step is to go from I_1 to I. Since $\mathsf{k}_{587} \in I$ and the coefficient of A^2 in h_{587} is $5e$, which is non-zero, it follows from the Extension Theorem 4.14 that there exists $a \in \mathbb{C}$ such that f has the form q^2l for some quadratic form q over $\mathbb{C}$ and linear form l over $\mathbb{C}$.

If on the other hand, all of the above listed coefficients are zero, then since $u = 0$,

$$\begin{aligned} f &= pX_0^5 + 5qX_0^4X_1 + 10rX_0^3X_1^2 + 10sX_0^2X_1^3 + 5tX_0X_1^4 \\ &= X_0(pX_0^4 + 5qX_0^3X_1 + 10rX_0^2X_1^2 + 10sX_0X_1^3 + 5tX_1^4), \end{aligned}$$

with $\mathsf{g}_1, \ldots, \mathsf{g}_7$ (listed in Appendix B.2) vanish for $P(f) = p, Q(f) = \frac{5q}{4}, R(f) = \frac{10r}{6}, S(f) = \frac{10s}{4}, T(f) = 5t$. Thus the Jacobian of the binary quartic form

$$(pX_0^4 + 5qX_0^3X_1 + 10rX_0^2X_1^2 + 10sX_0X_1^3 + 5tX_1^4)$$

is zero. Therefore, by Theorem 4.28 this binary quartic form is a square of a binary quadratic form, say g^2. This implies f has the form $g^2\,X_0$ for some quadratic form g over $\mathbb{C}$. Hence the result. □

A covariant for the ideal $\mathcal{I}(2,2,1)$

By working with the Gröbner basis of the elimination ideal $I_5 = \mathcal{I}(2,2,1)$, I have been able to determine a covariant

$$\Psi = -6(\mathcal{P},\mathcal{J})^{(1)} - 30\mathcal{J}\,(\mathcal{H},\mathcal{P})^{(2)} - 5\mathcal{P}^2\,\mathcal{J} + 3\mathcal{H}(\mathcal{J},\mathcal{P})^{(2)},$$

such that the radical of the coefficient ideal of Ψ is $\mathcal{I}(2,2,1)$.

The leading coefficient of any such covariant must satisfy

$$5\,degree - 2\,weight > 0.$$

This follows from the general theory of covariants of binary quintic forms (see [Schur 1968] page 59). Accordingly, the procedure is this:

1. Select the Gröbner basis polynomials which satisfy the above inequality;
2. From this selection, retain, for each degree only the polynomials with least weight;
3. Make up expressions involving the basic covariants of binary quintic forms (transvectants, Hessians, ...) with leading coefficients equal to one of the remaining list in step 2;
4. Checking the covariants resulting from step 3 in turn, turns up Ψ as the only one satisfying our requirements.

Notice that,

$$\frac{1}{10368\times 10^5}\left\{-6(\mathcal{P}(f),\mathcal{J}(f))^{(1)}-30f\,(\mathcal{H}(f),\mathcal{P}(f))^{(2)}-5\mathcal{P}^2(f)\,f+3\mathcal{H}(f)(f,\mathcal{P}(f))^{(2)}\right\} =$$

$$\mathsf{k}_{25}(f)X_0^9+\mathsf{k}_{23}(f)X_0^8X_1+\mathsf{k}_{22}(f)X_0^7X_1^2+\mathsf{k}_{18}(f)X_0^6X_1^3+\mathsf{k}_{15}(f)X_0^5X_1^4$$

$$+\mathsf{k}_{13}(f)X_0^4X_1^5+\mathsf{k}_{10}(f)X_0^3X_1^6+\mathsf{k}_{9}(f)X_0^2X_1^7+\mathsf{k}_{4}(f)X_0X_1^8+\mathsf{k}_{2}(f)X_1^9.$$

By the Gröbner basis of these polynomials with respect to lexicographic order (attached in Appendix B.8), we have

$$\langle \mathsf{k}_1,\ldots,\mathsf{k}_{25}\rangle=\langle \mathsf{k}_{25},\mathsf{k}_{23},\mathsf{k}_{22},\mathsf{k}_{18},\mathsf{k}_{15},\mathsf{k}_{13},\mathsf{k}_{10},\mathsf{k}_{9},\mathsf{k}_{4},\mathsf{k}_{2}\rangle.$$

Hence we have the following:

The following are equivalent for a binary quintic form f.

1. f has the form q^2l for some quadratic form q over $\mathbb{C}$ and linear form l over $\mathbb{C}$.
2. The covariant $-6(\mathcal{P},\mathcal{J})^{(1)}-30\mathcal{I}\,(\mathcal{H},\mathcal{P})^{(2)}-5\mathcal{P}^2\mathcal{I}+3\mathcal{H}(\mathcal{I},\mathcal{P})^{(2)}$ vanishes for f.

Now we shall give a direct proof of the above result.

Proof:

Every binary quintic form in $\mathcal{F}(2,2,1)$ is equivalent (with respect to the action by $GL_2(\mathbb{C})$)to one of the following $X_0^5, X_0^4X_1, X_0^3X_1^2, X_0^2X_1^2(X_0+X_1)$. We see from the Maple work sheet(attached in Appendix D) that $-6(\mathcal{P},\mathcal{J})^{(1)}-30\mathcal{I}\,(\mathcal{H},\mathcal{P})^{(2)}-5\mathcal{P}^2\mathcal{I}+3\mathcal{H}(\mathcal{I},\mathcal{P})^{(2)}$ vanishes for

$$X_0^5, X_0^4X_1, X_0^3X_1^2, X_0^2X_1^2(X_0+X_1),$$

and this covariant does not vanish for $X_0^3X_1(X_0+X_1)$. Since $-6(\mathcal{P},\mathcal{J})^{(1)}-30\mathcal{J}\,(\mathcal{H},\mathcal{P})^{(2)}-5\mathcal{P}^2\mathcal{J}+3\mathcal{H}(\mathcal{J},\mathcal{P})^{(2)}$ is covariant, it vanishes for every binary quintic form in $\mathcal{F}(2,2,1)$, and does not vanish for every binary quintic form in $\mathcal{F}(3,2)$, or in $\mathcal{F}(2,1,1,1)$. Hence the result. □

The figure Fig. 4.5 summarizes the results for binary quintic forms.

Remark 4.32 *1. It is a not a fluke that we were able to extend the partial solution in the above proofs using elimination theory. In fact, Prof. H. K. Farahat pointed out that we can use the Theorem of implicitation([Cox, Little, O'Shea 1996] page 54) to deduce that the ideal is generated by the gröbner basis, because of the fact that $\mathcal{F}(m_1,\ldots,m_s)$ is closed.*

2. My External Examiner Dr. A. W. Herman has pointed out to me two papers ([Rollero 1990], [Rollero 1988]) by Aldo Rollero related to my work, which I was not aware of. I have not yet looked at the papers. Mathematical Reviews (92g:11038 11E76, 90d:14044 14J40 (11E76)) contains only a summary review.

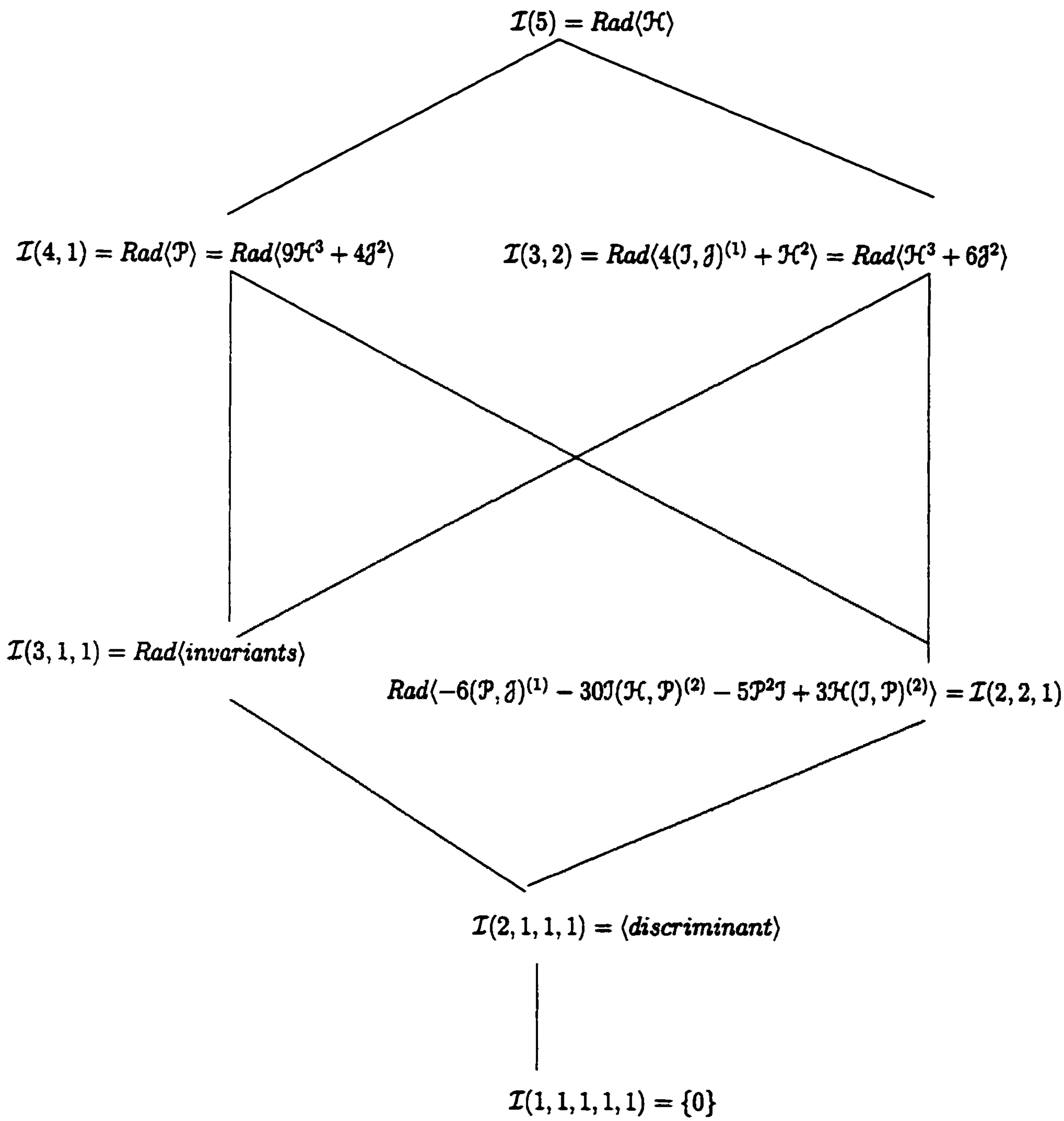

Figure 4.5: Ideals for binary quintic forms

Chapter 5

Transpose Systems of Binary Homogeneous Polynomial Equations

5.1 Some Topological Subsets of $\mathbb{C}^{(1)}_{r+1,r+1}$

Now we turn to the study of transpose systems of binary homogeneous polynomial equations which was introduced at the end of Chapter 3.

Recall that for $0 \leq l \leq (r+1)$,

- $$\begin{aligned} \mathbb{C}^{(l)}_{(r+1),(r+1)} &= \text{the set of all } (r+1)\times(r+1) \text{ matrices of rank less than} \\ &\quad\ \text{or equal to } l \\ &= \mathbf{V}(\text{ all } (l+1)\times(l+1) \text{ minors}). \end{aligned}$$

- $\mathcal{P}(C) = \{[X] = [X_0, X_1] \in \mathbb{P}^1 \mid CX^{[r]} = 0\}, C \in \mathbb{C}_{r+1,r+1}$.

- $\mathcal{E}^{(l)}(k) = \{C \in \mathbb{C}^{(l)}_{r+1,r+1} | \#\mathcal{P}(C) = \#\mathcal{P}(C^T) = k\}, k \geq 0$.

Since $\mathbb{C}$ is algebraically closed, $\mathcal{E}^{(1)}(0)$ is an empty set.

Notice that for $k > 0$,

$$\mathcal{E}^{(1)}(k) = \{C \in \mathbb{C}^{(1)}_{r+1,r+1} | \#\mathcal{P}(C) = k\} \cap \{C \in \mathbb{C}^{(1)}_{r+1,r+1} | \#\mathcal{P}(C^T) = k\}.$$

Definition 5.1 *1. Define for* $k \geq 1$, $\mathcal{S}(k)$ *to be the set of all* $(r+1) \times (r+1)$ *matrices* C *with rank* 1 *such that the system* $CX^{[r]} = 0$ *represents at most* k *projective points and the zero matrix. That is,*

$$\mathcal{S}(k) = \{C \in \mathbb{C}^{(1)}_{r+1,r+1} | \#\mathcal{P}(C) \leq k\} \cup \{0\}.$$

2. Define for $k \geq 1$, $\mathcal{S}^T(k)$ *to be the set of all* $(r+1) \times (r+1)$ *matrices* C *with rank* 1 *such that the system* $C^T X^{[r]} = 0$ *represents at most* k *projective points and the zero matrix. That is,*

$$\mathcal{S}^T(k) = \{C \in \mathbb{C}^{(1)}_{r+1,r+1} | \#\mathcal{P}(C^T) \leq k\} \cup \{0\}.$$

Now $\mathcal{E}^{(1)}(k)$ is the intersection of $(\mathcal{S}(k) \cap \mathcal{S}^T(k))$, with the complement of the set $\mathcal{S}(k-1) \cup \mathcal{S}^T(k-1)$, in $\mathbb{C}^{(1)}_{r+1,r+1}$.

It turns out that $\mathcal{S}(k)$ and $\mathcal{S}^T(k)$ are affine closed for each $k \geq 1$.

Theorem 5.2 *For all* $1 \leq k \leq r$,

1. *The set* $\mathcal{S}(k)$ *of all* $(r+1) \times (r+1)$ *matrices* C *with rank* 1 *such that the system* $CX^{[r]} = 0$ *represents at most* k *projective points and the zero matrix is an affine closed subset of* $\mathbb{C}^{(1)}_{r+1,r+1}$.
2. *The set* $\mathcal{S}^T(k)$ *of all* $(r+1) \times (r+1)$ *matrices* C *with rank* 1 *such that the system* $C^T X^{[r]} = 0$ *represents at most* k *projective points and the zero matrix is an affine closed subset of* $\mathbb{C}^{(1)}_{r+1,r+1}$.

<ins>Proof:</ins>

1. For each $i = 1, \ldots, r+1$, we have the polynomial mapping,

$$\rho_i : \mathbb{C}_{r+1,r+1} \to \mathbb{C}[X_0, X_1]_r$$

   where $\rho_i(C) = \sum_{j=1}^{r+1} c_{ij} X_0^{r-j+1} X_1^{j-1}$ for $C = (c_{ij}) \in \mathbb{C}_{r+1,r+1}$. Each ρ_i carries the set $\mathcal{S}(k)$ into the union $\mathcal{F}_k$ of the closed sets $\mathcal{F}(m_1, \ldots, m_k)$ with $m_1 + \ldots + m_k = r$. In fact, $\mathcal{S}(k)$ is the intersection of sets $\mathbb{C}^{(1)}_{r+1,r+1}, \rho_i^{-1}(\mathcal{F}_k), i = 1, \ldots, r+1$. Since $\mathcal{F}_k$ is closed, each of these sets is closed, hence $\mathcal{S}(k)$ is an affine closed subset of $\mathbb{C}^{(1)}_{r+1,r+1}$.

2. This follows by applying part 1 to C^T instead of C, noting that C^T also has rank 1. □

Thus we have the following ascending chains of affine closed sets:

$$\{0\} \subset \mathcal{S}(1) \subset \ldots \subset \mathcal{S}(r) = \mathbb{C}^{(1)}_{(r+1),(r+1)}$$

and

$$\{0\} \subset \mathcal{S}^T(1) \subset \ldots \subset \mathcal{S}^T(r) = \mathbb{C}^{(1)}_{(r+1),(r+1)}.$$

An interesting question about these sets is whether these affine closed sets are irreducible.

Since $\mathcal{S}(r) = \mathbb{C}^{(1)}_{r+1,r+1}$, it is irreducible.

We know that $\mathbb{C}_{1,r+1}$ and $\mathcal{F}(r)$ are irreducible (Theorem 4.3). Therefore, $\mathbb{C}_{1,(r+1)} \times \mathcal{F}(r)$ is irreducible (see [Shafarevich 1974] page 24). The closed set $\mathcal{S}(1)$ is the image of the polynomial mapping from $\mathbb{C}_{1,(r+1)} \times \mathcal{F}(r)$ to $\mathbb{C}^{(1)}_{r+1,r+1}$ which takes $(v, wX^{[r]})$

to $v^T w$, where $v, w \in \mathbb{C}_{1,r+1}$. Hence $\mathcal{S}(1)$ is irreducible.

Similarly, since the closed set $\mathcal{S}(r-1)$ is the image of the polynomial mapping from the irreducible closed set $\mathbb{C}_{1,(r+1)} \times \mathcal{F}(2, 1 \dots, 1)$ to $\mathbb{C}^{(1)}_{r+1,r+1}$ which takes $(v, wX^{[r]})$ to $v^T w$, where $v, w \in \mathbb{C}_{1,r+1}$, $\mathcal{S}(r-1)$ is irreducible.

Hence we have the following lemma.

Lemma 5.3 *1. The set* $\mathcal{S}(k)$ *of all* $(r+1) \times (r+1)$ *matrices* C *with rank* 1 *such that the system* $CX^{[r]} = 0$ *represents at most* k *projective points and the zero matrix is irreducible, when* $k = 1, r-1, r$.

2. The set $\mathcal{S}^T(k)$ *of all* $(r+1) \times (r+1)$ *matrices* C *with rank* 1 *such that the system* $C^T X^{[r]} = 0$ *represents at most* k *projective points and the zero matrix is irreducible, when* $k = 1, r-1, r$.

It turns out that when $r = 4$, the set $\mathcal{S}(2)$ of all 5×5 matrices C with rank 1 such that the system $CX^{[r]} = 0$ represents at most 2 projective points and the zero matrix is reducible. Indeed it is the union of the following affine closed non-empty proper subsets of $\mathcal{S}(2)$:

1. the intersection of all sets $\rho_i{}^{-1}\mathcal{F}(2,2), i = 1, \dots, r+1$
2. the intersection of all sets $\rho_i{}^{-1}\mathcal{F}(3,1), i = 1, \dots, r+1$.

By using Theorem 5.2 and the above remark about the sets $\mathcal{E}^{(1)}(k)$ we have the following lemma;

Lemma 5.4 *1. The set* $\mathcal{E}^{(1)}(r)$ *of all* $(r+1) \times (r+1)$ *matrices* C *with rank* 1 *such that both the systems* $CX^{[r]} = 0$ *and* $C^T X^{[r]} = 0$ *represent* r *projective points is* *a non-empty affine open subset of* $\mathbb{C}^{(1)}_{r+1,r+1}$. *Therefore* $\mathcal{E}^{(1)}(r)$ *is a dense subset of* $\mathbb{C}^{(1)}_{r+1,r+1}$.

