THE UNIVERSITY OF CALGARY

RANDOM BIVARIATE RAYS, STATISTICAL SOCIETIES AND BUFFON'S PI

by

TITUS S. MISI

A DISSERTATION
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF
MATHEMATICS AND STATISTICS

CALGARY, ALBERTA
JUNE, 1992

ISBN 0-315-79081-4

THE UNIVERSITY OF CALGARY
FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a dissertation entitled "*Random Bivariate Rays, Statistical Societies and Buffon's Pi* ", submitted by **Titus S. Misi** in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

E.G. Enns

Supervisor, Dr. E.G. Enns
Dept. of Mathematics & Statistics

M. D. Burke

Dr. M.D. Burke,
Dept. of Mathematics & Statistics

John R Collins

Dr. J.R. Collins
Dept. of Mathematics & Statistics

C. van de Panne

Dr. C. Van de Panne
Department of Economics

Jerome N. Sheahan

External Examiner, Dr. J.N. Sheahan
University of Alberta

16 June, 1992

ABSTRACT

This work comprises three different topics:

1. For an arbitrary convex body in n–dimensional Euclidean space, three types of random straight lines may be defined, namely : rays, secants and line segments both of whose terminal points are inside the body. After a review of univariate rays, bivariate rays are analyzed under different randomness assumptions. As an example, the sphere is examined.

2. Randomly generated points in $\mathbb{R}^d$ are connected to their nearest neighbours in terms of Euclidean distance. The resulting connected clusters of points are defined as societies. Questions related to the collection of societies formed, the internal structure of a society, and the relationships between individual points are examined. In particular, the one–dimensional societal structure is examined in detail.

3. Buffon's needle experiment and its variations yield empirical estimates of the value of π. Several of these estimates are derived and compared.

ACKNOWLEDGEMENTS

I wish to express my deepest appreciation to my supervisor, Dr. E.G. Enns, without whose wealth of ideas, continuous encouragement and careful guidance this research would not have come to fruition.

I am greatly indebted to Dr. P.F. Ehlers for being my interim supervisor and for all his invaluable advice and help during the preparation of this dissertation.

I owe a special word of thanks to Dr. E. Couch, Indy and Bona for all the pertinent discussions.

Sincere thanks are extended to Mrs. Maria Dourado for cheerfully and skillfully typing this work.

It is a pleasure to thank the Canadian Commonwealth Scholarship and Fellowship Plan for the financial support. The supplementary awards by the Department of Mathematics and Statistics at the University of Calgary are also gratefully acknowledged.

Lastly, and certainly not least, I wish to thank my wife Christina and my son Adrian Mphatso for their patience and support for the duration of this effort.

Dedicated to

my beloved parents

TABLE OF CONTENTS

CHAPTER

CHAPTER

LIST OF TABLES

LIST OF FIGURES

CHAPTER ONE

INTRODUCTION

1.1 RANDOM RAYS IN CONVEX BODIES

The theory of geometrical probabilities covers problems that arise when we ascribe probability distributions to geometric objects such as points, lines and planes (usually in Euclidean spaces), or to geometric operations such as rotations or projections. Historically, the first such problem appears to be that of Buffon in 1777 (which we discuss in Chapter 5).

Since the ascription of a measure to the geometric elements is not quite an obvious procedure, a number of "paradoxes" can be produced by the failure to distinguish the reference set. These are all based on a simple confusion of ideas but may be useful in illustrating the way in which geometric probabilities should be defined. The Bertrand paradox is such an example. In that paradox, the probability that a random chord in a circle exceeds the side of an inscribed equilateral triangle can be shown to be $\frac{1}{4}$, $\frac{1}{3}$, or $\frac{1}{2}$ for each of three different models by which the chord is drawn at random. Consequently, it is commonplace in geometrical probability that there may be more than one probability measure which may be used to describe a random geometrical element 'uniformly' distributed over some given set of such elements.

For an arbitrary convex body in n–dimensional Euclidean space, we may define three types of random straight lines, namely: rays, secants and line segments both of whose terminal points are inside the body. There is extensive literature on the distributions and moments of these random line segments.

We will concentrate on rays. After presenting the known results on univariate rays, we will derive expressions for distributions of bivariate rays. Here, two rays will be generated with one common terminal point. We will consider three randomness assumptions and their bivariate extensions. As an example, we will examine in detail the sphere and obtain various volume estimators.

APPLICATIONS:

(i) The subject of stereology represents a major part of the applications of geometrical probability (or stochastic geometry, as it is also known), see Stoyan et al (1987). The object of stereology is to draw inferences about the geometrical properties of three–dimensional structure when information is only available in some lower–dimensional form, such as random plane sections of an opaque solid, or randomly–oriented plane projections of a space curve. It is a multidisciplinary effort encompassing the biological, geological, material and mathematical sciences and in which geometrical probability plays a role.

(ii) Kellerer (1971) includes a discussion and reference to applications of chord (secant) length distributions (resulting from the random intersections of convex bodies by straight lines) to such different fields as acoustics, reactor design, ecology, microscopy, and radiation physics, just to mention a few.

1.2 STATISTICAL SOCIETIES

As far as we know, the problem we formulate and analyse in Chapters 3 and 4 concerning the nature of the connected clusters of points, has not been clearly formulated before (see Naus (1979) for a comprehensive bibliography of problems on clusters).

Consider randomly generated points in $\mathbb{R}^d$ connected to their nearest neighbours in terms of Euclidean distance. We define the resulting connected clusters of points as 'societies'. We may ask the following questions related to the collection of societies formed, the internal structure of a society, and the relationships between the individual points:

(a) Let M denote the number of societies formed. What is the distribution of M ? (This of course depends on the number of points generated).

In Chapter 4 we will examine the one–dimensional societal structure in detail. The points are assumed to be generated from a uniform distribution on the line. We will derive expressions for the distribution of M and the maximum number of societies. Moments are obtained through generating functions. We will also consider populations of societies.

The problem Glaz and Naus (1983) discuss is different. Given N events occurring over time, an n:t cluster is defined as n consecutive events all contained within an interval of length t. They derive the expectation, variance and approximate distribution of the number of n:t clusters.

(b) Let K denote the size of a society, i.e. the number of individuals (points) in a society. What is the distribution of K?

Define a Poisson ensemble as a set of points distributed in a d–dimensional space satisfying the following conditions:

(i) the probability that a point lies in an infinitesimal volume δV equals $\lambda\delta V + o(\delta V)$, where λ is the density of the Poisson law.

(ii) the probability that more than one point lies in an infinitesimal volume δV is close to zero.

Roberts (1967) considers an infinite plane on which are scattered discs of radius R whose centres are points distributed according to the Poisson law with density λ. A cluster of size n is defined as a set of n discs each of which overlaps at least one other member of the set and none of which overlaps a disc which is not a member of the set. The author uses a Monte Carlo technique to evaluate the expected size of a cluster. The three–dimensional version of this problem is discussed by Roberts and Storey (1968).

(c) Assume every point has a unique nearest neighbour. What proportion of points in a population are nearest neighbours to 0,1,2, or more other points? Determine p_n, the probability that a point is nearest neighbour to precisely n points.

In Chapter 3, we will classify individual points according to the number of other individuals that consider the particular individual as their nearest neighbour.

Clark and Evans (1955) define two points as reflexive nearest neighbours when each is the nearest point to the other. They obtain the proportion of reflexives in a population. We will use this to derive bounds for the classes of individual points. Dacey (1969) considers reflexive n^{th} order neighbours while Cox (1981) calculates the probability that an arbitrary event in a d–dimensional Poisson process is the m^{th} nearest neighbour to its own n^{th} nearest neighbour.

Roberts (1969) considers the problem of determining p_n for points in Poisson ensembles in 1, 2 and 3 dimensions. Except in the one–dimensional case, for which p_n is trivial, he only provides bounds and estimates of p_n using Monte Carlo methods.

Newman et al (1983) and Newman and Rinott (1985) study p_n in several models of random points processes (ensembles) and their limits.

(d) Form the convex hull or some other enclosure of each society and find:

(i) the content (area, volume) covered by a society,

(ii) the fraction of $\mathbb{R}^d$ that is "inhabited", i.e. contained in a society.

Various expectations concerning the convex hull of N independently and identically distributed random points in the plane or in space are evaluated by Efron (1965). Integral expressions are given for the expected area, expected perimeter, expected probability content and expected number of sides.

APPLICATIONS:

An obvious ecological application of clusters is in determining the spatial pattern of distribution of the individual members of a population of plants or of animals. This is of importance in the analysis of population behaviour. The distance between individuals (for example, the nearest neighbour distance) has been used as a variable in distance analysis.

1.3 BUFFON'S PI

Mathematical probability has been requisitioned for some unusual chores, including the proof – or disproof, according to the temper of the author – of biblical miracles, and the rationalization of our naive belief that the sun will rise tomorrow, (Gridgeman (1960)). Not the least strange is the estimation (in the sense of approximation to) the number π by means of geometrical probability.

In 1777, Georges Louis Leclerc, later Comte de Buffon, published his 'Essai d'arithmétique morale' in which he formulated the famous game now referred to as 'Buffon's needle'. This he solved elegantly by using the integral calculus. In essence he showed that when a straight line is drawn at random on a plane surface of parallels, the probability of an intersection is a simple function of π. The value of π can then be estimated by physical or computer simulation as has been done by a number of investigators.

Buffon described his problem as follows:
'Suppose that a thin rod is thrown in the air in a room whose floor consists of parallel boards. Of two players, one bets that the rod will not intersect any of the parallel floor joins, while the other bets the opposite, namely that the rod will intersect one of these joins. One may ask which of the two has the higher odds. This game can be played on a checker board with a sewing needle, or a headless pin.' This problem initiated the development of geometrical probability, an important subfield of probability theory in which concepts of randomness are applied to geometry.

We will first review Buffon's original experiment and the estimator obtained. Then we will consider a number of ways in which modern statistical procedures can yield estimates of π from other experimental designs with much better precision than the original Buffon procedure. In particular, the following variations will be discussed:

(i) The double grid with a short needle (Laplace's experiment).

(ii) The double grid with a long needle.

(iii) The single grid with a long needle.

The asymptotic sampling variances of the estimators for π, as a measure of efficiency, will be compared.

The new work we will do is to obtain the minimum variance unbiased estimator of π for the single grid with a long needle, and to compare its efficiency with that of the other estimators.

CHAPTER TWO

RANDOM RAYS IN CONVEX BODIES

2.1 INTRODUCTION

Random transversals of convex bodies have attracted considerable interest recently. Kendall and Moran in "Geometrical Probability" (1963) offer an extensive introduction to the subject. Subsequent literature includes Kingman (1969), Coleman (1969, 1973, 1989), Alagar (1976) and Enns and Ehlers (1978, 1980, 1981, 1988). The introduction to Kellerer's (1971) paper includes discussion and references to some applications.

In the literature mentioned, the straight line paths are either rays, secants or line segments within the convex body which do not terminate on the surface. Distributions and moments of these quantities are derived for various types of randomness.

In this chapter, we focus on rays. We present known results on univariate rays in Section 2.3. The distributions and moments have been obtained from or are written in terms of the normalized overlap volume and normalized overlap surface content of the convex body under consideration with its translated self, Enns and Ehlers (1978, 1980). We consider only three types of randomness.

In Section 2.4 we extend the results to bivariate rays in which case two rays are generated with one common terminal point.

In Section 2.5, we present the results when the convex body under consideration is a sphere. Estimators of volume are considered in Section 2.6.

Most of the computation was done with the aid of MACSYMA symbolic manipulation program.

2.2 DEFINITIONS AND NOTATION

We will provide the definitions we need for the work in this chapter.

K : an arbitrary convex body in n–dimensional Euclidean space.

$K(\ell,\theta)$: the body K translated a distance ℓ in direction θ.

$E_{X;D}(\cdot)$: the expected value of $(\cdot)$ when averaging X uniformly over its domain D.

$V(\cdot)$: the volume of $(\cdot)$.

$S(\cdot)$: the surface content of $(\cdot)$.

$\Omega_K(\ell,\theta) = \dfrac{V[K \cap K(\ell,\theta)]}{V(K)}$, the normalized overlap volume of K with $K(\ell,\theta)$,

(see Figure 2.1).

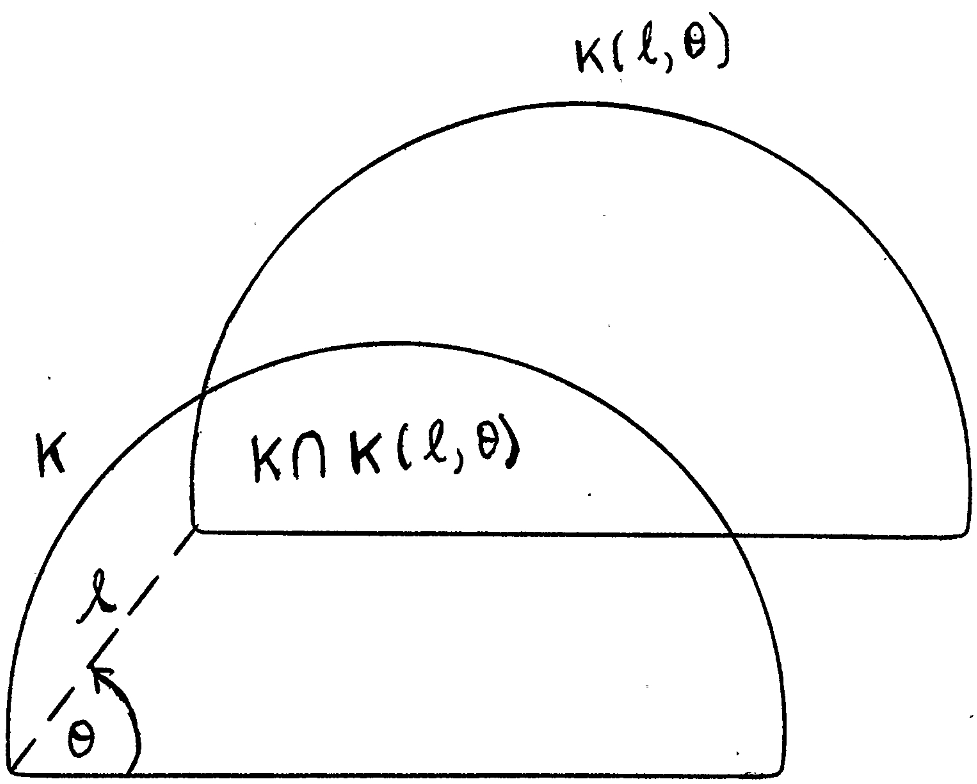

Figure 2.1

$$\Omega_K(\ell) = E_\theta\,[\Omega_K(\ell,\theta)],$$

θ is uniformly distributed over all possible directions.

$$\omega_K(\ell) = E_\theta \left\{\frac{S[(K \cap K(\ell,\theta)]}{S(K)}\right\}.$$

This represents the mean normalized overlap surface of K with its translated self by a distance ℓ.

$C(\ell,P)$: the n–sphere of radius ℓ centered at P.

$\pi(\ell,P)$: the conical subsets of $C(\ell,P)$ that fall partially outside K. In two–dimensions, these are pie–shaped slices. For example, the shaded area in Figure 2.2 represents $\pi(\ell,P)$.

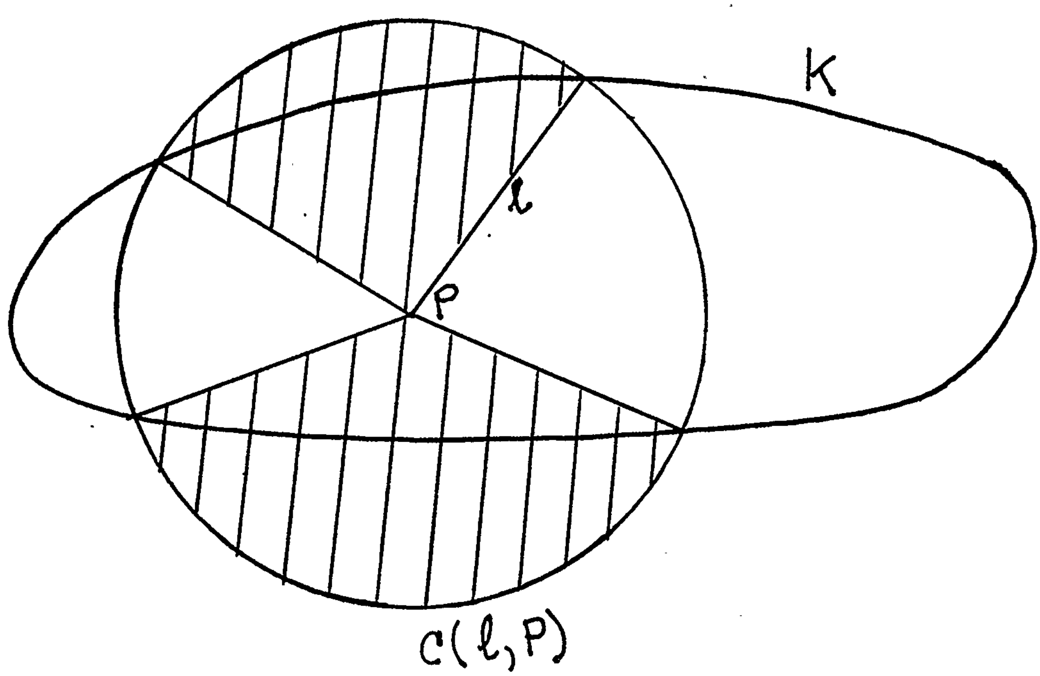

Figure 2.2

$\hat{\phi}(\ell,P)$: the total angle subtended at P by components of $\pi(\ell,P)$.

$\phi(\ell,P)$: the complement to $\hat{\phi}(\ell,P)$.

RAY : a ray is defined as a line segment with one terminal point on the surface of K and the other in the interior of K.

R_ρ : the random variable denoting ray length under ρ–randomness, $\rho \in \{\nu, \lambda, \alpha, \hat{\nu}, \hat{\lambda}, \hat{\alpha}\}$.

SECANT : a secant is defined as a line segment with both terminal points on the surface of K.

L_ρ : the random variable denoting secant length under ρ–randomness, $\rho \in \{\nu, \lambda, \alpha, \hat{\nu}, \hat{\lambda}, \hat{\alpha}\}$.

U : the random variable denoting the distance between two points chosen independently and uniformly from within K.

C_n : $\dfrac{2\pi^{\frac{n}{2}}}{n\Gamma\left[\frac{n}{2}\right]}$, the volume of the unit n–sphere.

Note that the subscripts will be omitted whenever there is no ambiguity.

<u>Type of randomness</u>:

ν–randomness: A point within K and a direction, each with independent uniform distributions, can be used to define a ray and a secant.

λ–randomness: Two points chosen independently within K, each with a uniform distribution. These points define three random lengths, namely a ray, a secant and the distance between the two points.

α–randomness: Two points are selected independently, one on the surface of K and the other inside K. Both points are obtained from uniform distributions over their respective domains. These points define a ray and a secant.

$\hat{\nu}$–randomness: A point P is selected at random within K. Two directions θ_1 and θ_2 are chosen each with independent uniform distributions and independent of P. P, θ_1 and θ_2 define two rays.

$\hat{\lambda}$–randomness: Three points P, Q and O are selected independently from within K, each with a uniform distribution. Two rays are defined from P, one through Q and the other through O.

$\hat{\alpha}_S$–randomness: One point, P, is selected randomly on the surface of K. Two points Q and O are chosen independently within K, each from a uniform distribution. Two rays can be defined with P as the common terminal point.

$\hat{\alpha}_K$–randomness: A point Q is chosen at random from within K. Two points P and O are selected at random on the surface of K, independently of each other. We may define two rays with Q as the common terminal point.

2.3 UNIVARIATE RAYS

The background results we present here are from a series of papers by Enns and Ehlers (1978, 1980, 1981, 1988) in which they derive the distributions and moments of the lengths of random rays and secants under various randomness assumptions.

