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UNIVERSITY OF CALGARY

Ramification, Structure and Ground

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE

DEGREE OF DOCTOR OF PHILOSOPHY

GRADUATE PROGRAM IN PHILOSOPHY

CALGARY, ALBERTA

JULY, 2023

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Abstract

This dissertation sets out to explore a deep interconnection between highly structured relational entities, various notions of grounding and ramified typed systems. It is argued that together these three form a powerful alliance that contributes to a unified picture of reality within which a cluster of recent puzzles of ground and grain can be resolved.

Preface

This manuscript-based thesis consists of four original papers composed by the author, that are published, under review or in progress. The following details the status of each paper and, if applicable, the permissions in using them here.

Chapter 2. This paper marks the start of my project—both historically and conceptually. While the paper could potentially be submitted for publication, at the moment I'm working on the open problem of the paper and will consider submitting it for publication when that's resolved.

Chapter 3 of this thesis has been published in the journal *Synthese*, volume 201, by Springer Nature, and reproduced here with their permission. The published manuscript can be found at the following address:

> Kiani, A. (2023). Structured Propositions and a Semantics for Unrestricted, Extended Impure Logics of Ground. Synthese, (4):141, https://doi.org/10.1007/s11229-023-04114-5

Chapter 4 is currently 5 months under review by *Journal of Philosophical Logic*, and Chapter 5 is still and progress, as its open problem is currently under investigation.

Note that since these papers are independent in nature, yet standalone and self-contained, some repetitions may occur throughout the thesis.

Acknowledgments

Over the years, many people have helped me, to various degrees, in the development of one or more of the papers appearing in this thesis. I would like to particularly thank some of these people that I can recall (in alphabetical order): Andrew Bacon, David Liebesman, Gabriel Uzquiano, Juhani Yli-Vakkuri, and Richard Zach.

I would also like to thank all the anonymous referees from different journals whose commentaries have improved my papers.

Special thanks to Peter Fritz for detailed discussions of various drafts of all of these papers.

To all philosophers who didn't dedicate their dissertations to themselves—in particular Bertrand Russell.

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Chapter 1

Introduction

This dissertation comprises four standalone papers that are interdependent in nature. Together, these papers bring about a grand picture of reality according to which propositions, as well as other relational entities, are highly structured and come in a certain infinite hierarchy, and as a result of this, a wide range of puzzles and paradoxes of ground and grain can be settled.

The papers that form this dissertation contribute to several areas in analytic philosophy, by relating to the relevant literature and their specific needs and questions. Below is a breakdown of the different chapters, what each one is about and how it relates to the existing literature.

In this chapter, I first lay down some necessary background and literature review, and map out the structure of the thesis.

Propositions and their Structures

Propositions are broadly used in contemporary philosophy. They're often considered the primary bearers of truth-value, the objects of belief, knowledge, doubt, and other propositional attitudes. They're also usually considered as the semantic value of sentences—what they refer to. For example, the English sentence 'the Sky is blue' and its German equivalent, 'der Himmel ist blau', seem to say the same thing; that 'thing' is the proposition that they both refer to.

A famous, widely used and formerly popular one is the *possible-worlds* account of propositions, also known as *intensionalism* about propositions.[1] This account treats propositions as sets of possible worlds in which they are true and provides a framework for analyzing various modal claims, such as necessity, possibility, and contingency.

One of the famous problems with the possible-worlds account of propositions is its coarse-grainedness: every pair of necessarily equivalent sentences express the same proposition. For example, the proposition that 2+2=4 is the same as the proposition that Kim is either bald or not bald, as they're presumably both necessary.

Alternatives to the flat-footed possible-worlds account of propositions have been proposed in the literature to address this or various other issues: its augmentation with impossible worlds (as in, e.g., Berto and Jago, 2019), truth-makers semantics (as in, e.g., Fine, 2017c), and Russellian propositions (Kaplan, 1977; King, 2009).

According to the *Russellian* account, propositions are highly struc-

[1] For a survey of some of the accounts propositions, their advantages and flaws, see King (2019); Berto and Jago (2019).

tured, somewhat reflecting the structure and identity conditions of the sentences that express them. In recent decades, many seminal works in the philosophy of language and metaphysics have assumed or argued for Russellian propositions in various contexts, ranging from attitude operators to different kinds of metaphysical priority, such as essence, ontological dependence and grounding (see, e.g., King, 1996, 2009; Soames, 1987; Fine, 1995, 1980, 1994; Kaplan, 1977; Salmon, 1986; Kiani, 2023). This all displays the influence and popularity of Russellian propositions in recent analytic philosophy.

Despite their popularity and heavy usage in contemporary metaphysics, Russellian propositions face a serious challenge: they contradict certain principles of higher-order logic (as shown in, e.g., Hodes, 2015; Uzquiano, 2015; Goodman, 2016). This type of inconsistency goes back to Russell (1908, Appendix B) and Myhill (1958), and is known in the literature as the 'Russell-Myhill' result.

The Russell-Myhill result has influenced many of the recent works in the metaphysics of propositions. In fact, a recent trend influenced by this result has been the higher-order cousins of intensionalism under brands such as Booleanism (Bacon, 2018; Dorr, 2016) and Classicism (Bacon and Dorr, 2023). According to *Booleanism*, logically equivalent formulas (in classical propositional logic) are identical and form the same relational entities; according to its generalization, *Classicism*, the same holds for a very expressive background higher-order logic. These trends embrace the coarse-grainedness of propositions while taking advantage of the expressive power of higher-order logic in the background to define modal notions from scratch and apply them to various contexts, such as propositional

attitudes, modality and grounding (see Goodman, 2022; Fritz, 2021; Dorr et al., 2021; Williamson, 2013, for some recent developments in this area).

While we address some of these works in our thesis, our approach takes a fundamentally different path by rejecting the design of the higher-order logics that these works adopt altogether, and introducing certain restrictions to them, along the lines of ramified type theory. See **Chapters 2 and 4** for more on this.

Type Theory

Type theory is an umbrella formal framework for reasoning about the structure and behavior of mathematical objects, programs as well as various other kinds of entities. It provides a set of rules and principles for classifying and manipulating expressions based on their types. The central idea of type theory is that every term or expression in a system is assigned a type, which describes the kind of object it represents and the operations that can be performed on it.

There are several variants of type theory, each with its own features and focus. Here are some notable ones:

Simple Typed Theory (STT): This is the simplest form of type theory, where terms are classified into types, and the types ensure the validity of operations performed on terms (see Ramsey, 1926; Church, 1940; Mitchell, 1996, for some formulations of STT). STT features a single type for each term and allows the definition of functions and applications.

Due to its expressiveness compared to simpler logics, such as first-order logic, STT has also lately found its way to the philosophical literature,

where an increasing number of works in the area adopt it for philosophical training. We mentioned some of these works earlier (such as Williamson, 2013; Bacon, 2018, 2023; Dorr et al., 2021; Dorr, 2016; Bacon and Dorr, 2023; Fritz, 2016, 2021).

On several occasions in this thesis (such as in **Chapter 2.2**), we introduce STT and its features.

Ramified type theory (RTT). RTT has old roots in the history of modern logic and mathematics. In the wake of the twentieth century, a cluster of paradoxes of similar nature emerged in mathematics and logic (see, e.g., Russell, 1908, for a list and an extensive discussion of these puzzles). One of the solutions offered for these paradoxes was to ban, in systematic ways, all instances of 'impredicativity'—a project that was mainly pursued by Russell and Whitehead in terms of a ramified theory of types (Russell, 1908; Whitehead and Russell, 1910). Loosely put, this means to disallow entities that have themselves as their own 'building blocks', so in effect, imposing some sort of hierarchy on the formation of entities that controls that.

Ramified type systems have seen many variations: starting from semi-formal presentation in Whitehead and Russell (1910, 1912), all the way to Church (1976) and Hodes (2013, 2015). Throughout this dissertation, I develop and work with a close cousin of Hodes' systems, which I believe to be more expressive and suitable for philosophical work, compared to some of the other variants.

According to the RTT introduced in this thesis, relational entities, such as propositions, properties and binary relations, come in certain infinitary hierarchies of levels and quantification over entities of level-n type leads

to an entity of type, at least, $n+1$. For instance, the proposition that all level-1 propositions are either true or false is itself a level-2 proposition.

On several occasions in this thesis (such as in **Chapter 2.3**), we introduce RTT, its features and applications in metaphysics.

System F (*Polymorphic Lambda Calculus*): System F introduces polymorphism to type theory, allowing the definition and manipulation of types that are parametric over other types (see, e.g., Girard et al., 1989; Nederpelt and Geuvers, 2014; Mitchell, 1996, for some expositions of these systems). This enables the development of reusable and generic programs.

In a recent, currently under progress manuscript (Kiani, MSa), I argue that polymorphic type systems should be adopted for metaphysical theorizing due to their expressive power. A footnote in one of the book chapters discusses this in more detail (footnote 22, **Chapter 4**).

These are just a few examples of type theories, and there are many more variations and extensions within each. Type theory has applications in various fields, including programming language design, formal verification, proof theory, the foundations of mathematics, and as far as the basic systems are concerned, philosophy.[2]

Fact Grounding

Fact Grounding (hereafter, just 'grounding', unless otherwise specified)

[2]This is mostly it as far as philosophers have come in adopting type theories. There are other, much more powerful variants that are more at the cutting edge of mathematical research these days, such as Dependent Type Theory, Homotopy Type Theory, and Intuitionistic Type Theory. We won't be covering these here due to the scope of our thesis, but we project that future philosophical research will lead towards working with these type systems, though that may not happen anytime soon due to their high level of technicality.

is a more recent notion in metaphysics, often taken to be a non-causal relation that holds between certain truths or facts and certain others, somehow reflecting a sense of 'fundamentality' or 'explanation' between them (see, e.g., Rosen, 2010; Woods, 2018; Fine, 2012a; deRosset, 2013; Sider, 2011; Maurin, 2019).

Various kinds of grounding relations, and their logics, are studied in the literature. The conception of ground which takes the relata of the grounding relation to be propositions is sometimes called *representational* or *conceptual*; the *worldly* conception concerns entities such as states of affairs or situations (Correia, 2017, p. 508). The kind of logics that take into account the logical structure of the relata of grounding relations are often called *impure*; *pure* logics ignore such complexities (Fine, 2012a, p. 54).

There is another important set of distinctions between grounding relations that has been studied in the literature, and we briefly introduce here (see, e.g., Fine, 2012a, pp. 52-4 for a detailed discussion of these variations and their differecnes). To start with, a number of truths are said to *fully* ground a truth if the latter somehow holds completely in virtue of the former and nothing else; a truth *partially* grounds another if it does so fully, standalone or together with other truths.

Another distinction is between mediate and intermediate grounds. Grounds of a truth are *immediate* if there's no mediating truth between them and what they ground; otherwise, they constitute *mediate* grounds, as if there's a 'chain' of immediate grounds involved. Finally, some truths are *strict* grounds of some others if they are, in a sense, more 'fundamental' or 'basic'; otherwise, the grounds are *weak*. Put in terms of explanation,

we can think of strict ground as, in the words of Fine (2012a), as one that "takes us down in the explanatory hierarchy," whereas weak grounds "may also move us sideways in the explanatory hierarchy" (*ibid*, p. 52). Finally, the conception of ground that allows *any* proposition, regardless of its truth value, as the relata of ground is called *non-factive*; the *factive* variant only works with truths, i.e., true propositions.

Now, while the semantics of pure logics of ground has been well studied and somewhat settled (see, e.g., Fine, 2012b), impure logics, and in particular, their representational variants, remain fairly underexplored, with only a few recent attempts on offer to semantically account for them (Correia, 2017; Krämer, 2018; deRosset and Fine, 2023). But, even though these works mark considerable progress in the study of the impure logics of ground, all these semantic accounts suffer from certain expressiveness limits, complying with the restricted languages or logics that they're supposed to capture.

In **Chapter 3**, I engage with this problem: I leverage Russellian propositions to provide a very expressive and unified semantics for a wide range of propositional logics of ground.

While this serves the propositional logics of ground at a large scale, quantificational principles of ground face their own challenges. For example, Fine (2010) and Krämer (2013) have put forward some puzzles regarding the interaction of some impeccable principles of classical logic with certain plausible principles of mediate partial ground. Other neighboring puzzles, in richer languages and settings, have also emerged in the literature that display similar behavior to Fine's and Krämer's puzzle (Donaldson, 2017).

Now, while there are a few proposals in the literature that are curated

for one or more of these puzzles, they mostly fail or lose relevance when applied to the other variants of these puzzles (see, e.g., Woods, 2018; Krämer, 2013; Fritz, 2020).

In **Chapter 5**, I discuss the quantificational puzzles of ground in the literature, and some new ones, and propose a novel, unified solution to them by means of deploying ramified typed systems in the background. Essentially, I show that all these puzzles are similar in nature, and a hierarchy of relational entities (propositions, relations, etc.) reminiscent of ramified type theory can circumvent the paradoxes. I also argue that this type of resolution for the puzzles of quantificational ground is more unified and powerful than most alternative proposals cited above.

Entity Grounding

The literature on metaphysical ground often conceives the relation of grounding as only concerning facts or fact-like entities that hold 'in virtue of' other such entities, manifesting the idea that the latter 'explain' or are 'more fundamental' than the former. However, a few exceptions to this tradition stand out, according to which entities of all kinds, such as individuals, propositions, facts, properties and relations, are capable of entering into grounding relations (as in, e.g., Schaffer, 2009; Wilhelm, 2020a; deRosset, 2013)—what is sometimes called 'entity grounding' (Wilhelm, 2020a).

Entity grounding is a relation of metaphysical priority that can hold between entities of any type. An individual may e-ground a proposition or fact, a proposition may e-ground a property, a property may e-ground a

relation or a proposition, and so on. To illustrate with examples along the lines of the literature: '[for any entity i,] $i = i$ is grounded in i' (Wilhelm, 2020a), 'Obama, the man in full, grounds the fact that Obama exists; Obama grounds his singleton; the property *being white* grounds *being white or square*; England grounds (in part) the property of *being queen of England*' (deRosset, 2013).

In **Chapter 4**, I lay down and defend certain plausible principles of entity grounding, along the lines of what's been explicitly or implicitly entertained in the literature, and argue that they require propositions, properties and other types of relations each to come in infinite levels, where, roughly put, the inhabitants of higher levels are obtained through quantification over the ones from lower levels. I then rigorously propose certain ramified type systems that best capture the talk of entity grounding and the infinitary hierarchies it calls for. The ultimate goal of this paper is to argue that certain considerations regarding entity grounding and structure call for infinitary hierarchies of relational entities such as propositions and properties and to rigorously devise a ramified type system that captures them.

In **Chapter 6**, I report on the project's accomplishments, and reiterate some of its main open problems and our predictions about them.

Chapter 2

Ramified Types and Metaphysical Structure

2.1 Introduction

According to the *structured* or *Russellian* picture of propositions (as presented in, e.g., Soames, 1987; King, 2009, 1996, 2019) propositions are structured entities that are built out of constituents such as individuals, properties, and operators, somewhat resembling and reflecting the overall syntactic structure of the sentences that express them.[1]

Structured propositions remain one of the main alternatives to the possible-worlds accounts of propositions, avoiding oddities that are caused due to the coarse-grainedness of possible worlds. They have also found their way into a large pool of seminal work in the philosophy of language—as in attitude operators—and metaphysics—as in the notions of essence

[1]Hereafter, I will use 'structured propositions' in the sense above, i.e., Russellian propositions. This notion will be rigorously defined later in the paper.

and ontological dependence (see, e.g., King, 1996, 2009; Soames, 1987; Fine, 1995, 1980, 1994; Kaplan, 1977; Salmon, 1986, as some of these works).

Recently there has been a renewed interest in questions of structure in metaphysics. It has been increasingly argued in different forms and shapes that flat-footed accounts of structured propositions face what is often called the Russell-Myhill Paradox (Deutsch, 2008; Dorr, 2016; Goodman, 2016; Hodes, 2015; Uzquiano, 2015), originally going back to the Appendix B of Russell (1908) and rediscovered in Myhill (1958).

This paper appeals to ramified type theory as a framework to put the assumptions of structure on a sound footing. More specifically, it will be argued that ramified type systems might have a chance of safeguarding assumptions of structure from the Russell-Myhill paradox.

The paper is one of a series of four in which I set out to explore a deep interconnection between the theory of structured propositions, the theory of metaphysical ground (fact-grounding and entity-grounding) and ramified type theory. The ultimate goal of the series is to show that these three together bring about a unified, elegant picture of reality within which a host of contemporary puzzles and paradoxes are resolved, and rejecting either of them will make the picture collapse in its entirety.

The task of this paper, in particular, is to show how ramified type theory can secure the notion of structured propositions from the Russell-Myhill paradox. The other three papers will explore how ramified type theory can settle some of the recent puzzles of the theory of fact-grounding (Kiani, MSe, i.e., Chapter 5 of this thesis), how it can be motivated by considerations of entity-grounding (Kiani, MSb, i.e., Chapter 4), and how

structured propositions can provide a powerful and unified semantics for a range of propositional logics of ground (Kiani, 2023, i.e., Chapter 3).

Here's how the paper is organized: In §2.2 I will lay down a minimal formal background on the syntax and the semantics of higher-order logic based on simple type theory. I will then briefly introduce the structured account of propositions and will give a diagnosis of the Russell-Myhill paradox in terms of the so-called 'impredicativity' of simple type theory. In §2.3 I will introduce a ramified type theory that is motivated by certain considerations of metaphysical priority in Kiani (MSb), and argue that it blocks the Russell-Myhill paradox. I will also prove the consistency of the system in question by imposing a ramified structure on the class of standard models of higher-order logic. In §2.4 I will explore the impact of our ramified type system in the presence of the structured account of propositions. The paper concludes in §2.5.

2.2 Simple Type Theory

To frame the problem of structured propositions with sufficient rigor, allow me to introduce a simple type theory (STT). In this section, I will lay down a minimal background on simple type theory and an assessment of the Russell-Myhill paradox.

Introduction

Simple types provide a way of tracking the grammatical categories of expressions. They are defined rigorously as follows:[2]

Definition 2.2.1 (Simple Types)**.** The set $\mathcal{T}^s$ of *simple types* is recursively defined as follows: (i) $e \in \mathcal{T}^s$, $\langle\rangle \in \mathcal{T}^s$, and (ii) for any types $t_1, ..., t_n$ (where $n \geq 1$), $\langle t_1, ..., t_n\rangle \in \mathcal{T}^s$.[3]

Before defining terms of the system, we assume that for any $t \in \mathcal{T}^s$ there's a denumerably infinite set of *variables* Var^t of type t and a (possibly empty) set of typed non-logical *constants* CST^t. For certain types there are also logical constants to be introduced below. (We will reserve CST^t for the set of all constants (logical or non-logical) of type t.) We define the sets of all variables and constants respectively as $\mathrm{Var} := \bigcup_{t \in \mathcal{T}^r} \mathrm{Var}^t$ and $\mathrm{CST} := \bigcup_{t \in \mathcal{T}^r} \mathrm{CST}^t$.

Treating the logical vocabulary as constants is the prevalent approach in higher-order logic (see, for example Church, 1940; Henkin, 1950; Mitchell, 1996; Bacon, 2018, 2019; Dorr, 2016; Dorr et al., 2021). Not only is it more elegant than giving them separate clauses in the definition of terms (e.g., "if p and q are terms of type $\langle\rangle$, then $p \wedge q$ is a term of type $\langle\rangle$"; see Hodes, 2013, as an example), but, as we will see shortly, it also has the metaphysical advantage of allowing us to intelligibly ask certain

[2]The type theories presented in this paper will be *relational* (as opposed to functional). Also, for higher readability, the style of typing will by *Church-typing* (as opposed to Curry-typing), where the types of variables are fixed and don't depend on 'contexts'. Alternative formulations are possible as well.

[3]In the literature, the second clause is sometimes more concisely expressed as this: for any types $t_1, ..., t_n$ (where $n \geq 0$), $\langle t_1, ..., t_n\rangle \in \mathcal{T}^s$. In this presentation, $\langle t_1, ..., t_n\rangle$ for $n = 0$ is just $\langle\rangle$.

questions and theorize about the granularity of the logical connectives and quantifiers—an option that is not available to the rival approach.

In any case, here's the list of our primitive, typed logical constants: negation, $\neg$, of type $\langle\langle\rangle\rangle$, implication, $\rightarrow$, of type $\langle\langle\rangle,\langle\rangle\rangle$, and for any type t, there is a constant for a (higher-order) universal quantifier $\forall^t$, which quantifies over properties of entities of type t; that is, $\forall^t$ is given the type $\langle\langle t\rangle\rangle$. After we introduce the set of terms of STT, we will see how the quantifiers function with the given type, and how other connectives will be defined in terms of the primitive constants above.

Definition 2.2.2 (Simple Terms)**.** The *terms* of STT are recursively defined as follows: **(i)** if x is a variable of type t, then x is a term of type t**; (ii)** if c is a constant of type t, then c is a term of type t**; (iii)** if ϕ is a terms of type $\langle\rangle$ and for $n \geq 1$, the variables $x_1, ..., x_n$ are pairwise distinct, and respectively of types $t_1, ... , t_n$, then $\lambda x_1^{t_1}, ..., x_n^{t_n}.\phi$ is a term of type $\langle t_1, ..., t_n\rangle$ **; (iv)** if τ is a term of type $\langle t_1, ..., t_n\rangle$, where $n \geq 1$, and for each $i = 1, ..., n$, σ_i is a term of type t_i, then $\tau(\sigma_1, ..., \sigma_n)$ is a term of type $\langle\rangle$.

We call a term of type $\langle\rangle$ a *formula*, and when it contains no free variables, a *sentence*. We use the letter t with or without subscripts as metavariables for types, lower-case Greek letters $\tau, \sigma, \phi, \psi, ...$ with or without subscripts as metavariables for general terms, and lower-case or capital English letters $x, y, z, p, q, X, Y, Z, P, Q$, with or without subscripts, as metavariables for variables; other letters may be used as metavariables for constants as well. Also, from now on, by convention, we write things like $\phi \vee \psi$ or $x = y$ to indicate the application instances $\vee(\phi, \psi)$ or $=(x, y)$, and so on. The notions of *free* and *bound* variables of terms, substitutions

of terms for variables, and *being free for a variable*, are defined as usual. We denote the set of free variables in a term σ by $FV(\sigma)$. Also, the set of all terms of STT is denoted by TERM_s.

Some examples: assuming that N^e and $E^{\langle e \rangle}$ are constants respectively standing for Napoleon and the property of being an emperor, the sentence 'Napoleon is an Emperor' is regimented in our language as $E(N)$, which is an instance of the rule (iv), also known as *application*. Now, suppose we have the sentence 'Napoleon is a young emperor' regimented with $Y(N) \wedge E(N)$, where Y^e is the constant standing for the property of being young. Now, the predicate that stands for the property of being a young emperor can be obtained by 'abstracting away' (using clause (iii), also called *abstraction*) from 'Napoleon', as $\lambda x^e. Y(x) \wedge E(x)$.

Finally, we understand the good old universal statements of the form $\forall x^t \phi$ as instances of term application $(\forall^t)(\lambda x^t.\phi)$; that is, by retyping the universal quantifier, we can understand universal claims in terms of abstraction and application, which is what we would need, when treating the quantifier as a constant.

Now we define a set of other connectives in terms of the primitive

constants $\neg$, $\rightarrow$ and $\forall^t$ and lambda-abstraction, as follows:[4,5]

$$\vee := \lambda p^{\langle\rangle} q^{\langle\rangle}.(\neg p \rightarrow q) \tag{2.1}$$

$$\leftrightarrow := \lambda p^{\langle\rangle} q^{\langle\rangle}.(p \rightarrow q) \wedge (q \rightarrow p) \tag{2.2}$$

$$\exists^t := \lambda X^{\langle t\rangle} \neg \forall^t x \neg X(x) \tag{2.3}$$

$$\wedge := \lambda p^{\langle\rangle} q^{\langle\rangle}.\neg(p \rightarrow \neg q) \tag{2.4}$$

$$=^t := \lambda x^t y^t.\forall^{\langle t\rangle} X(X(x) \rightarrow X(y)) \tag{2.5}$$

Notice that we're *not* taking the interdefined connectives above as the definitions of or shorthands for, say, the connectives disjunction, conjunction, etc.—although at the end of the day, they will produce propositions that are truth-functionally equivalent with the Boolean connectives, we would like to remain as neutral as possible regarding the status of the granularity of the latter conncetives and quantifiers. For instance, we would like to leave open the status of identities such as $\lambda p.\neg\neg\neg p =^{\langle\langle\rangle\rangle} \lambda p.\neg p$ or identifying disjunction, say, with $\lambda pq.\neg(\neg p \wedge \neg q)$ or $\lambda pq.(\neg p \rightarrow q)$, and intelligibly theorize about them.[6]

Our system is also relatively grain-neutral in another sense: for example, it does not guarantee identities like $\phi \vee \psi = \neg\phi \rightarrow \psi$; for that, we need a

[4]See for example Dorr et al. (2021) and Bacon (2023) for such formulations.

[5]Notice that, strictly speaking, in STT we don't need to consider a separate constant for negation; we could define it as $\neg := \lambda p^{\langle\rangle}.(p \rightarrow \bot)$, where $\bot$, the falsume, is itself defined as, e.g., $(\forall^{\langle\rangle})(\lambda p^{\langle\rangle}.p)$—the higher-order parallel of the proposition that every proposition is true. The reason that here we took $\neg$ as a primitive logical constant instead of taking the alternative approach just glossed is that in ramified type system that we will be developing in §3.1 the latter may involve some extra delicacies regarding $\bot$ that can be avoided if we just take negation as a constant. See footnote 17 for more on this.

[6]It's also remarkable that asking such questions is not an option that is available to the alternative path where the classic connectives are given separate clauses in the term-construction rules—we are able to ask such questions or otherwise theorize about the related identities only when we treat the connectives and quantifiers $\neg$, $\vee$ and $\forall^t$ as constants. This is one of the advantages of treating the logical vocabulary as constants, rather than introducing them via term-construction rules.

principle stronger than β_E, where the equivalence involved is replaced with an identity $=^{\langle\rangle}$. (We still have the very plausible equivalence $\phi\vee\psi \leftrightarrow \neg\phi\rightarrow\psi$, though.) Neither do identities like $\phi = (\phi\wedge\psi)\vee\phi$ which are imposed upon us by certain coarse-grained accounts of propositions necessarily hold.[7] The system, however, is *not* absolutely grain-neutral: as the following section will show, fine-grained assumptions about the structure of propositions will fail to hold.

Proof Theory (System $\mathcal{H}^-$)

In what follows, and as is expected, $[\sigma_i/x_i]$ stands for the simultaneous substitution of σ_i's with x's in τ. Also, in each case, it's been assumed that the substitutants are *free for* the substitutees. Intuitively, that guarantees that (i) no bound variable is allowed to be substituted (that is, the notion of substitution only applies to free variables), and (ii) no free variable can get bound after substitution.

Axioms:

1. $\vdash \phi \rightarrow (\psi\rightarrow\phi)$; $\vdash (\phi \rightarrow (\psi\rightarrow\gamma)) \rightarrow ((\phi\rightarrow\psi) \rightarrow (\phi\rightarrow\gamma))$;
   $\vdash (\neg\phi \rightarrow \neg\psi) \rightarrow (\psi\rightarrow\phi)$. PC[8]

2. $\vdash (\lambda x_1^{t_1}, ..., x_n^{t_n}.\phi)(\sigma_1, ..., \sigma_n) \leftrightarrow [\sigma_1/x_1, ..., \sigma_n/x_n]\phi$, where the type of σ_i is t_i, for each $i = 1, ..., n$. β_E

3. $\vdash \forall^t F\rightarrow F(\sigma)$, where F is a term of type $\langle t\rangle$, and σ or type t. UI

[7] See Bacon (2018) as an example of such a view.

[8] This choice of our axioms for propositional logic, which corresponds to the system P2 found in Church (1956), is more suitable for our purposes, given our choice of primitive Boolean connectives, namely, $\neg$ and $\rightarrow$.

4. $\vdash \forall^t(\lambda x^t.\ \phi{\to}\psi) \to (\phi \to \forall^t(\lambda x^t.\psi))$, where $x \notin FV(\phi)$ UD[9]

Rules of Inference:

5. If $\vdash \phi$ and $\vdash \phi{\to}\psi$, then $\vdash \psi$. MP

6. If $\vdash F(x)$, then $\vdash \forall^t(F)$, where x is of type t, and F is a term of type $\langle t \rangle$. GEN

The system $\mathcal{H}^-$ is a natural generalization of the proof system of first-order logic to higher types.[10] The new principle β_E is also an extremely plausible principle that gives us equivalences such as this: Napoleon was a French emperor iff Napoleon was French and Napoleon was an emperor—formally: $(\lambda x^e.F(x){\wedge}E(x))(N) \leftrightarrow F(N){\wedge}E(N)$.

Consistency of $\mathcal{H}^-$

Now I will introduce Henkin models, originally coming from Henkin (1950). Henkin's underlying type theory is functional. So, we'll have to introduce a relation version for our purposes. In what follows, I will propose a revised version of Gallin (1975)'s models which is based on a relational type theory.[11]

[9] One can replace this with a more general rule: $\vdash \forall^t(\lambda x^t.\phi \to F(x)) \to (\phi \to \forall^t F)$, (where F is of type $\langle t \rangle$ and $x \notin FV(\phi)$). This is, in effect, stronger than $\vdash \forall^t(\lambda x^t.\ \phi{\to}\psi) \to (\phi \to \forall^t(\lambda x^t.\psi))$. But in what follows I'll stick to the original formulation.

[10] Note that in UI and GEN, choosing F to be $\lambda x^t.\phi$ will lead to the more common, less general formulations of these principles: UI becomes "$\forall x^t\, \phi \to \phi(\sigma)$", and GEN becomes "If $\vdash \phi$, then $\vdash \forall x^t\, \phi$."

[11] Gallin (1975)'s systems doesn't have λ-abstaction, and he instead appeals to the axiom schema of comprehension. I will stick to our λ-calculus and, similar to Henkin (1950); Mitchell (1996) and others, do with λ-terms. Another difference with Gallin (1975) is that here we're taking the logical vocabulary as constants. See what follows after Definition 1 and before Definition 2, for an overview of this approach.

Definition 2.2.3 (Frames)**.** Let D be a non-empty set. A *frame* based on D is a set $F := \{M_t \,|\, t \in \mathcal{T}^s\}$ such that:

(i) $M_e := D$,

(ii) $\varnothing \neq M_{\langle\rangle} \subseteq \wp(\{\varnothing\}) = \{0, 1\}$,

(iii) $\varnothing \neq M_{\langle t_1,\ldots,t_n\rangle} \subseteq \wp(M_{t_1} \times \ldots \times M_{t_n})$, where $n \geq 1$.

Definition 2.2.4 (Henkin Models)**.** A *Henkin model* is a pair $M = (F, d)$, where F is a frame and d is a function $d : \bigcup_t \mathrm{CST}_t \to \bigcup_t M_t$ such that $m(c^t) \in M_t$. A model is *logical* if:

(i) $d(\neg) := \{0\}$,

(ii) $d(\rightarrow) = \{(0, 0), (0, 1), (1, 1)\}$.

(iv) $d(\forall^t) := \{M_t\}$.

An assignment function for a model is a function $g : \bigcup_t Var_t \rightarrow \bigcup_t M_t$ such that $g(x^t) \in M_t$ for each $x^t \in \mathrm{Var}_t$.

Definition 2.2.5 (Interpretations)**.** An *interpretation* for an assignment g based on a model M is a function $[\![.]\!]_g : \mathrm{TERM}_s \to \bigcup_t M_t$, such that:

(i) $[\![c^t]\!]_g = d(c^t)$, for $c^t \in \mathrm{CST}_t$,

(ii) $[\![x^t]\!]_g = g(x^t)$, for $c^t \in \mathrm{Var}_t$,

(iii) $[\![\lambda x_1^{t_1}, \ldots, x_n^{t_n}.\phi]\!]_g :=$ the set $X \subseteq M_{t_1} \times \ldots \times M_{t_n}$, such that $(d_1, \ldots, d_n) \in X$ iff $[\![\phi]\!]_{g[x_1 \mapsto d_1, \ldots, x_n \mapsto d_n]} = 1$,

(iv) $[\![\tau(\sigma_1, \ldots, \sigma_n)]\!]_g = 1$ iff $([\![\sigma_1]\!]_g, \ldots, [\![\sigma_n]\!]_g) \in [\![\tau]\!]_g$.

Notice that in our Henkin models it's not always granted to have the interpretation of terms belong to their intended domains. For example, there's nothing in our definitions that would reassure us that $[\![\lambda x_1^{t_1}, ..., x_n^{t_n}.\phi]\!]_g$ belongs to $M_{\langle t_1,...,t_n\rangle}$, which itself could be a proper subset of $\wp(M_{t_1} \times ... \times M_{t_n})$. To avoid this uncertainty, we therefore only work with the class of Henkin models that do grant the desired membership. We say that a Henkin model is *closed* if for every assignment g and every term σ of type t, we have $[\![\sigma]\!] \in M_t$.

For a model M, an assignment g based on M, and a sentence ϕ, we say that M *satisfies* ϕ *for the assignment* g, and write $M, g \vDash \phi$, if $[\![\phi]\!]_g = 1$. Notions of validity, etc., are defined as usual.

Theorem 2.2.1 (Soundness)**.** *The proof system* $\mathcal{H}^{-}$ *is sound with respect to the class of all closed Henkin models. That is, for every sentence* $\phi \in$ *TERM*$_s$*, we have: if* $\vdash \phi$*, then* $\vDash \phi$*.*

Proof. Easily verified. Also, see Henkin (1950); Mitchell (1996); Gallin (1975) for different variants of the proof system presented in different variants of the base type theory. □

Remark 2.2.1. Notice that $\mathcal{H}^{-}$ is *not* complete with respect to the class of Henkin Models. For example, we can show that for every model and every assignment function g, $\forall p^{\langle\rangle}\forall q^{\langle\rangle}(p \leftrightarrow q \rightarrow \forall F^{\langle\langle\rangle\rangle}(F(p) \leftrightarrow F(q)))$ is true. But this is not a theorem of the system $\mathcal{H}^{-}$. To derive the latter in our system, we will need to add the above sentence as a new axiom to our system—sometimes called the axiom of extensionality (Henkin, 1950) or functionality (Bacon, 2018). For reasons of granularity, however, we won't

adopt Extensionality.[12]

Remark 2.2.2. Following the previous remark, and since we're only considering the soundness of the system for our purposes, we could more conveniently just consider the class of *standard*, instead of Henkin models: the former replaces $\subseteq$ with $=$ in the clauses (ii) and (iii) of the definition of frames. In fact, for reasons similar to the ones from the previous remark, we will intentionally avoid the completeness of the proof system for RTT, and due to a complication regarding proving even the soundness of that system by using (ramified) Henkin models, there we'll just appeal to (ramified) standard models.

Structured Propositions and the Russell-Myhill Paradox

Now we are in good shape to discuss the Russell-Myhill paradox. The general principle that structured accounts of propositions commit to is along the lines of the following (simplified) schema, called Structure:

$$F(a) =^{\langle\rangle} G(b) \rightarrow F =^{\langle t\rangle} G \wedge a =^{t} b \tag{2.6}$$

In effect, it says that from a proposition expressed by a predication $F(a)$ we can always recover a property and an entity eligible for that property such that the former is uniquely attributed to the latter by the proposition.

Now, assuming ψ is a term of type $\langle\rangle$, then, by Definition 1, the term:

$$G := \lambda q^{\langle\rangle}.\exists X^{\langle\langle\rangle\rangle}((q =^{\langle\rangle} X(\psi)) \wedge \neg X(q)), \tag{2.7}$$

[12]In effect, this will go against the principle Structure which plays a canonical rule in this paper.

is well-formed in STT and is of type $\langle\langle\rangle\rangle$. Roughly, G stands for the property of being a proposition that lacks a property which it attributes to some proposition. By using β_E twice we can see that for any ψ of type $\langle\rangle$, $G(G(\psi))$ is well-formed and is equivalent to the following:

$$\exists X^{\langle\langle\rangle\rangle}(G(\psi)=^{\langle\rangle}X(\psi) \wedge \neg X(G(\psi))) \tag{2.8}$$

Now, in a broad-brush take, the known proofs of the paradox all assume Structure and then prove $G(G(\psi)) \leftrightarrow \neg G(G(\psi))$, which is a contradiction, hence the paradox. The proofs in particular crucially rely on the schemata UI and β_E to go through. Accordingly, a general strategy to avoid the paradox is to somehow weaken or perhaps even drop one or more of β_E, UI or Structure (or any of its consequences). A large portion of the recent literature on the resolutions of the Russell-Myhill paradox has especially shown interest in either rejecting or weakening Structure and leaving the other axioms of the system intact (Dorr, 2016; Goodman, 2016; Bacon, 2019; Fritz, 2019).

These resolutions, however, are all squarely committed to *simple* types. One could, however, take any of the paths above under a different picture of types, namely *ramified* types. To understand what ramified types are we need to first understand the motivation behind them. To start with, notice that in the presence of simple types it's possible to define predicates that pick out properties of, say, individuals, by 'assuming' or 'presupposing' all properties of individuals, which include the former too—something like a circular definition (a 'vicious circle'), also called *impredicative*. For example, the property of having all the properties that make a great general (i.e., all properties of type $\langle e\rangle$ which are had by great generals) is

itself a property of individuals (i.e., of type $\langle e \rangle$), and so is automatically 'assumed' by quantifying over *all* properties of individuals, in its definition. Similarly, and more relevant to the Russell-Myhill Paradox, G belongs to the properties that it assumes, or quantifies over, in its definition, i.e., properties of type $\langle\langle\rangle\rangle$.

Having closely examined a cluster of paradoxes of a similar nature (usually known as 'paradoxes of impredicativity'), Russell proposed ramified type theory as a means to block vicious circles all at once, as a unified solution to all the relevant paradoxes (Russell, 1908; Whitehead and Russell, 1912). The idea can be implemented in various ways, but the main thought is to manipulate the background type theory by assigning 'levels' to simple types and requiring the quantifiers to *raise* those levels in systematic ways: for example, if p is a sentential variable of level n, then $\exists p\, p$ stands for a proposition of level $n{+}1$. Moreover, properties of propositions may only apply to a range of propositions restricted to certain levels, not absolutely all of them. As a result, in the presence of ramified types, and because of the quantifier present in the definition of G, the proposition picked out by $G(\psi)$ in $G(G(\psi))$ will be of a level that is *strictly higher* than the level of propositions that G can possibly apply to, hence the application instance $G(G(\psi))$ will be ill-formed, and the proofs of the inconsistency won't go through anymore.[13]

The next section explores this primitive idea in more detail, by introducing a modern, rigorous reconstruction of Russell's implicit ideas of ramified types.

[13] As we will see in §4, and as Bacon et al. (2016) have also observed, this will essentially lead to a weakening of UI, only through manipulating the background type theory.

2.3 Ramified Type Theory

I will now lay out a formalization of the idea of type stratification based on the system presented in Church (1976), but also heavily influenced by the systems in Hodes (2013; 2015), where a λ-abstractor is at work instead of a comprehension schema. A version of this system can be found in Kiani (MSb), where it is argued that such ramified systems can be motivated by a recently emerged notion of metaphysical priority, namely entity-grounding.

