THE UNIVERSITY OF CALGARY

Commuting Matrices and Multiparameter Eigenvalue Problems

by

Tomaž Košir

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a dissertation entitled "Commuting Matrices and Multiparameter Eigenvalue Problems" submitted by Tomaž Košir in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Abstract

In this dissertation we study finite-dimensional multiparameter eigenvalue problems. The main objects considered are multiparameter systems, i.e., systems of n linear n-parameter pencils. To a multiparameter system we associate an n-tuple of commuting matrices called an associated system. The main problem considered is to describe a basis for the root subspaces of an associated system in terms of the underlying multiparameter system.

In Chapter 1 we study general n-tuples of commuting matrices, the motivation being the fact that the associated system is a special n-tuple of commuting matrices. Without loss of generality we may assume that the commuting matrices considered are nilpotent. We reduce an n-tuple of commuting nilpotent matrices to a special upper-triangular form using simultaneous similarities. The main two properties of this form are that certain column cross-sections are linearly independent and that certain products of row and column cross-sections are symmetric. This symmetry enables us to associate symmetric matrices and also symmetric tensors with the special upper-triangular form. We discuss this in detail for nonderogatory and simple cases, i.e., cases when the intersection of the kernels of the nilpotent commuting matrices has dimension one. The symmetric tensors appear as coefficients of decomposable tensors in the expansion of root vectors of associated systems.

In Chapter 2 we introduce multiparameter systems and their associated systems following the construction of F.V. Atkinson. We also describe a basis for the second root subspace of the associated system for general eigenvalues. For two-parameter systems this can be done in a canonical way. We describe this construction in Chapter 3.

In Sections 1.6, 2.3 and 2.4 we consider at various times the problem of the representation of commuting matrices by tensor products of matrices. This leads to

a similar problem of representation by the associated system of a multiparameter system.

The main results of the dissertation appear in Chapter 4. We describe bases for root subspaces of an associated system in terms of the underlying multiparameter system for nonderogatory and simple eigenvalues. These are eigenvalues for which the joint (geometric) eigenspace of the associated system is exactly one-dimensional.

Acknowledgements

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Many examples calculated on different computers using MATLAB and Mathematica software helped me to understand better the structure of root vectors described in this dissertation. I wish to thank Professors Len P. Bos and Larry Bates and my friend Rok Sosič who, at various times, allowed me the access to the software on their computers. I also wish to thank Rok Sosič and Andrej Brodnik for sending me by fax copies of articles not available in our library.

to my parents

Contents

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Chapter 0

Introduction

One way in which multiparameter eigenvalue problems arise is when the method of separation of variables is used to solve boundary value problems for partial differential equations. Each 'separation constant' gives rise to a different parameter. The resulting equations are simpler boundary value problems, for example of Sturm-Liouville type. Two-parameter problems of this type have been studied since the earliest days of the subject, and the following formulation is, for example, the main object of study in Faierman's monograph [69] :

$$\frac{d}{dx_i}\left(p_i(x_i)\frac{dy_i}{dx_i}\right) + (\lambda_1 A_i(x_i) + \lambda_2 B_i(x_i) - q_i(x_i))\, y_i = 0,\ i = 1, 2, \tag{0.1}$$

where $0 \leq x_i \leq 1$, and boundary conditions are

$$y_i(0)\cos\alpha_i - p_i(0)\frac{dy_i}{dx_i}(0)\sin\alpha_i = 0,\ 0 \leq \alpha_i < \pi,$$

and

$$y_i(1)\cos\beta_i - p_i(1)\frac{dy_i}{dx_i}(1)\sin\beta_i = 0,\ 0 < \beta_i \leq \pi,$$

for $i = 1, 2$. These and other problems have motivated the development of Multiparameter Spectral Theory. Atkinson [10] laid the foundations of Abstract Multiparameter Spectral Theory and he gave in [9] an overview of possible directions for further research that largely remain yet to be explored.

One of the main goals of Multiparameter Spectral Theory is to give completeness results for different multiparameter spectral problems. For example, one could try to expand functions defined on the domain of the partial differential equation in terms of Fourier-type series involving the eigenfunctions of the separated (say Sturm-Liouville) equations.

In the abstract theory, the main object studied is the n-tuple of n-parameter pencils

$$W_i(\boldsymbol{\lambda}) = \sum_{j=1}^{n} \lambda_j V_{ij} - V_{i0}, \ i = 1, 2, \ldots, n \ \ (n \geq 2),$$

also called the multiparameter system. Here V_{ij} are, for all j, linear operators on the Hilbert space H_i. In applications like (0.1), V_{ij}, $j = 1, 2, \ldots, n$, are multiplication operators and V_{i0} are differential operators. In the multiparameter eigenvalue problem we first find n-tuples of complex numbers $\boldsymbol{\lambda}$ such that all the operators $W_i(\boldsymbol{\lambda})$ are singular. This can be considered as a generalization of the ordinary eigenvalue problem.

One fundamental tool of Abstract Multiparameter Spectral theory is a tensor product construction. We consider the tensor product space $H = H_1 \otimes H_2 \otimes \cdots \otimes H_n$ and certain determinantal operators associated with V_{ij} acting in H. The concrete construction for our presentation is developed in Chapter 2. We limit our interest to so-called nonsingular multiparameter systems. Then we associate with a multiparameter system an n-tuple of commuting operators, called the associated n-tuple. Now, the completeness problem is to find a complete system of eigenvectors and root vectors for the associated system in terms of the underlying multiparameter system. Again this can be considered as a generalization of the completeness problem for one operator.

For example, it is well-known that, for an $N \times N$ matrix V, we can find a (Jordan) basis of $\mathbb{C}^N$ consisting of (Jordan) chains of vectors $z_0, z_1, \ldots, z_k$ such that

$$\begin{aligned} V z_0 &= \lambda z_0, \\ V z_i &= \lambda z_i + z_{i-1}, \quad i = 1, 2, \ldots, k. \end{aligned} \tag{0.2}$$

Then the vector z_0 is called an eigenvector, the vector z_1 is called a second root vector, the vector z_2 a third root vector, etc. Difficulties in proving multiparameter completeness results arise when the eigenvalues are not semisimple, i.e., when root vectors exist. Binding [23] gave the completeness result for real eigenvalues of self-adjoint multiparameter systems. Also Faierman in [69] gave a completeness result for real eigenvalues of the two-parameter spectral problem (0.1), while for the non-real eigenvalues he conjectured the structure of the general root functions. We return to his conjecture at the end of this dissertation.

Not much is known in the literature about nonself-adjoint multiparameter eigenvalue problems or even about non-real eigenvalues of self-adjoint multiparameter eigenvalue problems. Thus it seems natural first to consider the finite-dimensional setting. Atkinson dedicated most of his book [10] to the finite-dimensional setting, at the end generalizing it to compact operators on general Hilbert spaces using a limiting procedure. Multiparameter eigenvalue problems on Hilbert spaces can be approximated by finite-dimensional multiparameter eigenvalue problems using the finite difference method. Then results on Hilbert space can be proved using, as in Atkinson's case, a limiting procedure (see for example [66]). The germs of such finite-dimensional approximation ideas are found already in Carmichael's paper [50]. Another possible application of finite-dimensional results to the infinite-dimensional case is in connection with the discretization described by Müller [134, 135]. There are other problems in finite-dimensional Multiparameter Spectral Theory that are considered in the literature. For example, Browne and Sleeman considered in a series of papers [43, 44, 45] inverse multiparameter eigenvalue problem for matrices and Binding and Browne [26, 24] studied multiparameter eigenvalues for matrices, to mention a few. Finally, we remark that Isaev [112] stated the problem of describing root vectors of the associated system in terms of the underlying multiparameter system in the finite-dimensional setting.

In this dissertation we assume that Hilbert spaces H_i are finite-dimensional. Then V_{ij} can be considered as matrices. In the presentation we mostly use tools of Linear Algebra. There are two main foci of study in this dissertation. These are the structure of commuting matrices and the structure of root vectors. Even though both structures were developed simultaneously, each helping to reveal the other, it turned out that the understanding of the first one enabled us to construct root vectors, and eventually to prove completeness results.

In Chapter 1 we study n-tuples of nilpotent commuting matrices. As mentioned before, the completeness results are to be proven for the associated system, which is a special n-tuple of commuting matrices. This is our motivation to study commuting matrices. Without loss we can assume that they are all nilpotent. Then we can bring them to a special upper block triangular form (1.2). An important property is that it is reduced, i.e., certain columns in it are linearly independent. This linear independence ultimately enables us to prove the completeness result for simple eigenvalues. The commutativity of an n-tuple of nilpotent commuting matrices $\{A_1, A_2, \ldots, A_n\}$ is reflected in the symmetry of certain products. We explore these in further detail. For the simple case, i.e., when

$$\dim\left(\bigcap_{i=1}^{n} \ker A_i\right) = 1,$$

we are able to reconstruct commuting matrices in the form (1.2) from a special collection of symmetric matrices. Because (1.2) is reduced it follows that certain submatrices of these symmetric matrices are linearly independent. Later we prove that the isomorphic images of the submatrices are elements of the kernels of special matrices associated with the multiparameter system. Because we are also able to construct a set of linearly independent root vectors associated with a basis of the kernel of the special matrices, the completeness of root vectors follows.

It is the structure of these root vectors that is our second focus in this dissertation. The structure of root vectors for nonderogatory eigenvalues is the same as the structure of root vectors given in Binding's paper [23]. For simple eigenvalues the structure becomes more involved. The coefficients, that are all 1 in the nonderogatory case, of the decomposable tensors forming a root vector are now given by symmetric tensors that are associated with the special collection of symmetric matrices used to reconstruct a nilpotent n-tuple of commuting matrices. It turns out that in the two-parameter case this is a finite-dimensional simplified version of the structure conjectured by Faierman in [69]. A crucial tool in the study of the structure of root vectors is relation (2.7) that relates a multiparameter system with its associated system. Relation (2.7) is found in [10, Chapter 6].

We present the structure of the general second root vectors in Section 2.5.

In the two-parameter case these vectors are simpler and we can choose them so that the associated system is in a canonical form. We show this in Chapter 3. In Chapter 4 we prove a completeness result for eigenvalues $\boldsymbol{\lambda}_0 = (\lambda_{01}, \lambda_{02}, \ldots, \lambda_{0n})$ such that

$$\dim \ker W_i(\boldsymbol{\lambda}_0) = 1, \ i = 1, 2, \ldots, n.$$

These eigenvalues are of two types, nonderogatory and simple. (See page 78 for precise definitions.) We consider them separately.

We also study problems of representations of n-tuples of commuting matrices by tensor products (originally stated by Davis [57]) and by multiparameter systems, in Sections 1.6 and 2.4, respectively.

Let us mention that the results on commuting matrices of Chapter 1 are a major building block in the completeness results on root vectors for nonderogatory and simple eigenvalues in Chapter 4. It was the ability to obtain new completeness results that motivated us to work through some highly technical proofs. We include several examples[1] to illustrate the ongoing discussion at various times, especially after the technically involved proofs. At present we are not able to find a more elegant way to prove our results, though it appears almost certain that the application of the tools of Abstract Algebra should shed new light on them, and that the proofs might then become shorter and more elegant. Perhaps one should carry out the project of Atkinson motivated in [9].

[1]Examples in this dissertation which require longer calculations were done using the Mathematica software. In very long examples we do not include all the steps done by computer in the discussion.

Chapter 1

Commuting Matrices

1.1 Introduction

In this chapter we study n-tuples of commuting matrices. To each multiparameter system there is a special n-tuple of commuting matrices called the associated system. (A formal definition is given on page 62. Here we refer to this n-tuple as the associated n-tuple of commuting matrices.) This is our motivation to study the general case of commuting matrices. Our aim is to describe an n-tuple of commuting matrices by a special collection of matrices that reflect the commutativity in their structure.

The main results of this chapter are Theorems 1.13 and 1.18. Theorem 1.13 is the first step towards the construction of a special collection of matrices associated with an n-tuple of commuting matrices. Corollary 1.7 and Theorem 1.18 are used later in the construction of bases for root subspaces for nonderogatory and simple eigenvalues, respectively, of a multiparameter system.

A finite set of commutative matrices is considered as a cubic array. We restrict our interest to nilpotent commutative matrices. The general commutative case is easily deduced from the nilpotent one. In the next section we introduce some notation and define a basis in which the commutative matrices are simultaneously reduced to a special upper triangular form and so the corresponding cubic array is in a special upper triangular reduced form (1.2). Properties of the form (1.2) described in Proposition 1.2 and Corollary 1.3 are the main results of the section and as it turns out they are fundamental for most of the further presentation. They tell us that certain sets of columns in the reduced form (1.2) are linearly independent and that commutativity of the matrices is equivalent to certain symmetries in the products of these matrices. They also give rise to two sets of conditions that must hold for a special collection of matrices used to reconstruct (or build) a commutative array in the form (1.2). The two sets of conditions are the regularity conditions that are equivalent to the properties of Proposition 1.2 and the matching conditions that are equivalent to properties described in Corollary 1.3.

In Section 1.3 we study nonderogatory eigenvalues. It is well known that commuting nilpotent matrices can be brought simultaneously to upper Toeplitz form if one of them is nonderogatory (cf. [92, p.296] or [129, p.130]). This leads us to the definition of a nonderogatory eigenvalue for an n-tuple of commuting matrices.

Auxiliary results concerning matrices whose products are symmetric are presented in Section 1.4. They are needed in the proofs of the main two results of this chapter. We use a special collection of matrices to reconstruct the array in the form (1.2) inductively from the top left corner adding a row and a column at each step. The first important result in this direction is Theorem 1.13. It tells us how to reconstruct the array in the form (1.2) when there are only 3 columns. It turns out that the general case can be considered as a collection of cases with 3 columns which have to satisfy further regularity and matching conditions. It follows from Theorem 1.13, applied to the general case, that the entries on any block-diagonal of an array in the form (1.2) lie in the linear span of the entries of the first block row. (See Proposition 1.15.) Thus in the simple case all the entries are in the linear span of the first row. Furthermore, we can assume that all the nonzero entries of the first row are linearly independent. We refer here to the matrix which has these nonzero entries for its columns as the 'condensed first row'. The product of any row and any column of a commutative array in the form (1.2) is a symmetric matrix. In the simple case the product of the first row and any column is equal to the product of the condensed first row and the corresponding subcolumn. One result of Section 1.4 tells us that then this subcolumn is a product of the condensed first row and a unique symmetric matrix. Our goal is to expand this symmetric matrix to describe the complete column but to retain the symmetry and matching conditions. To prove the existence of the expanded matrix turns out to be technically very complex. Because of the length of this proof we include it in Appendix A. Theorem 1.18 and the preceding discussion tell us how to reconstruct the array in the form (1.2) in the simple case. This result is important in the construction of root vectors for the associated n-tuple of commuting matrices in the case of simple eigenvalues.

Section 1.6 of this chapter is not related to the preceding discussion. Rather it investigates the relation between an arbitrary and an associated n-tuple of commuting matrices.

1.2 Notation and Basic Properties of Commutative Arrays

Assuming that H is a Hilbert space we write $\mathcal{L}(H)$ for the algebra of all linear transformations $T : H \longrightarrow H$ and $\mathcal{R}(T)$ for the range of such a transformation T. A finite dimensional Hilbert space H_i, $i \in \underline{n}$ is equipped with a scalar product $y_i^* x_i$ for $x_i, y_i \in H_i$. The symbol $\underline{n}$ is used to denote the set of the first n positive integers, so $\underline{n} = \{1, 2, \ldots, n\}$. The tensor product space $H = H_1 \otimes H_2 \otimes \cdots \otimes H_n$ is then a Hilbert space under the scalar product defined by $(x, y) = \prod_{i=1}^n y_i^* x_i$ for decomposable tensors $x = x_1 \otimes x_2 \otimes \cdots \otimes x_n$ and $y = y_1 \otimes y_2 \otimes \cdots \otimes y_n$ and extended to all of H by linearity. For a linear transformation $V_i \in \mathcal{L}(H_i)$ we define the induced linear transformation $V_i^\dagger$ on the tensor product space H as follows : if $x_1 \otimes x_2 \otimes \ldots \otimes x_n \in H$ is a decomposable tensor then

$$V_i^\dagger (x_1 \otimes x_2 \otimes \ldots \otimes x_n) = x_1 \otimes x_2 \otimes \ldots \otimes V_i x_i \otimes \ldots \otimes x_n. \tag{1.1}$$

The action on all of H is then determined by linearity.

Let $\mathbf{A} = \{A_s;\ s \in \underline{n}\}$ be a set of n commuting matrices. Each matrix A_s is a $N \times N$ complex matrix. We also consider $\mathbf{A}$ as a cubic array of numbers of dimensions $N \times N \times n$. Such an array is called *commutative* (since A_s pairwise commute). Two arrays (or two sets of commuting matrices) $\mathbf{A}$ and $\mathbf{A}'$ are called *similar* if there is an $N \times N$ invertible matrix U such that $A_s = U^{-1}A'_sU$ for all s. For this collection of equations we also use the notation $\mathbf{A} = U^{-1}\mathbf{A}'U$.

The vector in $\mathbb{C}^n$ consisting of all the (i,j)-th entries of matrices in $\mathbf{A}$ is labelled

$$\mathbf{a}_{ij} = \begin{bmatrix} (A_1)_{ij} \\ (A_2)_{ij} \\ \vdots \\ (A_n)_{ij} \end{bmatrix}.$$

Then the *row* and *column cross-sections of* $\mathbf{A}$ are defined by

$$R_i = [\,\mathbf{a}_{i1} \quad \mathbf{a}_{i2} \quad \ldots \quad \mathbf{a}_{iN}\,]$$

and

$$C_j = \left[\ \mathbf{a}_{1j} \quad \mathbf{a}_{2j} \quad \cdots \quad \mathbf{a}_{Nj}\ \right],$$

where $i,j \in \underline{N}$. These are $n \times N$ complex matrices.

Definition. A complex $N \times N$ matrix is called *symmetric* if $A = A^T$, i.e. if it is equal to its transpose (without conjugation).

In this dissertation we reserve word *'symmetric'* for above definition. A matrix A such that $A = A^*$ will be called *'self-adjoint'*.

Lemma 1.1 *The array* $\mathbf{A}$ *is commutative if and only if the products* $R_iC_j^T$ *are symmetric for all* $i,j \in \underline{N}$.

Proof. The (i,j)-th entry of the product A_rA_s $(r,s \in \underline{n})$ is

$$(A_rA_s)_{ij} = \sum_{k=1}^{N} (A_r)_{ik}\,(A_s)_{kj} = \sum_{k=1}^{N} (R_i)_{rk}\,(C_j)_{sk} = \left(R_iC_j^T\right)_{rs}$$

Thus $A_rA_s = A_sA_r$ if and only if $\left(R_iC_j^T\right)_{rs} = \left(R_iC_j^T\right)_{sr}$, that is, if and only if $R_iC_j^T$ are symmetric. □

Our first and the main concern in this chapter is the spectral structure of a commutative array. For later reference we introduce our definitions of spectrum and related notions for a commutative array.

Definition. An n-tuple $\boldsymbol{\lambda}\in \mathbb{C}^n$ is *an eigenvalue of a commutative array* $\mathbf{A}$ if the intersection of kernels $\bigcap_{i=1}^n \ker(\lambda_i I - A_i)$ is nontrivial. The set of all the eigenvalues of $\mathbf{A}$ is called *the spectrum of* $\mathbf{A}$ and is labeled $\sigma(\mathbf{A})$.

For $i \in \underline{N}$ we write

$$\ker(\boldsymbol{\lambda}\mathbf{I} - \mathbf{A})^i = \bigcap_{\sum_{j=1}^n k_j = i, k_j \geq 0} \ker\left((\lambda_1 I - A_1)^{k_1}(\lambda_2 I - A_2)^{k_2}\cdots(\lambda_n I - A_n)^{k_n}\right).$$

Note that $\ker(\boldsymbol{\lambda}\mathbf{I} - \mathbf{A})^N = \bigcap_{i=1}^n \ker(\lambda_i I - A_i)^N$.

Definition. Suppose that $\boldsymbol{\lambda} \in \sigma(\mathbf{A})$. Then the subspace $\ker(\boldsymbol{\lambda}\mathbf{I} - \mathbf{A})$ is called an *eigenspace* (*of* $\mathbf{A}$ *at* $\boldsymbol{\lambda}$) and the subspace $\ker(\boldsymbol{\lambda}\mathbf{I} - \mathbf{A})^N$ is called a *root subspace* (*of* $\mathbf{A}$ *at* $\boldsymbol{\lambda}$). We call a nonzero element $x \in \bigcap_{i=1}^n \ker(\lambda_i I - A_i)$ an *eigenvector* and we call a nonzero element $x \in \bigcap_{i=1}^n \ker(\lambda_i I - A_i)^N = \ker(\boldsymbol{\lambda}\mathbf{I} - \mathbf{A})^N$ a *root vector.*

Note that according to the definition an eigenvector is also a root vector.

It is well known (see e.g. [92, p.298]) that commuting linear transformations A_s on $\mathbb{C}^n$ reduce the space into the direct sum of root subspaces of $\mathbf{A}$ (obviously a root subspace is invariant for all A_s). Replacing A_s by $\lambda_s I - A_s$, restricted to a root subspace of $\mathbf{A}$ at $\underline{\lambda}$, yield that all A_s have only one eigenvalue 0. Therefore we will assume in this and following three sections that the commuting matrices $\mathbf{A}$ have only one eigenvalue $\mathbf{0}$, or equivalently that they are all nilpotent.

Let M be the minimal number such that $A_1^{k_1}A_2^{k_2}\cdots A_n^{k_n} = 0$ for all collections of $k_j \geq 0$ such that $\sum_{j=1}^n k_j = M + 1$. There always exists a basis such that all the matrices A_s are upper triangular in this basis (cf. [92, Theorem 9.2.2, p. 303]). Because A_s are nilpotent they are strictly upper-triangular (i.e. the diagonal entries are also 0). Since the product of N upper triangular $N \times N$ matrices with zero diagonal is 0, it follows that $M < N$. (This idea can be found in the proof of Theorem 2 in [137] due to H. W. Lenstra Jr.) For $i \in \underline{M+1}$ we write $D_i = \dim \ker \mathbf{A}^i$ and $d_i = D_{i+1} - D_i$ for $i = 0, 1, \ldots, M$. It is assumed that $D_0 = 0$. We can choose a basis

$$\mathcal{B} = \left\{ z_0^1, z_0^2, \ldots, z_0^{d_0};\ z_1^1, z_1^2, \ldots, z_1^{d_1};\ \ \ldots\ \ ;\ z_M^1, z_M^2, \ldots, z_M^{d_M} \right\}$$

for $\mathbb{C}^N$ such that for every $i = 0, 1, \ldots, M$ the set

$$\mathcal{B}_i = \left\{ z_0^1, z_0^2, \ldots, z_0^{d_0};\ z_1^1, z_1^2, \ldots, z_1^{d_1};\ \ \ldots\ \ ;\ z_i^1, z_i^2, \ldots, z_i^{d_i} \right\}$$

is a basis for $\ker \mathbf{A}^{i+1}$.

Definition. The change of a basis $\mathcal{B}$ (corresponding to a commutative array $\mathbf{A}$ in the above described way) to a basis $\mathcal{B}'$ is called *admissible* if $\operatorname{span} \mathcal{B}_i = \operatorname{span} \mathcal{B}'_i$ for all i.

If we now consider $\mathbf{A}$ as a cubic array with slices consisting of matrices A_s, $s \in \underline{n}$, then $\mathbf{A}$ has the following representation on $\ker \mathbf{A}^{M+1} = \mathbb{C}^N$ in the basis $\mathcal{B}$:

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{A}^{01} & \mathbf{A}^{02} & \cdots & \mathbf{A}^{0,M} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}^{12} & \cdots & \mathbf{A}^{1,M} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}^{M-1,M} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \tag{1.2}$$

where

$$\mathbf{A}^{kl} = \begin{bmatrix} \mathbf{a}_{11}^{kl} & \mathbf{a}_{12}^{kl} & \cdots & \mathbf{a}_{1,d_l}^{kl} \\ \mathbf{a}_{21}^{kl} & \mathbf{a}_{22}^{kl} & \cdots & \mathbf{a}_{2,d_l}^{kl} \\ \vdots & \vdots & & \vdots \\ \mathbf{a}_{d_k,1}^{kl} & \mathbf{a}_{d_k,2}^{kl} & \cdots & \mathbf{a}_{d_k,d_l}^{kl} \end{bmatrix} \tag{1.3}$$

is a cubic array of dimensions $d_k \times d_l \times n$ and $\mathbf{a}_{ij}^{kl} \in \mathbb{C}^n$. The array (1.2) is block upper triangular with zero diagonal since $A_s (\ker \mathbf{A}^i) \subset \ker \mathbf{A}^{i-1}$ for all s. The last relation follows from the definition of $\ker \mathbf{A}^i$. If we expand the vector $A_s z_l^j$ in the basis $\mathcal{B}$ then

a^{kl}_{ijs} is the coefficient of z^i_k in this expansion, where

$$\mathrm{a}^{kl}_{ij} = \begin{bmatrix} a^{kl}_{ij1} \\ a^{kl}_{ij2} \\ \vdots \\ a^{kl}_{ijn} \end{bmatrix}. \tag{1.4}$$

The row and column cross-sections of $\mathbf{A}^{kl}$ are

$$R^{kl}_i = [\,\mathrm{a}^{kl}_{i1} \quad \mathrm{a}^{kl}_{i2} \quad \cdots \quad \mathrm{a}^{kl}_{i,d_l}\,],\ i \in \underline{d_k} \tag{1.5}$$

and

$$C^{kl}_j = [\,\mathrm{a}^{kl}_{1j} \quad \mathrm{a}^{kl}_{2j} \quad \cdots \quad \mathrm{a}^{kl}_{d_k,j}\,],\ j \in \underline{d_l}. \tag{1.6}$$

These are matrices of dimensions $n \times d_l$ and $n \times d_k$, respectively.