2. *For* $2 \leq k \leq r-1$, *the set* $\mathcal{E}^{(1)}(k)$ *of all* $(r+1)\times(r+1)$ *matrices with rank equal to* 1 *such that both the systems* $CX^{[r]} = 0$ *and* $C^T X^{[r]} = 0$ *represent* k *projective points is an intersection of an open subset and a closed subset (i.e. a locally closed subset) of* $\mathbb{C}^{(1)}_{r+1,r+1}$.

3. *The set* $\mathcal{E}^{(1)}(1) \cup \{0\}$ *of all* $(r+1)\times(r+1)$ *matrices* C *with rank* 1 *such that both the systems* $CX^{[r]} = 0$ *and* $C^T X^{[r]} = 0$ *represent* 1 *projective point with the zero matrix is an irreducible closed subset of* $\mathbb{C}^{(1)}_{r+1,r+1}$. *Moreover*

$$3 \leq dim(\mathcal{E}^{(1)}(1) \cup \{0\}) \leq 4.$$

Proof:

1. We know that $\mathcal{E}^{(1)}(r)$ is the intersection of $(\mathcal{S}(r) \cap \mathcal{S}^T(r))$, with the complement of the set $\mathcal{S}(r-1) \cup \mathcal{S}^T(r-1)$. Since $\mathcal{S}(r) = \mathcal{S}^T(r) = \mathbb{C}^{(1)}_{r+1,r+1}$, $\mathcal{E}^{(1)}(r)$ is the complement of the closed set $\mathcal{S}(r-1) \cup \mathcal{S}^T(r-1)$ (see Theorem 5.2). Thus $\mathcal{E}^{(1)}(r)$ is an open subset of $\mathbb{C}^{(1)}_{r+1,r+1}$.

   Since $\mathbb{C}^{(1)}_{r+1,r+1}$ is irreducible, every non-empty open subset of $\mathbb{C}^{(1)}_{r+1,r+1}$ is dense. Therefore, if $\mathcal{E}^{(1)}(r)$ is non-empty then $\mathcal{E}^{(1)}(r)$ is a dense subset of $\mathbb{C}^{(1)}_{r+1,r+1}$. It remains only to show that $\mathcal{E}^{(1)}(r)$ is non-empty. For that we shall show that

$$b = \begin{pmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & & & & \vdots \\ -1 & 0 & \dots & 0 & 1 \end{pmatrix} \in \mathcal{E}^{(1)}(r).$$

First of all $bX^{[r]} = b^T X^{[r]} = 0$, if and only if $X_0^r - X_1^r = 0$. Since $\mathbb{C}$ is algebraically closed and of characteristic zero, $X_0^r - X_1^r$ can be factored into r distinct linear forms. Hence $bX^{[r]} = 0$ represents exactly r projective points. Thus $b \in \mathcal{E}^{(1)}(r)$.

2. Since the intersection of two affine closed sets is affine closed and the union of two affine closed sets is affine closed, the result follows immediately from Theorem 5.2.

3. $\mathcal{E}^{(1)}(1) \cup \{0\} = (\mathcal{S}(1) \cap \mathcal{S}^T(1))$. Hence by Theorem 5.2, $\mathcal{E}^{(1)}(1) \cup \{0\}$ is an affine closed subset of $\mathbb{C}^{(1)}_{r+1,r+1}$. This closed set is the image of the polynomial mapping θ from $\mathcal{F}(r) \times \mathcal{F}(r)$ to $\mathbb{C}^{(1)}_{r+1,r+1}$ which takes $(vX^{[r]}, wX^{[r]})$ to $v^T w$, where $v, w \in \mathbb{C}_{1,r+1}$. By Theorem 4.3, $\mathcal{F}(r)$ is irreducible, so $\mathcal{F}(r) \times \mathcal{F}(r)$ is irreducible (see [Shafarevich 1974] page 24). Thus, the closed set $\mathcal{E}^{(1)}(1) \cup \{0\}$ is the image of the polynomial mapping from an irreducible closed set. Hence $\mathcal{E}^{(1)}(1) \cup \{0\}$ is irreducible. By Theorem 4.4 $dim(\mathcal{F}(r)) = 2$, therefore the dimension of $\mathcal{F}(r) \times \mathcal{F}(r)$ is 4. Now by the Theorem of Dimension of Fibres (Reference [Shafarevich 1974] p. 60), we have $dim(\mathcal{E}^{(1)}(1) \cup \{0\}) \leq 4$. Since $\theta^{-1}(v^T w) = \{(\alpha v X^{[r]}, \alpha^{-1} w X^{[r]}) | \alpha \neq 0\}$, the dimension of $\theta^{-1}(v^T w)$ is 1. Again

by the Theorem of Dimension of Fibres (Reference [Shafarevich 1974] p. 60), we have $3 \leq dim(\mathcal{E}^{(1)}(1) \cup \{0\})$. ◫

As a summary we have:

- $\mathcal{E}^{(1)}(r)$ is dense in $\mathbb{C}^{(1)}_{r+1,r+1}$.
- $\left.\begin{array}{c} \mathcal{E}^{(1)}(r-1) \\ \vdots \\ \mathcal{E}^{(1)}(2) \end{array}\right\}$ are locally closed in $\mathbb{C}^{(1)}_{r+1,r+1}$.
- $\mathcal{E}^{(1)}(1) \cup \{0\}$ is an affine closed subset of $\mathbb{C}^{(1)}_{r+1,r+1}$.

Figure 5.1: Some topological subsets of $\mathbb{C}^{(1)}_{r+1,r+1}$

5.2 An Ascending Chain of Dense Subsets

Theorem 5.5 *For $2 \leq l \leq (r+1)$, the set $\mathcal{E}^{(l)}(0)$ of all $(r+1)\times(r+1)$ matrices with rank less than or equal to l such that both the systems $CX^{[r]} = 0$, and $C^T X^{[r]} = 0$ have only the trivial solution is a dense subset of $\mathbb{C}^{(l)}_{r+1,r+1}$.*

Proof: Let $2 \leq l \leq (r+1)$. Since $\mathbb{C}^{(l)}_{r+1,r+1}$ is irreducible, every non-empty open subset of $\mathbb{C}^{(l)}_{r+1,r+1}$ is dense. And if a non-empty subset of $\mathcal{E}^{(l)}(0)$ is dense in $\mathbb{C}^{(l)}_{r+1,r+1}$ then $\mathcal{E}^{(l)}(0)$ is dense in $\mathbb{C}^{(l)}_{r+1,r+1}$. Hence it follows that in order to prove the above result, it suffices to find a non-empty subset of $\mathcal{E}^{(l)}(0)$ which is open in $\mathbb{C}^{(l)}_{r+1,r+1}$. We will consider two different cases: $2 \leq l \leq r$ and $l = (r+1)$.

First we shall define the following notation:

The k-rowed minor obtained from a matrix A by retaining only the elements belonging to rows with suffixes $r_1, \ldots, r_k$ and columns with suffixes $s_1, \ldots, s_k$ will be denoted by

$$|A(r_1, \ldots, r_k; s_1, \ldots, s_k)|.$$

Now assume that $2 \leq l \leq r$. Every matrix with rank l has at least one $l \times l$ submatrix with non-vanishing determinant.

Suppose A is an $(r+1) \times (r+1)$ matrix over $\mathbb{C}$ such that $|A(1, \ldots, l; 1, \ldots, l)| \neq 0$. Then the first l rows (columns) of A are linearly independent and every row (column) of A may be expressed linearly in terms of these l rows (columns).(Reference [Mirsky 1961] on page 137.)

Therefore $AX^{[r]} = 0$ is equivalent to the following system of equations,

$$A_{i1}X_0^r + A_{i2}X_0^{r-1}X_1 + \ldots + A_{i(r+1)}X_1^r = 0, \forall i = 1, \ldots, l. \tag{5.1}$$

Since $l \geq 2$ and $|A(1, \ldots, l; 1, \ldots, l)| \neq 0$,

$$A_{11}X_0^r + A_{12}X_0^{r-1}X_1 + \ldots + A_{1(r+1)}X_1^r,$$

$$A_{21}X_0^r + A_{22}X_0^{r-1}X_1 + \ldots + A_{2(r+1)}X_1^r$$

are binary forms of degree r. If the resultant of these two binary forms is non-zero, then these two binary forms have no common linear factor (see [Bôcher 1964] p.202). In that case the first two equations in the system (5.1) have no common non-trivial solution, and hence $AX^{[r]} = 0$ has no non-trivial solution.

In similar manner, if the resultant of the two binary forms

$$A_{11}X_0^r + \ldots + A_{(r+1)1}X_1^r, A_{12}X_0^r + \ldots + A_{(r+1)2}X_1^r$$

is non-zero then $A^T X^{[r]} = 0$ has no non-trivial solution.

Therefore we shall consider the following set,

$$W_l := \{A \in \mathbb{C}^{(l)}_{r+1,r+1} \| A(1,\ldots,l;1,\ldots,l)| Res(p,q) Res(p',q') \neq 0\},$$

where

$$\begin{aligned} p &= \sum_{i=1}^{r+1} A_{1i} X_0^{r-i+1} X_1^{i-1}, \\ q &= \sum_{i=1}^{r+1} A_{2i} X_0^{r-i+1} X_1^{i-1}, \\ p' &= \sum_{i=1}^{r+1} A_{i1} X_0^{r-i+1} X_1^{i-1}, \\ q' &= \sum_{i=1}^{r+1} A_{i2} X_0^{r-i+1} X_1^{i-1}. \end{aligned}$$

Then W_l is a subset of $\mathcal{E}^{(l)}(0)$ which is an affine open subset of $\mathbb{C}^{(l)}_{r+1,r+1}$.

We show that W_l is non-empty. Define the matrix A in the following manner,

$A_{ii} = 1$, for $i = 1, \ldots, l$,

$A_{1(r+1)} = A_{(r+1)1} = -1$,

$A_{ij} = 0$ otherwise.

Then $A^T X^{[r]} = AX^{[r]} = 0$ is equivalent to the system

$$X_0^r - X_1^r = 0, X_0^{r-1}X_1 = 0.$$

Clearly $X_0^r - X_1^r, X_0^{r-1}X_1$ have no common non-trivial zero. Hence their resultant, $Res(X_0^r + X_1^r, X_0^{r-1}X_1) \neq 0$ (see [Bôcher 1964] page 202). Also $|A(1,\ldots,l;1,\ldots,l)| = 1$. Therefore, $A \in W_l$.

Now assume that $l = r+1$. Define

$$W_{r+1} := \{A \in \mathbb{C}_{r+1,r+1} | det(A) \neq 0\}.$$

Let $A \in W_{r+1}$. Then A^{-1} exists. Hence

$$AX^{[r]} = 0,$$

and

$$A^T X^{[r]} = 0$$

have no solution in $\mathbb{P}^1$, which implies $A \in \mathcal{E}^{(r+1)}(0)$. Hence $W_{r+1} \subset \mathcal{E}^{(r+1)}(0)$. Since $I \in W_{r+1}$, W_{r+1} is a non-empty open subset of $\mathbb{C}_{r+1,r+1}$. □

From the above theorem we have the following ascending chain of subsets:

$$\begin{array}{ccc}
\mathcal{E}^{(r+1)}(0) & \text{dense in} & \mathbb{C}_{(r+1),(r+1)} \\
\vdots & \cdots & \vdots \\
\uparrow & & \uparrow \\
\mathcal{E}^{(l)}(0) & \text{dense in} & \mathbb{C}^{(l)}_{(r+1),(r+1)} \\
\uparrow & & \uparrow \\
\vdots & \cdots & \vdots \\
\uparrow & & \uparrow \\
\mathcal{E}^{(2)}(0) & \text{dense in} & \mathbb{C}^{(2)}_{(r+1),(r+1)} \\
\uparrow & & \uparrow \\
\mathcal{E}^{(1)}(0) = \emptyset & & \mathbb{C}^{(1)}_{(r+1),(r+1)}.
\end{array}$$

Figure 5.2: An ascending chain of subsets

5.3 Further Inquiry

As a further inquiry we shall state the following problems:

1. For a given partition $(m_1, \ldots, m_s)$ of r, and a binary form f of degree r, can we say that there exists a covariant whose vanishing for f is a necessary and sufficient condition that f has the form $l_1^{m_1} \ldots l_s^{m_s}$, for some linear forms $l_1, \ldots, l_s$ over $\mathbb{C}$?

   For the case of two part partition we have proved that this is true, by finding such a covariant. Even though Theorem 4.7 states that $\mathcal{I}(\frac{r}{2}, 1, \ldots, 1)$ is the radical of all invariants, when $r = 4$ we have found a covariant whose vanishing for f is a necessary and sufficient condition that f has the form $l_1^3 l_2$. My supervisor Prof.H.K. Farahat feels that such a covariant exists in general. Next project of mine is to find a proof.

2. What can be said about the sets $\mathcal{E}^{(l)}(k)$, for $l > 1$ and $1 \leq k \leq r$?

3. Consider the problem of transpose system of $n-$ary homogeneous polynomial equations: Find any relations that may exist between the solutions of the transpose systems of $n-$ary homogeneous polynomial equations

$$AX^{[r]} = 0$$

and

$$A^T X^{[r]} = 0,$$

where $r \geq 1, n > 2$, $A \in \mathbb{C}_{N(n,r),N(n,r)}$, and $X = \begin{pmatrix} X_0 \\ X_1 \\ \vdots \\ X_n \end{pmatrix}$.

List of Symbols

Abbreviations:

char	characteristic
dim	dimension
ker	kernel
Rad	radical
det	determinant

Set Theory:

$:=$	is defined as
$\{\}$	set consisting of
$\in$	is an element of
$\subset$	is a subset of
$\#A$	Number of elements in the set A
$\dot{\cup}$	disjoint union
$\square$	end of proof
$\mathbb{Z}$	set of integers
$\mathbb{Z}_{\geq 0}$	set of non-negative integers
$\mathbb{Z}^+$	set of positive integers
$\mathbb{C}$	field of complex numbers

Matrix:

A^T	transpose of the matrix A
$\lvert A(r_1, \ldots, r_k; s_1, \ldots, s_k)\rvert$	k-rowed minor obtained from A
$\mathbb{C}_{n,m}$	set of all $n \times m$ matrices over $\mathbb{C}$

Invariant theory:

$\mathbb{C}[X_0, X_1]_r$	sapce of all binary froms of degree r
$Res(f,g)$	Resultant of binary forms
$\mathcal{H}$	Hessian
$\mathcal{J}$	Jacobian
$\mathcal{P}$	fourth transvectant
$\partial_i f$	$\dfrac{\partial f}{\partial x_i}$
$\partial_i^2 f$	$\dfrac{\partial^2 f}{\partial x_i^2}$
$\partial_0\partial_1 f$	$\dfrac{\partial^2 f}{\partial x_0 \partial x_1}$

Algebraic geometry:

$\mathbb{C}[X_1, \ldots, X_n]$	polynomial ring over $\mathbb{C}$
$\mathbb{P}^n$	projective n-space over $\mathbb{C}$
$\mathbb{P}(V)$	projective space of V
M_n^r	set of all monomials in $x_1, \ldots, x_n$ of degree r
$N(n,r)$	number of elements in M_n^r
$X^{[r]}$	column matrix whose entries are the monomials $X_{i_1} \ldots X_{i_r}$
$\langle f_1, \ldots, f_s \rangle$	ideal generated by $f_1, \ldots, f_s$
I_l	l^{th} elimination ideal of I
$\mathbf{V}(f_1, \ldots, f_s)$	zero set of $f_1, \ldots, f_s$
$\mathbf{I}(V)$	vanishing ideal of the subset V

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Appendix A

Position map

In this section, for the sake of completeness, we will discuss formulas for the positioning monomial in the matrix $X^{[r]}$. First we shall define the position map.

Definition A.1 *1. Let $\mathcal{M}_r^n$ be the set of all monomials of degree r in $X_1, \ldots, X_n$. Then*

$$\mathcal{M}_0^n = \{1\},$$

$$\mathcal{M}_1^n = \{X_1, \ldots, X_n\},$$

$$\vdots$$

$$\mathcal{M}_r^n = \{X_1^{\alpha_1} \ldots X_n^{\alpha_n} | \alpha_1 + \ldots + \alpha_n = r, \alpha_1, \ldots, \alpha_n \geq 0, \}.$$

2. For every $r \geq 0$, the position map P is the function from $\mathcal{M}_r^n$ to $\{1, \ldots, N(n, r)\}$ defined by

$$P(X_{i_1} \ldots X_{i_r}) = \textit{position of } X_{i_1} \ldots X_{i_r} \textit{ in } X^{[r]}.$$

Position of $X_{i_1} \ldots X_{i_r}$ among all monomials of degree r, in $X^{[r]}$ is denoted by

$$P(i_1, \ldots, i_r; 1, \ldots, n),$$

where $1 \leq i_1 \leq \ldots \leq i_r \leq n$.

Example A.2 $P : \mathcal{M}_3^2 \to \{1, 2, 3, 4\}$

$$P(1,1,1;\ 1,2) = 1$$

$$P(1,1,2;\ 1,2) = 2$$

$$P(1,2,2;\ 1,2) = 3$$

$$P(2,2,2;\ 1,2) = 4.$$

Since $X^{[1]} = X$, the position of X_j in $X^{[1]}$, $P(j;\ 1,\dots,n) = j$, where $1 \le j \le n$.

The following lemma discusses the position of X_iX_j in $X^{[2]}$.

Lemma A.3 *Position of* X_iX_j *in* $X^{[2]}$,

$$P(i,j;\ 1,\dots,n) = \frac{(i-1)(2n-i+2)}{2} + (j-i+1), \ \textit{where}\ 1 \le i \le j \le n.$$

Proof: List the entries in $X^{[2]}$ in groups, those which start with X_1, then those which start with X_2 and so on. That is,

$$\begin{array}{cccccc}
X_1X_1 & X_1X_2 & \dots & \dots & \dots & X_1X_n \\
 & X_2X_2 & \dots & \dots & \dots & X_2X_n \\
 & & & \dots & \dots & \dots \\
 & & & X_iX_i & \dots & X_iX_n, \text{ etc.}
\end{array}$$

If $i \le j$, then X_iX_j appears as the $(j-i+1)^{th}$ element in the i^{th} group. The groups $1, 2, \dots, i-1$ contain

$$n + (n-1) + ..\ + (n-i+2) = \frac{(i-1)(2n-i+2)}{2}$$

elements. Hence the result. $\square$

Next lemma provides a formula for the inverse position function in $X^{[2]}$.

Lemma A.4 *(Formula for the inverse position function)*

The inverse position function

$$f:\left\{1,\ldots,\frac{n(n+1)}{2}\right\}\to\{(i,j):1\le i\le j\le n\},$$

is given as follows.

Let $r=P(i,j;\,1,\ldots,n), 1\le i\le j\le n$. To get (i,j) from r, define

$$f(r)=max\left\{i:\ \frac{(i-1)(2n-i+2)}{2}<r\right\}.$$

Then $i=f(r), j=r+i-1-\dfrac{(i-1)(2n-i+2)}{2}$.