Besides the conformity with our STT from the previous section, one advantage of employing λ-abstractors is that we get to construct the functions in our system (as opposed to merely claiming their existence), and carefully monitor their behavior and interaction with variables and the logical vocabulary and other terms—much like what is common in all modern type systems. Our system also has expressiveness advantages especially over Hodes's systems (See footnotes 14 and 16 for this).

Introduction

As in the case of STT, we start off our ramified type theory (henceforth: RTT) by introducing types. This is where all the difference begins: we will no longer have types of propositions or propositional functions *per se*, but 'leveled' such entities: each simple type will be assigned a level, somehow locating the entities of that type in the type hierarchy. We will see how term formation rules, especially quantification, interact with these levels in order to allow or disallow the formation of certain terms that proved troubling in the exposition of the Russell-Myhill paradox.

Definition 2.3.1 (Ramified Types)**.** The set $\mathcal{T}^r$ of *ramified types* is recursively defined as follows: $e \in \mathcal{T}^r$, $\langle\rangle/\!\!/m \in \mathcal{T}^r$ and $\langle t_1, ..., t_n\rangle/\!\!/m \in \mathcal{T}^r$, for $t_1, ..., t_n \in \mathcal{T}^r$, $(n \geq 1)$, for *any* $m \geq 1$.[14]

Definition 2.3.2 (Levels)**.** The *levels* of ramified types t, $\mathrm{l}(t)$, are defined as follows: $\mathrm{l}(e) = 0$, $\mathrm{l}(\langle\rangle/\!\!/m) = m$, and $\mathrm{l}(\langle t_1, ..., t_n\rangle/\!\!/m) = m$. A type $\langle t_1, ..., t_n\rangle/k$ is *directly lower* than a type $\langle t_1, ..., t_n\rangle/\!\!/m$ iff $k < m$.

In effect, e is the type of individuals, $\langle\rangle/\!\!/m$ is the type of level-m propositions, and for any types $t_1, ..., t_n$, where $n \geq 1$, $\langle t_1, ..., t_n\rangle/\!\!/m$ is the type of n-ary propositional functions of level m—functions that, as the term-formation rules below show, take arguments of types identical or directly lower than $t_1, ..., t_n$, and return a level-m proposition.[15] As a convention, from now on whenever we talk about the level of a term, we mean the level of the type that is assigned to it. For example, we may call a proposition ϕ of type $\langle\rangle/3$, a level-3 entity, and may represent this with $\mathrm{l}(\phi) = 3$.

[14]In the definition of ramified types, Hodes (Hodes (2013, 2015)), as opposed to Church (1976), adds constraints such as $m > \max\{\mathrm{l}(t_i)\}$ or $m \geq \max\{\mathrm{l}(t_i)\}$ in his systems (although our 'levels' conform to his 'heights'), but such constraints only impose expressive limits on the system which could otherwise be avoided. In particular, by dropping such constraints we can retain some forms of application and iteration that are not available in Hodes's systems: for example, if ϕ is a term of type $\langle\rangle/2$ and τ is a term of type $\langle\langle\rangle/2\rangle/1$ (the latter is allowed in our system, but not by any of Hodes' systems), then we are allowed to form $\tau(\tau(\phi))$, which would be of type $\langle\rangle/1$. Or suppose τ is a term of type $\langle\langle\rangle/m\rangle/m$ and ϕ is a term of type $\langle\rangle/m$. Then $\tau(\phi)$ is of type $\langle\rangle/m$ and the iterations $\tau(\tau(\phi))$, $\tau(\tau(\tau(\phi)))$, etc., are well-formed and of type $\langle\rangle/m$. But nothing like these are available for Hodes's System $\Rightarrow^r$, and although his System $\Rightarrow^{nr}$ can express the latter iterations, it still cannot express the first instance of application; so in that sense both of the systems $\Rightarrow^r$ and $\Rightarrow^{nr}$ are less expressive than our system.

[15]Notice that, at least formally speaking, we have the option of stratifying the individual type i too, as do Bacon et al. (2016). Here we don't have any philosophical applications for such a move, so we keep our system simpler without stratifying individual-type entities.

As before, for any ramified type $t \in \mathcal{T}^r$ we assume there's a denumerably infinite set of *variables* Var^t of type t and a (possibly empty) set of typed non-logical *constants* CST^t. For certain types there are also logical constants to be introduced below. (We will reserve CST^t for the set of all constants (logical or non-logical) of type t.) We define the sets of all variables and constants respectively as $\mathrm{Var} := \bigcup_{t \in \mathcal{T}^r} \mathrm{Var}^t$ and $\mathrm{CST} := \bigcup_{t \in \mathcal{T}^r} \mathrm{CST}^t$.

Also, for similar reasons about elegance and granularity that were mentioned in the case of STT, we assign typed constants to our logical vocabulary, instead of proposing separate clauses for them. In particular, and as before, we choose negation, implication, and universal quantification as our primitive constants, but now they need to be tailored to accommodate the talk of levels, and the intuition behind ramified-quantification: for example, we want a statement of the form $\forall p^{\langle\rangle/1} p$, which quantifies over level-1 propositions, to be of a level 2. We also want the quantifier to be treated as before, where higher-order quantification is understood in terms of application and abstraction.

To illustrate in particular how the constant for ramified higher-order quantification should be typed, suppose, instead of treating the quantifier as constant and defining universally quantified statements in terms of application and abstraction, we, much like Hodes (2013; 2015) wanted to give a separate clause for the universal quantifier in the term-formation rules. A suitable clause that would've satisfied our ramified picture is as follows: if x is a variable of type t and ϕ is a term of type $\langle\rangle/m$, then $\forall x^t \phi$ is a term of type $\langle\rangle/\max\{l(t)+1, m\}$, where $\max\{...\}$ is the function that picks out the maximum of the numbers in its scope. So, the corresponding

constant for each level m, denoted by $\forall^t_m$, should (a) apply to terms of the form $\lambda x^t.\phi$, and (b) return propositions of type $\langle\rangle/\max\{l(t)+1, m\}$. Given the working of λ in the definition below, it's easy to verify that the suitable type for this constant would therefore be $\langle\langle t\rangle/m\rangle/\max\{\mathrm{l}(t)+1, m\}$.

As for what the output level of the other connectives should be, first notice that the only means to raise levels in our RTT is via quantification. So, other operators can at most bring out the biggest levels in their arguments. With these in mind, we choose our typed, logical constants in RTT, thus: $\neg_m$ is of type $\langle\langle\rangle/m\rangle/m$; $\rightarrow_{m_1,m_2}$ of type $\langle\langle\rangle/m_1, \langle\rangle/m_2\rangle/\max\{m_1, m_2\}$; and, to repeat, for any ramified type t, $\forall^t_m$ is of type $\langle\langle t\rangle/m\rangle/\max\{\mathrm{l}(t)+1, m\}$.

Definition 2.3.3 (Ramified Terms)**.** The set of *terms* of type t are recursively defined as follows: **(i)** If $x \in \mathrm{Var}^t$, then x is a term of type t; **(ii)** if $c \in \mathrm{CST}^t$, then c is a term of type t; **(iii)** if $x_1, ..., x_n$ are pairwise distinct variables of respectively types $t_1, ..., t_n$, where $n \geq 1$, and ϕ is a term of type $\langle\rangle/m$, then $\lambda x_1^{t_1}, ..., x_n^{t_n}.\phi$ is a term of type $\langle t_1, ..., t_n\rangle/m$; **(iv)** if τ is a term of type $\langle t_1, ..., t_n\rangle/m$, where $n \geq 1$, and for each $i = 1, ..., n$, τ_i is a term of type identical to or directly lower than t_i, then $\tau(\tau_1, ..., \tau_n)$ is a term of type $\langle\rangle/m$.

The notions of *free* and *bound* variables of terms, *substitutions* of terms for variables, and *being free for a variable*, are defined as usual. We show the set of free variables in a term σ by $FV(\sigma)$. Also the set of all terms of RTT is denoted by TERM_r.

The feature of application rule above (iv) that allows for applying propositional functions to arguments of directly lower level is called the

cumulativity condition by Church (1976), which adds to the expressive power of our system.[16]

We adopt the previous conventions about metavariables for variables, terms, and types, except that now we may attach levels to them. Also, as expected, we understand the old-style $\forall x^t \phi$ as a shorthand for the application instance $(\forall^t_m)(\lambda x^t.\phi)$, where m is the level of the proposition ϕ. Finally, as before, we define a set of other connectives in terms of the existing ones in the familiar fashion and as follows, with remaining loyal to the previous caution that we don't mean to take them as the definitions of or abbreviations for disjunction, conjunction, etc. (Below, the types of the defining connectives are omitted for higher readability):

$$\vee_{m_1,m_2} := \lambda p^{\langle\rangle/m_1} q^{\langle\rangle/m_2}.(\neg p \to q) \tag{2.9}$$

$$\leftrightarrow_{m_1,m_2} := \lambda p^{\langle\rangle/m_1} q^{\langle\rangle/m_2}.(p \to q) \wedge (q \to p) \tag{2.10}$$

$$\wedge_{m_1,m_2} := \lambda p^{\langle\rangle/m_1} q^{\langle\rangle/m_2}.\neg(p \to \neg q) \tag{2.11}$$

$$\exists^t_m := \lambda X^{\langle t\rangle/m} \neg \forall^t_m x \neg X(x) \tag{2.12}$$

Notice that all the connectives above are of the same type as $\to_{m_1,m_2}$, namely, $\langle\langle\rangle/m_1, \langle\rangle/m_2\rangle/\max\{m_1, m_2\}$, and the constant $\exists^t_m$ is of type $\langle\langle t\rangle/m\rangle/\max\{\mathrm{l}(t)+1, m\}$, as expected.[17]

[16] In particular, thanks to cumulativity our is more expressive than Hodes's: for example, if τ is a term of type $\langle\langle\rangle/2\rangle/3$ and σ is a term of type $\langle\rangle/1$, then $\tau(\sigma)$ is expressible (and of type $\langle\rangle/3$) in our language, but not in Hodes's systems. That is, the least cumulativity does for us is that it retains certain forms of Application that aren't available in systems without cumulativity.

[17] As was mentioned in footnote 5, in STT we *could* in principle stop treating $\neg$ as a primitive constant and define it in terms of $\lambda p.(p \to \bot)$, where $\bot$ is itself a proposition like $\forall^{\langle\rangle}(\lambda p^{\langle\rangle}.p)$. But defining $\bot$ this way in RTT will make it a level-2 proposition, whereas, it is customary (as in, e.g., Hodes, 2013, 2015) and plausible to let $\bot$ to be of level 1: if not, then the Boolean operators may behave abnormally. Suppose, for example, that $\bot$ is of level 2, and assume $\neg_m := \lambda p^{\langle\rangle/m}.(p^{\langle\rangle/m} \to \bot)$; then the negation $\neg_1 p$ of a level-1 proposition will be a level-2 proposition, and the negation $\neg_m q$ of a level-m

As for the identity operator in RTT, notice that we cannot define it in the usual, Leibniz's-law form. To see this, remember that we defined higher-order identity in STT, as follows: $=^t := \lambda x^t y^t . \forall^{\langle t \rangle} X(X(x) \rightarrow X(y))$, encoding Leibniz's law of the identity of indiscernibles. But in RTT, we can't quantify over *all* properties. In other words: quantifiers in RTT only apply to variables of certain levels. For reasons of this sort which reveal the expressive weakness of RTT, Russell and Whitehead had to introduce the *reducibility schema* in *Principia*, positing, in effect, that any higher-level propositional function is coextensive with some level-1 propositional function:

$$\exists X^{\langle t_1, ..., t_m \rangle / 1} \forall y_1^{t_1}, ..., y_m^{t_m} (\tau(y_1, ..., y_m) \leftrightarrow X(y_1, ..., y_m)), \tag{2.13}$$

where τ is of type $\langle t_1, ..., t_m \rangle / n$, for any $n \geq 1$.

Now, if we grant the reducibility schema, we can define the ramified identity as sharing all level-1 properties:

$$=_r^t := \lambda x^t y^t . \forall X^{\langle t \rangle / 1} (X(x) \rightarrow X(y)), \tag{2.14}$$

which itself is of type $\langle t, t \rangle / 2$. Now, if σ and τ differ in some higher-level property, since by the reducibility schema there's a level-1, coextensive property with the higher-order property, σ and τ must differ in that level-1 property as well, hence, given the definition above, are not identical. And vice versa: if two things differ in some level-1 property, they cannot be identical, by definition.

proposition q will be of level m: so, $\neg_m$ sometimes will and sometimes will not raise the level of propositions, manifesting a philosophically unmotivated picture. (Similar stories hold for the other connectives as well.) So, if one chooses to define $\neg$ (and other connectives) in terms of $\perp$, one will most likely have to assume there's a level-1 falsum $\perp$ in the signature. But if one chooses to take $\neg$ as a primitive constant, as we have done here, one need not worry about overcoming the glossed worries concerning $\perp$.

But in the interest of avoiding the long-standing complications and allegations associated with the reducibility schema, we will not adopt it in the present paper.[18] Instead, I will take an alternative path and will take (ramified) identity as a primitive constant in our system. As for what will the level of identity statements have to be, first, remember that the only logical constant operator that raises levels in our RTT is the universal quantification (and its inter-defined dual), so we can take the level of the identity statements to be the level of the entities flanking the relation (see a related discussion for the connectives $\rightarrow$ and $\neg$ from earlier). But since identity statements are essentially formulae, the minimum level they can take should be 1. With these in mind, I propose that for any ramified type t we reserve a constant $=_r^t$ of type $\max\{1, \mathrm{l}(t)\}$.[19]

We can then consider the desired features of identity—reflexivity, transitivity, and symmetry—as axiom schemata of the system, defined in the usual ways. In what follows, I will omit the subscript r when it's clear from the context that we're working with the ramified identity.

Proof Theory (System $\mathcal{H}_r^-$)

In what follows I will use 'ϕ_m' to stand for any sentence of type $\langle\rangle/m$. Also, for higher readability, I won't write down the types of the classical

[18] Such as the one which says that adding the schema to RTT makes the hierarchy of levels collapse into level 1, and the paradoxes of impredicativity will be reinstantiated (Quine, 1971; Ramsey, 1926; Copi, 1950). For an attempted defense against this allegation see Myhill (1979).

[19] If we had defined identity in the presence of the reducibility schema as having the same level-1 properties, the level of every identity statement would've been 2. We could alternatively take the statements containing our primitive identity relation to always have level 2, but that won't make any difference in the following discussions: in fact, any level greater than or equal to 1 would do for us—see footnote 27 for more on this.

connectives involved in the axiom schemata below (see the remark below).

Axioms:

1. $\vdash \phi_m \to (\psi_n \to \phi_m)$; $\vdash (\phi_m \to (\psi_n \to \gamma_r)) \to ((\phi_m \to \psi_n) \to (\phi_m \to \gamma_r))$; $\vdash (\neg\phi_m \to \neg\psi_n) \to (\psi_n \to \phi_m)$. PC_r

2. $\vdash (\lambda x_1^{t_1}, ..., x_n^{t_n}.\phi_m)(\sigma_1, ..., \sigma_n) \leftrightarrow [\sigma_1/x_1, ..., \sigma_n/x_n]\phi_m$, where the type of σ_i is identical directly lower than t_i, for each $i = 1, ..., n$. β_{E_r}

3. $\vdash \forall_m^t F \to F(\sigma)$, where F is of type $\langle t \rangle / m^+$, $m^+ \leq m$, and the type of σ is identical to or directly lower than t. UI_r

4. $\vdash \forall_{n^*}^t(\lambda x^t.\ \phi_m \to \psi_n) \to (\phi_m \to \forall_n^t(\lambda x^t.\psi_n))$, where $n^* := \max\{m, n\}$, and $x \notin FV(\phi_m)$. UD_r

5. $\vdash \sigma =_r^t \sigma$, Ref_r

6. $\vdash \sigma =_r^t \tau \to (F(\sigma) \to F(\tau))$, where F is of type $\langle t^* \rangle$ for any t identical to or directly lower than t^*. LBZ_r

Rules of Inference:

7. If $\vdash \phi_m$ and $\vdash \phi_m \to \psi_n$, then $\vdash \psi_n$. MP_r

8. If $\vdash F(x)$, then $\vdash \forall_m^t(F)$, where the variable x is of type t, F is of type identical to or directly lower than $\langle t \rangle / m$. GEN_r

Notice that each of the axioms and rules of inference above are multiply schematic. For example in PC_r, the axioms hold for any sentence of any level, and the relevant instances of $\neg$ and $\to$ may differ in type and should be typed carefully. We will take such nuances into account when proving

the soundness of $\mathcal{H}_r^{\vdash}$ with respect to the class of all closed ramified standard models, which will be introduced shortly.

Notice also that the restrictions imposed on substitution instances in UI_r and β_{E_r} are quite natural. Take β_{E_r} for example. In RTT, this axiom schema would be intelligible only if for each $i = 1, ..., n$, the level of σ_i's type is identical to or directly lower than that of x_i's—this is necessitated upon us due to the constraints on the rule Application from Definition 2.3.3. As for UI_r, consider for example instances of the schema where we're quantifying over level-1 propositional variables: $\forall p^{\langle\rangle/1}\, \phi \rightarrow [\psi/p]\phi$, where ϕ is of type $\langle\rangle/m$. It would be implausible to expect, say, a level-2 proposition ψ to be ϕ just because all *level-1* propositions p are ϕ, a commitment that will be unavoidable if we don't restrict the instances of substitution in UI to cases where the level of ψ's type is identical or directly lower than that of p.[20]

Another way to motivate such restrictions might be along the lines of *aboutness*.[21] Here I will only sketch the strategy. Notice at first that, in general, and regardless of the background type theory, there is a sense in which a statement that contains quantification over entities of a particular type is *about* (some or all of) those entities: thus, for instance, $\forall p^{\langle\rangle} p$, stating that all propositions are true, is about all propositions, and $\exists x^e\, general(x)$, stating that someone is a general (or in its dual form: that not everyone is not a general), is about some individuals. In general, a plausible minimal assumption about aboutness and quantificational statements is that the

[20]This may we why Church (1976) imposes the exact same restriction on UI. Bacon et al. (2016) also offers a different systematic argument for the desired restriction: if we don't impose it, all quantifiers of different level imposed on simple types will collapse onto each other.

[21]This line of thinking has instigated another paper of mine, Kiani (MSc).

legitimate instances of a quantified statement have to, at the very least, belong to the entities that the statement is about.

Now, the rough idea to take home with us is this: if we put at work this minimal principle in the context of RTT, a statement of the form, say, $\exists p^{\langle\rangle/1} p$, if about anything, that thing should be a level-1 proposition. But the quantified statement itself is a level-2 proposition, so it can't be about itself, hence cannot instantiate itself. Similarly, $\forall p^{\langle\rangle/m}\, \phi$ will be about level-m propositions (and propositions of lower level, considering the cumulativity), but the statement itself is at least of level m+1, hence cannot be about itself. As a result, $\forall p^{\langle\rangle/m}\, \phi$ is not a legitimate instance of itself, hence the restriction in UI_r.

Consistency of $\mathcal{H}_r^-$

Below, and for convenience in proving soundness of $\mathcal{H}_r^-$, and our reluctance to demand completeness with respect to the class of (ramified) *Henkin* models (see the closing remarks of §2.1 for more on this) we will only impose the ramified hierarchy onto the class of standard, instead of Henkin models from STT.

In what follows, and for convenience, we write $t' \ll t$ as a shorthand for 'the type of t' is identical to or directly lower than t'.

Definition 2.3.4 (Ramified Frames)**.** Let D be a non-empty set. A *cumulative ramified frame* based on D is a set $F := \{M_t \,|\, t \in \mathcal{T}^r\}$ such that:

(i) $M_e := D$,

(ii) $\varnothing \neq M_{\langle\rangle/m} = \wp(\{\varnothing\}) \times \{1, ..., m\} = \bigcup_{i \leq m} \{(0, i), (1, i)\}$,

(iii) $\varnothing \neq M_{\langle t_1,\ldots,t_n\rangle/m} = \wp(M_{t_1} \times \ldots \times M_{t_n}) \times \{1,\ldots,m\}$, where $n \geq 1$,

Clauses (ii) and (iii) grant that the domains of leveled propositions and properties are cumulative (see the theorem below).

Lemma 1. *For any $t', t \in \mathcal{T}^r$, if $t' \ll t$ then $M_{t'} \subseteq M_t$.*

Proof. Induction on t. □

Definition 2.3.5 (Ramified Standard Models)**.** A *ramified standard model* is a pair $M = (F, d)$, where F is a ramified cumulative frame and d is a function $d : \bigcup_t \mathrm{CST}_t \to \bigcup_t M_t$ such that $d(c^t) \in M_t$, and for all $t \neq e$, if $\mathrm{l}(t) = m$ then $d(c^t) = (X, m)$ for an appropriate X. A model is *logical* if:[22]

(i) $d(\neg_m) := \big(\bigcup_{i \leq m} \{(0,i)\}, m\big)$.

(ii) $d(\rightarrow_{m,n}) := \Big(\bigcup\!\!\!\bigcup_{i \leq m, j \leq n} \Big\{\big((0,i),(1,j)\big), \big((0,i),(0,j)\big), \big((1,i),(1,j)\big)\Big\}, m^*\Big)$, where $m^* := \max\{m, n\}$,

(iii) $d(=^t_r) := \Big(\bigcup_{i \ll t} \big\{(d,d) | d \in M_{t_i}\big\}, m^*\Big)$, where $m^* = \max\{1, \mathrm{l}(t)\}$,

(iv) $d(\forall^t_m) := \Big(\bigcup_{i \leq m} \big\{(M_t, i)\big\}, m^*\Big)$, where $m^* := \max\{\mathrm{l}(t)+1, m\}$.

An assignment function for a model is a function $g : \bigcup_t Var_t \to \bigcup_t M_t$ such that $g(x^t) \in M_t$ for each $x^t \in \mathrm{Var}_t$.

Definition 2.3.6 (Interpretations)**.** An *interpretation* for an assignment g based on a model M is a function $[\![.]\!]_g : \mathrm{TERM}_r \to \bigcup_t M_t$, such that:

(i) $[\![c^t]\!]_g = m(c^t)$, for $c^t \in \mathrm{CST}_t$,

(ii) $[\![x^t]\!]_g = g(x^t)$, for $x^t \in \mathrm{Var}_t$,

[22] Later on we will illustrate how we came up with these types of the logical constants.

(iii) $[\![\lambda x_1^{t_1}, ..., x_n^{t_n}.\phi_m]\!]_g = (X, m)$, where $X \subseteq M_{t_1} \times ... \times M_{t_n}$ is the set such that $(d_1, ..., d_n) \in X$ iff $[\![\phi_m]\!]_{g[x_1 \mapsto d_1, ..., x_n \mapsto d_n]} = (1, m)$,

(iv) $[\![\tau(\sigma_1, ..., \sigma_n)]\!]_g = \begin{cases} (1, m) & \text{if } ([\![\sigma_1]\!]_g, ..., [\![\sigma_n]\!]_g) \in X \\ (0, m) & \text{if } o.w \end{cases}$, where τ is of type $\langle t_1, ..., t_n \rangle /\!\!/ m$, for each $i = 1, ..., n$, σ_i is of type identical to or directly lower than t_i, and $[\![\tau]\!]_g = (X, m)$.

As before, we show interest only in *closed* models—ramified models that for all assignments functions, the interpretation of all terms fall in the relevant domains. For a model M, an assignment g based on M, and an m-level sentence ϕ_m, we say that M *satisfies* ϕ_m *for the assignment* g, and write $M, g \vDash \phi_m$, iff $[\![\phi_m]\!]_g = (1, m)$.[23] Notions of validity, etc., are defined as usual.

Let's see an example of how the semantics works in practice.

Example 2.3.1. Let's verify that, as expected, for any model M, an assignment g based on M, and an m-level sentence ϕ_m we have $M, g \vDash \forall x^t \phi_m$ iff $[\![\phi_m]\!]_{g:[x \mapsto d]} = (1, m)$ for all $d \in M_t$. First notice that by convention we have $\forall x^t \phi_m$ as a shorthand for the application instance $\forall^t_m(\lambda x^t.\phi_m)$. Now, by clauses (iii) and (iv) of Definition 2.3.6 we have: $M, g \vDash \forall^t_m(\lambda x^t.\phi_m)$ iff $[\![\lambda x^t.\phi_m]\!]_g \in X$, where $[\![\forall^t_m]\!] = (X, \max\{\mathrm{l}(t)+1, m\})$. But by Definition 2.3.5 we have $X = \bigcup_{i \leq m} \{(M_t, i)\}$, and by Definition 2.3.6, we have $[\![\lambda x^t.\phi_m]\!]_g = (Y, m)$ in which $Y \subseteq M_t$ such that $d \in Y$ iff $[\![\phi_m]\!]_{g[x \mapsto d]} = (1, m)$. Therefore we have $Y = M_t$, and we get the desired result. Notice further that due to the cumulativity in our system and the fact that $\forall^t_m$—much like any other applicator—has encoded that in its interpretation,

[23] Notice that for any term ϕ_m of type $\langle\rangle /\!\!/ m$, and every pair of models and assignments (M, g), we have: $[\![\phi_m]\!]_g = (1, m)$ iff $[\![\phi_m]\!]_g \neq (0, m)$. (Proof: induction of ϕ_m.)

we could apply $\forall^t_m$ to propositions that are directly lower than $\langle\rangle\!/m$ and get a similar result. In other words, for any ψ_n of level $\langle\rangle\!/n$, where $n \leq m$, we have: $M, g \vDash \forall^t_m(\lambda x^t.\psi_n)$ iff $[\![\psi_n]\!]_{g:[x\mapsto d]} = (1, n)$, for all $d \in M_t$.

Lemma 2. *Suppose $\tau, \sigma_i \in TERM_r$, for each $i = 1, ..., n$ $(n \geq 0)$. We have* $[\![\tau]\!]_{g\left[x_1\mapsto[\![\sigma_1]\!]_g,...,x_n\mapsto[\![\sigma_n]\!]_g\right]} = \left[\!\left[[\sigma_1/x_1, ..., \sigma_n/x_n]\tau\right]\!\right]_g$.

Proof. Induction on the structure of τ. □

Theorem 2.3.1 (Soundness)**.** *The proof system $\mathcal{H}^{\vdash}_r$ is sound with respect to the class of all closed ramified models. That is, for every sentence $\phi \in TERM_r$, we have: if $\vdash \phi$ then $\vDash \phi$.*

Proof. We should prove all axioms of $\mathcal{H}^{\vdash}_r$ are valid in all ramified standard models, and the inference rules preserve validity. The proofs are straightforward, and I will only showcase some the items that represent new restrictions or diversity of types, compared to their replicas from STT. As expected, in what follows, we're focusing only on closed models.

(PC$_r$). Let's just prove the validity of the rule $\phi_m \to (\psi_n \to \phi_m)$. The most general form of this axiom, with levels construed as liberal as possible and cumulativity taken into account, is of the form $\phi_m \to_{m',m^*} (\psi_n \to \phi_m)$, where $m' \geq m$ and $m^* \geq \max\{m, n\}$. But for convenience we'll only prove the case where $m' = m$ and $m^* = \max\{m, n\}$; the more general case can be proved in a similar manner.

Let M and g be, respectively, an arbitrary model and an arbitrary assignment based on M. Then $M, g \vDash \phi_m \to_{m,m^*} (\psi_n \to \phi_m)$ (where, $m^* := \max\{m, n\}$) *iff* $[\![\phi_m \to_{m,m^*} (\psi_n \to \phi_m)]\!]_g = (1, m^*)$ *iff*

$([\![\phi_m]\!]_g, [\![(\psi_n \to \phi_m)]\!]_g)$ belongs to the first component of the ordered pair $[\![\to_{m,m^*}]\!]_g$ *iff* ... *iff* $[\![\phi_m]\!]_g = (0, m)$ or $[\![\phi_m]\!]_g = (1, m)$ or $[\![\psi_n]\!]_g = (1, n)$. But the latter obviously holds. Therefore: $\vDash \phi_m \to (\psi_n \to \phi_m)$.

(β_{E_r}). Let M and g be respectively an arbitrary model and an arbitrary assignment based on M. We have: $M, g \vDash (\lambda x_1^{t_1}, ..., x_n^{t_n}.\phi_m)(\sigma_1, ..., \sigma_n)$ *iff*

$([\![\sigma_1]\!]_g, ..., [\![\sigma_n]\!]_g) \in X$, where $[\![\lambda x_1^{t_1}, ..., x_n^{t_n}.\phi]\!]_g = (X, m)$ for the set $X \subseteq M_{t_1} \times ... \times M_{t_n}$ such that $(d_1, ..., d_n) \in X$ iff $[\![\phi_m]\!]_{g[\bar{x}_i \mapsto \bar{d}_i]} = (1, m)$. But the latter is equivalent to $[\![\phi_m]\!]_{g[\bar{x}_i \mapsto [\![\bar{\sigma}_i]\!]_g]} = (1, m)$, which, thanks to Lemma 2, is equivalent to $\left[\!\left[[\bar{\sigma}_i/\bar{x}_i]\phi\right]\!\right]_g = (1, m)$, which is in turn equivalent to $M, g \vDash [\bar{\sigma}_i/\bar{x}_i]\phi_m$. Therefore: $\vDash \beta_{E_r}$. Notice that, given that $X \subseteq M_{t_1} \times ... \times M_{t_n}$, then if for any $i = 1, ..., n$, σ_i is not of a type identical to or directly lower than that of t_i, then the n-tuple $([\![\sigma_1]\!]_g, ..., [\![\sigma_n]\!]_g)$ falls out of the domain of the interpretation of the λ-term, and we can't get the desired result. (This in turn reflects the ungrammaticality of such applications at the level of syntax.)

(UI_r). Suppose that F is of type $\langle t \rangle / m^+$, where $m^+ \leq m$. Now, for any model M and assignment g, we have: $M, g \vDash \forall^t_m(F)$ *iff* ... *iff* $[\![F]\!]_g = (\bigcup_{t_i \ll t} M_{t_i}, m^+)$. Now suppose that the type t' of σ is identical to or directly lower than t, that is, $t' \ll t$. From above we have: $M, g \vDash F(\sigma)$ *iff* $[\![\sigma]\!]_g \in M_t$, so, if $[\![\sigma]\!]_g \in M_t$, we'll be done. But of course $[\![\sigma]\!]_g \in M_t$, because $[\![\sigma]\!]_g \in M_{t'}$, and, thanks to Lemma 1, $M_{t'} \subseteq M_t$, so $M, g \vDash F(\sigma)$. Therefore: $\vDash$ UI_r. (See Remark 2.3.1 for a discussion of the restriction of substitution in UI_r.)

(UD_r). As in the case of PC_r, and for convenience, we'll only deal with cases of implication where the arguments of $\rightarrow_{m,n}$ are of type $\langle\rangle/m$, and $\langle\rangle/n$, respectively, and not directly lower. For an arbitrary model M, an assignment function g, and sentences ϕ_m and ψ_n, suppose $M, g \vDash \forall^t_{n^*}(\lambda x^t.\ \phi_m \rightarrow_{m,n} \psi_n)$ ($n^* := \max\{m, n\}$). We will prove that $M, g \vDash \phi_m \rightarrow_{m,m^+} \forall^t_n(\lambda x^t.\psi_n)$ ($m^+ := \max\{\mathrm{l}(t)+1, n\}$), if $x \notin FV(\phi_m)$. Now, we have: $M, g \vDash \forall^t_{n^*}(\lambda x^t.\ \phi_m \rightarrow_{m,n} \psi_n)$ *iff* ... (see Example 2.3.1) ... *iff* for all $d \in M_t$, $[\![\phi_m \rightarrow_{m,n} \psi_n]\!]_{g[x \mapsto d]} = (1, n^*)$ *iff* for all $d \in M_t$, we have $([\![\phi_m]\!]_{g[x \mapsto d]}, [\![\psi_n]\!]_{g[x \mapsto d]}) \in Z$, where $[\![\rightarrow_{m,n}]\!]_{g[x \mapsto d]} = (Z, n^*)$. So, the initial assumption that $M, g \vDash \forall^t_{n^*}(\lambda x^t.\ \phi_m \rightarrow_{m,n} \psi_n)$ is equivalent to the following:

$$\text{For all } d \in M_t,\ [\![\phi_m]\!]_{g[x \mapsto d]} = (0, m) \text{ or } [\![\psi_n]\!]_{g[x \mapsto d]} = (1, n) \qquad (*)$$

Now, on a separate note we have $M, g \vDash \phi_m \rightarrow_{m,m^+} \forall^t_n(\lambda x^t.\psi_n)$ *iff* $([\![\phi_m]\!]_g, [\![\forall^t_n(\lambda x^t.\psi_n)]\!]_g) \in Z'$, where $[\![\rightarrow_{m,m^+}]\!]_g = (Z', m^{++})$, and $m^{++} := \max\{m, m^+\}$. So, $M, g \vDash \phi_m \rightarrow_{m,m^+} \forall^t_n(\lambda x^t.\psi_n)$ is equivalent to the following:

$$[\![\phi_m]\!]_g = (1, m) \text{ or for all } d \in M_t,\ [\![\psi_n]\!]_{g[x \mapsto d]} = (1, n). \qquad (**)$$

Now, considering the (easily provable) fact that $[\![\phi_m]\!]_g = [\![\phi_m]\!]_{g[x \mapsto d]}$, when $x \notin FV(\phi_m)$, it's easy to see that (*) implies (**). Therefore: $\vDash \mathrm{UD}_r$.

(GEN_r). Let x be of type t, F be of type $\langle t\rangle/n$ for some $n \leq m$, and suppose $\vDash F(x)$. Then, for all models and assignments M and g, we have: $M, g \vDash F(x)$. So: for all M and g, $g(x) =: [\![x]\!]_g \in Y$, where $[\![F]\!]_g =$

(Y, n). But since by ranging over all assignments g, $g(x)$ can take any value $d \in M_t$, we have $M_t \subseteq Y$. So $Y = M_t$, and hence: $[\![F]\!]_g$ belongs to the first component of the ordered pair $[\![\forall^t_m]\!]_g$. So $[\![\forall^t_m(F)]\!]_g = (1, \max\{m, \mathrm{l}(t)+1\})$, which, by definition, is equivalent to $M, g \vDash \forall^t_m(F)$. Since the choice of M and g were arbitrary, we have proved that $\vDash \forall^t_m(F)$. Therefore, GEN_r preserves validity with respect to the class of all closed ramified standard models.

□

Remark 2.3.1. It would be instructive to see why soundness fails if we don't impose the restrictions of substitution in principles of $\mathcal{H}^-_r$. Take, for example, UI_r. I will now propose a way to construct a class of counter-models to cases where the intended restriction isn't in place: suppose $M, g \vDash \forall^t_m(F)$ and let t be directly lower than the type t' of a term σ. Then, although $M_t \subseteq M_{t'}$, there's no more a guarantee that $M_{t'} \subseteq M_t$. In fact, we have $M_{t'} \nsubseteq M_t$, due to clauses (iii) and (iv) of Definition 2.3.4. On the other hand, by definition, we have $M, g \vDash F(\sigma)$ iff $[\![\sigma]\!]_g$ is a member of the first element of the pair $[\![F]\!]_g$, which is just M_t (see the proof of $\vDash$ UI_r, above). Now, in cases where $[\![\sigma]\!]_g \in M'_t \setminus M_t$, we will have $M, g \nvDash F(\sigma)$, hence a counter-model to (the unrestricted) UI. It is also easy to check that GEN_r won't necessarily preserve validity, if x is of a level directly lower than t.

To see our system at work, consider the following example originally presented in *Principia*:[24]

[24]See Whitehead and Russell (1912), p.59.

Example 2.3.2. Consider the sentence 'Napoleon had all the qualities that make a great general'. Examples of such qualities would perhaps be *brave*, *strategic*, *foresight*, etc. Understanding these "qualities" of individuals as propositional functions, they will all presumably have the type $\langle e\rangle/1$, because they apply directly to individuals and no quantification is included in their definitions.[25] So does the quality of being a great general. Suppose G is the constant for the property of being a great general, i.e., has type $\langle e\rangle/1$. Now, the predicate standing for the quality of having all the qualities that make a great general can be regimented as:

$$\lambda x^e.\forall Y^{\langle e\rangle/1}\big(\forall z^e(G(z)\rightarrow Y(z)) \rightarrow Y(x)\big),$$

which is well-formed and of type $\langle e\rangle/2$. Accordingly, taking N to stand for Napoleon's name (so, N has type e), the sentence 'Napoleon had all the qualities that make a great general' will be obtained by the following instance application:

$$\Big(\lambda x^e.\forall Y^{\langle e\rangle/1}\big(\forall z^e(G(w)\rightarrow Y(z)) \rightarrow Y(x)\big)\Big)(N),$$

which, by the β_{E_r} is equivalent to:

$$\forall Y^{\langle e\rangle/1}\big(\forall z^e(G(z)\rightarrow Y(z)) \rightarrow Y(N)\big),$$

which is the closest regimentation of the original sentence in the language of RTT, and has type $\langle\rangle/2$. It's worth noting that the property of having all the qualities that make a great general, if regimented in STT, would

[25] Of course, this is an oversimplification of the matter for illustrative purposes: one might argue that among such properties is, say, the quality of thinking that one has all the qualities that make a great general, in which case the property of having all qualities that make a great general will become of a property of higher level than 2.

be picked by the following:

$$\lambda x^{e}.\forall Y^{\langle e\rangle}\big(\forall z^{e}(G(z)\rightarrow Y(z)) \rightarrow Y(x)\big),$$

which would be of the type $\langle e\rangle$, that is, the same type that any other quality that applies to someone has. As one can see, the definition of this property quantifies over the totality of all properties in which it belongs, displaying an instance of a vicious circle in the presence of simple types. As the previous paragraphs show, ramified type theory avoids this vicious circle by raising the level of the property with respect to the level of the properties over which it quantifies.

It's also easy to see how the ramified model theory prevents the properties of different level above from collapsing onto each other.

2.4 Blocking the Proofs of the Russell-Myhill Paradox

To close to section, we will now see how exactly RTT may help to disallow the proofs of the Russell-Myhill paradox to go through by banning the vicious circles involved in them. Before doing so, however, an important clarification is in order: what will be presented here will *not* show that RTT will block the paradox in all possible forms; all it shows is that the usual, known proofs of the inconsistency won't go through when naively reconstructed in the presence of ramified types.[26]

[26]Notice that those (e.g., Myhill, 1979) who have claimed to have blocked certain other paradoxes of impredicativity in the presence of ramified types in similar ways, strictly speaking, also have *not* really proved anything more than the fact that the proofs of those paradoxes won't go through if we naively try to reconstruct them in the presence of ramified types.

So, in a sense what we will do here secures assumptions of structure from the punch of the Russell-Myhill paradox, by avoiding a key feature involved in the proofs—namely, the existence of certain impredicative properties; for all we know, there may be ways of proving the paradox that are unbeknownst to us and cannot be helped by type-stratification techniques.

To show in full generality that the Russell-Myhill paradox is blocked in RTT, one needs to first lay down a full axiomatization of RTT and then establish its consistency with Structure, say, via models constructions. As of this moment, this is an open problem of the paper; we hope to attend to it in future work.