2.3.1 ν–randomness:

A point P is chosen at random from within K. A ray is defined from P in some random direction θ. If the body K is translated a distance r in direction θ, then denote the translated K by $K(r,\theta)$. If one now places a needle of length r in the body K in direction θ, then the tip of the needle must lie in $K \cap K(r,\theta)$ for the whole needle to lie within K (see Figure 2.1). Now since point P is chosen randomly in K, it must lie in $K \cap K(-r,\theta)$ for the ray R to be of length greater than r. Therefore,

$$\begin{aligned} \Pr(R > r \mid \theta) &= \frac{V[K \cap K(-r,\theta)]}{V(K)} \\ &= \frac{V[K \cap K(r,\theta)]}{V(K)}, \end{aligned}$$

where R now denotes the ray length.

Averaging over θ, one obtains:

$$\Pr(R > r) = E_\theta \left\{ \frac{V[K \cap K(r,\theta)]}{V(K)} \right\}$$

$$= \Omega(r) \ . \tag{2.1}$$

This defines the normalized overlap function $\Omega(r)$.

Equation (2.1) illustrates $\Omega(r)$ as an average over θ. However, since there are two random variables involved in generating R, we can also express $\Omega(r)$ as an average over the randomly chosen point P. We surround the point P with an n–sphere of radius r, C(r,P). Define the solid angles subtended by the body K as $\phi_1(r,P)$ and $\phi_2(r,P)$ (see Figure 2.3).

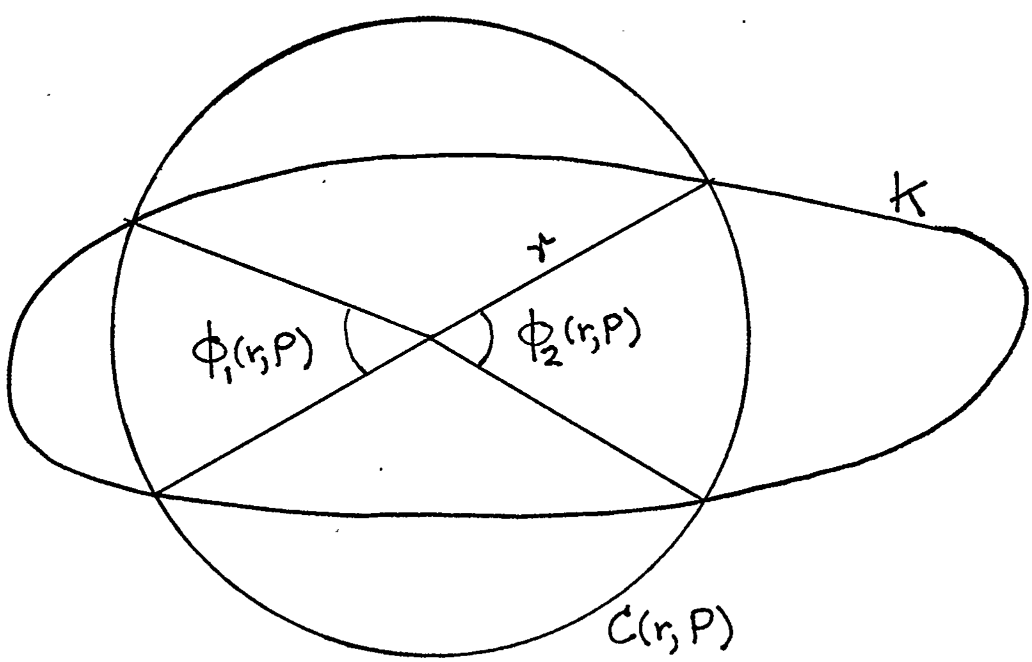

Figure 2.3

Let $\phi(r,P) = \sum_i \phi_i(r,P)$ be the sum of these internal angles. Then

$$\Pr(R > r|P) = \frac{\phi(r,P)}{nC_n} . \tag{2.2}$$

Equations (2.1) and (2.2), therefore, relate the two averaging procedures.

From (2.1) we get the probability density function of the ray–length R to be

$$f(r) = -\,\Omega'(r) \tag{2.3}$$

where the prime denotes differentiation with respect to r.

Let S be the random variable denoting the "backward" ray corresponding to R, namely R + S = L, the secant length. Then

$$\Pr(R > r,\ S > s) = \Omega(r + s) \tag{2.4}$$

and the marginal distributions are

$$\Pr(R > x) = \Pr(S > x) = \Omega(x) .$$

If $H(\ell) = \Pr(L < \ell)$ is the secant length distribution with corresponding probability density function $h(\ell)$, then

$$h(\ell) = \frac{\ell d^2\,\Omega(\ell)}{d\ell^2} . \tag{2.5}$$

The moments are

$$E(R^m) = m \int_0^\infty r^{m-1}\,\Omega(r)\,dr . \tag{2.6}$$

Hence

$$E(R) = E(S) = E(L)/2 \tag{2.7}$$

and

$$E(|R-S|) = E(L)/2 \ . \tag{2.8}$$

2.3.2 <u>λ–randomness</u>:

Two points are selected at random from within K. The probability density function of the ray formed under λ–randomness is

$$f(r) = \frac{-C_n \ r^n \ \Omega'(r)}{V(K)} \ . \tag{2.9}$$

The k^{th} moment is given by

$$E(R^k) = \frac{(n+k) \ C_n}{V(K)} \int_0^{\infty} r^{n+k-1} \ \Omega(r) \ dr \ . \tag{2.10}$$

2.3.3 <u>α–randomness</u>:

Let two points be chosen independently and at random, one from within K and the other on the surface K. The ray formed has the following probability density function:

$$f(r) = \frac{nC_n \ r^{n-1} \ \omega(r)}{2V(K)} \tag{2.11}$$

where $\omega(r)$ is the normalized overlap surface as defined in Section 2.2.

The k^{th} moment is given by

$$E(R^k) = \frac{n C_n}{2V(K)} \int_0^{\infty} r^{n+k-1} \omega_K(r) \, dr . \tag{2.12}$$

2.3.4 The Sphere

For the n–sphere of radius a, we have the normalized overlap volume

$$\Omega(r) = \frac{2C_{n-1}}{C_n} \int_{r/2a}^{1} (1-x^2)^{(n-1)/2} \, dx, \qquad 0 \leq r \leq 2a . \tag{2.13}$$

The overlap surface has normalized content of

$$\omega(r) = \Omega(r) + \frac{C_{n-1}}{n C_n} \left[\frac{r}{a}\right] \cdot \left[1 - \left[\frac{r}{2a}\right]^2\right]^{(n-1)/2} . \tag{2.14}$$

The moments are

$$E\left[R_\nu^k\right] = (2r)^k \left[\frac{n}{n+k}\right] \frac{\Gamma\left[\frac{n}{2}\right] \Gamma\left[\frac{k+1}{2}\right]}{\sqrt{\pi}\, \Gamma\left[\frac{n+k}{2}\right]} \qquad \text{for } k > -1 \tag{2.15}$$

$$E\left[R_\lambda^k\right] = 2^{n+k} r^k \left[\frac{n}{2n+k}\right] \frac{\Gamma\left[\frac{n}{2}\right] \Gamma\left[\frac{n+k+1}{2}\right]}{\sqrt{\pi}\, \Gamma\left[\frac{2n+k}{2}\right]} \qquad \text{for } k > -(n+1) \tag{2.16}$$

$$E\left[R_\alpha^k\right] = \left[\frac{2n+k}{2(n+k)}\right] E\left[R_\lambda^k\right] . \tag{2.17}$$

When $n = 3$, the first moments are

$$E(R_\nu) = 3r/4$$

$$E(R_\lambda) = 48r/35$$

$$E(R_\alpha) = 6r/5 \ .$$

2.4 BIVARIATE RAYS

2.4.1 $\hat{\nu}$-randomness

A point P is chosen at random from within K. let R and S be the random variables denoting the lengths of two rays defined from P by random directions θ_1 and θ_2 which are chosen independently. We will define the normalized overlap volume of K with $K(r,\theta_1)$ and $K(s,\theta_2)$ as

$$\begin{aligned}\psi(r,s) &= E_{\theta_1,\theta_2}\left\{\frac{V[K \cap K(r,\theta_1) \cap K(s,\theta_2)]}{V(K)}\right\} \\ &= \Pr(R > r,\ S > s)\ . \qquad (2.18)\end{aligned}$$

(See Figure 2.4).

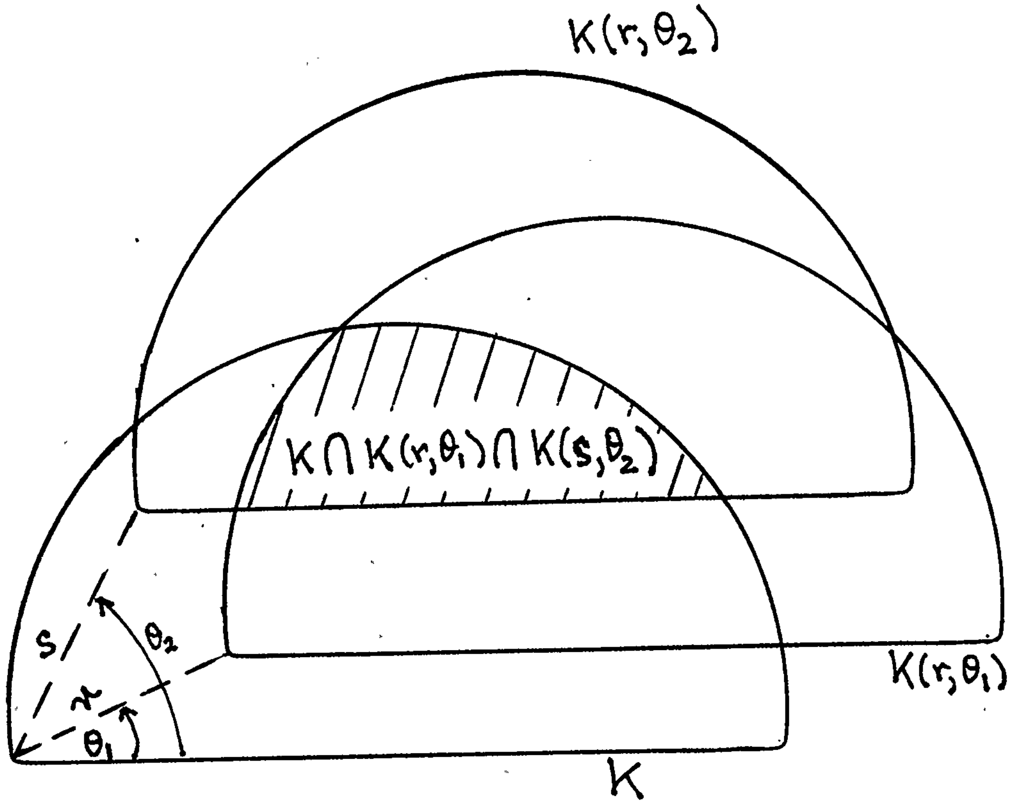

Figure 2.4

From equation (2.2) we have

$$\Pr(R > r) = E_P\left\{\frac{\phi(r,P)}{nC_n}\right\} .$$

The two rays R and S are independent when conditioned on P and the bivariate extension becomes

$$\begin{aligned}\Pr(R > r,\ S > s) &= E_P\left\{\frac{\phi(r,P)\,\phi(s,P)}{(nC_n)^2}\right\} \\ &= \psi(r,s) . \qquad (2.19)\end{aligned}$$

2.4.2 $\hat{\lambda}$-randomness

Two points P and Q are chosen independently and at random from within K. Let U be the random variable denoting the distance between P and Q. Then, the probability density function of U, $f_\lambda(u)$, can be written as

$$f_\lambda(u)\ du = E_P\left\{\frac{\phi(u,P)\ u^{n-1}\ du}{V(K)}\right\} \qquad (2.20)$$

(see Figure 2.5) .

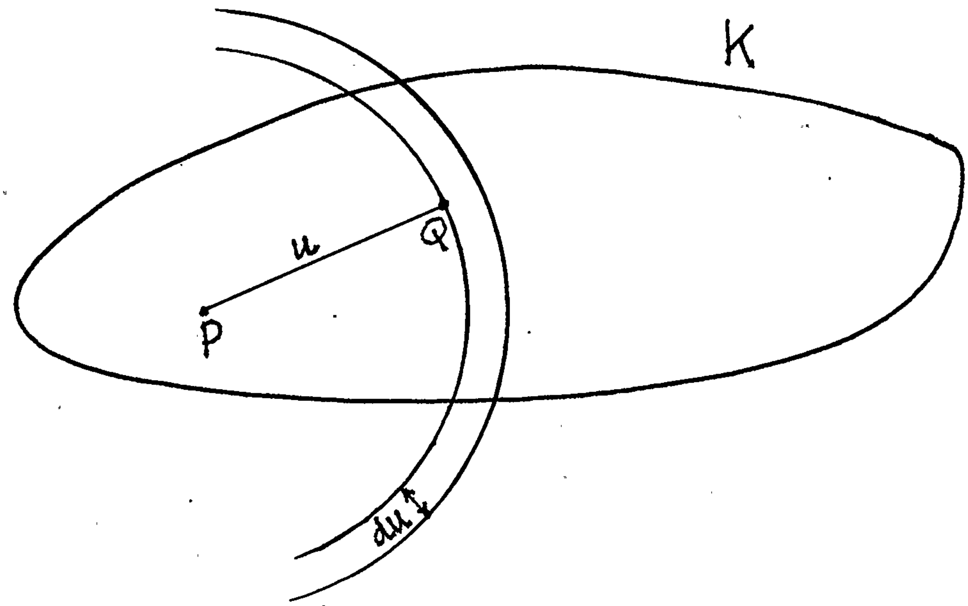

Figure 2.5

A third point O is generated at random within K, independently of P and Q. Let V be the random variable denoting the distance between P and O. Then

$$f_{\hat{\lambda}}(u,v)\ du\ dv = E_P\left\{\frac{\phi(u,P)\ u^{n-1}\ du}{V(K)}\ \frac{\phi(v,P)\ v^{n-1}\ dv}{V(K)}\right\}$$

$$= \frac{(uv)^{n-1}}{[V(K)]^2}\,(nC_n)^2\,\psi(u,v)\ du\ dv\ . \tag{2.21}$$

Hence

$$f_{\hat{\lambda}}(u,v) = \left[\frac{nC_n}{V(K)}\right]^2 (uv)^{n-1} \psi(u,v) . \tag{2.22}$$

By normalization, we obtain

$$\int_0^\infty \int_0^\infty (uv)^{n-1} \psi(u,v)\ dudv = \left[\frac{V(K)}{nC_n}\right]^2 . \tag{2.23}$$

Using the fact that $\psi(u,v) = \Pr(R > u,\ S > v)$ and integrating (2.23) twice, we get

$$E\left[(RS)^n\right] = \left\{\frac{V(K)}{C_n}\right\}^2 . \tag{2.24}$$

From univariate rays, we have

$$f(u) = \frac{nC_n}{V(K)} u^{n-1} \Omega(u) . \tag{2.25}$$

But

$$f(u) = \int_0^\infty f(u,v)\ dv$$

$$= \frac{nC_n\ u^{n-1}}{V(K)} \int_0^\infty \frac{nC_n}{V(K)} v^{n-1} \psi(u,v)\ dv . \tag{2.26}$$

Hence

$$\Omega(x) = \frac{nC_n}{V(K)} \int_0^\infty v^{n-1} \psi(x,v)\ dv . \tag{2.27}$$

Also

$$\Omega(x) = \psi(x,0) = \psi(0,x)$$

and $$(2.28)$$

$$\psi(0,0) = \Omega(0) = 1 \ .$$

Consider the following situation : Two points, P and Q, chosen independently and at random from within K define a line segment U. Suppose a ray, R, is defined from P under ν–randomness (see Figure 2.6).

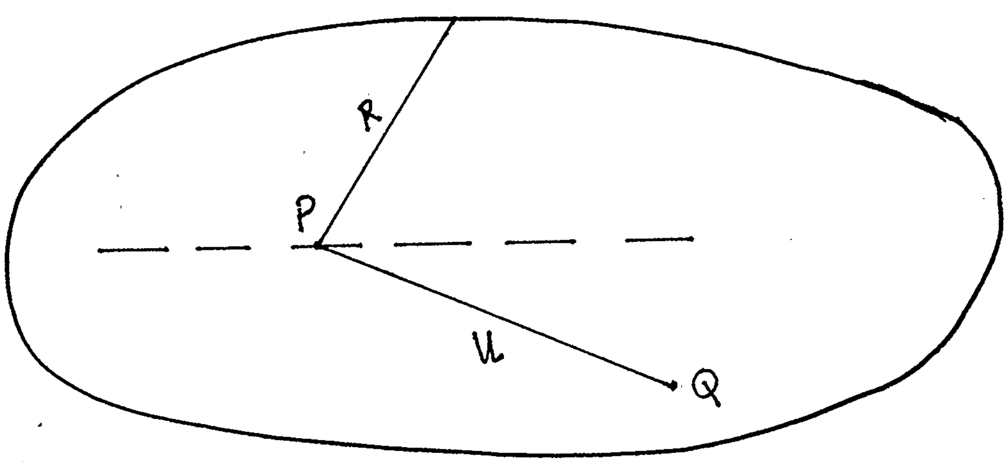

Figure 2.6

We obtain,

$$\Pr(R > r,\ u \le U \le u + du)$$

$$= E_P\left\{\frac{\phi(r,P)}{nC_n} \cdot \frac{\phi(u,P)\ u^{n-1}\ du}{V(K)}\right\}$$

$$= \frac{nC_n}{V(K)}\ u^{n-1}\ \psi(r,u)\ du\,. \tag{2.29}$$

Therefore

$$\Pr(R > r,\ U < u) = \frac{nC_n}{V(K)} \int_0^u x^{n-1}\ \psi(r,x)\ dx$$

$$= \frac{C_n}{V(K)} \left\{u^n\ \psi(r,u) - \int_0^u x^n\ \frac{\partial\,\psi(r,x)}{\partial x}\ dx\right\}. \tag{2.30}$$

THEOREM 2.1

Under $\hat{\lambda}$–randomness, the joint probability density function of R and S is

$$h(r,s) = \left[\frac{C_n}{V(K)}\right]^2 (rs)^n\ \frac{\partial^2\,\psi(r,s)}{\partial r\,\partial s}\,. \tag{2.31}$$

Proof:

To prove this result, we will need the following:

$$V[C(r,P)] - V[\pi(r,P)] = \phi(r,P)\ \frac{r^n}{n}. \tag{2.32}$$

$$V[C(r,P) \cap K] - V[\pi(r,P) \cap K] = \phi(r,P)\ \frac{r^n}{n} \tag{2.33}$$

Hence

$$V[\pi(r,P) \cap K] = V[C(r,P) \cap K] - \phi(r,P)\,\frac{r^n}{n} \tag{2.34}$$

and

$$(V[\pi(r,P) \cap K])\,(V[\pi(s,P) \cap K])$$

$$= \left[V[C(r,P) \cap K] - \phi(r,P)\,\frac{r^n}{n}\right]\left[V[C(s,P) \cap K] - \phi(s,P)\,\frac{s^n}{n}\right]$$

$$= (V[C(r,P) \cap K])\,(V[C(s,P) \cap K])$$

$$+ \phi(r,P)\,\phi(s,P)\,\frac{(rs)^n}{n^2}$$

$$- V[C(r,P) \cap K]\,\phi(s,P)\,\frac{s^n}{n}$$

$$- V[C(s,P) \cap K]\,\phi(r,P)\,\frac{r^n}{n}\,. \tag{2.35}$$

Also

$$E_P\left\{\frac{V[C(r,P) \cap K]}{V(K)}\;\frac{\phi(s,P)}{nC_n}\right\} = \Pr(U < r,\ R > s) \tag{2.36}$$

and

$$\Pr(U < r,\ V < s) = E_P\left\{\frac{V[C(r,P) \cap K]}{V(K)} \cdot \frac{V[C(s,P) \cap K]}{V(K)}\right\}. \tag{2.37}$$

Using equations (2.19), (2.35), (2.36) and (2.37), we obtain

$$\Pr(R < r,\ S < s) = E_P\left\{\frac{V[\pi(r,P) \cap K]}{V(K)} \cdot \frac{V[\pi(s,P) \cap K]}{V(K)}\right\}$$

$$= \Pr(U < r,\ V < s) + (rs)^n \left[\frac{C_n}{V(K)}\right]^2 \psi(r,s)$$

$$- \frac{s^n\ C_n}{V(K)} \Pr(R > s,\ U < r)$$

$$- \frac{r^n\ C_n}{V(K)} \Pr(R > r,\ U < s)\ . \qquad (2.38)$$

From (2.22), we can write

$$\Pr(U < r,\ V < s) = \left[\frac{nC_n}{V(K)}\right]^2 \int_0^r \int_0^s u^{n-1}\ v^{n-1}\ \psi(u,v)\ dudv$$

$$= \left[\frac{nC_n}{V(K)}\right]^2 \left\{\frac{(rs)^n}{n^2}\psi(r,s) - \frac{s^n}{n^2}\int_0^r u^n\ \frac{\partial\psi(u,s)}{\partial u}\ du\right.$$

$$- \frac{r^n}{n^2}\int_0^s v^n\ \frac{\partial\psi(r,v)}{\partial v}\ dv$$

$$\left. + \int_0^r \int_0^s \frac{(uv)^n}{n^2}\ \frac{\partial^2\psi(u,v)}{\partial u\partial v}\ dudv\right\}. \qquad (2.39)$$

Upon substituting (2.30) and (2.39) into (2.38) and simplifying, one obtains

$$\Pr(R < r,\ S < s) = \left[\frac{C_n}{V(K)}\right]^2 \int_0^r \int_0^s (uv)^n\ \frac{\partial^2\psi(u,v)}{\partial u\partial v}\ dudv$$

□

2.4.3 $\hat{\alpha}$–randomness

A point P is chosen at random on the surface of K. Another point Q is selected at random from within K. The distribution function for the length of the ray defined by the two points is given by

$$\Pr(R < r) = E_{P;S}\left\{\frac{V[C(r,P) \cap K]}{V(K)}\right\}, \tag{2.40}$$

(see Figure 2.7) .