<u>Proof</u>: We only need to check, if $i=f(r), j=r+i-1-\dfrac{(i-1)(2n-i+2)}{2}$ then $P(i,j;\,1,\ldots,n)=r$.

Consider

$$\begin{aligned} P(i,j;\,1,\ldots,n) &= j-i+1+\frac{(i-1)(2n-i+2)}{2} \\ &= r+i-1-\frac{(i-1)(2n-i+2)}{2}-i+1+\frac{(i-1)(2n-i+2)}{2} \\ &= r. \end{aligned}$$

$\square$

Now we shall state and prove a recurrence formula for positioning monomials of degree r in $X^{[r]}$, for any $r\ge 1$.

Lemma A.5 *(Basic Recurrence Formula)*

$$P(i_1, i_2, \ldots, i_r; 1, \ldots, n) = \binom{n+r-2}{r-1} + \ldots + \binom{n+r-j-1}{r-1} + \ldots + \binom{n+r-i_1-1}{r-1} +$$

$$+ P(i_2 - i_1 + 1, \ldots, i_r - i_1 + 1; 1, \ldots, n - i_1 + 1),$$

where $1 \leq i_1 \leq \ldots \leq i_r \leq n$, *with* $P(i_1; 1, \ldots, n) = i_1$.

Proof: Note that $\mathcal{M}_r^n$ and $X^{[r]}$ can be written as $\mathcal{M}_r[1, \ldots, n]$ and $X[1, \ldots, n]^{[r]}$ respectively.

With this notation we can list the entries in $X[1, \ldots, n]^{[r]}$ in groups, those which start with X_1, then those which start with X_2 and so on. That is,

$$X_1 \mathcal{M}_{(r-1)}[1, \ldots, n],$$

$$X_2 \mathcal{M}_{(r-1)}[2, \ldots, n],$$

$$\vdots$$

$$X_n \mathcal{M}_{(r-1)}[n].$$

If $1 \leq i_1 \leq \ldots \leq i_r \leq n$ then $X_{i_1} \ldots X_{i_r}$ appears in the i_1^{th} group. The groups $1, 2, \ldots, i_1 - 1$ contain

$$\binom{n + (r-1) - 1}{r-1} + \ldots + \binom{(n - i_1 + 1) + (r-1) - 1}{r-1}$$

elements. Hence, for $1 \leq i_1 \leq \ldots \leq i_r \leq n$, the position of $X_{i_1} \ldots X_{i_r}$ among all monomials in $X[1, \ldots, n]$ of degree r is

$$
\begin{aligned}
P(i_1, i_2, \ldots, i_r; 1, \ldots, n) &= \binom{n + (r-1) - 1}{r-1} + \ldots + \binom{(n - j + 1) + (r-1) - 1}{r-1} \\
&\quad + \ldots + \binom{(n - i_1 + 1) + (r-1) - 1}{r-1} \\
&\quad + \text{Position of } X_{i_2} \ldots X_{i_r} \text{in } X[i_1, \ldots, n]^{[r-1]} \\
&= \binom{n + r - 2}{r-1} + \ldots + \binom{n + r - i_1 - 1}{r-1} \\
&\quad + P(i_2, \ldots, i_r; i_1, \ldots, n)
\end{aligned}
$$

Now if we use the change of variable $Y_\beta = X_{\beta + i_1 - 1}$. Then

$$X_{i_1} = Y_1, X_{i_2} = Y_{i_2 - i_1 + 1}, \ldots, X_{i_r} = Y_{i_r - i_1 + 1}, \ldots, X_n = Y_{n - i_1 + 1}.$$

Thus we have

$$
\begin{aligned}
P(i_1, i_2, \ldots, i_r; 1, \ldots, n) &= \binom{n + r - 2}{r-1} + \ldots + \binom{n + r - i_1 - 1}{r-1} \\
&\quad + P(i_2 - i_1 + 1, \ldots, i_r - i_1 + 1; 1, \ldots, n - i_1 + 1).
\end{aligned}
$$

□

Appendix B

Gröbner Bases

We have used the computer algebra system Maple V to find the Gröbner basis for ideals, specifically, the Gröbner basis package. To access the commands in this package, type:

>with(Groebner);

(here > is the Maple prompt, and semi colon is the end of Maple command.)

In Maple, monomial ordering is called term order. Since monomial order depends also on how the variables are ordered, Maple needs to know both the term order and a list of variables. For example, to tell Maple to use lexicographic order with variables $A > B > C$, we need to input plex (for pure lexicographic) and $[A, B, C]$ (Maple encloses a list inside brackets [...]).

In Maple "gbasis" stands for Gröbner basis, and the syntax is as follows:

>gbasis(poly list,var list,term order);

this computes a Gröbner basis for the ideal generated by the polynomials in poly list with respect to the monomial ordering specified by the term order and var list.

In the following sections we state the codes to find Gröbner basis in the beginning. Then we list the polynomials which are needed for the proofs from ordered Gröbner basis (ordering is the position where those polynomials appeared in the Maple output).

B.1 A Gröbner basis

The Maple worksheet for finding a Gröbner basis for

$$I = \langle (r-m)A+mB-P, (r-m)A^2+mB^2+Q, (r-m)A^3+mB^3-S\rangle \subset \mathbb{K}(A,B,P,Q,S)$$

with respect to lexicographic order:

$> W := [(r-m)*A+m*B-P, (r-m)*A^2+m*B^2+Q, (r-m)*A^3+m*B^3-S];$

$W := [(r-m)\,A+m\,B-P,\ (r-m)\,A^2+m\,B^2+Q,\ (r-m)\,A^3+m\,B^3-S]$

Now we find the Gröbner basis for the above polynomials by using the lexicographic order on A, B, P, Q, S

$> gbasis(W, Plex(A, B, P, Q, S));$

$[3\,r\,Q\,P^4 - 4\,m^2\,S\,P^3 + 3\,Q^2\,P^2\,m^2 + m\,r^3\,S^2 - m^2\,r^2\,S^2 - 4\,m\,r^2\,Q^3 + 4\,m^2\,r\,Q^3$
$+ P^6 + 3\,Q^2\,P^2\,r^2 + r^3\,Q^3 + 4\,m\,S\,P^3\,r - 6\,m^2\,r\,Q\,S\,P + 6\,m\,r^2\,Q\,S\,P - 3\,Q^2\,P^2\,m\,r,$
$- m\,r^2\,Q^3\,B + 2\,m^2\,r\,Q^3\,B - m\,r^3\,S^2\,B + 2\,m^2\,r^2\,S^2\,B - Q\,P^5 + r\,S\,P^4 - 2\,r\,Q^2\,P^3$
$- 4\,Q\,S\,P^2\,m\,r + 2\,Q\,S\,P^2\,r^2 + 4\,Q\,S\,P^2\,m^2 - 3\,Q^3\,P\,m^2 + 2\,Q^3\,P\,m\,r - Q^3\,P\,r^2$
$+ 3\,r^2\,m\,S^2\,P - 4\,r\,m^2\,S^2\,P + 5\,r\,S\,Q^2\,m^2 - 5\,r^2\,S\,Q^2\,m + r^3\,S\,Q^2, -m\,r^2\,S\,B\,P$
$+ 2\,m^2\,r\,S\,B\,P + m\,r^2\,Q^2\,B - 2\,m^2\,r\,Q^2\,B + P^5 + 2\,r\,Q\,P^3 + 3\,m\,S\,P^2\,r - 4\,m^2\,S\,P^2$
$+ 3\,Q^2\,P\,m^2 - 2\,Q^2\,P\,m\,r + Q^2\,P\,r^2 + m\,r^2\,Q\,S - m^2\,r\,Q\,S, 2\,m^2\,r\,S\,B + 2\,m^2\,Q\,B\,P$
$+ 4\,m^2\,Q^2 - 4\,m^2\,P\,S - m\,r^2\,S\,B - m\,r\,Q\,B\,P + 3\,m\,P\,r\,S - 4\,m\,r\,Q^2 - m\,Q\,P^2 + r^2\,Q^2$
$+ 2\,Q\,P^2\,r + P^4,$
$- r^2\,S + r\,m\,S + 2\,r\,Q\,m\,B - 2\,P\,r\,Q - B\,r^2\,Q - B\,r\,P^2 + P\,m\,Q + 2\,m\,B\,P^2 - P^3,$
$r\,m\,B^2 + r\,Q - m\,Q - 2\,m\,B\,P + P^2,\ -A\,r + A\,m - m\,B + P]$

B.2 A Gröbner basis for the polynomials that make a binary quartic form a square of some binary quadratic form

$> L := [A^2 - P, A*B - Q, 2*B^2 + A*C - 3*R, B*C - S, C^2 - T];$

$$[A^2 - P, AB - Q, 2B^2 + AC - 3R, BC - S, -T + C^2]$$

$> gbasis(L, plex(A, B, C, P, Q, R, S, T));$

$$g_1 = QT^2 - 3RST + 2S^3, \tag{B.1}$$

$$g_2 = PT^2 + 2QST - 9R^2T + 6S^2R, \tag{B.2}$$

$$g_3 = PST - 3QRT + 2S^2Q, \tag{B.3}$$

$$g_4 = -Q^2T + S^2P, \tag{B.4}$$

$$g_5 = PQT - 3PRS + 2Q^2S, \tag{B.5}$$

$$g_6 = -9PR^2 + 6Q^2R + P^2T + 2PQS, \tag{B.6}$$

$$g_7 = -3PQR + 2Q^3 + P^2S, \tag{B.7}$$

$$g_8 = -T + C^2, \tag{B.8}$$

$$g_{20} = 2B^3 - 3BR + CQ, \tag{B.9}$$

$$g_{27} = A^2 - P. \tag{B.10}$$

B.3 A Gröbner basis for the parametrization of a binary quartic form with a linear factor of multiplicity at least 3

$> WL := [C*A^3-P, 3*C*B*A^2+D*A*A^2-4*Q, 3*A*B^2*C+3*A^2*B*D-6*R, 3*A*B^2*D+C*B^3-4*S, D*B*B^2-T];$

$> gbasis(WL, plex(A,B,C,D,P,Q,R,S,T));$

$$h_1 = 4QS^3-3S^2R^2+4TR^3-6TQRS+Q^2T^2, \tag{B.11}$$

$$h_2 = -4SQ+TP+3R^2, \tag{B.12}$$

$$h_3 = PS^2+4R^3-6QRS+Q^2T, \tag{B.13}$$

$$h_{20} = (-4QR+2PS)CD+D^2PR+C^2(4QS-3R^2), \tag{B.14}$$

$$h_{21} = (3PS-2QR)C^2+(-4Q^2+2PR)CD+QPD^2, \tag{B.15}$$

$$h_{22} = (9PR-8Q^2)C^2-2DPCQ+D^2P^2, \tag{B.16}$$

$$h_{23} = C^4T-4DC^3S+6D^2RC^2-4D^3CQ+D^4P, \tag{B.17}$$

$$h_{24} = DB^3-T, \tag{B.18}$$

$$h_{60} = 3CBA^2+DA^3-4Q, \tag{B.19}$$

B.4 A Gröbner basis for the parametrization of a binary quintic form with a linear factor having multiplicity at least 4

$> W := [A^4*C-P, (6*A^2*B^2*C+4*A^3*B*D)-10*R, (4*A^3*C*B+A^4*D)-5*Q, B^4*D-U, (4*A*B^3*D+B^4*C)-5*T, (4*A*B^3*C+6*A^2*B^2*D)-10*S]:$

$> gbasis(W, plex(A, B, C, D, P, Q, R, S, T, U));$

$$i_1 = 4T^3R - 3T^2S^2 - 6USTR + 4US^3 + U^2R^2, \tag{B.20}$$

$$i_2 = UQ - 4TR + 3S^2, \tag{B.21}$$

$$i_3 = QT^2 - 6STR + 4S^3 + UR^2, \tag{B.22}$$

$$i_4 = -3TQ + UP + 2RS, \tag{B.23}$$

$$i_5 = -4SQ + PT + 3R^2, \tag{B.24}$$

$$i_6 = TQ^2 - 6QRS + PS^2 + 4R^3, \tag{B.25}$$

$$\begin{aligned} i_{28} = {} & 4DCTQ - 6DCSR + 4D^2SQ - 3D^2R^2 \\ & + C^2(4TR - 3S^2), \end{aligned} \tag{B.26}$$

$$i_{29} = 7DSCQ - 9DCR^2 + SD^2P + C^2(3TQ - 2RS), \tag{B.27}$$

$$i_{30} = -5QRDC + PD^2R + 3DPCS + C^2(9SQ - 8R^2), \tag{B.28}$$

$$i_{31} = C^2(6PS - 5QR) - 5DQ^2C + PD^2Q + 3DPCR, \tag{B.29}$$

$$i_{32} = C^2(16PR - 15Q^2) - 2PDCQ + D^2P^2, \tag{B.30}$$

$$i_{33} = C^5U - 5DC^4T + 10C^3D^2S - 10D^3RC^2$$

$$+5\,D^4\,C\,Q - D^5\,P, \qquad \text{(B.31)}$$

$$\mathrm{i}_{34} \;=\; B^4\,D - U, \qquad \text{(B.32)}$$

$$\mathrm{i}_{87} \;=\; 4\,A^3\,B\,C + A^4\,D - 5\,Q, \qquad \text{(B.33)}$$

B.5 A Gröbner basis for the parametrization of a binary quintic form with linear factors of multiplicity either 2,3 or 5

$> W := [A^3 * C^2 - P, (A^3 * D^2 + 6 * A^2 * B * C * D + 3 * A * B^2 * C^2) - 10 * R, (2 * A^3 * C * D + 3 * A^2 * B * C^2) - 5 * Q, B^3 * D^2 - U, (3 * A * B^2 * D^2 + 2 * B^3 * C * D) - 5 * T, (3 * A^2 * B * D^2 + B^3 * C^2 + 6 * A * B^2 * C * D) - 10 * S]:$

$> gbasis(W, plex(A, B, C, D, P, Q, R, S, T, U));$

$$\begin{aligned}
\mathrm{j}_1 &= 108\,T^3\,U^2\,R + 219\,T^2\,U^2\,S^2 - 300\,T^4\,U\,S + 100\,T^6 \\
&\quad - 162\,T\,S\,U^3\,R + 27\,U^4\,R^2 + 8\,S^3\,U^3, && \text{(B.34)} \\
\mathrm{j}_2 &= 3\,U^3\,Q - 12\,T\,U^2\,R - 16\,U^2\,S^2 + 50\,U\,S\,T^2 - 25\,T^4, && \text{(B.35)} \\
\mathrm{j}_3 &= -162\,T\,S\,U^2\,R + 155\,U\,T^2\,S^2 - 100\,S\,T^4 + 12\,T^2\,U^2\,Q \\
&\quad + 60\,T^3\,R\,U + 27\,U^3\,R^2 + 8\,S^3\,U^2, && \text{(B.36)} \\
\mathrm{j}_4 &= -162\,T\,S^2\,U^2\,R + 227\,U\,T^2\,S^3 - 145\,S^2\,T^4 - 54\,S\,T^3\,R\,U \\
&\quad + 27\,S\,U^3\,R^2 + 8\,S^4\,U^2 \\
&\quad + 27\,T^2\,U^2\,R^2 + 60\,T^5\,R + 12\,T^4\,U\,Q, && \text{(B.37)} \\
\mathrm{j}_5 &= 324\,U^3\,R^3\,T + 783\,U^3\,R^2\,S^2 + 2484\,U^2\,R^2\,S\,T^2
\end{aligned}$$

$$-756\,U R^2 T^4 + 240\,Q T^6 - 4794\,R T S^3 U^2$$
$$-2736\,R T^3 S^2 U + 2520\,R T^5 S + 232\,S^5 U^2$$
$$+6511\,S^4 U T^2 - 4160\,S^3 T^4, \tag{B.38}$$

$$\mathrm{j}_6 = 9U^2 R^2 + 38\,T U R S - 20\,R T^3 - 24\,S^3 U + 4\,S U^2 Q$$
$$+15\,T^2 S^2 - 4\,T^2 U Q, \tag{B.39}$$

$$\mathrm{j}_7 = 116\,U Q S T^3 - 80\,T^5 Q + 108\,U^3 R^3 - 567\,T S U^2 R^2$$
$$+252\,T^3 R^2 U + 32\,R S^3 U^2$$
$$+390\,T^2 R U S^2 - 260\,T^4 R S + 24\,S^4 U T - 15\,S^3 T^3, \tag{B.40}$$

$$\mathrm{j}_8 = 48\,T R U^2 Q - 116\,S T^2 U Q - 81\,S U^2 R^2 - 12\,T^2 R^2 U$$
$$+230\,T U R S^2 - 140\,R S T^3 - 24\,S^4 U$$
$$+15\,T^2 S^3 + 80\,Q T^4, \tag{B.41}$$

$$\mathrm{j}_9 = 108\,T U^2 R^3 - 468\,T^2 U R^2 S + 240\,R^2 T^4 + 518\,S^3 T U R$$
$$-320\,R T^3 S^2 + 48\,Q T^3 R U - 116\,S^2 T^2 U Q - 81\,S^2 U^2 R^2$$
$$-24\,S^5 U + 15\,S^4 T^2 + 80\,Q S T^4, \tag{B.42}$$

$$\mathrm{j}_{10} = 960\,R T^5 Q - 3364\,T^2 S^3 Q U + 2320\,T^4 S^2 Q$$
$$-1296\,U^3 R^4 + 9936\,T S U^2 R^3 - 3024\,T^3 R^3 U$$
$$-2733\,R^2 S^3 U^2 - 18252\,T^2 R^2 U S^2$$
$$+10080\,T^4 R^2 S + 14734\,R S^4 U T - 9100\,R S^3 T^3$$
$$-696\,S^6 U + 435\,S^5 T^2, \tag{B.43}$$

$$\mathrm{j}_{11} = 8\,T U^2 Q^2 - 4\,Q T^2 R U - 46\,Q T S^2 U + 40\,Q S T^3 - 27\,U^2 R^3$$
$$+102\,T U R^2 S - 70\,R^2 T^3 - 8\,R S^3 U + 5\,R T^2 S^2, \tag{B.44}$$