With that in mind, let's see how RTT disallows the proofs of the paradox to go through. Remember from §2.2 that the paradox arose in the context of STT because terms like $G(G(\psi))$ were well-formed, in the first place, where

$$G := \lambda q^{\langle\rangle}.\exists X^{\langle\langle\rangle\rangle}((q =^{\langle\rangle} X(\psi)) \wedge \neg X(q))$$

is of type $\langle\langle\rangle\rangle$, and ψ is of type $\langle\rangle$. As we noticed before, the property picked out by G has the same type of the properties that it quantifies over—displaying an instance of a vicious circle in the presence of simple types.

But notice that we're now in the territory of ramified types, and the types $\langle\langle\rangle\rangle$ or $\langle\rangle$ no longer exist; accordingly, neither do G and ψ exist anymore. So there's a sense in which it's obvious that the proofs won't go through in RTT when the original G and ψ and their simple types are concerned. But nevertheless the worry is that in the new

setting there might still be *leveled* variants of G and ψ, G' and ψ', such that $G'('G'(\psi')) \leftrightarrow \neg G'('G'(\psi'))$ is provable and hence the paradox is reinstantiated. Below I will argue that that's not the case: for any intelligible candidates of G' and ψ', the term $G'('G'(\psi'))$ will be non-well-formed, due to avoidance of vicious circles.

The predicate G and its leveled variant G' are defined as follows:

$$G := \lambda q^{\langle\rangle}.\exists X^{\langle\langle\rangle\rangle}((q =^{\langle\rangle} X(\psi)) \wedge \neg X(q)), \tag{2.15}$$

$$G' := \lambda q^{\langle\rangle/m}.\exists X^{\langle\langle\rangle/m\rangle/m}((q =^{\langle\rangle/m} X(\psi)) \wedge \neg X(q)), \tag{2.16}$$

where ψ is a term of type $\langle\rangle/m$. Roughly, G' stands for the property of being a *level-m* proposition that lacks a *level-m* property which it attributes to some *level-m* proposition. Admittedly, if instances of the form $G'(G'(\psi'))$ (for a ψ' being of type $\langle\rangle/m$) were available, one might have succeeded to reinstantiate the paradox. But they aren't. Here's why: notice that the type of

$$\exists X^{\langle\langle\rangle/m\rangle/m}((q =^{\langle\rangle/m} X(\psi)) \wedge \neg X(q)) \tag{2.17}$$

is $\langle\rangle/\max\{r, m{+}1\}$, where r is the level of the type of $(q =^{\langle\rangle/m} X(\psi)) \wedge \neg X(q)$, whatever that might be.[27] Therefore, G' will be of type $\langle\langle\rangle/m\rangle/\max\{r, m{+}1\}$. So for any ψ' of type $\langle\rangle/m$ or lower, $G'(\psi')$ will be of type $\langle\rangle/\max\{r, m{+}1\}$. But notably $\max\{r, m{+}1\} > m$, so the application instance $G'(G'(\psi'))$ is not well-formed, because the level of $G'(\psi')$ is at least $m{+}1$, hence it is at least one more than the level of expressions that G' can possibly take as arguments, namely m. So the parallel of the troubling propositions $G(G(\psi))$

[27] The way we defined ramified identity makes $r = \max\{1, m\} = m$, and hence the level of the existential statement as well as G' is *exactly* $m{+}1$. However, as it should be clear, any other r would've worked equally well for us. So, even if we had typed ramified identities differently my arguments would've still gone through.

from STT are not well-formed if we reconstruct G and ψ in the presence of ramified types. Accordingly, under such a reconstruction we don't need to worry about contradictory equivalences like $G'(G'(\psi')) \leftrightarrow \neg G'(G'(\psi'))$ either: they're not even well-formed, let alone provable.[28]

In plain English: the property G' of being a level-m proposition that lacks a level-m property which it attributes to some level-m proposition is itself at least of *level m+1*. So when we attribute G' to a level-m proposition, we will have a proposition of at least level $m{+}1$, and the latter can no longer be attributed a property of *level-m* propositions, including the property G'. This is a clear example of avoiding the vicious circle inherent in the original G: unlike G, the property picked by G' doesn't belong to the properties that it quantifies over in its definition.

Notice that the cumulativity of our typing systems allows for more complicated reconstructions of the G, in RTT. Above, we chose the simplest kind, which were expressions of the form G', but a more general reconstruction would be of the form G^*, as follows:

$$G' := \lambda q^{\langle\rangle/m}.\exists X^{\langle\langle\rangle/m\rangle/m}((q{=}^{\langle\rangle/m}X(\psi)) \wedge \neg X(q)), \tag{2.18}$$

$$G^* := \lambda q^{\langle\rangle/m}.\exists X^{\langle\langle\rangle/n\rangle/m}((q{=}^{\langle\rangle/m}X(\psi)) \wedge \neg X(q)), \tag{2.19}$$

for *any* $n \geq m$, where ψ is a proposition of *any* level $s \leq n$.

But we're still safe: $\exists X^{\langle\langle\rangle/n\rangle/m}((q{=}^{\langle\rangle/m}X(\psi)) \wedge \neg X(q))$ will still be of type $\langle\rangle/\max\{r, m{+}1\}$, whatever r might be, and G^* will be of type

[28]Notice that even if $G'(G'(\psi'))$ was grammatical, the problem would've come back in unwelcoming ways: by the ramified β_E (see §4), $G'(G'(\psi'))$ would've been equivalent to $\exists X^{\langle\langle\rangle/m\rangle/m}((G'(\psi'){=}^{\langle\rangle/m}X(p)) \wedge \neg X(G'(\psi')))$, but the identity in the latter is ill-formed, as $G'(\psi')$ is of type $\langle\rangle/\max\{r, m{+}1\}$, whereas $X(p)$ is of type $\langle\rangle/m$, hence a type mismatch is in place. In short: if $G'(G'(\psi'))$ was grammatical, we would've encountered instances of extremely awkward equivalences that hold between grammatical and ungrammatical expressions.

$\langle\langle\rangle/m\rangle/\max\{r, m{+}1\}$. But just as before, $\max\{r, m{+}1\} > m$, so $G^*(\psi')$ (for any appropriate ψ') has a level strictly greater than m, hence cannot be an argument to G^*.

2.5 Conclusion

I proposed a consistent ramified type system and argued that deploying it as the background theory of relational entities, according to which they come in infinite levels in the specific ways predicted by ramified type theory, disallows the proof of the Russell-Myhill paradox to go through.

Admittedly, and as was noted, this falls short of rigorously establishing the consistency of structured propositions (i.e., the schema Structure) under the rein of ramified types. However, since there seems to be no other way to re-instantiate the paradox, and since other people, such as the original founders (Russell, 1908; Myhill, 1958), have speculated so, we hypothesize that RTT does save structured propositions from Russell-Myhill. As such, it remains open as to what models of RTT+Structure would look like, assuming they exist; we hope to explore these in future work.

Chapter 3

Structured Propositions and a Semantics for Unrestricted Impure Logics of Ground

3.1 Introduction

There is a popular view in analytic philosophy, going back to Russell (1903), according to which propositions are highly structured, somewhat reflecting the structure and identity conditions of the sentences that express them. Call the structured propositions along these lines *Russellian* (King, 2019; Kaplan, 1977). In recent decades, many seminal works in the philosophy of language and metaphysics have assumed or argued for Russellian propositions in various contexts, ranging from attitude operators to different kinds of metaphysical priority, such as essence and ontological dependence (see, e.g., King, 1996, 2009; Soames, 1987; Fine, 1995, 1980, 1994; Kaplan, 1977; Salmon, 1986).

Grounding, on the other hand, is a more recent notion in metaphysics, often taken to be a non-causal relation that holds between certain truths or facts and certain others, somehow reflecting a sense of 'fundamentality' or 'explanation' between them (see, e.g., Fine, 2012a; Rosen, 2010; Audi, 2012, for comprehensive introductions to the notion of ground).[1] The conception of ground which takes the relata of the grounding relation to be propositions is sometimes called *representational* or *conceptual*; the *worldly* conception concerns entities such as states of affairs or situations (Correia, 2017, p. 508). The kind of logics that take into account the logical structure of the relata of grounding relations are often called *impure*; *pure* logics ignore such complexities (Fine, 2012a, p. 54).

There is another important set of distinctions between grounding relations that has been studied in the literature, and we briefly introduce here (see, e.g., Fine, 2012a, pp. 52-4 for a detailed discussion of these variations and their differecnes). To start with, a number of truths are said to *fully* ground a truth if the latter somehow holds completely in virtue of the former and nothing else; a truth *partially* grounds another if it does so fully, standalone or together with other truths. Another distinction is between mediate and intermediate grounds. Grounds of a truth are *immediate* if there's no mediating truth between them and what they ground; otherwise, they constitute *mediate* grounds, as if there's a 'chain' of immediate grounds involved. Finally, some truths are *strict*

[1] While the nature of grounding and its relations to other notions such as explanation or fundamentality is an intricate issue that has been subject to extensive discussions in the literature (see, e.g., Rosen, 2010; Woods, 2018; Fine, 2012a; deRosset, 2013; Sider, 2011; Maurin, 2019, for different readings of ground based on explanation or fundamentality, their relationships with one another, and some of the complications involved in laying out those relationships), in this paper we stay fairly neutral in this regard, and appeal to either of readings mainly for illustrative purposes.

grounds of some others if they are, in a sense, more 'fundamental' or 'basic'; otherwise, the grounds are *weak*. Put in terms of explanation, we can think of strict ground as, in the words of Fine (2012a), is one that "takes us down in the explanatory hierarchy," whereas weak grounds "may also move us sideways in the explanatory hierarchy" (*ibid*, p. 52). Finally, the conception of ground that allows *any* proposition, regardless of its truth value, as the relata of ground is called *non-factive*; the *factive* variant only works with truths, i.e., true propositions.

In this paper, I study an intimate relationship between Russellian propositions and *impure* logics of representational ground. The main focus of the paper is on propositional logics of ground.[2] More specifically, we are concerned with *strict partial* grounding relations: non-factive immediate ($\prec$), factive immediate ($\prec_f$), non-factive mediate ($\prec_m$) and factive mediate ($\prec_{fm}$). (Hereafter we use 'ground' to indicate strict partial ground unless stated otherwise; the specific type will be mentioned as needed.) We take the notion of ground as a primitive, i.e., not reducible to any other notion.

I show that models of Russellian propositions can be used to semantically accommodate an infinitude of grounding facts that follow from unrestricted logics of impure ground, but are left unaccounted for in the existing semantics, found in Correia (2017); Krämer (2018); deRosset and Fine (2023), due to certain artificial restrictions inherited from the languages they work with. Moreover, it is shown that our models, unlike the ones in the literature, can be very easily extended to capture certain distinct philosophical views about, e.g., iterated as well as identity

[2]Some of seminal the works on the quantificational logics of ground are as follows: Fine (2012a); Korbmacher (2018b,a); Fritz (2019, 2021); Goodman (2022); Litland (2022).

grounding.[3]

The sensitivity of impure ground to the structure of propositions is easily detectable once the naive principles are laid down (see Fine, 2012a, Sections 1.6-1.7 for an early discussion of these principles). For instance, it is often argued that a conjunctive truth $\phi \wedge \psi$ is grounded by each of its conjuncts ϕ and ψ (temporarily call this CG_f: $\phi \prec_f (\phi \wedge \psi) \ \wedge \ \psi \prec_f (\phi \wedge \psi)$), a disjunctive truth $\phi \vee \psi$ by either of its true disjuncts ϕ or ψ (DG_f: $(\phi \prec_f \phi \vee \psi) \ \vee \ (\psi \prec_f \phi \vee \psi)$), a doubly negated truth $\neg\neg\phi$ is grounded by ϕ (NG_f: $\phi \prec_f \neg\neg\phi$), and that no proposition grounds itself (IG_f: $\phi \not\prec_f \phi$). Now, from IG_f and NG_f it follows that ϕ and $\neg\neg\phi$ can't be the same truth, and from CG_f and DG_f the same follows for any pair of sentences from ϕ, $\phi \vee \phi$ and $\phi \wedge \phi$.[4]

So, we quickly get a few boundaries surrounding the issue of propositional granularity under considerations of ground. As a result, certainly, coarse-grained accounts, such as the once-popular intensionalism which identifies necessarily equivalent propositions (see, e.g., Montague, 1969), and its close, more recently popularized cousin, Booleanism, which identifies propositions that are provably logically equivalent (see, e.g., Dorr, 2016) can't be consistently adopted under the principles above.

In general, as we will see along the way, from the propositional logics

[3]It might strike the reader, at this point, that Russellian propositions, as favored by the author cited earlier, have now been known to lead to the so-called Russell-Myhill paradox (as shown in, e.g., Goodman, 2016; Dorr, 2016; Hodes, 2015; Russell, 1903), and thus this might cast doubt on the conceptual value of the results to be explored in this paper. This, however, shouldn't worry us because (i) the Russell-Myhill result doesn't emerge at the level of propositional logic without quantification, so it shouldn't concern us in this paper, and (ii) the named paradox can be avoided under a different background type theory which is more ground-friendly and which saves Russellian propositions. We have discussed this issue at some length at the end of the paper.

[4]See Section 3.3 for more general formulations of these principles; the naming used here is temporary.

of impure ground, along with minimal principles of propositional identity, it follows that propositions ought to be significantly structured—in fact, sometimes as structured as Russellian propositions (see Theorem 3.3.1). Moreover, recently Fritz (2021) has shown that higher-order formulations of the principles of ground, in fact, entail certain higher-order instances of Russellian propositions. It can be said that the propositional and higher-order logics of immediate ground together portray a scattered picture of propositions that is most straightforwardly and systematically captured by Russellian propositions.[5]

Here's how the paper is organized. In Section 3.2, I informally address certain expressive shortcomings of the existing semantics of impure logics of ground. In Section 3.3, I rigorously introduce the language and lay down the immediate and mediate logics of ground, both non-factive and factive variants. In the same section, I establish certain structural results derived from logics of ground, lay down some identity principles for Russellian propositions and show that the latter systematically capture the former. Section 3.4 concerns semantics; it introduces propositional models for Russellian propositions, uses them to provide a semantics for the unrestricted logics of ground and discusses some meta-results such as soundness, consistency and completeness. Section 3.5 proposes various desirable extensions of logics of impure ground and their semantics

[5]Other works on the logic of ground that impose some kind of structural hierarchies on propositions are Poggiolesi (2016) and Correia (2017), though both end up with more relaxed structures on propositions in comparison to Russellian propositions. Poggiolesi (2016) appeals to the notion of 'g-complexity' for this but it's not clear if the resulting account fully appreciates the level of complexity of propositions that naturally emerges from the principles of grounding and the minimal principles of identity—i.e., the results in theorem 3.3.1. It's also unclear how Pogiolessi's account will perform when it comes to the semantic issues that are at stake in this paper. We will leave these open here. We will discuss Correia (2017) in more detail later in the paper.

and addresses some systematic difficulties of the existing semantics in the literature in undergoing similar extensions. Section 3.6 concludes the paper. The appendices collect all the principles of grounding and propositional identity and establish some of the technical results in the paper.

3.2 Present Semantics and their Shortcomings

While the semantics of pure logics of ground has been well studied and somewhat settled (see, e.g., Fine, 2012b), impure logics, and in particular, their representational variants, remain fairly underexplored, with only a few recent attempts on offer to semantically account for them (Correia, 2017; Krämer, 2018; deRosset and Fine, 2023). But, even though these works mark considerable progress in the study of the impure logics of ground, all these semantic accounts suffer from certain expressiveness limits, complying with the restricted languages or logics that they're supposed to capture. To see this, we should first see what limits are imposed on the languages and logics that these semantics attempt to capture.

In general, there is a tendency in the literature on the impure logics of ground to substantially impoverish the languages in which the principles are expressed, mainly allowing for statements of grounding in which the relata of ground contain truth-functional connectives, but not connectives such as grounding itself or propositional identity (see, e.g., Schnieder, 2011;

Krämer, 2018; Correia, 2017; Poggiolesi, 2020; Lovett, 2020; deRosset and Fine, 2023). As a result, an infinitude of grounding truths which live beyond these artificial restrictions are dismissed by these logics.

To illustrate this, suppose ϕ and ϕ are two sentences. Then, clearly, $\phi \prec_f (\phi \wedge \psi)$ follows from CG_f. But, by the same count, we would also expect the proposition expressed by ϕ to ground the one expressed by $\phi \wedge (\phi \prec_f (\phi \wedge \psi))$—i.e., $\phi \prec_f (\phi \wedge (\phi \prec_f (\phi \wedge \psi)))$; after all, ϕ *is* a conjunct of $\phi \wedge (\phi \prec_f (\phi \wedge \psi))$. In a similar fashion, we can consider consequences of CG_f, DG_f or IG_f where the relata of the grounding symbol are statements containing sentential identity $\approx$. For instance, by CG_f, the proposition expressed by $\phi \approx \phi$ grounds the one expressed by $(\phi \approx \phi) \wedge \psi$, and by DG_f, the latter grounds the proposition expressed by $(\phi \vee \phi) \vee ((\phi \approx \phi) \wedge \psi)$—thus: $(\phi \approx \phi) \prec_f ((\phi \approx \phi) \wedge \psi)$ and $\big((\phi \approx \phi) \wedge \psi\big) \prec_f \big((\phi \vee \phi) \vee ((\phi \approx \phi) \wedge \psi)\big)$. Finally, by IG_f, none of the propositions expressed by these grounding statements grounds itself. Clearly, an infinitude of examples such as these and even more complex ones can be given.

Now, as natural and plausible as these are, the current model theories (found in Krämer, 2018; Correia, 2017; deRosset and Fine, 2023) cannot capture them. The main reason for this is that they, much like the other works cited above, work with logics in which such grounding statements are not grammatically well-formed, so the principles are consequently restricted as well. But why impose such draconian, artificial restrictions on language or logic? As mentioned by some of the authors, nothing other than convenience in treatment seems to play role in such restrictions. In fact, there is no trace of such restrictions in some of the pioneering works that put forward and argue for these principles (e.g., Fine, 2012a).

Finally, some philosophers have put forward certain distinct views about iterated grounding and the grounds of identity statements. Consider, for instance, the view endorsed by, e.g., Bennett (2011), according to which a grounding truth like $\phi \prec_f \psi$ is grounded by its ground ϕ; that is: $\phi \prec_f (\phi \prec_f \psi)$. Or consider the view due to Wilhelm (2020a), according to which identity statements of the general form 'a is a' are (entity) grounded by a, where a can be any entity, such as an individual, fact, proposition, or relation. One might pick up this idea and apply it to the context of fact-grounding (e.g., by arguing that fact-grounding is a special form of entity-grounding, where the entities are limited to facts or propositions), to come up with a similar principle according to which the truth expressed by the (propositional) identity $\phi \approx \phi$ is grounded by the one expressed by ϕ; that is, $\phi \prec_f (\phi \approx \phi)$.[6]

Again, the existing models all fall short of capturing such views simply because their languages don't even allow for forming them.

In Section 3.5 we discuss these restrictions in the existing semantics and the prospects of lifting them in more detail.

3.3 Language and Logics

In this section, I lay down the language and logics of different variants of ground in a rigorous way and establish some structural results derived

[6]Note that Wilhelm (2020a) argues for the adoption of entity-grounding, where all kinds of entities can enter into grounding relations, *over* the more familiar notion of fact-grounding that is at stake in this paper, not their coexistence. Moreover, one might argue that fact-grounding isn't a form of entity-grounding (see, e.g., Chapter 4 for an argument on this). Regardless of these, all that matters to us here is the possibility of verifying or falsifying such principles at the level of semantics—something that the existing semantics in the literature seem to fail to do.

from the logics of ground. I also lay down some principles characterizing Russellian propositions, and show that they entail all the structural results derived from the logics of ground.

We assume that we have infinitely many sentential variables $p_1, p_2, ...$, and show the set that contains them all with AT. Here's a presentation of our language $\mathcal{L}$:

Definition 3.3.1 (Language $\mathcal{L}$)**.** The formulas of $\mathcal{L}$ are constructed as follows:

1. p_i is a formula, where $i \in \mathbb{N}$,

2. If ϕ and ψ are formulas, then so are $\neg\phi$ and $\phi \circ \psi$, where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \prec, \prec_m, \approx\}$.

Aside from the familiar Boolean cases, formulas of the form $\phi \approx \psi$, $\phi \prec \psi$ and $\phi \prec_m \psi$ respectively represent statements of *propositional identity*, *immediate* and *mediate* grounding.

Notice that our connectives are all given as *primitive* symbols of the language; thus, e.g., we don't have $\phi \rightarrow \psi$ as a shorthand for $\neg\phi \vee \psi$. (Of course, as expected, from our logic it will follow that these are truth-functionally equivalent.) We will return to the importance of this choice at the end of Section 3.5.

We've mentioned since the beginning of the paper that our models are going to treat propositional identity along the lines of Russellian propositions, according to which propositions exhibit the same structure and identity conditions that their underlying sentences do. Our background logic of propositional identity, accordingly, would be expected to capture

Russellian propositions, hence, e.g., considering all non-identities of the form $\phi \not\approx \neg\neg\phi$, $\phi \not\approx \phi \vee \phi$, $\neg\phi \not\approx (\psi \wedge \gamma)$ and $\neg\phi \not\approx (\psi \prec \gamma)$, all reflecting similar corresponding syntactic non-identities, as theorems.

We will eventually do so (Section 3.3), but for now, it's worth seeing that even under certain plausible, minimal principles of identity, in general, the logic of immediate ground formulated above entails a considerable amount of propositional structure, at times even conforming to Russellian propositions. When put together with the higher-order parallel result due to Fritz (2021), this portrays a picture of structured propositions implied by considerations of ground, which is most straightforwardly and systematically captured by Russellian propositions; a result that we will establish shortly.

The principles of propositional identity that we endorse are schematically stated as follows:

Minimal Principles of Propositional Identity (MPPI)

1. $\phi \approx \phi$ — Ref
2. $(\phi \approx \psi) \rightarrow (\psi \approx \phi)$ — Sym
3. $((\phi \approx \psi) \wedge (\psi \approx \gamma)) \rightarrow (\phi \approx \gamma)$ — Tr
4. $((\phi \approx \psi) \wedge \phi) \rightarrow \psi$ — IdTr
5. $(\phi \approx \psi) \rightarrow (\neg\phi \approx \neg\psi)$ — IdSt_1
6. $((\phi \approx \psi) \wedge (\gamma \approx \theta)) \rightarrow ((\phi \circ \gamma) \approx (\psi \circ \theta))$, where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \prec, \prec_m, \approx\}$ — IdSt_2

The principles will be given in the assumption of all theorems of classical propositional calculus in the background, which we cite as PC.[7] The first three principles are the standard principles of identity—reflexivity, symmetry and transitivity. IdTr says if two propositions are identical the truth of one implies the truth of the other, and the last two are schemata take our connectives to be functional in behavior: for example, if ϕ and ψ are the same propositions, their negations are the same as well.

Non-Factive Ground

We now introduce the notion of immediate grounding and its unrestricted non-factive logic. After that, we state some structural results that follow from the logic and the principles of identity stated above.

Immediate grounding concerns the relation of grounding that is intimate and holds between two propositions without any other propositions mediating this; *mediate* ground allows for such mediation and can be defined in terms of 'chains' of immediate grounding statements (see Fine, 2012a, pp. 50-1, for a discussion of mediate and immediate grounding). To illustrate, all the principles informally sketched in Section 3.2 (IG, CG, DG and NG) exhibit principles of immediate, as well as mediate, grounding. On the other hand, since, e.g., $\phi \prec (\phi \wedge \psi)$ and $(\phi \wedge \psi) \prec ((\phi \wedge \psi) \wedge \gamma)$ are both instances of immediate grounding obtained using CG, by forming a 'chain' one can deduce $\phi \prec_m ((\phi \wedge \psi) \wedge \gamma)$; though, in this case, a parallel

[7]Strictly, PC extends the theorems of the usual classical propositional calculus with Boolean connectives by allowing to express identity as well as grounding statements. So, for example, In PC, from $\phi \approx \psi$ and $(\phi \approx \psi) \rightarrow \gamma$ follows γ, using *Modus Ponens*. See Fritz (2021); Dorr et al. (2021) as examples of works that use extended versions of propositional calculus, similarly or even more generally than here, in formulating logics in languages with higher expressive power.

immediate grounding relation doesn't hold. As expected, mediate but not immediate grounding is transitive.

Earlier we mentioned the distinction made between factive and non-factive grounding. As is expected, *factive* grounding concerns only true propositions, i.e., truths, or facts, whereas *non-factive* grounding allows for the relata of ground to be false. While factive grounding is what the literature is often interested in, non-factive grounding represents a more fundamental notion in terms of which factive grounding can be defined, but not necessarily *vice versa* (see, e.g., Fine, 2012a, pp. 48-50, for an introduction to this distinction and a discussion their interdefinability).

Here's the unrestricted logic of non-factive grounding (see, e.g., Wilhelm, 2020b; Fritz, 2021; Correia, 2017, for factive variants of these):[8,9]

[8]It should be noted that not all of these works introduce new principles; for example, Wilhelm (2020b) only works with some of these principles to derive certain inconsistencies against a particular coarse-grained view of propositional identity. Nevertheless, these are some of the works that embrace such principles in their analyses.

[9]One might take issue with CG with an instance such as the following: $\phi \prec (\phi \wedge \neg\phi)$. It might be thought that even in non-factive grounding where we don't necessarily deal with facts, A grounds B if A *would've* grounded B, were they true, or that there is a possible world where A is true and explains B, etc. While such readings seem intuitive at first glance, it's not clear if they can be developed consistently, or if we should push for a factive interpretation of non-factive grounding in the first place. In fact, Fine (2012a, p. 49) attempts to reduce non-factive to factive ground in a similar way to the ones above and he runs into difficulties, which essentially leads him to leave the notion of non-factive ground as a primitive notion (while he earlier defines the notions of factive in terms of non-factive ground, somewhat like ours). In general, the literature doesn't seem to support such reductions. In fact, some explicitly argue for a primitive reading of non-factive grounding. For instance, Litland (2017) chooses non-factive over factive grounding as primitive and works towards solving the so-called 'problem of iterated ground' using the notion of 'zero-grounded'. One might respond this way: "But inconsistencies cannot be grounded; why would one want to account for such grounding relations? What would be the point?" Non-factive grounding can perhaps be explained as a type of relation between propositions, as it were, which could be explained in terms of facts whenever those propositions happen to be true. In fact, this idea doesn't seem too far-fetched; in the recent literature, many take factive grounding as a relation between *true* propositions (Correia, 2017; Fritz, 2021, 2019; Wilhelm, 2020b; Litland, 2022; Woods, 2018). It would only seem plausible to consider non-factive grounding as a relation between propositions. Indeed, Litland (2022, footnote 3) explicitly sketches a novel reading of non-factive grounding

Unrestricted Non-Factive Immediate Ground (UNIG)

1. $\phi \nless \phi$ IG
2. $(\phi \prec (\psi \wedge \gamma)) \leftrightarrow ((\phi \approx \psi) \vee (\phi \approx \gamma))$ CG
3. $(\phi \prec (\psi \vee \gamma)) \leftrightarrow ((\phi \approx \psi) \vee (\phi \approx \gamma))$ DG
4. $(\phi \prec \neg(\psi \wedge \gamma)) \leftrightarrow ((\phi \approx \neg\psi) \vee (\phi \approx \neg\gamma))$ NCG
5. $(\phi \prec \neg(\psi \vee \gamma)) \leftrightarrow ((\phi \approx \neg\psi) \vee (\phi \approx \neg\gamma))$ NDG
6. $(\phi \prec \neg\neg\psi) \leftrightarrow (\phi \approx \psi)$ NG

Here is the unrestricted propositional logic of non-factive partial mediate ground (see, e.g., Fine (2012a); Krämer (2018); Schnieder (2011) for these):

Unrestricted Non-Factive Mediate Ground (UNMG)

1. $\phi \nless_m \phi$ IG_m
2. $((\phi \prec_m \psi) \wedge (\psi \prec_m \theta)) \rightarrow \phi \prec_m \theta$ TRG_m
3. $(\phi \prec_m (\phi \wedge \psi)) \wedge (\psi \prec_m (\phi \wedge \psi))$ CG_m
4. $(\phi \prec_m (\phi \vee \psi)) \wedge (\psi \prec_m (\phi \vee \psi))$ DG_m

exactly for the kind of suspicious cases such as ours along these lines and in terms of "impossible grounds": "One might want to work with a yet wider notion of non-factive ground where contradictory propositions like $p\&{\sim}p$ and $q\&{\sim}q$ can be distinguished by their having different impossible grounds—$p, {\sim}p$ and $q, {\sim}q$ respectively". Finally, if someone is still unhappy with principles like CG at a conceptual level due to such cases, one can still appreciate them for their formal utility, as they can underlie the other notions of grounding, such as factive and mediate grounding, and also a provide a powerful formal semantics for the propositional logics of ground, which is our goal in this paper. Essentially, one can consider developing a logic for this relation that fairly behaves like the relation of fact-grounding but holds between propositions in order to formally underlie and capture the logics of fact-grounding—whether or not such a relation between propositions is metaphysically intelligible. Thanks to an anonymous referee for drawing my attention to this issue.

5. $(\neg\phi \prec_m \neg(\phi \wedge \psi)) \wedge (\neg\psi \prec_m \neg(\phi \wedge \psi))$ $\quad$ NCG_m

6. $(\neg\phi \prec_m \neg(\phi \vee \psi)) \wedge (\neg\psi \prec_m \neg(\phi \vee \psi))$ $\quad$ NDG_m

7. $\phi \prec_m \neg\neg\phi$ $\quad$ NG_m

We can now see how the immediate logic imposes a considerable amount of structure on propositions.

Theorem 3.3.1. *The followings are theorems of the non-factive logic of immediate ground plus PC and the minimal principles of identity stated above, where, in all cases* $\circ \in \{\wedge, \vee\}$*. In other words, the following can be derived from* MPPI $\cup$ UNIG*:*

1. $\phi \not\approx \neg\neg\phi$

2. $\phi \not\approx (\phi \circ \phi)$

3. $(\neg\phi \approx \neg\psi) \rightarrow (\phi \approx \psi)$

4. $(\gamma \not\approx \psi) \rightarrow (\neg\phi \not\approx (\gamma \circ \psi))$

5. $\neg\phi \not\approx (\phi \circ \phi)$

6. $\big((\phi \not\approx \psi) \wedge ((\phi \circ \psi) \approx (\gamma \circ \theta))\big) \rightarrow \big((\phi \approx \gamma) \wedge (\psi \approx \theta)\big) \vee \big((\phi \approx \theta) \wedge (\psi \approx \gamma)\big)$

7. $((\phi \circ \phi) \approx (\gamma \circ \gamma)) \rightarrow (\phi \approx \gamma)$

Notice that these theorems all express cases of non-identities where only conjunctive and disjunctive propositions are at stake (reflected by the condition that $\circ \in \{\wedge, \vee\}$). This is due to the fact that our logic only posits principles of conjunctive and disjunctive grounds; if we had similar principles regarding grounds of other types of propositions, we could've

easily extended these results to retain even more structure and get closer to Russellian propositions (we discuss extensions of this nature in Section 3.5).

Russellian Propositions

We now posit a set of principles that characterize Russellian propositions; we will see all the non-identities above follow from these principles.[10]

Russellian Propositions (RP)

Axioms

1. Theorems of propositional calculus — PC
2. $\phi \approx \phi$ — Ref
3. $(\phi \approx \psi) \rightarrow (\psi \approx \phi)$ — Sym
4. $((\phi \approx \psi) \wedge (\psi \approx \gamma)) \rightarrow (\phi \approx \gamma)$ — Tr
5. $((\phi \approx \psi) \wedge \phi) \rightarrow \psi$ — IdTr
6. $((\phi \circ \psi) \approx (\gamma \circ \theta)) \leftrightarrow ((\phi \approx \gamma) \wedge (\psi \approx \theta))$, where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \prec, \prec_m, \approx\}$ — Str_1
7. $(\neg\phi \approx \neg\psi) \leftrightarrow (\phi \approx \psi)$ — Str_2
8. $(\phi \circ_1 \psi) \not\approx (\gamma \circ_2 \theta)$, where $\circ_1 \neq \circ_2 \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \prec, \prec_m, \approx\}$ — Str_3

[10] It well may be the case that this isn't a complete axiomatization of Russellian propositions in our limited language, but the present principles arguably capture most if not all possible cases that come to mind, and in any case, are more than enough for our purposes here.

9. $\neg\phi \not\approx (\psi \circ \gamma)$, where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \prec, \prec_m, \approx\}$ $\textsc{Str}_4$

Inference Rules

10. If $\vdash \phi \rightarrow \psi$ and $\vdash \phi$, then $\vdash \psi$ MP

Notice that $\textsc{IdSt}_1$ and $\textsc{IdSt}_2$ from earlier are encapsulated as the right-to-left sides of the principles $\textsc{Str}_2$ and $\textsc{Str}_1$, respectively. Note also that $\textsc{Str}_3$ and $\textsc{Str}_4$ are just generalizations of structured propositions that the grounding principles entail with the minimal logic of identity in the background; the only reason that we couldn't derive the more general form is that, at least as of now, we don't have grounding principles for the other connectives, such as $\rightarrow$ (see Section 3.5 for more on such principles). As a result, this means that Russellian propositions, characterized by RP above, prove all the cases of propositional identity and non-identity stated in theorem 3.3.1, and of course many more. In other words, MPPI is a strict fragment of RP. So we have:

Theorem 3.3.2. *The unrestricted propositional calculus with identity proves all theorems stated in theorem 3.3.1. That is, the latter can be derived from* RP $\cup$ UNIG

Note that this observation holds at the level of propositional logics of grounding, without quantifiers taken into account. A similar situation holds for the *higher-order* logic of immediate ground (Fritz, 2021), where many instances of a general, higher-order formulation of a schema representing Russellian propositions are entailed. Thus, as it was claimed before, these altogether suggest that the Russellian account of propositions is the most systematic account that captures all the structure that emerges

from the principles of grounding. In the future sections, We will officially adopt RP as our logic of propositional identity to provide our semantics for the logics of grounding explored in this section.

Factive Ground

As was mentioned before, factive immediate grounding statements are just non-factive statements where the relata of the grounding relation are both true; similarly for mediate grounding. That is, we have:

- $\phi \prec_f \psi \coloneqq (\phi \wedge \psi) \wedge (\phi \prec \psi)$
- $\phi \prec_{fm} \psi \coloneqq (\phi \wedge \psi) \wedge (\phi \prec_m \psi)$

The unrestricted logic of factive immediate ground is as follows (see, e.g., Wilhelm, 2020b; Fritz, 2021, for these principles):

Unrestricted Factive Immediate Ground (UFIG)

1. $(\phi \prec_f \psi) \rightarrow (\phi \wedge \psi)$ $\quad$ FG_f
2. $\phi \not\prec_f \phi$ $\quad$ IG_f
3. $\big(\phi \prec_f (\psi \wedge \gamma)\big) \leftrightarrow ((\psi \wedge \gamma) \wedge ((\phi \approx \psi) \vee (\phi \approx \gamma)))$ $\quad$ CG_f
4. $\big(\phi \prec_f (\psi \vee \gamma)\big) \leftrightarrow (\phi \wedge ((\phi \approx \psi) \vee (\phi \approx \gamma)))$ $\quad$ DG_f
5. $\big(\phi \prec_f \neg(\psi \wedge \gamma)\big) \leftrightarrow (\phi \wedge ((\phi \approx \neg\psi) \vee (\phi \approx \neg\gamma)))$ $\quad$ NCG_f
6. $\big(\phi \prec_f \neg(\psi \vee \gamma)\big) \leftrightarrow \big(\neg(\psi \vee \gamma) \wedge ((\phi \approx \neg\psi) \vee (\phi \approx \neg\gamma))\big)$ $\quad$ NDG_f
7. $\big(\phi \prec_f \neg\neg\psi\big) \leftrightarrow (\phi \wedge (\phi \approx \psi))$ $\quad$ NG_f

The unrestricted logic of factive mediate ground is as follows:

Unrestricted Factive Mediate Ground (UFMG)

1. $(\phi \prec_{fm} \psi) \rightarrow (\phi \wedge \psi)$ $\quad$ FG_{fm}

2. $\phi \not\prec_{fm} \phi$ $\quad$ IG_{fm}

3. $((\phi \prec_{fm} \psi) \wedge (\psi \prec_{fm} \theta)) \rightarrow (\phi \prec_{fm} \theta)$ $\quad$ TRG_{fm}

4. $(\phi \wedge \psi) \rightarrow ((\phi \prec_{fm} (\phi \wedge \psi)) \wedge (\psi \prec_{fm} (\phi \wedge \psi)))$ $\quad$ CG_{fm}

5. $(\phi \rightarrow (\phi \prec_{fm} \phi \vee \psi)) \wedge (\psi \rightarrow (\psi \prec_{fm} \phi \vee \psi))$ $\quad$ DG_{fm}

6. $(\neg\phi \rightarrow (\neg\phi \prec_{fm} \neg(\phi \wedge \psi))) \wedge (\neg\psi \rightarrow (\neg\psi \prec_{fm} \neg(\phi \wedge \psi)))$ $\quad$ NCG_{fm}

7. $\neg(\phi \vee \psi) \rightarrow ((\neg\phi \prec_{fm} \neg(\phi \vee \psi)) \wedge (\neg\psi \prec_{fm} \neg(\phi \vee \psi)))$ $\quad$ NDG_{fm}

8. $\phi \rightarrow (\phi \prec_{fm} \neg\neg\phi)$ $\quad$ NG_{fm}

It is easy to check that the principles of the factive logics are just theorems of their corresponding non-factive logics. That is, we have the following, where $\vdash$ stands for or basic propositional logic in the background (PC), and $\bigwedge$ conjuncts all the principles of the relevant logics.:

Theorem 3.3.3. $\mathrm{NFIG} \vdash \bigwedge \mathrm{UFIG}$ *and* $\mathrm{NFMG} \vdash \bigwedge \mathrm{UFMG}$.

Since these are straightforward (they only depend on the definition of 'factive ground' in terms of 'non-factive ground'), we won't provide proofs.

3.4 Semantics

We start by introducing our Russellian propositions, which, as expected, perfectly mirror the structures of the sentences of our language (hereafter

we sometimes drop the qualification 'Russellian'). We use capital letters $P_1, P_2, ...$ to represent atomic *propositions*, as it were, and $A, B, C, ...$ as metavariables for general propositions. We also make bold the connectives; thus, e.g., we have the propositional connective $\boldsymbol{\prec}$ instead of the sentential connective $\prec$.[11]

Similar to our language, we assume to have infinitely many *atomic propositions* $P_1, P_2, ...$, and set the set of all of them with AT^*. Here's the definition of our propositions:

Definition 3.4.1 (Propositions)**.** *Propositions* are constructed as follows:

1. P_i is a proposition, where $i \in \mathbb{N}$,

2. If A and B are propositions, then so are $\boldsymbol{\neg} A$ and $A \circ B$, where $\circ \in \{\boldsymbol{\wedge}, \boldsymbol{\vee}, \boldsymbol{\rightarrow}, \boldsymbol{\leftrightarrow}, \boldsymbol{\prec}, \boldsymbol{\prec}_p, \boldsymbol{\approx}\}$.

We signify the set of all propositions constructed in this way with $\mathcal{D}_{\langle\rangle}$, and call it the *propositional domain*.[12,13]

[11] In effect, we could use our object language $\mathcal{L}$ in a new capacity now, but to avoid potential confusion, we proceed as in here.