Here the expectation is taken over the surface of K.

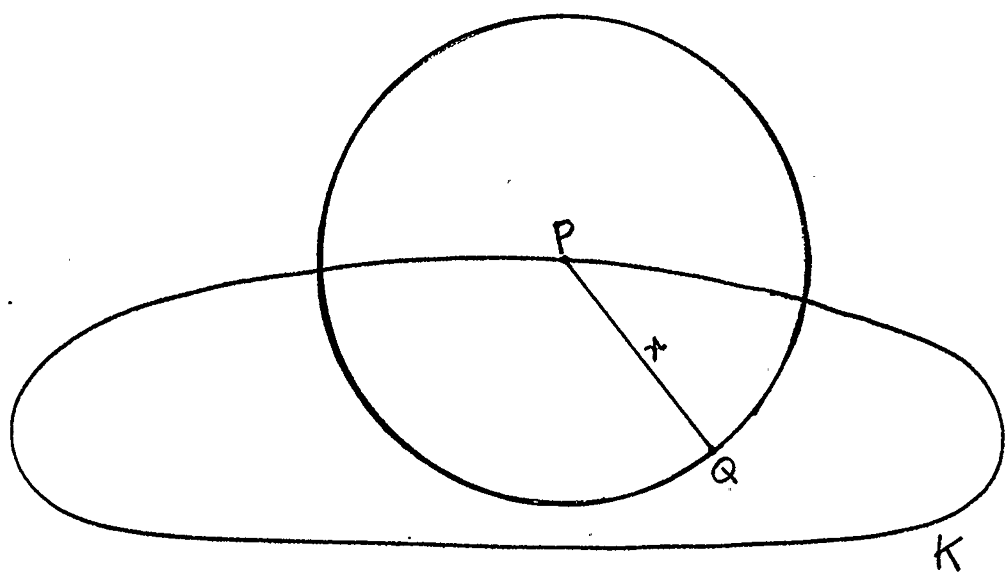

Figure 2.7

Note that we could write the distribution function with Q as the pivotal point instead of P (Figure 2.8).

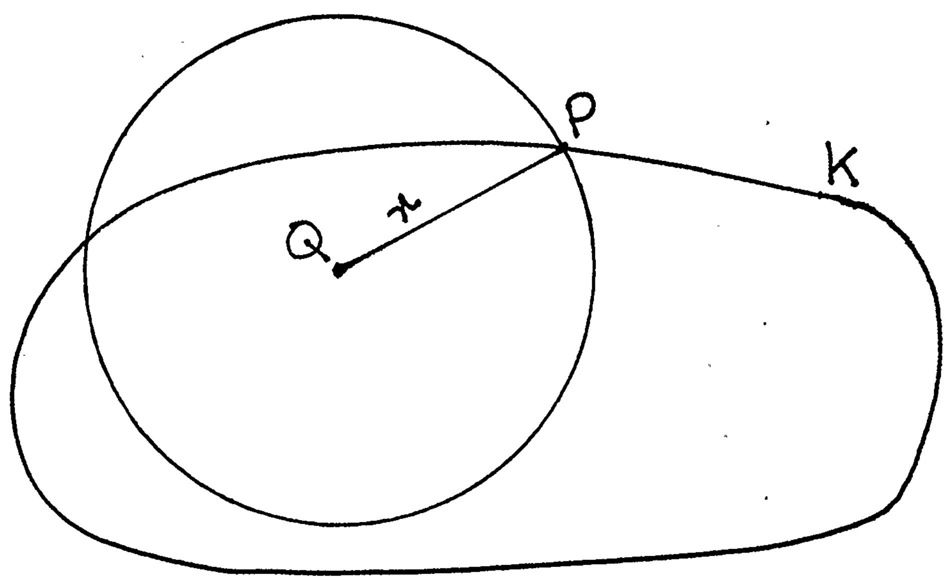

Figure 2.8

In this case

$$\Pr(R < r) = E_{Q;K}\left\{\frac{V[C(r,Q) \cap K]}{V(K)}\right\}. \tag{2.41}$$

Suppose a third point O is chosen at random from within K, independently of P and Q. We will call this $\hat{\alpha}_S$–randomness. Let S be the length of the ray from P to O. The bivariate equivalent of (2.40) is

$$\begin{aligned}\Pr(R < r,\ S < s) &= E_{P;S}\left\{\frac{V[C(r,P) \cap C(s,P) \cap K]}{V(K)}\right\} \\ &= E_{P;S}\left\{\frac{V[C(r_o,P) \cap K]}{V(K)}\right\}.\end{aligned} \tag{2.42}$$

where $r_o = \min(r,s)$.

If, instead, we select the third point at random on the surface of K, then the bivariate extension of (2.41) becomes

$$\begin{aligned}\Pr(R < r,\ S < s) &= E_{Q;K}\left\{\frac{V[C(r,Q) \cap C(s,Q) \cap K]}{V(K)}\right\} \\ &= E_{Q;K}\left\{\frac{V[C(r_o,Q) \cap K]}{V(K)}\right\}.\end{aligned} \tag{2.43}$$

We will call this $\hat{\alpha}_K$–randomness.

Equations (2.40) to (2.43) then provide a direct geometrical method for calculating the ray–length distributions.

2.5 THE SPHERE

Let K be a sphere (in three–dimensions) of radius a. We obtain the bivariate ray distributions through straightforward geometric methods which we first develop for the univariate cases.

2.5.1 ν– and $\hat{\nu}$–randomness

We will show

$$\Omega(r) = \frac{r^3 - 12ra^2 + 16a^3}{16a^3} \tag{2.44}$$

or

$$\Omega'(r) = \frac{3r^2 - 12a^2}{16a^3}$$

or

$$f(r) = -\,\Omega'(r) = \frac{12a^2 - 3r^2}{16a^3}\,. \tag{2.45}$$

Suppose the point P is at a distance t from the centre of the sphere (Figure 2.9). R is the random variable denoting a ray length from P in some random direction.

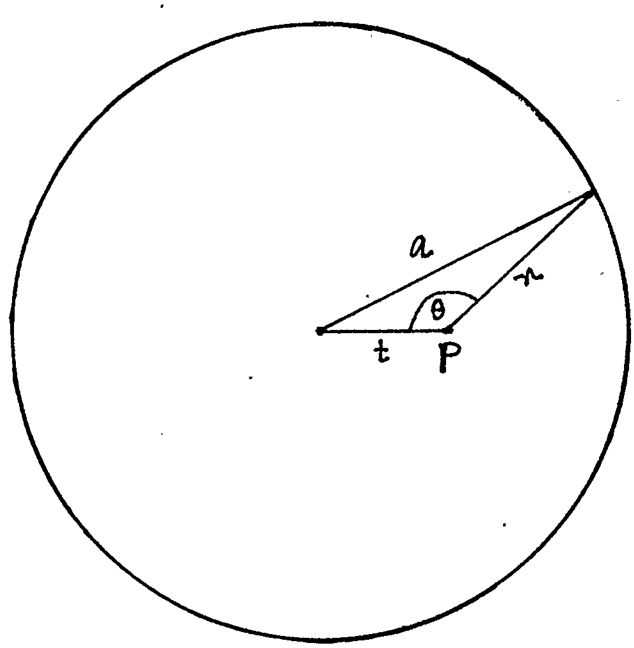

Figure 2.9

The solid angle, $2\pi(1 - \cos\theta)$, is the surface area subtended by the planar angle 2θ on a sphere of unit radius. Therefore,

$$\Pr(R > r|t) = \frac{\text{(Solid angle corresponding to the planar angle } 2\theta)}{\text{total solid angle}}$$

$$= \frac{2\pi(1 - \cos\theta)}{4\pi}$$

$$= \frac{1}{2}(1 - \cos\theta).$$

$$= \frac{1}{2}\left[1 - \left[\frac{r^2 + t^2 - a^2}{2rt}\right]\right] \tag{2.46}$$

where

$$\cos\theta = \frac{r^2 + t^2 - a^2}{2rt}.$$

Therefore,

$$f(r|t) = -\frac{d}{dr}\Pr(R > r|t)$$

$$= \frac{r^2 + a^2 - t^2}{4tr^2}. \tag{2.47}$$

The probability density function of T, the distance from P to the centre of the sphere is (see Appendix A1)

$$g(t) = \frac{3t^2}{a^3}. \tag{2.48}$$

The constraint on r is

$$a-t < r < a+t$$

which implies that

$$t > \max\,(a-r,\; r-a)\ .$$

Therefore

$$f(r) = \int_{a-r}^{a} f(r\,|\,t)\ g(t)\ dt \qquad \text{for} \quad r < a$$

$$= \int_{r-a}^{a} f(r\,|\,t)\ g(t)\ dt \qquad \text{for} \quad r > a \tag{2.49}$$

so that

$$f(r) = \frac{12a^2 - 3r^2}{16a^3} \qquad \text{which is (2.45).}$$

Consider another ray S defined from the same point P in some other random direction, independent of the first. We may write

$$f(r,s\,|\,t) = f(r\,|\,t)\ f(s\,|\,t)$$

$$= \frac{(r^2+a^2-t^2)(s^2+a^2-t^2)}{16t^2r^2s^2}\ . \tag{2.50}$$

Hence

$$f(r,s) = \int_{m}^{a} f(r,s\,|\,t)\ g(t)\ dt \tag{2.51}$$

where m is as shown in Figure 2.10.

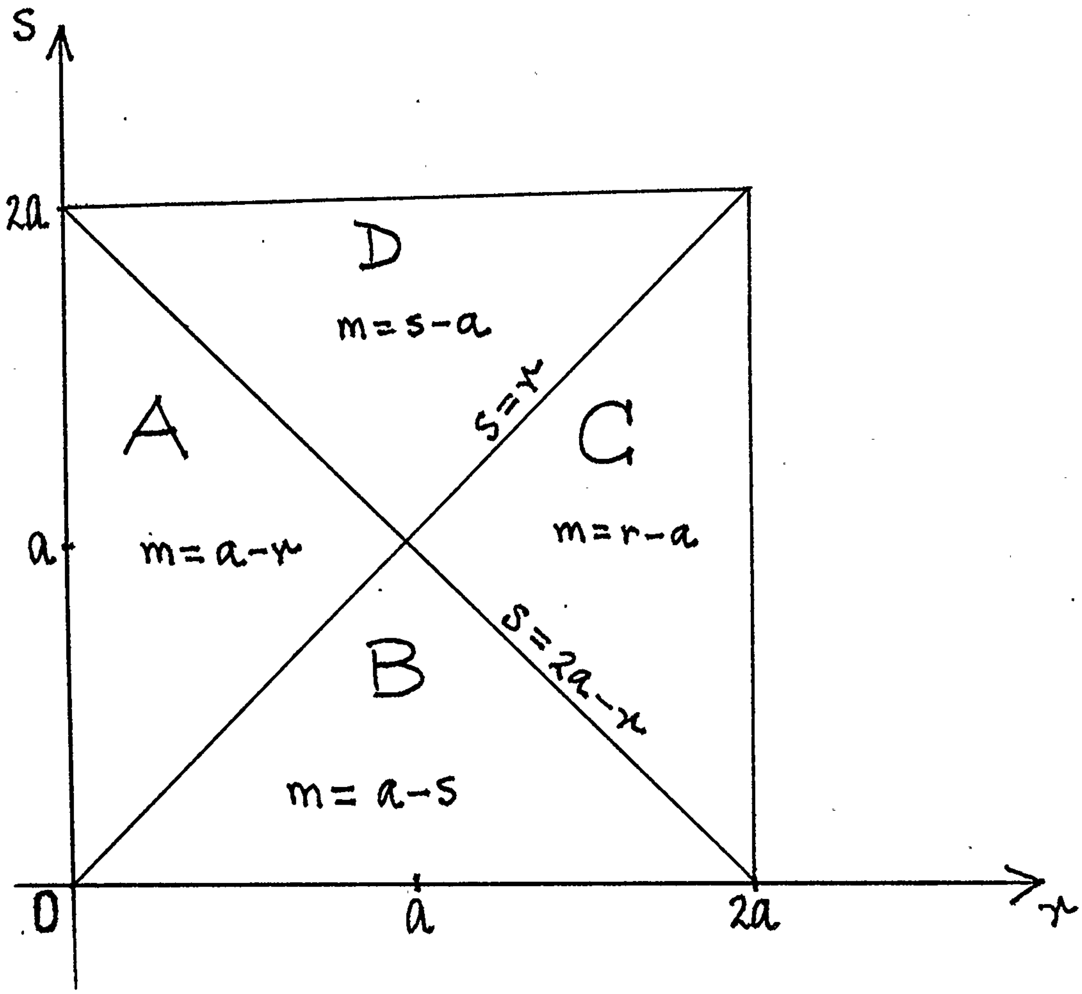

Figure 2.10

We obtain:

f(r,s)

$$= \frac{(10r+15a)s^2 - 2r^3 + 20a^2r}{80a^3s^2} \qquad \text{for} \qquad \begin{array}{l} 0 < r < a \\ r < s < 2a-r \end{array}$$

$$= \frac{15ar^2 + (10r^2+20a^2)s - 2s^3}{80a^3r^2} \qquad \text{for} \qquad \begin{array}{l} 0 < s < a \\ s < r < 2a-s \end{array}$$

$$= \frac{2r^5 - 20a^2r^3 + 20a^3r^2 - (10r^3 - 15ar^2 - 20a^3)s^2 + 16a^5}{80a^3r^2s^2}$$

for $\quad a < r < 2a$

$\quad 2a-r < s < r$

$$= \frac{2s^5 - (10r^2+20a^2)s^3 + (15ar^2+20a^3)s^2 + 20a^3r^2 + 16a^5}{80a^3r^2s^2}$$

for $\quad a < s < 2a$

$\quad 2a-s < r < s\ .$

(2.52)

If we let

$$f_1(r,s) = f(r,s) \text{ on } A$$

$$f_2(r,s) = f(r,s) \text{ on } B$$

(2.53)

$$f_3(r,s) = f(r,s) \text{ on } C$$

$$f_4(r,s) = f(r,s) \text{ on } D,$$

then the marginal probability density function

$$f(r) = \int_0^r f_2(r,s)ds + \int_r^{2a-r} f_1(r,s)ds + \int_{2a-r}^{2a} f_4(r,s,)ds \quad (\text{when } r < a)$$

$$= \frac{12a^2 - 3r^2}{16a^3} \qquad \text{which is (2.45).}$$

The same result is obtained for the whole region.

Knowing $f(r,s)$, we proceed to find $\psi(r,s)$. For example, if $a < s < 2a$ and $2a-s < r < s$, which is region D in Figure 2.10, we get

$$\begin{aligned}
\psi(r,s) &= \Pr(R > r,\ S > s) \\
&= \int_s^{2a} dy \left[\int_r^y f_4(x,y)dx + \int_y^{2a} f_3(x,y)dx \right] \\
&= \Big[5rs^4 - s^5 - (10r^2 - 20a^2)s^3 + (30ar^2 - 60a^2r - 40a^3)s^2 \\
&\qquad + 80a^3rs - 40a^3r^2 + 32a^5 \Big] / (160a^3rs).
\end{aligned}$$

In a similar manner we obtain:

$$\begin{aligned}
\psi(r,s) = \Big[&32a^5 - 40a^3r^2 + 20a^2r^3 - r^5 - (60a^2r^2 - 5r^4 - 80a^3r)s \\
&- (10r^3 - 30ar^2 + 40a^3)s^2 \Big] / (160a^3rs)
\end{aligned}$$

for $a < r < 2a$

$2a-r < s < r$

(region C)

$$\begin{aligned}
\psi(r,s) &= \Big[s^4 + 5rs^3 + (10r^2-20a^2)s^2 + (30ar^2 - 60a^2r)s \\
&\qquad + 10r^4 - 120a^2r^2 + 160a^3r \Big] / (160a^3r)
\end{aligned}$$

for $0 < s < a$

$s < r < 2a-s$

(region B)

$$\psi(r,s) = \left[10s^4 + (10r^2+30ar-120a^2)s^2 + r^4 - 20a^2r^2 + (5r^3 - 60a^2r + 160a^3)s\right]/(160a^3s)$$

for $0 < r < a$

$r < s < 2a-r$

(region A) . (2.54)

2.5.2 λ– and $\hat{\lambda}$–randomness

For univariate rays we will show

$$f(r) = \frac{-C_n r^n \, \Omega'(r)}{V(K)} = \frac{12a^2r^3 - 3r^5}{16a^6} \,. \tag{2.55}$$

Point P is at a distance t from the centre of the sphere. The ray R is defined from P through another random point Q also chosen randomly from within the sphere (Figure 2.11).

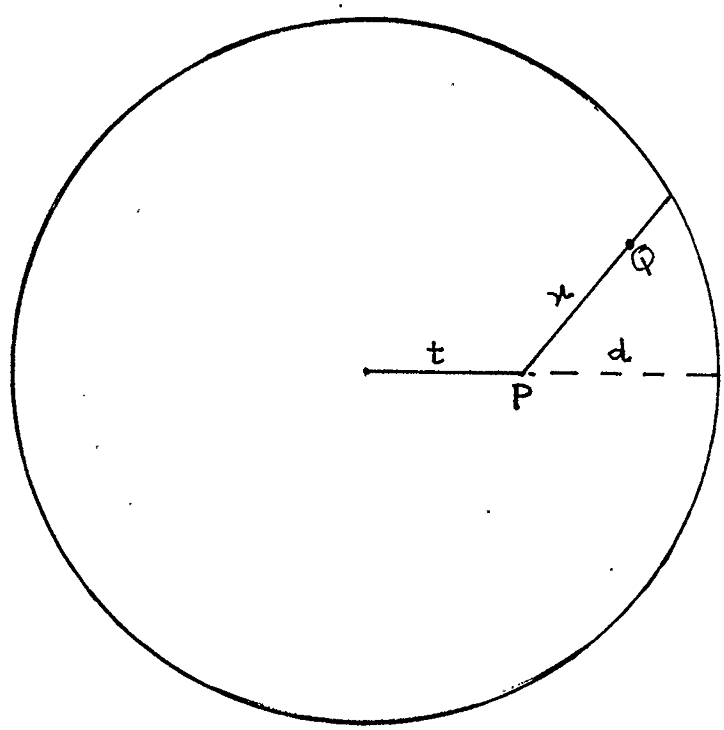

Figure 2.11

Then

$$F(r|t) = \Pr(R < r|t) = \frac{V^*}{V(K)}$$

$$\frac{3\left[\frac{2a^3}{3} - a^2\,(t+d) + \frac{1}{3}\,(t+d)^3 + \frac{1}{3}\,(r^2-d^2)d\right]}{4a^3} \tag{2.56}$$

where $d = \dfrac{a^2 - r^2 - t^2}{2t}$.