Definition. The array $\mathbf{A}$ in the form (1.2) is called *reduced* if the matrices $C^{k,k+1}_j$, $j \in \underline{d_{k+1}}$ are linearly independent for $k = 0, 1, \ldots, M-1$.

In the above setting we have

Proposition 1.2 *For a basis $\mathcal{B}$ as above the matrices $C^{k,k+1}_j$, $j \in \underline{d_{k+1}}$ are linearly independent for $k = 0, 1, \ldots, M-1$, or equivalently, the array $\mathbf{A}$ corresponding to a basis $\mathcal{B}$ is reduced.*

Proof. Let us assume the contrary to obtain a contradiction. If the matrices $C^{k,k+1}_j$ are linearly dependent, i.e. $\sum_{j=1}^{d_{k+1}} \alpha_j C^{k,k+1}_j = 0$ and not all α_j equal 0, then there exists a vector $x \in \ker \mathbf{A}^{k+1} \setminus \ker \mathbf{A}^k$, i.e. $x = \sum_{j=1}^{d_{k+1}} \alpha_j z^j_{k+1}$, such that $A_s x \in \ker \mathbf{A}^{k-1}$ for all s. But this yields $x \in \ker \mathbf{A}^k$ which contradicts $x \notin \ker \mathbf{A}^k$. $\square$

The above result will be crucial in the ultimate step of the proof of the completeness result for simple eigenvalues of a multiparameter system. Next we will restate Lemma 1.1 for the case when $\mathbf{A}$ is in the form (1.2).

Corollary 1.3 *An array* $\mathbf{A}$ *in the form* (1.2) *is commutative if and only if the matrices*

$$\sum_{h=k+1}^{l-1} R_i^{kh} \left(C_j^{hl}\right)^T,$$

$k = 0, 1, \ldots, M-2;\ l = k+2, k+3, \ldots, M;\ i \in \underline{d_k};\ j \in \underline{d_l}$, *are symmetric.*

Note that there is no condition on $\mathbf{A}^{0M}$. So an array $\mathbf{A}$ in the form (1.2) for $M = 1$ is always commutative.

In the examples we write a commutative array $\mathbf{A}$ as a two-dimensional array of column vectors.

Example 1.4 We consider a pair of commuting matrices

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then we find that $d_0 = 1,\ d_1 = d_2 = 2$ and $d_3 = 1$. Suppose that $\{e_i,\ i \in \underline{6}\}$ is the standard basis of $\mathbb{C}^6$. Then in the basis $\mathcal{B} = \{e_1;\ e_2, e_4;\ e_3, e_5;\ e_6\}$ the commutative array $\mathbf{A} = \{A_1, A_2\}$ is in the form (1.2), i.e.,

$$\mathbf{A} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 2 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 2 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix}. \tag{1.7}$$

□

1.3 Upper Toeplitz Form

The main results of this section are Corollaries 1.6 and 1.7. The preceding discussion is of its own interest and is necessary to prove the corollaries. These concern the nonderogatory eigenvalues that are the easiest special case of eigenvalues we discuss later.

Definition. An eigenvalue $\boldsymbol{\lambda} \in \sigma(\mathbf{A})$ is called *nonderogatory* if there exists an integer $k \geq 1$ such that

$$\dim\left(\bigcap_{i=1}^{n} \ker\left(\lambda_i I - A_i\right)^l\right) = l \qquad \text{for } l = 1, 2, \ldots, k$$

and

$$\dim\left(\bigcap_{i=1}^{n} \ker\left(\lambda_i I - A_i\right)^l\right) = k \qquad \text{for } l = k+1, k+2, \ldots, N.$$

Let us remark again that we assume matrices A_i are nilpotent, i.e., $\sigma(\mathbf{A}) = \{\mathbf{0}\}$.

Definition. Assume that $d_0 = d_1 = \cdots = d_M = 1$. Then $\mathbf{A}$ is in *upper Toeplitz form* if $\mathbf{A}^{kl} = \mathbf{A}^{k-1,l-1}$ for $k = 1, 2, \ldots, M; l > k$, and the other $\mathbf{A}^{kl}$ are $\mathbf{0}$.

Theorem 1.5 *Assume that $d_l = 1$ for some $l \geq 1$. Then $d_l = d_{l+1} = \cdots = d_M = 1$. By an admissible change of basis we can assume that*

$$\mathbf{A}^{l-1,l} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{a}^{l-1,l}_{d_{l-1}1} \end{bmatrix}$$

and the bottom right $(M - l + 1) \times (M - l + 1)$ block of $\mathbf{A}$ can be written in upper Toeplitz form.

Proof. By Corollary 1.3 the matrices

$$S_{i1} = \mathbf{a}^{l-1,l}_{i1} \cdot \left(\mathbf{a}^{l,l+1}_{11}\right)^T, \qquad i \in \underline{d_{l-1}}$$

are symmetric and by Proposition 1.2 they are not all 0. Thus there are complex numbers ϵ_{i1} not all 0 such that $\mathbf{a}^{l-1,l}_{i1} = \epsilon_{i1}\mathbf{a}^{l,l+1}_{11}$. If we replace $z^{d_{l-1}}_{l-1}$ in the basis $\mathcal{B}$ by the vector $\sum_{i=1}^{d_{l-1}} \epsilon_{i1} z^i_{l-1}$ we obtain a new basis in which the array $\mathbf{a}^{l-1,l}$ is of the required form

$$\mathbf{A}^{l-1,l} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{b} \end{bmatrix}$$

where $\mathbf{b} \neq \mathbf{0}$.

Now suppose that not all $d_j = 1$ for $j \geq l + 1$. Then say that h $(\geq l + 1)$ is the smallest number such that $d_h > 1$. If $h \geq l + 2$ then for $k = l, l + 1, \ldots, h - 2$ the arrays $\mathbf{A}^{k,k+1}$ are nonzero and of dimensions $1 \times 1 \times n$, so they can be considered as n–vectors. Thus we identify $\mathbf{A}^{k,k+1}$ with $\mathbf{a}^{k,k+1}_{11}$ and denote it $\mathbf{a}^{k,k+1}$. By Corollary 1.3 the matrices

$$S_{l-1} = \mathbf{b} \cdot \left(\mathbf{a}^{l,l+1}\right)^T \text{ and } S_k = \mathbf{a}^{k,k+1} \cdot \left(\mathbf{a}^{k+1,k+2}\right)^T; \; k = l, l + 1, \ldots, h - 3$$

are symmetric and since the vectors $\mathbf{b}$ and $\mathbf{a}^{k,k+1}$ are nonzero the ranks of the matrices S_k are exactly 1. Therefore, there exist nonzero complex numbers ϵ_k such that $\mathbf{a}^{k,k+1} = \epsilon_k \mathbf{b}$ for $k = l, l+1, \ldots, h-3$. Further, if $h = l+1$(resp. $h \geq l+2$) the matrices $S^j_{l-1} = \mathbf{b} \cdot \left(\mathbf{a}^{l,l+1}_{1j}\right)^T$(resp. $S^j_{h-2} = \mathbf{a}^{h-2,h-1} \cdot \left(\mathbf{a}^{h-1,h}_{1j}\right)^T$) are symmetric and of rank exactly 1 for $j = 1, 2$. They are not zero since by Proposition 1.2 the vectors $\mathbf{a}^{h-1,h}_{1j}$are linearly independent. Hence there exist nonzero numbers ϵ^j_{h-1} such that $\mathbf{a}^{h-1,h}_{1j} = \epsilon^j_{h-1}\mathbf{b}$. The vector $\epsilon^2_{h-1} z^1_h - \epsilon^1_{h-1} z^2_h$ is then in the subspace $\ker \mathbf{A}^{h-1}$. This contradicts the fact that the vectors z^k_i with index $i \leq h-1$ form a basis for $\ker \mathbf{A}^{h-1}$ and the vectors z^k_i with index $i \leq h$ form a basis for $\ker \mathbf{A}^h$. Thus $d_l = d_{l+1} = \cdots = d_M = 1$.

Now we restrict the matrices A_s to the quotient $\mathcal{Q} = \mathbb{C}^N|_{\mathbb{C}^{l-1} \times \{0\}}$. To finish the proof it has to be shown that there is a basis for $\mathcal{Q}$ such that all the restricted matrices $A_s \mid_{\mathcal{Q}}$ are in upper Toeplitz form. In the first part of the proof we showed that for $k = l, l+1, \ldots, M-1$ all the $\mathbf{a}^{k,k+1}$ are nonzero multiples of $\mathbf{b}$. Therefore there is a number r between 1 and n such that $A_s \mid_{\mathcal{Q}}$ has a Jordan chain of length $m - l + 1$. Then by [92, p. 296] or [129, p. 130] we can find a basis in which all $A_s \mid_{\mathcal{Q}}$ (and thus the bottom right $(M-l+1) \times (M-l+1)$ block of $\mathbf{A}$) are in upper Toeplitz form. □

The following are special cases of Theorem 1.5 and give another view of the results for the nonderogatory case in [92, p. 296] and [129, p. 130].

Corollary 1.6 *Assume that $d_0 = d_1 = 1$. Then for $j = 0, 1, \ldots, M$ each $d_j = 1$ and $\mathbf{A}$ has upper Toeplitz representation.*

Corollary 1.7 *The eigenvalue 0 of $\mathbf{A}$ is nonderogatory if and only if $d_0 = 1$ and $d_1 \leq 1$.*

Note that when 0 is nonderogatory eigenvalue at least one of the A_s is similar to the $N \times N$ Jordan matrix

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and $\ker \mathbf{A}^i = \bigcap_{s=1}^n \ker A_s^i$.

1.4 Matrices Whose Product is Symmetric

Before describing the structure of $\mathbf{A}$ further we will prove some auxiliary results which are of interest in themselves.

Lemma 1.8 *Let R and C be $p \times q$ complex matrices where $p \geq q$ and assume that* $\operatorname{rank} R = q$. *Then RC^T is symmetric if and only if there is a symmetric matrix $T \in \mathbb{C}^{q\times q}$ such that $C = RT$. The matrix T is unique.*

Proof. Assume first that the product RC^T is symmetric. Let $Y \in \mathbb{C}^{q\times p}$ be a left inverse for R. Then $C^T = YCR^T$ or $C = R\left(C^TY^T\right)$. Denoting $T = (YC)^T$, we have $T^T = YC = YRT = T$, thus T is symmetric.

Conversely, let $C = RT$ and $T = T^T$. Then

$$RC^T = RT^TR^T = RTR^T = CR^T$$

and thus the product RC^T is symmetric.

It remains to show that T is unique. Suppose that $C = RT_1 = RT_2$. Then by left invertibility of R it follows that $T_1 = T_2$. □

The next result will generalize Lemma 1.8 to the case where a set of k matrices R_j; $j \in \underline{k}$, is such that all the products R_jC^T are symmetric. We assume that $kp \geq q$ and that

$$\operatorname{rank}\begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_k \end{bmatrix} = q. \tag{1.8}$$

Let us remark that the set of row cross-sections of any of the arrays $\mathbf{A}^{k-1,k}$ of the form (1.2) and a column cross-section of the array $\mathbf{A}^{k,k+1}$ fit into the setting of the previous paragraph.

Next define $r = \operatorname{rank}\,[\,R_1 \quad R_2 \quad \cdots \quad R_k\,]$ and let the columns of the matrix $\tilde{R} \in \mathbb{C}^{p\times r}$ form a basis for the space spanned by the columns of $[\,R_1 \quad R_2 \quad \cdots \quad R_k\,]$. Then for $j \in \underline{k}$ there is a matrix $S_j \in \mathbb{C}^{r\times q}$ such that $R_j = \tilde{R}S_j$. Moreover (1.8) implies

$$\operatorname{rank}\begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_k \end{bmatrix} = q. \tag{1.9}$$

For every vector x in the intersection of the kernels of S_j it follows that $R_jx = \tilde{R}S_jx = 0$ whence $x \in \cap_{j=1}^{k} \ker R_j = \{0\}$ and so $x = 0$. Property (1.9) implies that the matrix $\begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_k \end{bmatrix}$ has a left inverse $[\,Z_1 \quad Z_2 \quad \cdots \quad Z_k\,]$ where all Z_j are $q \times r$ matrices. Using this notation we have

Lemma 1.9 *Assume that C and R_j; $j \in \underline{k}$ are $p \times q$ matrices, that $kp \geq q$ and that* (1.8) *holds. Then the matrices R_jC^T are all symmetric if and only if there exist k symmetric matrices $T_j \in \mathbb{C}^{r\times r}$ such that*

$$C = \tilde{R}\left(\sum_{j=1}^{k} Z_jT_j\right)^T \qquad \textit{and} \qquad S_l\left(\sum_{j=1}^{k} Z_jT_j\right) = T_l; \quad l \in \underline{k}. \tag{1.10}$$

Proof. Let R_jC^T be all symmetric. Then $R_jC^T = CR_j^T$ implies $\tilde{R}(S_jC^T) = \left(CS_j^T\right)R^T$, so matrices $\tilde{R}$ and CS_j^T satisfy the conditions of Lemma 1.8. Then there are symmetric matrices $T_j \in \mathbb{C}^{r\times r}$ such that $CS_j^T = \widetilde{R}T_j$. From the proof of Lemma 1.8 we see that $T_j = S_jC^TY^T$ where $Y \in \mathbb{C}^{r\times p}$ is a left inverse of $\widetilde{R}$. The above equations can be put together as

$$\begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_k \end{bmatrix} C^T = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_k \end{bmatrix} \widetilde{R}^T$$

Multiplying on the left by $[\, Z_1 \quad Z_2 \quad \cdots \quad Z_k \,]$ we get

$$\left(\sum_{j=1}^{k} Z_jS_j\right) C^T = \left(\sum_{j=1}^{k} Z_jT_j\right) \widetilde{R}^T$$

and so

$$C = \widetilde{R}\left(\sum_{j=1}^{k} Z_jT_j\right)^T$$

Finally, a simple calculation gives the second part of (1.10), viz.

$$S_l\left(\sum_{j=1}^{k} Z_jT_j\right) = S_l\left(\sum_{j=1}^{k} Z_jS_j\right) C^TY^T = S_lC^TY^T = T_l$$

for all $l = 1, 2, \ldots, k$.

Let us now prove the converse. We have symmetric matrices T_j which satisfy (1.10). Then

$$CR_l^T = \widetilde{R}\left(\sum_{j=1}^{k} T_jZ_j^TS_l^T\right) \widetilde{R}^T = \widetilde{R}T_l\widetilde{R}^T$$

and

$$R_lC^T = \widetilde{R}S_l\left(\sum_{j=1}^{n} Z_jT_j\right) \widetilde{R}^T = \widetilde{R}T_l\widetilde{R}^T.$$

Hence the products R_lC^T are all symmetric. □

Suppose that $r_1 = d_1$, $r_i \le d_i$, for $i \ge 2$ and that R_i , $i \in \underline{m}$ are $p \times r_i$ matrices such that

$$\operatorname{rank} \widetilde{R}_i = \operatorname{rank} \widehat{R}_i = \hat{r}_i$$

where $\hat{r}_i = \sum_{j=1}^{i} r_j$,

$$\widehat{R}_i = \left[\; R_1 \quad R_2 \quad \cdots \quad R_i \;\right]$$

is an $p \times \hat{r}_i$ matrix,

$$\tilde{R}_i = \begin{bmatrix} R_1 & \begin{pmatrix} R_2 & 0 \end{pmatrix} & \begin{pmatrix} R_3 & 0 \end{pmatrix} & \cdots & \begin{pmatrix} R_i & 0 \end{pmatrix} \end{bmatrix}$$

is an $p \times \hat{d}_i$ matrix, $\hat{d}_i = \sum_{j=1}^{i} d_j$ and the blocks $\begin{pmatrix} R_i & 0 \end{pmatrix}$ are of sizes $p \times d_i$. Thus we suppose in particular that the columns of $\widehat{R}_m$ are linearly independent.

Let us remark that when a commutative array **A** is in the form (1.2) and $d_0 = 1$ then its first row cross-section and any column cross-section can be assumed to fit the setting of the previous paragraph.

In the above setting we have the following lemma :

Lemma 1.10 *Suppose that* $C \in \mathbb{C}^{p \times d_m}$ *is such that* $\tilde{R}_m C^T$ *is symmetric and that*

$$\mathcal{R}\left(\begin{bmatrix} C_m & C_{m-1} & \cdots & C_{m-i+1} \end{bmatrix}\right) \subset \mathcal{R}\left(\widehat{R}_i\right) \tag{1.11}$$

where $C = \begin{bmatrix} C_1 & C_2 & \cdots & C_m \end{bmatrix}$ *and* $C_i \in \mathbb{C}^{p \times d_i}$. *Then there exists a unique* $\hat{r}_m \times \hat{d}_m$ *matrix*

$$T = \begin{bmatrix} T^{11} & T^{12} & \dots & T^{1,m-1} & T^{1m} \\ T^{21} & T^{22} & \dots & T^{2,m-1} & 0 \\ T^{31} & T^{32} & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ T^{m1} & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{1.12}$$

where $T^{ij} = \begin{bmatrix} \widehat{T}^{ij} & \overline{T}^{ij} \end{bmatrix} \in \mathbb{C}^{r_i \times d_j}$ *and* $\widehat{T}^{ij} \in \mathbb{C}^{r_i \times r_j}$, *such that*

$$C = \widehat{R}_m T$$

and $\widehat{T}^{ij} = \left(\widehat{T}^{ji}\right)^T$ *for all* i *and* j.

Proof. Write $C_i = \begin{bmatrix} C_{i1} & C_{i2} \end{bmatrix}$, where $C_{i1} \in \mathbb{C}^{p \times r_i}$ and $C_{i2} \in \mathbb{C}^{p \times (d_i - r_i)}$ and $\widehat{C} = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{m1} \end{bmatrix}$. Then the matrix $\sum_{i=1}^{m} R_i C_i^T = \widehat{R}_m \widehat{C}^T = \tilde{R}_m C^T$ is symmetric, the matrix $\widehat{R}_m$ has full rank and hence we can apply Lemma 1.8 to obtain a unique symmetric matrix $\widehat{T}$ such that $\widehat{C} = \widehat{R}_m \widehat{T}$. We partition $\widehat{T}$ blockwise according to the partition of $\widehat{R}_m$, viz.

$$\widehat{T} = \begin{bmatrix} \widehat{T}_{11} & \widehat{T}_{12} & \cdots & \widehat{T}_{1m} \\ \widehat{T}_{21} & \widehat{T}_{22} & \cdots & \widehat{T}_{2m} \\ \vdots & \vdots & & \vdots \\ \widehat{T}_{m1} & \widehat{T}_{m2} & \cdots & \widehat{T}_{mm} \end{bmatrix}, \ \widehat{T}_{ij} \in \mathbb{C}^{r_i \times r_j}.$$

The relations (1.11) imply that the blocks below the main antidiagonal in $\widehat{T}$ are 0, i.e. $\widehat{T}_{ij} = 0$ if $i + j > m + 1$, and that there exist unique matrices $\overline{T}_{ij} \in \mathbb{C}^{r_i \times (d_j - r_j)}$, $i = 1, 2, \ldots, m$,$j = 1, 2, \ldots, m - i + 1$ such that

$$C_{i2} = \widehat{R}_{m-i+1} \begin{bmatrix} \overline{T}_{1i} \\ \overline{T}_{2i} \\ \vdots \\ \overline{T}_{m-i+1,i} \end{bmatrix}.$$

The latter holds because every column of the matrix C_{i2} is linear combination of columns of $\widehat{R}_{m-i+1}$ that are linearly independent. Then we write $T^{ij} = \begin{bmatrix} \widehat{T}_{ij} & \overline{T}_{ij} \end{bmatrix}$ if $i+j \leq m+1$ and $T^{ij} = 0$ if $i+j > m+1$. The matrix T is then of the form (1.12), and by the construction of $\widehat{T}_{ij}$ and $\overline{T}_{ij}$ it follows that $C = \widehat{R}_m T$ and the matrix T is unique because the matrices $\widehat{T}_{ij}$ and $\overline{T}_{ij}$ are unique. □

1.5 Structure of Commuting Matrices

This is the main section in this chapter. Theorem 1.13, proved in the first subsection below, is the first important step towards the construction of a special collection of matrices that is used to reconstruct a commutative array in form (1.2). In the second subsection we give this collection for simple eigenvalues and discuss its properties. In particular, a set of symmetric tensors can be associated with the special collection of matrices. These tensors appear as an essential tool in the construction of root vectors for simple eigenvalues of a multiparameter system.

1.5.1 General Case

Proposition 1.11 *Denote the dimension of the span of the set*

$$\left\{\mathrm{a}_{ij}^{l,l+1};\ i \in \underline{d_l};\ \ j \in \underline{d_{l+1}}\right\}$$

by r_l, for $l = 0, 1, \ldots, M-1$. Then

$$\frac{d_{l+1}}{d_l} \leq r_l \leq \min\{n, d_l d_{l+1}\}$$

for $l = 0, 1, \ldots, M-1$ and $r_l \geq r_{l+1}$ for $l = 0, 1, \ldots, M-2$.

Proof. The array $\mathbf{A}^{l,l+1}$ is constructed so that $r_l \leq \min\{n, d_l d_{l+1}\}$. Furthermore, the rank of the matrix $\begin{bmatrix} R_1^{l,l+1} \\ R_2^{l,l+1} \\ \vdots \\ R_{d_l}^{l,l+1} \end{bmatrix} \in \mathbb{C}^{nd_l \times d_{l+1}}$ is d_{l+1} (cf. Proposition 1.2). Since $r_{lj} = \operatorname{rank}\left(R_j^{l,l+1}\right) \leq r_l$ for $j \in \underline{d_l}$ and $\operatorname{rank}\left(\sum_{j=1}^{d} R_j\right) \leq \sum_{j=1}^{d} \operatorname{rank} R_j$ for any matrices R_j of the same sizes it follows that

$$d_{l+1} \leq \sum_{j=1}^{d_l} r_{lj} \leq d_l r_l.$$

By Corollary 1.3 the matrices $R_i^{l,l+1}\left(C_j^{l+1,l+2}\right)^T$ are symmetric for $l = 0, 1, \ldots, M-2$ and by Proposition 1.2 the matrix $\begin{bmatrix} R_1^{l,l+1} \\ R_2^{l,l+1} \\ \vdots \\ R_{d_l}^{l,l+1} \end{bmatrix}$ has full rank. So for every j the matrices $C_j^{l+1,l+2}$ and $R_i^{l,l+1}$, $i \in \underline{d_l}$, satisfy the conditions of Lemma 1.9. Then by (1.10) the rows of $C_j^{l+1,l+2}$ are in the span of the columns of $R_i^{l,l+1}$ and so $r_l \geq r_{l+1}$. $\square$

Let us now consider the case $M = 2$. Then

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{A}^{01} & \mathbf{A}^{02} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}^{12} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}. \tag{1.13}$$

Commutativity imposes conditions only on the arrays $\mathbf{A}^{01}$ and $\mathbf{A}^{12}$. So we are only interested in these two arrays.

First we will discuss the special case when the row cross-sections of $\mathbf{A}^{01}$ span a one-dimensional subspace in $\mathbb{C}^{n\times d_1}$. By an admissible change of basis $\mathcal{B}$ we can assume that

$$\mathbf{A}^{01} = \begin{bmatrix} \mathbf{a}^{01}_{11} & \mathbf{a}^{01}_{12} & \cdots & \mathbf{a}^{01}_{1d_1} \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \tag{1.14}$$

Then we have a simpler version of the main result :

Theorem 1.12 *Assume that* $\mathbf{A}$ *is commutative with* $M = 2$ *and that* $\mathbf{A}^{01}$ *has the form* (1.14)*. Then the array* $\mathbf{A}^{12}$ *is generated by a set of* d_2 *symmetric matrices of sizes* $d_1 \times d_1$*.*

Proof. By Corollary 1.3 the products $R_1^{01}\left(C_j^{12}\right)^T$ are symmetric and by Proposition 1.2 the matrix R_1^{01} has full rank. Thus by Lemma 1.8 there exist symmetric matrices T_j such that $C_j^{12} = R_1^{01}T_j$ for all j. □

The above special case is important in the study of the simple eigenvalues which are significant in applications to Multiparameter Spectral Theory.