[12] Note that in both the definition of 'language' and 'proposition' we are constructing the relevant entities recursively, with the implicit assumption that a pair of sentences or propositions are identical if and only if they have the same structure. This really is enough in laying out the idea of unique readability, as is common in logic as well as philosophy texts. That said, the idea of unique readability can be more explicitly encoded in both Definitions 3.3.1 and 3.4.1 by using the notion of sets and set identity; we avoid this in the interest of simplicity. For example, one can take an approach along the following lines: first, take the P_is (atomic propositions) and all the propositional connectives ($\boldsymbol{\wedge}, \boldsymbol{\prec}$, etc.) as constituting an appropriate set of pairwise distinct sets (one suitable choice may be this: for each $i \in \mathbb{N}$, define $P_i := \mathbb{N} \times i$, $\boldsymbol{\neg} := \mathbb{R}$, $\boldsymbol{\wedge} := \mathbb{R}^2$, $\boldsymbol{\vee} := \mathbb{R}^3$, $\boldsymbol{\prec} := \mathbb{R}^4$, ...). Then, define the structured propositions using tuples—e.g., $A \circ B$ as $(A, \circ, B)$ and $\boldsymbol{\neg} A$ as $(\boldsymbol{\neg}, A)$. From this, all the Russellian propositions follow. For example, it follows that, e.g., $A \circ B = C \circ D$ if and only if $A = B$ and $C = D$, where $=$ is set identity. It also follows that no negative proposition of the form $\boldsymbol{\neg} A$ can be identical to a composite proposition of the form $B \circ C$, that is $\boldsymbol{\neg} A \neq (B \circ C)$; similarly, no proposition is identical to its double negation, i.e., $A \neq \boldsymbol{\neg}\boldsymbol{\neg} A$, and no proposition is identical to its composition with another proposition, i.e., $A \neq (A \circ B)$ and $A \neq (B \circ A)$.

[13] Our models assume that we have infinity many distinct atomic propositions, which

A crucial notion that plays role in our semantics of ground is that of grounding constituency. As we noticed in the previous section, logics of ground display certain structural patterns; grounding constituency encapsulates these patterns in their most general forms.

Definition 3.4.2 (Grounding Constituency)**.** We define *grounding constituency* as the relation $\sqsubset$ on the propositional domain $\mathcal{D}_{\langle\rangle}$, such that $A \sqsubset B$ (read: A is a grounding constituent of B) if, and only if, one of the following holds:

1. $B = A \circ C$ for some $C \in \mathcal{D}_{\langle\rangle}$ and $\circ \in \{\wedge, \vee\}$,
2. $B = C \circ A$ for some $C \in \mathcal{D}_{\langle\rangle}$ and $\circ \in \{\wedge, \vee\}$,
3. $A = \neg A^*$ & $B = \neg(A^* \circ B^*)$, for some $A^*, B^* \in \mathcal{D}_{\langle\rangle}$ and $\circ \in \{\wedge, \vee\}$,
4. $A = \neg A^*$ & $B = \neg(B^* \circ A^*)$, for some $A^*, B^* \in \mathcal{D}_{\langle\rangle}$ and $\circ \in \{\wedge, \vee\}$,
5. $B = \neg\neg A$.

As was mentioned before, to capture mediate grounding, we can think of statements of mediate grounding as obtained through 'chaining' immediate grounding statements. To implement this idea into our semantics, we define $\sqsubset^*$ as the transitive closure of $\sqsubset$.[14,15]

is a reasonable assumption, e.g., within a richer language that accommodates unary predicates and under the common view that any sentence of the form $F(a)$, where F is a unary, non-logical, predicate and a is an object, is an atomic proposition. There is, however, nothing crucial in what follows that hinges on this assumption; a finite base of atomic propositions will equally do and grant unique readability.

[14]The transitive closure of a binary relation R on a set X, in general, is the smallest relation R^* on X that contains R and is transitive, i.e., if aR^*b and bR^*c, then aR^*c. It's easy to prove that every binary relation has a transitive closure.

[15]Note that here we're not defining mediate grounding ($\prec_m$) as the transitive closure of immediate grounding ($\prec$) at the level of object language; we're merely *interpreting* the former as the transitive closure of the latter's interpretation, that is interdefining

Here's how we specify the truth value of our propositions:

Definition 3.4.3 (Atomic Truth Function; Truth Function)**.** An *atomic truth function* is a function $at : AT^* \to \{0, 1\}$. We define the *truth function* based on any atomic truth function at as the function $T_{at} : \mathcal{D}_{\langle\rangle} \to \{0, 1\}$ based on at as follows:

1. $T_{at}(P_i) = at(P_i)$, if and only if $P_i \in AT^*$
2. $T_{at}(\neg A) = 1$, if and only if $T_{at}(A) = 0$
3. Other Booleans as usual
4. $T_{at}(A \approx B) = 1$, if and only if $A = B$
5. $T_{at}(A \prec B) = 1$, if and only if $A \sqsubset B$
6. $T_{at}(A \prec_m B) = 1$, if and only if $A \sqsubset^* B$

Notice that the truth values of complex propositions are sensitive to the truth value of their constituents only in the case of Boolean propositions;

them at the level of semantics. The connectives themselves are treated as primitives in this paper, so aren't to be interdefined (see Definition 3.3.1). One might suggest that if we interdefine the connectives themselves, e.g., $\prec_m$ in terms of $\prec$, using the notion of transitive closure, the principles of UNMG will presumably just follow from those of UNIG. That might be true, but it would require higher-order quantification tools which are unavailable in our propositional language. For instance, one might define $\prec_m$ as follows: $\phi \prec_m \psi := \exists n \in \mathbb{N}\,(\phi = \phi_1 \prec \phi_2 \prec ... \prec \phi_n = \psi$. Alternatively, one might suggest just embracing the transitivity of mediate ground (i.e., TRG_m), and the rest of the principles of UNMG follow from that and UNIG. But even if some of the principles of UNMG follow, not all will—IG_m is an example. As for what *is* really the relationship between $\prec_m$ and $\prec$, i.e., of immediate and mediate grounding at the level of the object language, the answer seems unclear while we have no quantificational tools at our disposal. Our goal in this paper is to find a semantics for the principles of immediate and mediate ground, as entertained in the literature, and their interdefinability at the level of semantics does that for us. Thanks to an anonymous referee for drawing my attention to this issue.

what determines the truth value of identity and grounding statements is only the structure of the constituent propositions involved.

We now introduce our semantics by defining the 'interpretations' of statements of our language, which are essentially the propositions they denote.

Definition 3.4.4 (Assignment Function; Interpretation)**.** An *assignment function* is a function of the form $a : AT \to \mathcal{D}_{\langle\rangle}$. For any such function we define the *interpretation* based on a as the function $[\![.]\!]_a : \mathcal{L} \to \mathcal{D}_{\langle\rangle}$, such that:

1. $[\![p_i]\!]_a = a(p_i)$, if $p_i \in AT$,
2. $[\![\neg\phi]\!]_a = \boldsymbol{\neg}[\![\phi]\!]_a$,
3. $[\![\phi \circ \psi]\!]_a = [\![\phi]\!]_a \boldsymbol{\circ} [\![\psi]\!]_a$, where $\circ \in \{\wedge, \vee, \to, \leftrightarrow, \prec, \prec_m, \approx\}$ and $\boldsymbol{\circ}$ is the corresponding (emboldened) propositional operator.

We call any triple $(\mathcal{D}_{\langle\rangle}, at, a)$ a *propositional model*, where $\mathcal{D}_{\langle\rangle}$ is our propositional domain, at is an atomic truth function and a an assignment function. For a model $M := (\mathcal{D}_{\langle\rangle}, at, a)$ and sentence $\phi \in \mathcal{L}$, we say that ϕ is *true with respect to* M, written $M \vDash \phi$, if $T_{at}([\![\phi]\!]_a) = 1$. We call ϕ *valid* or a *truth*, written $\vDash \phi$, if it's true with respect to every model.[16]

[16]Note that from the semantics it follows that atomic propositions cannot ground one another. In particular, one might think "the fact that my shirt is maroon grounds the fact that it is red" (e.g., see Audi, 2012, p. 693), but our semantics doesn't accommodate that. This might be considered as a shortcoming of the semantics, however, our concern here is the logic of *impure* ground, with certain standard principles in mind. To the author's best knowledge, none of the alternative semantics in the literature (each imposing some kind of structure on propositions) can accommodate such claims, so even if they are correct, this won't be a unique problem to our semantics. Moreover, one might be able to somehow enrich the current semantics in a language where non-logical predicates are allowed, and take into account the inter-definability of properties in

Notice that in our models it's possible to assign *any* proposition whatsoever to any sentential letter of the language. This marks an important difference between our semantics with the one in Correia (2017): in the latter, 'crucially', sentential variables of the language cannot be assigned complex propositions. As Correia himself notes (see p. 517), this 'unorthodoxy' has a bearing on the applications of his logic to statements of natural language, and thus 'care is needed in order to apply the logic'.[17]

In any case, all of our logics of grounding, i.e., the unrestricted logic of immediate and mediate logics, both non-factive and factive, as well as Russellian Propositions, are sound with respect to our semantics (see Appendix II for a proof). Suppose $\vdash$ stands for derivability from UNIG $\cup$ UNMG $\cup$ UFIG $\cup$ UFMG $\cup$ RP.

Theorem 3.4.1 (Soundness)**.** *If* $\vdash \phi$*, then* $\vDash \phi$*.*

It is worth noticing that, as the proof of this theorem (as well as other soundness results from the next section) shows (see Appendix II), the fact that we are working with Russellian propositions plays a crucial role in our results.

Now, consider, e.g., the assignment function a such that $a(p_i) = P_i$ for all $p_i \in AT$, and the truth function at such that $at(P_i) = 0$. (Or consider any other pairs of assignment and truth functions, for that matter.) By

accounting for grounding statements containing them (Kiani, MSb, does this in a rigorous way for the neighboring notion of entity grounding)

[17]To give an example similar to Correia's: in natural languages, we can have the sentence 'Pluto is grue' (call it ϕ) to express the disjunctive proposition that Pluto is green or Pluto is blue. Assuming that the first and second disjuncts are respectively expressed by 'Pluto is green' (ψ) and 'Pluto is blue' (γ), we would normally want to consider the formal statements $\psi \prec \phi$ and $\gamma \prec \phi$ as true, but there's no way to get Correia's semantics to validate this judgment.

soundness, the induced model validates all the axioms of our logic. So, we have:

Corollary 3.4.1 (Consistency)**.** *The unrestricted propositional logics of immediate and mediate ground with identity, both non-factive and factive, are consistent. In other words,* $\mathrm{UNIG} \cup \mathrm{UNMG} \cup \mathrm{UFIG} \cup \mathrm{UFMG} \cup \mathrm{RP}$ *is consistent.*

Notice that our semantics can capture the infinitely many theorems of the kinds below which follow from our unrestricted logics, but which, as was noted in Section 3.2, the existing semantics in the literature fail to capture:

1. $(\phi \prec \psi) \prec ((\phi \prec \psi) \wedge \psi)$
2. $(\phi \approx \phi) \prec ((\phi \approx \phi) \vee \psi)$
3. $\big((\phi \approx \phi) \wedge \psi\big) \prec \big((\phi \vee \phi) \vee ((\phi \approx \phi) \wedge \psi)\big)$
4. $\Big((\phi \approx \phi) \prec ((\phi \approx \phi) \wedge \psi)\Big) \not\prec \Big((\phi \approx \phi) \prec ((\phi \approx \phi) \wedge \psi)\Big)$

$\vdots$

These are declared as true statements by our semantics, simply because $[\![\phi \prec \psi]\!]_a \sqsubset [\![(\phi \prec \psi) \wedge \psi]\!]_a$, $[\![\phi \approx \phi]\!]_a \sqsubset [\![(\phi \approx \phi) \vee \psi]\!]_a$ and so on, for any assignment function a.

We conclude the section by shedding light on the question of completeness. It's straightforward to see that our logics are not complete with respect to the proposed semantics. For instance, given the definition of grounding constituency, no model can validate a grounding statement

where the groundee is itself a grounding or identity statement, as they don't have the right structure to enter into the grounding constituency relation. That is, any statements of the form $\phi \prec (\psi \prec \gamma)$ or $\phi \prec (\psi \approx \gamma)$ is falsified by all models, no matter what propositions the schematic letters stand for; so their negations $\phi \not\prec (\psi \prec \gamma)$ and $\phi \not\prec (\psi \approx \gamma)$ must be valid. But there are no principles in our logic that would prove such statements.

At this point, one can choose between two options to achieve completeness: (i) add certain principles such as the ones above to the logic and make up for the gap, or (ii) extend the notion of grounding constituency in a way that, e.g., a proposition like A is considered as a grounding constituent of certain propositions like $C \prec D$ and $C \approx D$, thus avoiding the gap in a different way. (In the next section, we discuss some extensions along both lines in more detail.)

To be clear, although both of these options lead to filling some gap between our logic and semantics, that may or may not lead to completeness; we leave open how the gap is to be fully closed, and hence completeness achieved. However, we conclude the section by stating a close result, stating that our semantics, in its current form, gets right all the positive grounding claims (i.e., those that aren't in forms of negation); for a proof of this see Appendix II.

Theorem 3.4.2. *Every positive grounding and identity truth is provable:*

(i) If $\vDash \phi \prec \psi$, *then* $\vdash \phi \prec \psi$,

(ii) If $\vDash \phi \prec_f \psi$, *then* $\vdash \phi \prec_f \psi$,

(iii) If $\vDash \phi \prec_m \psi$, *then* $\vdash \phi \prec_m \psi$,

(iv) If $\vDash \phi \prec_{fm} \psi$, *then* $\vdash \phi \prec_{fm} \psi$,

(v) If $\vDash \phi \approx \psi$*, then* $\vdash \phi \approx \psi$*.*

3.5 Extensions

We now discuss two kinds of desirable extensions of our logics and semantics that aren't available to the existing semantic projects (Krämer, 2018; Correia, 2017; deRosset and Fine, 2023).[18]

Grounds of other Boolean Statements

There aren't many works on grounds of Boolean statements other than those that only contain instances of conjunction, disjunction and negation. An exception to this is Schnieder (2011), where, in laying down certain rules governing the logic of 'because' he proposes (the factive versions of) most of the rules that we have listed before under the factive logic of mediate ground, plus other Boolean cases. For instance, he offers the following principles regarding the grounds of conditional statements:

- $(\neg\phi \prec_m (\phi \to \psi)) \wedge (\psi \prec_m (\phi \to \psi))$ $\quad$ CoG$_m$
- $(\phi \prec_m \neg(\phi \to \psi)) \wedge (\neg\psi \prec_m \neg(\phi \to \psi))$ $\quad$ NCoG$_m$

As expected, the following are the corresponding immediate principles:

- $\gamma \prec (\phi \to \psi) \leftrightarrow (\gamma \approx \neg\phi \vee \gamma \approx \psi)$ $\quad$ CoG
- $\gamma \prec \neg(\phi \to \psi) \leftrightarrow (\gamma \approx \phi \vee \gamma \approx \neg\psi)$ $\quad$ NCoG

[18] See Poggiolesi and Francez (2021) for a tentative logic of 'exclusive' and 'ternary' notions of disjunction. While it is likely that these notions can also be captured in our approach as well, we don't attempt to establish that in this paper, as this paper is focused on the more urgent and widely used connectives that lack expressive semantics as shown.

To accommodate this, we can simply extend the notion of grounding constituency in a way that the desired principles of conditional grounding are accommodated. More specifically, we can add the following four clauses to the definition of $A \sqsubset B$ (Definition 3.4.2):

- $A = \neg A^*$ and $B = A^* \rightarrow C$ for some $A^*, C \in \mathcal{D}_{\langle\rangle}$,
- $B = C \rightarrow A$ for some $C \in \mathcal{D}_{\langle\rangle}$,
- $A = \neg A^*$ and $B = \neg(C \rightarrow A^*)$ for some $A^*, C \in \mathcal{D}_{\langle\rangle}$,
- $B = \neg(A \rightarrow C)$ for some $C \in \mathcal{D}_{\langle\rangle}$

Note that the semantics of ground expanded in this way also captures the logic of 'because' in Schnieder (2011), and in particular proves its consistency.[19] In general, assuming that we extend our models to accommodate the extended notion of grounding constituency, we have the following (see Appendix II for a proof):

Theorem 3.5.1. *CoG and NCoG are both valid.*

Consequently, the extended logics are all consistent as well.

Before moving on to the next type of extension, a remark is in order. Note that in stating the definition of our language $\mathcal{L}$ (Definition 3.4.1), we treated all the connectives from $\{\wedge, \vee, \rightarrow, \leftrightarrow, \prec, \prec_m, \approx\}$ as primitives, thus, in particular, avoided interdefining any of Boolean statements in terms of other ones. One might suggest otherwise, to deduce the relevant

[19]The general guiding principle behind Schnieder's logic is called 'core intuition', which he defines as follows (p. 448): 'A sentence governed by a classical truth-functional connective has its truth value because of the truth values of the embedded sentences.' The other Boolean cases dismissed here can be accounted for in a similar manner to the case of conditional grounding.

grounding principles from a smaller set of principles. For instance, it might be suggested to interdefine $\phi \to \psi$ as, e.g., $\neg\phi \vee \psi$ and deduce the principles of conditional grounding laid out above in terms of the principles of disjunctive grounds.

But this can't be easily done. To get a sense of complications attached to such identifications, suppose the identity above holds, thus $\phi \to \psi \approx \neg\phi \vee \psi$ is a theorem of our background identity logic. It then follows from DG and $\textsc{Str}_1$ that $\neg\phi \prec \phi \to \psi$ and $\psi \prec \phi \to \psi$. So far, so good: these in fact follow from CoG. But note that by NCoG, we have $\phi \prec \neg(\phi \to \psi)$. So, since from $\phi \to \psi \approx \neg\phi \vee \psi$ we have $\neg(\phi \to \psi) \approx \neg(\neg\phi \vee \psi)$, by $\textsc{Str}_1$ we have $\phi \prec \neg(\neg\phi \vee \psi)$. However, from NDG applied to $\neg(\neg\phi \vee \psi)$ it follows that the immediate grounds of $\neg(\neg\phi \vee \psi)$ are *only* $\neg\neg\phi$ and $\neg\psi$, hence, given that $\phi \not\approx \neg\neg\phi$, we must have $\phi \approx \neg\psi$. In other words, for any conditional $\phi \to \psi$ where $\phi \not\approx \neg\psi$, the identification of $\phi \to \psi$ with $\neg\phi \vee \psi$ leads to the inconsistency of extensions of our logical system with the plausible principles of immediate conditional ground due to Schnieder (2011).

In response to this, one might suggest rejecting one of the principles of, e.g., conditional grounding. But why do so? They are no less plausible than those governing conjunction or disjunction (also, see footnote 16 for a unified motivation behind them all). More importantly, as we noted above, the extended logic *is* provably consistent. So, the better option seems to be one that leaves the connectives alone and avoids reducing them to one another.

Iterated and Identity Grounding

We mentioned in Section 3.2 that, according to some authors (e.g., Bennett, 2011), a grounding fact like $\phi \prec \psi$ is grounded by its ground ϕ. If we take the immediate ground of $\phi \prec \psi$ to be only ϕ, then we have the following:

- $\gamma \prec (\phi \prec \psi) \leftrightarrow \gamma \approx \phi$ IDG

We also mentioned another plausible principle regarding grounds of statements of propositional identity: according to Wilhelm (2020a), e.g., identity statements of the general form $a \approx a$ are ('entity-')grounded by a, where a can be any entity. Someone might pick up this idea and argue for a fact-grounding counterpart of it, along the following lines (see footnote 5 for a proviso):

- $\psi \prec (\phi \approx \phi) \leftrightarrow \psi \approx \phi$ GG

Again, we can revise the notion of grounding constituency in a way that this is accounted for in our semantics, by adding the following clauses to the definition of $A \sqsubset B$ (Definition 3.4.2):

- $B = A \boldsymbol{\prec} C$ for some $C \in \mathcal{D}_{\langle\rangle}$,
- $B = A \boldsymbol{\approx} A$

Suppose we extend our conception of models to accommodate the extension of grounding constituency with these. Then we have the following (the proof is as straightforward as previous cases, so we omit them):

Theorem 3.5.2. *DIG and GG are both valid.*

As usual, the extended logics turn out to be consistent too.

Extending the Existing Semantics in the Literature

We conclude the section by reflecting on the status of the existing semantics (found in Correia, 2017; Krämer, 2018; deRosset and Fine, 2023) with regards to extensions of the logics and semantics along the lines above. We briefly noted in Section 3.2 that these semantics fail to accommodate an infinitude of grounding facts that follow from unrestricted impure logics of ground, as well as the distinct views on identity and iterated grounding glossed above, due to the artificial restrictions imposed on the languages in which the logics are formulated, where statements or propositions of iterated and identity grounding aren't allowed in the relata of ground.

Can these semantics be revised, though, to make up for these shortcomings? I charitably assume that *given the same linguistic limits*, all these semantics can be extended without any trouble, to capture the other Boolean extensions of their logics and semantics (though this really depends on certain details at play, I ignore that).

But what about the extensions that lift the restrictions of the language and logics to allow for statements of iterated and identity grounding to appear in the relata of grounding statements? In the case of Correia (2017), where he works with structured propositions of some sort, it might be possible to make certain revisions and generalizations to allow for the semantics to accommodate the unrestricted version of the principles he works with, though it's not clear if the semantic results in the paper that heavily rely on these notions stay intact under such extensions.[20] I leave

[20]The core notions upon which models are built in Correia (2017)—e.g., that of 'propositional structure' (which is the space of propositions and operators that construct them), 'degree' (measuring the complexity of propositions) and 'grounding relation' (holding between members of the propositional structure) (see pp. 518-19)—simply

this issue open here.

However, unlike Correia (2017), both Krämer (2018) and deRosset and Fine (2023) work with the *truthmaker* content of propositions. The idea of applying truthmaker semantics to logics of ground goes back to Fine (2012b), where he provides an elegant, sound and complete semantics for the pure logic of ground in terms of truthmakers. The original semantics of Fine (2012b) takes the semantic value of a statement to be its set of 'verifiers', i.e., the set of 'states' or 'facts' that make true the statement.

But while truthmakers work perfectly well in the case of pure logic, the impure logic soon displays various forms of resistance to the plain truthmaker semantics that Fine works with (see Fine, 2012a, footnote 22 for some early notes on this issue). For example, in order for the principles NG and IG to both go through, we need the truthmaker content of a sentence and its double negation to be, at the very least, distinct. Truthmaker semantics doesn't provide this: what verifies $\neg\phi$ falsifies ϕ, and what falsifies $\neg\phi$ verifies ϕ. So what verifies $\neg\neg\phi$ verifies ϕ, and what falsifies $\neg\neg\phi$ what falsifies ϕ; so ϕ and $\neg\neg\phi$ have the same truthmaker content, hence are identical. A similar problem holds for disjunction and conjunction: due to the standard way their truthmaker contents are defined, it turns out that ϕ, $\phi \vee \phi$, and $\phi \wedge \phi$ also have the same content, so the relevant instances of the principles CG and DG fail to hold.

So, to make truthmaker semantics work, certain revisions must be made on its standard workings. In particular, some sort of structure must

exclude cases of the sort where the relata of grounding are grounding or identity propositions. It's not clear which, if any, of the semantic results that heavily rely on these restrictions, such as soundness and completeness, can be retained under relevant extensions of these notions.

be imposed on the truthmaker content of sentences so that, e.g., ϕ, $\neg\neg\phi$, $\phi \vee \phi$ and $\phi \wedge \phi$ have pairwise distinct truthmaker contents; once that's done, the semantics of grounding must be given in a way that the desired principles of the impure logics are accounted for. This is exactly what both Krämer (2018) and deRosset and Fine (2023) do, though each in their own way. Krämer (2018) designs his 'mode-ified' semantics of ground, where his semantics relies on the 'modes of verification' of sentences—something which, according to himself, 'corresponds to a certain kind of answer to the question of how a truth is verified by a fact' (p. 786), and, in any case, leads to the required distinctions of truthmaker contents.

deRosset and Fine (2023), on the other hand, do a deeper dive into the semantics of a particular system of ground, called System GG, closely related to the logical system originally proposed in Fine (2012a). The semantics that this work proposes is both sound and complete, and captures total, as well as, partial grounds. It is also, like the semantics in Krämer (2018), based on a revised form truth-makers semantics which accommodates the appropriate structure that propositions need for the relevant principles of ground to go through. More specifically, deRosset and Fine adopt two notions of fusion that are more fine-grained than the usual one—'combination' (for conjunction) and 'choice (for disjunction)—and somehow elegantly capture the structured principles of System GG in a sound and complete way.

But as nice as the semantics in deRosset and Fine (2023) is in comparison to the other works in the literature, it still suffers from the exact same issue that the previous semantics do: the language is limited to the usual Boolean connectives (see, e.g., pp. 12-13 of the mentioned paper)

and the semantics is designed to exactly capture that, and nothing else. It's not clear when we extend the language, the semantics will be able to catch up.

In fact, aside from the artificial limitations of language and how that is built into the semantics, which is what all the existing semantics suffer from, there is a more profound issue with the semantics that particularly work with truthmakers. The issue is that even though it is straightforward to determine the verifiers and falsifiers of Boolean statements in whatever level of granularity, using truthmakers (in terms of 'fusions', 'manners of fusions', 'combination' or 'choice' of states; see, e.g., Fine, 2017c,a,b; Krämer, 2018; deRosset and Fine, 2023), it is, in general, not clear at all how to account for the verifiers and falsifiers of *grounding* or *identity* statements in terms of such fusions. This would require granting access to the truthmaker content of identity or grounding statements, and that's where truthmaker semantics hits the bottom.

In general, truthmaker semantics is in its infancy, and, to the best of our knowledge, there just isn't any work in the literature that would tell us what the truthmaker contents of statements other than Boolean, quantificational or modal statements look like—certainly not the truthmaker content of grounding or identity statements. And this problem doesn't seem to be easily resolvable: due to the hyperintensionality of ground and identity, it's very unlikely for the truthmaker content of grounding or identity statements to be definable using the truth-functionally behaved operation of 'fusion' on states, no matter how granular they are manipulated to become. Further operations on state spaces are likely required, and, as of now, it's not clear what they would look like or if they can

be philosophically motivated or formally developed in a plausible and consistent way.

The semantics that we provided in this paper, however, has in its premise the extreme flexibility that any extension of the language can ever want, because it simply reflects the structure of sentences in the language to the semantics, using the idea of Russellian propositions. Whatever operator one adds to the language will be mirrored to the semantics; all it takes for the semantics to capture the logic of the newly added connectives is to simply revise the definition of 'Grounding Constituency', in the way that was shown in this section.

3.6 Conclusion

I showed that models for sentence-like, Russellian propositions can be used to provide a unified, simple and highly expressive semantics for various unrestricted propositional logics of ground. I also showed that our semantics can be extended to accommodate certain distinct philosophical positions about grounds of grounding and identity statements. We noted that the existing semantics in the literature (Krämer, 2018; Correia, 2017; deRosset and Fine, 2023) fail to do either of these. More importantly, even though we left it open whether Correia's semantics is safely extendable to accommodate these, we noted that the prospects of extending the semantics in Krämer (2018) and deRosset and Fine (2023), and in general, any truthmaker semantics of ground is bleak and dependent on unexplored limitations of truthmaker semantics. Moreover, even though formally satisfying (within the usual artificial boundaries of the language that my

paper conveniently surpasses), it's not clear how satisfying these revisions of the truth-maker semantics are at a conceptual level, and how such diversity of truth-maker semantics found in the literature (each serving a specific philosophical purpose) and the levels of content granularity that follow from them can be summed up and explained in a bigger picture.

Also, aside from the expressiveness and predictable high flexibility of our semantics, there are certain advantages of our project over those that assume less granular accounts of propositions in accounting for the semantics of logics of ground (as in Correia, 2017) or addressing their consistencies (as in Wilhelm, 2020b). For one, our assumption of grain is much more systematic and widely entertained in the literature, ranging from attitude contexts in the philosophy of language (as in, e.g., Kaplan, 1977; King, 1996, 2009; Soames, 1987), to the neighboring notions of metaphysical priority, essence and ontological dependence (as in, e.g., Fine, 1995, 1980, 1994). Moreover, we noted that the Russellian view is arguably the most straightforward and systematic account of propositions that explains all the built-in structural commitments of the notion of ground explored in this paper and in Fritz (2021).

But as popular and useful as Russellian propositions are, basic Cantorian reasoning about cardinalities reveals that their assumption leads to certain inconsistencies—an issue that, interestingly, was first mentioned in the original work of Russell (1903) himself (see Appendix II), and later was re-discovered by Myhill (1958) (hence, the *Russell-Myhill* paradox), but surprisingly has been completely ignored in most recent works that assume or argue for Russellian propositions, as cited earlier (see Deutsch, 2008, on this ignorance and its consequences for philosophy). In fact,

only recently has this paradox been rediscovered within the background of simple type theory, thus bringing to light the inconsistency of Russellian propositions with the standard assumptions of higher-order logic (see, e.g., Hodes, 2015; Goodman, 2016). Accordingly, this might be taken to undermine the conceptual value of our project, suggesting that the models to be presented are merely mathematical constructs that by no means represent propositions.

The situation becomes even more dramatic when we realize that, as Fritz (2021) notes, the instances of Russellian propositions that the higher-order logic of immediate ground entails happen to be sufficient to reconstruct the Russell-Myhill result, effectively establishing the inconsistency of the relevant higher-order logic of ground in question. In other words, not every proposition has to be Russellian for the paradox to go through—a certain, smaller fragment of propositions that are so is sufficient to reconstruct the paradox. The higher-order logic of ground gives us just one such fragment and hence is inconsistent.

This portrays a rather bleak picture of the notion of ground when coupled with considerations of granularity, and leaves one wondering if there's a way to save logics of ground from the troubles of grain.

In response to these issues, in a series of broadly related papers I adopt a picture of propositions (as well as other relational entities, such as properties and relations), reminiscent of Russell (1908); Whitehead and Russell (1910), according to which they come in infinite levels, in a way that roughly put, the inhabitants of higher levels are systematically obtained through quantification over the ones from lower levels. We call this view the *ramified* account of propositions (similarly for other relational entities).

Once the ramified picture is deployed, one can consistently reformulate the Russellian view and avoid the Russell-Myhill paradox, as well as ground's higher-order inconsistency result due to Fritz.[21]

Aside from establishing the consistency of the ramified Russellian theory of propositions (explored in Kiani, MSd, i.e., Chapter 1), one aim of the series is to show that this view can itself be independently motivated via certain considerations having to do with a neighboring notion of metaphysical priority, namely 'entity grounding' (as introduced in, e.g., Wilhelm, 2020a; Schaffer, 2009; deRosset, 2013); this is shown in Kiani (MSb) (Chapter 4). Another part of the series shows that ramified Russellian propositions can be leveraged to semantically account for and establish the consistency of various logics of ground—ranging from propositional to higher-order—and provide a unified 'predicative' solution to a cluster of paradoxes of quantificational ground that have emerged in recent years (e.g., Donaldson, 2017; Fine, 2010; Krämer, 2013); a type of solution that has long been predicted but remained fairly underexplored (see, e.g., Fine, 2010; Krämer, 2013; Korbmacher, 2018a,b, for various forms of these puzzles and some solutions to certain variants of them).

It is this latter part with which the present paper was concerned: while higher-order logics of ramified ground, their semantics and consistency results, as well as their contribution to puzzles of quantificational ground, are all explored in Kiani (MSe) (Chapter 5), the task of this paper was to only explore how the assumption of Russellian propositions alone can

[21] Other solutions to the Russell-Myhill paradox can be given that are not based on ramified types. For instance, Deutsch (2014) proposes a solution that is based on set theory. But it's not clear how, aside from their mathematical use, such solutions fare in the context of grounding and, in general, metaphysics; we leave that open here.

be leveraged to semantically account for various *propositional* logics of ground and establish their consistencies. The semantic contributions of this paper prepare the groundwork based on which the more sophisticated, higher-order logics of ground from Kiani (MSe) (that is, Chapter 5) are semantically accounted for. As such, since here we only treat propositional logics of ground without quantification, implementing ramification isn't needed, and the assumption of Russellian propositions suffices for our purposes. As we have noticed, none of the results obtained in this paper rely on the other works in the series.

I left the questions of completeness open. Also, throughout the paper, I've only focused on strict partial grounds and their logics. As a result, other variants of grounding relations, such as total and weak grounding, as well as their logics, still need to be semantically accounted for. I wish to attend to these issues in the future.[22]

3.7 Appendix I - Logics of Identity and Ground

Here we repeat all the principles of grounding, as well as propositional identity, that we explored in the paper, for their accessibility and use in the formal proofs in the next appendix.

Minimal Principles of Propositional Identity (MPPI)

1. $\phi \approx \phi$ Ref

[22] **Acknowledgments:** I would like to thank Richard Zach for helpful commentary, advice and directions at different stages of writing this paper. Special thanks to Peter Fritz for many detailed and insightful discussions on various drafts of the paper.

2. $(\phi \approx \psi) \rightarrow (\psi \approx \phi)$ SYM

3. $((\phi \approx \psi) \wedge (\psi \approx \gamma)) \rightarrow (\phi \approx \gamma)$ TR

4. $((\phi \approx \psi) \wedge \phi) \rightarrow \psi$ IDTR

5. $(\phi \approx \psi) \rightarrow (\neg\phi \approx \neg\psi)$ IDST$_1$

6. $((\phi \approx \psi) \wedge (\gamma \approx \theta)) \rightarrow ((\phi \circ \gamma) \approx (\psi \circ \theta))$, where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \prec, \prec_m, \approx\}$ IDST$_2$

RUSSELLIAN PROPOSITIONS (RP)

Axioms

1. Theorems of propositional calculus PC

2. $\phi \approx \phi$ REF

3. $(\phi \approx \psi) \rightarrow (\psi \approx \phi)$ SYM

4. $((\phi \approx \psi) \wedge (\psi \approx \gamma)) \rightarrow (\phi \approx \gamma)$ TR

5. $((\phi \approx \psi) \wedge \phi) \rightarrow \psi$ IDTR

6. $((\phi \circ \psi) \approx (\gamma \circ \theta)) \leftrightarrow ((\phi \approx \gamma) \wedge (\psi \approx \theta))$, where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \prec, \prec_m, \approx\}$ STR$_1$

7. $(\neg\phi \approx \neg\psi) \leftrightarrow (\phi \approx \psi)$ STR$_2$

8. $(\phi \circ_1 \psi) \not\approx (\gamma \circ_2 \theta)$, where $\circ_1 \neq \circ_2 \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \prec, \prec_m, \approx\}$ STR$_3$

9. $\neg\phi \not\approx (\psi \circ \gamma)$, where $\circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow, \prec, \prec_m, \approx\}$ STR$_4$

Inference Rules

10. If $\vdash \phi \rightarrow \psi$ and $\vdash \phi$, then $\vdash \psi$ — MP

Unrestricted Non-Factive Immediate Ground (UNIG)

1. $\phi \nprec \phi$ — IG
2. $(\phi \prec (\psi \wedge \gamma)) \leftrightarrow ((\phi \approx \psi) \vee (\phi \approx \gamma))$ — CG
3. $(\phi \prec (\psi \vee \gamma)) \leftrightarrow ((\phi \approx \psi) \vee (\phi \approx \gamma))$ — DG
4. $(\phi \prec \neg(\psi \wedge \gamma)) \leftrightarrow ((\phi \approx \neg\psi) \vee (\phi \approx \neg\gamma))$ — NCG
5. $(\phi \prec \neg(\psi \vee \gamma)) \leftrightarrow ((\phi \approx \neg\psi) \vee (\phi \approx \neg\gamma))$ — NDG
6. $(\phi \prec \neg\neg\psi) \leftrightarrow (\phi \approx \psi)$ — NG

Unrestricted Non-Factive Mediate Ground (UNMG)

1. $\phi \nprec_m \phi$ — IG_m
2. $((\phi \prec_m \psi) \wedge (\psi \prec_m \theta)) \rightarrow \phi \prec_m \theta$ — TRG_m
3. $(\phi \prec_m (\phi \wedge \psi)) \wedge (\psi \prec_m (\phi \wedge \psi))$ — CG_m
4. $(\phi \prec_m (\phi \vee \psi)) \wedge (\psi \prec_m (\phi \vee \psi))$ — DG_m
5. $(\neg\phi \prec_m \neg(\phi \wedge \psi)) \wedge (\neg\psi \prec_m \neg(\phi \wedge \psi))$ — NCG_m
6. $(\neg\phi \prec_m \neg(\phi \vee \psi)) \wedge (\neg\psi \prec_m \neg(\phi \vee \psi))$ — NDG_m
7. $\phi \prec_m \neg\neg\phi$ — NG_m

Unrestricted Factive Immediate Ground (UFIG)

1. $(\phi \prec_f \psi) \rightarrow (\phi \wedge \psi)$ — FG_f

2. $\phi \not\prec_f \phi$ $\quad$ IG_f

3. $\big(\phi \prec_f (\psi \wedge \gamma)\big) \leftrightarrow ((\psi \wedge \gamma) \wedge ((\phi \approx \psi) \vee (\phi \approx \gamma)))$ $\quad$ CG_f

4. $\big(\phi \prec_f (\psi \vee \gamma)\big) \leftrightarrow (\phi \wedge ((\phi \approx \psi) \vee (\phi \approx \gamma)))$ $\quad$ DG_f

5. $\big(\phi \prec_f \neg(\psi \wedge \gamma)\big) \leftrightarrow (\phi \wedge ((\phi \approx \neg\psi) \vee (\phi \approx \neg\gamma)))$ $\quad$ NCG_f

6. $\big(\phi \prec_f \neg(\psi \vee \gamma)\big) \leftrightarrow \big(\neg(\psi \vee \gamma) \wedge ((\phi \approx \neg\psi) \vee (\phi \approx \neg\gamma))\big)$ $\quad$ NDG_f

7. $\big(\phi \prec_f \neg\neg\psi\big) \leftrightarrow (\phi \wedge (\phi \approx \psi))$ $\quad$ NG_f

Unrestricted Factive Mediate Ground (UFMG)

1. $(\phi \prec_{fm} \psi) \rightarrow (\phi \wedge \psi)$ $\quad$ FG_{fm}

2. $\phi \not\prec_{fm} \phi$ $\quad$ IG_{fm}

3. $((\phi \prec_{fm} \psi) \wedge (\psi \prec_{fm} \theta)) \rightarrow (\phi \prec_{fm} \theta)$ $\quad$ TRG_{fm}

4. $(\phi \wedge \psi) \rightarrow ((\phi \prec_{fm} (\phi \wedge \psi)) \wedge (\psi \prec_{fm} (\phi \wedge \psi)))$ $\quad$ CG_{fm}

5. $(\phi \rightarrow (\phi \prec_{fm} \phi \vee \psi)) \wedge (\psi \rightarrow (\psi \prec_{fm} \phi \vee \psi))$ $\quad$ DG_{fm}

6. $(\neg\phi \rightarrow (\neg\phi \prec_{fm} \neg(\phi \wedge \psi))) \wedge (\neg\psi \rightarrow (\neg\psi \prec_{fm} \neg(\phi \wedge \psi)))$ $\quad$ NCG_{fm}

7. $\neg(\phi \vee \psi) \rightarrow ((\neg\phi \prec_{fm} \neg(\phi \vee \psi)) \wedge (\neg\psi \prec_{fm} \neg(\phi \vee \psi)))$ $\quad$ NDG_{fm}

8. $\phi \rightarrow (\phi \prec_{fm} \neg\neg\phi)$ $\quad$ NG_{fm}

3.8 Appendix II - Technical Results

Proof (**theorem 3.3.1**). We only give proof for all items where $\circ$ is $\wedge$; the cases where $\circ$ is $\vee$ can be proved quite similarly. In the cases where

we prove the theorem by contradiction, we specify the assumption that is to be refuted in the end, for readability, and obtain a contradiction, $\bot$.[23]

- $\phi \not\approx \neg\neg\phi$

Proof.