(See Appendix A2 for V^*).

Thus

$$f(r|t) = \frac{d}{dr} F(r|t) = \frac{r^3 + (a^2-t^2)r}{4a^3t} . \qquad (2.57)$$

Using (2.48) and the same form as (2.49), we obtain

$$f(r) = \frac{12a^2r^3 - 3r^5}{16a^6} \qquad \text{which is (2.55).}$$

Another ray S is defined from the point P through some other random point O selected within the sphere. We then have

$$f(r,s|t) = \frac{[r^3+(a^2-t^2)r][s^3+(a^2-t^2)s]}{16a^6t^2} . \qquad (2.58)$$

Thus

$$f(r,s) = \int_m^a f(r,s|t)\ g(t)\ dt$$

where m and g(t) are as defined in Subsection 2.5.1.

Hence

$$f(r,s)$$

$$= \frac{(10r^4+15ar^3)s^3 + (20a^2r^4-2r^6)s}{80a^9} \qquad \text{for} \qquad \begin{array}{l} 0 < r < a \\ r < s < 2a-r \end{array}$$

$$= \frac{(10r^3+ 20a^2r)s^4 - 2rs^6+15ar^3s^3}{80a^9} \qquad \text{for} \qquad \begin{array}{l} 0 < s < a \\ s < r < 2a-s \end{array}$$

$$= \frac{(20a^3r+15ar^3-10r^4)s^3 + (2r^6-20a^2r^4+20a^3r^3+16a^5r)s}{80a^9}$$

for $a < r < 2a$

$2a-r < s < r$

$$= \frac{2rs^6 - (10r^3+20a^2r)s^4 + (15ar^3+20a^3r)s^3 + (20a^3r^3+16a^5r)s}{80a^9}$$

for $a < s < 2a$

$2a-s < r < s.$

(2.59)

The marginal probability density functions can be shown to be equal to (2.55).

2.5.3 α– and $\hat{\alpha}$–randomness

From univariate rays we have

$$\omega(r) = \Omega(r) + \frac{C_{n-1}}{C_n}\left[\frac{r}{a}\right]\left[1 - \left[\frac{r}{2a}\right]^2\right]^{(n-1)/2}$$

$$= 1 - \frac{r}{2a} \tag{2.60}$$

and

$$f(r) = \frac{nC_n\, r^{n-1}\,\omega(r)}{2V(K)}$$

$$= \frac{6ar^2 - 3r^3}{4a^4}. \tag{2.61}$$

Suppose point P is at a distance t from the centre of the sphere. Another point Q is chosen at random on the surface of the sphere such that P and Q define a ray R (Figure 2.12) .

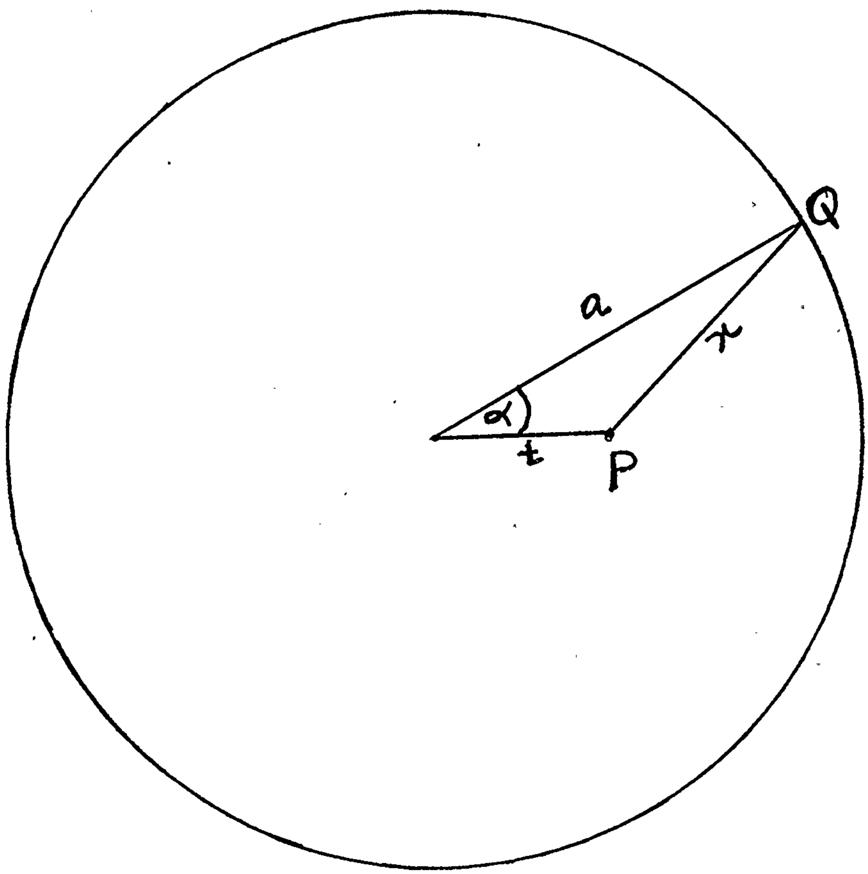

Figure 2.12

Let $S(c,\alpha)$ $= \text{surface area of the spherical cone with planar angle } \alpha$

$$= 2\pi a^2(1 - \cos\alpha)$$

$$= \pi a(r^2-t^2-a^2+2at)/t \ .$$

Then

$$F(r|t) = \Pr(R < r|t) = \frac{S(c\,,\alpha)}{S(K)}$$

$$= \frac{(r^2-t^2-a^2+2at)}{4at} \qquad (2.62)$$

and

$$f(r|t) = \frac{r}{2at} \ .$$

Use of (2.48) and 2.49) yields

$$f(r) = \frac{6ar^2 - 3r^3}{4a^4} \qquad \text{which is (2.61).}$$

If some other point O is chosen at random on the surface of the sphere, independent of the other two points, then P and O define another ray, say S. (Note that this is $\hat{\alpha}_K$–randomness as defined in Subsection 2.4.3). We then have

$$f(r,s|t) = \frac{rs}{4a^2t^2} \ . \qquad (2.63)$$

We use the same form as (2.51) to obtain

$$f(r,s) = \frac{3r^2s}{4a^5} \quad \text{for} \quad 0 < r < a, \quad r < s < 2a-r$$

$$= \frac{3rs^2}{4a^5} \quad \text{for} \quad 0 < s < a, \quad s < r < 2a-s$$

$$= \frac{3rs(2a-r)}{4a^5} \quad \text{for} \quad a < r < 2a, \quad 2a-r < s < r$$

$$= \frac{3rs(2a-s)}{4a^5} \quad \text{for} \quad a < s < 2a, \quad 2a-s < r < s . \qquad (2.64)$$

The marginal probability density function (2.61) is easily retrieved from f(r,s).

REMARK: With $f_i(r,s)$ as defined in (2.53), we observe the following to be true:

$$f_1(r,s) = f_2(s,r)$$
$$f_3(r,s) = f_4(s,r) . \qquad (2.65)$$

2.6 ESTIMATION OF VOLUME

2.6.1 Some Unbiased Estimators

In this section we generate various unbiased estimators of $V = (4\pi a^3/3)$, the volume of a sphere, under ν – and $\hat{\nu}$–randomness. We use different models and compare the efficiency of the estimators via their variances. Table 2.1 below summarizes the results.

Table 2.1: Unbiased Estimators for the volume of a sphere, $V = 4\pi a^3/3$

MODEL	ESTIMATOR	$\frac{\text{VARIANCE}}{a^6}$
I. R is a univariate ray under ν–randomness	$V_1 = \frac{4\pi}{3} R^3$	35.93
II. R_1 and R_2 are two univariate rays independently generated under ν–randomness	$V_2 = \frac{2\pi}{3} \left[R_1^3+R_2^3\right]$	17.96
III. R is a univariate ray under ν–randomness. S is its corresponding "backward" ray. L = R+S	$V_3 = \frac{\pi L^3}{3}$	5.85
	$V_4 = \frac{128}{9} (RS)^{3/2}$	13.28
	$V_5 = 4\pi(R^2S)$	14.54
IV. R and S are two rays generated under $\hat{\nu}$–randomness	$V_6 = C(RS)^{3/2}$ where $C = \frac{800\pi}{1056-225\pi}$	34.27
	$V_7 = \frac{4\pi}{17} (R+S)^3$	23.95
	$V_8 = \frac{24\pi}{11} R^2S$	33.36
	$\frac{1}{V_9} = K\left[\frac{1}{R} + \frac{1}{S}\right]^3$ OR $V_9 = 64.18 \left[\frac{RS}{R+S}\right]^3$	41.19

It is interesting to note that V_3, which is the only estimator to involve a secant, is the most efficient while V_9 is the least efficient. V_1, involving only one ray, is not much better than V_9.

2.6.1 Fixed–angle $\hat{\nu}$–randomness

In the quest to generate better estimators for the volume of the sphere, the following unusual type of randomness was suggested.

Point Q is generated at random from within a sphere. A ray R is defined from Q in some random direction θ. Suppose another ray S is made at a fixed angle, α, from R. One could conceivably obtain the joint distribution of R and S from which estimators of volume would be derived. These estimators would be functions of the fixed angle α. One might ask if there is an optimum value of α which would make this method better than the others we have considered in terms of the efficiency of the estimators.

We made a start on the problem and obtained expressions for R and S in terms of uniformly distributed random variables θ and β. We ran into some computational difficulties in our attempt to make the transformation to the joint distribution of R and S. We present the partial work on this in Appendix A3.

CHAPTER THREE

STATISTICAL SOCIETIES

3.1 INTRODUCTION

In the following two chapters, we discuss a problem of "societies", clusters of points formed by geometric nearest neighbour attachment rules. Related nearest–neighbour problems are treated by Roberts (1967, 1969), Roberts and Storey (1968), Newman et al (1983) and Newman and Rinott (1985), while Clark and Evans (1955), Dacey (1969) and Cox (1981) deal with reflexive nearest neighours.

Consider a population of n points generated by some random process in $\mathbb{R}^d$. From each point, we draw an arrow to its nearest neighbour (in terms of Euclidean distance), assumed to exist uniquely with probability one. We thereby generate clusters of points that are connected by arrows. We call these clusters "societies".

In Section 3.2 we introduce a classification of individuals in a society according to the number of other individuals that consider a particular individual as their nearest neighbour.

In Section 3.3, we derive the proportion of reflexive nearest neighbours and relate it to the classification of individuals.

3.2 CLASSES OF INDIVIDUALS

Within a society with k points or individuals, we will define three classes of individuals. Let V_i denote the number of individuals that consider the i^{th} individual to be their nearest neighbour. Suppose that an individual may visit only his nearest neighbour. Then we will say that the i^{th} individual is:

Lonely if $V_i = 0$

Normal if $V_i = 1$, and

Friendly if $V_i \geq 2$, $\qquad i = 1, 2, \ldots, k.$

The range of V_i depends on the dimension. In one–dimension, $V_i \leq 2$; in two–dimensions, $V_i \leq 5$; in three–dimensions, $V_i \leq 11$; and so forth.

Obviously, since each individual is joined to a unique nearest neighbour, we have that

$$\sum_{i=1}^{k} V_i = k \, . \tag{3.1}$$

If, in addition, we define L_k, N_k, and F_k as the number of lonely, normal, and friendly individuals, respectively, in a society of size k, then we also have:

$$L_k + N_k + F_k = k \, . \tag{3.2}$$

If $\mathscr{L}$, $\mathscr{N}$, and $\mathscr{F}$ are index sets corresponding to the three classes of individuals, then:

$$\sum_{i \in \mathscr{L}} V_i = 0, \qquad \sum_{i \in \mathscr{N}} V_i = N_k, \qquad \sum_{i \in \mathscr{F}} V_i \geq 2F_k . \tag{3.3}$$

From (3.1), (3.2), and (3.3) one obtains:

$$\sum_{i=1}^{k} V_i = L_k + N_k + F_k \geq N_k + 2F_k$$

or

$$F_k \leq L_k . \tag{3.4}$$

As one expects, there are more lonely than friendly individuals in a society!

In one dimension, since $V_i \leq 2$, every society will have an equal number of lonely and friendly individuals. More specifically, $F_2 = L_2 = 0$ and $F_k = L_k = \{1,2\}$ for $k > 2$.

It should be noted that the values of L_k and F_k (which define N_k) do not necessarily define the societal pattern of the individuals uniquely. For example, if we have a society with $k = 5$ individuals in $\mathbb{R}^2$, the patterns in Figure 3.1 both give us $L_5 = F_5 = 2$.

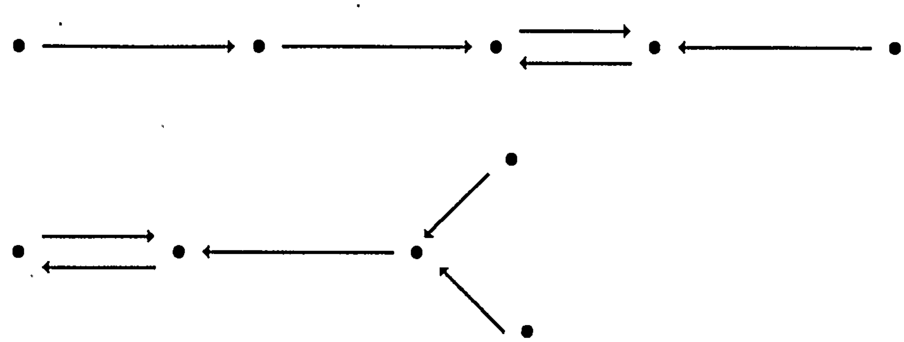

Figure 3.1

3.3 REFLEXIVE NEAREST NEIGHBOURS

In natural populations, many individuals are spatially related to one another in a "reflexive" manner; that is, in many cases two individuals are closer to each other than either one is to any other individual. When the distance between two such individuals is smaller compared with distances to other individuals, there is an obvious occurrence of pairs. It can be shown that in a randomly distributed population, the relation of nearest neighbour is reflexive for a definite proportion of individuals.

Consider an individual, I_1, in a population of density λ distributed at random in two-dimensional space. The probability that there is a point, I_2, at a distance between r and r + dr from I_1 can be written, for dr small, as

$$dp_1 = \lambda dA = \lambda(2\pi r dr) . \qquad (3.5)$$

The probability that I_1 and I_2 are reflexive nearest neighbours, given that I_2 is at a distance r from I_1, is the same as the probability that the region representing the union of the circles defined by I_1 and I_2 (see Figure 3.2) is empty of other points. This is

$$p_2 = e^{-\lambda A(\text{union})} \tag{3.6}$$

where $\quad A(\text{union}) \quad = r^2\left[\frac{\sqrt{3}}{2} + \frac{4\pi}{3}\right]$.

= the area of the union of the two circles in Figure 3.2.

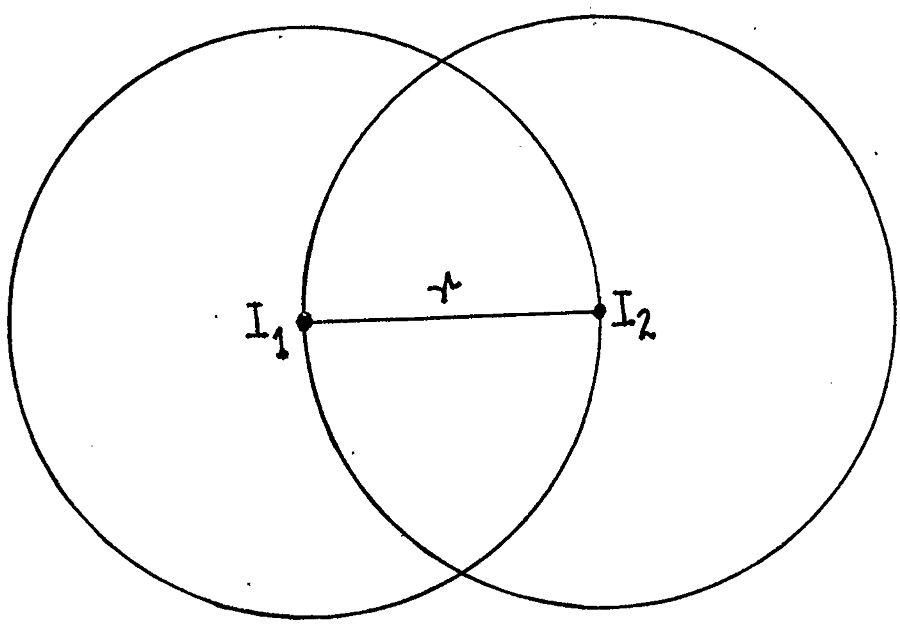

Figure 3.2

Therefore, the probability that I_1 and I_2 form a reflexive pair is

$$P^{(2)} = \int_0^{\infty} 2\pi r\lambda \exp\left[-\lambda r^2 \left[\frac{\sqrt{3}}{2} + \frac{4\pi}{3}\right] \right] dr$$

$$= \frac{6\pi}{3\sqrt{3} + 8\pi} = 0.6215 \ . \qquad (3.7)$$

This is also the expected proportion of individuals that belong to reflexive pairs in a population of random pattern.

Now, if an individual i belongs to a reflexive pair, then $V_i \geq 1$; i.e. individuals belonging to reflexive pairs are either normal or friendly. Thus

$$P(i \in \mathscr{N}) + P(i \in \mathscr{F}) \geq 0.6215$$

and

$$P(i \in \mathscr{L}) \leq 0.3785 \ . \qquad (3.8)$$

This holds for the two-dimensional case (see also Roberts (1969)).

We can extend this analysis to n-dimensional space. In place of the circles in Figure 3.2, we now have identical n-dimensional spheres with volume:

$$A = C_n r^n$$

$$= \frac{\pi^{n/2}}{\Gamma\left[\frac{n}{2} + 1\right]} r^n \ . \qquad (3.9)$$

(C_n is defined in Section 2.2).

The volume of the intersection of the two spheres is given by

$$B = \frac{2\pi^{(n-1)/2} r^n}{\Gamma\left[\frac{n+1}{2}\right]} \int_{\frac{1}{2}}^{1} (1-x^2)^{(n-1)/2} \, dx \ . \qquad (3.10)$$

Let

$$p^* = \frac{A-B}{A}$$

$$= \frac{2\Gamma\left[\frac{n}{2} + 1\right]}{\pi^{\frac{1}{2}}\Gamma\left[\frac{n+1}{2}\right]} I_n \qquad (3.11)$$

where

$$I_n = \int_0^{\frac{1}{2}} (1-x^2)^{(n-1)/2} \, dx \ .$$

Given an individual I_1 in n–dimensional space, the probability that there is a point I_2 in the space between A and A + dA from I_1, for dA small, is

$$dp_1 = \lambda dA . \tag{3.12}$$

The probability that I_1 and I_2 are reflexive nearest neighbours, given that I_2 is between A and A + dA from I_1, is

$$p_2 = e^{-\lambda A(\text{union})}. \tag{3.13}$$

where

$$A(\text{union}) = 2A - B = A(1+p^*) .$$

Hence the probability, $P^{(n)}$, that an individual is the nearest neighbour to its own nearest neighbour in n–dimensional space is given by

$$\begin{aligned} P^{(n)} &= \int_0^\infty \lambda e^{-\lambda A(1+p^*)}\, dA \\ &= \frac{1}{1+p^*} . \end{aligned} \tag{3.14}$$

From (3.11) and (3.14) we obtain

$$P^{(1)} = 0.6667$$

$$P^{(2)} = 0.6215$$

$$P^{(3)} = 0.5926 . \tag{3.15}$$

In one–dimension, if we define p_m as the probability that an individual is the nearest neighbour of exactly m other points, then obviously:

$$p_0 = p_2 = \frac{1}{4}$$

$$p_1 = \frac{1}{2}$$

$$p_m = 0 \qquad \text{for } m \geq 3 \,. \tag{3.16}$$

Most of the questions related to the collection of societies formed and to the internal structure of a society appear to be very difficult to tackle in general. In the following chapter, we provide a detailed examination of the one–dimensional societal structure.