Before we state the main result for the general case $M = 2$ let us introduce some further constructions. Proposition 1.11 makes the following definition sensible.

Definition. The set of integers $\mathcal{D} = \{d_0, d_1, d_2;\ r\}$, where all d_j and r are positive, is called an admissible set *if*

$$\sum_{j=0}^{2} d_j = N, \qquad r \le n \qquad \textit{and} \qquad \frac{d_{l+1}}{d_l} \le r \le d_l d_{l+1} \quad \text{for } l = 0, 1.$$

For the set of matrices $T_{ij} \in \mathbb{C}^{r\times s}$; $i \in \underline{d_0}$; $j \in \underline{d_1}$ we introduce the matrix

$$\mathbf{T}_j = \begin{bmatrix} T_{1j} \\ T_{2j} \\ \vdots \\ T_{d_0 j} \end{bmatrix}$$

and we denote by $\mathcal{S}$ the subspace in $\mathbb{C}^{d_1 r}$ spanned by the union of the ranges of $\mathbf{T}_j$ for all j. Similarly for a set of matrices $\left\{S_i \in \mathbb{C}^{r \times s};\ i \in \underline{d_0}\right\}$ we write

$$\mathbf{S} = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_{d_0} \end{bmatrix}$$

and

$$\mathbf{S}^{\mathbf{T}} = [\, S_1 \quad S_2 \quad \cdots \quad S_{d_0} \,].$$

Definition. For a given admissible set $\mathcal{D}$ the triple $(\tilde{R}, \mathcal{T}, P)$, where $\tilde{R}$ is a full rank $n \times r$ matrix, $\mathcal{T} = \left\{T_{ij};\ i \in \underline{d_0};\ j \in \underline{d_2}\right\}$ is a set of $r \times r$ symmetric matrices and P is a projection in $\mathbb{C}^{d_0 r \times d_0 r}$, is a *structure triple (*for $\mathcal{D}$*)* if it satisfies the conditions :

(i) $\mathbf{T}_j$, $j \in \underline{d_2}$ are linearly independent

(ii) the rank of P is d_1

(iii) $\mathcal{S}$ is a subspace of $\mathcal{R} = \mathcal{R}\,(P)$.

Theorem 1.13 *Given a structure triple we can describe (to within similarity) the arrays* $\mathbf{A}^{01}$ *and* $\mathbf{A}^{12}$ *of a commutative cubic array* $\mathbf{A}$ *with* $M = 2$. *(Commutativity does not depend on the choice of the array* $\mathbf{A}^{02}$*.)*

Conversely, for a given commutative array $\mathbf{A}$ *with* $M = 2$ *we can find a structure triple which generates the arrays* $\mathbf{A}^{01}$ *and* $\mathbf{A}^{12}$ *of* $\mathbf{A}$.

Proof. Suppose we are given a structure triple $(\tilde{R}, \mathcal{T}, P)$. Let $\ker P = \mathcal{K}$ and $\mathcal{R}\,(P) = \mathcal{R}$. The projection P can be written in the form

$$P = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_{d_0} \end{bmatrix} [\, Z_1 \quad Z_2 \quad \cdots \quad Z_{d_0} \,] \tag{1.15}$$

where $S_j, Z_k^T \in \mathbb{C}^{r\times d_2}$, $\sum_{j=1}^{d_1} Z_j S_j = I$ and $\mathcal{R}(\mathbf{S}) = \mathcal{R}$. The decomposition (1.15) can be obtained for example from the matrix $X = \begin{bmatrix} S_1 & S_{11} \\ S_2 & S_{21} \\ \vdots & \vdots \\ S_{d_0} & S_{d_0 1} \end{bmatrix}$, $S_{j1} \in \mathbb{C}^{r\times rd_0 - d_1}$ where the first d_1 columns form a basis for $\mathcal{R}$ and the rest form basis for $\mathcal{K}$. Then we choose $[\,Z_1 \quad Z_2 \quad \cdots \quad Z_{d_1}\,]$ to be the first d_1 rows of the inverse X^{-1}. Any other decomposition of P as in (1.15) is given by

$$P = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_{d_0} \end{bmatrix} UU^{-1}[\,Z_1 \quad Z_2 \quad \cdots \quad Z_{d_0}\,]$$

for some invertible matrix $U \in \mathbb{C}^{d_1\times d_1}$. Then an array $\mathbf{A}$ is generated as follows. The rows of $\mathbf{A}^{01}$ are given by

$$R_i = \tilde{R} S_i, \qquad i \in \underline{d_0}$$

and the columns of $\mathbf{A}^{12}$ are given by

$$C_j = \tilde{R} \sum_{i=1}^{d_0} T_{ij} Z_i^T, \qquad j \in \underline{d_2}. \tag{1.16}$$

First, the columns of $\mathbf{A}^{01}$ and $\mathbf{A}^{12}$ are linearly independent. The columns of $\mathbf{A}^{12}$ are linearly independent since $\mathbf{T}_j$ are linearly independent and the columns of $\mathbf{A}^{01}$ are linearly independent since the columns of $\mathbf{S}$ are linearly independent. In order to prove that $\mathbf{A}$ is commutative it remains to show by Corollary 1.3 and Lemma 1.9 that $S_l \left(\sum_{i=1}^{d_0} Z_i T_{ij} \right) = T_{lj}$ for all l and j. Since $\mathcal{S} \subset \mathcal{R}$ we have $P\mathbf{T}_j = \mathbf{T}_j$ or written by blocks $\sum_{i=1}^{d_0} S_l Z_i T_{ij} = T_{lj}$ for all l and j, which proves commutativity.

If we take another decomposition

$$P = \begin{bmatrix} S_1 U \\ S_2 U \\ \vdots \\ S_{d_0} U \end{bmatrix} [\,U^{-1}Z_1 \quad U^{-1}Z_2 \quad \cdots \quad U^{-1}Z_{d_0}\,]$$

we will get a similar array $\mathbf{A}_U$. The similarity transformation between $\mathbf{A}$ and $\mathbf{A}_U$ is given by

$$\begin{bmatrix} I & 0 & 0 \\ 0 & U & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Let us now explain how to obtain the structure triple from a commutative array $\mathbf{A}$. Since $\mathbf{A}$ is commutative the products $R_i^{01}\left(C_j^{12}\right)^T$ are symmetric for $i \in \underline{d_0}$; $j \in \underline{d_2}$ by Corollary 1.3. For every j the matrices R_i^{01}, $i \in d_0$ and C_j^{12} satisfy the conditions of Lemma 1.9. So there exist matrices $\widetilde{R}, T_{ij}, S_l$ and Z_l as in Lemma 1.9. We can choose the matrices $\widetilde{R}$, S_l and Z_l to be the same for all j since they depend only on R_i^{01}. Then the triple $(\widetilde{R}, \mathcal{T}, P)$ is a structure triple where $\mathcal{T} = \left\{T_{ij};\ i \in \underline{d_0};\ j \in \underline{d_2}\right\}$ and $P = \mathbf{S}\mathbf{Z}^T$. We need to check conditions (i)-(iii). Condition (i) holds since C_j are linearly independent. By the construction of $\mathbf{S}$ and $\mathbf{Z}$ the rank of P is equal to rank $\mathbf{S} = d_1$ and by the right-hand equations in (1.10) the span of the ranges of $\mathbf{T}_j$ is a subspace of Im P. □

To illustrate the preceding discussion we consider an example.

Example 1.14 Let

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then the (nilpotent) matrices that commute with A_1 have the form

$$A_2 = \begin{bmatrix} 0 & a_{11} & a_{12} & a_{21} & a_{22} \\ 0 & 0 & a_{11} & 0 & a_{21} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & a_{31} & a_{32} & 0 & a_{41} \\ 0 & 0 & a_{31} & 0 & 0 \end{bmatrix}.$$

where all a_{ij} are arbitrary. In order to construct the array $\mathbf{A}$ in the form (1.2) we need to look at different cases depending if some of a_{ij} are 0. There are two choices for

M, 2 and 4. In the case $M = 2$ there are two choices for admissible sets : $\{1,2,2;\ 2\}$ and $\{2,2,1;\ 2\}$. If $M = 4$ then $d_i = 1$ for all i. We here present the cubic array $\mathbf{A}$ as a two-dimensional array of column vectors.

(i) Let all a_{ij} in A_2 be nonzero. Then $M = 4$ and $d_0 = d_1 = d_2 = d_3 = d_4 = 1$. In the basis $\mathcal{B} = \{e_1, e_4, e_2, e_5, e_3\}$ the array $\mathbf{A}$ is

$$\mathbf{A} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ a_{21} \end{pmatrix} & \begin{pmatrix} 1 \\ a_{11} \end{pmatrix} & \begin{pmatrix} 0 \\ a_{22} \end{pmatrix} & \begin{pmatrix} 0 \\ a_{12} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ a_{31} \end{pmatrix} & \begin{pmatrix} 1 \\ a_{41} \end{pmatrix} & \begin{pmatrix} 0 \\ a_{32} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ a_{21} \end{pmatrix} & \begin{pmatrix} 1 \\ a_{11} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ a_{31} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix}$$

and in the basis $\mathcal{B}' = \{e_1, e_4, \alpha e_2, \alpha e_5 + \beta e_2, \alpha^2 e_3 + \beta e_5 + \gamma e_2\}$ where $\alpha = \frac{a_{21}}{a_{31}}$, $\beta = \frac{a_{21}}{a_{31}^2}(a_{11} - a_{41})$, $\gamma = \frac{a_{21}}{a_{31}^3}\left((a_{11} - a_{41})^2 + a_{22}a_{31} - a_{21}a_{32}\right)$ the array $\mathbf{A}$ is in the upper Toeplitz form

$$\mathbf{A} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ a_{21} \end{pmatrix} & \begin{pmatrix} \alpha \\ \delta \end{pmatrix} & \begin{pmatrix} \beta \\ \eta \end{pmatrix} & \begin{pmatrix} \gamma \\ \phi \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ a_{21} \end{pmatrix} & \begin{pmatrix} \alpha \\ \delta \end{pmatrix} & \begin{pmatrix} \beta \\ \eta \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ a_{21} \end{pmatrix} & \begin{pmatrix} \alpha \\ \delta \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ a_{21} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix}$$

where

$$\delta = \frac{a_{11}a_{21}}{a_{31}}, \quad \eta = \frac{a_{21}}{a_{31}^2}\left(a_{11}^2 + a_{22}a_{31} - a_{11}a_{41}\right)$$

and

$$\phi = \frac{a_{21}}{a_{31}^2}\left(a_{21}a_{12} + a_{11}a_{22} - a_{22}a_{41} + \frac{a_{11}}{a_{31}}\left((a_{11} - a_{41})^2 + a_{22}a_{31} - a_{21}a_{32}\right)\right).$$

(*ii*) Suppose now that $a_{21} = 0$ and the other a_{ij} are nonzero. Then $M = 2$ and $d_0 = 2$, $d_1 = 2$ and $d_2 = 1$. In the basis $\mathcal{B} = \{e_1, e_4;\ e_2, e_5;\ e_3\}$ we have

$$\mathbf{A} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ a_{11} \end{pmatrix} & \begin{pmatrix} 0 \\ a_{22} \end{pmatrix} & \begin{pmatrix} 0 \\ a_{12} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ a_{31} \end{pmatrix} & \begin{pmatrix} 1 \\ a_{41} \end{pmatrix} & \begin{pmatrix} 0 \\ a_{32} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ a_{11} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ a_{31} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix}.$$

We can choose the structure triple of $\mathbf{A}$ to be

$$\widetilde{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix};\ T_{11} = \begin{bmatrix} 1 & a_{11} \\ a_{11} & a_{11}^2 + a_{22}a_{31} \end{bmatrix},\ T_{21} = \begin{bmatrix} 0 & a_{31} \\ a_{31} & a_{31}(a_{11} + a_{41}) \end{bmatrix}$$

and

$$P = \mathbf{S}\mathbf{Z}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 \\ a_{11} & a_{22} \\ 0 & 1 \\ a_{31} & a_{41} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{a_{11}}{a_{22}} & \frac{1}{a_{22}} & 0 & 0 \end{bmatrix}.$$

The array $\mathbf{A}^{02}$ is

$$\mathbf{A}^{02} = \begin{bmatrix} \begin{pmatrix} 0 \\ a_{12} \end{pmatrix} \\ \begin{pmatrix} 0 \\ a_{32} \end{pmatrix} \end{bmatrix}.$$

(*iii*) The last case we will consider is $a_{31} = 0$ while the other $a_{ij} \neq 0$. Then $M = 2$ and $d_0 = 1$, $d_1 = 2$ and $d_2 = 2$. In the basis $\mathcal{B} = \{e_1;\ e_2, e_4;\ e_3, e_5\}$ we find

$$\mathbf{A} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ a_{11} \end{pmatrix} & \begin{pmatrix} 0 \\ a_{21} \end{pmatrix} & \begin{pmatrix} 0 \\ a_{12} \end{pmatrix} & \begin{pmatrix} 0 \\ a_{22} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ a_{11} \end{pmatrix} & \begin{pmatrix} 0 \\ a_{21} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ a_{32} \end{pmatrix} & \begin{pmatrix} 1 \\ a_{41} \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix}.$$

One possible choice for the structure triple is

$$\tilde{R} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \; T_{11} = \begin{bmatrix} 1 & a_{11} \\ a_{11} & a_{11}^2 + a_{21}a_{32} \end{bmatrix},$$

$$T_{12} = \begin{bmatrix} 0 & a_{21} \\ a_{21} & a_{21}(a_{11} - a_{41}) \end{bmatrix}, \; P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The decomposition (1.15) for P is

$$P = \begin{bmatrix} 1 & 0 \\ a_{11} & a_{21} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{a_{11}}{a_{21}} & \frac{1}{a_{21}} \end{bmatrix}$$

and the array $\mathbf{A}^{02}$ is

$$\mathbf{A}^{02} = \left[\begin{pmatrix} 0 \\ a_{12} \end{pmatrix} \; \begin{pmatrix} 0 \\ a_{22} \end{pmatrix} \right].$$

□

Theorem 1.13 tells us that in the case $M = 2$ the array (1.2) is commutative if the arrays $\mathbf{A}^{01}$ and $\mathbf{A}^{12}$ are given through a structure triple. So we ensure that the row-column products of Corollary 1.3 are symmetric. If $M \geq 3$ we can consider the array (1.2) as a collection of $\frac{M(M-1)}{2}$ cases with $M = 2$. Namely, for every pair of integers (k, l); $0 \leq k \leq l-2 \leq M-2$ we have the problem

$$\begin{bmatrix} \mathbf{0} & \begin{pmatrix} \mathbf{A}^{k,k+1} & \mathbf{A}^{k,k+2} & \cdots & \mathbf{A}^{k,l-1} \\ \mathbf{0} & \mathbf{A}^{k+1,k+2} & \cdots & \mathbf{A}^{k+1,l-1} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{A}^{l-2,l-1} \end{pmatrix} & * \\ \mathbf{0} & \mathbf{0} & \begin{pmatrix} \mathbf{A}^{k+1,l} \\ \mathbf{A}^{k+2,l} \\ \vdots \\ \mathbf{A}^{l-1,l} \end{pmatrix} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{1.17}$$

with

$$\mathcal{D}_{kl} = \left\{ \sum_{i=k}^{l-2} d_i, \sum_{i=k+1}^{l-1} d_i, d_l; \, r_{kl} \right\}.$$

The number r_{kl} is the dimension of the span of

$$\left\{ \mathrm{a}_{ij}^{kh}, \, h = k+1, k+2, \ldots, l-1; \, i \in \underline{d_k}; \, j \in \underline{d_h} \right\}.$$

The array
$$\begin{pmatrix} \mathbf{A}^{k,k+1} & \mathbf{A}^{k,k+2} & \dots & \mathbf{A}^{k,l-1} \\ \mathbf{0} & \mathbf{A}^{k+1,k+2} & \dots & \mathbf{A}^{k+1,l-1} \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{A}^{l-2,l-1} \end{pmatrix} \tag{1.18}$$
is acting as the array $\mathbf{A}^{01}$ in the case $M = 2$ and the array
$$\begin{pmatrix} \mathbf{A}^{k+1,l} \\ \mathbf{A}^{k+2,l} \\ \vdots \\ \mathbf{A}^{l-1,l} \end{pmatrix} \tag{1.19}$$
as the array $\mathbf{A}^{12}$ in the case $M = 2$. The sizes of $\mathbf{0}$ and $*$ in (1.17) are not important when we generate the arrays (1.18) and (1.19) from a structure triple as described in Theorem 1.13 for $\mathbf{A}^{01}$ and $\mathbf{A}^{12}$. The row-column products of the arrays (1.18) and (1.19) are exactly the products in Corollary 1.3. So $\mathbf{A}$ is commutative if and only if these products are symmetric. Then the structure triples of the above problems (1.17) (subject to appropriate matching conditions), together with an array $\mathbf{A}^{0,M}$, describe $\mathbf{A}$.

Before we proceed with the discussion of the simple case we state an observation. It shows that all the entries of the arrays $\mathbf{A}^{kl}$ of a commutative array $\mathbf{A}$ in the form (1.2) lie in the linear span of the entries of the first row of $\mathbf{A}$. Precisely, if we denote by $\mathcal{S}_k$ the linear span of the set
$$\left\{\mathrm{a}_{rs}^{0j},\ j \in \underline{k},\ r \in \underline{d_0},\ s \in \underline{d_j}\right\},$$
then we have :

Proposition 1.15 *For* $k = 1, 2, \dots, M-1,\ l = k+1, k+2, \dots, M,\ r \in \underline{d_k},\ s \in \underline{d_l}$ *it follows that* $\mathrm{a}_{rs}^{kl} \in \mathcal{S}_{l-k}$.

Proof. Theorem 1.13 and relation (1.16) imply that $\mathrm{a}_{rs}^{12} \in \mathcal{S}_1$. Similarly, we can apply Theorem 1.13 and relation (1.16) to the arrays $\mathbf{A}^{k-1,k}$ and $\mathbf{A}^{k,k+1}$, $k = 2, 3, \dots, M-1$ and obtain
$$\mathrm{a}_{rs}^{k,k+1} \in \mathcal{S}_{1k}$$

where $\mathcal{S}_{1k} = span\left\{\mathrm{a}_{rs}^{k-1,k},\ r=1,2,\ldots,d_{k-1},\ s=1,2,\ldots,d_{k+1}\right\}$. Therefore it follows $\mathrm{a}_{rs}^{k,k+1} \in \mathcal{S}_{1k} \subset \mathcal{S}_{1,k-1} \subset \cdots \subset \mathcal{S}_{11} = \mathcal{S}_1$. Next we apply Theorem 1.13 to the arrays $\begin{pmatrix} \mathbf{A}^{01} & \mathbf{A}^{02} \\ \mathbf{0} & \mathbf{A}^{12} \end{pmatrix}$ and $\begin{pmatrix} \mathbf{A}^{13} \\ \mathbf{A}^{23} \end{pmatrix}$. Then it follows $\mathrm{a}_{rs}^{13} \in \mathcal{S}_2$. Similarly as for $l-k=1$ we have $\mathrm{a}_{rs}^{k,k+2} \in \mathcal{S}_2$ and proceeding in the above manner for $l-k=3,4,\ldots,M-1$ we obtain that $\mathrm{a}_{rs}^{kl} \in \mathcal{S}_{l-k}$. □

1.5.2 Simple Case

Definition. An eigenvalue $\boldsymbol{\lambda}$ of a commuting n-tuple $\mathbf{A}$ is called *simple* if $d_0 = 1$ and $d_1 \geq 2$.

The above definition coincides with the terminology used in [23] except that we added the condition $d_1 \geq 2$ because we are not interested in nonderogatory eigenvalues when studying simple ones. Though the statements for the simple eigenvalues would remain valid if nonderogatory eigenvalues were included, we excluded them because we developed a more simple approach for them which could not be generalized for simple eigenvalues.

In this subsection we assume that the only eigenvalue $\boldsymbol{\lambda} = \mathbf{0}$ of $\mathbf{A}$ is simple. After an admissible change of basis $\mathcal{B}$ we can assume for $k=2,3,\ldots,M$ that $R^{0k} = \left[\begin{array}{cc} R_1^{0k} & 0 \end{array}\right]$ and $\hat{r}_k = \operatorname{rank}\widehat{R}_k = \operatorname{rank}\widetilde{R}_k$ where $R_1^{0k} \in \mathbb{C}^{n\times r_k}$, $r_k \leq d_k$, $\hat{r}_k = \sum_{j=1}^{k} r_j$,

$$\widehat{R}_k = \left[\begin{array}{cccc} R_1^{01} & R_1^{02} & \cdots & R_1^{0k} \end{array}\right] \tag{1.20}$$

and

$$\widetilde{R}_k = \left[\begin{array}{cccc} R^{01} & R^{02} & \cdots & R^{0k} \end{array}\right]. \tag{1.21}$$

Proposition 1.15 implies that

$$\mathcal{R}\left(\left[\begin{array}{cccc} C_f^{m-1,m} & C_f^{m-2,m} & \cdots & C_f^{m-k,m} \end{array}\right]\right) \subset \mathcal{R}\left(\widehat{R}_k\right)$$

for $m=2,3,\ldots,M$, $k=1,2,\ldots,m-1$ and $f=1,2,\ldots,d_m$. Then it follows from Lemma 1.10 that there exists a unique matrix $\widetilde{T}_f^m$ of the form (1.29) such that

$$\left[\begin{array}{cccc} C_f^{1m} & C_f^{2m} & \cdots & C_f^{m-1,m} \end{array}\right] = \widehat{R}_{m-1}\widetilde{T}_f^m.$$

We write

$$\widetilde{T}_f^m = \begin{bmatrix} \widetilde{T}_f^{m(11)} & \widetilde{T}_f^{m(12)} & \cdots & \widetilde{T}_f^{m(1,m-1)} \\ \widetilde{T}_f^{m(21)} & \widetilde{T}_f^{m(22)} & & 0 \\ \vdots & & & \vdots \\ \widetilde{T}_f^{m(m-1,1)} & 0 & \cdots & 0 \end{bmatrix} \tag{1.22}$$

and $\widetilde{T}_f^{m(l_1 l_2)} = \left[t_{f(h_1 h_2)}^{m(l_1 l_2)}\right]_{h_1=1,h_2=1}^{r_{l_1},d_{l_2}}$ for all l_1 and l_2 such that $l_1+l_2 \leq m$ and corresponding $h_1 \in \underline{d_{l_1}}$ and $h_2 \in \underline{d_{l_2}}$.We also have

$$\mathbf{a}_{h_1 f}^{l_1 m} = \sum_{l_2=1}^{m-l_1} \sum_{h_2=1}^{r_{l_2}} t_{f(h_1 h_2)}^{m(l_1 l_2)} \mathbf{a}_{h_2}^{0 l_2}. \tag{1.23}$$

Thus the commutative array $\mathbf{A}$ in the simple case can be given by a matrix $\widetilde{R}_M$ of the form (1.21) and matrices $\widetilde{T}_f^m$, $m = 2, 3, \ldots, M$ of the form (1.22) where $\widetilde{R}_M$ and $\widetilde{T}_f^m$ have to satisfy the *regularity* and *matching conditions.* The regularity conditions are :

– the matrix $\widehat{R}_M$ has full rank and

– the matrices $\widetilde{T}_f^{m(1,m-1)}$, $f \in \underline{d_m}$ are linearly independent.

The matching conditions are equivalent to those of Corollary 1.3.