(1)	$\phi \approx \neg\neg\phi$	Assumption (to be refuted)
(2)	$\phi \approx \phi$	Ref
(3)	$\phi \prec \neg\neg\phi$	NG
(4)	$(\phi \prec \phi) \approx (\phi \prec \neg\neg\phi)$	IdSt$_2$ 1, 2
(5)	$(\phi \prec \neg\neg\phi) \approx (\phi \prec \phi)$	Sym 4
(6)	$\phi \prec \phi$	IdTr 3, 5
(7)	$\phi \not\prec \phi$	IG
(8)	$\bot$	PC 6, 7

□

- $\phi \not\approx (\phi \wedge \phi)$

Proof.

(1)	$\phi \approx (\phi \wedge \phi)$	Assumption (to be refuted)
(2)	$\phi \approx \phi$	Ref
(3)	$\phi \prec (\phi \wedge \phi)$	CG
(4)	$(\phi \prec \phi) \approx (\phi \prec (\phi \wedge \phi))$	IdSt$_2$ 1, 2
(5)	$(\phi \prec (\phi \wedge \phi)) \approx (\phi \prec \phi)$	Sym 4
(6)	$\phi \prec \phi$	IdTr 3, 5
(7)	$\phi \not\prec \phi$	IG
(8)	$\bot$	PC 6, 7

□

[23]The proofs proceed in Hilbert-Style for higher rigor, where at each line the relevant axiom and potentially the previous lines or theorems are cited. One can offer an English reading of such proofs for higher readability, as is sometimes done in works of metaphysics (see, e.g., Bacon, 2018; Dorr, 2016; Dorr et al., 2021).

- $(\neg\phi \approx \neg\psi) \rightarrow (\phi \approx \psi)$

  Proof.

(1)	$\neg\phi \approx \neg\psi$	Assumption
(2)	$\phi \approx \phi$	Ref
(3)	$\neg\neg\phi \approx \neg\neg\psi$	IdSt_1 1
(4)	$(\phi \prec \neg\neg\phi) \approx (\phi \prec \neg\neg\psi)$	IdSt_2 2, 3
(5)	$\phi \prec \neg\neg\phi$	NG
(6)	$\phi \prec \neg\neg\psi$	IdTr 4,5
(7)	$\phi \prec \neg\neg\psi \rightarrow \phi \approx \psi$	NG
(8)	$\phi \approx \psi$	MP 6, 7

□

- $(\gamma \not\approx \psi) \rightarrow (\neg\phi \not\approx (\gamma \wedge \psi))$

  Proof.

(1)	$\gamma \not\approx \psi$	Assumption
(2)	$\neg\phi \approx \gamma \wedge \psi$	Assumption (to be refuted)
(3)	$\neg\neg\phi \approx \neg(\gamma \wedge \psi)$	IdSt$_1$ 2
(4)	$\neg(\gamma \wedge \psi) \approx \neg\neg\phi$	Sym 3
(5)	$\neg\gamma \approx \neg\gamma$	Ref
(6)	$\neg\gamma \prec \neg(\gamma \wedge \psi)$	NCG
(7)	$(\neg\gamma \prec \neg\neg\phi) \approx (\neg\gamma \prec \neg(\gamma \wedge \psi))$	IdSt$_2$ 5, 3
(8)	$(\neg\gamma \prec \neg(\gamma \wedge \psi)) \approx (\neg\gamma \prec \neg\neg\phi)$	Sym 7
(9)	$\neg\gamma \prec \neg\neg\phi$	IdTr 6, 8
(10)	$\neg\gamma \approx \phi$	NG 9
(11)	$\neg\psi \prec \neg(\gamma \wedge \psi)$	NCG
(12)	$\neg\psi \approx \neg\psi$	Ref
(13)	$(\neg\psi \prec \neg\neg\phi) \approx (\neg\psi \prec \neg(\gamma \wedge \psi))$	IdSt$_2$ 12, 3
(14)	$(\neg\psi \prec \neg(\gamma \wedge \psi)) \approx (\neg\psi \prec \neg\neg\phi)$	Sym 13
(15)	$(\neg\psi \prec \neg\neg\phi)$	IdTr 11, 14
(16)	$\neg\psi \approx \phi$	NG 15
(17)	$\phi \approx \neg\psi$	Sym 16
(18)	$\neg\gamma \approx \neg\psi$	TR 10, 17
(19)	$\gamma \approx \psi$	Theorem 3.3.1 3 18
(20)	$\perp$	PC 1, 19 $\square$

- $\neg\phi \not\approx (\phi \wedge \phi)$

Proof.

(1)	$\neg\phi \approx \phi \wedge \phi$	Assumption (to be refuted)
(2)	$\neg\neg\phi \approx \neg(\phi \wedge \phi)$	IdSt$_1$ 1
(3)	$\phi \approx \phi$	Ref
(4)	$(\phi \prec \neg\neg\phi) \approx (\phi \prec \neg(\phi \wedge \phi))$	IdSt$_2$ 3, 2
(5)	$\phi \prec \neg\neg\phi$	NG
(6)	$\phi \prec \neg(\phi \wedge \phi)$	IdTr 4, 5
(7)	$\phi \approx \neg\phi$	NCG 6
(8)	$\neg\phi \approx \neg\neg\phi$	IdSt$_1$ 7
(9)	$\phi \approx \neg\neg\phi$	TR 7, 8
(10)	$\phi \prec \neg\neg\phi$	NG
(11)	$(\phi \prec \phi) \approx (\phi \prec \neg\neg\phi)$	IdSt$_2$ 3, 10
(12)	$(\phi \prec \neg\neg\phi) \approx (\phi \prec \phi)$	Sym 11
(13)	$\phi \prec \phi$	IdTr 10, 12
(14)	$\phi \not\prec \phi$	NG
(15)	$\perp$	PC 13, 14 $\square$

- $\big((\phi \not\approx \psi) \wedge ((\phi \circ \psi) \approx (\gamma \circ \theta))\big) \rightarrow \big((\phi \approx \gamma) \wedge (\psi \approx \theta)\big) \vee \big((\phi \approx \theta) \wedge (\psi \approx \gamma)\big)$

  Proof.

(1)	$(\phi \not\approx \psi) \wedge ((\phi \wedge \psi) \approx (\gamma \wedge \theta))$	Assumption
(2)	$\phi \not\approx \psi$	PC 1
(3)	$(\phi \wedge \psi) \approx (\gamma \wedge \theta)$	PC 1
(4)	$\neg(\phi \wedge \psi) \approx \neg(\gamma \wedge \theta)$	IdSt$_1$ 3
(5)	$\neg\neg(\phi \wedge \psi) \approx \neg\neg(\gamma \wedge \theta)$	IdSt$_1$ 4
(6)	$\neg\phi \prec \neg(\phi \wedge \psi)$	NCG
(7)	$\neg\phi \approx \neg\phi$	Ref
(8)	$(\neg\phi \prec \neg(\phi \wedge \psi)) \approx (\neg\phi \prec \neg(\gamma \wedge \theta))$	IdSt$_2$ 7, 4
(9)	$\neg\phi \prec \neg(\gamma \wedge \theta)$	IdTr 6, 8
(10)	$(\neg\phi \approx \neg\gamma) \vee (\neg\phi \approx \neg\theta)$	NCG 9
(11)	$(\phi \approx \gamma) \vee (\phi \approx \theta)$	PC, Theorem 3.3.1 3 10
(12)	$\neg\psi \prec \neg(\phi \wedge \psi)$	NCG
(13)	$\neg\psi \approx \neg\psi$	Ref
(14)	$(\neg\psi \prec \neg(\phi \wedge \psi)) \approx (\neg\psi \prec \neg(\gamma \wedge \theta))$	IdSt$_2$ 13, 4
(15)	$\neg\psi \prec \neg(\gamma \wedge \theta)$	IdTr 12, 14
(16)	$(\neg\psi \approx \neg\gamma) \vee (\neg\psi \approx \neg\theta)$	NCG 15
(17)	$(\psi \approx \gamma) \vee (\psi \approx \theta)$	PC, Theorem 3.3.1 3 16
(18)	$((\phi \approx \gamma) \vee (\phi \approx \theta)) \wedge ((\psi \approx \gamma) \vee (\psi \approx \theta))$	PC 11, 17
(19)	$((\phi \approx \theta) \wedge (\psi \approx \gamma)) \vee ((\phi \approx \gamma) \wedge (\psi \approx \theta))$	PC 2, 18

□

Note that, applying basic laws of propositional calculus we find out that (17) is equivalent to $((\phi \approx \gamma) \wedge (\psi \approx \gamma)) \vee ((\phi \approx \theta) \wedge (\psi \approx \gamma)) \vee ((\phi \approx \gamma) \wedge (\psi \approx \theta)) \vee ((\phi \approx \theta) \wedge (\psi \approx \theta))$, but the first and the fourth disjuncts from this disjunction will not be the case due to (2); that's how we get to (19) by PC.

- $((\phi \wedge \phi) \approx (\gamma \wedge \gamma)) \rightarrow (\phi \approx \gamma)$

Proof.

(1)	$(\phi \wedge \phi) \approx (\gamma \wedge \gamma)$	Assumption
(2)	$\phi \approx \phi$	Ref
(3)	$(\phi \prec (\phi \wedge \phi)) \approx (\phi \prec (\gamma \wedge \gamma))$	IdSt$_2$ 2, 1
(4)	$(\phi \prec (\phi \wedge \phi)$	CG
(5)	$\phi \prec (\gamma \wedge \gamma)$	IdTr 3, 4
(6)	$(\phi \approx \gamma) \vee (\phi \approx \gamma)$	CG 5
(7)	$\phi \approx \gamma$	PC 6

□

□

Proof (**Theorem 3.4.1**). To save space, I mainly focus on the non-factive logic immediate ground (UNIG) and prove the validity of IG, CG, NDG and NG as samples; I also prove TRG$_m$ as a sample for mediate grounding principles. The rest of the principles and logics are done similarly, by direct use of the definitions.

IG. For an arbitrary model $M = (\mathcal{D}_{\langle\rangle}, at, a)$, suppose on the contrary that $M \vDash \phi \prec \phi$. Then $T_{at}([\![\phi \prec \phi]\!]_a) = 1$, hence $[\![\phi]\!]_a \sqsubset [\![\phi]\!]_a$. Let $[\![\phi]\!]_a := A$. According to Definition 3.4.2 (Grounding Constituency), the following constitute all the possible cases: **(i)** $A = A \circ C$ or $A = C \circ A$ for some $C \in \mathcal{D}_{\langle\rangle}$; **(ii)** $A = \neg A^*$ and either $A = \neg(A^* \circ C)$ or $A = \neg(C \circ A^*)$ for some $A^*, C \in \mathcal{D}_{\langle\rangle}$; **(iii)** $A = \neg\neg A$. All these cases are impossible due to the unique readability of the propositions. In other words, as was mentioned before (see footnote 10), since by

Definition 3.4.1, our propositions are as structured as sentences, all these five cases fail due to structural mismatch.

CG ($\Leftarrow$). For an arbitrary model $M = (\mathcal{D}_{\langle\rangle}, at, a)$, suppose $M \vDash \phi \approx \psi \vee \phi \approx \gamma$. Then $M \vDash \phi \approx \psi$ or $M \vDash \phi \approx \gamma$. In the first case we have $T_{at}([\![\phi \approx \psi]\!]_a) \coloneqq T_{at}([\![\phi]\!]_a \mathbin{\boldsymbol{\approx}} [\![\psi]\!]_a) = 1$, so $[\![\phi]\!]_a = [\![\psi]\!]_a$, thus $[\![\phi \wedge \gamma]\!]_a \coloneqq [\![\phi]\!]_a \mathbin{\boldsymbol{\wedge}} [\![\gamma]\!]_a = [\![\psi]\!]_a \mathbin{\boldsymbol{\wedge}} [\![\gamma]\!]_a \coloneqq [\![\psi \wedge \gamma]\!]_a$, and hence $[\![\phi]\!]_a \sqsubset [\![\psi \wedge \gamma]\!]_a$, because $[\![\phi]\!]_a \sqsubset [\![\phi \wedge \psi]\!]_a$. So, we have $T_{at}([\![\phi]\!]_a \mathbin{\boldsymbol{\prec}} [\![\psi \wedge \gamma]\!]_a) \coloneqq T_{at}([\![\phi \prec \psi \wedge \gamma]\!]_a) = 1$. Similarly for the second case. So, in either case we have $M \vDash \phi \prec (\psi \wedge \gamma)$. $\square$

CG ($\Rightarrow$). For an arbitrary model M, suppose $M \vDash \phi \prec (\psi \wedge \gamma)$. Then we have $[\![\phi]\!]_a \sqsubset [\![\psi \wedge \gamma]\!]_a$. By Definition 3.4.2, one of the following must hold: **(i)** $[\![\psi]\!]_a \mathbin{\boldsymbol{\wedge}} [\![\gamma]\!]_a = [\![\phi]\!]_a \mathbin{\boldsymbol{\circ}} B$ or $[\![\psi]\!]_a \mathbin{\boldsymbol{\wedge}} [\![\gamma]\!]_a = B \mathbin{\boldsymbol{\circ}} [\![\phi]\!]_a$, for some $\boldsymbol{\circ} \in \{\boldsymbol{\wedge}, \boldsymbol{\vee}\}$ and $B \in \mathcal{D}_{\langle\rangle}$; **(ii)** $[\![\phi]\!]_a = \boldsymbol{\neg} A^*$ and either $[\![\psi]\!]_a \mathbin{\boldsymbol{\wedge}} [\![\gamma]\!]_a = \boldsymbol{\neg}(A^* \mathbin{\boldsymbol{\circ}} B^*)$ or $[\![\psi]\!]_a \mathbin{\boldsymbol{\wedge}} [\![\gamma]\!]_a = \boldsymbol{\neg}(B^* \mathbin{\boldsymbol{\circ}} A^*)$, for some $\boldsymbol{\circ} \in \{\boldsymbol{\wedge}, \boldsymbol{\vee}\}$ and $A^*, B^* \in \mathcal{D}_{\langle\rangle}$; **(iii)** $[\![\psi]\!]_a \mathbin{\boldsymbol{\wedge}} [\![\gamma]\!]_a = \boldsymbol{\neg\neg}[\![\phi]\!]_a$. All options packed in **(ii)** and **(iii)** are immediately ruled out by the corresponding propositional non-identities due to structural differences. That leaves us with **(i)**. Again, the only possible identity for **(i)** is when $\boldsymbol{\circ} = \boldsymbol{\wedge}$. It follows that either $[\![\psi]\!]_a = [\![\phi]\!]_a$ or $[\![\gamma]\!]_a = [\![\phi]\!]_a$. Hence $M \vDash \psi \approx \phi \vee \gamma \approx \phi$.

NDG ($\Rightarrow$). For an arbitrary model M, suppose $M \vDash \phi \prec \neg(\psi \vee \gamma)$. Then we have $[\![\phi]\!]_a \sqsubset \boldsymbol{\neg}([\![\psi]\!]_a \mathbin{\boldsymbol{\vee}} [\![\gamma]\!]_a)$. From all the possible cases of constituency, only the following are structurally possible:

(iii) $[\![\phi]\!]_a = A^*$ and $\boldsymbol{\neg}([\![\psi]\!]_a \mathbin{\boldsymbol{\vee}} [\![\gamma]\!]_a) = \boldsymbol{\neg}(A^* \mathbin{\boldsymbol{\vee}} C)$ for some

$A^*, C \in \mathcal{D}_{\langle\rangle}$,

(iv) $[\![\phi]\!]_a = A^*$ and $\neg([\![\psi]\!]_a \vee [\![\gamma]\!]_a) = \neg(C \vee A^*)$ for some $A^*, C \in \mathcal{D}_{\langle\rangle}$.

So either of these two can hold. Now, from the first one it follows that $[\![\psi]\!]_a = A^*$, so $[\![\neg\psi]\!]_a := \neg[\![\psi]\!]_a = \neg A^* = [\![\phi]\!]_a$, and thus $M \vDash (\psi \approx \neg\phi)$. In a similar manner, from the second one it follows that $M \vDash (\gamma \approx \neg\phi)$. As a result, it follows that $M \vDash (\psi \approx \neg\phi) \vee (\gamma \approx \neg\phi)$.

NDG ($\Leftarrow$). This side holds because for any given assignment function a, we have $[\![\phi]\!]_a \sqsubset [\![\neg(\phi \vee \gamma)]\!]_a$ and $[\![\gamma]\!]_a \sqsubset [\![\neg(\phi \vee \gamma)]\!]_a$.

NG ($\Rightarrow$). For an arbitrary model M, suppose $M \vDash \psi \prec \neg\neg\phi$. The only structurally possible case is that $\neg\neg[\![\psi]\!]_a = \neg\neg[\![\phi]\!]_a$, thus $[\![\psi]\!]_a = [\![\phi]\!]_a$, and hence $M \vDash (\psi \approx \phi)$.

NG ($\Leftarrow$). This side holds, because $[\![\phi]\!]_a \sqsubset \neg\neg[\![\phi]\!]_a$ for any assignment a.

TRG$_m$. The aim is to show $M \vDash ((\phi \prec_m \psi) \wedge (\psi \prec_m \theta)) \rightarrow \phi \prec_m \theta$ for an arbitrary model $M = (\mathcal{D}_{\langle\rangle}, at, a)$. That is, we need to show that if $M \vDash ((\phi \prec_m \psi) \wedge (\psi \prec_m \theta))$, then $M \vDash \phi \prec_m \theta$. Suppose $M \vDash ((\phi \prec_m \psi) \wedge (\psi \prec_m \theta))$. It follows that $M \vDash \phi \prec_m \psi$ and $M \vDash \psi \prec_m \theta$. From the first relation it follows that $[\![\phi]\!]_a \sqsubset^* [\![\psi]\!]_a$, where $\sqsubset^*$ is the transitive closure of the grounding constituency relation $\sqsubset$; from the second it follows that $[\![\psi]\!]_a \sqsubset^* [\![\theta]\!]_a$. Since $\sqsubset^*$ is a transitive closure, it follows that $[\![\phi]\!]_a \sqsubset^* [\![\theta]\!]_a$ (see Footnote 14), which means that $M \vDash \phi \prec_m \theta$.

$\square$

To prove Theorem 3.4.2, we need the following lemma:

Lemma. *If $a^* : AT \to \mathcal{D}_{\langle\rangle}$ is the assignment function such that $a^*(p_i) = P_i$, the induced interpretation function $[\![.]\!]_{a^*} : \mathcal{L} \to \mathcal{D}_{\langle\rangle}$ is one to one.*

Proof. Induction on the structure of propositions in $\mathcal{D}_{\langle\rangle}$ (surjection) and on the structure of formulas in $\mathcal{L}$ (injection). □

We call the assignment function described in the lemma *straightforward*.

Proof (**Theorem 3.4.2**). **(i)** Suppose $\vDash \phi \prec \psi$. Then $[\![\phi]\!]_a \sqsubset [\![\psi]\!]_a$ for every assignment function a. Consider, in particular, the straightforward assignment function a^*. By Definition 3.4.2 there are 5 possible general cases. We only prove the claim for one of them; the rest are proved similarly, using the fact that a^* is 1-1 function.

> $[\![\psi]\!]_{a^*} = [\![\phi]\!]_{a^*} \circ B$ or $[\![\psi]\!]_{a^*} = B \circ [\![\phi]\!]_{a^*}$ for some $B \in \mathcal{D}_{\langle\rangle}$ and $\circ \in \{\wedge, \vee\}$. Consider the first case. By Lemma, $B = [\![\gamma]\!]_{a^*}$ for some $\gamma \in \mathcal{L}$. So we have $[\![\phi]\!]_{a^*} \circ B = [\![\phi]\!]_{a^*} \circ [\![\gamma]\!]_{a^*}$, and hence $[\![\psi]\!]_{a^*} = [\![\phi]\!]_{a^*} \circ [\![\gamma]\!]_{a^*} = [\![\phi \circ \gamma]\!]_{a^*}$. Since a injective (Lemma), we have it that ψ is syntactically identical to $\phi \circ \gamma$, hence by Ref, we have $\vdash \psi \approx \phi \circ \gamma$. Now, since by CG and DG (depending on the choice of $\circ$) we have $\vdash \phi \prec (\phi \circ \gamma)$, it follows by Str_1 that $\vdash \phi \prec (\phi \circ \gamma) \approx \phi \prec \psi$. From propositional calculus (PC) and Str_1 it follows that $\vdash \phi \prec \psi$. □

(iii) Suppose $\vDash \phi \prec_m \psi$. Then $[\![\phi]\!]_a \sqsubset^* [\![\psi]\!]_a$ for every assignment function a. So for every assignment a there are propositions $A_1, A_2, ...A_n \in \mathcal{D}_{\langle\rangle}$ such that $[\![\phi]\!]_a \sqsubset A_1 \sqsubset A_2 \sqsubset ... \sqsubset A_n \sqsubset [\![\psi]\!]_a$. Now, consider the

straightforward assignment function a^*. Due to Lemma, for each $i = 1, ..., n$, we have $A_i = [\![\gamma_i]\!]_{a^*}$ for some formula γ_i. Thus our 'chain' of immediate grounding relations turns into $[\![\phi]\!]_{a^*} \sqsubset [\![\gamma_1]\!]_{a^*} \sqsubset [\![\gamma_2]\!]_{a^*} \sqsubset ... \sqsubset [\![\gamma_n]\!]_{a^*} \sqsubset [\![\psi]\!]_{a^*}$. From the proof of case (i) above, we have $\vdash \phi \prec \gamma_1$, $\vdash \gamma_1 \prec \gamma_2$, ..., $\vdash \gamma_n \prec \psi$. By multiple applications of the transitivity schema (TRG_m) and *modus ponens* (MP), we obtain $\vdash \phi \prec \psi$.

□

Proof (**Theorem 3.5.1**). Straightforward. I only prove the left-to-right of CoG. For an arbitrary (extended) model M, suppose that $M \vDash \gamma \prec (\phi \to \psi)$. Then we have $[\![\gamma]\!]_a \sqsubset [\![\phi \to \psi]\!]_a := [\![\phi]\!]_a \to [\![\psi]\!]_a$. Given the extended definition of grounding constituency and the sentence-like structure of propositions, the only possible cases here are the following: **(i)** $[\![\gamma]\!]_a = \neg A^*$ and $[\![\phi]\!]_a \to [\![\psi]\!]_a = A^* \to C$ for some $A^*, C \in \mathcal{D}_{\langle\rangle}$, which results in $[\![\gamma]\!]_a = \neg[\![\phi]\!]_a$, or **(ii)** $[\![\phi]\!]_a \to [\![\psi]\!]_a = C \to [\![\gamma]\!]_a$ for some $C \in \mathcal{D}_{\langle\rangle}$, which entails $[\![\gamma]\!]_a = [\![\psi]\!]_a$. Thus either of the identities $[\![\gamma]\!]_a = [\![\neg\phi]\!]_a$ and $[\![\gamma]\!]_a = [\![\psi]\!]_a$ holds, hence $M \vDash \gamma \approx \neg\phi \vee \gamma \approx \psi$. □

Chapter 4

Entity Grounding, Structure and Ramification

4.1 Introduction

The literature on metaphysical ground often conceives the relation of grounding as only concerning facts or fact-like entities that hold 'in virtue of' other such entities, manifesting the idea that the latter 'explain' or are 'more fundamental' than the former (see, e.g., Fine, 2012a; Rosen, 2010; Audi, 2012, for such construals of ground). A few exceptions to this tradition stand out, however, according to which entities of all kinds, such as individuals, propositions, facts, properties and relations, are capable of entering into grounding relations (as in, e.g., Schaffer, 2009; Wilhelm, 2020a; deRosset, 2013)—what is sometimes called 'entity grounding' (Wilhelm, 2020a).

In this paper, I lay down and defend certain plausible principles of entity grounding, along the lines of what's been explicitly or implicitly

entertained in the literature, and argue that they require propositions, properties and other types of relations each to come in infinite levels, where, roughly put, the inhabitants of higher levels are obtained through quantification over the ones from lower levels. I then propose certain ramified type systems that best capture the talk of entity grounding and the infinitary hierarchies it calls for. The ultimate goal of this paper is to argue that certain considerations regarding entity grounding and structure call for infinitary hierarchies of relational entities such as propositions and properties, and to rigorously devise a ramified type system that captures them.

Here's how the paper is organized. In §4.2 I introduce the notion of entity grounding and argue for some minimal principles that characterize it. In §4.3 I argue that attempts in capturing the propositional fragment of the talk of entity grounding naturally lead to the fragmentation of the space of propositions into an infinite hierarchy of levels. §4.4 explicates notions of structure and constituency for relational entities and argues for stratification of all relational types. In §4.5 I introduce a general ramified type system motivated by discussions taking place in the preceding sections. The paper concludes in §4.6.

4.2 Entity Grounding and Its Principles

In this section, I introduce the notion of entity grounding and lay down some plausible principles that characterize it. The next two sections will utilize these principles and argue for the fragmentation of propositions, as well as other relational entities, into certain infinitary hierarchies of levels

that are best captured by ramified type systems.

Entity grounding (hereafter: *e-grounding*) is a relation of metaphysical priority that can hold between entities of any type. An individual may e-ground a proposition or fact, a proposition may e-ground a property, a property may e-ground a relation or a proposition, and so on. To illustrate with examples along the lines of the literature: '[for any entity i,] $i = i$ is grounded in i' (Wilhelm, 2020a), 'Obama, the man in full, grounds the fact that Obama exists; Obama grounds his singleton; the property *being white* grounds *being white or square*; England grounds (in part) the property of *being queen of England*' (deRosset, 2013).[1]

The examples above, and many more in the literature, highlight an implicit or explicit sense of structural complexity that statements of e-grounding exhibit. Thus, correctly saying that a e-grounds b reflects the fact that a is, as it were, a 'building block' of b, or b is somehow 'constructed' in part from a.[2] It is this construction, it would seem, that

[1]Traces of the idea that Russell's ramified type theory can be motivated by a sense of 'construction', and the relation of this to vicious circle principles, can be found in Gödel (1944) and Quine (1963), though the resolution there is that only a *constructivist* worldview, according to which, e.g., mathematical entities are pure constructions of mind, can accommodate ramified types (see, e.g., Gödel, 1944, p. 456). However, Jung (1999) and Hylton (2008) argue through textual evidence that Russell's notion of 'presupposition' does exactly carry the relevant sense of construction, though in a completely *realist* background. My notion of entity grounding is, in fact, motivated by and close kin of Russell's 'presupposition', and most of the principles applying to the former (as introduced below) correspond to similar principles governing the latter (as laid down in the references above). However, unlike entity grounding, Russell's 'presuppositions' aren't given primitively, and in fact are closer in nature to naive, 'modal-existential' conceptions of ontological dependence which have been heavily criticized in the recent literature (see, e.g., Fine, 1995). Another difference is our departure from Russell's rather contentious assumptions on the nature of propositional functions and how they're related to presuppositions (see Hylton, 2008, for a discussion of the latter). In what follows I will not address these historical remarks due to the scope limits of the paper

[2]One might think the talk of 'building blocks' presupposes some sense of unique decomposition of entities into parts—*the* parts that constitute them, as it were—and that there might be more than one way to decompose a relational entity into parts,

puts a prior to b in a metaphysical sense. We will explicate the sense of construction at stake further, along the way.[3]

We mentioned that e-grounding is a relation of metaphysical priority. One might wonder at the outset whether the notion of e-grounding is the same as that of fact grounding (henceforth: f-grounding). But while there are an infinitude of f-grounding examples that also establish instances of e-grounding (for instance, a conjunctive fact is both f-grounded and e-grounded by its conjuncts; more examples will become evident as we further explore e-grounding), these relations are not the same. The most straightforward reason for this is that, as was mentioned earlier, f-grounding, as opposed to e-grounding, is much more restricted in its scope, being only concerned with entities like propositions and facts.[4] Also, and as it will become clear along the way, even if one narrows down the scope of e-grounding to fact-like entities, the notion does not have anything to do with the truth of the entities involved, but rather, somehow with their structural complexity.

In what follows, I will argue for some plausible principles that characterize the notion of e-grounding.[5] We start by taking, along with Schaffer

much like there are many ways to cut a sphere into two hemispheres. While it well might be so, the assumption of either a unique or plural decomposition of entities into constituents would seem to serve our purposes in this paper equally well; in effect, we can run our arguments for those decompositions that are relevant to our purposes.

[3]It is to be noted that while in this paper we mostly take interest in and focus on structurally related entities in our explication of e-grounding, some of the examples in the literature may not necessarily carry that sense. For instance, according to Schaffer (2009), a Swiss cheese grounds its holes, or natural properties ground moral properties. Whether or not such examples can count as instances of true e-grounding statements, we find the 'constructional' intuition behind e-grounding plausible and somewhat crucial in the discussions to follow.

[4]That said, however, it's not at all obvious whether or not f-grounding is a special kind of e-grounding. We will return to this point when we introduce the e-grounding principle S, blow.

[5]As we will see, most of these principles are either explicitly or implicitly, and

(2009), the notion of e-grounding as a primitive, that is not analyzable in terms of any other notions. Also, following Schaffer (2009); deRosset (2013), I take the relation of e-grounding to be transitive and irreflexive (hence a strict order).

Thus we have our first two principles:

- *Nothing e-grounds itself.* IR
- *If a e-grounds b and b e-grounds c, then a e-grounds c.* TR

These requirements are especially natural when, as suggested earlier, we come to think about the relation as somehow measuring the 'constructional' profile of entities. Surely nothing is a 'building block' of itself. Also, if a is a 'building block' of b, and b is itself a 'building block' of yet another thing c, then there is a sense in which a is a 'building block' of c.

Another principle that we would like to entertain is along the lines of this: propositions, properties and relations are e-grounded by their constituents, assuming that a suitable sense of relational constituency is in place. For instance, the proposition that Joe drinks soda is e-grounded by Joe, and the property of drinking soda, and the property of being friends with Geoff is e-grounded by Geoff and the relation of friendship. As the examples from the beginning of the section show, assumptions similar to this can also be found, in implicit or explicit forms, in the literature.

We would like to entertain a principle along these lines, but since one of the goals of this paper is to rigorously account for the talk of

in part or fully entertained by other authors in the literature, and otherwise quite naturally build upon them. At the end of the paper, we will also mention a few works where wider applications of entity grounding in metaphysics, particularly its power to settle a wide range of puzzles of ground and grain, are explored. The results in this paper, however, in no way depend on those, and this paper can be read independently.

e-grounding in suitable formal languages, it would be desirable to recover the constituents of propositions, properties or relations, from the syntactic structure of the expressions that refer to them. That is, first we would like to find a suitable notion of constituency for relational entities in our formal language that captures our intuitions about the constituency of relational entities (which we rigorously will at §5); once we have such a notion pinned down, our principle will be as follows:

- *Entities picked by expressions are e-grounded by the things that are picked by the constituents of those expressions.* S

Thus, for instance, we would like to say that the proposition *Joe is sleeping* is e-grounded by Joe and the property of sleeping because the sentence expressing that proposition—'Joe is sleeping'—has as constituents the name 'Joe' and the predicate 'is sleeping'.

Some qualifications about S, and in particular its propositional fragment, are in order. First, notice that in S we're not taking the e-grounds of an entity to consist *only* of the things denoted by its constitutive expressions. (Call the version of S that does so the *strong* variant.) We're only *including* those things in the list of entities that e-ground the proposition, but we are open, and in many cases, obligated to, allowing for more things to count as e-grounds of the proposition. (Call the more liberal version of S the *weak* variant.) We will return to the importance of this choice after we introduce our next principle.

Second, consider the propositional instance of S, according to which propositions are e-grounded by the denotations of the expressions which constitute the sentences expressing them. This principle, in either of its

strong or weak readings, clearly imposes *some* structure on propositions. But how much structure do we really need for this to go through? The most granular account of propositions available in the literature takes propositions to almost exactly reflect the syntactic structure of the sentences that express them, in a way that two propositions $F(a)$ and $G(b)$ are the same only if F and G are the same, and a and b are the same; call this identity condition Structure (see, e.g., Kaplan, 1977; Soames, 1987; Russell, 1903; King, 1996, 2009, for such structured accounts of propositions). But endorsing the weak variant of S doesn't necessarily lead to Structure. For consider the pair of propositions *Sarah lives in LA or John is happy* and *John is happy or Sarah lives in LA*. By Structure, these two are different propositions (because *Sarah lives in LA* and *John is happy* are different propositions), but by S, they have the same grounds; there's nothing else that tells us whether or not they are identical.[6] So as long as we adopt the weak variant of S, we don't need to endorse highly structured accounts of propositions along the lines of Structure.[7]

That said, however, even weak S would presumably require enough

[6]This is best explained by a categorematic reading of connectives, as is the common reading in the literature in higher-order metaphysics (see, e.g., Bacon, 2018), where 'and' denotes a binary relation that takes propositions as arguments. A syncategorematic approach can also be accommodated by Structure.

[7]On the other hand, it can be readily seen that the strong variant of S leads to the highly granular picture of propositions manifested by Structure. For by strong S, $F(a)$ is e-grounded *only* by a and F, and $G(b)$ is e-grounded *only* by b and G. Now, if $F(a)$ and $G(b)$ are the same, they should have the same grounds; in particular, $F(a)$ also *only* has b and G as its grounds. It then follows that $\{a, F\} = \{b, G\}$, which, only plausibly results in F being G, and a being b. (Other possibilities manifest type mismatch in any language where predicates and names are considered as members of different syntactic categories.) Now, it can be shown that Structure is inconsistent with simple type theory due to the Russell and Myhill paradox (Myhill, 1958; Russell, 1903; Hodes, 2015; Uzquiano, 2015; Goodman, 2016). But even so, we can still argue that Structure since it is consistent with the ramified type system which we will eventually adopt in this paper. A rigorous consistency proof is yet to be found; a step towards a proof can be found in Kiani (MSd).

structure that coarse-grained views of propositions, such as Booleanism (Dorr, 2016; Bacon, 2018), would become difficult to maintain. For example, according to Booleanism, the sentence 'John is happy' and its self-conjunction, 'John is happy and John is happy', express the same proposition because they're provably equivalent, but under any plausible sense of syntactic constituency, the former sentence is a constituent of the latter, so, by S, the proposition expressed by the latter is e-grounded by the one expressed by the former, hence they have to be distinct due to IR. This is one of the major conflicts of our project and some of the rival views in recent metaphysics, where such coarse-grained accounts of propositions stand out.[8]

So, if the instances of S that concern propositions are true, then propositions cannot be too coarse-grained. That said, however, we will see in the next two sections that it's still possible to argue that propositions, properties and relations have to come in levels, *even if* propositions aren't structured at all. In fact, in §5 we will see that it's only essential for non-propositional types of relational entities to be structured in certain ways, for ramification to go through unless certain assumptions regarding quantificational statements are in place (more on this shortly). In any case, we find the propositional instances of S plausible, and in the rest of the paper, we put them out in the open and leave it to the reader to choose whether or not to accept it. We will revisit and discuss this choice and its implications later in the paper.

[8]Another related conflict consists in the type systems that we use: while views like Booleanism extract metaphysics from simple type systems, we cannot capture the talk of entity grounding within such systems (as this paper shows) and have to deploy ramified type systems, in the end.

Finally, earlier we argued that due to scope differences, f-grounding and e-grounding cannot be identified. But we also brought up the natural question of whether or not f-grounding can be considered as an instance of e-grounding. Our principle S answers to this negatively. Consider again the fact that John is happy. Assuming that *John is happy* is a constituent of *John is not happy*, from *S* it follows that the former e-grounds the latter. But of course, we can't say the same thing about f-grounding, not at least under a *factive* conception of f-grounding, according to which the relata of grounding statements should be true.[9,10]

The final assumption that we make about e-grounding is this:

- *Quantificational propositions are e-grounded by all the entities they non-vacuously quantify over.* Q

For example, consider the proposition *every individual is self-identical*. By Q this proposition is e-grounded by every individual. On the other hand, we don't want to say that *every property of individuals is such that Mike lives in Chicago* is e-grounded by every property of individuals.

[9] Accordingly, non-factive grounding might have a different status in this regard, and in fact, be a special kind of e-grounding. See Fine (2012a) for more on the distinction between factive and none-factive grounding.

[10] This example also elaborates on another distinction between fact grounding and entity grounding, in that the former, as is standardly assumed, seems to depict an 'explanation' element between facts, where a fact grounding another in certain ways 'explains' it. On the other hand, entity grounding, if understood as a reflection of explanation, could lead to the counter-intuitive result that a fact explains its negation. This is why despite the temptation we shouldn't try hard to understand entity grounding relations as depicting some sort of explanatory element (not to mention that such reading wouldn't make much sense when it comes to non-fact entities such as properties and relations). The better intuition seems to be along the lines of 'construction', as the examples suggest. Another potential reading of e-grounding is along the lines of essentialist or definitional 'dependence', where certain entities depend on certain others via their essences etc. While this reading seems plausible and particularly rooted in Russell's realist original motivations of ramified type theory, I won't investigate this reading here.

There's a sense in which *Mike lives in Chicago* in no way uses every property of individuals as a 'building block'. For instance, the property of jumping off a cliff seems to play no contribution in the construction of the proposition in question. This is why it's important to assure that Q concerns non-vacuous quantification.

A natural way to motivate Q is via construing universal and existential quantification as 'long', possibly infinite conjunctions and disjunctions, respectively. In that case, Q will become a special case of a more general version of S, where structured propositions with infinite constituents are allowed.[11]

Notice, however, that such construals aren't necessary for committing to Q. It just seems quite natural and intuitive to think of, e.g., universal quantification as somehow built out of the things it quantifies over, even if it's not construed as a conjunction. Phenomena like this, where an entity that uses up, as it were, all of a kind in its construction falls out of the range of the things it uses, aren't unheard of. Set theory is a good source of such examples. For instance assuming that A is a set, the singleton $\{A\}$, which, in a way is 'constructed' out of A with a set-formation operation, doesn't belong to A.[12]

As another example, consider the way ordinal numbers are constructed in

[11]The idea of reducing quantificational sentences to infinitary conjunctions or disjunctions goes back to the school of logical atomism (see, e.g., Russell, 1918, lecture 5, for an early objection to the idea). For some recent discussions of the problems such construals face, in particular, what's known as the 'totality problem', see Fine (2012a, 2017c). Throughout the paper, we sometimes make such reductionist assumptions about quantification, but mainly heuristically; there are, however, ways to make rigorous these assumptions. See the relevant remark at the end of Section 5, for more on this comparison.