CHAPTER FOUR

ONE–DIMENSIONAL SOCIETIES : UNIFORM DISTRIBUTION

4.1 INTRODUCTION

In this chapter, we look at the simplest societal structure which is one–dimensional. We will consider a random process which generates points on the line from a uniform distribution.

Glaz and Naus (1983) discuss a somewhat different problem of multiple clusters on the line. A review of the distribution theory of spacings is done by Pyke (1965, 1972) while David and Groeneveld (1982) study measures of variation of the distribution of spacings.

If M is the number of societies formed by n points, then let $P_n(m) = P\{M{=}m|n \text{ points}\}$. We derive an expression for $P_n(m)$ and solve it for several extremal cases. In general, we present a generating function representation from which moments are obtained.

We derive expressions for the distribution of the maximum number of societies and we discuss the extension of this work to populations of societies (super–societies) in place of individuals.

4.2 FORMULATION

We consider n individual locations X_i, $i = 1,\ldots,n$, which are i.i.d. random variables from a uniform distribution on some interval. Let $X_{(i)}$, $i = 1,\ldots,n$

denote the corresponding order statistics and let $A_i = X_{(i+1)} - X_{(i)}$. $i = 1,\ldots,n-1$ denote the lengths of the spacings between adjacent positions. The A_i are identically distributed and all orders of ranks of the A_i are equally likely. We need only know their relative magnitudes to determine the number of societies these points form.

We now consider the number of societies M formed by a population of n points. Clearly, M satisfies $1 \leq M \leq \left[\frac{n}{2}\right]$, where [x] denotes the greatest integer less than or equal to x. Now obviously,

$$P_2(1) = P_3(1) = 1 . \tag{4.1}$$

If we define $A^* = \max \left[A_1,\ldots,A_{n-1}\right]$, then this maximum interval may be used to partition the distribution of M.

In particular,

$$P_n(1) = \sum_{i=1}^{n-1} P_n\left[M=1 \mid A_i=A^*\right] P\left[A_i=A^*\right]. \tag{4.2}$$

Now

$$\begin{aligned} P_n\left[M=1 \mid A_i=A^*\right] &= P_{n-1}(1) && \text{if } i = 1, n-1 \\ &= 0 && \text{otherwise.} \end{aligned}$$

Also

$$P\left[A_i=A^*\right] = \frac{1}{n-1} . \tag{4.3}$$

Hence

$$P_n(1) = \frac{2}{n-1} P_{n-1}(1). \tag{4.4}$$

By iteration one obtains:

$$P_n(1) = \frac{2^{n-2}}{(n-1)!}\,. \tag{4.5}$$

In general, the partition according to A^* yields

$$P_n(m) = \sum_{i=1}^{n-1} P_n\left[M=m \mid A_i=A^*\right] P\left[A_i=A^*\right] \tag{4.6}$$

where

$$P_n\left[M=m \mid A_i=A^*\right] = P_{n-1}(m) \qquad \text{if } i = 1,\ n-1$$

$$= \sum_{i=1}^{m-1} P_i(j)\, P_{n-i}(m-j) \qquad \text{if } 2 \le i \le n-2.$$

Hence, for $n \geq 4$

$$P_n(m) = \frac{2}{n-1} P_{n-1}(m) + \sum_{i=2}^{n-2} \sum_{j=1}^{m-1} \frac{P_i(j)\, P_{n-i}(m-j)}{n-1} \tag{4.7}$$

where we know that

$$P_2(1) = P_3(1) = 1.$$

4.3 ITERATIVE METHOD

The distribution of M may be found by solving equation (4.7) recursively. For small values of m, many of the terms in the double summation are zero. For example,

$$(n-1)\, P_n(2) = 2P_{n-1}(2) + \sum_{i=2}^{n-2} P_i(1)\, P_{n-i}(1). \tag{4.8}$$

Use of (4.5) yields

$$P_n(2) = \frac{2}{n-1} P_{n-1}(2) + \frac{2^{n-3}(2^{n-3}-1)}{(n-1)!}, \qquad \text{for } n \geq 4, \tag{4.9}$$

where

$$P_3(2) = 0.$$

Recursively solving one finds:

$$P_n(2) = \frac{2^{n-3}}{(n-1)!}\left\{2^{n-2} + 1-n\right\} \tag{4.10}$$

which is valid for $n \geq 2$.

Similarly, for $n \geq 6$,

$$\begin{aligned}(n-1)\, P_n(3) &= 2P_{n-1}(3) + 2P_2(1)\, P_{n-2}(2) \\ &\quad + \sum_{i=3}^{n-3}\left[P_i(1)\, P_{n-i}(2) + P_i(2)\, P_{n-i}(1)\right]\end{aligned} \tag{4.11}$$

where

$$P_5(3) = 0\ .$$

Using (4.5) and (4.10), we obtain

$$P_n(3) = \frac{2}{n-1} P_{n-1}(3) + \frac{2^{n-5}}{(n-1)!}\left\{2n-5+3^{n-2} - (n-1)\, 2^{n-2}\right\}. \tag{4.12}$$

This yields:

$$P_n(3) = \frac{2^{n-5}}{(n-1)!} \left\{ \frac{1}{2} \left(3^{n-1} - 1\right) - (n-2)\, 2^{n-1} + (n-1)(n-3) \right\} . \tag{4.13}$$

This formula is valid for $n \geq 3$.

4.4 GENERATING FUNCTION METHODS

To solve (4.7), we introduce two generating functions:

$$G_n(s) = \sum_{m=1}^{\left[\frac{n}{2}\right]} s^m \, P_n(M=m) \tag{4.14}$$

$$\psi(z,s) = \sum_{n=2}^{\infty} z^n \, G_n(s) . \tag{4.15}$$

Multiplying (4.7) by s^m and summing, one finds

$$(n-1)\, G_n(s) - 2G_{n-1}(s) = \sum_{i=2}^{n-2} G_i(s)\, G_{n-i}(s), \qquad n \geq 4 \tag{4.16}$$

where

$$G_2(s) = G_3(s) = s.$$

Multiplying (4.16) by z^n and summing, one finds

$$z \frac{d}{dz} \psi(z,s) - (1+2z)\, \psi(z,s) - sz^2 = \psi^2(z,s). \tag{4.17}$$

A simpler form of (4.17) may be obtained by letting $\psi(z,s) = z\, \phi(z,s)$.

Then (4.17) becomes:

$$\frac{d}{dz}\phi(z,s) = s + 2\phi(z,s) + \phi^2(z,s) \ . \tag{4.18}$$

This is a Ricatti differential equation. We substitute

$$\phi(z,s) = -\frac{1}{\omega(z,s)}\frac{d}{dz}\omega(z,s)$$

to obtain:

$$\frac{d^2}{dz^2}\omega(z,s) - 2\frac{d}{dz}\omega(z,s) + s\,\omega(z,s) = 0. \tag{4.19}$$

This factors into:

$$(D - 1 - \sqrt{1-s})\,(D - 1 + \sqrt{1-s})\,\omega(z,s) = 0 \quad \text{where} \quad D = \frac{d}{dz} \ .$$

Hence

$$\omega(z,s) = ae^{(1+\sqrt{1-s})z} + be^{(1-\sqrt{1-s})z}$$

where a and b are constants to be determined. Now the boundary conditions $\phi(0,s) = 0$ and $\phi'(0,s) = s$ imply that $\omega(0,s) = 1$ and $\omega'(0,s) = 0$.

Therefore

$$\omega(z,s) = \frac{(1+\sqrt{1-s})\ e^{(1-\sqrt{1-s})z} - (1-\sqrt{1-s})\ e^{(1+\sqrt{1-s})z}}{2\sqrt{1-s}}$$

or

$$\omega(z,s) = \frac{e^z}{\sqrt{1-s}} \left[\sqrt{1-s}\,\cosh\,((\sqrt{1-s})z) - \sinh((\sqrt{1-s})z)\right]. \tag{4.20}$$

From $\omega(z,s)$ we find $\phi(z,s)$ and hence $\psi(z,s)$ to be:

$$\psi(z,s) = \frac{sz}{\sqrt{1-s}\,\coth\,((\sqrt{1-s})z) - 1}. \tag{4.21}$$

We proceed to expand $\psi(z,s)$ in a series in z, the coefficients of which are $G_n(s)$, $n = 1,2,\ldots$.

$$\psi(z,s) = \sum_{n=2}^{\infty} G_n(s)\, z^n = \frac{sz}{\sqrt{1-s}\,\coth\,((\sqrt{1-s})z) - 1}.$$

Therefore

$$\frac{1}{\psi(z,s)} = \frac{(\sqrt{1-s})\,\coth((\sqrt{1-s})z)-1}{sz}. \tag{4.22}$$

Now,

$$\mathrm{Coth}\ x = \sum_{n=0}^{\infty} \frac{2^{2n}\, B_{2n}\, x^{2n-1}}{(2n)!} \tag{4.23}$$

where B_i, $i = 0,1,2,\ldots$ are Bernoulli numbers (see Appendix A4).

Hence

$$\frac{1}{\psi(z,s)} = \sum_{n=-2}^{\infty} a_n z^n \tag{4.24}$$

where

$$a_{-2} = \frac{1}{s}$$

$$a_{-1} = -\frac{1}{s}$$

$$a_{2n-2} = \frac{2^{2n} B_{2n} (1-s)^n}{(2n)!s} \qquad n = 1,2,...$$

$$a_{2n-1} = 0 . \qquad n = 1,2,....$$

Hence

$$\psi(z,s) = \sum_{n=2}^{\infty} G_n(s)\, z^n = \left[\sum_{n=-2}^{\infty} a_n\, z^n\right]^{-1}$$

or

$$\left[\sum_{n=-2}^{\infty} a_n\, z^n\right] \left[\sum_{n=2}^{\infty} G_n(s)\, z^n\right] = 1. \tag{4.25}$$

$G_n(s)$ is obtained by solving the following system of equations:

$$\sum_{j=0}^{n-2} a_{j-2}\, G_{n-j} = 0$$

$$a_{-2}\, G_2 = 1 \tag{4.26}$$

or, in matrix form:

$$\begin{bmatrix} a_{-2} & a_{-1} & a_0 & a_1 & \cdots\cdots & a_{n-3} \\ 0 & a_{-2} & a_{-1} & a_0 & \cdots\cdots & a_{n-4} \\ 0 & 0 & a_{-2} & \cdots\cdots & & \\ 0 & 0 & 0 & \cdots\cdots & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 \cdots\cdots 0 & & a_{-2} & a_{-1} & & a_0 \\ 0 \cdots\cdots & & \cdots 0 & & a_{-2} & a_{-1} \\ 0 \cdots\cdots & & & \cdots 0 & & a_{-2} \end{bmatrix} \begin{bmatrix} G_n \\ G_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ G_4 \\ G_3 \\ G_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The following table gives $G_n(s)$ for some values of n

TABLE 4.1: $G_n(s)$ for $n = 2, 3, 4, 5, 6, 7, 8$

n	$G_n(s)$
2,3	s
4	$s\left[1 - \left[\frac{2^2B_2}{2!}\right](1-s)\right]$
5	$s\left[1 - 2\left[\frac{2^2B_2}{2!}\right](1-s)\right]$
6	$s\left[1 - 3\left[\frac{2^2B_2}{2!}\right](1-s) + (1-s)^2\left\{\left[\frac{2^2B_2}{2!}\right]^2 - \left[\frac{2^4B_4}{4!}\right]\right\}\right]$
7	$s\left[1 - 4\left[\frac{2^2B_2}{2!}\right](1-s) + (1-s)^2\left[3\left[\frac{2^2B_2}{2!}\right]^2 - 2\left[\frac{2^4B_4}{4!}\right]\right]\right]$
8	$s\left[1 - 5\left[\frac{2^2B_2}{2!}\right](1-s) + (1-s)^2\left[6\left[\frac{2^2B_2}{2!}\right]^2 - 3\left[\frac{2^4B_4}{4!}\right]\right]\right.$ $\left. + (1-s)^3\left\{2\left[\frac{2^6B_2B_4}{2!4!}\right] - \left[\frac{2^2B_2}{2!}\right]^3 - \left[\frac{2^6B_6}{6!}\right]\right\}\right]$

A closed–form representation for $G_n(s)$ seems rather involved.

4.5 MOMENTS

Given G(s), the probability generating function for some random variable X, the mean and variance of X may be obtained from G(s), namely

$$E(X) = G'(1)$$

$$Var(X) = G''(1) + G'(1) - [G'(1)]^2 .$$

Let $E_n(M)$ and $Var_n(M)$ denote the mean and variance of the number of societies formed in a population of n individuals.

Differentiating (4.15), we obtain

$$\left.\frac{\partial \psi(z,s)}{\partial s}\right|_{s=1} = \sum_{n=2}^{\infty} G_n'(1)z^n = \sum_{n=2}^{\infty} E_n(M)z^n \tag{4.27}$$

and

$$\left.\frac{\partial^2\psi(z,s)}{\partial s^2}\right|_{s=1} = \sum_{n=2}^{\infty} G_n''(1)z^n . \tag{4.28}$$

Hence

$$\sum_{n=2}^{\infty} Var_n(M)z^n = \sum_{n=2}^{\infty} G_n''(1)z^n + \sum_{n=2}^{\infty} G_n'(1)z^n - \sum_{n=2}^{\infty} [G_n'(1)]^2 z^n . \tag{4.29}$$

From (4.24) we obtain

$$\left.\left[\sum_{n=-2}^{\infty} a_n z^n\right]\right|_{s=1} = \frac{1-z}{z^2} \tag{4.30}$$

$$\frac{d}{ds}\left[\sum_{n=-2}^{\infty} a_n\ z^n\right]\Bigg|_{s=1} = \frac{(3z-3-z^2)}{3z^2} \tag{4.31}$$

$$\frac{d^2}{ds^2}\left[\sum_{n=-2}^{\infty} a_n\ z^n\right]\Bigg|_{s=1} = \frac{90-90z+30z^2-2z^4}{45z^2}\ . \tag{4.32}$$

Using (4.30), (4.31) and (4.32) we obtain

$$\frac{\partial\ \psi(z,s)}{\partial s}\Bigg|_{s=1} = z^2 + \sum_{n=3}^{\infty}\left[\frac{n}{3}\right] z^n\ . \tag{4.33}$$

Thus

$$\begin{aligned} E_n(M) &= 1 \qquad\qquad n = 2 \\ &= \frac{n}{3} \qquad\qquad n \geq 3\ . \end{aligned} \tag{4.34}$$

Similarly,

$$\frac{\partial^2\psi(z,s)}{\partial s^2}\Bigg|_{s=1} = \left[\frac{2z^8-14z^7+42z^6-60z^5+30z^4}{45}\right]\left[\sum_{n=0}^{\infty}\frac{(n+3)!}{6\,n!}\ z^n\right]\ . \tag{4.35}$$

Hence

$$\begin{aligned} \sum_{n=2}^{\infty} Var_n(M)z^n &= \left[\frac{2z^8-14z^7+42z^6-60z^5+30z^4}{45}\right]\left[\sum_{n=0}^{\infty}\frac{(n+3)!}{6\,n!}\ z^n\right] \\ &\quad + z^2 + \sum_{n=3}^{\infty}\left[\frac{n}{3}\right] z^n - z^2 - \sum_{n=3}^{\infty}\left[\frac{n}{3}\right]^2 z^n \\ &= \frac{2}{9}\ z^4 + \frac{2}{45}\sum_{n=5}^{\infty} nz^n\ . \end{aligned} \tag{4.36}$$

Thus

$$\begin{aligned} \mathrm{Var}_n(M) &= 0 & n &= 2,\ 3 \\ &= \frac{2}{9} & n &= 4 \\ &= \frac{2n}{45} & n &\geq 5\ . \end{aligned} \tag{4.37}$$

4.6 ALTERNATIVE FORMULATION

In this section we consider an alternative recursive formula for $P_n(m)$ which, for computational purposes, is preferable to equation (4.7). This formula was conjectured after observing the patterns that emerge from computing $P_n(m)$ for some values of n and m using (4.7).

<u>Theorem 4.1</u>.

$$P_n(m) = \frac{2m}{n-1} P_{n-1}(m) + \left[1 - \frac{2(m-1)}{n-1}\right] P_{n-1}(m-1) \tag{4.38}$$

for $n \geq 3$ and $m \geq 1$. [Note that $P_n(0) = 0$].

<u>Proof</u>: With $G_n(s)$ and $\psi(z,s)$ as defined in Section 4.4, we obtain

$$2s(1-s) \frac{\partial}{\partial s} G_{n-1} = (n-1) \cdot \left[G_n - s\, G_{n-1}\right] \tag{4.39}$$

and therefore

$$2s(1-s)\, z \frac{\partial \psi}{\partial s} - z(1-zs) \frac{\partial \psi}{\partial z} + z^2 s + \psi = 0\ . \tag{4.40}$$

$\psi(z,s)$ as given by equation (4.21), and its derivatives, satisfy (4.40). This proves the equivalence of (4.7) and (4.38). □

The heuristic interpretation of (4.38) is as follows. Denote by $E_{k,m}$ the event that the left–most k points defined by A_1, A_2, ..., A_{k-1}, taken as a population, form m societies. Then

$$P(E_{k,m}) = P_k(m).$$

Equation (4.38) then represents the partition

$$E_{n,m} = \left[E_{n,m} \cap E_{n-1,m}\right] \cup \left[E_{n,m} \cap E_{n-1,m-1}\right] \cup \left[E_{n,m} \cap E_{n-1,m+1}\right],$$

where, however, $E_{n,m} \cap E_{n-1,m+1} = \phi$. It follows that

$$P\left[E_{n,m} | E_{n-1,m}\right] = \frac{2m}{n-1},$$

which is the conditional probability that addition of the right–most point to the population formed by the left–most n–1 points does not increase the number of societies. The <u>unconditional</u> probability that addition of $X_{(n)}$ does not increase the number of societies is

$$\begin{aligned}\sum_m P\left[E_{n-1,m} \cap E_{n,m}\right] &= \frac{2}{n-1} \sum_m m\, P_{n-1}(m) \\ &= \frac{2}{n-1} E_{n-1}(m) \\ &= \frac{2}{3}\end{aligned}$$

using results of Sections 4.4 and 4.5.

For m = 1, equation (4.38) becomes

$$P_n(1) = \frac{2}{n-1} P_{n-1}(1)$$

which is equation (4.4).

Tables 4.2, 4.3 and 4.4 in Section 4.9 give $P_n(m)$ for selected values of n and m.

4.7 MAXIMUM NUMBER OF SOCIETIES

In this section, we derive an expression for

P_{2r} = the probability that a population of size $n = 2r$ consists of r societies,

= the probability that a population of even size consists entirely of two–element societies.

Equation (4.38) provides the corresponding probabilities for populations of odd size:

$$P_{2r-1}(r-1) = (2r-1)\ P_{2r}(r) \tag{4.40a}$$

which is the probability that a population of odd size consists entirely of two–element societies except for a single three–element society. The intuitive argument for this result is as follows:

Consider a population of 2r elements split into r societies. Now remove any element, forming a new population of size 2r–1. The erstwhile partner of the discarded element must then join one of its adjacent societies to form a new three–element society. Clearly, the number of ways $N_{2r-1}(r-1)$ of splitting the new population into r–1 societies is equal to the number of ways $N_{2r}(r)$ of splitting the original population into r societies. The total number of ways of splitting n individuals into societies equals the total number of ways of arranging the n–1 spacings between them; A_1, ..., A_{n-1}, which in turn equals $(n-1)!$. Since $P_n(m) = \frac{N_n(m)}{(n-1)!}$, the result follows.