Example 1.16 Let us consider again the matrices A_1 and A_2 of Example 1.4. The eigenvalue $\mathbf{0}$ is simple because $d_0 = 1$ and $d_1 = 2$. The columns of the first row of the array (1.7) are not linearly independent, but to make them so we perform an admissible change of basis substituting $e_5 - \frac{1}{2}e_4$ for the vector e_5 in basis $\mathcal{B}$. The array $\mathbf{A}$ in the new basis is

$$\mathbf{A} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 2 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 2 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ -1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{bmatrix}.$$

We find that the unique matrices $\widetilde{T}_f^m$ are *:*

$$\widetilde{T}_1^2 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}, \ \widetilde{T}_2^2 = \begin{bmatrix} 0 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix} \text{ and } \widetilde{T}_1^3 = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 1 \\ \frac{1}{2} & -\frac{1}{4} & 1 & -\frac{1}{2} \end{bmatrix}.$$

Note that the matrix $T_1^{3(11)}$ is symmetric, and that also the products

$$R_1^{12}\left(C_1^{23}\right)^T = R_1^{01}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}\left(R_1^{01}\right)^T = R_1^{01}\begin{bmatrix} 0 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}\left(R_1^{01}\right)^T$$

and

$$R_1^{12}\left(C_2^{23}\right)^T = R_1^{01}\begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & -\frac{1}{2} \end{bmatrix}\left(R_1^{01}\right)^T = R_1^{01}\begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{4} \end{bmatrix}\left(R_1^{01}\right)^T$$

are symmetric. □

Before we state the next result we introduce some further notation. For $m = 3, 4, \ldots, M$ we denote by Φ_m the set of indices $\{(l_1, l_2, l_3)\,;\ l_i \geq 1,\ l_1 + l_2 + l_3 \leq m\}$ and for $\mathbf{l} = (l_1, l_2, l_3) \in \Phi_m$ we define $\rho_\mathbf{l} = \underline{d_{l_1}} \times \underline{r_{l_2}} \times \underline{r_{l_3}}$ and $\chi_\mathbf{l} = \underline{d_{l_1}} \times \underline{d_{l_2}} \times \underline{d_{l_3}}$. We also write

$$s_{f\mathbf{h}}^{m\mathbf{l}} = \sum_{k=l_1+l_2}^{m-l_3} \sum_{g=1}^{d_k} t_{g(h_1 h_2)}^{k(l_1 l_2)}\, t_{f(h_3 g)}^{m(l_3,k)} \tag{1.24}$$

where $\mathbf{h} = (h_1, h_2, h_3) \in \rho_{\mathbf{l}}$ and $f \in \underline{d_m}$.

The following result then tells that a commutative array $\mathbf{A}$ in the simple case can be reconstructed from the matrices $\widetilde{R}_M$ and $\widetilde{T}_f^m$ which satisfy the regularity and matching conditions.

Proposition 1.17 *Suppose that a matrix $\widetilde{R}_M$ and matrices $\widetilde{T}_f^m$, $m = 2, 3, \ldots, M$, $f \in \underline{d_m}$ are given in the form (1.22) and that an array $\mathbf{A}$ is described by relations (1.23). Then $\mathbf{A}$ is commutative if and only if*

$$s_{f(h_1,h_2,h_3)}^{m(l_1,l_2,l_3)} = s_{f(h_1,h_3,h_2)}^{m(l_1,l_3,l_2)} \tag{1.25}$$

for $m = 3, 4, \ldots, M$, $f \in \underline{d_m}$, $\mathbf{l} \in \Phi_m$ and $\mathbf{h} \in \rho_{\mathbf{l}}$. The array $\mathbf{A}$ is reduced if and only if the matrix $\widehat{R}_M$ has full rank and the matrices $\widetilde{T}_f^{m(1,m-1)}$, $f \in \underline{d_m}$ are linearly independent for $m = 2, 3, \ldots, M$.

Proof. By Corollary 1.3 it follows that the array $\mathbf{A}$ is commutative if and only if the matrices

$$\sum_{k=l_1+1}^{m-1} R_{h_1}^{l_1 k} C_f^{km} \tag{1.26}$$

are symmetric for $l_1 = 0, 1, 2, \ldots, M-2$, $m = l_1+2, l_1+3, \ldots, M$, $h_1 \in \underline{d_{l_1}}$ and $f \in \underline{d_m}$. The nonzero blocks of a matrix $\widetilde{T}_f^m$ are of the form $\widetilde{T}_f^{m(ij)} = \left[\begin{array}{cc} \widehat{T}_f^{m(ij)} & \overline{T}_f^{m(ij)} \end{array} \right]$ where $\widehat{T}_f^{m(ij)} \in \mathbb{C}^{r_i \times r_j}$ and $\widehat{T}_f^{m(ij)} = \left(\widehat{T}_f^{m(ij)}\right)^T$. Then the matrix

$$\widehat{T}_f^m = \left[\begin{array}{cccc} \widehat{T}_f^{m(11)} & \widehat{T}_f^{m(12)} & \cdots & \widehat{T}_f^{m(1,m-1)} \\ \widehat{T}_f^{m(21)} & \widehat{T}_f^{m(22)} & & 0 \\ \vdots & & & \vdots \\ \widehat{T}_f^{m(m-1,1)} & 0 & \cdots & 0 \end{array} \right] \tag{1.27}$$

is symmetric. Therefore it follows that the matrices (1.26) for $l_1 = 0$ are equal to

$$\widetilde{R}_M \left(\widetilde{T}_f^m\right)^T \left(\widetilde{R}_M\right)^T = \widehat{R}_M \widehat{T}_f^m \left(\widehat{R}_M\right)^T$$

and hence are symmetric. Here the matrix $\widetilde{R}_M$ is defined as in (1.21), i.e.,

$$\widetilde{R}_M = \left[\begin{array}{cccc} R_1^{01} & \left(\begin{array}{cc} R_1^{02} & 0 \end{array} \right) & \cdots & \left(\begin{array}{cc} R_1^{0M} & 0 \end{array} \right) \end{array} \right].$$

It follows then from (1.23) that (1.26) are symmetric for $l_1 \in \underline{M-2}$ if and only if

$$\widehat{R}_{m-l_1-1}\widetilde{T}_{h_1R}^{l_1(m)}\left(\widetilde{T}_{mC}^{m(l_1)}\right)^T\widehat{R}_{m-l_1-1}$$

are symmetric or equivalently, if and only if

$$\widetilde{T}_{h_1R}^{l_1(m)}\left(\widetilde{T}_{fC}^{m(l_1)}\right)^T \tag{1.28}$$

are symmetric. Here

$$\widetilde{T}_{h_1R}^{l_1(m)} = \begin{bmatrix} \widetilde{T}_{h_1R}^{l_1(1,l_1+1)} & \widetilde{T}_{h_1R}^{l_1(1,l_1+2)} & \widetilde{T}_{h_1R}^{l_1(1,l_1+3)} & \cdots & \widetilde{T}_{h_1R}^{l_1(1,m-1)} \\ 0 & \widetilde{T}_{h_1R}^{l_1(2,l_1+2)} & \widetilde{T}_{h_1R}^{l_1(2,l_1+3)} & \cdots & \widetilde{T}_{h_1R}^{l_1(2,m-1)} \\ 0 & 0 & \widetilde{T}_{h_1R}^{l_1(3,l_1+3)} & \cdots & \widetilde{T}_{h_1R}^{l_1(3,m-1)} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & & \widetilde{T}_{h_1R}^{l_1(m-2,m-1)} \end{bmatrix},$$

$$\widetilde{T}_{fC}^{m(l_1)} = \begin{bmatrix} \widetilde{T}_f^{m(1,l_1+1)} & \widetilde{T}_f^{m(1,l_1+2)} & \cdots & \widetilde{T}_f^{m(1,m-2)} & \widetilde{T}_f^{m(1,m-1)} \\ \widetilde{T}_f^{m(2,l_1+1)} & \widetilde{T}_f^{m(2,l_1+2)} & \cdots & \widetilde{T}_f^{m(2,m-2)} & 0 \\ \widetilde{T}_f^{m(3,l_1+1)} & \widetilde{T}_f^{m(3,l_1+2)} & & 0 & 0 \\ \vdots & & & \vdots & \vdots \\ \widetilde{T}_f^{m(m-l_1-1,l_1+1)} & 0 & \cdots & 0 & 0 \end{bmatrix},$$

$\widetilde{T}_{h_1R}^{l_1(l_2k)} = \left[t_{g(h_1h_2)}^{k(l_1l_2)}\right]_{h_2=1,g=1}^{r_{l_2},d_{l_1}}$ and $\widetilde{T}_f^{m(l_3k)} = \left[t_{f(h_3g)}^{m(l_3k)}\right]_{h_3=1,g=1}^{r_{l_3},d_k}$.

We use the letters C and R in the subscripts above to indicate which of the matrices corresponds to a *r*ow cross-section of the array $\mathbf{A}$ and which one to the *c*olumn cross-section.

It follows from above that the matrices (1.28) are symmetric if and only if

$$\sum_{k=l_1+l_2}^{m-l_3}\sum_{g=1}^{d_k} t_{g(h_1h_2)}^{k(l_1l_2)}\, t_{f(h_3g)}^{m(l_3k)} = \sum_{k=l_1+l_3}^{m-l_2}\sum_{g=1}^{d_k} t_{g(h_1h_3)}^{k(l_1l_3)}\, t_{f(h_2g)}^{m(l_2k)}$$

and, by definition (1.24), if and only if relations (1.25) hold.

By definition, the array $\mathbf{A}$ is reduced if and only if the matrices $C_f^{m,m+1}$, $f \in \underline{d_m}$ are linearly independent for $m = 1, 2, \ldots, M$. Since $\mathrm{a}_{h_1f}^{l_1m}$ are given by (1.23) this holds if and only if R_1 has full rank and $T_f^{m(1,m-1)}$, $f \in \underline{d_m}$ are linearly independent for $m = 2, 3, \ldots, M$. $\square$

In the next theorem we expand the matrices $\widetilde{T}_f^m$ to symmetric matrices T_f^m of sizes $\hat{d}_M \times \hat{d}_M$, where as before $\hat{d}_M = \sum_{i=1}^M d_i$, and the form

$$T_f^m = \begin{bmatrix} T_f^{m(11)} & T_f^{m(12)} & \cdots & T_f^{m(1,m-1)} \\ T_f^{m(21)} & T_f^{m(22)} & & 0 \\ \vdots & & & \vdots \\ T_f^{m(m-1,1)} & 0 & \cdots & 0 \end{bmatrix} \tag{1.29}$$

where $T_f^{m(l_1 l_2)} = \left[t_{f(h_1 h_2)}^{m(l_1 l_2)}\right]_{h_1=1, h_2=1}^{d_{l_1}\ d_{l_2}}$. It is crucial for the proof of the completeness result in Chapter 4 that we prove that the expanded matrices T_f^m are symmetric and that the matching conditions 1.31 hold for them.

Theorem 1.18 *Suppose that an array* $\mathbf{A}$ *is in the form* (1.2), $d_0 = 1$ *and the nonzero elements in the set* $\left\{\mathrm{a}_f^{0m},\ m \in \underline{M},\ f \in \underline{d_m}\right\}$ *are linearly independent. Then there exist symmetric matrices* T_f^m, $m = 2, 3, \ldots, M$, $f \in \underline{d_m}$ *in the form* (1.29) *such that the relations*

$$\mathrm{a}_{h_1 f}^{l_1 m} = \sum_{l_2=1}^{m-l_1} \sum_{h_2=1}^{r_{l_2}} t_{f(h_1 h_2)}^{m(l_1 l_2)} \mathrm{a}_{h_2}^{0 l_2} \tag{1.30}$$

hold, where $l_1 \in \underline{m-1}$, $h_1 \in \underline{d_{l_1}}$, *and also*

$$\sum_{k=l_1+l_2}^{m-l_3} \sum_{g=1}^{d_k} t_{g(h_1 h_2)}^{k(l_1 l_2)} t_{f(h_3 g)}^{m(l_3 k)} = \sum_{k=l_1+l_3}^{m-l_2} \sum_{g=1}^{d_k} t_{g(h_1 h_3)}^{k(l_1 l_3)} t_{f(h_2 g)}^{m(l_2 k)}, \tag{1.31}$$

where $\mathbf{l} \in \Phi_m$, $\mathbf{h} \in \chi_{\mathbf{l}}$, $k \in \underline{m-2}$ *and* $g \in \underline{d_k}$. *Moreover the matrices* $T_f^{m(1,m-1)}$, $f \in \underline{d_m}$ *are linearly independent for* $m = 2, 3, \ldots, M$.

The proof of this theorem is long and technically complicated. To preserve continuity of the presentation we include it in Appendix A. Here we only explain how the matrices T_f^m are constructed.

The matrices $\widetilde{T}_f^m$ are as in the previous proposition. Then the matrices T_f^m are constructed inductively. First we set $T_f^{m(l_1,l_2)} = 0$ if $l_1 + l_2 > m$. Because $d_1 = r_1$

we can define

$$T_f^{m(1l)} = \widetilde{T}_f^{m(1l)} \quad \text{and} \quad T_f^{m(l1)} = \left(\widetilde{T}_f^{m(1l)}\right)^T, \tag{1.32}$$

so the matrices T_f^m for $m = 2, 3$ are determined.

Next we inductively define matrices $T_f^{m(l_1 l_2)}$ for $m \geq 4$, $f \in \underline{d_m}$ and $l_1 + l_2 \leq m$. Suppose that we already have matrices $T_f^{m'}$ for $m' = 2, 3, \ldots, m-1$, $f \in \underline{d_{m'}}$ and thus we also know the matrices $T_{gR}^{k(m)}$ for $k \in \underline{m-2}$, $g \in \underline{d_k}$ where

$$T_{gR}^{k(m)} = \begin{bmatrix} T_{gR}^{k(1,k+1)} & T_{gR}^{k(1,k+2)} & T_{gR}^{k(1,k+3)} & \cdots & T_{gR}^{k(1,m-1)} \\ 0 & T_{gR}^{k(2,k+2)} & T_{h_1R}^{lk(2,k+3)} & \cdots & T_{gR}^{k(2,m-1)} \\ 0 & 0 & T_{h_1R}^{k(3,k+3)} & \cdots & T_{gR}^{k(3,m-1)} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & & T_{gR}^{k(m-2,m-1)} \end{bmatrix} \tag{1.33}$$

and

$$T_{gR}^{k(l_1 l_2)} = \left[t_{h_2(h_1 g)}^{l_2(l_1 k)}\right]_{h_1=1,h_2=1}^{d_{l_1}\ \ d_{l_2}}.$$

Note that it follows from Proposition 1.2 that the columns of the matrices

$$\begin{bmatrix} T_{1R}^{1(l,l+1)} \\ T_{2R}^{1(l,l+1)} \\ \vdots \\ T_{d_1R}^{1(l,l+1)} \end{bmatrix}$$

are linearly independent for $l \in \underline{m-2}$ and therefore the matrix

$$\mathbf{T}_R^{1(m)} = \begin{bmatrix} T_{1R}^{1(m)} \\ T_{2R}^{1(m)} \\ \vdots \\ T_{d_1R}^{1(m)} \end{bmatrix}$$

is left invertible. We write

$$\mathbf{Z}_R^{1(m)} = \left[\ Z_{1R}^{1(m)} \quad Z_{2R}^{1(m)} \quad \cdots \quad Z_{d_1R}^{1(m)}\ \right]$$

for a left inverse of $\mathbf{T}_R^{1(m)}$ where

$$Z_g^{1(m)} = \begin{bmatrix} Z_{gR}^{1(12)} & Z_{gR}^{1(13)} & \cdots & Z_{gR}^{1(1,m-1)} \\ 0 & Z_{gR}^{1(23)} & \cdots & Z_{gR}^{1(2,m-1)} \\ \vdots & & \ddots & \cdots \\ 0 & 0 & & Z_{gR}^{1(m-2,m-1)} \end{bmatrix}$$

for $g \in \underline{d_1}$ and $Z_{gR}^{1(l_1 l_2)} \in \mathbb{C}^{d_{l_2} \times d_{l_1}}$. Now we are ready to define the matrices $T_f^{m(l_1 l_2)}$. For $l_1 = 2, 3, \ldots, \left[\frac{m+1}{2}\right]$ we write

$$T_f^{m(l_1,m-l_1)} = \sum_{h=1}^{d_1} Z_{hR}^{1(l_1-1,l_1)} T_f^{m(l_1-1,m-l_1+1)} \left(T_{hR}^{1(m-l_1,m-l_1+1)}\right)^T \tag{1.34}$$

where $\left[\frac{m+1}{2}\right]$ is the integer part of the fraction $\frac{m+1}{2}$. For $l_1 = \left[\frac{m+1}{2}\right]+1, \left[\frac{m+1}{2}\right]+2, \ldots, m-2$ we write $T_f^{m(l_1,m-l_1)} = \left(T_f^{m(m-l_1,l_1)}\right)^T$. Next we define inductively for $l_2 = 1, 2, \ldots, m-4$ matrices $T_f^{m(l_1,m-l_1-l_2)}$. These matrices for $l_2 = 0$ were just defined above and matrices $T_f^{m(1l)}$ are already known. Then we can define inductively for $l_1 = 2, 3, \ldots, \left[\frac{m-l_2+1}{2}\right]$

$$T_f^{m(l_1,m-l)} = \sum_{h=1}^{d_1} \left[\begin{array}{cccc} Z_{hR}^{1(l_1-1,l_1)} & Z_{hR}^{1(l_1-1,l_1+1)} & \cdots & Z_{hR}^{1(l_1-1,l)} \end{array}\right] \cdot$$

$$\cdot \begin{bmatrix} T_f^{m(l_1-1,m-l+1)} & T_f^{m(l_1-1,m-l+2)} & \cdots & T_f^{m(l_1-1,m-l_1+1)} \\ T_f^{m(l_1,m-l+1)} & T_f^{m(l_1,m-l+2)} & & 0 \\ \vdots & & & \vdots \\ T^{m(l-1,m-l+1)} & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} \left(T_{hR}^{1(m-l,m-l+1)}\right)^T \\ \left(T_{hR}^{1(m-l,m-l+2)}\right)^T \\ \vdots \\ \left(T_{hR}^{1(m-l,m-l_1+1)}\right)^T \end{bmatrix}.$$

Here we write $l = l_1 + l_2$. For $l_1 = \left[\frac{m-l_2+1}{2}\right]+1, \left[\frac{m-l_2+1}{2}\right]+2, \ldots, m-l_2-2$ we define $T_f^{m(l_1,m-l_1-l_2)} = \left(T_f^{m(m-l_1-l_2,l_1)}\right)^T$.

We continue inductively until $m' = M$.

To illustrate the above construction we include three examples.

Example 1.19 Suppose that $n = 2$. We would like to find two nilpotent commuting matrices A_1 and A_2 with $d_0 = 1$ such that the value of d_1, d_2 and d_3 is the greatest possible.

Because $\operatorname{rank} \widetilde{R}_3 \leq 2$ the greatest possible choice is $d_1 = 2$. Then we can take

$$\mathbf{A}^{01} = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \; \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right].$$

The array $\mathbf{A}^{12}$ is described by a set of 2×2 symmetric, linearly independent matrices. Since the space of these matrices is 3 dimensional we have $d_2 = 3$. We choose

$$T_1^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad T_2^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad T_3^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Next it follows from (1.23) that

$$\mathbf{A}^{12} = \begin{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix}.$$

The matrices $T_{1R}^{1(12)}$ and $T_{2R}^{1(12)}$ are

$$T_{1R}^{1(12)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } T_{2R}^{1(12)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since the products

$$T_{gR}^{1(12)} \left(T_f^{3(12)}\right)^T, \; g = 1, 2, \tag{1.35}$$

are 2×2 matrices and since the matrices $T_{gR}^{1(12)}$, $g = 1, 2$, are linearly independent, it follows that the space of 2×3 matrices $T_f^{3(12)}$ such that (1.35) are symmetric is 4-dimensional. We can choose

$$T_1^{3(12)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; T_2^{3(12)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

$$T_3^{3(12)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } T_4^{3(12)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and $T_f^{3(11)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for all f. Then all the matching and regularity conditions hold and we have

$$\mathbf{A}^{23} = \begin{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix}.$$

The array

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{A}^{01} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}^{12} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{A}^{23} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}$$

is commutative and we have $d_i = i+1$, $i = 0, 1, 2, 3$. In general, if $n = 2$ and $d_0 = 1$, it follows that $d_i \le i+1$. If $d_i = i+1$ for $i = 0, 1, \ldots, M-1$ then the corresponding commutative array $\mathbf{A}$ is similar to

$$\begin{bmatrix} \mathbf{0} & \mathbf{A}^{01} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}^{12} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{A}^{M-1,M} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix},$$

where $\mathbf{A}^{i,i+1}$ has dimensions $(i+1) \times (i+2) \times 2$ and

$$\mathbf{A}^{i,i+1} = \begin{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \cdots & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \ddots & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \ddots & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \cdots & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix}.$$

We do not prove the last statement, but the proof is easy. Note that the array $\mathbf{A}$ for $M = 3$ was constructed above. □

Example 1.20 Suppose that $n = 3$ and that $d_0 = 1,\ d_1 = 2,\ d_2 = 3,\ d_3 = 2$ and $d_4 = 1$. We are given

$$\widetilde{R}_4 = \begin{bmatrix} 1 & 0 & \quad 0 & 0 & 0 & \quad 0 & 0 & \quad 0 \\ 0 & -1 & \quad 0 & 0 & 0 & \quad 3 & 0 & \quad 0 \\ 0 & 2 & \quad 0 & 0 & 0 & \quad -1 & 0 & \quad 0 \end{bmatrix},$$

where the spaces indicate the partition according to the d_i, and

$$T_1^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ T_2^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ T_3^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$T_1^{3(12)} = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},\ T_2^{3(12)} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \text{ and } T_1^{3(11)} = T_2^{3(11)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

It is easy to verify that the matching conditions (1.28) and regularity conditions (described on page 32) hold for this collection of matrices for $m = 2, 3$. So the above collection describes the first three columns of a commutative array $\mathbf{A}$. To illustrate how the matching conditions work let us find a symmetric matrix T_1^4 which will define the last column of $\mathbf{A}$. The matrices $T_{gR}^{2(4)}$, $g = 1, 2, 3$, are

$$T_{1R}^{2(4)} = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix},\ T_{2R}^{2(4)} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \text{ and } T_{3R}^{2(4)} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

We write $T_1^{4(13)} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}$. Then the products $T_{gR}^{2(13)} \left(T_1^{4(13)}\right)^T$ are symmetric matrices if

$$\begin{aligned} t_{11} + t_{12} - 3t_{21} &= 0 \\ t_{12} + t_{21} + t_{22} &= 0\,. \\ t_{11} + t_{22} &= 0 \end{aligned}$$

There is a one-parameter family of solutions of this linear system of equations. The matrix $T_1^{4(13)}$ has to be nonzero, so we choose $t_{21} = 1$ for convenience. Then it follows that $T_1^{4(13)} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$. Next we have

$$T_{1R}^{1(4)} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \text{ and } T_{2R}^{1(4)} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

The symmetry of the matrices $T_{gR}^{1(4)} \left(T^{4(1)}\right)^T$, $g = 1, 2$, implies that

$$T_1^{4(22)} = \begin{bmatrix} 6 & 3 & -1 \\ 3 & -1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$$

and, if we write

$$T_1^{4(12)} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \end{bmatrix}$$

then $u_{22} = u_{13}$ and $u_{12} = u_{21}$. Here we can choose 4 of the entries u_{ij}, and we can choose the symmetric matrix $T_1^{4(11)}$, arbitrarily. We take, for example,

$$T_1^{4(12)} = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \text{ and } T_1^{4(11)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then the matrix T_1^4 is determined by symmetry. Finally we use the relations (1.23) to find the array $\mathbf{A} = \{A_1, A_2, A_3\}$. The matrices

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$A_3 = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

then commute. $\square$

Example 1.21 In this last example we are given 3 commuting matrices such that $d_0 = 1$ and we will find the corresponding matrices $\widetilde{R}_M$ and T_f^m. The commuting matrices, written in a basis $\{e_i;\ i \in \underline{10}\}$, are :

$$A_1 = \begin{bmatrix} J_4 & 0 & 0 \\ 0 & J_3 & 0 \\ 0 & 0 & J_3 \end{bmatrix},$$

where J_i is $i \times i$ nilpotent Jordan matrix,

$$A_2 = \begin{bmatrix} 0 & 1 & 1 & -1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 1 & 1 & 2 & -3 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 3 & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$A_3 = \begin{bmatrix} 0 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & -4 & 0 & -2 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 6 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In a basis $\mathcal{B} = \{e_1; e_2, e_5; e_8, e_6, e_3; e_4, e_7, e_9; e_{10}\}$ the array $\mathbf{A} = \{A_1, A_2, A_3\}$ is in the form (1.2). We have $d_1 = 2,\ d_2 = d_3 = 3$ and $d_4 = 1$. The first row of $\mathbf{A}$ is :

$$\left[\begin{pmatrix}0\\0\\0\end{pmatrix}\begin{pmatrix}1\\1\\2\end{pmatrix}\begin{pmatrix}0\\1\\2\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\2\end{pmatrix}\begin{pmatrix}0\\-1\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\end{pmatrix}\right].$$

To make its nonzero columns linearly independent we substitute the vectors $e_4 - e_3$ and $e_7 + e_6$ for e_4 and e_7, respectively, in $\mathcal{B}$. Then we have

$$\widetilde{R}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\mathbf{A}^{12} = \left[\begin{matrix}\begin{pmatrix}0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\1\\2\end{pmatrix} & \begin{pmatrix}1\\1\\2\end{pmatrix}\\ \begin{pmatrix}0\\1\\2\end{pmatrix} & \begin{pmatrix}1\\-1\\-2\end{pmatrix} & \begin{pmatrix}0\\-1\\-2\end{pmatrix}\end{matrix}\right].$$