[12]This is a particularly suggestive example because singletons are canonically taken to be e-grounded by their elements (deRosset, 2013; Schaffer, 2009). See Fine (1995) for a similar view regarding the ontological dependence of singletons on their elements.

set theory: ω is constructed through a union over *all* natural numbers, and itself falls out of their realm; $\omega+\omega$ is constructed by consuming *all* ordinals of the form $\omega + i$, for natural i, and itself falls out of them, and so on. This hierarchical construction of transfinite ordinals by unioning over all numbers beneath them needn't be cashed out in terms of 'infinitary sums'; it is an independently plausible and useful construction. Yet another rich source of such objects is category theory. In general *accessible categories* are (possibly) large categories that are in a certain way constructed by small categories; e.g., the objects of the former are colimits of small objects from the latter and fall out of their range.[13]

In any case, now that we have introduced Q, we're also in a position to see why we chose the weak variant of S over its strong variant: this is mandated upon us by Q. For example, by Q, every individual e-grounds the proposition *some individual is distinct from John*. Now, it follows, for instance, that Sarah e-grounds *some individual is distinct from John*, but Sarah is not picked by any of the syntactic constituents of the sentence, 'Some individual is distinct from John'. That said, however, if take Q as an instance of S, then we end up committing to strong S.

To conclude the section, here's a summary of the list of our principles of e-grounding:

1. *Nothing e-grounds itself.* IR

2. *If a e-grounds b and b e-grounds c, then a e-grounds c.* TR

3. *Entities denoted by expressions are e-grounded by the things that are picked by the constituents of those expressions.* *S*

[13]See, e.g., Adamek and Rosicky (1994) for an introduction to accessible categories.

4. *Quantificational propositions are e-grounded by all the entities they non-vacuously quantify over.* Q

4.3 Ramifying Propositions

I now argue that for the principles of e-grounding to be accommodated we need an infinite hierarchy of propositions, where, roughly put, tenants of each level are obtained through quantification from those of the lower levels. In line with this, we develop a formal language and logic that capture such a stratified universe of propositions as well as the talk of e-grounding based upon it.

We start with a simple, informal argument showing that our principles, in fact, two of them, lead to an inconsistency. Then we argue that the particular inconsistency involved is most naturally and efficiently resolved if there are at least two kinds of propositions, where the members of the second kind are obtained through quantification over all the members of the first kind. Ensuing arguments suggest that there must at least three kinds of hierarchical propositions, at least four kinds, ..., and for any natural number n, at least n kinds of them. In general, we will soon realize that the principles of e-grounding are most naturally and efficiently captured, without running into those generic inconsistencies, if propositions come in an infinite hierarchy of levels as described above.

Throughout this section, we focus only on the propositional fragment of the principles above—instances that are concerned with the relation of e-grounding between propositions. By doing so we can pinpoint the issues at stake in a setting where there aren't many types of entities available. Not

only this avoids certain heavy-handed metaphysical commitments to all sorts of higher-order entities, but it also helps us to see in the clearest way why we need propositions to come in a ramified hierarchy, without being distracted by other entities and the structural complexities associated with them. Moreover, as we will see, capturing the propositional fragment is considerably easier when it comes to developing formal languages and logics for e-grounding. The next section will look up to this section as a role model and argue for stratifying other relational types into infinitary levels.

Before we start, there's an important methodological remark that needs to be explicated. Throughout this section, as well as the rest of the paper, we assume that the principles of e-grounding from the previous section articulate substantive constraints on e-grounding that we aim not to abandon. In other words, we take these principles true by stipulation and set out to explore their implications as well as the formal systems that can capture them. Aside from the plausibility and naturality of the principles, which we discussed in the previous section, it is this loyalty that, as we will see shortly, quite naturally leads to hierarchical propositions, and later on, other relational entities in a way that is best captured by ramified type systems. On the other hand, and as has been advertised several times, the latter is what provides a unified and natural solution to a cluster of puzzles and paradoxes of ground and grain. The intuitive appeal of our principles, espoused with other abductive, large-scale considerations surrounding them, gives us enough confidence to hold onto these principles as much as possible.

With this methodological remark out of the way, we now offer a series

of arguments to the effect that there must be infinitely many kinds of propositions that behave in the way explained earlier. The arguments here are informal; a formalization of the talk propositional e-grounding and the arguments here will follow later in the section.

We first start by giving an informal presentation of a paradox that arises from our principles of e-grounding. By Q, any proposition p which states that every proposition is such and such (e.g., is true or false) is e-grounded by *every* proposition. So p cannot be among the propositions in its range of quantification, otherwise, contradiction ensues by IR. So the range of quantification in p consists of *all* propositions, and yet p cannot belong to it, hence contradiction.

There are two choice options in response to this argument: (i) at least one of IR or Q is false, or (ii) there is no such proposition as *all propositions are such and such*, or rather, p doesn't express any proposition. In line with the methodological remark above, we avoid option (i). But option (ii), with no further explanation attached to it, doesn't sound satisfactory either: the sentence 'Every proposition is such and such' supposedly denotes *something*. In fact, anyone who commits to second-order quantification as a reliable source for doing metaphysics would admit that p expresses a proposition (e.g., Williamson, 2013; Fine, 1970; Kaplan, 1970). So, something needs to fill in the gap if p is taken not to be express a proposition.

To get around this tension, we could posit certain new types of entities that behave very much like the good old propositions but aren't propositions, strictly speaking. That is, in line with option (ii), we take it that p doesn't denote to a *proposition*, but it does denote to something

akin to a proposition, only more sensitive, in its nature, to quantification. So, there should be at least two kinds of proposition-like entities, one of which is obtained by quantification over all members of the other. Call the latter *level-1*, and the former *level-2* propositions. These namings seem appropriate: as we will see later, leveled propositions interact with each other, and exhibit truth-functional behaviors very similar to how the good old propositions do. In fact, leveled propositions together will play the theoretical roles that propositions are supposed to play alone, but (it would seem) also accommodate the talk of e-grounding, without running into inconsistencies. In light of this, if one still insists on keeping the term 'proposition' in their metaphysical vocabulary, one can then take the term to ambiguously refer to either level-1 or level-2 propositions.

This segregation also gains support from our intuitions about the notion of e-grounding in terms of 'construction' and the constructional profile of entities: if a e-grounds b, then a is, in a sense, a 'building block' of b. In particular, we took quantificational propositions to be 'constructed' out of the things they quantify over. By considering $\forall qq$ as a member of the collection that it ranges over, however, we'd be treating $\forall qq$ as if it's one of its own 'building blocks,' which doesn't sit well with that intuition.[14]

[14]Another helpful way to see the issue at stake is to revisit our heuristic way of construing universal quantifications as infinitary conjunctions (similarly for existential quantifications construed as infinitary disjunctions). Suppose $\forall p\phi$ is just a 'long', infinitary conjunction $\phi(\psi) \wedge \phi(\gamma) \wedge ...$ in a language with infinitary conjunctions and enough constants for all propositions. Then, in general, instantiating a universal statement $\forall p\phi$ with itself will amount to considering the long conjunction $\phi(\psi)\wedge\phi(\gamma)\wedge...$ as one of its own conjuncts. Under a sufficiently structured view of propositions, which is presupposed by our principle S, it's no more appropriate to consider the 'long' conjunction as one of its own conjuncts than it is to take the 'short' conjunction $\phi \wedge \psi$ as one of its own conjuncts.

But the satisfaction that a bi-level account of propositions brings is only temporary. For we can similarly argue that there are at least *three* kinds of propositions. Suppose, on the contrary, that there are exactly two kinds of propositions, level-1 and level-2 propositions, as described above. Suppose also that Q is naturally revised for the new propositions: propositions that quantify over level-i propositions are e-grounded by all level-i propositions (where $i = 1, 2$). Given our assumption, the proposition p that every level-2 proposition is such and such should be either of level 1 or 2. Also, by Q, p is e-grounded by all level-2 propositions. Now, suppose p is of level 1. But since at least one level-2 proposition q (e.g., the proposition that all level-1 propositions are true or false) is e-grounded by *all* level-1 propositions, the proposition q must also be e-grounded by p. By TR, p e-grounds itself, which goes against IR. And if p is of level 2, then again it e-grounds itself. So p must be denoting a *third* kind of entity that's akin to leveled propositions, but isn't one of them; though it's obtained from level-2 propositions via quantification. Call this new, proposition-like entity a *level-3* proposition.

This line of argument clearly can be generalized to the effect that there are at least four, five, ..., n levels of propositions, for any natural number n, that behave expectedly, as explained above. So there must be infinitely many levels of propositions with the expected hierarchical construction. Notice that, given the similar e-grounding behavior of existential and universal propositions (both being e-grounded by the things they quantify over), similar arguments can be given, using existential propositions, to the effect that propositions have to come in levels.

In the rest of this section, we aim at crafting formal languages that

rigorously capture the propositional fragment of the talk of e-grounding and the arguments above.[15] Our starting point is the simple language of second-order logic, where quantification over propositions (construed as nullary predicates) is permitted. Formal languages that allow for quantification into sentential position appear in a number of places in philosophy, none of which have anything to do with notions of metaphysical priority such as e-grounding (see Fine, 1970; Kaplan, 1970; Williamson, 2013, for some works along these lines). We, in particular, will be working with a further impoverished language with propositional quantification, where no first-order entities (i.e., individuals) play any role. So the only instances of structured propositions are going to be Boolean combinations of propositions, or when we add an operator for e-grounding, propositions that say of e-grounding relations that hold between propositions; no (non-nullary) predicates are available in the present language.

So, let $\mathcal{L}_1$ be the language of propositional logic with the addition of propositional variables $p, q, r...$ and universal quantification over them. We assume that formulas are closed under Boolean connectives and, for the sake of simplicity, that our language doesn't have any non-logical constants. Here's the abstract syntax of $\mathcal{L}_1$ in Backus–Naur form:

$\mathcal{L}_1 ::= p \,|\, \neg\phi \,|\, \phi \circ \psi \,|\, \forall p\, \phi \,|\, \exists p\, \phi$, where $\circ \in \{\rightarrow, \leftrightarrow, \vee, \wedge\}$[16]

[15]The approach employed here is somewhat similar in spirit to the way Fritz (MS) motivates, in a step-by-step manner, simple type theory as a way of capturing some plausible talk of properties in English. What we will be doing, instead, is to motivate certain formal languages, also in a step-by-step manner, that aim at capturing our talk of e-grounding. There's a slight difference in our approaches, though: Fritz (MS) works with a more abstract sense of language, where he fixates upon certain desiderata that his desired languages need to satisfy. We, on the other hand, start with some concrete examples of languages that already exist in the literature and have gained traction by some philosophers and start improving upon them, step by step.

[16]In this paper we assume that all Boolean and quantified statements come as

Each legal term of this language is called a *formula*. *Free* and *bound* variables are defined in the usual way, and represent the set of all free variables of a formula ϕ with $FV(\phi)$. A formula with no free variable is called a *sentence*.

PROOF SYSTEM $\vdash^{\mathcal{L}_1}$:

Axioms:

1. Axioms of propositional logic — PC
2. $\forall p\, \phi \to \phi[\psi/p]$ — UI
3. $\phi[\psi/p] \to \exists p\, \phi$ — EG
4. $\forall p\, (\phi \to \psi) \to (\phi \to \forall p\, \psi)$, where $p \notin FV(\phi)$ — UD
5. $\forall p\, (\phi \to \psi) \to (\exists p\, \phi \to \psi)$, where $p \notin FV(\psi)$ — ED

Inference Rules:

6 If $\vdash \phi$ and $\vdash \phi \to \psi$, then $\vdash \psi$ — MP

7 If $\vdash \phi$ then $\vdash \forall p\, \phi$ — UG

We add an entity grounding operator $\ll$ to $\mathcal{L}_1$, to be able to express our desired principles of e-grounding in the extended language:

primitives, and not interdefined in terms of the other ones (e.g., defining $\wedge$ in terms of $\vee$ and $\neg$). One reason for this is to remain maximally neutral about the nature and granularity of logical connectives and quantifiers, without committing to any prejudices about their granularity. Moreover, even though we keep our connectives insensitive to the level of sentences in this section, this is mainly due to convenience; in later sections we choose a categorematic approach to present the logical vocabulary (coming in the form of constants, instead of clauses) and assign levels to them.

$\mathcal{L}_1^{\ll} ::= p \,|\, \neg\phi \,|\, \phi \circ \psi \,|\, \forall p\, \phi$, where $\circ \in \{\rightarrow, \leftrightarrow, \vee, \wedge, \ll\}$.

We can now express our informal principles of e-grounding from the previous section in the language $\mathcal{L}_1^{\ll}$.

Proof System $\vdash^{\mathcal{L}_1^{\ll}}$:

The extended proof system $\vdash^{\mathcal{L}_1^{\ll}}$ is just $\vdash^{\mathcal{L}_1}$ plus the following axioms:

1. $(\phi \ll \psi \;\wedge\; \psi \ll \gamma) \rightarrow \phi \ll \gamma$ $\qquad$ TR_p
2. $\neg(\phi \ll \phi)$ $\qquad$ IR_p
3. $\phi \ll (\phi \circ \psi) \wedge \psi \ll (\phi \circ \psi) \wedge (\phi \ll \neg\phi)$, where $\circ \in \{\rightarrow, \leftrightarrow, \vee, \wedge, \ll\}$ $\qquad$ S_p
4. $\psi \ll \forall p\, \phi \;\wedge\; \psi \ll \exists p\, \phi$, where $p \in FV(\phi)$ $\qquad$ Q_p

Notice that, in the statement of S_p, each choice of $\circ$ amounts to a separate schema of the logic; we have packed them all together only for convenience and higher readability. Notice also that in all of the principles above, ϕ, ψ and γ schematically stand for formulas.

We can now see rigorously where things go wrong in this system. (In what follows, we replace the schematic ϕ with $\forall q\, q$.)

Theorem 4.3.1. $\varnothing \vdash^{\mathcal{L}_1^{\ll}} \bot$

Proof.

(1)	$\forall q\, q \ll \forall q\, q$	Q_p
(2)	$\neg(\forall q\, q \ll \forall q\, q)$	IR_p
(3)	$\bot$	PC 1, 2

□

An immediate reaction to this contradiction is to undermine at least one of the two principles of e-grounding that led to it, that is, IR_p and Q_p. But remember the methodological remark from the beginning of the section: we take our principles of e-grounding to articulate substantive constraints on e-grounding that we should aim not to abandon. Instead, we try to find suitable formal languages and logics that can accommodate them. In the present case, we only started by *assuming* that $\mathcal{L}_1$, a relatively well-known and simple language that seems suitable for our purposes, can do the job when enriched with an e-grounding operator (hence the language $\mathcal{L}_1^{\ll}$), and we faced an inconsistency using our minimal background logic. So, we do not conclude that any of the principles of e-grounding involved are false; rather, we question our choice of language in modeling the informal, stipulatively endorsed IR and Q. But how do we improve on our languages?

Notice that Theorem 1 essentially formalizes the first informal paradox that we proposed at the beginning of this section, in response to which we posited two kinds of propositions—level-1 and level-2 propositions. An improvement of the language that goes hand in hand with this solution, therefore, is desirable. Since we have it that $\forall q\, q$ expresses a level-2 proposition (obtained by quantification over all level-1 propositions), we may impose a similar structure on the sentences of $\mathcal{L}_1$. More specifically, if we assume that the sentential variable q is of level 1, then we can take the level of the sentence $\forall q\, q$ to be 2. Logical rules such as UI will need to be revised accordingly, accommodating leveled sentences. As a result of such level assignments to our sentences, we no longer will be able to instantiate $\forall q\, q$, which ranges over all *level-1* sentences, with itself, as it is of level 2, and the proof of Theorem 1 breaks down in its second step.

To accommodate all of this more rigorously in a formal setting, we explicitly assign types to our sentences, along the lines of type theory. In simple type theory (higher-order logic), it is a common practice to distinguish different kinds of expressions by assigning to them *types*. For example, individual terms are assigned type e, propositional terms type $\langle\rangle$, and n-ary relational terms type $\langle t_1, ..., t_n\rangle$, where $t_1, ..., t_n$ are themselves types. For various reasons, however, we started our project with languages that only have sentential types. That is, so far we have only worked with terms of type $\langle\rangle$, so we didn't need to write down the types of our terms. But now we have found a basis for distinguishing two kinds of propositions and sentences that correspond to them. On the other hand, if we hold onto $\langle\rangle$ as the only symbol for types, we won't be able to syntactically distinguish sentences that stand for different levels of propositions.

So we extend our second-order language by adding a new sentential type. More specifically, we now index the old sentential type with numbers 1 and 2 to explicitly indicate which kind of propositions they stand for. We reserve the type $\langle\rangle/2$ for formulas that stand for level-2 propositions, i.e., ones that are 'constructed from' *all* members of the other kind propositions, and $\langle\rangle/1$ for formulas that stand for the 'building blocks' of the former kind of propositions. A corresponding revision of the proof system $\vdash^{\mathcal{L}_1}$ is also required. In particular, we replace UI_p with two similar principles UI_i, one for each $i, j \in \{1, 2\}$: $\forall p^{\langle\rangle/i}\phi_j \to \phi_j[\psi_i/p]$, where ψ_i schematically stands for any formula of level i. The logic $\vdash^{\mathcal{L}_1^*}$ of e-grounding also needs to be revised in such a way that leveled formulas are accommodated. In particular Q_p needs to be replaced by two parallel principles Q_i, one for each $i, j \in \{1, 2\}$: $\psi_j \ll \forall q^{\langle\rangle/i}\,\phi_j$, where ϕ_j and ψ_j schematically stand for

formulas of levels j and i, respectively.

Now, since we're working with leveled formulas, it should be rigorously decided by the syntax how the levels of Boolean and quantificational formulas are determined by the level of their constituents. For example, what is the level of the negation of a level-1 sentence, or the conjunction of a level-1 and a level-2 sentence? We take any combination $\phi \circ \psi$ of two leveled sentences ϕ and ψ to be of the *maximum* level of them. The reason for this is that we motivated the talk of 'levels' directly via quantification: for example, we took $\forall p^{\langle\rangle/1} p$ to be of type $\langle\rangle/2$. So there's no other way for quantification to lift levels. As for quantified statements, we can say that the type of $\forall p^{\langle\rangle/1} \phi_j$, for a formula ϕ_j of level j, is of type $\langle\rangle/\max\{2, j\}$, which is just $\langle\rangle/2$, where $j \in \{1, 2\}$.

But what about formulas of the form $\forall p^{\langle\rangle/2} \phi$, where we quantify over *level-2* sentences? For all we know at this stage, such formulas will have to be either of level 1 or 2. But similar to before, it can be readily verified that either of these options leads to inconsistencies like the one above. Here's why. (What follows is a more rigorous reconstruction of the second informal argument that was given at the beginning of the section.) Suppose, say, $\forall p^{\langle\rangle/2}(p \vee \neg p)$ is of level 2. Then since it quantifies over all level-2 propositions, by Q_2 it should be e-grounded by itself, which contradicts irreflexivity. So $\forall p^{\langle\rangle/2}(p \vee \neg p)$ must be of level 1. But now on the one hand, according to Q_2, $\forall p^{\langle\rangle/2}(p \vee \neg p)$ is e-grounded by all level-2 propositions, and on the other hand, at least one of these propositions (e.g., $\forall p^{\langle\rangle/1}(p \vee \neg p)$) is, by Q_1, itself e-grounded by all level-1 propositions, including $\forall p^{\langle\rangle/2}(p \vee \neg p)$. By transitivity of e-grounding it follows that $\forall p^{\langle\rangle/2}(p \vee \neg p)$ e-grounds itself, which again contradicts irreflexivity.

In line with the resolution of the informal, corresponding argument at the beginning of the section, we posit a *third* kind of propositional type to avoid the present inconsistencies. The syntax of the resulting language, as well as the principles of the logic governing it, also need revisions similar to the ones offered at the previous stage. It can easily be seen that similar inconsistencies arise as before and that we need a *fourth* kind of proposition-like entity to avoid the ensuing inconsistencies, and so on. In general, as expected, this process goes on and on *ad infinitum*. That is, at every level n, we are going to need to posit a sentential type $\langle\rangle\!\!/n{+}1$ for level-n sentences. In general, continuing the process of improving our languages and their logics leads to an infinite sequence of language pairs $\mathcal{L}_1, \mathcal{L}_1^{\ll}, \mathcal{L}_2, \mathcal{L}_2^{\ll}, \mathcal{L}_3, \mathcal{L}_3^{\ll}, ...$ and corresponding logic pairs $\vdash^{\mathcal{L}_1}, \vdash^{\mathcal{L}_1^{\ll}}, \vdash^{\mathcal{L}_2}, \vdash^{\mathcal{L}_2^{\ll}}, \vdash^{\mathcal{L}_3}, \vdash^{\mathcal{L}_3^{\ll}}, ...$, all attempting to capture the notion of e-grounding and its principles at a certain stage, but facing a familiar inconsistency.

At this point, a minimal language $\mathcal{L}_\infty^{\ll}$ and a corresponding proof system $\vdash^{\mathcal{L}_\infty^{\ll}}$ that encompasses all the useful type distinctions and rules that these languages had to offer, but without running into similar, generic inconsistencies can be proposed here. I won't introduce these here. Instead, I leave this to Section 5, where I propose a much more general system that captures not only these, but also systems designed to accommodate similar e-ground considerations having to do with general relations, not just propositions (see Systems $\mathcal{R}$ and $\mathcal{G}$).

We conclude this section with an important remark concerning the relationship between the principles Q and S. We mentioned earlier that Q can be endorsed plausibly and independently from S. In other words,

accepting Q doesn't hinge on construing statements of quantification as some 'long' Boolean sentences. If one is on board with us in this, then one can be neutral about, or even against any structured picture of propositions. But even if, for whatever reason, one accepts Q only as an instance of S, we noticed in §3 that it's still not necessary to commit to highly granular propositions that are susceptible to paradoxes of grain, such as the Russell-Myhill result. Either way, this puts those who reject structured propositions based on such inconsistencies (e.g., Goodman, 2016; Uzquiano, 2015; Dorr, 2016) in an awkward position: they are now offered independent reasons to admit a ramified reality, which, if they do, they end up having access to the resources that allow for highly structured propositions, as well.

In the next section, we will expand the scope of our project from the propositional fragment of e-grounding to all that can be said about it. Accordingly, we will expand our linguistic resources by adding, among other things, variables and quantificational tools for individuals as well as relational entities of different types and arities. In a similar way to this section, we will also try to find appropriate languages that can capture the talk of e-grounding and its logic.

4.4 Relational Structure and Ramification

We now extend the scope of our project by aiming at capturing the talk of e-grounding in its entirety. In particular, we will allow for statements of e-grounding that hold between entities of any pair of types: individuals e-grounding properties, propositions e-grounding operators, relations and

individuals grounding propositions, etc. We can argue that, as long as properties and relations are properly structured, they should come in infinitary hierarchies of levels. The argument is similar in its spirit to the one from the previous section to the effect that propositions should come in infinite levels. First, we assume in the background that properties and other types of relations can have structure. For instance, it seems plausible to say that the property of being loved by everyone has the relation of loving as a constituent, or the property of being identical to Mike and such that Mike has some property has both Mike and the proposition that Mike has some property as a constituent.

Later in the section, we will propose a rigorous account of relational structure and constituency, but for now, consider the latter property, namely, *being identical to Mike and such that Mike has some property.* We can argue that this property is e-grounded by itself. Here's how: by S, the property is e-grounded by its constituent proposition, *Mike has some property.* On the other hand, by Q, that proposition itself is e-grounded by *all* properties of individuals. So by TR, the property of being identical to Mike and such that Mike has some property is e-grounded by *all* individual properties, including itself, which goes against IR.

Similar to the case of the propositional fragment, and given our strong commitment to the principles of e-grounding, it can be argued that the most viable option to resolve this contradiction while retaining those principles is to posit a new kind of individual properties whose inhabitants are obtained through quantification over all members of the other kind. The rest of the story is also similar to the one from before: we can run analogous arguments to the effect that there has to be at least 3, at least

4, and for any natural number n, at least n kinds of individual properties. As before, and to make these kinds traceable, we assign levels to these relations. Other relational entities can be argued to come in infinitary hierarchies of levels.

Throughout the rest of this section, we will devise formal languages that can rigorously express statements of e-grounding in their full generality and capture the line of argument for segregating relational entities into infinitely many levels. Now, while the spirit of the project here is quite similar to the one from the previous section, the syntactic complexities involved are considerably more complicated than the ones found there. In particular, we will see that given the way many predicates are constructed via lambda abstraction in higher-order languages, and certain complications attached to free and bound variables in λ-terms, finding a notion of constituency that properly and rigorously capture our intuitions of structure and constituency for properties and relations is, by no count, a trivial task and deserves special attention.

In what follows, we add variables and quantifiers of different types into our language. In the previous section, we were only interested in the propositional fragment of the talk of e-grounding, so we only focused on propositional variables and quantifiers. But now we want to capture everything that can be said about e-grounding. So, we add variables and quantifiers for individuals, propositions and relations of different arities.

We start with types. Simple types provide a way of tracking the grammatical categories of terms.[17]

[17]The type theories presented in this paper will be *relational* (as opposed to functional). Also, for higher readability, the style of typing will by *Church-typing* (as opposed to Curry-typing), where the types of variables are fixed and attached to

Definition 4.4.1 (Simple Types)**.** The set $\mathcal{T}^s$ of *simple types* is recursively defined as follows: $e \in \mathcal{T}^s$, and for any $t_0, ..., t_n \in \mathcal{T}^s$, $\langle t_0, ..., t_n \rangle \in \mathcal{T}^s$.

When $n = 0$, the relational type is shown by $\langle\rangle$, which is the type of propositions. We assume that for any $t \in \mathcal{T}^s$ there's a denumerably infinite set of *variables* Var^t of type t and a (possibly empty) set of typed *constants* CST^t. We will reserve CST^t for the set of all constants of type t. We define the sets of all variables and constants respectively as $\mathrm{Var} := \bigcup_{t \in \mathcal{T}^r} \mathrm{Var}^t$ and $\mathrm{Var} := \bigcup_{t \in \mathcal{T}^r} \mathrm{CST}^t$.

In the previous section, we introduced the logical statements of our languages through clauses—what's sometimes called a 'syncategorematic' representation of logical statements. For instance, in Definition 1 we took it that whenever ϕ is a formula, then so is $\neg\phi$ (similarly for other connectives and quantifiers). Introducing the logical vocabulary via clauses is common in many textbooks and papers on logic, but there's an alternative, categorematic approach that especially dominates the literature on simple type theory (see, e.g., Church, 1940; Henkin, 1950; Mitchell, 1996; Dorr, 2016). According to the alternative approach, logical connectives and quantifiers are constants of certain types, and logical statements are formed using a certain operation called *application*. For instance, we take negation to be represented by a constant $\neg$ of type $\langle\langle\rangle\rangle$, and a negative statement like $\neg\phi$ to be a shorthand for application of the constant $\neg$ to a term ϕ of the appropriate type $\langle\rangle$, which is shown by $\neg(\phi)$. (A similar attitude can be taken for other connectives and quantifiers.) Below we will discuss some of the advantages of treating the logical vocabulary

them as superscripts, instead of depending on 'contexts'. Alternative formulations are possible as well.

categorematically, using typed constants.

In any case, here's the list of our primitive, typed logical constants: negation $\neg$ of type $\langle\langle\rangle\rangle$, implication $\rightarrow$, disjunction $\vee$ and conjunction $\wedge$ each of type $\langle\langle\rangle,\langle\rangle\rangle$, and for any type t, there is a constant $=^t$ for identities between t-type entities and two constants for quantification, one for (higher-order) universal quantifier $\forall^t$ and one for (higher-order) existential quantifier $\exists^t$, each being of type $\langle\langle t\rangle\rangle$. Notice that our quantifier constants apply to *predicates* of t-type entities, not those entities themselves. As will become clear through the proof system, however, there won't make any difference in the truth-conditional behavior of the logical statements in the constant-based and the clausal approach.

Definition 4.4.2 (Simple Terms)**.** The *terms* of simple type theory (STT) are recursively defined as follows: **(i)** if $x^t \in Var^t$, then x^t is a term of type t**; (ii)** if $c \in CST^t$, then c is a term of type t**; (iii)** if ϕ is a terms of type $\langle\rangle$ and for $n \geq 1$, the variables $x_1^{t_1},...,x_n^{t_n}$ are pairwise distinct, then $\lambda x_1^{t_1},...,x_n^{t_n}.\phi$ is a term of type $\langle t_1,...,t_n\rangle$**; (iv)** if τ is a term of type $\langle t_1,...,t_n\rangle$, where $n \geq 1$, and for each $i = 1,...,n$, σ_i is a term of type t_i, then $\tau(\sigma_1,...,\sigma_n)$ is a term of type $\langle\rangle$.

The operations at (iii) and (iv) are called, *application* and *abstraction*, respectively. We sometimes drop parentheses when no risk of ambiguity, and write e.g., Fa instead of $F(a)$. We call a term of type $\langle\rangle$ a *formula*, and when it contains no free variables, a *sentence*. We use the letter t with or without subscripts as metavariables for types, lower-case Greek letters $\tau,\sigma,\phi,\psi,...$ with or without subscripts as metavariables for general terms, and lower-case or capital English letters x,y,z,p,q,X,Y,Z,P,Q,

with or without subscripts, as metavariables for variables. The notions of *free* and *bound* variables of terms, substitutions of terms for variables, and *being free for a variable* are defined as usual. We show the set of free variables in a term σ by $\mathrm{FV}(\sigma)$. Also, the set of all terms of STT is denoted by TERM_s.

From now on, by convention, we write things like $\phi \vee \psi$ or $x = y$ to indicate the application instances $\vee(\phi, \psi)$ or $=(x, y)$, respectively. Similarly, quantified statements of the forms $\forall x^t \phi$ and $\exists x^t \phi$ are now construed as shorthands for application instances $\forall^t(\lambda x^t.\phi)$ and $\exists^t(\lambda x^t.\phi)$.

Before attending to the logic of our typed language, let's briefly discuss some of the advantages of our categorematic, constant-based approach to logical statements. Not only this approach is more elegant than the alternative, syncategorematic approach, with fewer axioms or term-formation rules in place and a unified way (i.e., application) to produce logical statements, but it also has the metaphysical advantage of allowing us to intelligibly ask certain questions and theorize about the granularity of the logical connectives and quantifiers—an option that is not available to the rival approach.

For instance, one could theorize about whether the operation of disjunction should be treated as a primitive relation or identified with truth-functionally similar properties such as $\lambda pq.\neg(\neg p \wedge \neg q)$ or $\lambda pq.(\neg p \rightarrow q)$. To pre-theoretically settle this question is to prejudge matters of grain. But more importantly, to be unable to even ask such questions rigorously would be a loss of expressiveness. It is only the categorematic approach that allows for expressing and defending any of the positions above. This, in effect, constitutes our main reason to choose a categorematic vs. syn-

categorematic treatment of logical statements in this paper.[18]

We now spell out the proof system for our simple type theory. In what follows, expressions like $\bar{\sigma}_i$ will stand for tuples like $(\sigma_1, ..., \sigma_n)$, and $[\bar{\sigma}_i/\bar{x}_i]$ stands for the simultaneous substitution of σ_i's with x's in τ.[19] Also, in each case, it's been assumed that the substitutants are *free for* the substituent. Intuitively, that guarantees that (i) no bound variable is allowed to be substituted (that is, the notion of substitution only applies to free variables), and (ii) no free variable can get bound after substitution.

Proof System $\mathcal{S}$:

Axioms:

1. Axioms of propositional logic PC
2. $(\lambda x_1^{t_1}, ..., x_n^{t_n}.\phi)(\bar{\sigma}_i) \leftrightarrow [\bar{\sigma}_i/\bar{x}_i]\phi$, where the type of σ_i is t_i $(i = 1, ..., n)$ β_E
3. $\forall^t F \rightarrow F\sigma$, where F and σ are, respectively, of types $\langle t \rangle$ and t UI
4. $F\sigma \rightarrow \exists^t F$, where F and σ are, respectively, of types $\langle t \rangle$ and t EG
5. $\forall^t(\lambda x^t.\phi \rightarrow Fx) \rightarrow (\phi \rightarrow \forall^t F)$, where F is of type $\langle t \rangle$ and $x \notin FV(\phi)$ UD

[18]With that in mind, one can object to the categorematic approach by saying that, in English, the talk of, e.g., identity, quantification and many other relations and logical operators doesn't seem to be bound to types—we seem to use the same locution of 'is identical to' or 'for all' for most if not all claims regarding individual, properties and relations. So, contra to the popular view, the thought goes, the categorematic might fall short of capturing the talk of properties and quantifiers in English. See footnote 21 for a related discussion regarding the categorematic vs. syncategorematic treatment of e-grounding statements, and a potential, novel reply to these sorts of objections.

[19]For a rigorous definition of substitution, see Mitchell (1996, p. 53). Mitchell's definition is given for functional type theory. Similar definitions can be given for relational type theory.

6. $\forall^t(\lambda x^t.Fx{\rightarrow}\psi) \rightarrow (\exists^t F \rightarrow \psi)$, where F is of type $\langle t\rangle$ and $x \notin FV(\psi)$ ED

7. $\sigma{=}^t\sigma$, where σ is of type t REF

8. $\sigma{=}^t\tau \rightarrow (F\sigma{\rightarrow}F\tau)$, where F is of type $\langle t\rangle$ LBZ

Rules of Inference:

9. If $\vdash \phi$ and $\vdash \phi \rightarrow \psi$, then $\vdash \psi$ MP

10. If $\vdash Fx^t$, then $\vdash \forall^t F$, where F is of type $\langle t\rangle$ UG

Now that we have our primary language set up, we need to find a way to rigorously define a suitable notion of syntactic constituency that reflects the sense of constituency that we've seen earlier through examples of structured properties and relations. In other words, we want a syntactic criterion that, whenever applied to any relational term reveals the structure of the thing denoted by the term.

For example, we want to find the constituents of the property of being friends with Geoff by 'scanning through' the predicate that expresses it in our language, namely $\lambda x^e.F(x, g)$, where F is a constant standing for the relation of friendship, and g is Geoff's name. In this specific case, we want to systematically recognize g and the F to be among the constituents of the predicate, because that would reflect the fact that the property in question has Geoff and the relation of friendship as a constituent. Such rigorous syntactic specifications matter to us in particular due to the way we have introduced our e-grounding principle S, as relying on the syntax-semantics interplay. Recall that S, in its most general form, says

that entities denoted by syntactic expressions are e-grounded by the things picked by the constituents of those expressions.

In general, we take it that many properties and relations that are expressible in our language by the lambda device are structured, and we want to find a general way to specify their constituents through the λ-terms that denote to them. But implementing the ideas of a structuredness and constituency can be perplexing in the presence of λ. For example, even though we may convincingly find the constituents of the property of being friends with Geoff from the corresponding λ-term, $\lambda x^e.F(x, g)$, it's not clear how we can pinpoint the constituents (if any) of the property of having every property of individuals from its corresponding λ-term, i.e., the predicate $\lambda x^e.\forall Y^{\langle e\rangle}Y(x)$. An immediate, though naive thought is to take $\forall Y^{\langle e\rangle}Y(x)$ to be a constituent of that predicate. But that sentence doesn't express a unique proposition: depending on what value x takes by an assignment function, it expresses a different proposition.

In general, it's not clear what's the contribution of abstraction or free and bound variables involved in the determination of the constituency of λ-terms. In the rest of the section, we will explore three main options regarding constituency, and choose one of them as the correct definition of constituency. But as will become clear in the end, *all* of them can be used equally well to motivate examples where stratification of relational types is needed. The rest of this section mainly attempts to find the best account of relational constituency among the three options that will be discussed.

We start with the broadest sense of relational constituency, which is the same as being a sub-expression. The idea is to take constituents of

terms, in general, just to be their sub-expressions. (Sub-expressions of terms are the terms that contribute to the recursive definition of them, as expressed for all terms in Definition 3.) For instance, by Sub we have it that the predicate $\lambda x^e.F(x,g)$ has F, g and x as constituents. Or that $\lambda x^t.((\forall y^{t'}Fy)\wedge(x=a))$ has as constituents the sentences $(\forall y^t Fy)\wedge(x=a)$, $\forall y^t Fy$, $x=a$; plus all of their constituents, i.e., $\wedge$, x, a, $\forall^t, y, Fy$ and F, as well. Similarly, $\lambda x^t.\forall z^t R(z,x)$ has as constituents, $\forall z^t R(z,x)$, $\forall^t$, $z, R(z,x), R$ and x.

Thus here's the first attempt:

- τ is a *constituent* of σ iff τ is a sub-expression of σ. Sub

Sub is the most liberal account of constituency. To motivate Sub, remember that in the case of the propositional fragment of e-grounding (see §4), the constituents of a Boolean expression were taken to be the things connected by the relevant connectives: constituents of a conjunctive sentence were taken to be its conjuncts, etc. We can generalize the idea for applicational terms in STT, by taking the constituents of an application term $F(a)$ to be F, a and their respective constituents. One might further expand the notion of constituency of terms, including λ-terms of the general form $\lambda x_1^{t_1},...,x_n^{t_n}.\phi$, as well.

But the sub-expressional sense of constituency is too liberal, and in some cases, unmotivated by our metaphysical considerations. While it seems natural to say that the property of being friends with Geoff, expressed by $\lambda x^e.F(x,g)$, has Geoff and the property of friendship as constituents, it seems implausible to say the same for the variable x, as due to its boundness, none of the values assigned to it seem to have to do

anything with the property of being friends with Geoff. For example, x could be assigned Mike, but Mike doesn't seem to have anything to do with structure of the property in question—it certainly doesn't seem to be a constituent of it.[20]

So we need to impose some restrictions on our initial definition of constituency, for the sake of metaphysical plausibility. The examples above suggest that we need to rule out bound occurrences of variables as constituents. More generally, they suggest that every free occurrence of a variable in the constituent term should also occur free in the term that it's a constituent of. Clearly, limiting constituents to *closed* sub-expressions satisfies this:

- τ is a *constituent* of σ iff τ is a closed sub-expression of σ.

Closed Sub

From Closed Sub it follows that the property of being an individual such that Mike is drinking—expressed by $\lambda x^e.Dm$—has the proposition Dm that Mike is drinking, and accordingly, both Mike and the property of drinking as constituents. We will also have it that the property

[20]Of course, one may want to take, e.g., any property to be e-grounded by all the propositions obtained from it. In that case, the property in question will in particular be e-grounded by all propositions obtained from it, and that includes the proposition that Mike is friends with Geoff. From S and propositional TR, it will follow that Mike e-grounds the property of being friends with Geoff. But this option doesn't seem to sit well with our constructional intuitions of e-grounding. Remember that we took e-grounding to somehow reflect the sense of 'construction' involved in entities; clearly, no such sense can plausibly be given to justify the claim that Mike e-grounds the property of being friends with Geoff, or the tentative principle that properties are e-grounded by their propositional values. Similarly, it sounds unmotivated to say that the property denoted by $\lambda x^e.\forall z^e L(z,x)$ is e-grounded by $\forall z^e L(z,x)$: for any assignment of values to variables, this formula returns an entirely different sentence. Suppose L stands for the property of loving. Then for any value a of x, $\forall z^e L(z,a)$ expresses the proposition that everyone loves a. But no such proposition seems to have anything with the 'construction' of the property of being loved by everyone.

$\lambda x^e.\forall Y^{\langle e\rangle}Y(m)$ of being an individual such that Mike has every individual property is structured and has as a constituent the proposition $\forall Y^{\langle e\rangle}Y(m)$ that Mike has every individual property.