For $n = 2r$ and $m = r$, equation (4.7) reduces to

$$(2r-1)\, P_{2r}(r) = \sum_{k=1}^{r-1} P_{2k}(k)\, P_{2(r-k)}(r-k), \tag{4.41}$$

where we have used $P_{2r-1}(r) = 0$ and the fact that $P_i(j)\, P_{2r-i}(r-j)$ is non–zero only for $j = \frac{i}{2}$ = integer. The generating function

$$H(s) = \sum_{r=1}^{\infty} s^r\, P_{2r}(r)$$

satisfies

$$2s \frac{d}{ds} H(s) - s = H(s) + H^2(s), \qquad H(0) = 0,\ H'(0) = 1,$$

which has solution

$$H(s) = \sqrt{s}\, \tan \sqrt{s}\ . \tag{4.42}$$

From the power series expansion of H(s) we obtain

$$P_{2r}(r) = \frac{2^{2r}(2^{2r}-1)}{(2r)!} \left| B_{2r} \right|, \tag{4.43}$$

where B_{2r} is a Bernoulli number. Using equation (4.40a), we obtain the corresponding probability for odd–sized populations:

$$P_{2r-1}(r-1) = (2r-1)\, \frac{2^{2r}(2^{2r}-1)}{(2r)!} \left| B_{2r} \right| . \tag{4.44}$$

Since $\dfrac{|B_{2r}|}{(2r)!} \approx \dfrac{2}{(2\pi)^{2r}}$ for large r, we find that, for large populations, $P_{2r}(r) \approx 2 \left[\frac{2}{\pi}\right]^{2r}$ and $P_{2r+1}(r) \approx 4r \left[\frac{2}{\pi}\right]^{2r+2}$.

It is reasonable, of course, that odd populations have a greater probability of achieving the maximum number of societies since the single three–element society can occur in $\frac{n}{2}$ ways.

4.8 SUPER–SOCIETIES

This work may be extended to populations of societies in place of individuals. For example, if a population of n individuals forms m societies, these may be joined by some attachment criterion to form a collection of M_s super–societies. Two possible attachment criteria are:

(a) the nearest distance between individuals in different societies;

(b) the nearest distance between convex hulls enclosing societies. (More generally, one might use the nearest distance between any enclosure of societies).

Consider a situation where $n = 2r$ individuals form $m = r$ societies. These r societies would in turn form 1, 2, ..., $\left[\frac{r}{2}\right]$ super–societies. If we use criterion (a) above to connect societies, we arrive at the following result:

Theorem 4.2.

$$P_{2r}(M_s=1,\ M=r) = \frac{2^{2r-3}(r-1)!}{(2r-1)!}\ . \tag{4.45}$$

Proof: We will carry out a proof by induction.

Let $n = 8$, $r = 4$. Substituting into (4.45) we get

$$P(M_s=1,\ M=4) = \frac{4}{105}\ ,$$

which can easily be obtained by direct computation.

Suppose (4.45) is true for $r = k$. Thus for $r = k+1$ we have

$$\begin{aligned} P_{2(k+1)}(M_s=1,\ M=k+1) &= \frac{2^{2(k+1)-3}\,k!}{[2(k+1)-1]!} \\ &= \frac{2^{2k-1}\,k!}{(2k+1)!} \end{aligned}$$

But

$$\begin{aligned} P_{2(k+1)}(M_s=1,\ M=k+1) &= 2\ P(A_2=A^*)\ P_{2k}(M_s=1,\ M=k) \\ &= \frac{2}{[2(k+1)-1]} \cdot \frac{2^{2k-3}\ (k-1)!}{(2k-1)!} \\ &= \frac{2^{2k-1}\,k!}{(2k+1)!}\ . \end{aligned}$$

□

Using (4.43) and (4.45) we have:

$$P_{2r}(M_s=1|M=r) = \frac{r!}{4(2^{2r}-1)|B_{2r}|}\ . \tag{4.46}$$

For general n, we can write

$$P_n(M_s=1,\ M=m) = \frac{2}{n-1}\left[P_{n-1}(M_s=1,\ M=m) + \sum_{i=2}^{n-2} P_i(M=1)\ P_{n-i}(M_s=1,\ M=m-1)\right] \quad (2.47)$$

$$m = 1, 2, \ldots, \left[\frac{n}{2}\right].$$

Hence

$$P_n(M_s=1) = \frac{2}{n-1} \sum_{m=1}^{\left[\frac{n}{2}\right]} \left\{P_{n-1}(M_s=1,\ M=m) + \sum_{i=2}^{n-2} P_i(M=1)\ P_{n-i}(M_s=1,\ M=m-1)\right\}. \quad (4.48)$$

Tables 4.5, 4.6 and 4.7 in Section 4.9 give $P_n(M_s=k,\ M=m)$ and $P_n(M_s=k)$ for some values of n, m and k.

4.9 TABLES

TABLE 4.2: $P_n(m)$ for $4 \leq n \leq 12$

n \ m	1	2	3	4	5	6
4	0.6667	0.3333				
5	0.3333	0.6667				
6	0.1333	0.7333	0.1333			
7	0.0444	0.5778	0.3778			
8	0.0127	0.3619	0.5714	0.0540		
9	0.0032	0.1905	0.6095	0.1968		
10	0.0007	0.0871	0.5122	0.3781	0.0219	
11	0.0001	0.0354	0.3596	0.5074	0.0975	
12	0.0000	0.0130	0.2187	0.5324	0.2270	0.0089

Note: $P_2(1) = P_3(1) = 1$

TABLE 4.3: $P_n(m)$ for n = 20, 30, 40, 80.

n=20		n=30		n=40		n = 80	
m	$P_n(m)$	m	$P_n(m)$	m	$P_n(m)$	m	$P_n(m)$
2	< .0001	5	< .0001	8	< .0001	19	< .0001
3	0.0002	6	0.0008	9	0.0016	20	0.0004
4	0.0079	7	0.0123	10	0.0136	21	0.0024
5	0.0915	8	0.0786	11	0.0657	22	0.0102
6	0.3245	9	0.2351	12	0.1803	23	0.0324
7	0.3959	10	0.3420	13	0.2869	24	0.0779
8	0.1618	11	0.2413	14	0.2651	25	0.1422
9	0.0180	12	0.0788	15	0.1399	26	0.1974
10	0.0002	13	0.0106	16	0.0406	27	0.2080
		14	0.0005	17	0.0060	28	0.1659
		15	< .0001	18	0.0004	29	0.0996
				19	< .0001	30	0.0447
						31	0.0148
						32	0.0035
						33	0.0006
						34	< .0001

TABLE 4.4: $P_n(m)$ for m = 2, 5, 10, 20.

m=2

n	$P_n(m)$
4	0.3333
5	0.6667
6	0.7333
7	0.5778
8	0.3619
9	0.1905
10	0.0871
11	0.0354
12	0.0130
13	0.0044
14	0.0013
15	0.0004
16	0.0001
17	< .0001

m=5

n	$P_n(m)$
10	0.0219
11	0.0975
12	0.2270
13	0.3667
14	0.4606
15	0.4787
16	0.4276
17	0.3369
18	0.2385
19	0.1539
20	0.0915
21	0.0505
22	0.0261
23	0.0127
24	0.0058
25	0.0025
26	0.0011
27	0.0004
28	0.0002
29	< .0001

m=10

n	$P_n(m)$
20	0.0002
21	0.0020
22	0.0089
23	0.0264
24	0.0601
25	0.1119
26	0.1770
27	0.2446
28	0.3012
29	0.3355
30	0.3420
31	0.3220
32	0.2823
33	0.2319
34	0.1794
35	0.1313
36	0.0913
37	0.0606
38	0.0384
39	0.0233
40	0.0136
41	0.0077
42	0.0042
43	0.0022
44	0.0011
45	0.0005
46	0.0003
47	0.0001
48	< .0001

m = 20

n	$P_n(m)$
44	< .0001
45	0.0003
46	0.0010
47	0.0026
48	0.0058
49	0.0117
50	0.0215
51	0.0364
52	0.0570
53	0.0832
54	0.1138
55	0.1467
56	0.1789
57	0.2071
58	0.2284
59	0.2407
60	0.2431
61	0.2356
62	0.2198
63	0.1977
64	0.1717
65	0.1442
66	0.1174
67	0.0926
68	0.0710
69	0.0529
70	0.0384
71	0.0272
72	0.0187
73	0.0126
74	0.0083
75	0.0053
76	0.0033
77	< .0030

TABLE 4.5: $P_n(M_s=1, M=m)$ for $8 \le n \le 12$

n \ m	1	2	3	4	5	6	$P_n(M_s=1)$
8	0.0127	0.3619	0.5714	0.0381			0.9841
9	0.0032	0.1905	0.6095	0.1373			0.9405
10	0.0007	0.0871	0.5122	0.2612	0.0085		0.8697
11	0.0001	0.0354	0.3596	0.4277	0.0368		0.8596
12	0.0000	0.0130	0.2187	0.3771	0.0838	0.0015	0.6941

TABLE 4.6: $P_n(M_s=2, M=m)$ for $8 \le n \le 12$

n \ m	4	5	6	$P_n(M_s=2)$
8	0.0159			0.0159
9	0.0595			0.0595
10	0.1169	0.0134		0.1303
11	0.0797	0.0607		0.1404
12	0.1553	0.1432	0.0064	0.3049

$P_n(M_s=3, M=6) = 0.0010$ for $n = 12$

TABLE 4.7: $P_n(M_s=k)$ for $8 \le n \le 12$

n \ k	1	2	3
8	0.9841	0.0159	
9	0.9405	0.0595	
10	0.8697	0.1303	
11	0.8596	0.1404	
12	0.6941	0.3049	0.0010

$P_n(M_s=1) = 1$ for $n \le 7$

$P_n(M_s=1,\ M=m) = P_n(M=m)$ for $m = 1, 2, 3.$

CHAPTER FIVE

BUFFON'S PI

5.1 INTRODUCTION

The famous needle experiment of Buffon and its variations provide empirical estimates of the value of π. Some of the estimators are discussed in Mantel (1953), Gridgeman (1960), Schuster (1974), Perlman and Wichura (1975) and Solomon (1978).

In Section 5.2, we review Buffon's original experiment and the estimator obtained. We go on to investigate estimators from variations involving short and long needles, single and double grids.

Our approach is to estimate $\theta = \frac{1}{\pi}$ first. In this way, we avoid some pitfalls in treating asymptotic variances of our estimates, yet estimating $\frac{1}{\pi}$ or π gives us the same information. We obtain estimators which utilise the available statistical information as fully as possible. Efficiency comparison is done through asymptotic variances.

5.2 THE BUFFON NEEDLE PROBLEM

In the classical formulation of the Buffon needle problem, a needle (line segment) of length ℓ is dropped at random on a set of equidistant parallel lines in the plane that are d units apart, $\ell \leq d$. One asks for the probability p of an intersection. Let x denote the distance of the needle's midpoint to the nearest line and ϕ, the acute angle formed by the needle and a perpendicular from the midpoint to the line (see Figure 5.1).

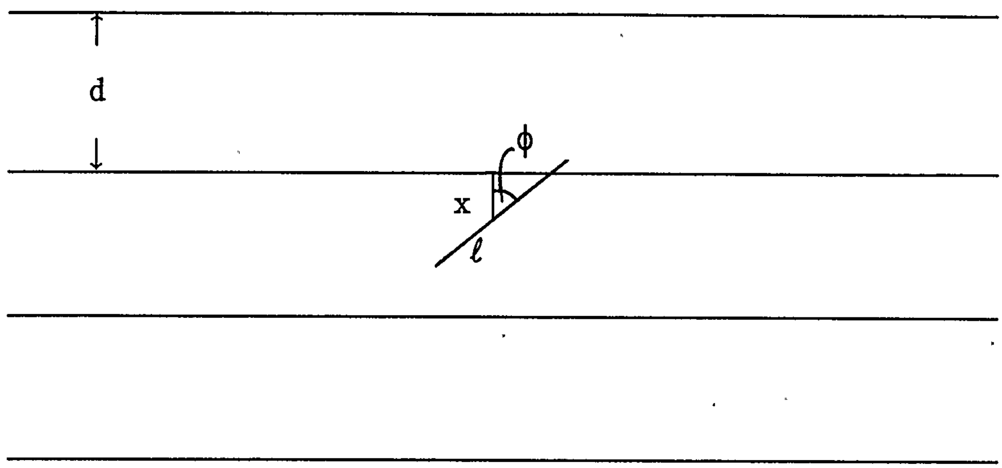

Figure 5.1

Consider the possible positions for the needle as equally likely outcomes. Then the measure of the set of total outcomes is

$$\int_0^{\pi/2} \int_0^{d/2} dx d\phi = \frac{\pi d}{4} . \tag{5.1}$$

From Figure 5.1 we evaluate the measure of the set of favourable cases (intersections) as

$$\int_0^{\pi/2} \int_0^{(\ell/2)\cos\phi} dx d\phi = \frac{\ell}{2} . \tag{5.2}$$

Therefore

$$p = \frac{2\ell}{\pi d} \ . \tag{5.3}$$

Suppose we let $d = 1$ and $\theta = \frac{1}{\pi}$. Then

$$p = 2\ell\theta \ . \tag{5.4}$$

Since $\ell \leq 1$, then $0 \leq p \leq 2\theta \leq 1$ and $0 \leq \theta \leq \frac{1}{2}$. If n independent throws of the needle, ℓ, $0 \leq \ell \leq 1$ result in N crossings on the single grid, then N is binomially distributed with parameters n, p and

$$\hat{\theta}_1 = \frac{N}{2\ell n} \tag{5.5}$$

is an unbiased estimator of θ (i.e. $E(\hat{\theta}_1) = \theta$). N is a sufficient statistic for θ and $\hat{\theta}_1$ is a minimum variance unbiased estimator (MVUE) of θ. Furthermore, $\hat{\theta}_1$ is the maximum likelihood estimator (MLE) of θ and therefore has 100% asymptotic efficiency (see Appendix A5) in this experiment. Now

$$\mathrm{Var}(\hat{\theta}_1) = \frac{p(1-p)}{4\ell^2 n} = \frac{\theta^2}{n}\left[\frac{1}{p} - 1\right] . \tag{5.6}$$

The efficiency of $\hat{\theta}_1$, as measured by the reciprocal of its variance, is maximized by taking p as close to 1 as possible. In this case $p = 2\theta$, and therefore,

$$\mathrm{nVar}(\hat{\theta}_1) = \theta^2\left[\frac{1}{2\theta} - 1\right] . \tag{5.7}$$

An application of the δ–method (see Appendix A5) shows that Buffon's estimator

$$\hat{\pi}_1 = \frac{1}{\hat{\theta}_1} \tag{5.8}$$

is an asymptotically unbiased 100% efficient estimator of $\frac{1}{\theta}$ with asymptotic variance

$$\mathrm{AVar}(\hat{\pi}_1) = \pi^4 \mathrm{var}(\hat{\theta}_1) = \frac{\pi^2}{n}\left[\frac{\pi}{2} - 1\right] = \frac{5.63}{n} . \tag{5.9}$$

Here, as in the rest of the chapter, the asymptotic variance has been numerically evaluated at the "true" value 3.1416 of π.

5.3 LAPLACE'S EXPERIMENT – THE DOUBLE GRID

Consider two sets of parallel lines over the plane where one set is orthogonal to the other, call them A–lines and B–lines. Suppose the lines are separated by unit distance. A needle of length $\ell \leq 1$ is now thrown onto this grid of lines (see Figure 5.2). This is the Laplace extension.

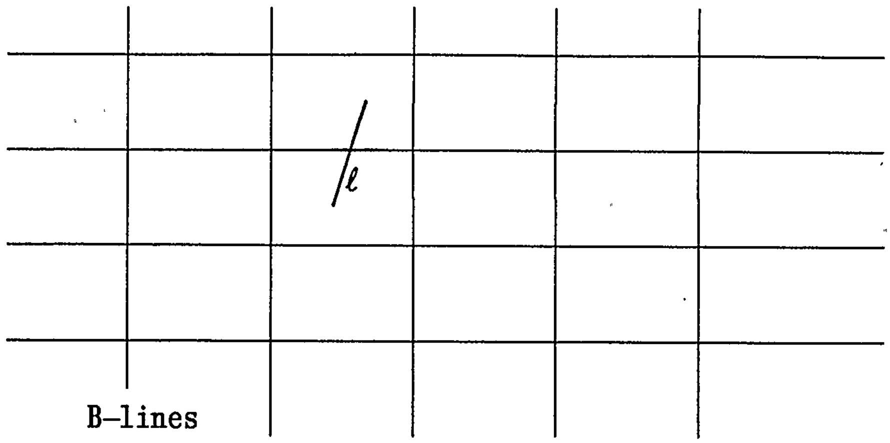

Figure 5.2

Denote by

p_A : the probability of crossing an A–line,

p_{AB} : the probability of simultaneously crossing an A–line and a B–line, and

$p_{A\bar{B}}$: the probability of crossing an A–line but not a B–line.

Similarly, define p_B, $p_{\bar{A}B}$ and $p_{\bar{A}\bar{B}}$. Then these crossing probabilities, originally obtained by Laplace, are

$$
\begin{aligned}
p_A &= p_B = \frac{2\ell}{\pi} = 2\ell\theta \\
p_{AB} &= \frac{\ell^2}{\pi} = \ell^2\theta \\
p_{\bar{A}B} &= p_{A\bar{B}} = \ell(2-\ell)/\pi = \ell(2-\ell)\theta \\
p_{\bar{A}\bar{B}} &= 1 - \frac{4\ell}{\pi} + \frac{\ell^2}{\pi} = 1-4\ell\theta+\ell^2\theta \ . \qquad (5.10)
\end{aligned}
$$

The needle is thrown n times, resulting in N_A crossings of the A–lines and N_B crossings of the B–lines. Then both

$$\hat{\theta}_A = \frac{N_A}{2\,\ell n} \text{ and } \hat{\theta}_B = \frac{N_B}{2\,\ell n} \tag{5.11}$$

are unbiased estimators of θ and have the same distribution as $\hat{\theta}_1$. Schuster (1974) proposed the combined estimator

$$\hat{\theta}_2 = \frac{\hat{\theta}_A+\hat{\theta}_B}{2} = \frac{N_A+N_B}{4\,\ell\,n} \tag{5.12}$$

and posed the interesting question of whether the efficiency of $\hat{\theta}_2$ (based on n throws of the needle onto the double grid) is twice that of $\hat{\theta}_1$ (based on n throws onto the single grid). This would indeed be the case if the event that the needle crosses an A–line were independent of the event that it crosses a B–line, for then $\hat{\theta}_A$ and $\hat{\theta}_B$ would be independent. A little reflection, however, shows that these events, and therefore $\hat{\theta}_A$ and $\hat{\theta}_B$, are negatively correlated, so in fact the efficiency of the combined estimator $\hat{\theta}_2$ will be greater than twice the efficiency of $\hat{\theta}_1$. This idea of combining antithetic (i.e. negatively correlated) variates to obtain an estimator with reduced variance is well–known to statisticians (see, for example, Hammersley and Morton (1956)). Using the crossing probabilities, we can readily calculate the variance of $\hat{\theta}_2$ (see Perlman and Wichura (1975) and Solomon (1978)). Introduce the indicator random variables

$$I_i(A) = \begin{cases} 1 & \text{if an A–line is crossed on the } i^{\text{th}} \text{ throw} \\ 0 & \text{if not}, \end{cases}$$

and similarly define $I_i(B)$, so that

$$N_A = \sum_{i=1}^{n} I_i(A) , \qquad N_B = \sum_{i=1}^{n} I_i(B) \qquad \text{and}$$

$$\hat{\theta}_2 = \frac{1}{4\ell n} \sum_{i=1}^{n} \left[I_i(A) + I_i(B)\right] .$$

The n pairs $\left[I_i(A),\ I_i(B)\right]$ are independent but $I_i(A)$ and $I_i(B)$ are dependent and so

$$\begin{aligned} \mathrm{Var}(\hat{\theta}_2) &= \frac{1}{16\ell^2 n} \left[\mathrm{Var}\ I_i(A) + \mathrm{Var}\ I_i(B) + 2\ \mathrm{Cov}\left[I_i(A),\ I_i(B)\right]\right] \\ &= \frac{\theta}{n} \left[\frac{1}{4\ell} + \frac{1}{8} - \theta\right] . \end{aligned} \tag{5.13}$$

It is apparent that the estimator $\hat{\theta}_2$ has greatest efficiency when $\ell = 1$. In this case, $p_{\bar{A}\bar{B}} = 1-3\theta$ which imposes the tighter constraint $0 \le \theta \le \frac{1}{3}$ or $\pi \ge 3$. Also

$$n\mathrm{Var}(\hat{\theta}_2) = \theta^2 \left[\frac{3}{8\theta} - 1\right] \tag{5.14}$$

and so

$$\mathrm{AVar}(\hat{\pi}_2) = \frac{\pi^2}{n} \left[\frac{3\pi}{8} - 1\right] = \frac{1.76}{n} . \tag{5.15}$$

Comparing (5.9) and (5.15), it is seen that by doubling the grid we have obtained an estimator $\hat{\pi}_2$ which is $\frac{5.63}{1.76} =$ 3.20 times as efficient as $\hat{\pi}_1$ where $\theta = \frac{1}{\pi}$. In a large number of throws, the double grid experiment contains approximately 3.20 times as much statistical information about the value of π per throw as the single grid experiment.