So the matrices T_1^2, T_2^2 and T_3^2 are :

$$T_1^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ T_2^2 = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \text{ and } T_3^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Next we find the matrices $\widetilde{T}_1^3$, $\widetilde{T}_2^3$ and $\widetilde{T}_3^3$ that are associated with the arrays

$$\mathbf{A}^{13} = \left[\begin{matrix}\begin{pmatrix}0\\1\\2\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix} & \begin{pmatrix}0\\1\\0\end{pmatrix}\\ \begin{pmatrix}1\\2\\0\end{pmatrix} & \begin{pmatrix}0\\2\\0\end{pmatrix} & \begin{pmatrix}0\\2\\0\end{pmatrix}\end{matrix}\right] \text{ and } \mathbf{A}^{23} = \left[\begin{matrix}\begin{pmatrix}0\\2\\4\end{pmatrix} & \begin{pmatrix}0\\2\\4\end{pmatrix} & \begin{pmatrix}1\\3\\6\end{pmatrix}\\ \begin{pmatrix}0\\-1\\-2\end{pmatrix} & \begin{pmatrix}1\\-1\\-2\end{pmatrix} & \begin{pmatrix}0\\1\\2\end{pmatrix}\\ \begin{pmatrix}1\\1\\2\end{pmatrix} & \begin{pmatrix}0\\1\\2\end{pmatrix} & \begin{pmatrix}0\\0\\0\end{pmatrix}\end{matrix}\right].$$

They are

$$\widetilde{T}_1^3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & 2 & -1 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix}, \ \widetilde{T}_2^3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -2 & 1 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix} \ \widetilde{T}_3^3 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 \end{bmatrix}.$$

Note that their top 2×3 right corner blocks are linearly independent and we use symmetry to expand them to 5×5 symmetric matrices T_1^3, T_2^3 and T_3^3. Finally we will find the matrix T_1^4 which will describe the structure of the last column of $\mathbf{A}$. The arrays in this column are :

$$\mathbf{A}^{14} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \end{bmatrix}, \ \mathbf{A}^{24} = \begin{bmatrix} \begin{pmatrix} 0 \\ 9 \\ -4 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \end{bmatrix} \text{ and } \mathbf{A}^{34} = \begin{bmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \end{bmatrix}.$$

Then we have

$$\widetilde{T}_1^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 2 \\ 0 & -2 & 11 & 2 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

In order to expand it to the matrix T_1^4 we need to find the matrix $T_1^{4(22)}$, defined in (1.34) by

$$T_1^{4(22)} = \sum_{h=1}^{2} Z_{hR}^{1(12)} T_1^{4(13)} \left(T_{hR}^{1(23)}\right)^T, \tag{1.36}$$

while the matrices $T_1^{4(21)}$ and $T_1^{4(31)}$ are determined by symmetry. In our case we have

$$T_R^{1(12)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & -1 \end{bmatrix},$$

we choose its left inverse

$$Z_R^{1(12)} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

and we have

$$T_{1R}^{1(23)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad T_{2R}^{1(23)} = \begin{bmatrix} 2 & 2 & 2 \\ -1 & -2 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

From (1.36) it follows that

$$T_1^{4(22)} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 2 & -2 & 1 \\ 2 & 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 11 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and so

$$T_1^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 & 0 & 0 & 1 & 2 \\ 0 & -2 & 11 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now it is easy to check the matching conditions, for instance

$$T_{2R}^{1(4)} \left(T_1^{4(1)}\right)^T = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & -2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & -1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -2 & 11 & 2 & 1 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 2 & 1 & 0 \\ 0 & -2 & 6 & 0 & 1 \\ 2 & 6 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

etc. □

The commutativity of the array $\mathbf{A}$ implies further symmetries on the products of the elements of matrices T_f^m. Thus we obtain symmetric tensors that will play an essential role in the formulation of the joint root vectors for the associated n-tuple of commuting matrices of a multiparameter system. First we need some further notation. For $m = 2, 3, \ldots, M$ and $2 \le q \le m$ we denote by $\Phi_{m,q}$ the set of multiindices $\{(l_1, l_2, \ldots, l_q)\,;\ l_i \ge 1,\ \sum_{i=1}^q l_i \le m\}$. For $\mathbf{l} = (l_1, l_2, \ldots, l_q) \in \Phi_{m,q}$ we define a set $\chi_\mathbf{l} = \underline{d_{l_1}} \times \underline{d_{l_2}} \times \cdots \times \underline{d_{l_q}}$. The set of all permutations of the set $\underline{q}$ is denoted by Π_q. For a permutation $\sigma \in \Pi_q$ and multiindices $\mathbf{l} \in \Phi_{m,q}$ and $\mathbf{h} = (h_1, h_2, \ldots, h_q)$ we write $\mathbf{l}_\sigma = \left(l_{\sigma(1)}, l_{\sigma(2)}, \ldots, l_{\sigma(q)}\right)$ and $\mathbf{h}_\sigma = \left(h_{\sigma(1)}, h_{\sigma(2)}, \ldots, h_{\sigma(q)}\right)$. Then we define recursively numbers $s_{f\mathbf{h}}^{m\mathbf{l}}$: for $\mathbf{l} \in \Phi_{m,2}$ and $\mathbf{h} \in \chi_\mathbf{l}$ we write $s_{f\mathbf{h}}^{m\mathbf{l}} = t_{f(h_1h_2)}^{m(l_1l_2)}$ and for $q > 2$ and $\mathbf{l} \in \Phi_{m,q}$ and $\mathbf{h} \in \chi_\mathbf{l}$ we write

$$s_{f\mathbf{h}}^{m\mathbf{l}} = \sum_{k=l_1+l_2}^{m-\sum_{i=3}^q l_i} \sum_{g=1}^{d_k} t_{g(h_1h_2)}^{k(l_1l_2)}\, s_{f(g,h_1,h_2,\ldots,h_q)}^{m(k,l_3,l_4,\ldots,l_q)}. \tag{1.37}$$

Suppose that

$$\left\{y_h^l;\ l = 1, 2, \ldots, M,\ h = 1, 2, \ldots, r_l\right\}$$

is a basis of the vector space $\mathbb{C}^{\hat{r}_M}$. Then we define for every m, f and q a tensor

$$S_{fq}^m = \sum_{\mathbf{l}\in\Phi_{m,q}} \sum_{\mathbf{h}\in\chi_\mathbf{l}} s_{f\mathbf{h}}^{m\mathbf{l}} y_{h_1}^{l_1} \otimes y_{h_2}^{l_2} \otimes \cdots \otimes y_{h_q}^{l_q}.$$

The first observation concerning the tensors S_{fq}^m is :

Proposition 1.22 *For $m, f, q, \mathbf{l}$ and $\mathbf{h}$ as above it follows that*

$$s_{f\mathbf{h}}^{m\mathbf{l}} = \sum_{k=\sum_{i=1}^r l_i}^{m-\sum_{i=r+1}^q l_i} \sum_{g=1}^{d_k} s_{g(h_1h_2,\ldots,h_r)}^{k(l_1l_2,\ldots,l_r)}\, s_{f(g,h_{r+1},h_{r+2},\ldots,h_q)}^{m(k,l_{r+1},l_{r+2},\ldots,l_q)}$$

where $2 \leq r \leq q-1$.

Proof. It follows from definition (1.37) that

$$s_{f\mathbf{h}}^{m\mathbf{l}} = \sum_{k_1=l_1+l_2}^{m-\sum_{i=3}^{q} l_i} \sum_{k_2=k_1+l_3}^{m-\sum_{i=4}^{q} l_i} \cdots \cdot \sum_{k_{r-1}=k_{r-2}+l_r}^{m-\sum_{i=r+1}^{q} l_i} \sum_{g_1=1}^{d_{k_1}} \sum_{g_2=1}^{d_{k_2}} \cdots \cdot \sum_{g_{r-1}=1}^{d_{l_{r-1}}} t_{g_1(h_1h_2)}^{k_1(l_1l_2)} \cdot$$

$$\cdot t_{g_2(g_1h_3)}^{k_2(k_1l_3)} \cdot \ldots \cdot t_{g_{r-1}(g_{r-2}h_r)}^{k_{r-1}(k_{r-2}l_r)} s_{f(g_{r-1},h_{r+1},h_{r+2},\ldots,h_q)}^{m(k_{r-1},l_{r+1},l_{r+2},\ldots,l_q)}.$$

Interchanging the order of summation we observe that

$$\sum_{k_{r-2}=k_{r-3}+l_{r-1}}^{m-\sum_{i=r}^{q} l_i} \sum_{k_{r-1}=k_{r-2}+l_r}^{m-\sum_{i=r+1}^{q} l_i} = \sum_{k_{r-1}=k_{r-3}+l_{r-1}+l_r}^{m-\sum_{i=r+1}^{q} l_i} \sum_{k_{r-2}=k_{r-3}+l_{r-1}}^{k_{r-1}-l_r} .$$

We obtain similar rules when interchanging the order of summation between k_{r-1} and $k_{r-3}, k_{r-4}, \ldots, k_1$ respectively. Thus it follows that

$$s_{f\mathbf{h}}^{m\mathbf{l}} = \sum_{k_{r-1}=\sum_{i=1}^{r} l_i}^{m-\sum_{i=r+1}^{q} l_i} \sum_{g_{r-1}=1}^{d_{l_{r-1}}} s_{f(g_{r-1},h_{r+1},h_{r+2},\ldots,h_q)}^{m(k_{r-1},l_{r+1},l_{r+2},\ldots,l_q)} \cdot \sum_{k_1=l_1+l_2}^{k_{r-1}-\sum_{i=3}^{r} l_i} \sum_{k_2=k_1+l_3}^{k_{r-1}-\sum_{i=4}^{r} l_i} \cdots \cdot$$

$$\cdot \sum_{k_{r-2}=k_{r-3}+l_{r-1}}^{k_{r-1}-l_r} \sum_{g_1=1}^{d_{k_1}} \sum_{g_2=1}^{d_{k_2}} \cdots \cdot \sum_{g_{r-2}=1}^{d_{l_{r-2}}} t_{g_1(h_1h_2)}^{k_1(l_1l_2)} t_{g_2(g_1h_3)}^{k_2(k_1l_3)} \cdots t_{g_{r-1}(g_{r-2}h_r)}^{k_{r-1}(k_{r-2}l_r)} =$$

$$= \sum_{k=\sum_{i=1}^{r} l_i}^{m-\sum_{i=r+1}^{q} l_i} \sum_{g=1}^{d_k} s_{f(g,h_{r+1},h_{r+2},\ldots,h_q)}^{m(k,l_{r+1},l_{r+2},\ldots,l_q)} s_{g(h_1h_2,\ldots,h_r)}^{k(l_1l_2,\ldots,l_r)}.$$

Hence the result follows. □

Using the same notation as above we have

Lemma 1.23 *For every permutation* $\sigma \in \Pi_q$ *it follows that*

$$s_{f\mathbf{h}}^{m\mathbf{l}} = s_{f\mathbf{h}_\sigma}^{m\mathbf{l}_\sigma} \tag{1.38}$$

or, equivalently, the tensors S_{fq}^{m} *are symmetric.*

Proof. We prove the lemma by induction on q. For $q=2$ the result follows by definition of T_f^m (cf. relation (1.32)). Proposition 1.17 together with the relation

(1.31) give the result for $q = 3$. We suppose now that the result holds for $q-1$ (where $q \geq 4$) and we prove it for q. Since every permutation is a product of transposition it is enough to check (1.38) for transpositions and further more it is enough to check the result for transpositions of the form $(i, i+1)$. From the definition of $s^{m\mathbf{l}}_{f\mathbf{h}}$ in (1.37) and the inductive assumption it follows that the result holds if $i = 1, 3, 4, \ldots, q-1$. Therefore we only need to check (1.38) for the transposition $\sigma = (2,3)$. We showed in Proposition 1.22 that

$$s^{m\mathbf{l}}_{f\mathbf{h}} = \sum_{p=l_1+l_2+l_3}^{m-\sum_{i=4}^{q} l_i} \sum_{j=1}^{d_p} s^{p(l_1,l_2,l_3)}_{j(h_1,h_2,h_3)} \, s^{m(p,l_4,l_5,\ldots,l_q)}_{f(j,h_4,h_5,\ldots,h_q)}.$$

Then by the inductive assumption the result follows also for the transposition $\sigma = (2,3)$. □

Later we use the following two results that are consequences of Proposition 1.22 and Lemma 1.23.

Corollary 1.24 *Suppose that* $m \geq q \geq 3$, $\mathbf{l} \in \Phi_{m,q}$, $f \in \underline{d_m}$ *and* $\mathbf{h} \in \chi_{\mathbf{l}}$. *Then :*

(i)

$$\sum_{k=l_1+l_2}^{m-\sum_{i=3}^{q} d_k} \sum_{g=1} t^{k(l_1l_2)}_{g(h_1h_2)} \, s^{m(k,l_3,l_4,\ldots,l_q)}_{f(g,h_3,h_4,\ldots,h_q)} = \sum_{k=\sum_{i=2}^{q} l_i}^{m-l_1} \sum_{g=1}^{d_k} s^{k(l_2,l_3,\ldots,l_q)}_{g(h_2,h_3,\ldots,h_q)} t^{m(kl_1)}_{f(gh_1)},$$

(ii)

$$s^{m\mathbf{l}}_{f\mathbf{h}} = \sum_{k=\sum_{i=2}^{q} l_i}^{m-l_1} \sum_{g=1}^{d_k} s^{k(l_2,l_3,\ldots,l_q)}_{g(h_2,h_3,\ldots,h_q)} t^{m(kl_1)}_{f(gh_1)}.$$

Proof. By the defining relations (1.37) it follows that

$$\sum_{k=l_1+l_2}^{m-\sum_{i=3}^{q} l_i} \sum_{g=1}^{d_k} t^{k(l_1l_2)}_{g(h_1h_2)} \, s^{m(k,l_3,l_4,\ldots,l_q)}_{f(g,h_1,h_2,\ldots,h_q)} = s^{m\mathbf{l}}_{f\mathbf{h}} \tag{1.39}$$

and by Lemma 1.23

$$s^{m\mathbf{l}}_{f\mathbf{h}} = s^{m\mathbf{l}_\sigma}_{f\mathbf{h}_\sigma} \tag{1.40}$$

holds where $\sigma = (1q)$ is the transposition of 1 and q. Next we apply Proposition 1.22 to give

$$s_{f\mathbf{h}_\sigma}^{m\mathbf{l}_\sigma} = \sum_{k=\sum_{i=2}^{q} l_i}^{m-l_1} \sum_{g=1}^{d_k} s_{g(h_2,h_3,\ldots,h_q)}^{k(l_2,l_3,\ldots,l_q)} t_{f(gh_1)}^{m(kl_1)}. \tag{1.41}$$

Now assertion (i) follows because (1.39) and (1.41) are equal and assertion (ii) follows because (1.40) and (1.41) are equal. □

1.6 Representation of Commuting Matrices by Tensor Products

The material in the previous sections of this chapter was mostly developed for better understanding of the structure of commuting matrices. Our main motivation for this comes from Multiparameter Spectral Theory where the main tool that helps us understand the spectral structure of a given multiparameter system is a special n-tuple of commuting transformations. In this section our study comes closer to Multiparameter Spectral Theory. It will be seen later that the matrices we use to represent an n-tuple of commuting matrices are a special case of commuting matrices studied in Multiparameter Spectral Theory. For the definition of the induced linear transformation see page 8.

Definition. An n-tuple of commuting operators $\mathbf{A}$ on a finite dimensional Hilbert space K has a *representation by tensor products* if there exist finite dimensional Hilbert spaces H_i, operators $B_i \in \mathcal{L}(H_i)$, $i \in \underline{n}$, a subspace $\mathcal{M} \subset H = H_1 \otimes H_2 \otimes \ldots \otimes H_n$ invariant for all induced transformations $B_i^\dagger \in \mathcal{L}(H)$ and an invertible linear transformation $T : K \longrightarrow \mathcal{M}$ such that

$$A_i = T^{-1} B_i^\dagger T \qquad \text{for } i \in \underline{n}.$$

The following result was proved by Davis [57]. He and later Fong and Sourour [75] proved a similar result for a commuting n-tuple of operators in general Hilbert space. We give the complete proof that was already outlined by Davis

[57]. The reason for reproducing the proof of Davis is the close relation between the representation by tensor products and multiparameter systems studied later.

Theorem 1.25 *Every n-tuple* $\mathbf{A}$ *of commuting linear operators on a finite dimensional Hilbert space* K *has a representation by tensor products.*

Proof. Assume that $K = K_1 \otimes K_2 \otimes \cdots \otimes K_n$. This is not a restriction since we can always take, for example, $K_1 = K$ and $K_2 = K_3 = \cdots = K_n = \mathbb{C}$. Let x_i, $i \in \underline{n}$ be indeterminants. Then we write P_i for the vector space of polynomials in x_i. Denote by $\varphi_i(x_i)$ the minimal polynomial of A_i, by J_i the ideal in P_i generated by $\varphi_i(x_i)$ and by Q_i the quotient space P_i/J_i. Then we choose $H_i = \mathcal{L}(Q_i, K_i)$ to be the space of all linear transformations of Q_i into K_i. Write P for the space of all polynomials in indeterminants $x_1, x_2, \ldots, x_n$ and J for the ideal in P generated by all the polynomials $\varphi_i(x_i)$. Then the Hilbert space H can be identified with the space $\mathcal{L}(Q, K)$ where $Q = P/J$. The quotient projections are $q_i : P_i \longrightarrow Q_i$, $i \in \underline{n}$ and $q : P \longrightarrow Q$. Choose a transformation $U_i \in H_i$ and a polynomial $p_i(x_i) \in P_i$. Then the transformation $B_i \in \mathcal{L}(H_i)$ is defined by

$$B_i(U_i)\left(q_i\left(p_i(x_i)\right)\right) = U_i\left(q_i\left(x_i p_i(x_i)\right)\right).$$

This transformation is well defined since for every $p_i(x_i) \in J_i$ the product $x_i p_i(x_i) \in J_i$ also. It is easy to verify that B_i is linear. Let us mention that then the induced transformation is

$$B_i^\dagger(U)\left(q\left(p(x_1, x_2, \ldots, x_n)\right)\right) = U\left(q\left(x_i p(x_1, x_2, \ldots, x_n)\right)\right)$$

where $U \in H = \mathcal{L}(Q, K)$ and $p \in P$.

Next we define the transformation $T : K \longrightarrow H$ by

$$(Tu)\left(q\left(p(x_1, x_2, \ldots, x_n)\right)\right) = p\left(A_1, A_2, \ldots, A_n\right) u$$

where $u \in K$ and $p \in P$. It is well defined since for every $p \in J$ we can find polynomials $\psi_i \in P$, $i \in \underline{n}$, so that

$$p(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} \psi_i(x_i, x_2, \ldots, x_n)\varphi_i(x_i)$$

and so $p(A_1, A_2, \ldots, A_n) = 0$. It is an easy exercise to show that T is linear and that it maps K into H. Now choose a vector $u \in \ker T$. Then it follows that $p(A_1, A_2, \ldots, A_n)u = 0$ for all $p \in P$ including $p \equiv 1$. Therefore $Iu = u = 0$ and we have showed that operator T is one-to-one. It remains to prove that

$$TA_i = B_i^\dagger T \qquad \text{for } i \in \underline{n}. \tag{1.42}$$

Then it will follow that the subspace $\mathcal{M} = \mathcal{R}(T) \subset H$ is invariant for all operators $B_i^\dagger$, that $T : K \longrightarrow \mathcal{M}$ is invertible and hence $\mathbf{A}$ has a representation by tensor products. To verify relations (1.42) we choose a vector $u \in K$ and a polynomial $p \in P$. Then we have

$$(TA_iu)\left(q\left(p(x_1, x_2, \ldots, x_n\right)\right) = p\left(A_1, A_2, \ldots, A_n\right) A_iu = A_ip\left(A_1, A_2, \ldots, A_n\right) u =$$

$$(Tu)\left(q\left(x_ip(x_1, x_2, \ldots, x_n)\right)\right) = B_i^\dagger(Tu)\left(q\left(p(x_1, x_2, \ldots, x_n\right)\right).$$

Therefore the relations (1.42) hold and the proof is complete. □

The dimension of the space H on which the above representing operators $B_i^\dagger$ act equals $(\dim\ K)^3$. We will call it the *dimension of a representation by tensor products.* The question of minimal dimension of representation by tensor products for a given n-tuple of commuting matrices was already posed by Davis [57]. It remains an open problem. To motivate the interested reader we give two examples where the dimension of the space H is less than in the construction given in the proof of Theorem 1.25. The first example is taken from the work of De Boor and Rice [37]. They studied the approximation of partial differential equations by partial difference equations.

Example 1.26 A matrix $A = [a_{ij}]_{i,j=1}^n$ is said to be *connected* if for every $i = 2, 3, \ldots, n$ there exists a sequence of integers $1 = j_1, j_2, \ldots, j_{k_i} = i$ such that

$$d_i = \prod_{l=2}^{k_i} a_{j_{l-1}, j_l} \neq 0. \tag{1.43}$$

A matrix $B = [B_{ij}]_{i,j=1}^{m}$, where $B_{ij} \in \mathbb{C}^{n\times n}$ is said to be *blockwise connected* if for every $i = 2, 3, \ldots, n$ there exists a sequence of integers $1 = j_1, j_2, \ldots, j_{k_i} = i$ such that

$$D_i = \prod_{l=2}^{k_i} B_{j_{l-1}, j_l} \tag{1.44}$$

is nonsingular. Let $P \in \mathbb{C}^{mn\times mn}$ be the unique permutation matrix such that $P(A \otimes B)P^{-1} = B \otimes A$ for every $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$.

Using the above notation we can state the result of De Boor and Rice [37, Theorem 1] :

Proposition 1.27 *Suppose that*

$$A = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & A_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_m \end{bmatrix} \quad \textit{and} \quad B = [B_{ij}]_{i,j=1}^{m} = P \begin{bmatrix} C_1 & 0 & \cdots & 0 \\ 0 & C_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & C_n \end{bmatrix} P^{-1} \tag{1.45}$$

commute. Here we assume that $A_i, B_{ij} \in \mathbb{C}^{n\times n}$ and $C_i \in \mathbb{C}^{m\times m}$. Suppose also that A_1 is connected and B is blockwise connected. Then there exist a matrix $E \in \mathbb{C}^{m\times m}$ and a nonsingular diagonal matrix $D \in \mathbb{C}^{mn\times mn}$ such that

$$A = D^{-1}(I \otimes A_1)\, D \quad \textit{and} \quad B = D^{-1}(E \otimes I)\, D.$$

Proof. The commutativity of A and B implies $A_i C_{ij} = C_{ij} A_j$ for all indices i and j. The fact that B is blockwise connected yields $A_i = D_i A_1 D_i^{-1}$ where $D_1 = I$ and other D_i are given by (1.44). If we write $D = \begin{bmatrix} D_1^{-1} & 0 & \cdots & 0 \\ 0 & D_2^{-1} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & D_m^{-1} \end{bmatrix}$ then $A = D^{-1}(I \otimes A_1)\, D$. Observe also that all the matrices B_{ij}, and hence D_i, are diagonal. The last expression for A and commutativity implies $A_1 \left(D_i^{-1} B_{ij} D_j\right) = \left(D_i^{-1} B_{ij} D_j\right) A_1$. Since the matrix A_1 is connected and the matrices $D_i^{-1} B_{ij} D_j$ are diagonal, say $D_i^{-1}B_{ij}D_j = \begin{bmatrix} e_{ij}^1 & 0 & \cdots & 0 \\ 0 & e_{ij}^2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & e_{ij}^n \end{bmatrix}$, we have $e_{ij}^k = d_k^{-1}e_{ij}^1 d_k = e_{ij}^1$, where d_k are defined in (1.43). The blocks $D_i^{-1}B_{ij}D_j$ of the matrix DBD^{-1} are scalar multiples of identity $e_{ij}^1 I$ and so $B = D^{-1}(E \otimes I)D$ where $E = \left[e_{ij}^1\right]_{i,j=1}^m$. □

The special matrices A and B defined in (1.45) then have a representation by tensor products on the same space as they act on. □

Example 1.28 It is a well known fact that the general matrix that commutes with the Jordan block

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

is an upper Toeplitz matrix

$$B = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ 0 & \alpha_1 & \alpha_2 & & \alpha_{n-1} \\ 0 & 0 & \alpha_1 & \ddots & \alpha_{n-2} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_1 \end{bmatrix}.$$

(See for example [129, pp. 130-131].) Now choose the Hilbert space $H = \mathbb{C}^n \otimes \mathbb{C}^n$ and the subspace $\mathcal{M} \subset H$ spanned by the set $\mathcal{B} = \left\{\mathbf{x}_j = \sum_{i=1}^{j-1} \mathbf{e}_i \otimes \mathbf{e}_{j-i},\ j \in \underline{n}\right\}$ where $\mathbf{e}_i$ denote the standard basis vectors in $\mathbb{C}^n$. Then we have $(A \otimes I)\mathbf{x}_j = \mathbf{x}_{j-1}$, $j = 1, 2, \ldots, n$, where $\mathbf{x}_0 = 0$ and $(I \otimes B)\mathbf{x}_j = \sum_{l=1}^{j} \alpha_{j-l+1}\mathbf{x}_j$, $j \in \underline{n}$. Next define a transformation $T : \mathbb{C}^n \longrightarrow \mathcal{M}$ *by* $T(\mathbf{e}_j) = \mathbf{x}_j$, $j \in \underline{n}$. Then $A = T^{-1}(A \otimes I)T$ and $B = T^{-1}(I \otimes B)T$ is a representation by tensor products for A and B. It is minimal since on the tensor product space $\mathbb{C}^p \otimes \mathbb{C}^q$ where $p < n$ or $q < n$ there do not exist two matrices C and D such that both $C \otimes I$ and $I \otimes D$ have a Jordan chain of length n. □

Proposition 1.27 states that the matrices A and B of Example 1.26 are a special case of matrices that already have a representation by tensor products in the space they act. The consequence of Example 1.28 is that in general we can not expect the minimal dimension of a representation by tensor products to be of dimension less than $(\dim\ K)^2$.