$$
\begin{aligned}
j_{12} &= 12T^2UQ^2 + 3R^2U^2Q - 62RQTSU + 20RQT^3 + 3QS^3U \\
&\quad + 15S^2QT^2 + 6TR^3U + 38R^2US^2 - 35R^2ST^2, \qquad \text{(B.45)} \\
j_{13} &= 48Q^2T^4 - 24R^2T^2UQ - 56S^2QTUR \\
&\quad - 112SQRT^3 + 3S^4QU + 96S^3QT^2 + 27U^2R^4 \\
&\quad - 36R^3TUS + 96R^3T^3 + 62R^2S^3U - 104R^2T^2S^2, \qquad \text{(B.46)} \\
j_{14} &= -12QU^2R^3 + 56QTUR^2S - 20R^2QT^3 - 6QRS^3U \\
&\quad + 132QRT^2S^2 + 12Q^2UTS^2 - 48Q^2ST^3 - 81QS^4T \\
&\quad - 44R^3US^2 - 46R^3ST^2 + 54R^2TS^3 + 3R^4TU, \qquad \text{(B.47)} \\
j_{15} &= 4RU^2Q^2 - 8Q^2TSU + 32Q^2T^3 - 20QTR^2U \\
&\quad - 19QRUS^2 - 68QRST^2 + 54QS^3T \\
&\quad + 42UR^3S + 19T^2R^3 - 36R^2TS^2, \qquad \text{(B.48)} \\
j_{16} &= 16Q^2TURS + 32Q^2RT^3 + 12Q^2S^3U - 48Q^2T^2S^2 \\
&\quad - 64QUR^2S^2 - 68SR^2QT^2 + 216RQTS^3 + 63SUR^4 \\
&\quad - 132R^3TS^2 + 54R^2S^4 \\
&\quad - 81QS^5 - 24QTR^3U + 24T^2R^4, \qquad \text{(B.49)} \\
j_{17} &= 36Q^2S^4U - 144Q^2T^2S^3 - 12R^5TU + 365R^4US^2 \\
&\quad + 256R^4ST^2 - 612R^3TS^3 - 732QR^2T^2S^2 \\
&\quad + 288RQ^2ST^3 + 972RQS^4T + 48QU^2R^4 \\
&\quad + 80R^3QT^3 - 168QR^2S^3U \\
&\quad + 162R^2S^5 - 243QS^6 - 296QTUR^3S, \qquad \text{(B.50)} \\
j_{18} &= 16Q^3U^2 - 96RTUQ^2 - 168Q^2S^2U
\end{aligned}
$$

$$
\begin{aligned}
& +384\,S T^2 Q^2 + 304\,Q U R^2 S - 160\,R^2 Q T^2 - 936\,R Q T S^2 \\
& -63\,U R^4 + 548\,R^3 T S - 342\,R^2 S^3 + 513\,Q S^4, \qquad \text{(B.51)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{j}_{19} \;=\; & 32\,Q^3 T S U - 128\,Q^3 T^3 - 16\,Q^2 T R^2 U - 92\,S^2 Q^2 R U \\
& +656\,Q^2 R S T^2 - 216\,Q^2 S^3 T \\
& +136\,Q U R^3 S - 236\,Q T^2 R^3 - 792\,Q R^2 T S^2 \\
& -63\,U R^5 + 548\,R^4 T S - 342\,R^3 S^3 + 513\,R Q S^4, \qquad \text{(B.52)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{j}_{20} \;=\; & 342\,R^4 S^3 + 63\,U R^6 - 162\,Q^2 S^5 - 548\,R^5 T S \\
& +528\,Q R^3 T S^2 - 10\,Q S U R^4 \\
& +24\,Q^3 S^3 U + 648\,R Q^2 T S^3 - 96\,Q^3 T^2 S^2 \\
& -32\,Q^2 T R^3 U - 405\,Q R^2 S^4 + 284\,Q T^2 R^4 \\
& +192\,Q^3 R T^3 - 36\,Q^2 U R^2 S^2 - 792\,S R^2 Q^2 T^2, \qquad \text{(B.53)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{j}_{21} \;=\; & -48\,S U^2 R + 80\,U T S^2 - 50\,S T^3 + U^3 P \\
& +7\,T U^2 Q + 10\,T^2 R U, \qquad \text{(B.54)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{j}_{22} \;=\; & 2\,T U^2 P + 2\,T^2 U Q - 27\,U^2 R^2 + 66\,T U R S - 40\,R T^3 \\
& -8\,S^3 U + 5\,T^2 S^2, \qquad \text{(B.55)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{j}_{23} \;=\; & 3\,P T^2 U + 5\,Q T^3 - 12\,T R^2 U - 20\,R S T^2 \\
& +32\,R U S^2 - 6\,R U^2 Q - 2\,Q T S U, \qquad \text{(B.56)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{j}_{24} \;=\; & -10\,R T U Q + 3\,P T^3 + 6\,Q S^2 U - U^2 Q^2 \\
& +12\,U R^2 S - 10\,T^2 R^2, \qquad \text{(B.57)}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{j}_{25} \;=\; & 3\,S U^2 P - 24\,R U^2 Q + 61\,Q T S U - 40\,Q T^3 \\
& +6\,T R^2 U - 16\,R U S^2 + 10\,R S T^2, \qquad \text{(B.58)}
\end{aligned}
$$

$$\begin{aligned}
j_{26} &= 21QS^2U - 4U^2Q^2 - 15SQT^2 - 4RTUQ \\
&\quad -6UR^2S + 5T^2R^2 + 3STUP, &\text{(B.59)}\\
j_{27} &= -80QURS - 24SPT^2 + 32TUQ^2 \\
&\quad +40RQT^2 + 9S^2UP \\
&\quad +15QTS^2 + 18UR^3 - 10R^2TS, &\text{(B.60)}\\
j_{28} &= 3PT^2S^2 + 12Q^2T^3 - 9QTR^2U - 38QRST^2 \\
&\quad +QRUS^2 - 4Q^2TSU + 18UR^3S + 9T^2R^3 \\
&\quad +24QS^3T - 16R^2TS^2, &\text{(B.61)}\\
j_{29} &= 6PTS^3 - 16RTUQ^2 - 12Q^2S^2U \\
&\quad +40ST^2Q^2 + 46QUR^2S \\
&\quad -20R^2QT^2 - 140RQTS^2 + 75QS^4 - 9UR^4 \\
&\quad +80R^3TS - 50R^2S^3, &\text{(B.62)}\\
j_{30} &= 54PS^5 + 128Q^3T^3 - 144Q^2TR^2U \\
&\quad -108S^2Q^2RU - 240Q^2RST^2 + 216Q^2S^3T \\
&\quad +360QUR^3S + 36QT^2R^3 - 552QR^2TS^2 + 189RQS^4 \\
&\quad -27UR^5 + 234R^4TS - 146R^3S^3, &\text{(B.63)}\\
j_{31} &= U^2PR - 4U^2Q^2 + 3RTUQ + 23QS^2U - 20SQT^2 \\
&\quad -18UR^2S + 15T^2R^2, &\text{(B.64)}\\
j_{32} &= 9RTUP - 24SPT^2 + 8TUQ^2 - 2QURS \\
&\quad +55RQT^2 - 30QTS^2 - 36UR^3 + 20R^2TS, &\text{(B.65)}\\
j_{33} &= 12RPT^2 - 15QUR^2 + 122QRTS - 66TR^3
\end{aligned}$$

$$+44\,R^2\,S^2 - 15\,P\,T\,S^2$$

$$+12\,S\,U\,Q^2 - 66\,Q\,S^3 - 28\,Q^2\,T^2, \tag{B.66}$$

$$j_{34} = 3\,R\,S\,U\,P - 6\,P\,T\,S^2 + 8\,Q^2\,T^2 - 6\,Q\,U\,R^2$$

$$-7\,Q\,R\,T\,S + 6\,Q\,S^3 + 6\,T\,R^3 - 4\,R^2\,S^2, \tag{B.67}$$

$$j_{35} = -16\,T\,U\,Q^3 + 40\,Q^2\,U\,R\,S - 8\,R\,Q^2\,T^2 - 18\,Q\,U\,R^3$$

$$-28\,Q\,R^2\,T\,S - 27\,P\,S^4 + 30\,R\,P\,T\,S^2$$

$$+24\,Q\,R\,S^3 + 9\,T\,R^4 - 6\,R^3\,S^2, \tag{B.68}$$

$$j_{36} = -368\,R\,Q^3\,T\,U - 240\,Q^3\,S^2\,U + 960\,Q^3\,S\,T^2$$

$$+1040\,R^2\,Q^2\,U\,S - 464\,R^2\,Q^2\,T^2$$

$$-3240\,R\,Q^2\,T\,S^2 + 1620\,Q^2\,S^4$$

$$-204\,R^4\,Q\,U + 1896\,R^3\,Q\,T\,S - 1188\,R^2\,Q\,S^3 - 3\,R^5\,T$$

$$+2\,R^4\,S^2 + 189\,R\,P\,S^4, \tag{B.69}$$

$$j_{37} = 8\,R^2\,U\,Q^2 - 216\,R\,S\,T\,Q^2 + 153\,Q\,T\,R^3$$

$$,-102\,Q\,R^2\,S^2 - 16\,S\,U\,Q^3$$

$$+108\,Q^2\,S^3 + 64\,Q^3\,T^2 + 54\,P\,R\,S^3$$

$$-60\,P\,R^2\,T\,S + 7\,P\,U\,R^3, \tag{B.70}$$

$$j_{38} = R^4\,P\,T\,S - 32\,R\,Q^4\,T\,U - 16\,Q^4\,S^2\,U + 64\,Q^4\,S\,T^2$$

$$+92\,R^2\,Q^3\,U\,S - 32\,R^2\,Q^3\,T^2$$

$$-216\,R\,Q^3\,T\,S^2 + 108\,Q^3\,S^4 - 24\,R^4\,Q^2\,U \tag{B.71}$$

$$+116\,R^3\,Q^2\,T\,S - 81\,R^2\,Q^2\,S^3 + 6\,R^5\,Q\,T$$

$$-4\,R^4\,Q\,S^2 + 18\,R^3\,P\,S^3, \tag{B.72}$$

$$j_{39} = 9QU^2P - 114SPT^2 + 191TUQ^2$$
$$-428QURS + 250RQT^2$$
$$+60QTS^2 + 72UR^3 - 40R^2TS, \quad \text{(B.73)}$$

$$j_{40} = 6QTUP - 21PTS^2 + 22Q^2T^2 - 21QUR^2$$
$$+22QRTS - 6QS^3 - 6TR^3 + 4R^2S^2, \quad \text{(B.74)}$$

$$j_{41} = 18PS^3 - 80RPTS + 40STQ^2 + 9PR^2U - 24RUQ^2$$
$$+15QTR^2 - 10QRS^2 + 32PQT^2, \quad \text{(B.75)}$$

$$j_{42} = 4SQUP - PR^2U + 8RPTS - 18PS^3 - 8RUQ^2$$
$$+20STQ^2 - 15QTR^2 + 10QRS^2, \quad \text{(B.76)}$$

$$j_{43} = 42PQS^2T - 3PR^2TS - 54PRS^3 - 12SUQ^3$$
$$-8Q^3T^2 + 6R^2UQ^2 - 8RSTQ^2$$
$$+39Q^2S^3 - 6QTR^3 + 4QR^2S^2, \quad \text{(B.77)}$$

$$j_{44} = 189PQS^4 - 15SR^3TP - 270PR^2S^3 + 112TUQ^4$$
$$-340Q^3URS + 16RQ^3T^2 + 156Q^2UR^3 + 156Q^2R^2TS$$
$$+27Q^2RS^3 - 93QR^4T + 62QR^3S^2, \quad \text{(B.78)}$$

$$j_{45} = PRUQ + 12PQTS - 4Q^3U - 5Q^2TR$$
$$+15Q^2S^2 - 18RPS^2 - R^2TP, \quad \text{(B.79)}$$

$$j_{46} = 216Q^3RTS - 188Q^2TR^3 + 207Q^2R^2S^2$$
$$+16Q^4SU - 108Q^3S^3 - 64Q^4T^2$$
$$+144PQR^2TS - 7R^4TP - 126PR^3S^2$$
$$-54RPQS^3 - 36R^2UQ^3, \quad \text{(B.80)}$$

$$
\begin{aligned}
j_{47} &= -3420\,R^4\,Q^3\,U + 9216\,Q^5\,S\,T^2 - 2304\,Q^5\,S^2\,U \\
&\quad +126\,R^5\,P\,S^2 - 11556\,R^2\,Q^3\,S^3 - 4544\,R^2\,Q^4\,T^2 \\
&\quad -31104\,R\,Q^4\,T\,S^2 + 1052\,R^5\,Q^2\,T - 783\,R^4\,Q^2\,S^2 \\
&\quad +13232\,R^2\,Q^4\,S\,U + 2646\,R^3\,P\,Q\,S^3 \\
&\quad -4608\,R\,Q^5\,T\,U + 16488\,R^3\,Q^3\,T\,S \\
&\quad +7\,R^6\,T\,P + 15552\,Q^4\,S^4, \qquad \text{(B.81)} \\
j_{48} &= 3\,P^2\,U^2 - 57\,P\,T\,S^2 + 217\,Q^2\,T^2 - 57\,Q\,U\,R^2 \\
&\quad -578\,Q\,R\,T\,S + 354\,Q\,S^3 + 354\,T\,R^3 - 236\,R^2\,S^2, \qquad \text{(B.82)} \\
j_{49} &= 32\,T\,P^2\,U - 191\,P\,R^2\,U + 176\,R\,P\,T\,S \\
&\quad -126\,P\,S^3 + 104\,R\,U\,Q^2 \\
&\quad +40\,S\,T\,Q^2 - 105\,Q\,T\,R^2 + 70\,Q\,R\,S^2, \qquad \text{(B.83)} \\
j_{50} &= P^2\,T^2 + 10\,P\,Q\,T\,S - 6\,R^2\,T\,P \\
&\quad -12\,R\,P\,S^2 - 3\,Q^3\,U + 10\,Q^2\,S^2, \qquad \text{(B.84)} \\
j_{51} &= S\,U\,P^2 + 43\,P\,Q\,T\,S - R^2\,T\,P - 66\,R\,P\,S^2 \\
&\quad -12\,Q^3\,U - 20\,Q^2\,T\,R + 55\,Q^2\,S^2, \qquad \text{(B.85)} \\
j_{52} &= 12\,Q\,P\,S^2 + 2\,Q\,R\,T\,P - 3\,P\,U\,Q^2 - 5\,T\,Q^3 \\
&\quad -32\,S\,R^2\,P + 20\,R\,Q^2\,S + 6\,P^2\,T\,S, \qquad \text{(B.86)} \\
j_{53} &= 27\,P^2\,S^3 + 84\,P\,Q^2\,T\,S - 2\,Q\,R^2\,T\,P - 198\,P\,Q\,R\,S^2 \\
&\quad +8\,S\,R^3\,P - 24\,Q^4\,U - 40\,Q^3\,T\,R \\
&\quad +150\,Q^3\,S^2 - 5\,R^2\,Q^2\,S, \qquad \text{(B.87)} \\
j_{54} &= R\,P^2\,U - 4\,P\,U\,Q^2 + 23\,Q\,R\,T\,P + 18\,Q\,P\,S^2
\end{aligned}
$$

$$-48\,S\,R^2\,P - 20\,T\,Q^3 + 30\,R\,Q^2\,S, \tag{B.88}$$

$$j_{55} = -24\,P\,R^3 + 15\,R^2\,Q^2 + 38\,Q\,S\,R\,P + 4\,R\,T\,P^2$$
$$-4\,P\,T\,Q^2 - 20\,Q^3\,S - 9\,P^2\,S^2, \tag{B.89}$$

$$j_{56} = -8\,P\,U\,Q^3 + 44\,Q^2\,R\,T\,P - 40\,T\,Q^4 + 36\,P\,Q^2\,S^2$$
$$+27\,R\,P^2\,S^2 - 162\,Q\,S\,R^2\,P + 100\,R\,Q^3\,S$$
$$+8\,P\,R^4 - 5\,R^3\,Q^2, \tag{B.90}$$

$$j_{57} = 2\,Q\,U\,P^2 - 27\,P^2\,S^2 + 2\,P\,T\,Q^2 + 66\,Q\,S\,R\,P$$
$$-8\,P\,R^3 - 40\,Q^3\,S + 5\,R^2\,Q^2, \tag{B.91}$$

$$j_{58} = P^3\,U + 7\,Q\,T\,P^2 - 48\,S\,R\,P^2 + 10\,P\,Q^2\,S$$
$$+80\,Q\,P\,R^2 - 50\,R\,Q^3, \tag{B.92}$$

$$j_{59} = 50\,Q^2\,P\,R - 25\,Q^4 - 12\,Q\,P^2\,S \tag{B.93}$$
$$+3\,P^3\,T - 16\,P^2\,R^2, \tag{B.94}$$

$$j_{60} = 27\,P^3\,S^2 + 12\,P^2\,T\,Q^2 - 162\,Q\,S\,R\,P^2 + 60\,P\,Q^3\,S \tag{B.95}$$
$$+155\,Q^2\,P\,R^2 - 100\,R\,Q^4 + 8\,P^2\,R^3, \tag{B.96}$$

$$j_{96} = U^2\,C^2 + 6\,D^2\,U\,S - 5\,D^2\,T^2 - 2\,D\,C\,T\,U, \tag{B.97}$$

$$j_{105} = -7\,D\,Q\,C\,R + 3\,P\,C\,D\,S + 2\,D^2\,Q^2$$
$$+(-4\,R^2 + 6\,Q\,S)C^2, \tag{B.98}$$

$$j_{108} = (9\,P\,S - 5\,R\,Q)C^2 + 2\,P\,D\,C\,R + 4\,P\,D^2\,Q$$
$$-10\,C\,D\,Q^2, \tag{B.99}$$

$$j_{109} = (6\,P\,R - 5\,Q^2)\,C^2 - 2\,P\,C\,D\,Q + D^2\,P^2 - 5\,C^2\,Q^2, \tag{B.100}$$

$$j_{112} = B^3\,D \tag{B.101}$$

$$j_{187} = A^3 D^2 + 6 A^2 B C D + 3 A B^2 C^2 - 10 R, \tag{B.102}$$

B.6 A Gröbner basis for a binary quintic form which is a factor of a square of a quadratic form and a linear form

$> W := [(2*A*C*D+4*B^2*D+4*A*B*E)-10*R, (A^2*E+4*A*B*D)-5*Q, (4*B*C*E+C^2*D)-5*T, A^2*D-P, (4*B^2*E+4*B*C*D+2*A*C*E)-10*S, C^2*E-U];$

$> gbasis(W, plex(A,B,C,D,P,Q,R,S,T,U));$

There are 588 polynomials in the Gröbne basis of I.