However, the estimator $\hat{\theta}_2$ does not fully utilize all the information about θ provided by the double grid experiment. The full information obtained from n throws of the needle onto the double grid is summarized by the statistic $\underset{\sim}{N} = \left[N_{AB}, N_{A\bar{B}}, N_{\bar{A}B}, N_{\bar{A}\bar{B}}\right]$, where N_{AB} is the number of times the needle simultaneously crosses an A–line and a B–line, etc. Clearly, $\underset{\sim}{N}$ has the multinomial distribution with cell probabilities $\left[p_{AB}, p_{A\bar{B}}, p_{\bar{A}B}, p_{\bar{A}\bar{B}}\right]$ given in (5.10). Thus, the probability distribution of $\underset{\sim}{N}$ is given by

$$P_\theta(\underset{\sim}{N} = \underset{\sim}{n}) = C(\underset{\sim}{n})\left[p_{AB}\right]^{n_{AB}} \left[p_{A\bar{B}}\right]^{n_{A\bar{B}}} \left[p_{\bar{A}B}\right]^{n_{\bar{A}B}} \left[p_{\bar{A}\bar{B}}\right]^{n_{\bar{A}\bar{B}}}$$

$$= C(\underset{\sim}{n})\ h(\underset{\sim}{n})\ \theta^{(n_{AB}+n_{A\bar{B}}+n_{\bar{A}B})}\ [\ 1-m\theta\ ]^{n_{\bar{A}\bar{B}}} \tag{5.16}$$

$$\text{where}\quad C(\underset{\sim}{n}) = \frac{n!}{\left[n_{AB}\right]!\ \left[n_{A\bar{B}}\right]!\ \left[n_{\bar{A}B}\right]!\ \left[n_{\bar{A}\bar{B}}\right]!}$$

$$h(\underset{\sim}{n}) = \ell^{2n_{AB}}\ [\ell(2-\ell)]^{(n_{A\bar{B}}+n_{\bar{A}B})}$$

$$m = 4\ell - \ell^2\ . \tag{5.17}$$

Since

$$n_{\bar{A}\bar{B}} = n - \left[n_{AB} + n_{A\bar{B}} + n_{\bar{A}B}\right]\ , \tag{5.18}$$

the Factorization Criterion for sufficiency (see Appendix A5) and (5.16) imply that $N_{AB} + N_{A\bar{B}} + N_{\bar{A}B}$ is a sufficient statistic for θ.

If we define N_j to be the number of times in n throws that the needle crosses exactly j lines (j = 0,1,2), we have $N_0 = N_{\bar{A}\bar{B}}$, $N_1 = N_{A\bar{B}} + N_{\bar{A}B}$, $N_2 = N_{AB}$ and $\sum_j N_j = n$, then the sufficient statistic can be expressed as $N_1 + N_2$, the number of times in n throws that the needle crosses at least one line.

Now

$$N_1 + N_2 \sim \text{Binomial}\ (n,p^*) \text{ where } p^* = m\theta = (4\ell-\ell^2)\theta$$

(since $P_{\bar{A}\bar{B}} = 1 - [(4\ell-\ell^2)\theta]$), so $N_1 + N_2$ is a sufficient statistic for θ. The estimator,

$$\hat{\theta}_3 = \frac{N_1 + N_2}{mn} \tag{5.19}$$

is MVUE and, being the MLE of θ, has 100% asymptotic efficiency in the double grid experiment. Its variance is

$$\text{Var}(\hat{\theta}_3) = \frac{\theta}{n}\left[\frac{1}{m} - \theta\right] \tag{5.20}$$

which, by (5.17), is minimized by the needle length $\ell = 1$. In this case $m = 3$, $p^* = 3\theta$,

$$n\text{Var}(\hat{\theta}_3) = \theta^2\left[\frac{1}{3\theta} - 1\right] \tag{5.21}$$

and

$$\text{AVar}(\hat{\pi}_3) = \frac{\pi^2}{n}\left[\frac{\pi}{3} - 1\right] = \frac{0.466}{n} \tag{5.22}$$

where $\theta = \frac{1}{\pi}$ and $\hat{\pi}_3 = \frac{1}{\hat{\theta}_3}$.

Comparing (5.15) and (5.22) we see that the fully efficient estimator $\hat{\pi}_3$ is $\frac{1.76}{.466} = 3.77$ times as efficient as $\hat{\pi}_2$, reflecting the fact that $\hat{\theta}_2$ is based on

$$\begin{aligned} N_A + N_B &= N_{AB} + N_{A\bar{B}} + N_{AB} + N_{\bar{A}B} \\ &= N_1 + 2N_2 \end{aligned} \tag{5.23}$$

which is not a function of the sufficient statistic $N_1 + N_2$. A moral here is that the method of antithetic variates, advocated for a wide variety of problems, should not be applied before a careful search for a sufficient statistic. Furthermore, comparing (5.9) and (5.22) we see that, in a large number of throws, one throw of the needle onto the double grid approximately contains not 3.20 but actually $\frac{5.63}{0.466} = 12.08$ times the statistical information about the value of π as one throw onto the single grid.

5.4 DOUBLE GRID, LONG NEEDLE

Consider a unit–spaced square grid. Let $\ell > 1$. The expected number of intersections of the needle with the grid, per fall, is

$$E_o = \frac{4\ell}{\pi} . \tag{5.24}$$

Assuming n throws of the needle, we can get say c_i intersections at the i^{th} fall; $i = 1,2,\ldots,n$ and write

$$\hat{\pi}_4 = \frac{4\ell}{\bar{c}} \tag{5.25}$$

as an estimate of $\pi = \dfrac{4\ell}{E_o}$ (5.26)

where $\bar{c}$ is the average number of intersections per fall. By the δ method, we can get

$$E(E_o - \bar{c})^2 \cong E(\pi - \hat{\pi}_4)^2 \frac{16\ell^2}{\pi^4} \tag{5.27}$$

and

$$AVar(\hat{\pi}_4) = \frac{\pi^4}{16\ell^2} \frac{\sigma_c^2}{n} \tag{5.28}$$

where σ_c^2 is the variance of the number of intersections obtained at the fall of the needle.

Let $\ell >> 1$ so that certain marginal effects can be disregarded. These marginal effects arise from the actual location of the end of the line within the squares in which it falls and they would slightly increase the value of σ_c^2 over what is now developed here but would have no effect on E(c).

For any given angle ϕ at which the needle of length ℓ falls, there would be $\ell \sin\phi$ intersections with vertical lines and $\ell \cos\phi$ intersections with horizontal lines. Thus, the expected number of intersections per fall is given by

$$E(c) = \frac{2}{\pi} \int_0^{\pi/2} \ell(\sin\phi + \cos\phi)\, d\phi = \frac{4\ell}{\pi}. \tag{5.29}$$

The expected square of the number of intersections is

$$E(c^2) = \frac{2}{\pi}\int_0^{\pi/2} \ell^2(\sin\phi + \cos\phi)^2\, d\phi$$

$$= \ell^2\left[1 + \frac{2}{\pi}\right] . \tag{5.30}$$

Thus

$$\hat{\sigma}_c^2 = \ell^2\left[1 + \frac{2}{\pi} - \frac{16}{\pi^2}\right] \tag{5.31}$$

and

$$\mathrm{AVar}\hat{\pi}_4 = \frac{\pi^2}{16n}(\pi^2 + 2\pi - 16)$$

$$= \frac{0.094}{n} . \tag{5.32}$$

Comparing (5.9) with (5.32), we see that the precision in estimating π from the double grid with a long needle is about 60 times as good as the single grid with a short needle. Equivalently, the information in one fall in a large number of throws of the long needle here is about the same as in 60 falls of the needle in the original Buffon needle problem when the length of the needle is equal to the distance between the parallel lines.

5.5 SINGLE GRID, LONG NEEDLE

A needle of length $\ell > 1$ is thrown onto a plane ruled by unit–spaced parallel lines. The number of intersections can range from 0 to M where $M = [\ \ell\ ] + 1$ and $[\ \ell\ ]$ is the greatest integer less than or equal to ℓ. The distribution of the number of intersections is (Diaconis (1976)):

$$p_0 = 1 - \frac{2\ell}{\pi} + \delta_1$$

$$p_i = \delta_{i-1} + \delta_{i+1} - 2\delta_i \qquad \text{for} \quad 1 \le i \le M-2$$

$$p_{M-1} = \delta_{M-2} - 2\delta_{M-1}$$

$$p_M = \delta_{M-1} \tag{5.33}$$

where

$$p_i = \text{the probability of i intersections}$$

$$\delta_i = \frac{2}{\pi}\left\{\ell \sin\alpha_i - i\alpha_i\right\} = \frac{2}{\pi}\left\{\sqrt{\ell^2 - i^2} - i\cos^{-1}\left[\frac{i}{\ell}\right]\right\},$$

$$\cos\alpha_i = \frac{i}{\ell}.$$

Letting $\theta = \frac{1}{\pi}$, we obtain

$$p_0 = 1 - \frac{2}{\pi}\left\{\ell - \sqrt{\ell^2-1} + \cos^{-1}\left[\frac{1}{\ell}\right]\right\} = 1 - 2\theta f_0, \text{ say.}$$

$$p_i = \frac{2}{\pi}\left\{\sqrt{\ell^2-(i-1)^2} - (i-1)\cos^{-1}\left[\frac{i-1}{\ell}\right] + \sqrt{\ell^2-(i+1)^2}\right.$$

$$\left. - (i+1)\cos^{-1}\left[\frac{i+1}{\ell}\right] - 2\sqrt{\ell^2-i^2} + 2i\cos^{-1}\left[\frac{i}{\ell}\right]\right\}$$

$$= 2\theta f_i, \text{ say} \qquad \text{for } 1 \le i \le M-1 .$$

$$p_{M-1} = \frac{2}{\pi}\left\{\sqrt{\ell^2-(M-2)^2} - (M-2)\cos^{-1}\left[\frac{M-2}{\ell}\right] + 2\sqrt{\ell^2-(M-1)^2}\right.$$

$$\left. + 2(M-1)\cos^{-1}\left[\frac{M-1}{\ell}\right]\right\}.$$

$$= 2\theta f_{M-1}, \text{ say}$$

$$p_M = \frac{2}{\pi}\left\{\sqrt{\ell^2-(M-1)^2} - (M-1)\cos^{-1}\left[\frac{M-1}{\ell}\right]\right\} = 2\theta f_M, \text{ say.} \qquad (5.34)$$

Thus

$$p_0 = 1 - 2\theta f_0$$

$$p_i = 2\theta f_i, \qquad 1 \le i \le M$$

where the f_i's are free of θ, hence π.

Consider an experiment with n throws of the needle. Full information is given by

$$\underset{\sim}{N} = (N_0, N_1, \ldots, N_M) \qquad (5.35)$$

where N_i = the number of times i lines were intersected in the n throws.

$$\underset{\sim}{N} \sim \text{multinomial } (p_0, p_1, \ldots, p_M).$$

Thus

$$P(\underset{\sim}{N} = \underset{\sim}{n}) = \frac{n!}{N_0!\, N_1! \cdots N_M!} \; p_0^{N_0}\, p_1^{N_1} \cdots p_M^{N_M}$$

$$= c(\underset{\sim}{n})\; (1 - 2\theta f_o)^{n-T}\; (2\theta)^T\, f_1^{N_1}\, f_2^{N_2} \cdots f_M^{N_M} \tag{5.36}$$

where $c(\underset{\sim}{n}) = \dfrac{n\,!}{\left[n-(N_1+N_2+\cdots+N_M)\right]!\; N_1! \cdots N_M!}$

$$T = \sum_{i=1}^{M} N_i\,.$$

By the Factorization theorem, T is a sufficient statistic for θ and

$$T \sim \text{Binomial}\left[n,\; p^* = \sum_{i=1}^{M} p_i\right]. \tag{5.37}$$

Now,

$$E(T) = np^* = 2n\theta f_0 \tag{5.38}$$

Therefore

$$\hat{\theta}_5 = \frac{T}{2nf_0} \tag{5.39}$$

is unbiased for θ. The estimator $\hat{\theta}_5$ is MVUE for θ and

$$\mathrm{Var}(\hat{\theta}_5) = \frac{\theta}{2n}\left[\frac{1}{f_0} - 2\theta\right] . \tag{5.40}$$

Minimizing $\mathrm{Var}(\hat{\theta}_5)$ is equivalent to maximizing f_0. Since f_0 is an increasing function of ℓ, this is achieved by lengthening the needle. Note that f_0 approaches $\pi/2$ as ℓ becomes large, in which case $\mathrm{Var}(\hat{\theta}_5)$ approaches zero. Also

$$\mathrm{AVar}(\hat{\pi}_5) = \frac{\pi^2}{n}\left[\frac{\pi}{2f_0} - 1\right] \tag{5.41}$$

where $\hat{\pi}_5 = \dfrac{1}{\hat{\theta}_5}$.

5.6 EFFICIENCY COMPARISON

In Table 5.1, we compare the efficiencies of the estimators we have investigated via their asymptotic variances. For $\hat{\pi}_5$ we consider several values of ℓ.

Table 5.1: Comparing asymptotic variances of the estimators of π

EXPERIMENT (WITH LINES SEPARATED BY UNIT DISTANCE)	ESTIMATOR ($\hat{\pi}_i$)	n(AVar($\hat{\pi}_i$))
Single grid, $\ell \leq 1$	$\hat{\pi}_1 = \frac{2\ell n}{N}$	5.63
Double grid, $\ell \leq 1$	$\hat{\pi}_2 = \frac{2\ell n}{N_A + N_B}$	1.76
	$\hat{\pi}_3 = \frac{(4\ell-\ell^2)n}{N_1 + N_2}$	0.466
Double grid, $\ell > 1$	$\hat{\pi}_4 = \frac{4\ell}{\bar{c}}$ where $\bar{c}$ is the average number of intersections per fall.	0.094
Single grid, $\ell \geq 1$	$\hat{\pi}_5 = (2nf_0)/T$	0.325 (ℓ = 10)
	where $T = \sum_{i=1}^{M} N_i$ and	
	$f_0 = \ell-\sqrt{(\ell^2-1)}+\cos^{-1}\left[\frac{1}{\ell}\right]$	0.003 (ℓ = 1000)

We observe that the best estimators are from long needle experiments. In particular, $\hat{\pi}_5$ with $\ell = 1000$ is about 1900 times as efficient as $\hat{\pi}_1$. In other words, the statistical information in one fall in a large number of throws of the long needle, $\ell = 1000$, onto the single grid is about the same as 1900 falls of the needle in the original Buffon needle problem when the length of the needle is equal to the distance between the parallel lines.

REFERENCES

1. ABRAMOWITZ, M. and STEGUN, I.A. (1968). *Handbook of Mathematical Functions.* National Bureau of Standards. Washington, D.C.

2. ALAGAR, V.S. (1976). The distribution of the distance between random points. *J. Appl. Prob.* 13, 558–566.

3. BUFFON, G. (1977). Essai d'arithmetique morale. *Supplément a l'Histoire Naturelle*, Vol. 4.

4. CLARK, P.J. and EVANS, F.C. (1955). On some aspects of spatial pattern in biological populations. *Science* 121, 397–398.

5. COLEMAN, R. (1969). Random paths through convex bodies. *J. Appl. Prob.* 6, 430–441.

6. COLEMAN, R. (1973). Random paths through rectangles and cubes. *Metallography* 6, 103–114.

7. COLEMAN, R. (1989). Random sections of a sphere. *Cdn. J. Statist.* 17(1), 27–39.

8. COX, T.F. (1981). Reflexive nearest neighbours. *Biometrics* 37, 367–369.

9. DACEY, M.F. (1969). Proportion of reflexive n^{th} order neighbours in a spatial distribution. *Geographical Analysis* 1, 385–388.

10. DAVID, H.A. and GROENEVELD, R.A. (1982). Measures of local variation in a distribution : Expected length of spacings and variances of order statistics. *Biometrika* 69(1), 227–232.

11. DIACONIS, P. (1976). Buffon's problem with a long needle. *J. Appl. Prob.* 13, 614–618.

12. EFRON, B. (1965). The convex hull of a random set of points. *Biometrika* 52, 331–343.

13. EHLERS, P.F. and ENNS, E.G. (1981). Random secants of a convex body generated by surface randomness. *J. Appl. Prob.* 18, 157–166.

14. ENNS, E.G. and EHLERS, P.F. (1978). Random paths through a convex region. *J. Appl. Prob.* 15, 144–152.

15. ENNS, E.G. and EHLERS, P.F. (1980). Random paths originating within a convex region and terminating on its surface. *Austral. J. Statist.* 22(1), 60–68.

16. ENNS, E.G., EHLERS, P.F. and STUHR, S. (1981). Every body has its moments. *Statistical Distributions in Scientific Work* 5, 387–396.

17. ENNS, E.G. and EHLERS, P.F. (1988). Chords through a convex body generated from within an embedded body. *J. Appl. Prob.* 25, 700–707.

18. FELLER, W. (1971). *An Introduction to Probability Theory and its Applications.* Vol. I & II. John Wiley, New York.

19. GANI, J. (1980). The problem of Buffon's needle. *Math. Spectrum* 13, 14–18.

20. GLAZ, J. and NAUS, J. (1983). Multiple clusters on the line. *Commun. Statist. – Theor. Meth.* 12(17), 1961–1986.

21. GRIDGEMAN, N.T. (1960). Geometric probability and the number π. *Scripta Mathematica* 25, 183–195.

22. HAMMERSLEY, J.M. and MORTON, K.W. (1956). A new Monte Carlo technique : Antithetic variables. *Proceedings of the Cambridge Philosophical Society* 52, 449–475.

23. KELLERER, A.M. (1971). Considerations on the random transversal of convex bodies and solutions for general cylinders. *Radiation Research* 47, 359–376.

24. KENDALL, M.G. and MORAN, P.A.P. (1963). *Geometrical Probability.* Griffin, London.

25. KINGMAN, J.F.C. (1969). Random secants of a convex body. *J. Appl. Prob.* 6, 660–672.

26. LEHMANN, E.L. (1983). *Theory of Point Estimation.* John Wiley, New York.

27. MANTEL, N. (1953). An extension of the Buffon needle problem. *Ann. Math. Statist.* 24, 674–677.

28. MATHLAB GROUP (1983). *MACSYMA Reference Manual.* MIT, Cambridge, Mass.

29. NAUS, J.I. (1979). An indexed bibliography of clusters, clumps, and coincidences. *Internat. Statist. Review.* 47, 47–78.

30. NEWMAN, C.M., RINOTT, Y. and TVERSKY, A. (1983). Nearest neighbours and Voronoi regions in certain point processes. *Adv. Appl. Prob.* 15, 726–751.

31. NEWMAN, C.M. and RINOTT, Y. (1985). Nearest neighbours and Voronoi volumes in high–dimensional point processes with various distance functions. *Adv. Appl. Prob.* 17, 794–809.