1.7 Comments

Commutative matrices have been studied since the second half of the last century. Some of the related results were discussed in the works of Frobenius [79, 80, 81], Sylvester [158], Taber [159, 160], Plemelj [140] and Schur [148]. For a more detailed discussion of the early developments compare [128, pp. 93-94]. It follows from results of Voss [169] (see the remark about Schur's lectures in [146]) that an n-tuple of commuting matrices can be simultaneously reduced to upper-triangular form. Rutherford [146] described further properties that can be attained by this upper-triangular form. Also Trump [164] and Egan and Ingram [62] discussed simultaneous reduction of pairwise commuting matrices. The reduced form (1.2) that we use in our presentation often appears in works on algebras of commuting matrices. See, for example, the monograph of Suprunenko and Tyshkevich [154]. The authors in [154, p.66] also noticed that commutativity of matrices is equivalent to certain symmetries in the products of these matrices. We explore this property in greater detail. It was shown by Gel'fand and Ponomarev [87] that the problem of a canonical form for an n-tuple of commuting matrices contains as a subproblem the description of a canonical form for a (not necessarily commuting) m-tuple of matrices. By a canonical form for n-tuples of (commuting) matrices we mean a collection of n-tuples of (commuting) matrices such that every n-tuple of (commuting) matrices would be simultaneously similar to exactly one n-tuple in the collection. Gel'fand and Ponomarev gave a canonical form for a class of pairs of commuting matrices in [86]. Though it would be more elegant to have a canonical form for an n-tuple of commuting matrices at hand it turns out that the reduced form (1.2) serves well for our purposes. We refer the interested reader to [15, 77, 78, 88, 96, 121, 170] for other more recent discussions on commuting matrices and on canonical forms for matrices.

Our motivation for studying commutative matrices comes from Multiparameter Spectral Theory where a special n-tuple of commuting matrices is associated with a multiparameter system. The results of this chapter were developed simultaneously with the results on the structure of root vectors for the associated n-tuple of commuting matrices. Some of the results here were suggested by the structure of root vectors although they are independent from Multiparameter Spectral Theory and conversely, now form a very important building block in the theory of root vectors developed later in this dissertation.

Our discussion of nonderogatory eigenvalues gives a different view of the previously known results (cf. [92, p.296] or [129, p.130]). Corollary 1.7 seems to be an interesting new observation.

The matrices $\widetilde{R}_M$ and T_f^m that appear in Theorem 1.18 can be described as solutions of linear equations in terms of the underlying multiparameter system. Commutativity implies further symmetries on the products of matrices T_f^m. We associate with these products higher order tensors which are then symmetric as shown in Lemma 1.23. These symmetric tensors appear as a coefficients in the expansion of root vectors and their symmetry enables us to prove our main result on root vectors for simple eigenvalues.

We remark that the results of the second section and subsection 1.5.1 together with the necessary auxiliary results are presented in [118]. We also remark that it appears to be possible to reconstruct an arbitrary array in the form (1.2) (not necessarily simple) from a matrix $\widehat{R}_M$ and matrices T_f^m (which are now not necessarily symmetric) where $m = 1, 2, \ldots, M$ and $f \in \underline{d_m}$. These matrices have to satisfy regularity and matching conditions similar to those in Theorem 1.18. Because the proof of the results in this case appears to be a lengthy exercise in simple calculations and because it seems highly technically involved to apply the eventual results to the theory of root vectors for the associated n-tuple of commuting matrices we do not proceed with this further discussion. Let us mention however that a possible canonical form for the matrices T_f^m or a canonical form for the symmetric tensors S_f^m could give a canonical form for the n-tuple of commuting matrices. (See [172] for a canonical form for a special case of symmetric tensors.) The investigation of these relations is beyond the scope of this dissertation.

In Section 1.6 we investigate the relation between an arbitrary and an associated n-tuple of commuting matrices. Davis in [57] was first to consider the representation of an n-tuple of commuting matrices by tensor products, which is a special case of a representation by a multiparameter system that we define and discuss in the next chapter. Also Fong and Sourour [75] studied the representation by tensor products on an arbitrary Hilbert space and De Boor and Rice [37] considered a related problem. As will be shown later not every n-tuple of commuting matrices is an associated n-tuple of commuting matrices. It follows from the result of Davis [57, Theorem 1] that every n-tuple of commuting matrices is a restriction of an associated n-tuple of commuting matrices. The problem of the minimal representation by tensor products was stated by Davis in [57] and we will state later an analogue for minimal representation by a multiparameter system. These problems have not yet been solved.

Chapter 2

Multiparameter Systems

2.1 Introduction

We begin our discussion in this chapter by introducing the finite-dimensional abstract setting for Multiparameter Spectral Theory in Sections 2.2 and 2.3. We follow Atkinson [10, Chapter 6] who laid the fundamental tensor space construction. Partly we also follow Isaev [112, Lecture 1]. A set of determinantal operators on the tensor product space is induced by a multiparameter system. We assume that a multiparameter system is nonsingular, i.e., one of the induced determinantal operators, called Δ_0, is invertible. Then we associate an n-tuple of linear operators with a multiparameter system called an associated system. The basic property of the associated system is that it is an commutative n-tuple. We also have the basic relation (2.7) that connects a multiparameter system with its associated system.

At the end of Section 2.3 we include a discussion on some basic relations between a general n-tuple of commuting matrices and the associated system of a multiparameter system. This is closely connected with the presentation in Section 1.6. Also Example 2.13 in Section 2.4 is related to this discussion.

In Section 2.4 we define the notions of spectra, eigenvectors and root vectors for multiparameter system. They are defined so that they correspond to the equivalent notions for the associated system.

In Section 2.5 we study root subspaces for multiparameter systems. Here we describe a basis for the second root subspace. In the first subsection we focus on simple and nonderogatory eigenvalues and in the second subsection we study the general case. Relation (2.7) relating a multiparameter system with its associated system, and commutativity of the associated system, play crucial roles in this development. Relation (2.7) leads us to equalities of the type (2.13) and (2.23). In this way we find vectors $x_{i1}^{\mathbf{k}}$ that are used to construct a basis for the second root subspace. Technically the most difficult part is to prove that the vectors we construct are actually root vectors. We perform a direct calculation using properties of determinantal operators and relations that hold for the vectors $x_{i1}^{\mathbf{k}}$. To prove completeness, i.e., the fact that a particular collection of root vectors is a basis, we use the theory of commuting matrices, developed in the first chapter, applied to the associated system. Certain columns in an array in the form (1.2) are linearly independent. It turns out that they are elements of the kernel of a special matrix that we associate with an eigenvalue of a multiparameter system, that is the matrix B_0 in the simple case (or $\mathcal{D}_0^{\lambda}$ in the general case). Because we are able to associate with a basis for the kernel of B_0 (or $\mathcal{D}_0^{\lambda}$) a set of linearly independent vectors in the second root subspace it follows therefore that this set is a basis for the latter.

2.2 Notation

Assume that H_i ($i \in \underline{n}$) are finite dimensional Hilbert spaces, the dimensions of H_i are n_i and that $V_{ij} \in \mathcal{L}(H_i)$ for $j = 0, 1, \ldots, n$. Then a system of operators

$$W_i(\boldsymbol{\lambda}) = \sum_{j=1}^{n} \lambda_j V_{ij} - V_{i0} \qquad i \in \underline{n}$$

is called a *multiparameter system* and is denoted by $\mathbf{W}$. Here $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n$. We write $U_i(\boldsymbol{\lambda}) = \sum_{j=1}^{n} \lambda_j V_{ij}$. A multiparameter system is called *diagonal* if $V_{ij} = 0$ for $1 \leq i, j \leq n$, $i \neq j$ and it is called *upper-triangular* if $V_{ij} = 0$ for $1 \leq j < i \leq n$. As before (cf. page 8) the transformation $V_i \in \mathcal{L}(H_i)$ induces

$V_i^\dagger \in \mathcal{L}(H)$ where $H = H_1 \otimes H_2 \otimes \cdots \otimes H_n$. Note that $\dim H = \prod_{i=1}^n n_i$. We write $\dim H = N$.

The *determinantal operator* $\Delta_0 \in \mathcal{L}(H)$ is defined by

$$\Delta_0 = \begin{vmatrix} V_{11}^\dagger & V_{12}^\dagger & \cdots & V_{1n}^\dagger \\ V_{21}^\dagger & V_{22}^\dagger & \cdots & V_{2n}^\dagger \\ \vdots & \vdots & & \vdots \\ V_{n1}^\dagger & V_{n2}^\dagger & \cdots & V_{nn}^\dagger \end{vmatrix}. \tag{2.1}$$

The operator Δ_0 is well defined since the operators from different rows in above determinant commute. It can be also written

$$\Delta_0 = \sum_{\sigma \in \Pi_n} (-1)^{\operatorname{sgn}(\sigma)} V_{1\sigma(1)} \otimes V_{2\sigma(2)} \otimes \cdots \otimes V_{n\sigma(n)}$$

where $\operatorname{sgn}(\sigma)$ is the signature of a permutation σ.

Given a decomposable tensor $x = x_1 \otimes x_2 \otimes \ldots \otimes x_n \in H$ we use the notation

$$\Delta_0 x = \sum_{\sigma \in \Pi_n} (-1)^{\operatorname{sgn}(\sigma)} V_{1\sigma(1)} x_1 \otimes V_{2\sigma(2)} x_2 \otimes \cdots \otimes V_{n\sigma(n)} x_n =$$

$$= \begin{vmatrix} V_{11}x_1 & V_{12}x_1 & \cdots & V_{1n}x_1 \\ V_{21}x_2 & V_{22}x_2 & \cdots & V_{2n}x_2 \\ \vdots & \vdots & & \vdots \\ V_{n1}x_n & V_{n2}x_n & \cdots & V_{nn}x_n \end{vmatrix}^{\otimes}. \tag{2.2}$$

If $y = y_1 \otimes y_2 \otimes \cdots \otimes y_n \in H$ is another decomposable tensor then it follows that

$$(\Delta_0 x, y) = \begin{vmatrix} y_1^* V_{11} x_1 & y_1^* V_{12} x_1 & \cdots & y_1^* V_{1n} x_1 \\ y_2^* V_{21} x_2 & y_2^* V_{22} x_2 & \cdots & y_2^* V_{2n} x_2 \\ \vdots & \vdots & & \vdots \\ y_n^* V_{n1} x_n & y_n^* V_{n2} x_n & \cdots & y_n^* V_{nn} x_n \end{vmatrix}.$$

(The scalar product $(\cdot,\cdot)$ is defined on page 8.)

We define further operators Δ_i for $i \in \underline{n}$ by

$$\Delta_i = \begin{vmatrix} V_{11}^\dagger & \cdots & V_{1,i-1}^\dagger & V_{10}^\dagger & V_{1,i+1}^\dagger & \cdots & V_{1n}^\dagger \\ V_{21}^\dagger & \cdots & V_{2,i-1}^\dagger & V_{20}^\dagger & V_{2,i+1}^\dagger & \cdots & V_{2n}^\dagger \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1}^\dagger & \cdots & V_{n,i-1}^\dagger & V_{n0}^\dagger & V_{n,i+1}^\dagger & \cdots & V_{nn}^\dagger \end{vmatrix}. \tag{2.3}$$

Definition. A multiparameter system $\mathbf{W}$ is called *nonsingular* if the associated operator Δ_0 is invertible in $\mathcal{L}(H)$.

In this dissertation we study nonsingular multiparameter systems. Let us remark at this point that the assumption 'Δ_0 is invertible in $\mathcal{L}(H)$' could be replaced by the weaker assumption that 'there exist $n+1$ complex numbers τ_i such that the operator $\sum_{i=0}^n \tau_i \Delta_i$ is invertible'. Note that the latter case can be converted to the former by a suitable substitution of parameters $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \ldots, \lambda_n)$ in the homogeneous formulation of the multiparameter system

$$W_i(\boldsymbol{\lambda}) = \sum_{j=0}^n \lambda_j V_{ij}, \qquad i \in \underline{n},$$

i.e. substitution $\boldsymbol{\lambda} = T\boldsymbol{\lambda}'$ where T is an $(n+1) \times (n+1)$ invertible matrix. So our theory can be applied also in the latter case with minor notational changes.

Definition. The set of operators $\Gamma_i = \Delta_0^{-1}\Delta_i \in \mathcal{L}(H)$, $i \in \underline{n}$, is called the *associated system* (of a multiparameter system $\mathbf{W}$), and is denoted by $\boldsymbol{\Gamma}$.

2.3 Determinantal Operators

Determinantal operators retain the properties of scalar determinants when we perform column operations.

Lemma 2.1 *(i) If two columns of a determinantal operator* (2.1) *are interchanged, the operator is multiplied by* -1.

(ii) If two columns of (2.1) *are identical, the operator equals* 0.

(iii) The value of a determinantal operator is unchanged if a scalar multiple of one column is added to another column.

The proof uses the same method as in the scalar case. As a consequence of Lemma 2.1 it follows that properties *(i)*–*(iii)* hold also for the tensor determinant (2.2). Further we have (Laplace) expansion identities as in the scalar case. We will use only the column versions. We write Δ_{ijk} $(i = 0, 1, \ldots, n;\ j, k \in \underline{n})$ for the cofactor of $V_{jk}^\dagger$ in Δ_i.

Lemma 2.2 *(i) For $i, k \in \underline{n}$ we have*

$$\sum_{j=1}^{n} \Delta_{0jk} V_{ji}^\dagger = \sum_{j=1}^{n} V_{ji}^\dagger \Delta_{0jk} = \begin{cases} \Delta_0, & if\ i = k \\ 0, & if\ i \neq k \end{cases}$$

and

$$\sum_{j=1}^{n} V_{j0}^\dagger \Delta_{0jk} = \sum_{j=1}^{n} \Delta_{0jk} V_{j0}^\dagger = \Delta_k.$$

(ii) For $i, k, l \in \underline{n}$

$$\sum_{j=1}^{n} \Delta_{ijl} V_{jk}^\dagger = \sum_{j=1}^{n} V_{jk}^\dagger \Delta_{ijl} = \begin{cases} \Delta_0, & if\ k = l = i \\ \Delta_i, & if\ k \neq i, k = l \\ -\Delta_l, & if\ k = i, l \neq i \\ 0, & if\ k \neq l, k \neq i \end{cases}$$

and when $k = 0$

$$\sum_{j=1}^{n} \Delta_{ijl} V_{j0}^\dagger = \sum_{j=1}^{n} V_{j0}^\dagger \Delta_{ijl} = \begin{cases} \Delta_i, & if\ i = l \\ 0, & if\ i \neq l \end{cases}.$$

The method used to prove this result is the same as in scalar case. In particular we use assertion *(ii)* of Lemma 2.1. Before we proceed with further properties of the multiparameter system we need the following definition :

Definition. The *decomposability set* $\mathcal{R}(\mathbf{W})$ of a multiparameter system $\mathbf{W}$ is defined as the set of all vectors $x \in H$ for which there exist vectors $x_1, x_2, \ldots, x_n \in H$ such that

$$\sum_{j=1}^{n} V_{ij}^\dagger x_j = V_{i0}^\dagger x \qquad \text{for } i \in \underline{n}. \tag{2.4}$$

Lemma 2.3 *Choose $x \in \mathcal{R}(\mathrm{W})$ and $x_1, x_2, \ldots, x_n$ such that relations* (2.4) *hold. Put $x_0 = -x$. Then we have the identities*

$$\Delta_i x_j = \Delta_j x_i \qquad \text{for } i,j = 0,1,\ldots,n. \tag{2.5}$$

Furthermore we have $x_i = \Gamma_i x$ and so $\sum_{j=1}^n V_{ij}^\dagger \Gamma_i x = V_{i0}^\dagger x$ for $i = 1,2,\ldots,n$ and

$$\Delta_i \Delta_0^{-1} \Delta_j x = \Delta_j \Delta_0^{-1} \Delta_i x \tag{2.6}$$

for $i,j = 1,2,\ldots,n$.

Proof. By the definition of the decomposability set the vectors $x_0, x_1, \ldots, x_n$ satisfy the relations

$$\sum_{l=0}^{n} V_{kl}^\dagger x_l = 0 \qquad \textit{for } k = 1,2,\ldots,n.$$

Now we apply the operator Δ_{ikj} on the left-hand side and sum over k to obtain

$$\sum_{k=1}^{n} \sum_{l=0}^{n} \Delta_{ikj} V_{kl}^\dagger x_l = 0 \qquad \textit{for } i,j = 0,1,\ldots,n.$$

Next the identities of Lemma 2.2 imply

$$0 = \sum_{k=1}^{n} \sum_{l=0}^{n} \Delta_{ikj} V_{kl}^\dagger x_l = -\Delta_i x_j + \Delta_j x_i.$$

Since Δ_0 is invertible we get from the above relations for $j = 0$ that $x_i = \Delta_0^{-1} \Delta_i x = \Gamma_i x$ and $\sum_{j=1}^n V_{ij}^\dagger \Gamma_i x = V_{i0}^\dagger x$. It also follows that $\Delta_j \Delta_0^{-1} \Delta_i x = \Delta_j x_i = \Delta_i x_j = \Delta_i \Delta_0^{-1} \Delta_j x$. The proof is complete. □

The array

$$\mathcal{D} = \begin{bmatrix} V_{11}^\dagger & V_{12}^\dagger & \cdots & V_{1n}^\dagger \\ V_{21}^\dagger & V_{22}^\dagger & \cdots & V_{2n}^\dagger \\ \vdots & \vdots & & \vdots \\ V_{n1}^\dagger & V_{n2}^\dagger & \cdots & V_{nn}^\dagger \end{bmatrix}$$

defines a linear transformation $\mathcal{D} : H^n \longrightarrow H^n$. Here we write H^n for a direct sum of n copies of space H. Suppose for the moment that transformation $\mathcal{D}$ is not invertible. Then the following result describes the relation between the kernels of Δ_0 and $\mathcal{D}$. We will consider this relation more closely for the special case $n = 2$ in Chapter 3.

Lemma 2.4 *Suppose that* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \ker \mathcal{D}$. *Then* $x_k \in \ker \Delta_0$ *for all* k.

Proof. We have $\sum_{j=1}^{n} V_{ij}^{\dagger} x_j = 0$ for $i \in \underline{n}$. From Lemma 2.2, (i) it follows that

$$0 = \sum_{i=1}^{n} \sum_{j=1}^{n} \Delta_{0ik} V_{ij}^{\dagger} x_j = \Delta_0 x_k$$

and the result follows. □

Before we state the main result of this section we make an observation that follows from the definition of the decomposability set.

Lemma 2.5 *A vector* $x \in H$ *is an element of* $\mathcal{R}(\mathbf{W})$ *if and only if*

$$(V_{10}^{\dagger} x, V_{20}^{\dagger} x, \ldots, V_{n0}^{\dagger} x) \in \mathcal{R}(\mathcal{D}).$$

Theorem 2.6 *Let* H_i, $i = 1, 2, \ldots, n$, *be finite dimensional vector spaces and let* $\mathbf{W}$ *be a nonsingular multiparameter system. Then* $\mathcal{R}(\mathbf{W}) = H$, *the associated operators* $\Gamma_i, i = 1, 2, \ldots, n$ *commute and they satisfy the relations*

$$\sum_{j=1}^{n} V_{ij}^{\dagger} \Gamma_j = V_{i0}^{\dagger} \qquad \text{for } i \in \underline{n}. \tag{2.7}$$

Proof. We only need to show that $\mathcal{R}(\mathbf{W}) = H$. The other assertions then follow from Lemma 2.3.

The adjugate operator of $\mathcal{D}$ is defined as

$$\mathcal{B}_{\mathcal{D}} = \begin{bmatrix} \Delta_{011} & \Delta_{012} & \cdots & \Delta_{01n} \\ \Delta_{021} & \Delta_{022} & \cdots & \Delta_{02n} \\ \vdots & \vdots & & \vdots \\ \Delta_{0n1} & \Delta_{0n2} & \cdots & \Delta_{0nn} \end{bmatrix}.$$

The equalities of Lemma 2.2 imply that

$$\mathcal{D} \cdot \mathcal{B}_{\mathcal{D}} = \mathcal{B}_{\mathcal{D}} \cdot \mathcal{D} = \begin{bmatrix} \Delta_0 & 0 & \cdots & 0 \\ 0 & \Delta_0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \Delta_0 \end{bmatrix} \tag{2.8}$$

and hence the operator $\mathcal{D}$ is invertible. Then the equality $H^n = \mathcal{R}(\mathcal{D})$ holds and thus it follows from Lemma 2.5 that $\mathcal{R}(\mathbf{W}) = H$. The proof is complete. □

The following results are closely related to the discussion in Section 1.6. We describe elementary relations between a general n-tuple of commuting matrices $\mathbf{A}$ and the associated system $\mathbf{\Gamma}$ of a multiparameter system. Further research on these relations might give an interesting new view point on commuting matrices.

The following Corollary tells us that the n-tuple of commuting matrices that satisfies the relations (2.7) is uniquely defined.

Corollary 2.7 *If $A_1, A_2, \ldots, A_n$ is an n-tuple of commuting linear transformations on a finite-dimensional vector space H and $\mathbf{W}$ is a nonsingular multiparameter system such that*

$$\sum_{j=1}^{n} V_{ij}^{\dagger} A_j = V_{i0}^{\dagger} \qquad \textit{for } i \in \underline{n} \tag{2.9}$$

then $A_j = \Gamma_j$ for $j \in \underline{n}$, where $\mathbf{\Gamma}$ is associated with $\mathbf{W}$.

Proof. For any vector $x \in H$ the relations (2.7) and (2.9) imply

$$\mathcal{D}\left(\left(A_1 - \Gamma_1\right)x, \left(A_2 - \Gamma_2\right)x, \ldots, \left(A_n - \Gamma_n\right)x\right) = 0.$$

Since $\mathbf{W}$ is a nonsingular multiparameter system it follows from (2.8) that the operator $\mathcal{D}$ is invertible. Hence we have $A_j x = \Gamma_j x$ for $j \in \underline{n}$ and the result is proved. □

As a corollary of Theorem 1.25 we have the following result concerning an n-tuple of commuting operators on an arbitrary finite-dimensional vector space.

Corollary 2.8 *Every n-tuple $\mathbf{A}$ of commuting linear transformations on a finite dimensional vector space K is similar to a restriction of the associated system of a multiparameter system.*

Proof. By Theorem 1.25 there exists a representation by tensor products for $\mathbf{A}$. Using the notation of the proof of Theorem 1.25 we write

$$A_i = T^{-1} B_i^{\dagger} T \quad \text{or} \quad B_i^{\dagger}\Big|_{\mathcal{M}} = T A_i T^{-1} \qquad \text{for } i \in \underline{n}. \tag{2.10}$$

If we define the multiparameter system $\mathbf{W}$ by $W_i(\boldsymbol{\lambda}) = I\lambda_i - B_i$, $i = 1, 2, \ldots, n$, then it follows that $\Gamma_i = B_i^\dagger$. Therefore it follows from the relations (2.10) that $\mathbf{A}$ is similar to a restriction of the associated system $\mathbf{\Gamma}$. □

Corollary 2.9 *Let $A_1, A_2, \ldots, A_n$ be an n-tuple of linear transformations on a finite-dimensional vector space K. Then the following are equivalent :*

(i) the transformations A_i commute

(ii) there exist a multiparameter system $\mathbf{W}$, a common invariant subspace $\mathcal{M}$ for associated transformations Γ_i and an invertible linear transformation $T : K \longrightarrow \mathcal{M}$ such that

$$\sum_{j=1}^{n} V_{ij}^\dagger T A_j = V_{i0}^\dagger T \qquad \text{for } i \in \underline{n}.$$

Proof. It follows from the previous corollary that assertion (i) implies assertion (ii).