$$
\begin{aligned}
k_1 \;=\;& 2234\,T^2\,S^2\,U^3\,R^2 - 320\,S^6\,U^3 + 500\,T^8\,Q + 500\,T^6\,S^3 \\
& +27\,U^5\,R^4 + 4\,U^6\,Q^3 + 200\,T\,S^2\,R\,U^4\,Q \\
& -1048\,T^3\,R\,S\,U^3\,Q - 3300\,T^3\,U^2\,S^3\,R + 156\,R^2\,U^4\,T^2\,Q \\
& +160\,T\,S^4\,U^3\,R + 36\,Q\,S\,U^5\,R^2 \\
& -48\,U^5\,Q^2\,T\,R - 468\,T\,S\,U^4\,R^3 + 176\,T^2\,S\,Q^2\,U^4 \\
& -1196\,T^2\,S^3\,Q\,U^3 + 3600\,T^5\,U\,S^2\,R \\
& +2580\,T^4\,Q\,S^2\,U^2 + 104\,T^3\,U^3\,R^3 + 100\,U\,T^6\,R^2 \\
& -176\,U^4\,S^3\,R^2 - 2000\,T^6\,U\,Q\,S \\
& +1680\,U^2\,S^5\,T^2 + 560\,T^5\,U^2\,Q\,R + 224\,Q\,U^4\,S^4 \\
& -52\,Q^2\,S^2\,U^5 - 1420\,T^4\,S\,U^2\,R^2 \\
& -88\,T^4\,U^3\,Q^2 - 1000\,T^7\,S\,R - 1725\,T^4\,U\,S^4, \qquad \text{(B.103)} \\
k_2 \;=\;& 50\,T^2\,S^3 - 54\,S\,U^2\,R^2 - 80\,U\,S^4 - 4\,U^3\,Q^2 \\
& +50\,Q\,T^4 - 3\,U^2\,S\,T\,P + 2\,U\,P\,T^3 + 27\,U^2\,Q\,T\,R \\
& -105\,U\,T^2\,Q\,S + R\,U^3\,P + 36\,Q\,S^2\,U^2
\end{aligned}
$$

$$+180\,T\,U\,S^2\,R-100\,T^3\,S\,R, \tag{B.104}$$

$$\begin{aligned}
\mathsf{k}_3 \;=\;& 925\,T^4\,S^4-4\,U^5\,Q^3-27\,U^4\,R^4+320\,S^6\,U^2-100\,T^6\,R^2\\
&+1048\,T^3\,R\,S\,U^2\,Q-2234\,T^2\,S^2\,U^2\,R^2+20\,T^7\,P\\
&+1196\,T^2\,S^3\,Q\,U^2+468\,T\,S\,U^3\,R^3\\
&-176\,T^2\,S\,Q^2\,U^3-200\,T\,S^2\,R\,U^3\,Q-160\,T\,S^4\,U^2\,R\\
&-36\,Q\,S\,U^4\,R^2+48\,U^4\,Q^2\,T\,R\\
&+52\,Q^2\,S^2\,U^4-224\,Q\,U^3\,S^4-1680\,U\,S^5\,T^2\\
&-156\,R^2\,U^3\,T^2\,Q+10\,T^4\,R\,P\,U^2-30\,T^5\,S\,U\,P\\
&+176\,U^3\,S^3\,R^2-290\,T^5\,R\,U\,Q-2220\,T^4\,Q\,U\,S^2\\
&+880\,T^4\,U\,S\,R^2+3300\,T^3\,U\,S^3\,R\\
&+48\,T^4\,Q^2\,U^2+950\,T^6\,S\,Q-104\,T^3\,U^2\,R^3-1800\,T^5\,S^2\,R,
\end{aligned} \tag{B.105}$$

$$\begin{aligned}
\mathsf{k}_4 \;=\;& 2\,Q\,U^3\,P-11\,R\,T\,P\,U^2\\
&-8\,S^2\,U^2\,P+37\,S\,U\,P\,T^2-20\,T^4\,P+2\,T\,Q^2\,U^2\\
&-36\,R\,S\,U^2\,Q+15\,R\,U\,T^2\,Q+40\,Q\,U\,T\,S^2\\
&-25\,S\,Q\,T^3+54\,U^2\,R^3-180\,U\,T\,S\,R^2+100\,R^2\,T^3\\
&+80\,U\,S^3\,R-50\,T^2\,S^2\,R,
\end{aligned} \tag{B.106}$$

$$\begin{aligned}
\mathsf{k}_5 \;=\;& -312\,R^2\,U^2\,T^2\,Q+352\,U^2\,S^3\,R^2-100\,U\,R^3\,T^3\\
&+104\,Q^2\,S^2\,U^3-448\,Q\,U^2\,S^4+100\,U\,Q^2\,T^4\\
&+96\,U^3\,Q^2\,T\,R-72\,Q\,S\,U^3\,R^2+4\,Q\,U^2\,P\,T^3\\
&-320\,T\,S^4\,U\,R-2\,U\,R\,T^4\,P-2470\,T^2\,S^2\,U\,R^2\\
&+100\,T^3\,S^3\,R-475\,T^4\,S^2\,Q+1400\,T^4\,S\,R^2
\end{aligned}$$

$$
\begin{aligned}
& +1060\,T^2\,S^3\,Q\,U - 60\,T^5\,S\,P + 936\,T\,S\,U^2\,R^3 \\
& -204\,T^2\,S\,Q^2\,U^2 + 95\,T^3\,S^2\,U\,P + 640\,S^6\,U \\
& -400\,S^5\,T^2 - 550\,T^5\,R\,Q - 54\,U^3\,R^4 \\
& +1025\,T^3\,S\,R\,U\,Q - 400\,T\,S^2\,R\,U^2\,Q \\
& -37\,T^2\,S\,R\,P\,U^2 - 8\,U^4\,Q^3, \qquad \text{(B.107)}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{k}_6 \;=\; & 300\,T^3\,S^4\,R - 1344\,Q\,U^2\,S^5 + 312\,Q^2\,S^3\,U^3 \\
& +1056\,U^2\,S^4\,R^2 + 285\,T^3\,S^3\,U\,P + 2808\,T\,S^2\,U^2\,R^3 \\
& -1225\,T^4\,Q\,S^3 - 960\,T\,S^5\,U\,R + 4300\,T^4\,S^2\,R^2 \\
& +40\,T^6\,R\,P - 216\,Q\,S^2\,U^3\,R^2 \\
& -1080\,T^2\,Q\,S\,U^2\,R^2 - 80\,T^4\,R\,S\,U\,P + 3715\,T^3\,Q\,U\,S^2\,R \\
& -95\,T^2\,R\,S^2\,U^2\,P - 200\,T^5\,R^3 \\
& +200\,T^6\,Q^2 + 1920\,S^7\,U - 1200\,S^6\,T^2 - 16\,T^2\,U^3\,Q^3 \\
& +288\,S\,U^3\,Q^2\,T\,R - 1200\,T\,S^3\,R\,U^2\,Q \\
& -468\,T^2\,Q^2\,S^2\,U^2 + 104\,T^3\,U^2\,Q^2\,R - 120\,T^4\,U\,Q^2\,S \\
& +2860\,T^2\,Q\,U\,S^4 - 7570\,T^2\,U\,S^3\,R^2 \\
& -108\,T^2\,U^2\,R^4 + 8\,T^5\,Q\,U\,P + 22\,T^3\,R^2\,P\,U^2 \\
& -2000\,T^5\,Q\,S\,R + 60\,T^3\,U\,S\,R^3 - 30\,T^4\,R^2\,U\,Q \\
& -24\,S\,U^4\,Q^3 - 162\,S\,U^3\,R^4 - 180\,T^5\,S^2\,P, \qquad \text{(B.108)}
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{k}_7 \;=\; & 8500\,T^3\,R\,S^5 - 2430\,S^2\,U^3\,R^4 - 2700\,T^5\,S^3\,P \\
& -10680\,T^4\,Q^2\,U\,S^2 - 18000\,S^7\,T^2 + 28800\,S^8\,U \\
& -944\,T^2\,S\,Q^3\,U^3 - 144\,Q^2\,S\,U^4\,R^2
\end{aligned}
$$

$$
\begin{aligned}
& -2236\,T^2\,S^3\,Q^2\,U^2+63500\,T^4\,R^2\,S^3+272\,T\,R\,U^4\,Q^3 \\
& -20800\,T\,R\,S^6\,U+1000\,T^4\,R^4\,U \\
& -17000\,T^5\,R^3\,S-10980\,T^2\,R^4\,S\,U^2+1200\,T^6\,R\,S\,P \\
& -32450\,T^5\,R\,S^2\,Q+25600\,T^3\,R^3\,S^2\,U+20\,T^5\,R^2\,U\,P \\
& -110350\,T^2\,R^2\,S^4\,U-1584\,T^2\,R^2\,U^3\,Q^2-2160\,T^5\,R\,U\,Q^2 \\
& +38600\,T\,R^3\,U^2\,S^3+2704\,T^3\,R^3\,U^2\,Q+15840\,U^2\,S^5\,R^2 \\
& +4275\,T^3\,S^4\,U\,P-16\,U^5\,Q^4 \\
& +7792\,T^3\,R\,S\,Q^2\,U^2+58325\,T^3\,R\,S^3\,Q\,U \\
& +2592\,T\,R^3\,Q\,S\,U^3-14160\,T\,R\,Q\,U^2\,S^4 \\
& +2480\,T\,R\,Q^2\,S^2\,U^3+700\,T^3\,R^2\,S\,P\,U^2 \\
& -21136\,T^2\,R^2\,S^2\,U^2\,Q-7180\,T^4\,R^2\,S\,U\,Q \\
& -2150\,T^4\,R\,S^2\,U\,P-1425\,T^2\,R\,S^3\,U^2\,P+80\,Q\,T^7\,P \\
& -18880\,Q\,S^6\,U^2-108\,Q\,U^4\,R^4 \\
& +192\,T^4\,Q^3\,U^2+6800\,T^6\,S\,Q^2-14675\,Q\,T^4\,S^4 \\
& +3784\,Q^2\,U^3\,S^4-152\,Q^3\,S^2\,U^4 \\
& -2536\,Q\,U^3\,S^3\,R^2+540\,T\,R^5\,U^3+5100\,T^6\,R^2\,Q \\
& +36180\,Q\,U\,S^5\,T^2, \qquad\qquad \text{(B.109)} \\
\mathrm{k}_8 \;=\; & -8\,R\,S^2\,U^2\,P-180\,U\,T\,S\,R^3+15\,R^2\,U\,T^2\,Q \\
& -11\,R^2\,T\,P\,U^2+175\,Q\,T^3\,S\,R+6\,Q\,U^2\,S\,T\,P \\
& -320\,Q\,T\,U\,S^2\,R+37\,R\,S\,U\,P\,T^2+54\,U^2\,R^4 \\
& +100\,R^3\,T^3+8\,U^3\,Q^3-100\,Q^2\,T^4
\end{aligned}
$$

$$-50T^2S^2R^2 - 20RT^4P + 80US^3R^2$$
$$-4QUPT^3 + 210UT^2Q^2S + 72QSU^2R^2$$
$$+160QUS^4 - 52U^2Q^2TR - 72Q^2S^2U^2 - 100QT^2S^3, \qquad \text{(B.110)}$$

$$\mathrm{k}_9 = -80ST^3P + 200QT^3R + 108R^2U^2Q - 100S^2QT^2$$
$$-64RSU^2P + 8T^2RUP + 6TQU^2P + 128TS^2PU$$
$$-360TSQRU - 32SU^2Q^2 + 25T^2Q^2U$$
$$+U^3P^2 + 160S^3QU, \qquad \text{(B.111)}$$

$$\mathrm{k}_{10} = P^2TU^2 - 8SQU^2P + 6T^2PQU - 4R^2PU^2$$
$$-8RPTSU + 32S^3PU - 20S^2T^2P + 16RQ^2U^2$$
$$-40STQ^2U + 25T^3Q^2, \qquad \text{(B.112)}$$

$$\mathrm{k}_{11} = -4S^2QU^2P - 13ST^2PQU - 2SR^2PU^2$$
$$+28RPTS^2U + 16S^4PU$$
$$-10S^3T^2P + 8SRQ^2U^2 - 40S^2TQ^2U$$
$$+25ST^3Q^2 - 8T^2R^2UP + 10T^4PQ$$
$$+2P^2T^3U + 4TU^2Q^3 + 5RT^2Q^2U$$
$$-20RPT^3S - TRQU^2P, \qquad \text{(B.113)}$$

$$\mathrm{k}_{12} = -540STR^4U + 8T^2Q^2U^2P + 8STU^2Q^3$$
$$+40ST^4PQ - 80RPT^3S^2 - 124TR^2Q^2U^2 - 41TR^3PU^2$$
$$+32S^5PU - 200RT^4Q^2 - 960QUTS^2R^2$$
$$-80S^3TQ^2U + 50S^2T^3Q^2 - 140PR^2T^4$$
$$-20S^4T^2P - 64RT^3PQU + 480SRT^2Q^2U$$

$$-58\,S^2\,T^2\,P\,Q\,U + 120\,R\,P\,T\,S^3\,U$$
$$+207\,S\,T^2\,R^2\,U\,P - 200\,S^2\,R\,Q^2\,U^2 - 8\,S^3\,Q\,U^2\,P$$
$$-28\,S^2\,R^2\,P\,U^2 + 40\,T^3\,Q^3\,U + 300\,R^4\,T^3$$
$$+16\,P^2\,T^5 + 162\,U^2\,R^5 + 480\,Q\,U\,S^4\,R + 216\,Q\,S\,U^2\,R^3$$
$$+525\,Q\,S\,R^2\,T^3 - 300\,Q\,T^2\,S^3\,R$$
$$+45\,Q\,T^2\,R^3\,U - 150\,T^2\,S^2\,R^3 + 240\,U\,S^3\,R^3 + 24\,R\,U^3\,Q^3, \qquad \text{(B.114)}$$

$$\mathrm{k}_{13} = 16\,T\,R^2\,U\,P - 25\,Q^2\,S\,T^2 - 20\,T^3\,P\,Q - 4\,P^2\,T^2\,U$$
$$+40\,Q^2\,S^2\,U - 8\,U^2\,Q^3 + S\,U^2\,P^2 - 10\,R\,T\,Q^2\,U + 32\,S\,T\,Q\,U\,P$$
$$-64\,S^2\,P\,U\,R + 40\,R\,P\,T^2\,S + 2\,R\,Q\,U^2\,P, \qquad \text{(B.115)}$$

$$\mathrm{k}_{14} = 4\,R\,S\,Q\,U^2\,P + 13\,R\,T^2\,P\,Q\,U + 2\,R^3\,P\,U^2$$
$$-28\,R^2\,P\,T\,S\,U - 16\,R\,S^3\,P\,U + 10\,R\,S^2\,T^2\,P$$
$$-8\,R^2\,Q^2\,U^2 + 40\,R\,S\,T\,Q^2\,U - 25\,R\,T^3\,Q^2$$
$$+P^2\,U\,T^2\,S - 4\,P^2\,T^4 - 2\,T\,Q^2\,U^2\,P$$
$$+8\,T\,Q\,S^2\,P\,U - 5\,Q\,S\,T^3\,P + 20\,P\,R^2\,T^3 - 10\,T^2\,Q^3\,U, \qquad \text{(B.116)}$$

$$\mathrm{k}_{15} = P^2\,U^2\,R + 2\,P^2\,U\,T\,S - 8\,P^2\,T^3 - 4\,Q^2\,U^2\,P$$
$$+32\,P\,Q\,T\,R\,U + 16\,Q\,S^2\,P\,U - 10\,Q\,S\,T^2\,P$$
$$-64\,S\,R^2\,P\,U + 40\,P\,R^2\,T^2 - 20\,T\,Q^3\,U$$
$$+40\,S\,Q^2\,R\,U - 25\,Q^2\,T^2\,R, \qquad \text{(B.117)}$$

$$\mathrm{k}_{16} = -8\,T\,R^3\,U\,P + 10\,R\,T^3\,P\,Q + 2\,R\,P^2\,T^2\,U$$
$$+4\,R\,U^2\,Q^3 + 5\,R^2\,T\,Q^2\,U - R^2\,Q\,U^2\,P + P^2\,U\,T\,S^2 - 4\,S\,P^2\,T^3$$
$$-2\,S\,Q^2\,U^2\,P + 8\,Q\,S^3\,P\,U - 5\,Q\,S^2\,T^2\,P - 10\,S\,T\,Q^3\,U, \qquad \text{(B.118)}$$

$$\begin{aligned}
\mathsf{k}_{17} &= -4\,S^2\,P^2\,T^3 - 2\,S^2\,Q^2\,U^2\,P + P^2\,U\,T\,S^3 + 20\,R\,T^2\,Q^3\,U \\
&\quad +4\,S\,R\,U^2\,Q^3 - 10\,S^2\,T\,Q^3\,U - 5\,Q\,S^3\,T^2\,P + 8\,Q\,S^4\,P\,U \\
&\quad -20\,R^2\,S^2\,T^2\,P - 4\,R^4\,P\,U^2 + 50\,R^2\,T^3\,Q^2 \\
&\quad -40\,P\,R^3\,T^3 + 16\,R^3\,Q^2\,U^2 + 8\,R\,P^2\,T^4 \\
&\quad +32\,R^2\,S^3\,P\,U + 20\,R\,Q\,S\,T^3\,P - 16\,R\,T\,Q\,S^2\,P\,U \\
&\quad +4\,R\,T\,Q^2\,U^2\,P + 48\,R^3\,P\,T\,S\,U \\
&\quad -75\,R^2\,S\,T\,Q^2\,U - 9\,R^2\,S\,Q\,U^2\,P - 26\,R^2\,T^2\,P\,Q\,U, \qquad \text{(B.119)} \\
\mathsf{k}_{18} &= Q\,U^2\,P^2 - 8\,R\,P^2\,T\,U - 4\,S^2\,P^2\,U \\
&\quad +16\,S\,T^2\,P^2 + 6\,P\,T\,Q^2\,U - 8\,P\,S\,Q\,R\,U \\
&\quad -40\,P\,Q\,T^2\,R + 32\,P\,R^3\,U + 25\,T^2\,Q^3 - 20\,Q^2\,R^2\,U, \qquad \text{(B.120)} \\
\mathsf{k}_{19} &= -8\,Q\,T\,R^2\,U\,P + 25\,Q^3\,S\,T^2 + 10\,T^3\,P\,Q^2 \\
&\quad +2\,Q\,P^2\,T^2\,U - 20\,Q^3\,S^2\,U + 4\,U^2\,Q^4 + 5\,R\,T\,Q^3\,U \\
&\quad -13\,S\,T\,Q^2\,U\,P + 28\,Q\,S^2\,P\,U\,R - 40\,Q\,R\,P\,T^2\,S \\
&\quad -R\,Q^2\,U^2\,P - 4\,S\,R\,P^2\,T\,U - 2\,S^3\,P^2\,U \\
&\quad +8\,S^2\,T^2\,P^2 + 16\,S\,P\,R^3\,U - 10\,S\,Q^2\,R^2\,U, \qquad \text{(B.121)} \\
\mathsf{k}_{20} &= -10\,S^2\,Q^2\,R^2\,U + 16\,S^2\,P\,R^3\,U - 4\,Q\,R^3\,P\,U^2 \\
&\quad +20\,Q^2\,S\,T^3\,P + 16\,R^2\,Q^3\,U^2 + 4\,T\,Q^3\,U^2\,P \\
&\quad -75\,R\,S\,T\,Q^3\,U - 29\,T\,Q^2\,S^2\,P\,U - 4\,S^2\,R\,P^2\,T\,U \\
&\quad -2\,S^4\,P^2\,U + 8\,S^3\,T^2\,P^2 + 48\,Q\,R^2\,P\,T\,S\,U \\
&\quad +60\,Q\,R\,S^3\,P\,U - 60\,Q\,R\,S^2\,T^2\,P - 40\,Q\,P\,R^2\,T^3 \\
&\quad -9\,R\,S\,Q^2\,U^2\,P - 26\,R\,T^2\,P\,Q^2\,U + 4\,S\,U^2\,Q^4
\end{aligned}$$