32. PERLMAN, M. and WICHURA, M. (1975). On sharpening Buffon's needle. *Amer. Statist.* 19, 157–163.

33. PIELOU, E.C. (1977). *Mathematical Ecology.* John Wiley, New York.

34. PYKE, R. (1965). Spacings. *J. Roy. Statist. Soc. Ser. B* 27, 395–449.

35. PYKE, R. (1972). Spacings revised. *Proc. Sixth Berkeley Symp. Math. Stat. Probability* 1, 417–427.

36. RAO, C.R. (1973). *Linear Statistical Inference and Its Applications.* Second edition. John Wiley, New York.

37. ROBERTS, F.D.K. (1967). A Monte Carlo solution of a two–dimensional unrestricted cluster problem. *Biometrika* 54, 625–628.

38. ROBERTS, F.D.K. and STOREY, S.H. (1968). A three–dimensional cluster problem. *Biometrika* 55, 258–260.

39. ROBERTS, F.D.K. (1969). Nearest neighbours in a Poisson ensemble. *Biometrika* 56, 401–406.

40. ROUSSAS, G.G. (1973). *A First Course in Mathematical Statistics.* Addison–Wesley, Reading, Mass.

41. SCHUSTER, E.F. (1974). Buffon's needle experiment. *Amer. Math. Monthly* 81, 26–29.

42. SOLOMON, H. (1978). *Geometric Probability.* SIAM, Philadelphia, Penn.

43. STOYAN, D., KENDALL, W.S. and MECKE, J. (1987). *Stochastic Geometry and Its Applications.* John Wiley, New York.

44. UPENSKY, J.V. (1937). *Introduction to Mathematical Probability.* McGraw–Hill, New York.

APPENDICES

A1: DISTRIBUTION OF THE DISTANCE FROM A RANDOM POINT WITHIN A SPHERE TO THE CENTRE OF THE SPHERE.

Consider a point P chosen at random inside a sphere of radius a. Let T be the random variable denoting the distance from P to the centre of the sphere (see Figure A1.1).

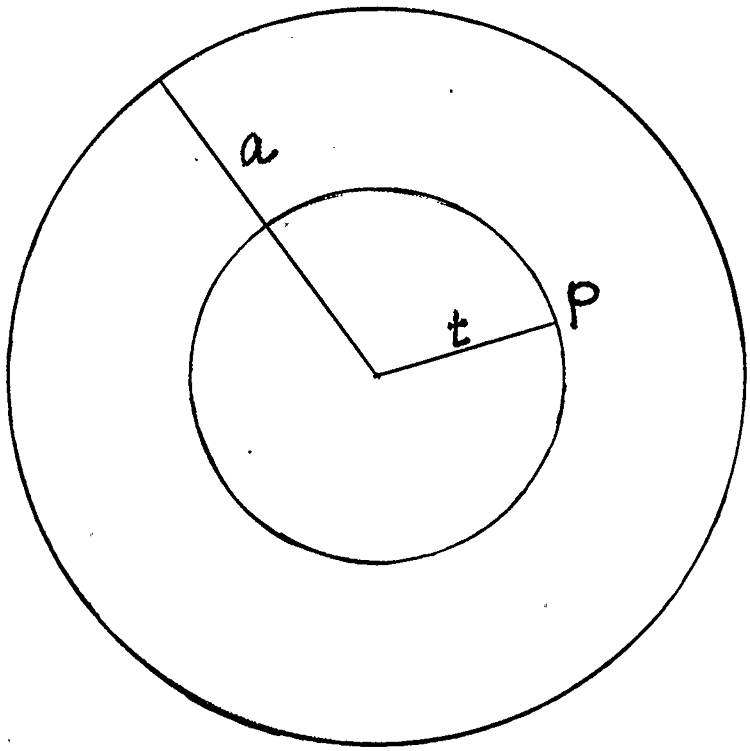

Figure A1.1

Let V(x) be the volume of a sphere of radius x. Then

$$G(t) = \Pr(T \le t) = \frac{V(t)}{V(a)} = \frac{t^3}{a^3} . \tag{A1.1}$$

Hence

$$g(t) = \frac{3t^2}{a^3} . \tag{A1.2}$$

A2: VOLUME OF A SPHERICAL CONE

Consider the sphere in Figure A2.1.

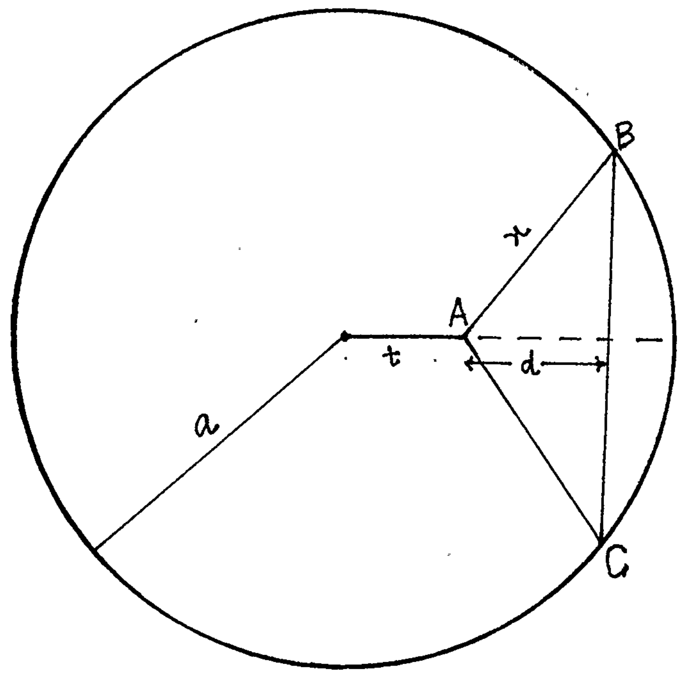

Figure A2.1

From elementary calculus, we know that the volume of a spherical cap, height h, $0 \leq h \leq 2a$, is:

$$V[\text{cap, h}] = \pi \left\{\frac{2a^3}{3} - a^2(a-h) + \frac{1}{3}(a-h)^3\right\} . \tag{A2.1}$$

Therefore

$$V[\text{cap, a-t-d}] = \pi \left\{\frac{2a^3}{3} - a^2(t+d) + \frac{1}{3}(t+d)^3\right\} . \tag{A2.2}$$

Also

$$V_{\text{cone ABC}} = \frac{\pi}{3}\,(r^2 - d^2)d \qquad \text{(A2.3)}$$

where

$$d = \frac{(a^2 - t^2 - r^2)}{2t}\ .$$

Hence

$$V* \ = V[\text{cap},\ a-t-d] + V_{\text{cone ABC}}$$

$$= \pi \left\{\frac{2a^3}{3} - a^2(t+d) + \frac{1}{3}\,(t+d)^3 + \frac{1}{3}\,(r^2 - d^2)d\right\}\ . \qquad \text{(A2.4)}$$

A3: FIXED–ANGLE $\hat{\nu}$–RANDOMNESS

The randomness assumption we consider here is the one mentioned in Subsection 2.6.1.

We will use the following notation from analytic geometry:

$\overrightarrow{AB}$: a vector directed from point A to point B

$\underset{\sim}{a}$: alternative notation for a vector.

$\|\underset{\sim}{a}\|$: the length of vector $\underset{\sim}{a}$.

$\underset{\sim}{a}\cdot\underset{\sim}{b}$: the scalar product of $\underset{\sim}{a}$ and $\underset{\sim}{b}$ which is the product of their lengths and the cosine of the angle between them.

A point Q is chosen at random inside the sphere. A ray R is defined from Q in some random direction θ. Suppose another ray S is made at a fixed angle α from R (Figure A3.1).

Let $\|\overrightarrow{QB}\| = 1$, so that

$$\overrightarrow{QB} = (0,\ \cos\theta,\ \sin\theta).$$

Also let

β be the angle on the cone–circle (Figure A3.2) and $\underset{\sim}{d}$ be perpendicular to $\overrightarrow{QB}$.

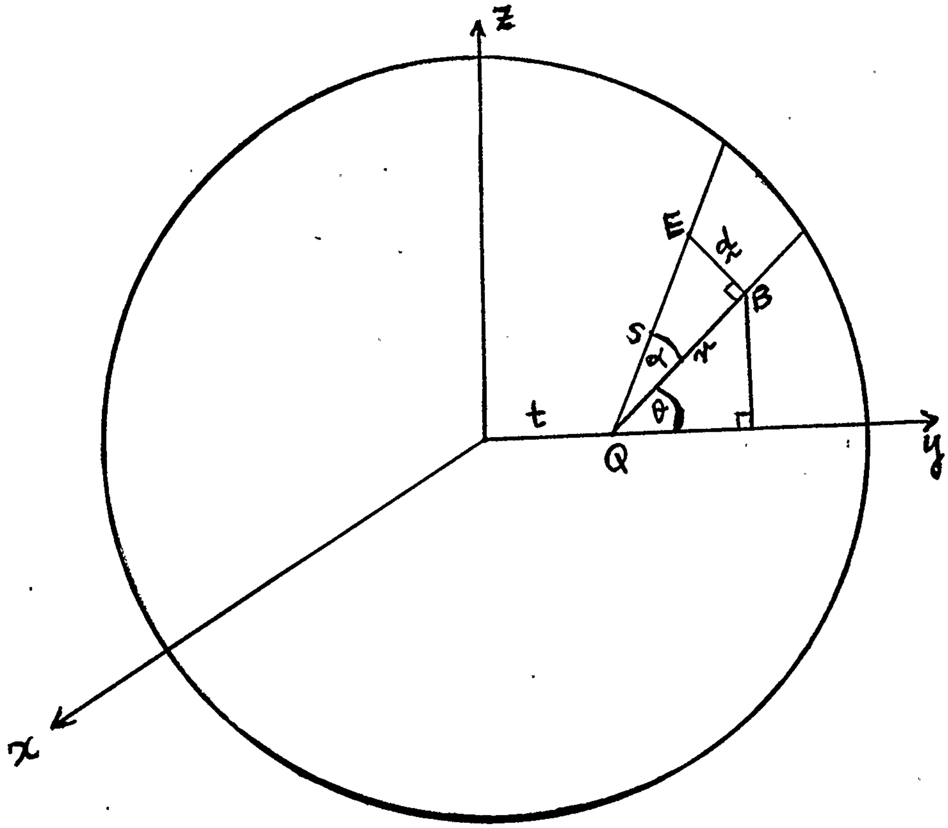

Figure A3.1

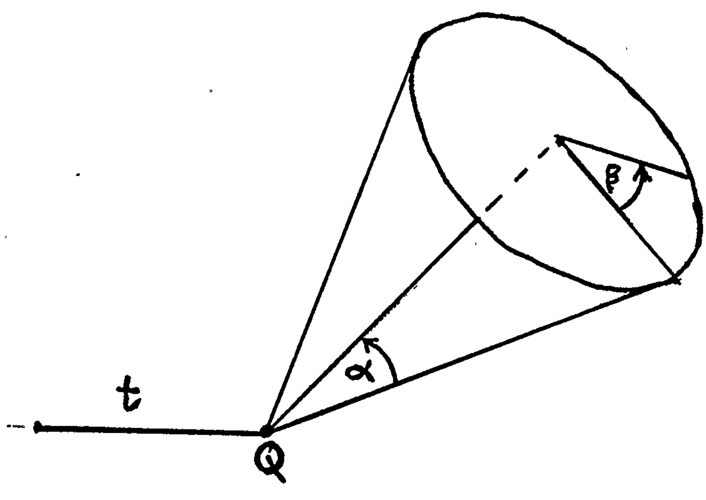

Figure A3.2

Then

$$\underset{\sim}{d} = \tan\alpha\ (\sin\beta,\ y,\ z) \text{ for some } y,\ z \text{ and } \underset{\sim}{d} \cdot \overrightarrow{QB} = 0$$

so that

$$y \cos\theta + z \sin\theta = 0$$

or

$$y = - z \tan\theta\ . \tag{A3.1}$$

Note that $\|\underset{\sim}{d}\| = \tan\alpha$, so that

$$\|(\sin\beta,\ y,\ z)\| = 1$$

hence

$$\sin^2\beta + y^2 + z^2 = 1$$

or

$$y^2 + z^2 = \cos^2\beta\ . \tag{A3.2}$$

Substituting (A3.1) into (A3.2) yields

$$z^2 \tan^2\theta + z^2 = \cos^2\beta$$

or

$$z = \cos\theta \cos\beta\ .$$

Thus

$$y = - z \tan\theta = - \sin\theta \cos\beta$$

and

$$\underset{\sim}{d} = \tan\alpha\ (\sin\beta,\quad - \sin\theta \cos\beta,\quad \cos\theta \cos\beta)\ .$$

Therefore

$$\vec{QE} = (\sin\beta \tan\alpha, \quad \cos\theta - \sin\theta \cos\beta \tan\alpha,$$

$$\sin\theta + \cos\theta \cos\beta \tan\alpha) \ .$$

The parametric equations of the straight line passing through the point Q: (0,t,0) and having direction $\vec{QE}$ are:

$$\begin{aligned} x &= \ell \sin\beta \tan\alpha & &= \ell x_o \\ y &= t + \ell(\cos\theta - \sin\theta \cos\beta \tan\alpha) & &= t + \ell y_o \\ z &= \ell(\sin\theta + \cos\theta \cos\beta \tan\alpha) & &= \ell z_o \end{aligned}$$

where ℓ is an arbitrary variable parameter.

This line cuts the sphere $x^2 + y^2 + z^2 = a^2$ at

$$(\ell x_o)^2 + (t + \ell y_o)^2 + (\ell z_o)^2 = a^2$$

or

$$\ell^2(x_o^2 + y_o^2 + z_o^2) + 2t\ell y_o + t^2 - a^2 = 0 \ .$$

But $x_o^2 + y_o^2 + z_o^2 = 1 + \tan^2\alpha = \sec^2\alpha$.

Therefore

$$\ell^2 \sec^2\alpha + 2t\ell y_o + t^2 - a^2 = 0$$

and

$$\ell = \left[\sqrt{t^2 y_o^2 + (a^2 - t^2)\sec^2\alpha} - t y_o\right] / \sec^2\alpha \ .$$

Hence

> A general point on the curve of intersection between the cone and the sphere has co-ordinates:
>
> $$x = \ell x_o$$
> $$y = t + \ell y_o$$
> $$z = \ell z_o$$
>
> where $$\ell = \left[\sqrt{t^2 y_o^2 + (a^2-t^2)\sec^2\alpha} - ty_o\right]/\sec^2\alpha$$
> $$x_o = \sin\beta \tan\alpha$$
> $$y_o = \cos\theta - \sin\theta \cos\beta \tan\alpha$$
> $$z_o = \sin\theta + \cos\theta \cos\beta \tan\alpha$$

(A3.3)

Let F be this point.

Then

$$(\|QF\|)^2 = s^2 = \ell^2 \sec^2\alpha$$

or

$$s = t\cos\alpha\,(\sin\theta\cos\beta\tan\alpha - \cos\theta) + \cos\alpha\sqrt{t^2(\cos\theta - \sin\theta\cos\beta\tan\alpha)^2 + (a^2-t^2)\sec^2\alpha}\ . \qquad \text{(A3.4)}$$

Note that the intersection curve is not planar.

The parametric equations of the straight line passing through Q: (0,t,0) with direction $\overrightarrow{QB}$ are

$$x = 0$$
$$y = t + m \cos\theta$$
$$z = m \sin\theta$$

where m is an arbitrary, variable parameter. The line cuts the sphere at

$$(t + m \cos\theta)^2 + (m \sin\theta)^2 = a^2 .$$

Hence

$$m = \sqrt{a^2 - t^2 \sin^2\theta} - t \cos\theta . \quad \text{(A3.5)}$$

The point of intersection of this line with the sphere has co-ordinates:

$$x = 0$$
$$y = t + m \cos\theta$$
$$z = m \sin\theta$$

where $m = \sqrt{a^2 - t^2 \sin^2\theta} - t \cos\theta$

(A3.6)

Let P be this point.

Then

$$\|\overrightarrow{QP}\| = r = \sqrt{a^2 - t^2 \sin\theta} - t \cos\theta . \quad \text{(A3.7)}$$

We have, therefore, obtained r and s in terms of the random variables θ and β which are both uniformly distributed.

$$s = t\cos\alpha(\sin\theta\cos\beta\tan\alpha - \cos\theta)$$
$$\quad + \cos\alpha\sqrt{t^2(\cos\theta - \sin\theta\cos\beta\tan\alpha)^2 + (a^2-t^2)\sec^2\alpha}$$
$$r = \sqrt{a^2 - t^2\sin^2\theta} - t\cos\theta$$
$$\theta \sim U(0,\pi)$$
$$\beta \sim U(0,2\pi)$$
α is a constant

We may use these relationships to derive the probability density functions for r and s. For general α the derivations are rather involved. But, for example, if $\alpha = \pi$, we easily retrieve the expressions for the "forward" and "backward" rays.

A4: BERNOULLI NUMBERS

The rational numbers B_n $(n \geq 1)$ defined by

$$\frac{t}{e^t-1} = 1 + \sum_{n=1}^{\infty} \frac{B_n}{n!} t^n \tag{A4.1}$$

are called Bernoulli numbers.

All Bernoulli numbers with odd index, except for $B_1 = -\frac{1}{2}$, equal zero.
We give values of the first 6 Bernoulli numbers with even index:

$$B_2 = \frac{1}{6}, \qquad B_4 = -\frac{1}{30}, \qquad B_6 = \frac{1}{42},$$

$$B_8 = -\frac{1}{30}, \qquad B_{10} = \frac{5}{66}, \qquad B_{12} = -\frac{691}{2730}.$$

A5: ON STATISTICAL ESTIMATION

Let $X_1, X_2, \ldots, X_n$ denote a random sample of size n from a distribution that has probability density function $f(x;\theta)$ which depends on the parameter $\theta \in \Omega$.

1. SUFFICIENCY:

The statistic T is said to be a sufficient estimator of the parameter θ if and only if for each value of T, the conditional distribution of the random sample, given T=t, is independent of θ. A sufficient statistic summarizes all the relevant information supplied by the sample.

2. FACTORIZATION CRITERION:

The statistic T is a sufficient estimator θ if and only if the joint density or probability distribution of the random sample can be factored so that

$$f(x_1, \ldots, x_n; \theta) = g\left[T(x_1, \ldots, x_n); \theta\right] h(x_1, \ldots, x_n) \qquad \text{(A5.1)}$$

where g depends on $x_1, \ldots, x_n$ only through T and h is independent of θ.

3. MAXIMUM LIKELIHOOD

A statistic T is said to be a maximum likelihood estimator (MLE) of θ if it maximizes the likelihood function

$$L(\theta) = f(x_1, \ldots, x_n; \theta) \ .$$

4. ASYMPTOTIC EFFICIENCY:

If standard regularity conditions are satisfied, then the asymptotic efficiency of an estimator δ of θ is

$$\text{AEff}\ (\delta) \equiv \frac{1 / n\ I(\theta)}{\text{AVar}(\delta)} \leq 1 \tag{A5.2}$$

where AVar(δ) is the asymptotic variance of δ ,

I(θ) is the Fisher information number defined by

$$I(\theta) = E_\theta\left\{\left[\frac{\partial}{\partial\theta} \log f(X;\theta)\right]^2\right\} \tag{A5.3}$$

and n I(θ) is the information (about θ) contained in the sample $X_1,\ldots,X_n$.
If AEff(δ) = 1, then δ is said to be 100% asymptotically efficient.

5. THE δ-METHOD:

From C.R. Rao (1973) pp 385, we have the following:

Let (T_n), $n = 1, 2, \ldots$, be a sequence of statistics such that

$$\sqrt{n}\,(T_n - \theta) \xrightarrow{L} X \sim N[0,\ \sigma^2(\theta)]\ .$$

Let g be a function of a single variable admitting the first derivative g'. Then

$$\sqrt{n}\,[g(T_n) - g(\theta)] \xrightarrow{L} X \sim N[0,\ (g'(\theta)\ \sigma(\theta))^2] \qquad \text{if } g'(\theta) \neq 0.$$

Thus

$$\mathrm{AVar}[g(T_n)] = [g'(\theta)]^2\ \mathrm{AVar}(T_n)\ .$$

($X_n \xrightarrow{L} X$ means convergence in distribution or in law).