Now assume that (ii) holds. As in the proof of Corollary 2.7 it follows that $TA_jT^{-1}x = \Gamma_j x$ for all j and all $x \in \mathcal{M}$. So A_j are simultaneously similar to restrictions $\Gamma_j\,|_{\mathcal{M}}$ and therefore they commute. □

We say that an n-tuple $\mathbf{A}$ of commuting matrices on Hilbert space K has a *representation by a multiparameter system* if there exist a multiparameter system $\mathbf{W}$ with associated system $\mathbf{\Gamma}$ acting on a Hilbert space H, a subspace $\mathcal{M} \subset H$ invariant for all Γ_i and an one-to-one map $T : K \longrightarrow \mathcal{M}$ such that

$$T^{-1}\Gamma_i T = A_i \qquad \text{for } i \in \underline{n}.$$

Theorem 1.25 tells us that every n-tuple of commuting matrices has a representation by tensor products which is actually a representation by a (special) diagonal multiparameter system. The natural question arises : what is the minimal dimension of the space H on which $\mathbf{A}$ has a representation by a multiparameter system ? This is still an open problem. Example 2.13 below shows that $\mathbf{A}$ does not always have a representation by a multiparameter system on the original space K. First we need to establish some more properties of multiparameter systems.

2.4 Eigenvalues, Eigenvectors and Root Vectors of a Multiparameter System

Definition. An n-tuple $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n$ is an eigenvalue of a multiparameter system $\mathbf{W}$ if all the operators $-V_{i0} + \sum_{j=1}^{n} \lambda_j V_{ij}$, $i \in \underline{n}$ are singular. The collection of all the eigenvalues of the system $\mathbf{W}$ is the spectrum of $\mathbf{W}$ and is denoted by $\sigma(\mathbf{W})$.

The following proposition is a standard fact. We state it because we later refer to it. It can be proved in the finite-dimensional case using a dimensional argument (cf. Atkinson [10, pp. 72-73]).

Proposition 2.10 *Suppose that $V_i \in \mathcal{L}(H_i)$, $i \in \underline{n}$, and $H = H_1 \otimes H_2 \otimes \cdots \otimes H_n$. Then it follows that*

$$\bigcap_{i=1}^{n} \ker V_i^{\dagger} = \bigotimes_{i=1}^{n} \ker V_i \ .$$

Atkinson [10] proved the next important result connecting the spectrum of a multiparameter system $\mathbf{W}$ and the spectrum of its associated system. It also describes the eigenspace of the associated system $\boldsymbol{\Gamma}$. We include, for the completeness of presentation, the proof following Isaev [112].

Theorem 2.11 *The spectra of a multiparameter system $\mathbf{W}$ and its associated system $\boldsymbol{\Gamma}$ coincide. If $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ then the space of common eigenvectors for $\boldsymbol{\Gamma}$ at the eigenvalue $\boldsymbol{\lambda}$ is*

$$\bigcap_{i=1}^{n} \ker \ (\lambda_i I - \Gamma_i) = \bigotimes_{i=1}^{n} \ker W_i(\boldsymbol{\lambda}). \tag{2.11}$$

Proof. Assume that $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ and that $y_i \in \ker W_i \setminus \{0\}$, $i \in \underline{n}$. Write $x = y_1 \otimes y_2 \otimes \ldots \otimes y_n$ and $x_j = \lambda_j x$. Then $V_{i0}^{\dagger} x - \sum_{j=1}^{n} V_{ij}^{\dagger} x_j = 0$ for all i and hence $x \in \mathcal{R}(\mathbf{W})$. Lemma 2.3 then implies that $\Gamma_j x = x_j = \lambda_j x$. So we have $\boldsymbol{\lambda} \in \sigma(\boldsymbol{\Gamma})$ and $\bigotimes_{i=1}^{n} \ker W_i(\boldsymbol{\lambda}) \subset \bigcap_{i=1}^{n} \ker \ (\lambda_i I - \Gamma_i)$.

Now assume that $\boldsymbol{\lambda} \in \sigma(\boldsymbol{\Gamma})$ and that a vector $x \in \bigcap_{i=1}^n \ker\ (\lambda_i I - \Gamma_i) \setminus \{0\}$. Then relations (2.7) yield

$$0 = \sum_{j=1}^n V_{ij}^\dagger \Gamma_j x - V_{i0}^\dagger x = \sum_{j=1}^n V_{ij}^\dagger \lambda_j x - V_{i0}^\dagger x = W_i(\boldsymbol{\lambda})^\dagger x.$$

Proposition 2.10 implies that $\bigcap_{i=1}^n \ker W_i(\boldsymbol{\lambda})^\dagger = \bigotimes_{i=1}^n \ker W_i(\boldsymbol{\lambda})$ and therefore it follows that $x \in \bigotimes_{i=1}^n \ker W_i(\boldsymbol{\lambda})$ and $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$. □

Next we choose, for every i, a subspace $H_i' \subset H_i$ such that $H_i = \ker W_i(\boldsymbol{\lambda}) \oplus H_i'$. For later reference we also need the next lemma.

Lemma 2.12 *Suppose that a vector $x_i \in (\ker W_i(\boldsymbol{\lambda})^*)^\perp$. Then there exists a vector $y_i \in H_i'$ such that $x_i = W_i(\boldsymbol{\lambda})\, y_i$.*

Proof. Suppose that $x_i \in (\ker W_i(\boldsymbol{\lambda})^*)^\perp$. Because

$$(\ker W_i(\boldsymbol{\lambda})^*)^\perp = \mathcal{R}(W_i(\boldsymbol{\lambda}))$$

it follows that there exists a vector $z_i \in H_i$ such that $x_i = W_i(\boldsymbol{\lambda})\, z_i$. By the definition of the direct sum of vector spaces we can find vectors $y_i \in H_i'$ and $w_i \in \ker W_i(\boldsymbol{\lambda})$ such that $y_i + w_i = z_i$. Then it follows that $x_i = W_i(\boldsymbol{\lambda})\, y_i$. □

We now state the example, promised on page 67, of commuting matrices that do not have a representation by a multiparameter system on the original space.

Example 2.13 Assume that

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 0 & 0 & \alpha_1 \\ 0 & 0 & 0 & \alpha_2 \\ 0 & 0 & 0 & \alpha_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and that $\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \neq 0$. It is easy to see that $AB = BA = 0$. We will show that A and B do not form an associated system of a (nonsingular) multiparameter system.

Assume the contrary. Since A and B have a three-dimensional subspace of common eigenvectors they could arise from a multiparameter system $\mathbf{W}$ only with $n_1 = 1$ and $n_2 = 4$ (see Theorem 2.11). Write $\mathbf{W}$ as

$$\begin{aligned} W_1(\boldsymbol{\lambda}) &= v_1\lambda_1 + v_2\lambda_2 - v_0 \\ W_2(\boldsymbol{\lambda}) &= V_1\lambda_1 + V_2\lambda_2 - V_0 \end{aligned}.$$

Then we have

$$\Gamma_1 = (v_1V_2 - v_2V_1)^{-1}(v_0V_2 - v_2V_0) \quad \text{and} \quad \Gamma_2 = (v_1V_2 - v_2V_1)^{-1}(v_1V_0 - v_0V_1).$$

The equation (2.7) for $i = 1$ is $v_1\Gamma_1 + v_2\Gamma_2 = v_0I$. If $\Gamma_1 = A$ and $\Gamma_2 = B$ then it follows that $v_0 = 0$, $\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} v_2 = 0$ implies $v_2 = 0$ and also $v_1 + \alpha_3 v_2 = 0$ implies $v_1 = 0$. This contradicts the assumption that $\mathbf{W}$ is nonsingular. Hence the matrices A and B are not the associated system of a multiparameter system. □

The main topic of our study is to describe a basis for the root subspaces of the associated system of commuting matrices $\mathbf{\Gamma}$ in terms of the corresponding multiparameter system $\mathbf{W}$. The results of Chapter 1 showed that two extreme cases of commuting matrices that are easily understood are the case when the commuting matrices are represented by tensor products (i.e., $\mathbf{A}$ acts on a space $H_1 \otimes H_2 \otimes \cdots \otimes H_n$ and $A_i = B_i^\dagger$ where $B_i \in \mathcal{L}(H_i)$) and the nonderogatory case. The first case corresponds to a diagonal multiparameter system with identity matrices on the diagonal (see Example 2.14) and is easy to deal with. The nonderogatory case is yet to be defined, but for the moment we call an eigenvalue $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ *nonderogatory* if it is a nonderogatory eigenvalue for a commutative n-tuple $\mathbf{\Gamma}$. Later we will be able to define a nonderogatory eigenvalue completely in terms of the multiparameter system $\mathbf{W}$.

Example 2.14 Assume that a multiparameter system is diagonal with $V_{ii} = I$, for all i. Then we have $\Gamma_i = V_{i0}^\dagger$. Choose an eigenvalue $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ and suppose that

$\{x_{ik_i l};\ k_i \in \underline{p_i},\ l \in \underline{q_{k_i}}\}$ is a complete system of Jordan chains for V_{i0} at λ_i. Proposition 2.10 implies

$$\bigcap_{i=1}^{n} \ker (\lambda_i I - \Gamma_i)^N = \bigcap_{i=1}^{n} \ker ((\lambda_i I - V_{i0})^{n_i})^\dagger = \bigotimes_{i=1}^{n} \ker (\lambda_i I - V_{i0})^{n_i}$$

and therefore

$$\left\{x_{1k_1 l_{k_1}} \otimes x_{2k_2 l_{k_2}} \otimes \cdots \otimes x_{nk_n l_{k_n}};\ l_{k_i} \in \underline{q_{k_i}},\ k_i \in \underline{p_i},\ i \in \underline{n}\right\}$$

is a basis for $\bigcap_{i=1}^{n} \ker (\lambda_i I - \Gamma_i)^N$. Write $\mathbf{k} = (k_1, k_2, \ldots, k_n)$ and $\mathbf{l} = (l_1, l_2, \ldots, l_n)$. The action of $\lambda_i I - \Gamma_i$ on $z_{\mathbf{kl}} = x_{1k_1 l_1} \otimes x_{2k_2 l_2} \otimes \cdots \otimes x_{nk_n l_n}$ is $(\lambda_i I - \Gamma_i)\, z_{\mathbf{kl}} = z'_{\mathbf{kl}}$ where $\mathbf{l}' = (l_1, \ldots, l_{i-1}, l_i - 1, l_{i+1}, \ldots, l_n)$ and $z_{\mathbf{kl}'} = 0$ if $l_i = 1$. Each Γ_i has $\sum_{j=1,\, j\neq i}^{n} \sum_{k=1}^{p_j} q_k$ Jordan chains of lengths q_{k_i} , $k_i \in \underline{p_i}$ at eigenvalue λ_i. □

Before we state the next definition we recall that the subspace $\ker (\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^i$ for a set of commuting linear transformation $\boldsymbol{\Gamma}$ was defined on page 10.

Definition. Suppose that $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$. Then we call the subspace $\ker (\lambda\lambda\mathbf{I} - \boldsymbol{\Gamma})^N$ the *root subspace* (*of* $\mathbf{W}$ *at* $\boldsymbol{\lambda}$). We call the subspace $\ker (\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^m$ for $m = 2, 3, \ldots$ the *m-th root subspace* (*for the multiparameter system* $\mathbf{W}$ *at the eigenvalue* $\boldsymbol{\lambda}$) and for $m = 1$ the *eigenspace of* $\mathbf{W}$ *at* $\boldsymbol{\lambda}$.

A nonzero element of $\ker (\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^N$ is called a *root vector.* Also a nonzero element of the m-th root subspace is called an *m-th root vector* if $m \geq 1$ and a nonzero element of the eigenspace is called an *eigenvector.*

Note that a k-th root vector is also an l-th root vector if $k < l$.

2.5 A Basis for the Second Root Subspace

Our main objective in this dissertation is to describe a basis for the root subspace using the multiparameter system $\mathbf{W}$ directly, i.e., without using the tensor product constructions Δ_i etc. This is particularly important in infinite dimensions, but also for matrices it offers an advantage on dimensional grounds. In this section we describe a basis for the second root subspace.

2.5.1 Simple Case

First we consider the space $\ker(\boldsymbol{\lambda}\mathbf{I}-\boldsymbol{\Gamma})^2$ for eigenvalues that have one-dimensional eigenspaces, i.e., when $d_0 = \dim\ker(\boldsymbol{\lambda}\mathbf{I}-\boldsymbol{\Gamma}) = 1$. By Theorem 2.11 we have $d_0 = 1$ if and only if

$$\dim\ker \mathbf{W}_i(\boldsymbol{\lambda}) = 1 \tag{2.12}$$

for all i. Then we choose $x_{i0} \in \ker W_i(\boldsymbol{\lambda}) \setminus \{0\}$ and $y_{i0} \in \ker W_i(\boldsymbol{\lambda})^* \setminus \{0\}$, so $W_i(\boldsymbol{\lambda})x_{i0} = 0$ and $y_{i0}^* W_i(\boldsymbol{\lambda}) = 0$. The vectors $z_0 = x_{10} \otimes x_{20} \otimes \cdots \otimes x_{n0}$ and $w_0^* = y_{10}^* \otimes y_{20}^* \otimes \cdots \otimes y_{n0}^*$ are, respectively, right and left eigenvectors for all Γ_i. The following proposition gives some necessary and sufficient conditions for a root vector of $\boldsymbol{\Gamma}$, i.e. a vector $z_1 \in H\setminus\{0\}$ such that $(\lambda_i I - \Gamma_i) z_1 = a_i z_0$ for each i and not all $a_i \in \mathbb{C}$ are zero, to exist. As before (see page 69) we denote by H_i' a direct complement of the kernel of $W_i(\boldsymbol{\lambda})$ in H_i.

Proposition 2.15 *The following statements are equivalent :*

(i) There are $\mathbf{a} \in \mathbb{C}^n\setminus\{0\}$ *and* $x_{i1} \in H_i'$ *such that*

$$U_i(\mathbf{a})x_{i0} = W_i(\boldsymbol{\lambda})\, x_{i1} \tag{2.13}$$

for $i \in \underline{n}$.

(ii) There is $\mathbf{a} \in \mathbb{C}^n\setminus\{0\}$ *such that*

$$y_{i0}^* U_i(\mathbf{a})\, x_{i0} = 0 \tag{2.14}$$

for $i \in \underline{n}$.

(iii) $w_0^* \Delta_0 z_0 = 0$.

(iv) There exists an index i such that Γ_i has a root vector at λ_i corresponding to z_0. (That is there exists a vector $z_1 \in H\setminus\{0\}$ such that $(\lambda_i I - \Gamma_i) z_1 = z_0$.)

(v) $\ker(\boldsymbol{\lambda}\mathbf{I}-\boldsymbol{\Gamma}) \neq \ker(\boldsymbol{\lambda}\mathbf{I}-\boldsymbol{\Gamma})^2$.

(vi) There are $\mathbf{a} \in \mathbb{C}^n \setminus \{0\}$ *and* $y_{i1} \in H_i$ *such that* $y_{i0}^* U_i(\mathbf{a}) = y_{i1}^* W_i(\boldsymbol{\lambda})$ *for* $i \in \underline{n}$.

Proof. It is easy to observe that (i) implies (ii) and (vi) implies (ii).

Assume (ii). Then it follows $U_i(\mathbf{a})\, x_{i0} \in (\ker (W_i(\boldsymbol{\lambda})^*))^{\perp}$ and then by Lemma 2.12 there exists a vector $x_{i1} \in H_i'$ such that $U_i(\mathbf{a})\, x_{i0} = W_i(\boldsymbol{\lambda})\, x_{i1}$. This proves (i), and similarly (ii) implies (vi). So (i), (ii) and (vi) are equivalent.

The system of equations (2.14) has a nontrivial solution $\mathbf{a}$ if and only if the determinant of the system

$$\begin{vmatrix} y_{10}^* V_{11} x_{10} & y_{10}^* V_{12} x_{10} & \cdots & y_{10}^* V_{1n} x_{10} \\ y_{20}^* V_{21} x_{20} & y_{20}^* V_{22} x_{20} & \cdots & y_{20}^* V_{2n} x_{20} \\ \vdots & \vdots & & \vdots \\ y_{n0}^* V_{n1} x_{n0} & y_{n0}^* V_{n2} x_{n0} & \cdots & y_{n0}^* V_{nn} x_{n0} \end{vmatrix} = w_0^* \Delta_0 z_0$$

equals 0. Thus (ii) and (iii) are equivalent.

Suppose now that (i) holds. Write

$$z_1 = \sum_{i=1}^{n} x_{10} \otimes \cdots \otimes x_{i-1,0} \otimes x_{i1} \otimes x_{i+1,0} \otimes \cdots \otimes x_{n0}. \tag{2.15}$$

Using the properties established in Lemma 2.1 and assumption (i) we can make the following calculation :

$$(\lambda_i \Delta_0 - \Delta_i)\, z_1 = \begin{vmatrix} V_{11}^{\dagger} & \cdots & V_{1,i-1}^{\dagger} & W_1(\boldsymbol{\lambda})^{\dagger} & V_{1,i+1}^{\dagger} & \cdots & V_{1n}^{\dagger} \\ V_{21}^{\dagger} & \cdots & V_{2,i-1}^{\dagger} & W_2(\boldsymbol{\lambda})^{\dagger} & V_{2,i+1}^{\dagger} & \cdots & V_{2n}^{\dagger} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1}^{\dagger} & \cdots & V_{n,i-1}^{\dagger} & W_n(\boldsymbol{\lambda})^{\dagger} & V_{n,i+1}^{\dagger} & \cdots & V_{nn}^{\dagger} \end{vmatrix} z_1 =$$

$$\sum_{j=1}^{n} \begin{vmatrix} V_{11} x_{10} & \cdots & V_{1,i-1} x_{10} & 0 & V_{1,i+1} x_{10} & \cdots & V_{1n} x_{10} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{j-1,1} x_{j-1,0} & \cdots & V_{j-1,i-1} x_{j-1,0} & 0 & V_{j-1,i+1} x_{j-1,0} & \cdots & V_{j-1,n} x_{j-1,0} \\ V_{j1} x_{j1} & \cdots & V_{j,i-1} x_{j1} & U_j(\mathbf{a})\, x_{j0} & V_{j,i+1} x_{j1} & \cdots & V_{jn} x_{j1} \\ V_{j+1,1} x_{j+1,0} & \cdots & V_{j+1,i-1} x_{j+1,0} & 0 & V_{j+1,i+1} x_{j+1,0} & \cdots & V_{j+1,n} x_{j+1,0} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1} x_{n0} & \cdots & V_{n,i-1} x_{n0} & 0 & V_{n,i+1} x_{n0} & \cdots & V_{nn} x_{n0} \end{vmatrix}^{\otimes} . \tag{2.16}$$

The value of the determinants above remains the same if we replace the terms $V_{jk}x_{j1}$, $k = 1, \ldots, i-1, i+1, \ldots, n$, by $V_{jk}x_{j0}$. Then by Lemma 2.1 the sum (2.16) equals

$$\begin{vmatrix} V_{11}x_{10} & \cdots & V_{1,i-1}x_{10} & U_1(\mathbf{a})\,x_{10} & V_{1,i+1}x_{10} & \cdots & V_{1n}x_{10} \\ V_{21}x_{20} & \cdots & V_{2,i-1}x_{20} & U_2(\mathbf{a})\,x_{20} & V_{2,i+1}x_{20} & \cdots & V_{2n}x_{20} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n0}x_{n0} & \cdots & V_{n.i-1}x_{n0} & U_n(\mathbf{a})\,x_{n0} & V_{n,i+1}x_{n0} & \cdots & V_{nn}x_{n0} \end{vmatrix}^{\otimes} = a_i\Delta_0 z_0,$$

where $\mathbf{a} = [a_1, a_2, \ldots, a_n]^T$. Therefore $z_1 \in \ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^2 / \ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})$, i.e., (v) holds.

It is trivial that (v) implies (iv), so assume now that (iv) holds, that is

$$(\lambda_i\Delta_0 - \Delta_i)\, z_1 = \Delta_0 z_0.$$

We also have $w_0^*(\lambda_i\Delta_0 - \Delta_i) = 0$ and hence property (iii) follows since

$$0 = w_0^*(\lambda_i\Delta_0 - \Delta_i)\, z_1 = w_0^*\Delta_0 z_0.$$

The proof is complete because we have established the implications $(i) \Rightarrow (v) \Rightarrow (iv) \Rightarrow (iii)$ and the equivalence between (i), (ii), (iii) and (vi). $\square$

Remark. Suppose that the conditions of Proposition 2.15 hold. Because $\mathbf{a} \neq 0$ there is at least one index h such that $a_h \neq 0$. Then $(\lambda_h I - \Gamma_h)\, z_1 \neq 0$ and $(\lambda_h I - \Gamma_h)\, z_0 = 0$ and therefore the vectors z_1 from (2.15) and z_0 are linearly independent. This cannot be established when Δ_0 is singular as shown in the following example. Therefore the assumption that the multiparameter system $\mathbf{W}$ is nonsingular is essential for our discussion. $\square$

In the next example we identify the tensor product $x_1 \otimes x_2$ of two vectors $x_1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix} \in \mathbb{C}^2$ and $x_2 = \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix} \in \mathbb{C}^2$ with the vector $\begin{bmatrix} x_{11}x_{21} \\ x_{11}x_{22} \\ x_{12}x_{21} \\ x_{12}x_{22} \end{bmatrix}$, and similarly we identify the tensor product $V_{11} \otimes V_{22}$ of two matrices $V_{11}, V_{22} \in \mathbb{C}^{2\times 2}$, $V_{11} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$ with the matrix $\begin{bmatrix} v_{11}V_{22} & v_{12}V_{22} \\ v_{21}V_{22} & v_{22}V_{22} \end{bmatrix}$. Later we use this construction, that is sometimes called the *Kronecker product* of two vectors and two matrices, respectively, also for the tensor product of two vectors from arbitrary vector spaces $\mathbb{C}^n$ and $\mathbb{C}^m$ and for two arbitrary, $k \times l$ and $m \times n$, matrices.

Example 2.16 Suppose that

$$W_1(\boldsymbol{\lambda}) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \lambda_1 + \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \lambda_2 - \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$W_2(\boldsymbol{\lambda}) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \lambda_1 + \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \lambda_2 - \begin{bmatrix} 0 & -1 \\ 0 & -2 \end{bmatrix}.$$

Then

$$\Delta_0 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$

is a singular matrix and so $\mathbf{W}$ is a singular multiparameter system. Suppose that $\boldsymbol{\lambda}_0 = (1, 1)$. Then

$$W_1(\boldsymbol{\lambda}_0) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \text{ and } W_2(\boldsymbol{\lambda}_0) = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix},$$

and thus $\boldsymbol{\lambda}_0 \in \sigma(\mathbf{W})$. We choose

$$x_{10} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, y_{10} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, x_{20} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } y_{20} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

The pair of equations (2.14) for $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ reduces to a single equation $a_1 - a_2 = 0$. We choose $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The vectors $x_{11} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $x_{21} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ are such that $U_i(\mathrm{a})\, x_{i0} = W_i(\boldsymbol{\lambda})\, x_{i1}$ for $i = 1, 2$, and then the vectors $z_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $z_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ are linearly dependent, but $(\Delta_0 - \Delta_1)\, z_1 = (\Delta_0 - \Delta_2)\, z_1 = \Delta_0 z_0$. □

Suppose that $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ is such that the relations (2.12) hold. We restrict our attention to the root subspace $\mathcal{N} = \ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^N$ and we bring the restricted transformations $(\lambda_i I - \Gamma_i)|_{\mathcal{N}}$, that are commuting and nilpotent, to the form (1.2), i.e.,

$$(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})|_{\mathcal{N}} = \begin{bmatrix} \mathbf{0} & \mathbf{A}^{01} & \mathbf{A}^{02} & \cdots & \mathbf{A}^{0M} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}^{12} & \cdots & \mathbf{A}^{1M} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ddots & \mathbf{A}^{M-1,M} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix},$$

where the array $\mathbf{A}^{kl}$ has sizes $d_k \times d_l \times n$, and, by definition,

$$d_k = \dim\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^{k+1} - \dim\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^k \tag{2.17}$$

for $k \geq 1$ and $d_0 = \dim\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})$. Because we assumed (2.12) holds for $\boldsymbol{\lambda}$ it follows that $d_0 = 1$. We use the notation (1.3) for the arrays $\mathbf{A}^{kl}$ and the notation (1.4) for the n-tuples $\mathbf{a}_{ij}^{kl}$.