$$-20\,Q^3\,S^3\,U + 25\,Q^3\,S^2\,T^2 + 8\,Q\,P^2\,T^4$$

$$+20\,T^2\,Q^4\,U + 50\,R\,T^3\,Q^3, \tag{B.122}$$

$$\begin{aligned}
\mathrm{k}_{21} \;=\;& Q\,P^2\,U\,T\,S - 4\,Q\,P^2\,T^3 - 2\,Q^3\,U^2\,P \\
&+13\,P\,Q^2\,T\,R\,U + 8\,Q^2\,S^2\,P\,U \\
&-5\,Q^2\,S\,T^2\,P - 28\,Q\,S\,R^2\,P\,U + 40\,Q\,P\,R^2\,T^2 \\
&-10\,T\,Q^4\,U + 20\,S\,Q^3\,R\,U - 25\,Q^3\,T^2\,R \\
&+4\,R^2\,P^2\,T\,U + 2\,R\,S^2\,P^2\,U - 8\,R\,S\,T^2\,P^2 \\
&-16\,P\,R^4\,U + 10\,Q^2\,R^3\,U,
\end{aligned} \tag{B.123}$$

$$\begin{aligned}
\mathrm{k}_{22} \;=\;& 128\,Q\,R^2\,P\,U + 8\,S\,P\,Q^2\,U - 64\,P^2\,S\,R\,U \\
&+6\,P^2\,T\,Q\,U - 80\,Q^3\,R\,U + 25\,P\,T^2\,Q^2 + 108\,S^2\,P^2\,T \\
&-100\,Q^2\,R^2\,T + 200\,S\,Q^3\,T + 160\,P\,R^3\,T \\
&+P^3\,U^2 - 32\,R\,P^2\,T^2 - 360\,P\,S\,R\,Q\,T,
\end{aligned} \tag{B.124}$$

$$\begin{aligned}
\mathrm{k}_{23} \;=\;& -11\,QSUP^2 + 15\,STPQ^2 + 37\,PRUQ^2 + 2\,QP^2T^2 \\
&+40\,PQTR^2 - 36\,RSTP^2 + 54\,S^3P^2 + 100\,Q^3S^2 \\
&-20\,Q^4U - 25\,Q^3TR + 80\,SPR^3 - 180\,PS^2RQ \\
&-50\,Q^2SR^2 + 2\,P^3TU - 8\,R^2UP^2,
\end{aligned} \tag{B.125}$$

$$\begin{aligned}
\mathrm{k}_{24} \;=\;& 80\,S^2PR^3 - 52\,QST^2P^2 + 210\,PQ^2T^2R - 4\,PTQ^3U \\
&+15\,S^2TPQ^2 + 160\,PR^4T - 100\,Q^2R^3T + 175\,RSQ^3T \\
&+72\,RS^2P^2T + 8\,P^3T^3 - 8\,SR^2UP^2 - 180\,PS^3RQ \\
&-320\,PSR^2QT + 6\,P^2QTRU - 11\,QS^2UP^2 - 100\,T^2Q^4 \\
&-50\,Q^2S^2R^2 - 20\,SQ^4U - 72\,P^2R^2T^2 + 54\,S^4P^2
\end{aligned}$$

$$+100\,Q^3S^3+37\,SPRUQ^2, \tag{B.126}$$

$$\begin{aligned}
\mathrm{k}_{25} &= SUP^3-4\,P^3T^2-3\,QRUP^2+27\,QSTP^2\\
&+36\,R^2P^2T-54\,RS^2P^2+2\,PUQ^3-105\,PQ^2TR+\\
&180\,PSR^2Q-80\,PR^4+50\,TQ^4-100\,SQ^3R+50\,Q^2R^3
\end{aligned} \tag{B.127}$$

$$\begin{aligned}
\mathrm{k}_{26} &= 27\,U^5\,D\,R^2-162\,U^4\,D\,R\,T\,S+108\,U^3\,D\,T^3\,R+8\,U^4\,D\,S^3\\
&+219\,U^3\,D\,S^2\,T^2-300\,U^2\,D\,T^4\,S+100\,D\,T^6\,U-2\,U^5\,E\,S\,P\\
&+2\,U^4\,E\,P\,T^2+36\,U^5\,E\,R\,Q-98\,U^4\,E\,Q\,T\,S\\
&+62\,U^3\,E\,Q\,T^3-171\,U^4\,E\,R^2\,T-96\,U^4\,E\,S^2\,R\\
&+982\,U^3\,E\,T^2\,S\,R-580\,U^2\,E\,R\,T^4+280\,U^3\,E\,T\,S^3\\
&-1615\,U^2\,E\,S^2\,T^3+1700\,E\,U\,T^5\,S-500\,E\,T^7,
\end{aligned} \tag{B.128}$$

$$\begin{aligned}
\mathrm{k}_{218} &= (3\,T^2\,Q\,-\,6\,R\,T\,S\,+\,3\,S^3)\,D^3\,-\,E\,D^2\,P\,S\,U\\
&+E\,D^2\,T^2\,P\,+\,2\,E\,R\,Q\,D^2\,U\,-\,7\,E\,Q\,D^2\,S\,T\,-\,2\,E\,D^2\,R^2\,T\\
&+7\,E\,R\,S^2\,D^2\,+\,E^2\,R\,D\,U\,P\,-\,2\,E^2\,D\,S\,T\,P\,-\,E^2\,Q^2\,D\,U\\
&+7\,E^2\,R\,T\,D\,Q\,+\,2\,S^2\,E^2\,D\,Q\,-\,7\,D\,E^2\,S\,R^2\,-\,3\,E^3\,T\,Q^2\\
&+6\,Q\,E^3\,S\,R\,-\,3\,E^3\,R^3,
\end{aligned} \tag{B.129}$$

$$\begin{aligned}
\mathrm{k}_{224} &= (24T^2\,P\,+\,30\,Q\,S\,T\,-\,120\,R^2\,T+\,60\,R\,S^2\,)\,D^3\\
&+4\,E\,R\,P\,D^2\,U\,-\,46\,D^2\,E\,S\,P\,T\,+\,10\,E\,Q^2\,D^2\,U\,-\,50\,T\,Q\,D^2\,R\,E\\
&-20\,E\,S^2\,D^2\,Q\,+\,120\,E\,D^2\,S\,R^2\,-\,E^2\,Q\,D\,U\,P\,+\,30\,D\,E^2\,T\,P\,R\\
&-12\,E^2\,D\,S^2\,P\,+\,5\,Q^2\,E^2\,T\,D\,+\,140\,E^2\,R\,D\,Q\,S\,-\,180\,D\,E^2\,R^3\\
&-U\,E^3\,P^2\,-\,37\,Q\,E^3\,T\,P\,+44\,R\,E^3\,P\,S,
\end{aligned} \tag{B.130}$$

$$\mathrm{k}_{226} = (36\,P\,S\,T\,-\,90\,Q\,R\,T\,+\,45\,Q\,S^2\,)D^3\,+\,3\,E\,P\,Q\,D^2\,U$$

$$-18\,E\,R\,P\,D^2\,T - 33\,E\,P\,S^2\,D^2 - 15\,E\,Q^2\,D^2\,T$$

$$+90\,E\,Q\,D^2\,S\,R - E^2\,D\,U\,P^2 - 11\,E^2\,P\,T\,D\,Q$$

$$+70\,E^2\,P\,D\,S\,R + 50\,E^2\,S\,D\,Q^2 - 135\,Q\,D\,E^2\,R^2 - 8\,T\,E^3\,P^2$$

$$-10\,Q\,S\,E^3\,P + 27\,P\,E^3\,R^2, \qquad (B.131)$$

$$\mathrm{k}_{227} = (4\,P\,S^2 - 5\,Q^2\,T)\,D^3 - 2\,E\,P\,T\,D^2\,Q + 5\,S\,E\,D^2\,Q^2$$

$$-D\,P^2\,T\,E^2 + 4\,D\,S\,Q\,E^2\,P + 4\,D\,E^2\,R^2\,P - 10\,E^2\,R\,D\,Q^2$$

$$-E^3\,P^2\,S + 2\,Q\,R\,E^3\,P, \qquad (B.132)$$

$$\mathrm{k}_{230} = (12\,P\,T\,Q - 24\,P\,S\,R + 15\,S\,Q^2)\,D^3$$

$$+4\,E\,D^2\,T\,P^2 - 10\,E\,P\,Q\,D^2\,S - 8\,E\,D^2\,R^2\,P$$

$$+5\,E\,R\,Q^2\,D^2 + 3\,D\,S\,E^2\,P^2 - 14\,P\,E^2\,R\,D\,Q$$

$$+20\,E^2\,Q^3\,D + R\,E^3\,P^2 - 4\,Q^2\,E^3\,P, \qquad (B.133)$$

$$\mathrm{k}_{232} = (8\,P^2\,T + 10\,P\,Q\,S - 40\,R^2\,P + 25\,R\,Q^2)\,D^3$$

$$+6\,E\,D^2\,P^2\,S - 40\,E\,Q\,R\,P\,D^2$$

$$+25\,E\,Q^3\,D^2 - D\,R\,E^2\,P^2 + 10\,D\,Q^2\,E^2\,P - 3\,Q\,E^3\,P^2, \qquad (B.134)$$

$$\mathrm{k}_{233} = (16\,P^2\,S + 25\,Q^3 - 40\,Q\,R\,P)\,D^3$$

$$-8\,P^2\,D^2\,E\,R + 3\,P^2\,E^2\,D\,Q - P^3\,E^3 + 5\,D^2\,Q^2\,E\,P, \qquad (B.135)$$

$$\mathrm{k}_{234} = D^5\,U - 5\,E\,D^4\,T + 10\,E^2\,D^3\,S -$$

$$10\,D^2\,E^3\,R + 5\,E^4\,D\,Q - E^5\,P, \qquad (B.136)$$

$$\mathrm{k}_{235} = C^2\,E - U, \qquad (B.137)$$

$$\mathrm{k}_{550} = 4\,B^3\,E^2 + B\,C^2\,D^2 + 5\,B\,D\,T - 10\,E\,B\,S$$

$$-5\,S\,C\,D + 5\,C\,E\,R, \qquad (B.138)$$

$$\mathsf{k}_{587} = 5A^2E + 4ABD - 5Q, \tag{B.139}$$

B.7 A Gröbner basis for $j_{59}, j_{58}, j_{57}, j_{55}, j_{54}, j_{52}, j_{51}, j_{50}, j_{49}, j_{48}, j_{45}, j_{42}, j_{41}, j_{40}, j_{39}, j_{34}, j_{33}, j_{32}, j_{31}, j_{27}, j_{26}, j_{25}, j_{24}, j_{23}, j_{22}, j_{21}, j_6, j_2$

A Gröebner basis for $j_{59}, j_{58}, j_{57}, j_{55}, j_{54}, j_{52}, j_{51}, j_{50}, j_{49}, j_{48}, j_{45}, j_{42}, j_{41}, j_{40}, j_{39}, j_{34}, j_{33}, j_{32}, j_{31}, j_{27}, j_{26}, j_{25}, j_{24}, j_{23}, j_{22}, j_{21}, j_6, j_2$ with respect to lexicographic order is

$$[108\,t^3u^2r + 219\,s^2t^2u^2 - 300\,t^4us + 100\,t^6 + 27\,u^4r^2 - 162\,tu^3rs + 8\,s^3u^3,$$

$$3\,u^3q - 12\,tu^2r - 16\,u^2s^2 + 50\,ust^2 - 25\,t^4,$$

$$27\,u^3r^2 - 162\,tu^2rs + 60\,urt^3 + 8\,s^3u^2 + 155\,ut^2s^2 + 12\,t^2u^2q - 100\,st^4,$$

$$27\,t^2u^2r^2 - 54\,t^3urs + 60\,t^5r + 227\,t^2s^3u - 145\,t^4s^2 + 12\,t^4uq + 27\,su^3r^2$$

$$- 162\,tu^2rs^2 + 8\,s^4u^2, 2484\,st^2u^2r^2 - 2736\,t^3urs^2 + 2520\,st^5r + 6511\,t^2s^4u$$

$$- 4160\,t^4s^3 + 783\,s^2u^3r^2 - 4794\,tu^2rs^3 + 232\,s^5u^2 - 324\,tu^3r^3 - 756\,ur^2t^4$$

$$+ 240\,qt^6,\ -9\,u^2r^2 + 38\,turs - 20\,rt^3 - 24\,s^3u + 4\,su^2q + 15\,t^2s^2 - 4\,t^2uq,$$

$$108\,u^3r^3 - 567\,tu^2r^2s + 252\,ur^2t^3 + 32\,rs^3u^2 + 390\,rut^2s^2 - 260\,rst^4$$

$$+ 116\,qt^3su - 80\,qt^5 + 24\,ts^4u - 15\,t^3s^3, 48\,rtu^2q - 116\,qt^2su + 80\,qt^4$$

$$- 81\,u^2r^2s - 12\,ut^2r^2 + 230\,trus^2 - 140\,rst^3 - 24\,s^4u + 15\,t^2s^3, 108\,r^3tu^2$$

$$- 468\,r^2t^2us + 240\,r^2t^4 + 518\,rts^3u - 320\,rt^3s^2 + 48\,rt^3uq - 116\,qt^2s^2u$$

$$+ 80\,sqt^4 - 81\,u^2r^2s^2 - 24\,s^5u + 15\,t^2s^4, -1296\,u^3r^4 + 9936\,tu^2r^3s$$

$$- 3024\,ur^3t^3 - 2733\,r^2s^3u^2 - 18252\,r^2ut^2s^2 + 10080\,r^2st^4 + 960\,rqt^5$$

$$+ 14734\,rts^4u - 9100\,rt^3s^3 - 3364\,qt^2s^3u + 2320\,s^2qt^4 - 696\,s^6u + 435\,t^2s^5,$$

$$8\,tu^2q^2 - 4\,rt^2uq - 46\,tqs^2u + 40\,sqt^3 - 27\,u^2r^3 + 102\,tur^2s - 70\,t^3r^2$$

$$- 8\,rus^3 + 5\,rs^2t^2, 12\,t^2uq^2 + 3\,qu^2r^2 - 62\,tqurs + 20\,rqt^3 + 3\,qs^3u$$

$+ 15qt^2s^2 + 6tur^3 + 38ur^2s^2 - 35r^2t^2s, 48q^2t^4 - 24qut^2r^2 - 56qtrus^2$
$- 112qrst^3 + 3qs^4u + 96qt^2s^3 + 27u^2r^4 - 36tur^3s + 96t^3r^3 + 62r^2us^3$
$- 104r^2s^2t^2, -12r^3qu^2 + 56r^2qtus - 20r^2qt^3 - 6rqs^3u + 132rqt^2s^2$
$+ 12ts^2uq^2 - 81tqs^4 - 44ur^3s^2 + 54tr^2s^3 - 48sq^2t^3 - 46st^2r^3 + 3tur^4,$
$- 8tsuq^2 - 19rqs^2u - 68qrt^2s + 54tqs^3 + 42ur^3s - 36tr^2s^2 + 32q^2t^3$
$- 20tqur^2 + 19t^2r^3 + 4u^2q^2r, 16q^2turs + 32q^2rt^3 + 12s^3uq^2 - 48q^2t^2s^2$
$- 68sr^2qt^2 + 216rqts^3 - 64qur^2s^2 + 63sur^4 - 132r^3ts^2 + 54r^2s^4$
$- 81qs^5 - 24qtur^3 + 24t^2r^4, 288rsq^2t^3 - 168r^2qs^3u - 732r^2qt^2s^2$
$+ 36s^4uq^2 - 144q^2t^2s^3 + 162r^2s^5 - 243qs^6 + 48r^4qu^2 + 80r^3qt^3$
$+ 365ur^4s^2 - 612tr^3s^3 + 256st^2r^4 - 12tur^5 + 972rtqs^4 - 296r^3qtus,$
$384sq^2t^2 - 96rtuq^2 - 160r^2qt^2 - 936rqts^2 - 168s^2uq^2 + 16u^2q^3$
$+ 304qur^2s - 63ur^4 + 548r^3ts - 342r^2s^3 + 513qs^4, 32tsuq^3 - 92s^2uq^2r$
$+ 656q^2rt^2s - 216tq^2s^3 + 136qur^3s - 792qtr^2s^2 - 128q^3t^3 - 16tq^2ur^2$
$- 236qt^2r^3 - 63ur^5 + 548r^4ts - 342r^3s^3 + 513rqs^4, -548r^5ts + 284qt^2r^4$
$- 405qr^2s^4 - 96q^3t^2s^2 + 24s^3uq^3 + 192q^3rt^3 - 792sr^2q^2t^2 + 648rq^2ts^3$
$- 36q^2ur^2s^2 - 32q^2tur^3 - 10qsur^4 + 528qr^3ts^2 + 63ur^6 - 162q^2s^5$
$+ 342r^4s^3, -48su^2r + 80uts^2 - 50st^3 + u^3p + 7tu^2q + 10t^2ru,$
$2tu^2p + 2t^2uq - 27u^2r^2 + 66turs - 40rt^3 - 8s^3u + 5t^2s^2,$
$3pt^2u + 5qt^3 - 12tr^2u - 20rst^2 + 32rus^2 - 6ru^2q - 2qtsu,$
$- 10rtuq + 3pt^3 + 6qs^2u - u^2q^2 + 12ur^2s - 10t^2r^2,$
$3su^2p - 24ru^2q + 61qtsu - 40qt^3 + 6tr^2u - 16rus^2 + 10rst^2,$
$21qs^2u - 4u^2q^2 - 15sqt^2 - 4rtuq - 6ur^2s + 5t^2r^2 + 3stup, -80qurs$
$- 24spt^2 + 32tuq^2 + 40rqt^2 + 9s^2up + 15qts^2 + 18ur^3 - 10r^2ts, 3pt^2s^2$
$- 4tsuq^2 + 12q^2t^3 - 9tqur^2 + rqs^2u - 38qrt^2s + 24tqs^3 + 18ur^3s$

$+ 9t^2r^3 - 16tr^2s^2, 46qur^2s - 16rtuq^2 - 20r^2qt^2 - 140rqts^2 - 9ur^4$

$+ 80r^3ts + 6pts^3 + 40sq^2t^2 + 75qs^4 - 50r^2s^3 - 12s^2uq^2, 54ps^5 + 128q^3t^3$

$- 144tq^2ur^2 - 108s^2uq^2r - 240q^2rt^2s + 216tq^2s^3 + 360qur^3s$

$+ 36qt^2r^3 - 552qtr^2s^2 + 189rqs^4 - 27ur^5 + 234r^4ts - 146r^3s^3,$

$u^2pr - 4u^2q^2 + 3rtuq + 23qs^2u - 20sqt^2 - 18ur^2s + 15t^2r^2,$

$9rtup - 24spt^2 + 8tuq^2 - 2qurs + 55rqt^2 - 30qts^2 - 36ur^3 + 20r^2ts,$

$12rpt^2 - 15qur^2 + 122qrts - 66tr^3 + 44r^2s^2 - 15pts^2 + 12suq^2 - 66qs^3$

$- 28q^2t^2, 3rsup - 6pts^2 + 8q^2t^2 - 6qur^2 - 7qrts + 6qs^3 + 6tr^3 - 4r^2s^2,$