But the closed conception of constituency is somewhat too restrictive. Consider, for example, $\sigma \coloneqq \lambda x^t.L(y,x)$, where y is a variable of some type t' and L is a constant of type $\langle t', t\rangle$. With Closed Sub we can say that L is a constituent of σ, but we can't say that about y. But we *would* want the free variable y to be a constituent of σ, because for any value a that y is assigned, a is in fact a constituent of the property picked by $\lambda x^t.L(a,x)$. For instance, if L stands for the relation of loving, the property of being loved by Sarah, $\sigma \coloneqq \lambda x^t.L(s,x)$, seems to have Sarah ($s$) as a constituent.

So perhaps the best idea is to just hold onto or sharpen the two restrictions that we had ended up with in discussing Sub, as what determines a syntactic notion of constituency that suitably accommodates our favorite sense of constituency that holds between real entities. Let's see some more examples. Suppose $\sigma \coloneqq \lambda y^{t'}.((\forall x^t Fx) \wedge Gx)$. We would like the universal statement $\forall x^t Fx$ to be a constituent of σ, because it's a closed term. But not Fx, because the free occurrence of x in Fx doesn't occur free in σ. Neither is the occurrence of x in Fx a constituent of σ, for the same reason. On the other hand, Gx, G and the occurrence of x in Gx are all to be construed as constituents of σ because, whatever value they take, that value would seem to be a constituent of the property denoted by σ. Accordingly, for any assignment of values to variables, the entities picked by Gx and x will e-ground the property picked by σ, the latter in virtue of x being a constituent of σ through its free occurrence in Gx.[21]

[21] This is plausible, especially because due to the principle of 'α-conversion', according

Below is the general definition of constituency that suitably accommodates all the examples of relational e-grounding that we have been discussing so far:

• An occurrence of a term τ in a term σ is a *constituent occurrence* of τ in σ if τ is a sub-expression of σ and every free occurrence of a variable in τ occurs freely in σ. The term τ is a *constituent* of σ, written $\tau \in \mathbf{c}(\sigma)$, if τ has a constituent occurrence in σ. CONS

Notice that this definition encompasses the sense constituency for sentences as well. That is, a sentence of the form $R(a_1, ..., a_n)$ has as constituents R and all a_i's, simply because they're all sub-expressions of $R(a_1, ..., a_n)$ and every free occurrence of a variable in each of them occurs freely in $R(a_1, ..., a_n)$.

It can be shown that each of SUB, CLOSED SUB and CONS can motivates the idea of type-stratification for relational types. This means that as soon as we settle on a notion of syntactic constituency for λ-terms from among these three major candidates, we can motivate our desired type stratification. Of course, for the reasons given earlier, our favorite account of constituency will be CONS and we will use examples along those lines. First, we add to the terms language of STT entity grounding statements between any pair of types t_1 and t_2, to obtain $STT^{\ll}$. The relevant clause is as follows:

• If τ and σ are terms then $\tau \ll \sigma$ is a term of type $\langle\rangle$.

to which terms with corresponding bound variables of different names are the same, σ is can be identified with $\lambda y^{t'}.((\forall z^t Fz) \wedge Gx)$. In this α-equivalent variant of σ, x, but not z, still plausibly is a constituent of the term. Such α-equivalent representations of terms may well allow for redefining constituency in terms of variables, instead of variable occurrences.

Notice that here we are treating statements of e-grounding syncategorematically. Alternatively, we could treat them categorematically and take statements of e-grounding to be obtained by, e.g., typed constants (standing for relations of e-grounding) that apply to entities of appropriate types. More specifically, for any pair of types t_1 and t_2 we could associate a constant $\ll_{t_1 t_2}$ of type $\langle t_1, t_2 \rangle$ and construe statements of e-grounding $a \ll_{t_1 t_2} b$ are in fact abbreviations for applications of the form $\ll_{t_1 t_2} (a, b)$, similar to what we did for the logical vocabulary.

Our syncategorematic treatment of e-grounding statements is mainly due to the fact that our pre-theoretic talk of e-grounding (as introduced in §3) doesn't discriminate against entities of different types; it appeals to a unified notion that runs across reality. It's the same locution all over as if we are talking about the same relation that holds between entities of different types. So a syncategorematic treatment of e-grounding statements seems closer to our pre-theoretic conception and use of the notion.[22]

[22]One might object to the syncategorematic formulation of e-grounding statements by saying that there is no such relation out there in the reality, after all, that, e.g., would contribute to the truth of statements of e-grounding; at best, there are infinitely many such relations that do the job (and that has to be cashed out on the alternative, categorematic approach). In response to this, although one should admit the structure of the type theories in this paper, and in general in the philosophical literature, don't allow for a unique relation that ignores type differences, that's hardly a unique problem for e-grounding, or any other notion of grounding, for that matter. The same issue can be raised regarding categorematic vs. syncategorematic treatments of the logical vocabulary or identity. This has to do with the design of the kind of type systems that most philosophers use, such as simple and ramified type systems, where no cross-type term can be expressed. There are, however, more recent type theories, though so far mostly in the service of mathematicians and computer scientists, that allow for such entities. For instance, $\lambda 2$ or *System F* is among such type systems, otherwise known as *Dependent Type Theories*. These type systems raise above the type restrictions built into the kind of type theories entertained here, and in general by philosophers, and allow for terms and their denotations that aren't sensitive to the choice of simple types. We believe that the notion of e-grounding, as well as various other notions of grounding that call for ramification, and, in fact, many other relations outside the context of

In any case, we can now express our desired principles of e-grounding in the extended language, to obtain $\vdash^{STT^{\ll}}$, which is just $\vdash^{STT}$ plus the following axiom schemata:

1. $(\tau \ll \sigma \wedge \sigma \ll \gamma) \rightarrow \tau \ll \gamma$ TR
2. $\neg(\tau \ll \tau)$ IR
3. $\tau \ll \sigma \rightarrow \neg\sigma \ll \tau$ AS
4. $\tau \ll \sigma$, if $\tau \in \mathbf{c}(\sigma)$ S
5. $\tau \ll \forall x^t \phi \wedge \tau \ll \exists x^t \phi$, where τ is of type t and $x \in FV(\phi)$ Q

We can now see exactly why we need relational ramification. Consider, for example, the property P of being Mike such that Mike has some property, expressed by $\lambda x^e.(x = m \wedge \exists Y^{\langle e \rangle} Y(m))$. We argued at the beginning of this section that the structure of this property calls for an infinitary hierarchy of individual properties. Using Cons and the principles of e-grounding above, this can be shown more rigorously. Notice that according to our definition of relational constituency, P has the proposition $\exists Y^{\langle e \rangle} Y(m)$ that Mike has some property as a constituent, so by S they are e-grounded by it. On the other hand by Q the proposition itself is e-grounded by *all* properties of individuals. A contradiction follows from applying UI and TR. Put formally, we have the following:[23]

metaphysical priority (e.g., identity and existence), are best captured by such general systems. For a recent argument in favor of System F as the right framework to capture the talk of identity, existence, quantification and various other typed relations, see Kiani (MSa). For a general introduction to dependent type theories see Nederpelt and Geuvers (2014). More detailed discussions of System F and their applications can be found at Girard et al. (1989) and Mitchell (1996, Chapter 9).

[23]One who goes only as far as committing to the restrictive Closed Sub could use the term $P := \lambda x^e.\exists Y^{\langle e \rangle} Y(m)$ and run similar arguments.

Theorem 4.4.1. $\varnothing \vdash^{STT^{\ll}} \bot$

Proof.

(1)	$\lambda x^e.(x = m \wedge \exists Y^{\langle e \rangle} Y(m)) \ll \exists Y^{\langle e \rangle} Y(m)$	Q
(2)	$\exists Y^{\langle e \rangle} Y(m) \ll \lambda x^e.(x = m \wedge \exists Y^{\langle e \rangle} Y(m))$	S
(3)	$\lambda x^e.(x = m \wedge \exists Y^{\langle e \rangle} Y(m)) \ll \lambda x^e.(x = m \wedge \exists Y^{\langle e \rangle} Y(m))$	TR 1, 2
(4)	$\neg[\lambda x^e.(x = m \wedge \exists Y^{\langle e \rangle} Y(m)) \ll \lambda x^e.(x = m \wedge \exists Y^{\langle e \rangle} Y(m))]$	IR
(5)	$\bot$	PC 3, 4

□

This Theorem essentially formalizes the inconsistency result outlined at the beginning of the present section. The rest of the story is similar to the previous section. We have *assumed* that the language of simple type theory can capture our stipulative talk of e-grounding. We have then run into contradictions when formulating our desired principles in this language. In line with our arguments at the beginning of the section, the most e-ground-friendly resolution to the problem at stake is to segregate the property picked by $\lambda x^e.\exists Y^{\langle e \rangle} Y(m)$ and the ones it quantifies over, so we implement similar revisions in our syntax.

More specifically, we replace the type $\langle e \rangle$ with two types $\langle e \rangle /\!\!/ 1$ and $\langle e \rangle /\!\!/ 2$, the first one assigned to predicates that pick the 'building-block' individual properties, and the second one for the predicates that pick the 'buildings'. As a result, we will be able to revise our term-formation rules in a way that, e.g., $\lambda x^e.\exists Y^{\langle e \rangle /\!\!/ 1} Y(m)$ will be of type $\langle e \rangle /\!\!/ 2$, and so on. As expected, the proof system needs to also be calibrated, accordingly.

As expected, this process improving upon languages and running into inconsistencies leads to positing an infinite array of newer and newer leveled types $\langle e \rangle /\!\!/ 1, \langle e \rangle /\!\!/ 2, \langle e \rangle /\!\!/ 3, \ldots$ for predicates that pick different kinds

of individual properties, and $\langle\rangle/\!\!/1, \langle\rangle/\!\!/2, \langle\rangle/\!\!/3, ...$ for sentences that express different kinds of propositions. In general, we can run similar arguments for different expressions of different relational types of the form $\langle t_1, ..., t_n\rangle$, for any $n \geq 0$, and end up with an infinite hierarchy of types $\langle t_1, ..., t_n\rangle/\!\!/1, \langle t_1, ..., t_n\rangle/\!\!/2, \langle t_1, ..., t_n\rangle/\!\!/3, ...$ that behave in the way expected. In the next section, we propose a formal language and logic that fully accommodates the syntactic changes glossed here.

4.5 Ramified Type Theory

We now introduce a system ramified types based on the previous discussions, in its most general form. First, let's introduce ramified types and their levels:

Definition 4.5.1 (Ramified Types and Levels)**.** The set $\mathcal{T}^r$ of *ramified types* t and their levels $\mathrm{l}(t)$ are simultaneously defined as follows: $e \in \mathcal{T}^r$ with $\mathrm{l}(e) = 0$, and for $t_0, ..., t_n \in \mathcal{T}^r$ and $m \geq 1$, if $\mathrm{l}(t_i) \leq m$ for each $i = 0, ..., n$, then $\langle t_0, ..., t_n\rangle/\!\!/m \in \mathcal{T}^r$, with $\mathrm{l}(\langle t_0, ..., t_n\rangle/\!\!/m) = m$.

In effect, e is the type of individuals, and for any types $t_0, ..., t_n$, where $n \geq 0$, $\langle t_0, ..., t_n\rangle/\!\!/m$ is the type of n-ary propositional functions of level m—functions that, as the term-formation rules below show, take arguments of types $t_0, ..., t_n$, respectively, and return an level-m proposition. The type of level-m propositions is obtained as the limiting case of the relational types, when $n = 0$, and is represented by $\langle\rangle/\!\!/m$.

As before, for any ramified type $t \in \mathcal{T}^r$ we assume there's a denumerably infinite set of *variables* Var^t of type t and a (possibly empty) set of

typed *constants* CST^t. We reserve CST^t for the set of all constants) of type t. We define the sets of all variables and constants respectively as $\mathrm{Var} := \bigcup_{t \in \mathcal{T}^r} \mathrm{Var}^t$ and $\mathrm{CST} := \bigcup_{t \in \mathcal{T}^r} \mathrm{CST}^t$. We also represent the set of t-type terms with TERM^t.

As before, we choose the constant-based approach to introduce our logical vocabulary. In line with our discussions of levels from before, we choose our typed, logical constants in RTT, as follows: $\neg_m$ is of type $\langle\langle\rangle\!/m\rangle\!/m$; $\rightarrow_{m_1,m_2}$, $\leftrightarrow_{m_1,m_2}$, $\vee_{m_1,m_2}$ and $\wedge_{m_1,m_2}$, each of type $\langle\langle\rangle\!/m_1, \langle\rangle\!/m_2\rangle\!/\max\{m_1, m_2\}$; and, to repeat, for any ramified type t, $\forall^t_m$ is of type $\langle\langle t\rangle\!/m\rangle\!/\max\{\mathrm{l}(t)+1, m\}$. As for the identity operator in RTT, for any t we reserve a constant $=^t_r$ of type $\langle t, t\rangle\!/\max\{1, \mathrm{l}(t)\}$. Notice that since identity statements are essentially formulae, the minimum level they can take should be 1. Notice also that the type of the universal quantifier constant $\forall^t_m$ is determined through the convention $\forall x^t\, \phi := \forall^t_m(\lambda x^t.\phi_m)$ and level conventions of λ-terms (as introduced below).

Definition 4.5.2 (Ramified Terms)**.** The *terms* of RTT are recursively defined as follows: **(i)** If $x^t \in Var^t$, then x^t is a term of type t; **(ii)** if $c \in CST^t$, then c is a term of type t; **(iii)** if $x_1 \in Var^{t_1}, ..., x_n \in Var^{t_n}$ are pairwise distinct, where $n \geq 1$ and $\mathrm{l}(t_i) \leq m$ for each t_i, and ϕ is a term of type $\langle\rangle\!/m$, then $\lambda x_1^{t_1}, ..., x_n^{t_n}.\phi$ is a term of type $\langle t_1, ..., t_n\rangle\!/m$; **(iv)** if τ is a term of type $\langle t_1, ..., t_n\rangle\!/m$, where $n \geq 1$, and for each $i = 1, ..., n$, τ_i is a term of type t_i, then $\tau(\tau_1, ..., \tau_n)$ is a term of type $\langle\rangle\!/m$.

The notions of *free* and *bound* variables of terms, *substitutions* of terms for variables and *being free for a variable* are defined as usual. We denote the set of free variables in a term σ by $FV(\sigma)$, and the set of all terms

of ramified type theory by TERM_r.[24] We also adopt similar conventions about meta-variables for variables, terms and types as before.

We now introduce the proof system for our ramified language, which we named *System* $\mathcal{R}$.

Proof System $\mathcal{R}$:

Axioms:

1. Leveled appropriate axioms of propositional logic[25] PC_r
2. $(\lambda x_1^{t_1}, ..., x_n^{t_n}.\phi_m)(\bar{\sigma}_i) \leftrightarrow [\bar{\sigma}_i/\bar{x}_i]\phi_m$, where the type of σ_i is t_i β_{E_r}
3. $\forall^t_m F \rightarrow F\sigma$, where F and σ are, respectively, of types $\langle t \rangle /\!\!/ m$ and t UI_r
4. $F\sigma \rightarrow \exists^t_m F$, where F and σ are, respectively, of types $\langle t \rangle /\!\!/ m$ and t EG_r
5. $\forall^t_{n^*}(\lambda x^t.\ \phi_m \rightarrow Fx) \rightarrow (\phi_m \rightarrow \forall^t_n F)$, where F is of type $\langle t \rangle /\!\!/ n$, $n^* = \max\{m, n\}$ and $x \notin FV(\phi_m)$ UD_r
6. $\forall^t_{n^*}(\lambda x^t.Fx \rightarrow \psi_m) \rightarrow (\exists^t_n F \rightarrow \psi_m)$, where F is of type $\langle t \rangle /\!\!/ n$, $n^* = \max\{m, n\}$ and $x \notin FV(\psi_m)$ ED_r
7. $\sigma =^t_r \sigma$, where σ is of type t Ref_r
8. $\sigma =^t_r \tau \rightarrow (F\sigma \rightarrow F\tau)$, where F is of type $\langle t \rangle /\!\!/ m$ LBZ_r

[24] Notice that our ramified types and terms, as introduced here are very similar to Harold Hodes's System $\Rightarrow^{nr}$, as introduced in Hodes (2013). What we consider as level here is called 'order' by Hodes, and that in our system, but not Hodes's vacuous lambda abstraction is possible.

[25] For instance, $\phi_m \rightarrow (\psi_n \rightarrow \phi_m)$ in place of $\phi \rightarrow (\psi \rightarrow \phi)$, where each indexed letter doubly schematically stands for a formula of a certain level.

Rules of Inference:

9. If $\vdash \phi_m$ and $\vdash \phi_m \rightarrow \psi_n$, then $\vdash \psi_n$ $\quad$ MP_r

10. If $\vdash Fx^t$, then $\vdash \forall^t_m F$, where F is of type $\langle t \rangle /\!\!/ m$ $\quad$ UG_r

Notice that each of the axioms and rules of inference above are multiply schematic. For example in PC_r, the axioms hold for any sentence of any level, and the relevant instances of $\neg$ and $\rightarrow$ may differ in type and should be typed carefully. In particular, notice that in UI_r, LBZ_r and UG_r are all schematic in multiple ways: in the occurrence of the terms, types t *and* the level m of the relational types $\langle t \rangle /\!\!/ m$.

Finally, we express the principles of e-grounding in RTT. To do this, we first extend RTT to $\mathrm{RTT}^{\ll}$ by adding the following clause:

- If τ and σ are terms, then $\tau \ll \sigma$ if a term of type $\langle \rangle$.

The e-grounding system looks like what we introduced at the beginning of this section, but now with the types being schematic for different types (individuals, propositional and relational) *and* levels (1, 2, 3, ...). We call the augmentation of $\mathcal{R}$ with the following axiom schemata, resulting in $\mathcal{R}^{\ll}$ or what we also call *System* $\mathcal{G}$:

1. $(\tau \ll \sigma \wedge \sigma \ll \gamma) \rightarrow \tau \ll \gamma$ $\quad$ TR_r

2. $\neg(\tau \ll \tau)$ $\quad$ IR_r

3. $\tau \ll \sigma$, if $\tau \in \mathbf{c}(\sigma)$ $\quad$ S_r

4. $\tau \ll \forall x^t \phi \wedge \tau \ll \exists x^t \phi$, where τ is of type t and $x \in FV(\phi)$ $\quad$ Q_r

We conclude the paper with some final remarks. First, notice that in Definition 3 we assumed that the level of the arguments $\tau_0, ..., \tau_n$ of a relational type $\langle \tau_0, ...\tau_n \rangle /\!\!/ m$ are no higher than the level of type, i.e., m. We can see now why we had to make this choice. Suppose, on the contrary, that $\langle\langle\rangle /\!\!/ 3\rangle /\!\!/ 2$ is a legit type and entities of this type could apply to entities of type $\langle\rangle /\!\!/ 3$ in order to produce $\langle\rangle /\!\!/ 2$-type propositions. Now, let $F := \lambda p^{\langle\rangle /\!\!/ 3}.\phi_2$, where ϕ_2 is any sentence of type $\langle\rangle /\!\!/ 2$, and let $\phi := \forall p^{\langle\rangle /\!\!/ 2} p$. Then $F(\phi)$ will be of type $\langle\rangle /\!\!/ 2$. But by S, we have $\phi \ll F(\phi)$, and by Q, all level-2 propositions would e-ground ϕ. A contradiction then follows from IR and TR: by TR all level-2 propositions would e-ground $F(\phi)$, and that includes $F(\phi)$ itself, which goes against IR. Similar examples can be given if types like $\langle\langle\rangle /\!\!/ 3\rangle /\!\!/ 1$ were allowed, whereas types of the form $\langle\langle\rangle /\!\!/ 3\rangle /\!\!/ n$ for any $n \geq 3$ are safe to inhabit.

The second remark has to do with Q_r. One might think that it's intuitive, particularly along the lines of our 'construction' analogy of e-grounding, to have a proposition of the form, say, $\forall p^{\langle\rangle /\!\!/ i} \phi_j$ not only e-grounded by all level-i propositions, but *also* by all propositions of levels lower than i; yet Q_r only considers i-level propositions. This is a valid worry, but our system does accommodate it eventually. For example, $\forall p^{\langle\rangle /\!\!/ 3} p$ is e-grounded by all level-3 propositions, but one of those propositions is $\forall p^{\langle\rangle /\!\!/ 2} p$, which is e-grounded by all level-2 propositions. By the transitivity of e-grounding, it follows that all level-2 propositions also ground $\forall p^{\langle\rangle /\!\!/ 3} p$. Similarly, all level-1 propositions ground $\forall p^{\langle\rangle /\!\!/ 3} p$.

The third remark has to do with the proposed intuition of quantification as 'long' conjunction or disjunction (hence considering Q_r as an instance of S_r), in a literal sense, which we offered earlier in the paper: the statements

of quantification but not conjunction shift levels of sentences, so the former cannot be construed or identified as the latter, strictly speaking. Another reason one might resist such identifications is about aboutness: for instance, one might think that differentiates an existentially quantified proposition from the corresponding disjunction of instances is that the disjunction is 'about' or depends on the disjuncts, but that this doesn't seem to be the case for the parallel quantified proposition. For reasons like these, in this paper, we rely on the construal of Q_r as an instance of S_r mostly for heuristic purposes. That said, however, there are ways to make the analogy more appropriate, at least from a technical standpoint, without running into such level mismatches. For instance, instead of taking the level of, say, the conjunction $\phi_i \wedge \psi_j$ of a level-i formula ϕ_i and a level-j formula ψ_j to be $\max\{i, j\}$, we could take it to have the level $\max\{i, j\} + 1$. In that case, we can add an infinitary conjunction operation to our language and just generalize this level assignment to it to get our desired level assignment for quantified statements. This won't have any serious effects on our path from entity grounding to ramified type systems—the path only ends up with a slightly different ramified system than what we did.

The next remark concerns our constant-based, categorematic presentation of the logical vocabulary in this section and the previous one. We mentioned that there are advantages in this approach, both concerning presentation (more elegance and convenience in defining terms and specifying axioms) and expressiveness for metaphysical theorizing. That said, however, one might be skeptical if the constant-based approach is the best one when, in particular, it comes to *ramified* type systems principles.

For instance, given the abundance of levels, the constant-based approach requires a very big ontology, with infinitely many entities sitting out there to just do the job of, say, negation. Similarly, we need considerably more axioms, compared to the syncategorematic approach, each crafted for certain levels, in laying out the proof system. These considerations might make the constant-based approach in ramified type theory lose attraction to sparser ontologies within certain big-picture considerations. This might also be why most of the works in the literature on ramified type systems (including Russell's original works) choose the syncategorematic approach to the logical vocabulary. Moreover, from a purely e-ground-theoretic perspective, it may sound somewhat mysterious that some but not other constants raise levels of the things they apply to.

One might think that these go against one of the primary motives of adapting a categorematic approach towards ramified type theory, and wonder if it's possible to present our ramified system syncategorematically while somehow retaining the relevant formalizations regardless of e-grounding. This seems possible. In fact, this is the approach we took in §4 to build up our leveled languages, though mostly for convenience. Here we can also associate in our term-formation rules separate clauses for logical terms. But then we will have to make sure that the notion of constituency (Cons) is also extended with enough clauses concerning logical statements and their constituents. It should be noted, however, that this approach will no longer allow for the attractive thought that logical entities can enter into e-grounding relations. For instance, there is no longer a relation of conjunction that can be said to e-ground a conjunctive proposition. In any case, we leave it open which choice is

more appropriate here, all things considered.

Finally, in this paper we argued that simple *relational* entities each have to come in certain infinitary hierarchies of levels, for the principles of e-grounding to go through without facing immediate inconsistencies. One might wonder if individuals should also be fragmented into levels, similarly to relations. Even though this is formally possible (for instance, Bacon et al., 2016, do this), such a move seems unmotivated by the metaphysical views that we have been appealing to, for if a propositional function contains a proposition that quantifies over individuals, then whatever it refers to is *not* an individual: it's a proposition, property or relation. That is to say, the type of propositional functions are *already* different from the type e of individuals, whether or not they quantify over individuals in their structure. That said, however, some might have independent reasons to stratify individuals into levels, as well, e.g., by considering certain mereological relations that hold between them as instances of e-grounding. We also leave that possibility open for future investigations.

4.6 Conclusion

I argued that considerations of e-grounding, as presented in this paper, naturally call for fragmentation of relational entities into certain infinitary hierarchies of levels, in a way that is best captured by ramified type systems. I proposed a natural ramified type system that nicely captures the principles of e-grounding.

Several problems remain open, which we hope to attend to in the future. We haven't yet verified the consistency of the systems $\mathcal{R}$ and $\mathcal{G}$. In

particular, while the consistency of the former is proved in Kiani (MSd), the consistency of the latter still remains open. Another issue concerns the choice between categorematic vs. syncategorematic representations of logical statements as well as e-grounding statements. Even though throughout the paper, and for a combination of reasons, we made a certain choice in this regard, namely, a categorematic treatment of logical, and a syncategorematic treatment of e-grounding statements, we also mentioned that alternative options are available, without any serious impact on our arguments for the ramification of relational types, though, each choice has its own pros and cons. We, however, leave it open which choice of options is more appropriate under large-scale considerations.

We also only worked with entity grounding as if it is a partial relation between entities, similar to how partial fact-rounding relations work: an entity can be e-grounded by many entities, e.g., its constituents. This can be compared to partial fact-grounding relations, where a fact is partially grounded by some other fact, while could need other facts too, in order to hold. It would be interesting to see how one can expand the notion to a more general one that acts as a total relation between entities—again, the way total fact-grounding relations work.

A somewhat larger open problem concerns the general choice between ramified versus simple type theories as the 'correct' framework in pursuing philosophical, and in particular, metaphysical inquiry. We noted that the ramified approach is naturally motivated by considerations of e-grounding. But ramified type theory can be shown to do much more. For one, as hinted by Deutsch (2008), it can secure highly structured accounts of propositions from paradoxes of grain, such as the Russell-Myhill paradox (see e.g.,

Goodman, 2016; Hodes, 2015; Myhill, 1958; Russell, 1903; Uzquiano, 2015, for various versions of the paradox); this is shown in Kiani (MSd). Moreover, Kiani (MSe) leverages the ramified hierarchy and takes a step in settling some of the prominent puzzles of quantificational ground (as explored in, e.g., Fine, 2010; Krämer, 2013; Donaldson, 2017; Fritz, 2021), providing a unified solution to them in one sweep.[26] Kiani (2023), on the other hand, uses highly structured propositions to lay down a novel and extremely expressive semantics for unrestricted impure propositional logics of truth-functional, iterated and identity grounding (as explored in, e.g., Wilhelm, 2020b; Fine, 2012a; Correia, 2017; Krämer, 2018; Bennett, 2011; Schnieder, 2011).[27]

Essentially, there seems to be a deep unexplored interconnection between structured views of reality, different notions of metaphysical priority, and ramified type systems, in that together they bring about a uniform, elegant picture of reality within which a cluster of puzzles and paradoxes of ground and grain in contemporary metaphysics are settled. This speaks to the immense and unified explanatory power of the ramified approach in doing philosophy, and in particular, metaphysics. But simple type theory has been proved more fruitful in certain other areas, such as in mathematics, where, r.g., those systems can be enriched with certain axioms to serve as a foundation for classical mathematics (see, e.g., Church, 1940, as an early work along these lines). Also, some major projects, especially in the

[26] In effect, our ramified approach can be said to constitute a 'predicative' solution to these puzzles which has long been speculated but remained fairly underexplored; see Fine (2010); Krämer (2013) for a brief mention of such solutions. See Korbmacher (2018b,a) for a full-blooded account og predicative solutions to some of the puzzles of quantificational ground along a hierarchical account of Tarskian 'truths'.

[27] Others, such as Prior (1961) and Kripke (2011), have argued that ramified types can provide solutions to certain paradoxes of intensionality as well.

recent literature on the metaphysics of modality, have been carried out in simple type systems and supposedly seriously rely on their full expressive power (e.g., Williamson, 2013; Bacon, 2018).

Finally, we noted that unless we construe quantificational statements as 'long' conjunctions or disjunctions, and accordingly the principle Q as an instance of S (which we found reasons not to), one doesn't need to embrace structured propositions in order to be receptive to the idea that propositions, along with properties and relations, have to come in infinitary levels, as described by ramified type theory. All that's needed is that other, non-propositional types of relational entities are structured in certain plausible ways and that principles of e-grounding are true. But, and perhaps more importantly, we noted that even if propositions are structured, the principles of e-grounding don't require them to be too structured to be susceptible to paradoxes of grain such as the Russell-Myhill theorem. This makes it possible to have coarse-grained views about propositions, and yet admit that they have to come in a ramified hierarchy. It also puts those who reject structured propositions based on paradoxes of grain in an awkward position, as they now have independent motives to endorse a ramified space of propositions, which if they do, they can secure highly structured propositions, as well.

Only future work on both ramified and simple type systems and their large-scale metaphysical implications will determine which one, if any, is to be preferred as the correct framework for pursuing metaphysical inquiry.

Chapter 5

Towards a Unified Predicative Solution to Puzzles of Quantificational Ground

5.1 Introduction

In recent years, the theory of metaphysical ground (as presented in, e.g., Rosen, 2010; Fine, 2012a; Audi, 2012) has dominated a notable portion of the work in contemporary metaphysics, trying to explain in principled ways how certain facts or truths may hold 'in virtue' of certain others.

Numerous kinds of grounding relations have been introduced to the literature.[1] One important distinction is between partial and full grounds. *Partial* grounds are truths in virtue of which a truth holds, but they don't necessarily suffice in doing so; otherwise, they constitute *full* grounds of that truth. Another distinction is between mediate and intermediate

[1] See Fine (2012a) for more detailed discussions of the variants glossed here.

grounds. Grounds of a truth are *immediate* if there's no mediating truth between them and what they ground; otherwise, they constitute *mediate* grounds. Finally, some facts are *strict* grounds of some others if they are, in a sense, more 'fundamental'; otherwise, the grounds are *weak*. Put in terms of explanation, we can think of strict grounds as, in the words of Fine (2012a), ones that takes us "down in the explanatory hierarchy," whereas weak grounds "may also move us sideways in the explanatory hierarchy."

Recently, the theory of strict partial ground (henceforth: just ground, if no risk of ambiguity) has faced a range of puzzles and paradoxes. For example, Fine (2010) and Krämer (2013) have put forward some puzzles regarding the interaction of some impeccable principles of classical logic with certain plausible principles of mediate partial ground.[2] The variant put forward at Krämer (2013) is stated in a language where quantification into sentence position is permitted, and principles of ground are generalized to statements involving sentential variables. Quantifying into sentential position not only simplifies Fine's puzzles but also allows us to straightforwardly apply the idea of type-stratification to the particular entities at stake, namely propositions.[3]

In this paper, I discuss Krämer's puzzle, along with some other neigh-

[2]Immediate strict partial ground has also been argued to face certain inconsistency results (Fritz, 2021; Wilhelm, 2020b). This paper proposes a unified solution to the puzzles of mediate partial ground. We will, however, mention at the end how our solution may have the promise of addressing the latter, as well.

[3]Nothing in our discussions, however, will crucially hinge on the use of languages with resources that allow for second- or higher-order quantification. In fact, Fine's original puzzles (as stated in Fine, 2010) are all stated in first-order languages. That is, we can implement the same ideas explored in this paper in first-order languages, as well. Similarly, Korbmacher (2018b) offers his predicative solution in a first-order language.

boring puzzles of quantificational ground, and propose a novel, 'predicative' solution to them by means of deploying ramified typed systems in the background.

Here's how the paper is organized. In §5.1, I discuss Krämer's Puzzle. §5.2 proposes a solution to this puzzle along the lines of ramified type theory. In §5.3 I assess some of the alternative solutions to Krämer's Puzzle and in each case will reveal their shortcomings in comparison to the ramified solution. §5.4 addresses some neighboring puzzles to Krämer's Puzzle, and shows yet again, ramified type theory has the upper hand in resolving them than the typical solutions found in the literature. §5.5 discusses a higher-order variant of Krämer's Puzzle due to Thomas Donaldson, the ramified solution to it, as well as a promising, recently emerged resolution in the literature. The paper concludes at §5.6. The technical appendix attends to the rigorous presentation of ramified types and the theory of ground built on top of it.

5.2 Krämer's Puzzle of Ground

This puzzle, as presented by Krämer (2013), is second order, in that it allows for quantification in sentential position.

The puzzle has two assumptions. The first says that ground an irreflexive relation: no fact grounds itself. Formally put:

$$\neg(\phi \prec \phi) \qquad \text{IR}$$

where $\prec$ is a binary sentential connective that stands for the grounding relation, and ϕ schematically stands for any sentence.

The other principle states that existential statements are grounded by their true instances. Thus the fact that someone is the president of the US is grounded by the fact that, say, Joe Biden is the president of the US.

In a setting where we can quantify into sentence position, a version of the principle above will be thus: any true instance of the fact $\exists p\, \phi$ that some proposition is ϕ grounds it.[4] Put formally:

$$[\psi/p]\phi \rightarrow [\psi/p]\phi \prec \exists p\, \phi, \qquad \text{EG}$$

where $[\psi/p]\phi$ stands for the uniform 'substitution' of all occurrences of p in ϕ with ψ. Such a substitution presupposes that ψ is one of the things in the range of the quantifier in $\exists p\, \phi$; that is, it belongs to the range of the quantifier in question. Now, in particular, by replacing the schematic ϕ and ψ with p and $\exists p\, p$, respectively, it follows that:

$$\exists p\, p \rightarrow \exists p\, p \prec \exists p\, p, \qquad \text{EG}'$$

which is to say that if there is a truth, then the fact that there is a truth grounds itself.

But of course, there is a truth; that is, $\exists p\, p$ holds.[5] It follows then that the fact that there is some truth grounds itself; that is $\exists p\, p \prec \exists p\, p$, which goes against IR. Call this Krämer's Puzzle.

[4]In what follows I will take facts to be true propositions. This is not an entirely uncontentious assumption (see, e.g., Fine, 1982), but is the assumption at play in Krämer (2013) and some other recent responses to it (e.g., as in Correia, 2017; Fritz, 2021, 2019; Wilhelm, 2020b; Litland, 2022; Woods, 2018; Kiani, 2023). This, although makes it much more convenient to treat the puzzle at hand, is not a substantial premise for advancing predicative solution to the puzzle. In fact, and related to the previous footnote, Fine (2010) himself works in a setting where facts and true propositions are distinct entities, with their accommodating principles separately spelled out, and, as mentioned before, he contends that the problem can nevertheless be treated predicatively.

[5]That $\exists p\, p$ is true is both extremely plausible and in any case, a theorem of our background logic.

Krämer's Puzzle essentially shows that the set {EG, IR} is inconsistent, given the background propositional logic with quantification.

Two main options to reject the argument at Krämer's Puzzle have been pursued in the literature. They contain rejecting either EG or IR in its full generality.

One way to implement the first option is due to Fine (2010) (originally proposed for a close, first-order variant of it) which is to weaken EG to some but not all instances. This, as Woods (2018) puts it, is an "inegalitarian (and rather puzzling) thought": it weakens EG in an *ad hoc* manner just to avoid a contradiction. Moreover, as Fine (2010) himself observes, there are other variants of the puzzle that would still go through even under the tentative restriction in question. We will see two such variants later.

Woods (2018) pursues the second option. Woods's approach rests upon a construal of the grounding relation along the lines of 'explanation'. The idea is that in a genuine grounding statement, the particular content of the grounding fact has to contribute to the explanation of the content of the grounded fact. In such cases, the thought goes, grounding should be irreflexive, as nothing can substantively explain itself; for instance, we can't convincingly say that ϕ holds 'because' ϕ holds. However, sometimes the explanatory connection is lost, and any other fact could do in place of the ground. In other words, the grounding relation holds "vacuously". For example, in Krämer's Puzzle, *any* fact can ground the fact $\exists p\, p$ that some proposition is true: surely itself is a ground of it, but so is the fact that monkeys like bananas. According to Woods (2018), in such cases where the explanatory link is broken, instances of reflexive ground are unproblematic.

But this resolution, as Woods himself observes, at best applies only if we construe grounding entirely in its explanatory capacity. In particular, it doesn't seem applicable to the construals of grounding according to which grounds are "more fundamental" than the grounded, or that reality comes in "grounding layers"—a very widespread construals of the notion of ground.[6,7]

Another group of solutions propose predicative treatments of the matter, which impose certain restrictions on what goes in the domain of quantifiers (commonly known as predicativity). Both Fine (2010); Krämer (2013) mention predicative solutions in passing, but never explore them. Korbmacher (2018b) proposes one such solution in great detail, but only for some of the original puzzles in Fine (2010) that only concern the grounds of sentences and their truth.

In the next two sections, we propose a predicate solution to Krämer's Puzzle and some neighboring puzzles in terms of ramified type theory.

5.3 Ramified Types and Krämer's Puzzle

I will now attend to an informal exposition of ramified type theory and how it can block Krämer's Puzzle, in two different ways.

While there is no consensus on what 'predicativity' exactly means in the literature, typically, and loosely put, an entity that belongs to the

[6]As in, e.g.,(Rosen, 2010; Schaffer, 2009; Correia, 2021a,b; Leuenberger, 2020; Werner, 2020)

[7]It is worth mentioning that even construing grounding purely in its explanatory capacity doesn't seem obviously sufficient in allowing for instances of reflexive grounding. In fact, Fine (2010, p. 105) offers exactly an explanatory characterization of ground and claims that all of his grounding principles (including IR and EG) will hold. See also Trogdon (2013) for a similar view.

range of a quantifier that occurs in its 'definition' or 'construction' is said to be defined or specified "impredicatively"; otherwise it's "predicatively" defined or specified. Predicative definitions, in essence, impose some sort of hierarchy to what constructs what, banning certain circularities that could lead to certain inconsistencies similar to the case of the Liar's Paradox.

Now, since in deriving EG′, we are instantiating a quantified proposition (namely, $\exists p\, p$) with itself (by choosing ψ in $[\psi/p]$ to be $\exists p\, p$), which means that we are taking $\exists p\, p$ to be one of the propositions that it quantifies over in its definition, a predicative solution to the Krämer's Puzzle is to disallow just that, hence breaking the argument and avoiding the contradiction.

As Krämer puts this point: "Thus we might say, firstly, that in the sense that is relevant to EG, being an instance of quantification is not a purely syntactic matter. Rather, the expression generalized upon also has to satisfy a semantic condition: roughly, that of determining, or picking out, a value in the range of the corresponding existential quantifier. Secondly, we say that a sentence that itself contains a given sentential quantifier does not determine a value in the range of that quantifier. A simple implementation of that idea restricts [EG] to cases in which $[\phi]$ is free of sentential quantifiers; a less restrictive option is to introduce a hierarchy of sentential quantifiers and postulating a version of [EG] for each of them, requiring in each case that $[\phi]$ contain only quantifiers lower in the hierarchy." (*ibid*; p. 88)

Our solution to Krämer's Puzzle and other neighboring puzzles is closer to the second suggestion in the passage above and is cashed out in terms of the theory of ramified types. According to the latter,

propositions are stratified into a hierarchy of levels, in such a way that high-level propositions are obtained via quantification over lower-level ones. Moreover, quantified statements about propositions of a certain level can only be instantiated with propositions of that particular level. It then follows that the problematic instances of EG are no longer available to the ramifier: if $\exists pp$ is taken to range over, say, level n propositions, and is itself of a higher level, e.g., $n+1$, then $\exists pp$ won't be one of the propositions quantified over, hence the instantiation of $\exists pp$ with itself, which leads to EG$'$, will be illegal.