$40q^2urs - 16tuq^3 - 8rq^2t^2 - 18qur^3 - 28qr^2ts + 30rpts^2 - 27ps^4$

$+ 24qrs^3 + 9tr^4 - 6r^3s^2, 189rps^4 - 368rtuq^3 - 240s^2uq^3 + 960sq^3t^2$

$+ 1040sr^2uq^2 - 464r^2q^2t^2 - 3240rs^2tq^2 + 1620q^2s^4 - 204r^4qu$

$+ 1896r^3qts - 1188r^2qs^3 - 3r^5t + 2r^4s^2, 64q^3t^2 + 8r^2uq^2 - 216rstq^2$

$+ 108q^2s^3 + 153qtr^3 - 102qr^2s^2 - 16suq^3 + 54rps^3 - 60r^2pts + 7pr^3u,$

$- 32rtuq^4 - 16s^2uq^4 + 64sq^4t^2 + 92sr^2uq^3 - 32r^2q^3t^2 - 216rs^2tq^3$

$+ 108q^3s^4 - 24r^4q^2u + 116r^3q^2ts - 81r^2q^2s^3 + 6qr^5t - 4qr^4s^2 + r^4pts$

$+ 18r^3ps^3, 9qu^2p - 114spt^2 + 191tuq^2 - 428qurs + 250rqt^2 + 60qts^2$

$+ 72ur^3 - 40r^2ts,$

$6qtup - 21pts^2 + 22q^2t^2 - 21qur^2 + 22qrts - 6qs^3 - 6tr^3 + 4r^2s^2,$

$18ps^3 - 80rpts + 40stq^2 + 9pr^2u - 24ruq^2 + 15qtr^2 - 10qrs^2$

$+ 32pqt^2,$

$4squp - pr^2u + 8rpts - 18ps^3 - 8ruq^2 + 20stq^2 - 15qtr^2 + 10qrs^2,$

$42qpts^2 - 3r^2pts - 54rps^3 - 12suq^3 - 8q^3t^2 + 6r^2uq^2 - 8rstq^2$

$+ 39q^2s^3 - 6qtr^3 + 4qr^2s^2, 189qps^4 - 15r^3pts - 270r^2ps^3 + 112tuq^4$

$- 340q^3urs + 16rq^3t^2 + 156q^2ur^3 + 156q^2r^2ts + 27q^2rs^3 - 93qtr^4$

$+62\,q\,r^3\,s^2,\,p\,r\,u\,q+12\,p\,q\,t\,s-4\,q^3\,u-5\,q^2\,t\,r+15\,q^2\,s^2-18\,r\,p\,s^2-r^2\,t\,p,$

$144\,r^2\,p\,q\,t\,s-36\,r^2\,q^3\,u-188\,r^3\,q^2\,t+207\,r^2\,q^2\,s^2-126\,r^3\,p\,s^2-7\,r^4\,t\,p$

$-64\,q^4\,t^2+216\,r\,s\,t\,q^3-108\,q^3\,s^3+16\,s\,u\,q^4-54\,q\,r\,p\,s^3,-31104\,r\,s^2\,t\,q^4$

$+15552\,q^4\,s^4+1052\,r^5\,q^2\,t-783\,r^4\,q^2\,s^2+126\,r^5\,p\,s^2+7\,r^6\,t\,p-4544\,r^2\,q^4\,t^2$

$-11556\,r^2\,q^3\,s^3-3420\,r^4\,q^3\,u+16488\,r^3\,s\,t\,q^3+13232\,r^2\,s\,u\,q^4+2646\,r^3\,q\,p\,s^3$

$-4608\,r\,t\,u\,q^5-2304\,s^2\,u\,q^5+9216\,s\,q^5\,t^2,3\,p^2\,u^2-57\,p\,t\,s^2+217\,q^2\,t^2$

$-57\,q\,u\,r^2-578\,q\,r\,t\,s+354\,q\,s^3+354\,t\,r^3-236\,r^2\,s^2,32\,t\,p^2\,u-191\,p\,r^2\,u$

$+176\,r\,p\,t\,s-126\,p\,s^3+104\,r\,u\,q^2+40\,s\,t\,q^2-105\,q\,t\,r^2+70\,q\,r\,s^2,$

$p^2\,t^2+10\,p\,q\,t\,s-6\,r^2\,t\,p-12\,r\,p\,s^2-3\,q^3\,u+10\,q^2\,s^2,$

$s\,u\,p^2+43\,p\,q\,t\,s-r^2\,t\,p-66\,r\,p\,s^2-12\,q^3\,u-20\,q^2\,t\,r+55\,q^2\,s^2,$

$12\,q\,p\,s^2+2\,q\,r\,t\,p-3\,p\,u\,q^2-5\,t\,q^3-32\,s\,r^2\,p+20\,r\,q^2\,s+6\,p^2\,t\,s,27\,p^2\,s^3$

$+84\,s\,p\,t\,q^2-2\,q\,r^2\,t\,p-198\,q\,s^2\,r\,p+8\,s\,p\,r^3-24\,q^4\,u-40\,r\,t\,q^3+150\,q^3\,s^2$

$-5\,s\,r^2\,q^2,\,r\,p^2\,u-4\,p\,u\,q^2+23\,q\,r\,t\,p+18\,q\,p\,s^2-48\,s\,r^2\,p-20\,t\,q^3+30\,r\,q^2\,s,$

$-24\,p\,r^3+15\,r^2\,q^2+38\,q\,s\,r\,p+4\,r\,t\,p^2-4\,p\,t\,q^2-20\,q^3\,s-9\,p^2\,s^2,27\,r\,p^2\,s^2$

$+44\,r\,p\,t\,q^2-162\,q\,s\,r^2\,p+8\,p\,r^4+100\,r\,q^3\,s-5\,r^3\,q^2-8\,p\,q^3\,u+36\,p\,q^2\,s^2$

$-40\,t\,q^4,\,2\,q\,u\,p^2-27\,p^2\,s^2+2\,p\,t\,q^2+66\,q\,s\,r\,p-8\,p\,r^3-40\,q^3\,s+5\,r^2\,q^2,$

$p^3\,u+7\,q\,t\,p^2-48\,s\,r\,p^2+10\,p\,q^2\,s+80\,q\,p\,r^2-50\,r\,q^3,$

$50\,q^2\,p\,r-25\,q^4-12\,q\,p^2\,s+3\,p^3\,t-16\,p^2\,r^2,$

$8\,r^3\,p^2+155\,p\,r^2\,q^2-162\,q\,s\,r\,p^2+12\,p^2\,t\,q^2+60\,p\,q^3\,s+27\,p^3\,s^2-100\,r\,q^4]$

B.8 A Gröbner basis for $\mathsf{k}_{25}, \mathsf{k}_{23}, \mathsf{k}_{22}, \mathsf{k}_{18}, \mathsf{k}_{15}, \mathsf{k}_{13}, \mathsf{k}_{10}, \mathsf{k}_9, \mathsf{k}_4, \mathsf{k}_2$

$W := [\mathsf{k}_{25}, \mathsf{k}_{23}, \mathsf{k}_{22}, \mathsf{k}_{18}, \mathsf{k}_{15}, \mathsf{k}_{13}, \mathsf{k}_{10}, \mathsf{k}_9, \mathsf{k}_4, \mathsf{k}_2]$

> $> gbasis(W, plex(p,q,r,s,t,u));$

$$[-1000\,t^7\,s\,r - 1048\,t^3\,u^3\,q\,r\,s + 200\,s^2\,t\,u^4\,q\,r - 88\,t^4\,u^3\,q^2 + 560\,t^5\,q\,u^2\,r$$
$$- 1420\,t^4\,u^2\,r^2\,s + 2580\,t^4\,s^2\,u^2\,q - 2000\,t^6\,s\,q\,u + 3600\,t^5\,s^2\,u\,r - 3300\,t^3\,s^3\,u^2\,r$$
$$- 468\,s\,t\,u^4\,r^3 + 176\,s\,t^2\,u^4\,q^2 - 1196\,s^3\,t^2\,q\,u^3 + 2234\,s^2\,t^2\,r^2\,u^3 + 160\,s^4\,t\,u^3\,r$$
$$+ 156\,r^2\,t^2\,q\,u^4 + 36\,q\,u^5\,r^2\,s - 48\,u^5\,q^2\,r\,t - 1725\,t^4\,s^4\,u + 1680\,u^2\,s^5\,t^2$$
$$+ 104\,t^3\,u^3\,r^3 + 224\,q\,s^4\,u^4 - 176\,s^3\,u^4\,r^2 - 52\,s^2\,u^5\,q^2 + 100\,u\,t^6\,r^2 + 500\,t^6\,s^3$$
$$+ 500\,t^8\,q - 320\,s^6\,u^3 + 4\,u^6\,q^3 + 27\,u^5\,r^4, -4\,u^3\,q^2 + 50\,s^3\,t^2 - 54\,u^2\,r^2\,s$$
$$- 100\,s\,t^3\,r + 36\,s^2\,u^2\,q - 80\,s^4\,u + u^3\,r\,p + 50\,t^4\,q - 3\,t\,s\,u^2\,p + 2\,t^3\,p\,u$$
$$- 105\,s\,t^2\,q\,u + 27\,u^2\,q\,r\,t + 180\,s^2\,u\,r\,t, 48\,t^4\,u^2\,q^2 - 1680\,u\,s^5\,t^2 + 950\,t^6\,s\,q$$
$$- 104\,t^3\,u^2\,r^3 - 290\,t^5\,r\,q\,u - 100\,t^6\,r^2 + 20\,t^7\,p + 320\,s^6\,u^2 - 4\,u^5\,q^3 - 27\,u^4\,r^4$$
$$+ 925\,s^4\,t^4 - 1800\,t^5\,r\,s^2 - 224\,q\,s^4\,u^3 + 176\,s^3\,u^3\,r^2 + 52\,s^2\,u^4\,q^2 + 10\,t^4\,u^2\,p\,r$$
$$- 30\,t^5\,s\,u\,p - 2220\,t^4\,s^2\,q\,u + 880\,t^4\,r^2\,s\,u + 1048\,t^3\,u^2\,q\,r\,s + 3300\,t^3\,s^3\,u\,r$$
$$+ 468\,s\,t\,u^3\,r^3 - 176\,s\,t^2\,u^3\,q^2 + 1196\,s^3\,t^2\,q\,u^2 - 2234\,s^2\,t^2\,r^2\,u^2 - 160\,s^4\,t\,u^2\,r$$
$$- 156\,r^2\,t^2\,q\,u^3 - 36\,q\,u^4\,r^2\,s + 48\,u^4\,q^2\,r\,t - 200\,s^2\,t\,u^3\,q\,r, 54\,u^2\,r^3 - 11\,u^2\,p\,r\,t$$
$$- 8\,s^2\,u^2\,p + 37\,s\,t^2\,u\,p + 2\,u^3\,p\,q - 25\,s\,t^3\,q + 2\,u^2\,q^2\,t + 100\,r^2\,t^3 + 40\,s^2\,t\,q\,u$$
$$- 180\,r^2\,s\,u\,t + 15\,r\,t^2\,q\,u - 50\,r\,t^2\,s^2 - 36\,u^2\,q\,r\,s - 20\,t^4\,p + 80\,s^3\,u\,r, 100\,u\,t^4\,q^2$$
$$- 475\,s^2\,t^4\,q - 448\,q\,s^4\,u^2 + 936\,s\,t\,u^2\,r^3 - 400\,s^5\,t^2 + 640\,s^6\,u - 8\,u^4\,q^3 - 54\,u^3\,r^4$$
$$- 400\,s^2\,t\,u^2\,q\,r - 37\,s\,t^2\,u^2\,p\,r + 95\,s^2\,t^3\,u\,p - 204\,s\,t^2\,u^2\,q^2 + 1060\,s^3\,t^2\,q\,u$$
$$- 2470\,s^2\,t^2\,r^2\,u - 320\,s^4\,t\,u\,r + 4\,q\,t^3\,p\,u^2 - 312\,r^2\,t^2\,q\,u^2 - 72\,q\,u^3\,r^2\,s$$
$$+ 96\,u^3\,q^2\,r\,t - 2\,u\,r\,t^4\,p + 1025\,s\,t^3\,r\,q\,u + 1400\,s\,t^4\,r^2 + 100\,s^3\,t^3\,r - 60\,s\,t^5\,p$$
$$+ 352\,s^3\,u^2\,r^2 + 104\,s^2\,u^3\,q^2 - 100\,u\,r^3\,t^3 - 550\,r\,t^5\,q, -162\,s\,u^3\,r^4 - 24\,s\,u^4\,q^3$$
$$+ 312\,s^3\,u^3\,q^2 + 1056\,s^4\,u^2\,r^2 - 180\,s^2\,t^5\,p + 300\,s^4\,t^3\,r - 1344\,q\,s^5\,u^2$$
$$- 108\,t^2\,u^2\,r^4 - 16\,t^2\,u^3\,q^3 - 1225\,t^4\,q\,s^3 + 4300\,t^4\,r^2\,s^2 + 40\,t^6\,r\,p + 8\,t^5\,q\,p\,u$$
$$+ 2860\,t^2\,q\,s^4\,u - 7570\,t^2\,s^3\,u\,r^2 - 468\,t^2\,s^2\,u^2\,q^2 - 2000\,t^5\,q\,s\,r - 95\,t^2\,r\,s^2\,u^2\,p$$
$$- 1080\,t^2\,q\,u^2\,r^2\,s + 104\,t^3\,u^2\,q^2\,r + 3715\,t^3\,q\,s^2\,u\,r - 80\,t^4\,r\,s\,u\,p + 200\,t^6\,q^2$$

$$
\begin{aligned}
&- 1200\, s^6 t^2 + 1920\, s^7 u - 200\, t^5 r^3 + 288\, s u^3 q^2 r t - 1200\, s^3 t u^2 q r + 60\, t^3 r^3 s u \\
&+ 22\, t^3 u^2 p r^2 - 216\, q u^3 r^2 s^2 - 960\, s^5 t u r + 285\, s^3 t^3 u p + 2808\, s^2 t u^2 r^3 \\
&- 120\, t^4 s q^2 u - 30\, t^4 r^2 q u, -16\, u^5 q^4 + 36180\, q u s^5 t^2 - 2536\, q s^3 u^3 r^2 \\
&- 144\, q^2 u^4 r^2 s - 10680\, t^4 s^2 q^2 u - 944\, s t^2 u^3 q^3 - 2236\, s^3 t^2 q^2 u^2 \\
&+ 38600\, t r^3 s^3 u^2 + 2480\, t r s^2 u^3 q^2 - 7180\, t^4 r^2 s q u - 21136\, t^2 r^2 s^2 u^2 q \\
&+ 700\, t^3 r^2 s u^2 p - 2150\, t^4 r s^2 u p + 2592\, t r^3 q u^3 s - 1584\, t^2 r^2 u^3 q^2 \\
&+ 1200\, t^6 r s p + 20\, t^5 r^2 u p + 7792\, t^3 r s u^2 q^2 + 58325\, t^3 r s^3 q u \\
&- 110350\, t^2 r^2 s^4 u + 2704\, t^3 r^3 q u^2 - 14160\, t r q s^4 u^2 + 272\, t r u^4 q^3 \\
&+ 25600\, t^3 r^3 s^2 u - 32450\, t^5 r s^2 q - 10980\, t^2 r^4 s u^2 - 20800\, t r s^6 u \\
&- 2160\, t^5 r u q^2 + 28800\, s^8 u - 1425\, t^2 r s^3 u^2 p + 4275\, s^4 t^3 u p - 18000\, s^7 t^2 \\
&- 2430\, s^2 u^3 r^4 + 15840\, s^5 u^2 r^2 - 2700\, s^3 t^5 p + 5100\, t^6 r^2 q + 1000\, t^4 r^4 u \\
&+ 63500\, t^4 r^2 s^3 - 17000\, t^5 r^3 s + 540\, t r^5 u^3 + 8500\, t^3 r s^5 - 14675\, q s^4 t^4 \\
&- 108\, q u^4 r^4 - 18880\, q s^6 u^2 + 80\, q t^7 p - 152\, s^2 u^4 q^3 + 3784\, q^2 s^4 u^3 \\
&+ 6800\, t^6 s q^2 + 192\, t^4 u^2 q^3, -20\, r t^4 p - 50\, r^2 t^2 s^2 - 100\, q s^3 t^2 + 160\, q s^4 u \\
&+ 37\, r s t^2 u p + 80\, s^3 u r^2 + 8\, u^3 q^3 - 100\, t^4 q^2 + 54\, u^2 r^4 + 100\, r^3 t^3 - 72\, s^2 u^2 q^2 \\
&+ 6\, q t s u^2 p - 320\, q s^2 u r t + 175\, q s t^3 r - 4\, q t^3 p u - 11\, u^2 p r^2 t - 180\, r^3 s u t \\
&+ 15\, r^2 t^2 q u - 8\, r s^2 u^2 p + 72\, q u^2 r^2 s - 52\, u^2 q^2 r t + 210\, s t^2 q^2 u, 8\, r t^2 u p \\
&+ u^3 p^2 + 25\, q^2 t^2 u + 200\, q t^3 r - 64\, u^2 p r s - 80\, s t^3 p - 100\, q t^2 s^2 - 360\, q s u r t \\
&+ 108\, r^2 u^2 q + 128\, s^2 t u p + 160\, s^3 q u - 32\, q^2 s u^2 + 6\, u^2 p q t, -4\, r^2 u^2 p \\
&- 8\, p s u^2 q - 40\, q^2 s t u + 6\, p t^2 q u - 8\, p s u r t + 16\, q^2 r u^2 - 20\, p t^2 s^2 \\
&+ 32\, s^3 u p + t p^2 u^2 + 25\, q^2 t^3, -2\, s r^2 u^2 p - 4\, p s^2 u^2 q - 40\, q^2 s^2 t u \\
&- 13\, s p t^2 q u + 28\, p s^2 u r t + 8\, s q^2 r u^2 - 10\, p t^2 s^3 + 16\, s^4 u p + 25\, s q^2 t^3 \\
&+ 5\, q^2 u r t^2 - 20\, p t^3 r s - t p r u^2 q - 8\, r^2 t^2 u p + 10\, p t^4 q + 2\, p^2 t^3 u + 4\, t q^3 u^2, \\
&50\, s^2 q^2 t^3 - 200\, q^2 t^4 r - 140\, p r^2 t^4 + 32\, s^5 u p - 20\, p t^2 s^4 + 40\, q^3 t^3 u
\end{aligned}
$$

$$-28s^2r^2u^2p+16p^2t^5-8ps^3u^2q-80q^2s^3tu-200s^2q^2ru^2-80pt^3rs^2$$
$$+40spt^4q-58s^2pt^2qu+120ps^3urt+480sq^2urt^2+207sr^2t^2up$$
$$+8stq^3u^2+8t^2q^2u^2p-41tr^3u^2p-124tq^2r^2u^2+525qst^3r^2$$
$$-540r^4sut+45r^3t^2qu+216qu^2r^3s-300rqs^3t^2+480rqs^4u$$
$$-64pqurt^3+162u^2r^5+300r^4t^3-150r^3t^2s^2+240s^3ur^3+24ru^3q^3$$
$$-960qs^2ur^2t, -10q^2urt-64rs^2up+40pt^2rs+40q^2s^2u-25q^2st^2$$
$$+2pru^2q+32pstqu+p^2su^2+16r^2tup-20pt^3q-4p^2t^2u-8q^3u^2,$$
$$-5pst^3q+8tps^2qu-2tq^2u^2p+40tq^2urs-10q^3t^2u-25q^2t^3r-4p^2t^4$$
$$+20pr^2t^3+p^2st^2u+13pqurt^2-28tr^2sup+2r^3u^2p+4rpsu^2q$$
$$-8q^2r^2u^2+10rpt^2s^2-16rs^3up, -10pst^2q+16ps^2qu-4q^2u^2p$$
$$+40q^2urs-20q^3tu-25q^2t^2r-8p^2t^3+40pr^2t^2+2p^2stu+32pqurt$$
$$+p^2ru^2-64r^2sup, -5ps^2t^2q+8ps^3qu-2sq^2u^2p-10sq^3tu-4sp^2t^3$$
$$+p^2s^2tu+5q^2ur^2t-pr^2u^2q-8r^3tup+10rpt^3q+2rp^2t^2u+4rq^3u^2,$$
$$-4s^2p^2t^3+8rp^2t^4-5ps^3t^2q+8ps^4qu-20r^2pt^2s^2+32r^2s^3up$$
$$+50q^2t^3r^2-40pr^3t^3-4r^4u^2p+p^2s^3tu-10s^2q^3tu-2s^2q^2u^2p$$
$$-16rtps^2qu+20rq^3t^2u+20rpst^3q+4rtq^2u^2p-26pqur^2t^2$$
$$+48tr^3sup-75tq^2ur^2s+4srq^3u^2-9r^2psu^2q+16q^2r^3u^2, --8pqurs$$
$$-8p^2urt+25q^3t^2-40pqt^2r-20q^2r^2u+p^2u^2q+6q^2tup+16p^2st^2$$
$$+32r^3up-4p^2s^2u, 28pqurs^2-4sp^2urt+25sq^3t^2-40spqt^2r$$
$$-10sq^2r^2u-13sq^2tup+8p^2s^2t^2+16sr^3up-2p^2s^3u+5q^3urt$$
$$-20q^3s^2u-pru^2q^2-8qr^2tup+10pt^3q^2+2qp^2t^2u+4q^4u^2, 25s^2q^3t^2$$
$$+20q^4t^2u+50q^3t^3r+16q^3r^2u^2+8qp^2t^4+8p^2s^3t^2-2p^2s^4u-20q^3s^3u$$
$$+4sq^4u^2+20pst^3q^2+4tq^3u^2p-40qpr^2t^3-4qr^3u^2p-4s^2p^2urt$$
$$-10s^2q^2r^2u+16s^2r^3up+48qtr^2sup-60qrpt^2s^2-9rpsu^2q^2$$

$$- 26pq^2urt^2 + 60qrs^3up - 29tps^2q^2u - 75tq^3urs, -5pst^2q^2$$
$$+ 8ps^2q^2u - 2q^3u^2p + 20q^3urs - 10q^4tu - 25q^3t^2r - 4qp^2t^3$$
$$+ 40qpr^2t^2 + qp^2stu + 13pq^2urt - 28qr^2sup + 4p^2ur^2t + 10q^2r^3u$$
$$- 8rp^2st^2 - 16r^4up + 2rp^2s^2u, -100q^2r^2t - 64p^2urs - 32p^2t^2r$$
$$- 80q^3ru + 200q^3st + 108p^2ts^2 + 8q^2sup + 160pr^3t - 360pqsrt + p^3u^2$$
$$+ 6p^2tqu + 128pr^2qu + 25pq^2t^2, 37pq^2ru + 100q^3s^2 - 20q^4u$$
$$- 50q^2r^2s + 54p^2s^3 - 36p^2srt - 25q^3rt + 15pq^2st - 11p^2squ + 80pr^3s$$
$$+ 40pr^2qt - 180pqrs^2 - 8p^2r^2u + 2p^2t^2q + 2p^3tu, -100tq^2r^3$$
$$+ 160tpr^4 + 37spq^2ru - 11p^2s^2qu - 8sp^2r^2u - 180pqrs^3 + 54p^2s^4$$
$$+ 100q^3s^3 - 50q^2r^2s^2 + 80pr^3s^2 - 20sq^4u - 320tpqr^2s - 4tq^3up$$
$$+ 175tq^3rs + 72tp^2s^2r + 15pq^2s^2t + 6tp^2qru + 8p^3t^3 - 100q^4t^2$$
$$- 72p^2r^2t^2 - 52p^2sqt^2 + 210pq^2rt^2, -4p^3t^2 + 50q^2r^3 + 180pqr^2s$$
$$- 80pr^4 + 50q^4t + 2q^3up - 100q^3rs + 36p^2r^2t + 27p^2sqt - 54p^2s^2r$$
$$- 3p^2qru + p^3su - 105pq^2rt]$$

Appendix C

MAPLE Work Sheet

Here we use the variables x and y instead of X_0 and X_1.

First evaluate G stated in Lemma 4.16, by substituting

$$\begin{aligned} P &= \frac{1}{f}\frac{\partial f}{\partial x}, \\ Q &= \frac{\partial}{\partial x}\left(\frac{1}{f}\frac{\partial f}{\partial x}\right), \\ S &= \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(\frac{1}{f}\frac{\partial f}{\partial x}\right). \end{aligned}$$

$> G := eval(3*r*Q*P^4-4*m^2*S*P^3+3*Q^2*P^2*m^2+m*r^3*S^2-m^2*r^2*S^2-4*m$
$r^2*Q^3+4*m^2*r*Q^3+P^6+3*Q^2*P^2*r^2+r^3*Q^3+4*m*S*P^3*r-6*m^2*r*Q*S$
$P+6*m*r^2*Q*S*P-3*Q^2*P^2*m*r, \{P = diff(f(x,y),x)/f(x,y),$
$Q = diff(f(x,y),x,x)/f(x,y) - diff(f(x,y),x)^2/f(x,y)^2,$
$S = (2*diff(f(x,y),x)^3/f(x,y)^3 + diff(f(x,y),x,x,x)/f(x,y)$
$-3*diff(f(x,y),x)*diff(f(x,y),x,x)/f(x,y)^2)/2, m = m, r = r\});$

$$G := 3\,\frac{r\,\%1\,(\frac{\partial}{\partial x}\,\mathrm{f}(x,\,y))^4}{\mathrm{f}(x,\,y)^4} - 4\,\frac{m^2\,\%2\,(\frac{\partial}{\partial x}\,\mathrm{f}(x,\,y))^3}{\mathrm{f}(x,\,y)^3} + 3\,\frac{\%1^2\,(\frac{\partial}{\partial x}\,\mathrm{f}(x,\,y))^2\,m^2}{\mathrm{f}(x,\,y)^2} + m\,r^3\,\%2^2$$
$$-\,m^2\,r^2\,\%2^2 - 4\,m\,r^2\,\%1^3 + 4\,m^2\,r\,\%1^3 + \frac{(\frac{\partial}{\partial x}\,\mathrm{f}(x,\,y))^6}{\mathrm{f}(x,\,y)^6} + 3\,\frac{\%1^2\,(\frac{\partial}{\partial x}\,\mathrm{f}(x,\,y))^2\,r^2}{\mathrm{f}(x,\,y)^2}$$

$$+r^3\,\%1^3+4\,\frac{m\,\%2\,(\frac{\partial}{\partial x}\,\mathrm{f}(x,\,y))^3\,r}{\mathrm{f}(x,\,y)^3}-6\,\frac{m^2\,r\,\%1\,\%2\,(\frac{\partial}{\partial x}\,\mathrm{f}(x,\,y))}{\mathrm{f}(x,\,y)}$$
$$+6\,\frac{m\,r^2\,\%1\,\%2\,(\frac{\partial}{\partial x}\,\mathrm{f}(x,\,y))}{\mathrm{f}(x,\,y)}-3\,\frac{\%1^2\,(\frac{\partial}{\partial x}\,\mathrm{f}(x,\,y))^2\,m\,r}{\mathrm{f}(x,\,y)^2}$$
$$\%1:=\frac{\frac{\partial^2}{\partial x^2}\,\mathrm{f}(x,\,y)}{\mathrm{f}(x,\,y)}-\frac{(\frac{\partial}{\partial x}\,\mathrm{f}(x,\,y))^2}{\mathrm{f}(x,\,y)^2}$$
$$\%2:=\frac{(\frac{\partial}{\partial x}\,\mathrm{f}(x,\,y))^3}{\mathrm{f}(x,\,y)^3}+\frac{1}{2}\,\frac{\frac{\partial^3}{\partial x^3}\,\mathrm{f}(x,\,y)}{\mathrm{f}(x,\,y)}-\frac{3}{2}\,\frac{(\frac{\partial}{\partial x}\,\mathrm{f}(x,\,y))\,(\frac{\partial^2}{\partial x^2}\,\mathrm{f}(x,\,y))}{\mathrm{f}(x,\,y)^2}$$

Now we collect the terms of G.

> *collect(expand(4*f(x,y)^6*G), [f(x,y), (diff(f(x,y),x)), (diff(f(x,y),x,x)), (diff(f(x,y),x,x,x)), (diff(f(x,y),x,x,x,x))], distributed);*

$$(16\,m^2\,r+4\,r^3-16\,m\,r^2)\,(\tfrac{\partial^2}{\partial x^2}\,\mathrm{f}(x,\,y))^3\,\mathrm{f}(x,\,y)^3+(r^3\,m-m^2\,r^2)\,(\tfrac{\partial^3}{\partial x^3}\,\mathrm{f}(x,\,y))^2\,\mathrm{f}(x,\,y)^4+$$

$$(12\,r-12\,r^3\,m+12\,r^3+12\,m^2\,r^2+12\,m\,r^2-24\,r^2-12\,m^2\,r)\,(\tfrac{\partial^2}{\partial x^2}\,\mathrm{f}(x,\,y))$$
$$(\tfrac{\partial}{\partial x}\,\mathrm{f}(x,\,y))^4\,\mathrm{f}(x,\,y)+$$
$$(-12\,m^2\,r-12\,m\,r-9\,m^2\,r^2-12\,r^3+12\,m\,r^2+12\,m^2+9\,r^3\,m+12\,r^2)$$
$$(\tfrac{\partial^2}{\partial x^2}\,\mathrm{f}(x,\,y))^2\,(\tfrac{\partial}{\partial x}\,\mathrm{f}(x,\,y))^2\,\mathrm{f}(x,\,y)^2+$$
$$(8\,m\,r-4\,m^2\,r^2+4\,r^3\,m+12\,m^2\,r-12\,m\,r^2-8\,m^2)\,(\tfrac{\partial^3}{\partial x^3}\,\mathrm{f}(x,\,y))\,(\tfrac{\partial}{\partial x}\,\mathrm{f}(x,\,y))^3$$
$$\mathrm{f}(x,\,y)^2+$$
$$(6\,m^2\,r^2+12\,m\,r^2-6\,r^3\,m-12\,m^2\,r)\,(\tfrac{\partial^2}{\partial x^2}\,\mathrm{f}(x,\,y))\,(\tfrac{\partial}{\partial x}\,\mathrm{f}(x,\,y))\,(\tfrac{\partial^3}{\partial x^3}\,\mathrm{f}(x,\,y))\,\mathrm{f}(x,\,y)^3$$
$$+(4\,m\,r-4\,r^3+8\,m^2\,r-4\,m^2\,r^2+4\,r^3\,m+12\,r^2-12\,r-4\,m^2+4-8\,m\,r^2)$$
$$(\tfrac{\partial}{\partial x}\,\mathrm{f}(x,\,y))^6$$

Now we rewrite this differential equation according to our notations, meaning

$$\begin{aligned}\partial_0 f &= \frac{\partial f}{\partial x},\\ \partial_0^2 f &= \frac{\partial^2 f}{\partial x^2},\\ \partial_0^3 f &= \frac{\partial^3 f}{\partial x^3}\end{aligned}$$

we get,

$$4f^6 g = (16\,m^2\,r + 4\,r^3 - 16\,m\,r^2)\,(\partial_0^2 f)^3\,f^3 + (r^3\,m - m^2\,r^2)\,(\partial_0^3\,f)^2\,f^4$$

$$+ \,(12\,r - 12\,r^3\,m + 12\,r^3 + 12\,m^2\,r^2 + 12\,m\,r^2 - 24\,r^2 - 12\,m^2\,r)\,(\partial_0^2 f)(\partial_0 f)^4\,f$$

$$+(-12\,m^2\,r - 12\,m\,r - 9\,m^2\,r^2 - 12\,r^3 + 12\,m\,r^2 + 12\,m^2 + 9\,r^3\,m + 12\,r^2)(\partial_0^2 f)^2\,(\partial_0 f)^2\,f^2$$

$$+ \,(8\,m\,r - 4\,m^2\,r^2 + 4\,r^3\,m + 12\,m^2\,r - 12\,m\,r^2 - 8\,m^2)\,(\partial_0^3 f)\,(\partial_0\,f)^3 f^2$$

$$+ \,(6\,m^2\,r^2 + 12\,m\,r^2 - 6\,r^3\,m - 12\,m^2\,r)\,(\partial_0^2 f)\,(\partial_0 f)\,(\partial_0^3 f)\,f^3$$

$$+ \,(4\,m\,r - 4\,r^3 + 8\,m^2\,r - 4\,m^2\,r^2 + 4\,r^3\,m + 12\,r^2 - 12\,r - 4\,m^2 + 4 - 8\,m\,r^2)(\partial_0 f)^6.$$

Appendix D

Covariant calculations for binary quintic forms

```
> f:=x^2*y^2*(x+y);
```

$$f := x^2\, y^2\, (x+y)$$

We shall calculate $\mathcal{P}(f)$:

```
> p:=diff(f,x,x,x,x)*diff(f,y,y,y,y)-4*diff(f,x,x,x,y)*diff(f,x,y,y,y)+
6*diff(f,x,x,y,y)*diff(f,x,x,y,y)-4*diff(f,x,y,y,y)*diff(f,x,x,x,y)+di
ff(f,y,y,y,y)*diff(f,x,x,x,x);
```

$$p := -1152\, x\, y + 6\, (12\, x + 12\, y)^2$$

```
> with(linalg):
```

```
Warning, new definition for norm
```

```
Warning, new definition for trace
```

Next we calculate $\mathcal{H}(f)$:

```
> h:=det(array( [[diff(f,x,x),diff(f,x,y)],[diff(f,x,y),diff(f,y,y)]]
));
```

$$h := -24\, x^4\, y^2 - 32\, x^3\, y^3 - 24\, y^4\, x^2$$

Next we calculate $\mathcal{J}(f)$:

```
> j:=det(array( [[diff(f,x),diff(f,y)],[diff(h,x),diff(h,y)]]) );
```

$$j := 48\,x^6\,y^3 + 96\,x^5\,y^4 - 96\,x^4\,y^5 - 48\,x^3\,y^6$$

Next we calculate $(\mathcal{J}, \mathcal{P})^{(1)}(f)$:

```
> pj1:=det(array( [[diff(j,x),diff(j,y)],[diff(p,x),diff(p,y)]] ));
```

$$pj1 := -580608\,x^6\,y^3 + 1382400\,x^5\,y^4 + 1382400\,x^4\,y^5 - 580608\,x^3\,y^6 - 248832\,x^2\,y^7 - 248832\,x^7\,y^2$$

Next we calculate $(\mathfrak{I}, \mathcal{P})^{(2)}(f)$:

```
> fp2:=diff(f,x,x)*diff(p,y,y)-2*diff(f,x,y)*diff(p,x,y)+diff(f,y,y)*di
ff(p,x,x);
```

$$fp2 := 3456\,y^2\,(x+y) + 4608\,x\,y^2 - 4608\,x\,y\,(x+y) + 4608\,x^2\,y + 3456\,x^2\,(x+y)$$

Next we calculate $(\mathcal{H}, \mathcal{P})^{(2)}(f)$:

```
> ph2:=diff(h,x,x)*diff(p,y,y)-2*diff(h,x,y)*diff(p,x,y)+diff(h,y,y)*di
ff(p,x,x);
```

$$ph2 := -663552\,x^2\,y^2 - 110592\,x\,y^3 - 82944\,y^4 - 110592\,x^3\,y - 82944\,x^4$$

Next we calculate $(-(1/5) * (\mathcal{P}, \mathcal{J})^{(1)}(f) - (\mathcal{P}, \mathcal{H})^{(2)}(f) * f - (1/6) * \mathcal{P}^2(f) * f + (1/10) * \mathcal{H} * (\mathfrak{I}, \mathcal{P})^{(2)}(f)))$:

```
> f221:=collect(expand((-(1/5)*pj1-ph2*f-(1/6)*p^2*f+(1/10)*h*fp2)),[x,
y],distributed);
```

$$f221 := 0$$

```
> g:=x^3*y*(x+y);
```

$$g := x^3\, y\, (x + y)$$

Now we will follow the same calculations for g :

```
> p1:=diff(g,x,x,x,x)*diff(g,y,y,y,y)-4*diff(g,x,x,x,y)*diff(g,x,y,y,y)
+6*diff(g,x,x,y,y)*diff(g,x,x,y,y)-4*diff(g,x,y,y,y)*diff(g,x,x,x,y)+d
iff(g,y,y,y,y)*diff(g,x,x,x,x);
```

$$p1 := 864\, x^2$$

```
> h1:=det(array( [[diff(g,x,x),diff(g,x,y)],[diff(g,x,y),diff(g,y,y)]]
));
```

$$h1 := -24\, x^5\, y - 24\, x^4\, y^2 - 16\, x^6$$

```
> j1:=det(array( [[diff(g,x),diff(g,y)],[diff(h1,x),diff(h1,y)]]) );
```

$$j1 := 216\, x^8\, y + 72\, x^7\, y^2 + 48\, x^6\, y^3 + 96\, x^9$$

```
> pj11:=det(array( [[diff(j1,x),diff(j1,y)],[diff(p1,x),diff(p1,y)]]
));
```

$$pj11 := -1728\, (216\, x^8 + 144\, x^7\, y + 144\, x^6\, y^2)\, x$$

```
> gp2:=diff(g,x,x)*diff(p1,y,y)-2*diff(g,x,y)*diff(p1,x,y)+diff(g,y,y)*
diff(p1,x,x);
```

$$gp2 := 3456\,x^3$$

```
> p1h12:=diff(h1,x,x)*diff(i1,y,y)-2*diff(h1,x,y)*diff(p1,x,y)+diff(h1,
y,y)*diff(p1,x,x);
```

$$p1h12 := -82944\,x^4$$

```
> g221:=collect(expand((-(1/5)*pj11-p1h12*f-(1/6)*p1^2*g+(1/10)*h1*gp2)
),[x,y],distributed);
```

$$g221 := 69120\,x^9 - 82944\,x^8\,y + 82944\,x^6\,y^3$$

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