As we mentioned, in general, predicative responses to the puzzles such as Krämer's Puzzle remain fairly underexplored in the literature. Both Fine (2010) and Krämer (2013) gesture at such solutions, but none of them explores them in detail. Korbmacher (2018b) seems to have taken the first step in detailing such a solution to certain variants of the puzzle that have to do with *sentences* and their grounds, and not worldly entities such as propositions or facts, which is what is at stake as in Krämer's Puzzle (and the ones from the next section).

As a result, and importantly, the formal system that Korbmacher relies on only works if the relata of the grounding relation are literally sentences, and not the real entities such as facts or propositions which are expressed or represented by sentences. Although interesting on its own right, as we've noticed our puzzles of interest concern grounds of facts or propositions, and in any case, the majority of work on ground seems to conceive the relation as something that holds between facts or propositions, not their syntactic representations.[8]

[8]Here's the puzzle that Korbmacher (2018b) is concerned with: assuming that

5.4 Next-Door Puzzles

There are various other puzzles in the vicinity that are often overlooked in the discussions of Krämer's Puzzle and solutions to it. In this section I will introduce two such puzzles and show that, just as in the case of Krämer's Puzzle, ramified types block the inconsistencies involved in them. I will then discuss some potential ways to resolve these puzzles, including the responses to Krämer's Puzzle, glossed above, and will argue that most of these options are either irrelevant or implausible. This highlights the superiority of our ramified approach to most alternative approaches in the literature.

Our first puzzle is a simplified version of one of the puzzles in Fine (2010), called "Universal Argument for Propositions", that concerns grounds of universal statements. It's a simplification in the same way that Krämer's Puzzle is a simplification of another related puzzle in Fine's paper which relates grounds of existential statements. In particular, we

the truth of a sentence S grounds the truth of a sentence which says that S is true ($S \rightarrow S \prec \mathrm{T}(\ulcorner S \urcorner)$, where T is a truth predicate and $\ulcorner S \urcorner$ is the name of the sentence S), then the true sentence $\exists x \mathrm{T}(x)$ which says that there is at least one true sentence grounds $\mathrm{T}(\ulcorner \exists x \mathrm{T}(x) \urcorner)$. That is: $\exists x \mathrm{T}(x) \prec \mathrm{T}(\ulcorner \exists x \mathrm{T}(x) \urcorner)$. On the other hand, by the sentential version of the principle EG, $\exists x \mathrm{T}(x)$ is grounded by any of its true instances, including $\mathrm{T}(\ulcorner \exists x \mathrm{T}(x) \urcorner)$. That is: $\mathrm{T}(\ulcorner \exists x \mathrm{T}(x) \urcorner) \prec \exists x \mathrm{T}(x)$. By the transitivity for sentential grounds, we get to contradiction with irreflexivity. Korbmacher resolves this puzzle by appealing to a Tarskian hierarchy of truth: if we take truth predicates to come in levels, and that no truth predicate of any given level can appear in its scope, this puzzle can be shown to block the contradiction. While we don't treat these kinds of puzzles in our system, it remains an intriguing question whether ramified type theory can straightforwardly be used in resolving them. This sounds possible, especially because there seem to be intimate relations between ramified type theory and Tarskian hierarchies of truth—something that has been explored in detail at Church (1976). In any case, even if the ramified approach falls short of accommodating such variants of the puzzles, we may consider it as a complement rather than a competition to Korbmacher's approach, both living under the same roof, namely "predicative" solutions to the puzzles of mediate ground. We wish to leave investigating these matters for another occasion.

continue working in Krämer's setting where quantification into sentential is allowed, and assume that the principles of ground apply there.

The first assumption involved is one that usually comes in the same package with EG: a true universal claim is partially grounded by each and every one of its instances. Thus the fact that everyone eventually dies is grounded by each and every one of the facts that John will eventually die, Marry will eventually die, etc. In our setting this is formally put as follows:

$$\forall p\, \phi \rightarrow [\psi/p]\phi \prec \forall p\, \phi \qquad \text{UG}$$

The next assumption is about the grounds of true disjunctions, and is structurally very similar to EG. The idea is that a true disjunction is grounded by each of its true disjnucts. Thus the fact that either $2 + 2 = 4$ or Vancouver is located in Brazil is grounded by the fact that $2 + 2 = 4$. Formally put, we have:

$$\phi \vee \psi \rightarrow (\phi \rightarrow \phi \prec \phi \vee \psi) \wedge (\psi \rightarrow \psi \prec \phi \vee \psi) \qquad \text{DG}$$

Finally, mediate grounding is taken to be transitive. This is, in particular, plausible if we take (along with Fine, 2012a) statements of mediate grounding as chains of statements if immediate grounding, in the sense that $\phi \prec \psi$ stands for a chain like $\phi \ll \phi_1 \ll ... \ll \phi_n \ll \psi$, where $\ll$ stands for the relation of immediate ground.[9] Thus we have:

$$(\phi \prec \psi) \wedge (\psi \prec \theta) \rightarrow \phi \prec \theta \qquad \text{TR}$$

[9] In general, from σ being an immediate ground of τ (in symbols: $\sigma \ll \tau$) it follows that no fact is needed in between the latter two to make τ hold in virtue of σ.

Now the puzzle, which we call Finean Puzzle after Kit Fine, goes as follows:

Finean Puzzle. **(1)** from UG it follows that the fact that every proposition is either true or false is grounded by its self-instantiation. That is, if we substitute ϕ and ψ with $p \vee \neg p$ and $\forall p\,(p \vee \neg p) \vee \neg \forall p\,(p \vee \neg p)$, respectively, we have: $[\forall p\,(p \vee \neg p) \vee \neg \forall p\,(p \vee \neg p)] \prec \forall p\,(p \vee \neg p)$. **(2)** From DG it follows that the fact that every proposition is either true or false is grounded by its self-instantiation. That is, if we substitute ϕ and ψ with $p \vee \neg p$ and $\forall p\,(p \vee \neg p) \vee \neg \forall p\,(p \vee \neg p)$, respectively, we have: $[\forall p\,(p \vee \neg p) \vee \neg \forall p\,(p \vee \neg p)] \prec \forall p\,(p \vee \neg p)$.[10] **(3)** By TR it follows both that the fact that every proposition is either true or false grounds itself ($\forall p\,(p \vee \neg p) \prec \forall p\,(p \vee \neg p)$), and also that the fact that every proposition is either true or false, or every proposition is either true or false grounds itself. That is: $[\forall p\,(p \vee \neg p) \vee \neg \forall p\,(p \vee \neg p)] \prec [\forall p\,(p \vee \neg p) \vee \neg \forall p\,(p \vee \neg p)]$. Both of the latter are counterexamples to IR.

Finean Puzzle essentially shows that {UG, DG, TR, IR} is inconsistent. Now, clearly weakening or even rejecting EG, which was inspired by Fine (2010) as a potential response to Krämer's Puzzle, is orthogonal to this puzzle (Fine (2010) himself makes a similar point). To invalidate this argument, one has to now weaken or reject one of the principles involved in deriving the contradiction, i.e., DG, UG, TR and IR.

But as Fine (2010) observes, weakening DG (e.g., to a principle which states that *some* true disjuncts ground a true disjunction, not necessarily

[10] That every proposition is either true or false ($\forall p\,(p \vee \neg p)$) is just the logical law of excluded middle, and in any case a theorem of our background propositional logic with quantification.

all) is unhelpful in such situations because even if not all true disjuncts ground a true disjunction, in the case where there is only one true disjunct (as in all instances of $\phi \vee \neg\phi$), that disjunct *will* be a ground, and that's enough to preserve the puzzle.

Weakening UG is also implausible because, as Fine (2010) puts it, "we cannot properly take a universal truth to be grounded by some of its instances to the exclusion of others." In other words, the truth of all instances is needed to account for the truth of a universal statement. Rejecting either DG or UG completely is also implausible, and *ad hoc*.

There seems to be, therefore, only two plausible options to potentially avoid the inconsistency in Finean Puzzle: to reject transitivity (TR), or irreflexivity (IR) in their full generality. Despite its intuitive appeal, some people have, in fact, cast doubts on the transitivity of ground, and this might be a good time to leverage such results or add to them.[11] We already discussed proposals for rejecting reflexivity in its full generality in the case of Krämer's Puzzle.[12]

But let's assume these two options *are* properly defensible anyway. This would mean that rejecting either TR or IR might still be a viable option to some people, as a response to Finean Puzzle or Krämer's Puzzle. That may be. But there's a puzzle closer to Finean Puzzle that will go through even in a weaker system where irreflexivity and transitivity are *both* rejected. The idea is to trade irreflexivity and transitivity together

[11] See Tahko (2013); Schaffer (2012) for some works along these lines. See Litland (2013) for a defense of transitivity.

[12] For example, we learned that Woods (2018) allows for instances of reflexivity in cases of "vacuous" grounding statements. In particular, someone like Woods might still be able to claim that given all the new assumptions in Finean Puzzle, the grounding statements $\forall p\,(p \vee \neg p) \prec \forall p\,(p \vee \neg p)$ and $[\forall p\,(p \vee \neg p) \vee \neg\forall p\,(p \vee \neg p)] \prec [\forall p\,(p \vee \neg p) \vee \neg\forall p\,(p \vee \neg p)]$ hold 'vacuously'.

with asymmetry—a principle that seems to enjoy a more robust status in the literature than the pair of irreflexivity and transitivity with all the noise surrounding them. Asymmetry would disallow ϕ to be a ground of ψ, under the assumption that ψ is a ground of ϕ. Formally put:

$$(\phi \prec \psi) \rightarrow \neg(\psi \prec \phi) \qquad \text{AS}$$

The new puzzle is simple. It borrows the same first two lines from Finean Puzzle, but line (3) just highlights the contradiction of lines (1) and (2) with AS. Call the resulting puzzle New Puzzle.

New Puzzle. **(1)** from UG it follows that the fact that every proposition is either true or false is grounded by its self-instantiation. That is, if we substitute ϕ and ψ with $p \vee \neg p$ and $\forall p\,(p \vee \neg p) \vee \neg \forall p\,(p \vee \neg p)$, respectively, we have: $[\forall p\,(p \vee \neg p) \vee \neg \forall p\,(p \vee \neg p)] \prec \forall p\,(p \vee \neg p)$. **(2)** From DG it follows that the fact that every proposition is either true or false is grounded by its self-instantiation. That is, if we substitute ϕ and ψ with $p \vee \neg p$ and $\forall p\,(p \vee \neg p) \vee \neg \forall p\,(p \vee \neg p)$, respectively, we have: $[\forall p\,(p \vee \neg p) \vee \neg \forall p\,(p \vee \neg p)] \prec \forall p\,(p \vee \neg p)$.[13] **(3)** By AS lines 2 and 3 contradict.

This puzzle essentially shows the inconsistency of {UG, DG, AS}. This means that none of the plausible options for rejecting Finean Puzzle or Krämer's Puzzle are available for rejecting New Puzzle.[14]

[13] That every proposition is either true or false ($\forall p\,(p \vee \neg p)$) is just the logical law of excluded middle, and in any case a theorem of our background propositional logic with quantification.

[14] In particular, and concerning the resolution of Woods (2018), notice that by rejecting transitivity, there seems to be no way to say that $\forall p\,(p \vee \neg p) \prec \forall p\,(p \vee \neg p)$ holds vacuously, because for that to be the case one needs to say that *any* fact can be replaced with the ground on the left-hand side of the statement in question; that is, that

Finally, note that Finean Puzzle and New Puzzle can both be resolved by our ramified approach with the exact line reasoning as in the case of Krämer's Puzzle. The former two both rely on the principle UG, which, in the presence of ramified types, should be restricted on the same grounds as before. That is, assuming that in UG the quantifier ranges over propositions of level n, only *level-n* propositions can instantiate UG. In particular, this implies that $\forall p^{\langle\rangle/n}\,(p \vee \neg p)$ cannot be an instance of itself because if p is of level n, then $\forall p\,(p \vee \neg p)$ will be of level $n{+}1$, which makes it an illegitimate instance of itself.

5.5 Upper-Floor Puzzles

Donaldson (2017) proposes a variant of the puzzle close to Krämer's Puzzle, in a second-order language that is richer than what we've been working with so far, where variables for properties are available and predicate-making devices such as lambda abstraction are available. In this section, I introduce a *higher-order* variant of puzzles of that sort that really encompass any relational entities, including propositions (as in Krämer's Puzzle) and properties (as in Donaldson, 2017). (Donaldson's puzzle will fall under this more general variant.) I will, in particular, consider

for any fact γ we need $\gamma \prec \forall p\,(p \vee \neg p)$ to follow from $\forall p\,(p \vee \neg p) \prec \forall p\,(p \vee \neg p)$. Notice the reason that we could correctly say this in in the case of Finean Puzzle was that for our arbitrary fact γ, by DG we had $\gamma \prec \gamma \vee \neg\gamma$ and by UG we had $\gamma \vee \neg\gamma \prec \forall p\,(p \vee \neg p)$, which then by TR we could get $\gamma \prec \forall p\,(p \vee \neg p)$. But TR is no longer available, so the link that would connect the arbitrary γ to $\forall p\,(p \vee \neg p)$ is broken. Therefore, the resolution of Woods (2018) is also inapplicable to this puzzle. In fact, the damage to Woods's strategy seems to be even more serious: the whole distinction of vacuous vs. non-vacuous grounding statements was supposed to allow for some unproblematic instances of *irreflexivity*, the one present at Krämer's Puzzle being such a case. But now that we have also rejected irreflexivity, it's not at all clear how the mentioned distinction is still relevant, or if it can be generalized to the case of asymmetry.

the variant that goes through with AS, instead of the pair of IR and TR. We will see that the ramified approach will, just as conveniently and naturally as before, will settle this puzzle as well. All that is needed is to use richer ramified languages that have the relevant resources such as all the relational types as well as term-forming rules such as lambda abstraction.

Some background regarding higher-order quantification first. (See the appendix for the rigorous presentation of the higher-order ramified system at use in this section.) We assume there are entities such as individuals, properties of individuals, propositions, polyadic relations between such entities, and all sorts of properties and relations that hold for or between these things, etc. We distinguish these entities at the level of syntax of associating *types* to the relevant terms that stand for them: type e for individuals, $\langle\rangle$ for propositions and $\langle t_1, ..., t_n\rangle$ for n-ary relations ($n \geq 1$) that between entities of types, t_1, ..., t_n, respectively.

One way to form sentences in higher-order logic is through *application*: for any given type t, if F and a are, respectively, terms of type $\langle t\rangle$ and t, then $F(a)$ is a term of type $\langle\rangle$, i.e., a formula. Similarly any relational term of type $\langle t_1, ..., t_n\rangle$ can simultaneously apply to entities of types t_1, ..., t_n, respectively, to create sentences. Another way to create terms in higher-order logic is through *abstraction*, which creates predicates out of sentences. For instance, from the sentence 'Someone likes John', formally represented by $\exists x^e L(x, j)$ (with L being a constant of type $\langle e, e\rangle$ standing for the relation of loving, and j of type e a name for John), we can create the predicate 'being loved by someone' by abstracting from the name of John (j), using lambda abstraction: $\lambda y^e.\exists x^e L(x, y)$. The predicate

is taken to stand for the property be being loved by someone. We can similarly create predicates with regards entities of any arbitrary type t. Thus the property of being a proposition that has all properties of propositions can be said to picked by the predicate $\lambda p^{\langle\rangle}.\forall X^{\langle\langle\rangle\rangle}X(p)$, which itself has type $\langle\langle\rangle\rangle$.

Now, in higher-order logic (simple type theory), besides very natural generalizations of the principles of 'lower'-order logics. In particular, for the first-order logic and the axiom schema Existential Introduction—EI: $[a/x]\phi \rightarrow \exists x\, \phi$—the higher-order version is this:

$$[a/x]\phi \rightarrow \exists x^t\, \phi \qquad \text{EI}^t$$

where a is any term of type t.

We have a principle that governs λ-terms:

$$(\lambda x^t.\phi)(\psi) \leftrightarrow \phi[\psi/x] \qquad \beta_E^t$$

This principle is extremely plausible that gives us equivalences such as this: Napoleon was a French emperor if and only if Napoleon was French and Napoleon was an emperor: $(\lambda x^e.F(x)\wedge E(x))(N) \leftrightarrow F(N)\wedge E(N)$, with the relevant conventions regarding the constants used in place.

As for the grounds of sentences encompassing λ-terms, a principle that is commonly used is as follows (Fine, 2012a; Dorr, 2016):

$$[a/x]\phi \rightarrow ([a/x]\phi \prec (\lambda x^t.\phi)(a)) \qquad \lambda G^t$$

Thus granting that to be a bachelor is to be an unmarried man (i.e., $B \coloneqq \lambda x^e.(M(x) \wedge U(x))$), this principle implies that the fact $B(g)$ that Gary is a bachelor is grounded by the fact $M(g) \wedge U(g)$ that Gary is an unmarried man: $(M(g) \wedge U(g)) \prec (\lambda x^e.(M(x) \wedge U(x)))(g)$.

Finally, to get the contradiction, we need that EG is generalized to existential quantification over types of *any* entity (including propositions, as before). Thus for any type t we have the following principle:

$$[a/x]\phi \to [a/x]\phi \prec \exists x^t\, \phi, \text{ where } a \text{ is a term of type } t, \qquad \mathrm{EG}^t$$

The puzzle, which I call Donaldsonian Puzzle for obvious reasons, goes as follows:[15]

Donaldsonian Puzzle. **(1)** Consider an arbitrary true sentence like $F(a)$, where F is of type $\langle t\rangle$ and a is of type t.[16] **(2)** By EI^t we have $\exists X^{\langle t\rangle} X(a)$. By β_E, the latter is equivalent to $\lambda y^t.\exists X^{\langle t\rangle} X(y)$, which then has to be also true. **(3)** By λG: $\exists X^{\langle t\rangle} X(a) \prec (\lambda y^t.\exists X^{\langle t\rangle} X(y))(a)$. **(4)** Independently, given that $\lambda y^t.\exists X^{\langle t\rangle} X(y)$ is of type $\langle t\rangle$, from (2) and EG^t it follows that $(\lambda y^t.\exists X^{\langle t\rangle} X(y))(a) \prec \exists X^{\langle t\rangle} X(a)$. **(5)** (3) and (4) together go against AS.

Now, in a language where *all* relational entities are stratified into levels (including propositions, as in the previous sections) the ramified solution can accommodate this puzzle by a similar restriction on EG^t: we should only allow for subsitutees a that are of the same level with x. But then Donaldsonian Puzzle will be unfounded: assuming X is of level n, then $(\lambda y^t.\exists X^{\langle t\rangle} X(y))$, now represented as $(\lambda y^t.\exists X^{\langle t\rangle/n} X(y))$ will be of type $\langle t\rangle/n{+}1$, and $(\lambda y^t.\exists X^{\langle t\rangle/n} X(y))(a)$ of type $\langle\rangle/n{+}1$. So the latter won't be an instance of the existential statement $\exists X^{\langle t\rangle/n} X(a)$.

[15] The original puzzle in Donaldson (2017) or the ones close to it are special instances of the following, for t being $\langle e\rangle$

[16] For instance, let t be $\langle\langle\rangle\rangle$, F stand the property of knowablity and a be the truth constant $\top$; then $F(a)$ stands for the proposition that $\top$ is knowable, which is true.

Before concluding the paper, notice that in what passed, we didn't discuss the kind of strategy gestured at Fritz (2020) for resolving Krämer's Puzzle. This is mostly due to the generally different approach that he takes, which appeals to the kind of higher-order systems, which we briefly introduced here and now have the muscles to engage with.

The idea is to treat, as common in higher-order logic, quantificational statements not as unanalyzable terms of the language (given by e.g., a clause like this: "if ϕ is a term of type $\langle\rangle$ and x a variable of type t, then $\forall x^t\,\phi$ is of a term of type $\langle\rangle$") but as instances of the application of certain constants standing for existential or universal quantifiers to λ-terms. For instance, therefore, a universal statement like $\forall x^t\,\phi$ would be a shorthand for the application $(\forall^t)(\lambda x^t\,\phi)$, where $\forall^t$ is constant of type $\langle t\rangle$, which is reserved for quantification over t type entities as above.

With this change of our construal of quantification, EG is replaced by the following:

$$(F\psi \to (F\psi \prec \exists^{\langle\rangle} F)), \qquad \exists G$$

where F and ψ are, respectively, of types $\langle\langle\rangle\rangle$ and $\langle\rangle$

Now if we define $T := \lambda p^{\langle\rangle}.p$ (the property of being true), by (higher-order) Universal Instantiation (UI^t: $\forall x^t\phi \to \phi[a/p]$), where a is of type t) we have:

$$T(\exists^{\langle\rangle}T) \to (T(\exists^{\langle\rangle}T) \prec \exists^{\langle\rangle}T) \qquad (*)$$

and since $T(\exists^{\langle\rangle}T)$ is true (it's true that there's a truth), then $T(\exists^{\langle\rangle}T) \prec \exists^{\langle\rangle}T$. But this is *not* necessarily an abuse of irreflexivity, unless, in general, we have something stronger than β_R, where the material equivalence $\leftrightarrow$ is

replaced with identity =. In the simple type theory that we've endorsed here, there is no such strong principle. One might, therefore, think that that's all it takes to avoid the variant of Krämer's Puzzle in a higher-order language with EG replaced by $\exists G$. But that's not the case. For even $\lambda G^{\langle\rangle}$ itself accompanied with $(*)$ goes against IR: since $T(\exists^{\langle\rangle}T)$ is true, from $\lambda G^{\langle\rangle}$ it follows that $\exists^{\langle\rangle}T \prec T(\exists^{\langle\rangle}T)$, which together with $T(\exists^{\langle\rangle}T) \prec \exists^{\langle\rangle}T$ (obtained from $(*)$), go against AS.

Throughout the paper, Fritz (2020) tries to motivate rejecting $\lambda G^{\langle\rangle}$ (as well as a strengthening of β_E where $\leftrightarrow$ is replaced by =; something that hasn't been endorsed here) in systematic, independent ways. Fritz himself observes, his solution to Krämer's Puzzle is applicable to puzzles where grounds of universal statements are dealt with (and admittedly, that includes our Finean Puzzle as well as New Puzzle). Clearly, his approach also would avoid the inconsistency involved in Donaldsonian Puzzle. This puts Fritz's approach in much better position to most of the rivals approaches that we glossed in previous sections, and certainly this strategy is worth further exploring. Goodman (2022) takes a step in exploring Fritz's ideas in more detail.

Lastly, recently Fritz (2021) has argued that the notion of immediate ground is somewhat too fine-grained, in a way that its principles make it susceptible to paradoxes of structured propositions along the lines of Russell-Myhill paradox, an issue that goes back to Russell (1903) and Myhill (1958), and has recently been revived and initiated new debates about the granularity of reality.[17]

[17]For some of the recent works along these lines, see Hodes (2015); Goodman (2016); Uzquiano (2015); Dorr (2016).

Since, as was mentioned before, ramified type theory also blocks the Russell-Myhill result in its most general form in simple type theory (as argued for in Chapter 2, i.e., Kiani, MSd), one would expect the particular instances found in Fritz (2021) would as well be rejected; all that is needed is to find the relevant models for the ramified versions of the principles of immediate ground. We leave this for future investigation. But importantly, as Fritz (2020) observes, the ideas he sketches for resolving his higher-order version of Krämer's Puzzle seem to also resolve the paradox of immediate ground introduced in Fritz (2020)—at least the one that is obtained using simple type theory in the background.

To conclude: the proposal sketched at Fritz (2020), and explored in Goodman (2022), might well be the most promising alternative to the ramified approach: they both offer unified solutions to a range of neighboring puzzles of mediate and immediate ground. The main advantage of Fritz's approach is that its background logic, which is based on simple types, is more expressive than the higher-order logic-based ramified types.

The ramified approach, however, has other advantages over this approach: first, we can maintain the attractive and plausible schemata λG^t while resolving higher-order puzzles of mediate ground. Second, and as has been mentioned on several occasions, the ramified system used in resolving all the puzzles in this paper has independent metaphysical motivations along the lines of entity grounding Kiani (MSb). Finally, as is historically known, the ramified approach has the capacity to resolve a host of other paradoxes of intensionality.

5.6 Conclusion

We noticed that ramified type theory offers an elegant, unified approach to solve four neighboring puzzles (and clearly different variants of them expressed within similar languages) that concern the relationship between mediate partial ground and logical operators (as presented, e.g., in Fine, 2010; Krämer, 2013; Donaldson, 2017) in a way that most of the rival accounts in the literature don't. This on its own, we believe, shows the superiority of the ramified approach. But as we've mentioned several times, ramified type theory itself can be motivated as a way to accommodate a neighboring, much vaster notion of metaphysical priority, namely entity-grounding (Kiani, MSb). Finally, we briefly mentioned that the ramified approach could potentially be used in proving the consistency of the notion of *immediate* ground, against the kind of paradoxes explored in Fritz (2021)—something that we conjecture but leave for future investigation in future works.

We noticed a particular solution to the puzzles, sketched by Fritz (2020), where he construes the existential quantification along the lines of higher-order logic (as instances of applications of quantificational constants to λ-terms), which, when implemented, the ensuing principle of existential grounds ($\exists G$) will avoid the puzzles—at least as long as principles like $\lambda G^{\langle\rangle}$ are rejected. While the proposal of Fritz's remains to be further explored, it seems the strongest contender of the ramified approach, in that they both, if properly laid down, cover a range of puzzles of mediate and immediate ground in systematic ways. We mentioned that while the Fritzian approach enjoys the more expressive language of simple type

theory, the ramified approach retains attractive principles such as λG^t (for any type t), and has independent metaphysical motivations along the lines of the neighboring, much broader notion of entity grounding—something that has been explored in a sequel to this paper (that is, Chapter 4). Only future work on both approaches will, hopefully, will shed light on their status and the 'correct' choice between them.

Finally, it is of utmost importance for the reader to understand that, while all the principles of higher-order logics of ramified ground can be properly formalized (as in the appendix), all the arguments in favor of ramified type systems in the context of grounding puzzles that were given throughout the paper were semi-formal, or rather informal. In other words, we argued but didn't exactly prove that the inconsistency proofs won't go through if we replaced propositions with leveled propositions, in the way predicated by ramified type theory. However, to show rigorously that the inconsistencies are in fact blocked in the presence of ramified types, one would have to prove the consistency results for the ramified versions of the principles above, e.g., through model constructions. We leave this as the open problem of our paper and hope to investigate it in the future.

5.7 Appendix - The Technical Appendix

Ramified Type Theory

Here we introduce a formal, $\mathcal{R}$, which underlies our informal discussions in the paper.[18]

[18]See Kiani (MSd) for a comprehensive study of $\mathcal{R}$ and related consistency results.

First, let's introduce ramified *types* and their *levels*:

Definition 5.7.1 (Ramified Types and Levels)**.** The set $\mathcal{T}^r$ of *ramified types* t and their levels $\mathrm{l}(t)$ are simultaneously defined as follows: $e \in \mathcal{T}^r$ with $\mathrm{l}(e) = 0$, and for $t_0, ..., t_n \in \mathcal{T}^r$, if $\mathrm{l}(t_i) \leq m$ for each $i = 0, ..., n$, then $\langle t_0, ..., t_n \rangle\!\rangle m \in \mathcal{T}^r$, with $\mathrm{l}(\langle t_1, ..., t_n \rangle\!\rangle m) = m$.

For any ramified type $t \in \mathcal{T}^r$ we assume there's a denumerably infinite set of *variables* Var^t of type t and a (possibly empty) set of typed non-logical *constants* CST^t. For certain types there are also logical constants to be introduced below. (We will reserve CST^t for the set of all constants (logical or non-logical) of type t.) We define the sets of all variables and constants respectively as $\mathrm{Var} := \bigcup_{t \in \mathcal{T}^r} \mathrm{Var}^t$ and $\mathrm{CST} := \bigcup_{t \in \mathcal{T}^r} \mathrm{CST}^t$.

We now define the language of $\mathcal{R}$. Later we will equip $\mathcal{R}$ with a grounding operator $\prec$, and propose a logic of higher-order ramified partial ground, which is the formal system that underlies the solution to the puzzles that were discussed in §§5.1-5.4.[19]

Definition 5.7.2 (Terms of $\mathcal{R}$)**.** The *terms* of $\mathcal{R}$ are defined as follows:

1. If σ is a variable or constant of type t, then is a term of type t,

2. If $x_1, ..., x_n$ are pairwise distinct variables of respectively types $t_1, ..., t_n$, where $n \geq 1$ and $\mathrm{l}(t_i) \leq m$ for each t_i, and ϕ is a term of type $\langle\rangle\!\rangle m$, then $\lambda x_1^{t_1}, ..., x_n^{t_n}.\phi$ is a term of type $\langle t_1, ..., t_n \rangle\!\rangle m$,

3. If τ is a term of type $\langle t_1, ..., t_n \rangle\!\rangle m$, where $n \geq 1$, and for each $i = 1, ..., n$, τ_i is a term of type t_i, then $\tau(\tau_1, ..., \tau_n)$ is a term of type $\langle\rangle\!\rangle m$,

[19] In this definition we're proposing separate clauses for the logical vocabulary. That is, we're treating the logical vocabulary asyncategormatically. This is mostly for convenience, but also in line with the common works in the theory of ramified types.

4. If ϕ is of type $\langle\rangle/i$, then $\neg\phi$ is also a formula and of type $\langle\rangle/i$,

5. If ϕ and ψ are respectively formulas of types $\langle\rangle/i$ and $\langle\rangle/j$, then $\phi \rightarrow \psi$ is a formula and is of type $\langle\rangle/\max\{i, j\}$,

6. If ϕ is a formula of type $\langle\rangle/j$, then $\forall x^t\phi$ is of type $\langle\rangle/\max\{\mathrm{l}(t)+1, j\}$,

The notions of *free* and *bound* variables of terms, *substitutions* of terms for variables, and *being free for a variable*, are defined as usual. We denote the set of free variables in a term σ by $FV(\sigma)$, and the set of all terms of ramified type theory by TERM_r.[20]

We can now introduce the proof system $\vdash^{\mathcal{R}_p}$. In what follows, subindexing a formula is intended to mean that the formula schematically stands for any sentence of that level. Thus ϕ_i schematically stands for any level-i sentence.

Proof System $\vdash^{\mathcal{R}_p}$:

Axioms:

- $\phi_i \rightarrow (\psi_j \rightarrow \phi_i)$; $(\phi_i \rightarrow (\psi_j \rightarrow \gamma_k)) \rightarrow ((\phi_i \rightarrow \psi_j) \rightarrow (\phi_i \rightarrow \gamma_k))$; $(\neg\phi_i \rightarrow \neg\psi_j) \rightarrow (\psi_j \rightarrow \phi_i)$. $\quad$ Taut_r[21]

- $\forall x^t\phi_j \rightarrow \phi_j[a/p]$, where a is of type t $\quad$ UI_r^t

[20]Notice that our ramified types and terms, as introduced in the definitions above, are very similar to Halod Hodes's System $\Rightarrow^{nr}$, as introduced in Hodes (2013). The main differences are that what we consider as level is called "order" by Hodes, and that in our system, but not Hodes', vacuous lambda abstraction is possible.

[21]Theorems of classical propositional logic can be derived with these axioms (obviously with levels dropped) (See system P2 in Church, 1956). Or choice of these axioms over other existing axiomatizations of propositional logic is due to our choice of primitive Boolean connectives, namely, $\neg$ and $\rightarrow$.

- $(\lambda x_1^{t_1}, ..., x_n^{t_n}.\phi_m)(\sigma_1, ..., \sigma_n) \leftrightarrow [\sigma_1/x_1, ..., \sigma_n/x_n]\phi_m$, where the type of σ_i is identical to t_i, for each $i = 1, ..., n$. $\beta^t_{E_r}$

Inference Rules:

- $\phi_i, \phi_i \rightarrow \psi_j / \psi_j$ MP_r
- $\phi_i \rightarrow \psi_j / \phi_i \rightarrow \forall x^t\, \psi_j$, where x doesn't occur free in ϕ_j UG^t_r

Notice that each of the axioms and rules of inference above are multiply schematic. For example in $\textsc{Taut}_r$, the axioms hold for any sentence of any level, and the relevant instances of $\neg$ and $\rightarrow$ may differ in type and should be typed carefully. Also, notice that UI^t_r is schematic in its occurrence of the terms, types t *and* the level m of the relational types $\langle t \rangle\!\!/m$ involved.

Higher-Order Logic of Ramified Partial Ground

I will now lay down a tentative formulation of the logic of partial ground under ramified type theory that accommodates the hierarchical structure of reality imposed upon us by type stratification. The principles are just the ones standardly attributed to the notion, many of which are familiar from earlier sections in the paper.[22] I will only take into account the notion of strict, mediate partial ground, which is the one that's at work in our favorite puzzles.

We first need to introduce the language. We only add the following clause to our ramified language $\mathcal{R}$, to obtain the language $\mathcal{R}^{\prec}$:

[22]See also Fine (2010); Fritz (2021); Fine (2012a) for those and the rest of the principles below.

If ϕ and ψ are formulas respectively types $\langle\rangle\!/\!m$ and $\langle\rangle\!/\!n$, then $\phi \prec \psi$ is a formula of type $\langle\rangle\!/\!r$, where $r := \max\{m, n\}$.

Here's the logic of ramified partial ground. As expected, the system builds upon $\vdash^{\mathcal{R}}$, so it encompasses the latter with the addition of the following principles. We call the resulting system $\mathcal{G}_f$.

1. $\neg(\phi_m \prec \phi_m)$ — IR_r
2. $(\phi_m \prec \psi_n) \wedge (\psi_n \prec \theta_r) \rightarrow \phi_m \prec \theta_r$ — TR_r
3. $(\phi_m \prec \psi_n) \rightarrow \neg(\psi_n \prec \phi_m)$ — AS_r
4. $\phi_m \wedge \psi_n \rightarrow (\phi_m \prec \phi_m \wedge \psi_n) \wedge (\psi_n \prec \phi_m \wedge \psi_n)$ — CG_r
5. $\phi_m \vee \psi_n \rightarrow (\phi_m \rightarrow \phi_m \prec \phi_m \vee \psi_n) \wedge (\psi_n \rightarrow \psi_n \prec \phi_m \vee \psi_n)$ — DG_r
6. $[a/x]\phi_m \rightarrow [a/x]\phi_m \prec \exists x^t \phi_m$, where a is of type t — EG_r^t
7. $\forall x^t \phi_m \rightarrow [a/x]\phi_m \prec \forall x^t \phi_m$, where a is of type t — AG_r^t
8. $\neg\phi_m \rightarrow \neg\phi_m \prec \neg(\phi_m \wedge \psi_n)$ — $\mathrm{NG}_{r,1}$
9. $\neg\psi_n \rightarrow \neg\psi_n \prec \neg(\phi_m \wedge \psi_n)$ — $\mathrm{NG}_{r,2}$
10. $\neg\phi_m \rightarrow \neg\phi_m \prec \neg(\phi_m \vee \psi_n)$ — $\mathrm{NG}_{r,3}$
11. $\neg\psi_n \rightarrow \neg\psi_n \prec \neg(\phi_m \vee \psi_n)$ — $\mathrm{NG}_{r,4}$
12. $\phi_m \prec \neg\neg\phi_m$ — $\mathrm{NG}_{r,5}$
13. $\neg[\psi_n/p]\phi_m \rightarrow \neg[\psi_n/p]\phi_m \prec \neg\forall p^{\langle\rangle\!/\!n}\phi_m$ — $\mathrm{NG}_{r,6}$
14. $\neg\exists p^{\langle\rangle\!/\!n}\phi_m \rightarrow \neg[\psi_n/p]\phi \prec \neg\exists p^{\langle\rangle\!/\!n}\phi_m$ — $\mathrm{NG}_{r,7}$

15. $[a/x]\phi_m \rightarrow [a/x]\phi_m \prec (\lambda x^t.\phi_m)(a)$. $\lambda G^t_{r,1}$

16. $\neg[a/x]\phi_m \rightarrow \neg[a/x]\phi_m \prec \neg((\lambda x^t.\phi_m)(a))$. $\lambda G^t_{r,2}$

We learned in §5.3 Krämer's Puzzle essentially shows that the set {IR, TR, EG} is inconsistent. We also noticed in §5.4 that Finean Puzzle and New Puzzle show that {IR, TR, DG, UG} and {AS, DG, UG} are, respectively, inconsistent. Similarly, for any type t, the higher-order Donaldsonian Puzzle shows the inconsistency of $\{\mathrm{EG}^t, \lambda G^t, \beta^t_E, \mathrm{AS}\}$.

While we argued rather informally, showing rigorously that Krämer's Puzzle, Finean Puzzle and New Puzzle are all blocked in our ramified setting, in particular, amounts to proving that the sets $\{\mathrm{IR}_r, \mathrm{EG}^{\langle\rangle}_r\}$, $\{\mathrm{IR}_r, \mathrm{TR}_r, \mathrm{DG}_r, \mathrm{UG}^{\langle\rangle}_r\}$ and $\{\mathrm{AS}_r, \mathrm{DG}_r, \mathrm{UG}^{\langle\rangle}_r\}$ are, respectively, consistent. Similarly, for any type t, showing that Donaldsonian Puzzle is blocked amounts to proving the consistency of $\{\mathrm{EG}^t_r, \lambda G^t_r, \beta^t_{E_r}, \mathrm{AS}_r\}$.

In general, all of these will follow by showing that the theory $\mathcal{G}_f$ of ramified ground proposed above is consistent—a result that we conjecture but leave for future work to explore.

Chapter 6

Conclusion

In this thesis, which is a bundle of four interconnected papers, we took the first step in exploring certain deep relationships between ramified type systems, highly structured relational entities, and various notions of grounding.

While these efforts are hoped to shed light on these topics, their relationships to one another and the picture of reality that they bring about, we would like to acknowledge the fact that a lot more work needs to be done for this picture to achieve a higher resolution, and for the doctrine to reach to an industry-level adoption.

Particularly, we believe some of the open problems that we encountered throughout these papers first need to be settled for these works to have the intended impact in the philosophical community.

Some of these open problems were as follows:

1. While models of ramified-type systems have been offered (Chapter 2), we only informally argued that the Russell-Myhill result won't

hold once we adopt ramified types. Models of these systems have yet to be explored to count as decisive proof for this claim.

2. Models for the formal system of entity-grounding (Chapter 4) need to be found in order to prove the consistency of the system.

3. Models of higher-order ramified logics of partial ground (Chapter 5) need yet to be explored.

In summary, we have proposed several well-motivated and seemingly well-behaving formalisms that have intuitive and systematic motivations and applications in philosophy, and the typical ways their alternative, simple-typed, systems lead to inconsistency seem to be unavailable for them, but the relevant models yet have to be explored.

We leave these as the open problems of this thesis, conjecturing that all these models exist and await discovery. In fact, if this conjecture is true, it implies that there is one 'mega-model', so to speak, that captures ramified type theory plus all the logical relevant logical augmentations explored in this thesis.

Our recent attempts suggest such a model may not be far from our reach, only requiring carefully implementing certain twitches in some of the recent simple-typed models proposed in the literature on ground and grain—the question seems to be about *what* twitches need to be made, mainly. We hope to have the opportunity to explore these in the future.

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