Next we write

$$B_0 = \begin{bmatrix} y_{10}^* V_{11} x_{10} & y_{10}^* V_{12} x_{10} & \cdots & y_{10}^* V_{1n} x_{10} \\ y_{20}^* V_{21} x_{20} & y_{20}^* V_{22} x_{20} & \cdots & y_{20}^* V_{2n} x_{20} \\ \vdots & \vdots & & \vdots \\ y_{n0}^* V_{n1} x_{n0} & y_{n0}^* V_{n2} x_{n0} & \cdots & y_{n0}^* V_{nn} x_{n0} \end{bmatrix}. \tag{2.18}$$

This $n \times n$ matrix will play an important role in the following proposition. We recall that the subspace $H_i^! \subset H_i$ is a direct complement of the kernel of $W_i(\boldsymbol{\lambda})$. It was introduced on page 69.

Proposition 2.17 *Suppose that $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ and that $\dim\ker W_i(\boldsymbol{\lambda}) = 1$ for $i = 1, 2, \ldots, n$. Then $d_1 = \dim\ker B_0$ and the set $\left\{\mathbf{a}_k^{01};\ k \in \underline{d_1}\right\}$ is a basis for $\ker B_0$. Furthermore there exist vectors $x_{i1}^k \in H_i^!$ such that*

(a) $U_i(\mathbf{a}_k^{01})\, x_{i0} = W_i(\boldsymbol{\lambda})\, x_{i1}^k$,

(b) *the vectors* $z_1^k = \sum_{i=1}^n x_{10} \otimes \cdots \otimes x_{i-1,0} \otimes x_{i1}^k \otimes x_{i+1,0} \otimes \cdots \otimes x_{n0}$, $k \in \underline{d_1}$ *together with the vector* z_0 *form a basis for* $\ker (\boldsymbol{\lambda} \mathbf{I} - \boldsymbol{\Gamma})^2$, *and*

(c) $(\lambda_i I - \Gamma_i)\, z_1^k = a_{ki}^{01} z_0$.

Proof. Write $\dim \ker B_0 = d$ and assume that $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_d \in \mathbb{C}^n$ form a basis for $\ker B_0$. So we have

$$\sum_{j=1}^n y_{i0}^* V_{ij} a_{kj} x_{i0} = 0, \qquad \text{for } i \in \underline{n} \text{ and } k \in \underline{d}.$$

Because the statements (*i*) and (*ii*) of Proposition 2.15 are equivalent we can find vectors $x_{i1}^k \in H_i'$ such that

$$W_i(\boldsymbol{\lambda})\, x_{i1}^k = U_i(\mathbf{a}_k)\, x_{i0} \tag{2.19}$$

for all i and k.

The same calculation as in the proof of Proposition 2.15 which showed that (*i*) implies (*v*) proves that

$$(\lambda_i I - \Gamma_i)\, z_1^k = a_{ki} z_0, \qquad \text{for } i \in \underline{n} \text{ and } k \in \underline{d}.$$

Here $z_1^k = \sum_{i=1}^n x_{10} \otimes \cdots \otimes x_{i-1,0} \otimes x_{i1}^k \otimes x_{i+1,0} \otimes \cdots \otimes x_{n0}$.

Let $\beta_0 z_0 + \sum_{k=1}^d \beta_k z_1^k = 0$. Then $0 = (\lambda_i I - \Gamma_i)\left(\beta_0 z_0 + \sum_{k=1}^d \beta_k z_1^k\right) = \sum_{k=1}^d a_{ki} \beta_k z_0$ implies $\beta_k = 0$ for $k \in \underline{d}$, and then $\beta_0 z_0 = 0$ implies $\beta_0 = 0$. So $\left\{z_0, z_1^1, \ldots, z_1^d\right\}$ are linearly independent, whence $d \leq d_1$ and we can assume that $\mathbf{a}_k^{01} = \mathbf{a}_k$ for $k \in \underline{d}$.

To complete the proof it suffices to prove the opposite inequality, i.e., $d \geq d_1$. We choose vectors $z_1^1, z_1^2, \ldots, z_1^{d_1}$ so that they form together with the vector z_0 a basis for $\ker (\boldsymbol{\lambda} \mathbf{I} - \boldsymbol{\Gamma})^2$. By Proposition 1.2 it follows that n-tuples

$$\mathbf{a}_k^{01} = \begin{bmatrix} a_{k1}^{01} & a_{k2}^{01} & \cdots & a_{kn}^{01} \end{bmatrix}^T,$$

$k \in \underline{d_1}$ are linearly independent. We also have

$$(\lambda_i I - \Gamma_i)\, z_1^k = a_{ki}^{01} z_0, \qquad \text{for } i \in \underline{n} \text{ and } k \in \underline{d_1}. \tag{2.20}$$

The relation (2.7) proved in Theorem 2.6 implies that

$$\sum_{j=1}^{n} V_{ij}^{\dagger}\Gamma_j z_1^k = V_{i0}^{\dagger} z_1^k$$

for all i and k. It follows then from (2.20) that

$$W_i(\boldsymbol{\lambda})^{\dagger} z_1^k = U_i\left(\mathrm{a}_k^{01}\right)^{\dagger} z_0. \tag{2.21}$$

Now we choose for every i an element $u_i^* \in H_i$ such that $u_i^* x_{i0} = 1$. This is possible because $x_{i0} \neq 0$. If we multiply the relation (2.21) by

$$u_1^* \otimes \cdots \otimes u_{i-1}^* \otimes y_{i0}^* \otimes u_{i+1}^* \otimes \cdots \otimes u_n^*$$

on the left-hand side it follows that $y_{i0}^* U_i\left(\mathrm{a}_k^{01}\right) x_{i0} = 0$ for all i and k. This proves that the n-tuples $\mathrm{a}_1^{01}, \mathrm{a}_2^{01}, \ldots, \mathrm{a}_{d_1}^{01} \in \ker B_0$. Since they are linearly independent it follows that $d \geq d_1$ as required. □

An immediate consequence of Proposition 2.17 is the next corollary.

Corollary 2.18 *An eigenvalue $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ is nonderogatory if and only if*

$$\dim\ker W_i(\boldsymbol{\lambda}) = 1 \qquad \text{for } i \in \underline{n}$$

and

$$\operatorname{rank} B_0 \geq n-1.$$

Proof. Theorem 2.11 implies that $d_0 = 1$ if and only if $\dim\ker W_i(\boldsymbol{\lambda}) = 1$ for all i and Proposition 2.17 implies that $d_1 \leq 1$ if and only if rank $B_0 \geq n-1$. From Corollary 1.7 it follows that an n-tuple $\boldsymbol{\lambda}$ is a nonderogatory eigenvalue for $\boldsymbol{\Gamma}$ if and only if $d_0 = 1$ and $d_1 \leq 1$. Hence the result is established. □

Using the result of Corollary 2.18 we are able to make the following definitions :

Definition. An n-tuple $\boldsymbol{\lambda} \in \mathbb{C}^n$ is called *a nonderogatory eigenvalue for a multiparameter system* $\mathbf{W}$ if $\dim\ker W_i(\boldsymbol{\lambda}) = 1$ for all i and

$$\operatorname{rank} B_0 \geq n-1. \tag{2.22}$$

The matrix B_0 is defined in (2.18).

Definition. An n-tuple $\boldsymbol{\lambda} \in \mathbb{C}^n$ is called *a simple eigenvalue for a multiparameter system* $\mathbf{W}$ if $\dim\ker W_i(\boldsymbol{\lambda}) = 1$ for all i and

$$\operatorname{rank} B_0 \leq n-2.$$

Note that an eigenvalue $\boldsymbol{\lambda}$ is simple eigenvalue for a multiparameter system $\mathbf{W}$ if and only if it is a simple eigenvalue for the associated system $\boldsymbol{\Gamma}$.

2.5.2 General Case

In this subsection we omit the assumption (2.12), i.e., the dimensions $q_i = \dim\ker W_i(\boldsymbol{\lambda})$ are now arbitrary. First we need an auxiliary result.

Lemma 2.19 *Let the vectors $x_{ij_i} \in H_i$, $j_i \in \underline{q_i}$, $i = 1,2,\ldots,k-1,k+1,\ldots,n$ be linearly independent and suppose that*

$$z = \sum_{\mathbf{j}'} x_{1j_1} \otimes \cdots \otimes x_{k-1,j_{k-1}} \otimes x_k^{\mathbf{j}'} \otimes x_{k+1,j_{k+1}} \otimes \cdots \otimes x_{nj_n} = 0$$

where $x_k^{\mathbf{j}'} \in H_k$ and the summation runs over all multiindices $\mathbf{j}' = (j_1,\ldots,j_{k-1}, j_{k+1},\ldots,j_n)$, $j_i \in \underline{q_i}$, $i = 1,2,\ldots,k-1,k+1,\ldots,n$. Then $x_k^{\mathbf{j}'} = 0$ for all $\mathbf{j}'$.

Proof. Suppose that x_{kj_k}, $j_k \in \underline{n_k}$ is a basis for H_k. Write $q_k = n_k$ and $x^{\mathbf{j}'} = \sum_{j_k=1}^{q_k} \alpha^{\mathbf{j}} x_{kj_k}$ for every $\mathbf{j}'$, where $\mathbf{j} = (j_1, j_2, \ldots, j_{k-1}, j_k, j_{k+1}, \ldots, j_n)$. Then

$$z = \sum_{\mathbf{j}} \alpha^{\mathbf{j}} x_{1j_1} \otimes \cdots \otimes x_{k-1,j_{k-1}} \otimes x_{kj_k} \otimes x_{k+1,j_{k+1}} \otimes \cdots \otimes x_{nj_n},$$

where summation runs over all multiindices $\mathbf{j} \in \underline{q_1} \times \underline{q_2} \times \cdots \times \underline{q_n}$, and since vectors $x_{1j_1} \otimes x_{2j_2} \otimes \cdots \otimes x_{nj_n}$, $j_i \in \underline{q_i}$, $i \in \underline{n}$ are linearly independent it follows that $\alpha^{\mathbf{j}} = 0$ for all $\mathbf{j}$. Thus it also follows that $x_k^{\mathbf{j}'} = 0$ for all $\mathbf{j}'$. $\square$

Let us introduce some notation. Assume that the vectors $x_{i0}^k \in H_i$, $k \in \underline{q_i}$ form a basis for the kernel $\ker W_i(\boldsymbol{\lambda})$, $i \in \underline{n}$. We define the set of integer n-tuples

$\mathbf{Q}_0 = \underline{q_1} \times \underline{q_2} \times \cdots \times \underline{q_n}$ and for $i \in \underline{n}$ we write $\mathbf{Q}_i = \underline{q_1} \times \cdots \times \underline{q_{i-1}} \times \underline{q_{i+1}} \times \cdots \times \underline{q_n}$. We call elements $\mathbf{k} = (k_1, k_2, \ldots, k_n) \in \mathbf{Q}_0$ and $\mathbf{l}' = (l_1, \ldots, l_{i-1}, l_{i+1}, \ldots, l_n) \in \mathbf{Q}_i$ multiindices. The notation $\mathbf{l}' \subset \mathbf{k}$ is used when $k_j = l_j$ for $j = 1, \ldots, i-1, i+1, \ldots, n$ and for $l \in \underline{q_i}$ we write $\mathbf{l}' \cup_i l = (l_1, \ldots, l_{i-1}, l, l_{i+1}, \ldots, l_n) \in \mathbf{Q}_0$. We introduce vectors $z_0^{\mathbf{k}} = x_{10}^{k_1} \otimes x_{20}^{k_2} \otimes \cdots \otimes x_{n0}^{k_n}$. Next we choose vectors $y_{i0}^{k_i} \in H_i$, $k_i \in \underline{q_i}$ so that they form a basis for $\ker W_i(\boldsymbol{\lambda})^*$ and write $X_{i0} = \begin{bmatrix} x_{i0}^1 & x_{i0}^2 & \cdots & x_{i0}^{q_i} \end{bmatrix}$ and $Y_{i0} = \begin{bmatrix} y_{i0}^1 & y_{i0}^2 & \cdots & y_{i0}^{q_i} \end{bmatrix}$. We restrict the transformations $(\lambda_i I - \Gamma_i)$ to the common spectral subspace $\mathcal{N} = \bigcap_{i=1}^n \ker (\lambda_i I - \Gamma_i)^2$. Then the transformations $\lambda_i I - \Gamma_i \mid_{\mathcal{N}}$ commute and are nilpotent. Hence we can choose a basis $\mathcal{B}$ for the subspace $\mathcal{N}$ as in Section 1.2, page 11. By Theorem 2.11 we can assume that $\mathcal{B}_0 = \left\{ z_0^{\mathbf{k}},\ \mathbf{k} \in \mathbf{Q}_0 \right\}$ and we reduce $\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma} \mid_{\mathcal{N}}$ to the form (1.2). Then we have $(\lambda_i I - \Gamma_i)\, z_1^l = \sum_{\mathbf{k} \in \mathbf{Q}_0} a_i^{\mathbf{k}l} z_0^{\mathbf{k}}$ for $i \in \underline{n}$ and $l \in \underline{d_1}$. We write $\mathbf{a}_i^l = \left(a_i^{\mathbf{k}l}\right)_{\mathbf{k} \in \mathbf{Q}_0}$ and we regard $\mathbf{a}_i^l$ as an element of H_λ, where $H_\lambda = \mathbb{C}^{q_1} \otimes \mathbb{C}^{q_2} \otimes \cdots \otimes \mathbb{C}^{q_n}$, and we regard $\mathbf{a}^l = \left(\mathbf{a}_1^l, \mathbf{a}_2^l, \ldots, \mathbf{a}_n^l\right)$ as an element of the n-tuple direct sum H_λ^n. Note that the n-tuple $\mathbf{a}^l$ corresponds to the column cross-section C_l^{01} of the array $\mathbf{A}^{01}$. See (1.6) for the definition of a column cross-section.

We also use the notation $V_{ij}^\lambda = Y_{i0}^* V_{ij} X_{i0} \in \mathbb{C}^{q_i \times q_i}$ for $i, j \in \underline{n}$. The matrix V_{ij}^λ induces a transformation $V_{ij}^{\lambda\dagger}$, which is defined by (1.1), on the tensor product space H_λ. Finally, the array

$$\mathcal{D}_0^\lambda = \begin{bmatrix} V_{11}^{\lambda\dagger} & V_{12}^{\lambda\dagger} & \cdots & V_{1n}^{\lambda\dagger} \\ V_{21}^{\lambda\dagger} & V_{22}^{\lambda\dagger} & \cdots & V_{2n}^{\lambda\dagger} \\ \vdots & \vdots & & \vdots \\ V_{n1}^{\lambda\dagger} & V_{n2}^{\lambda\dagger} & \cdots & V_{nn}^{\lambda\dagger} \end{bmatrix}$$

defines a transformation on space H_λ^n. This will play a very important role in the construction of the basis $\mathcal{B}_1$.

The following theorem describes the general form of a root vector in the second root subspace that is not an eigenvector, i.e., a vector

$$z \in \ker (\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^2 / \ker (\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma}).$$

Theorem 2.20 *A vector z is in* $\ker(\boldsymbol{\lambda}\mathbf{I}-\boldsymbol{\Gamma})^2/\ker(\boldsymbol{\lambda}\mathbf{I}-\boldsymbol{\Gamma})$ *if and only if there exist n-tuples $\mathbf{a}^{\mathbf{k}}\in\mathbb{C}^n$, not all 0, for $\mathbf{k}\in\mathbf{Q}_0$ and vectors $x_{i1}^{\mathbf{k}'}\in H_i$, $\mathbf{k}'\in\mathbf{Q}_i$, $i\in\underline{n}$ such that*

$$\sum_{k_i=1}^{q_i}U_i\left(\mathbf{a}^{\mathbf{k}'\cup_i k_i}\right)x_{i0}^{k_i}=W_i(\boldsymbol{\lambda})\,x_{i1}^{\mathbf{k}'},\tag{2.23}$$

and

$$z=\sum_{i=1}^{n}\sum_{\mathbf{k}'\in\mathbf{Q}_i}x_{10}^{k_1}\otimes\cdots\otimes x_{i-1,0}^{k_{i-1}}\otimes x_{i1}^{\mathbf{k}'}\otimes x_{i+1,0}^{k_{i+1}}\otimes\cdots\otimes x_{n0}^{k_n}.\tag{2.24}$$

It then follows for all i that

$$(\lambda_i I-\Gamma_i)\,z=\sum_{\mathbf{k}\in\mathbf{Q}_0}a_i^{\mathbf{k}}z_0^{\mathbf{k}}.\tag{2.25}$$

Proof. Suppose that z is in the form (2.24). Then the following direct calculation shows that (2.25) holds and since not all $\mathbf{a}^{\mathbf{k}}$ are 0 it then follows that $z\in\ker(\boldsymbol{\lambda}\mathbf{I}-\boldsymbol{\Gamma})^2/\ker(\boldsymbol{\lambda}\mathbf{I}-\boldsymbol{\Gamma})$. In the calculation we use the elementary properties of determinantal operators from Lemma 2.1 and relations (2.23), and proceed similarly as in the proof of Proposition 2.12, (i) implies (v) :

$$(\lambda_i\Delta_0-\Delta_i)\,z=\sum_{j=1}^{n}\sum_{\mathbf{k}'\in\mathbf{Q}_j}\begin{vmatrix}V_{11}x_{10}^{k_1} & \cdots & 0 & \cdots & V_{n1}x_{10}^{k_1}\\ \vdots & & \vdots & & \vdots\\ 0 & \cdots & \sum_{k_j=1}^{q_j}U_j\left(\mathbf{a}^{\mathbf{k}}\right)x_{j0}^{k_j} & \cdots & 0\\ \vdots & & \vdots & & \vdots\\ V_{n1}x_{n0}^{j_n} & \cdots & 0 & \cdots & V_{nn}x_{n0}^{j_n}\end{vmatrix}=$$

$$=\sum_{\mathbf{k}\in\mathbf{Q}_0}\begin{vmatrix}V_{11}x_{10}^{j_1} & \cdots & U_1\left(\mathbf{a}^{\mathbf{k}}\right)x_{10}^{j_1} & \cdots & V_{1n}x_{10}^{j_1}\\ V_{21}x_{20}^{j_2} & \cdots & U_2\left(\mathbf{a}^{\mathbf{k}}\right)x_{20}^{j_2} & \cdots & V_{2n}x_{20}^{j_2}\\ \vdots & & \vdots & & \vdots\\ V_{n1}x_{n0}^{j_n} & \cdots & U_n\left(\mathbf{a}^{\mathbf{k}}\right)x_{n0}^{j_n} & \cdots & V_{nn}x_{n0}^{j_n}\end{vmatrix}=\sum_{\mathbf{k}\in\mathbf{Q}_0}a_i^{\mathbf{k}}\Delta_0 z_0^{\mathbf{k}}.$$

Now assume that $z\in\ker(\boldsymbol{\lambda}\mathbf{I}-\boldsymbol{\Gamma})^2/\ker(\boldsymbol{\lambda}\mathbf{I}-\boldsymbol{\Gamma})$. Then we have (2.25) for some $a_i{}^{\mathbf{k}}\in\mathbb{C}$, $\mathbf{k}\in\mathbf{Q}_0$ and $i=1,2,\ldots,n$. We also write $\mathbf{a}^{\mathbf{k}}=\begin{bmatrix}a_1^{\mathbf{k}} & a_2^{\mathbf{k}} & \cdots & a_n^{\mathbf{k}}\end{bmatrix}^T\in\mathbb{C}^n$. The relation (2.7) then implies

$$W_i(\boldsymbol{\lambda})^\dagger z=\sum_{j=1}^{n}V_{ij}^\dagger(\lambda_i I-\Gamma_i)\,z=\sum_{j=1}^{n}V_{ij}^\dagger\sum_{\mathbf{k}\in\mathbf{Q}_0}a_j^{\mathbf{k}}z_0^{\mathbf{k}}=\sum_{\mathbf{k}\in\mathbf{Q}_0}U_i\left(\mathbf{a}^{\mathbf{k}}\right)^\dagger z_0^{\mathbf{k}}\tag{2.26}$$

for all i. After we multiply (2.26) with $Y_{i0}^{*\dagger}$ on the left-hand side it follows that

$$\sum_{\mathbf{k}\in\mathbf{Q}_0} x_{10}^{k_1}\otimes\cdots\otimes x_{i-1,0}^{k_{i-1}}\otimes\sum_{j=1}^{n} a_j^{\mathbf{k}} Y_{i0}^{*} V_{ij} x_{i0}^{k_i}\otimes x_{i+1,0}^{k_{i+1}}\otimes\cdots\otimes x_{n0}^{k_n}=0.$$

Since $x_{i0}^{k_i}$ are linearly independent it follows from Lemma 2.19 that

$$\sum_{k_i=1}^{q_i}\sum_{j=1}^{n} a_j^{\mathbf{k}'\cup_i k_i} Y_{i0}^{*} V_{ij} x_{i0}^{k_i}=0$$

for all $\mathbf{k}'\in\mathbf{Q}_i$ and every i. Thus we have

$$\sum_{k_i=1}^{q_i} U_i\left(\mathbf{a}^{\mathbf{k}'\cup_i k_i}\right) x_{i0}^{k_i}\in\left(\ker W_i(\boldsymbol{\lambda})^*\right)^{\perp}$$

and then, by Lemma 2.12, there exist vectors $x_{i1}^{\mathbf{k}'}\in H_i'$ such that relations (2.23) hold. Here the subspace $H_i'\subset H_i$ is a complement of the kernel $\ker W_i(\boldsymbol{\lambda})$ as defined on page 69. Now we can construct a vector

$$z'=\sum_{j=1}^{n}\sum_{\mathbf{k}'\in\mathbf{Q}_j} x_{10}^{k_1}\otimes\cdots\otimes x_{j-1,0}^{k_{j-1}}\otimes x_{j1}^{\mathbf{k}'}\otimes x_{j+1,0}^{k_{j+1}}\otimes\cdots\otimes x_{n0}^{j_n}. \tag{2.27}$$

The same calculation as above shows that $(\lambda_i I-\Gamma_i)\,z'=\sum_{\mathbf{k}\in\mathbf{Q}_0} a_i^{\mathbf{k}} z_0^{\mathbf{k}}$. Then we have $z-z'\in\ker(\boldsymbol{\lambda}\mathbf{I}-\boldsymbol{\Gamma})$ and thus there exist complex numbers $\beta_{\mathbf{k}}$, $\mathbf{k}\in\mathbf{Q}_0$ such that $z=z'+\sum_{\mathbf{k}\in\mathbf{Q}_0}\beta_{\mathbf{k}} z_0^{\mathbf{k}}$. If we substitute the vectors $x_{11}^{\mathbf{k}'}+\sum_{k_1=1}^{q_1}\beta_{\mathbf{k}'\cup k_1} x_{10}^{k_1}$ for the vectors $x_{11}^{\mathbf{k}'}$ in the expression (2.27) it follows that

$$z=\sum_{i=1}^{n}\sum_{\mathbf{k}'\in\mathbf{Q}_i} x_{10}^{k_1}\otimes\cdots\otimes x_{i-1,0}^{k_{i-1}}\otimes x_{i1}^{\mathbf{k}'}\otimes x_{i+1,0}^{k_{i+1}}\otimes\cdots\otimes x_{n0}^{k_n}$$

and, since (2.23) and (2.25) are unaffected by this substitution, the proof is complete. □

The following theorem extends Proposition 2.17 to the general case. As before we restrict our attention to the second root subspace $\mathcal{N}=\ker(\boldsymbol{\lambda}\mathbf{I}-\boldsymbol{\Gamma})^2$ and we bring the restricted transformations $(\lambda_i I-\Gamma_i)\,|_{\mathcal{N}}$, that are commuting and nilpotent, to the form (1.2). See also page 76 for details. We also recall that the $n\times\left(\prod_{i=1}^{n} q_i\right)$ column cross-section of the array $\mathbf{A}^{01}$ is regarded as an element of H_{λ}^{n}. (Cf. page 80.)

Theorem 2.21 *Suppose that* $\mathbf{a}^l \in H_\lambda^n$, $l = 1, 2, \dots, d_1$ *are the columns of the array* $\mathbf{A}^{01}$. *Then they form a basis of* $\ker \mathcal{D}_0^\lambda$.

Conversely, to any basis $\left\{\mathbf{a}^l,\ l = 1, 2, \dots, d_1\right\} \subset H_\lambda^n$ *of* $\ker \mathcal{D}_0^\lambda$ *we can associate a set of vectors* $\mathcal{B}' = \left\{z_1^l,\ l = 1, 2, \dots, d_1\right\} \subset H$ *such that* $\mathcal{B}_0 \cup \mathcal{B}'$ *is a basis for* $\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^2$ *and* $(\lambda_i I - \Gamma_i) z_1^l = \sum_{\mathbf{k}\in\mathbf{Q}_0} a_i^{\mathbf{k}l} z_0^{\mathbf{k}}$ *for all* i *and* l.

Proof. Suppose that $\left\{z_1^l,\ l = 1, 2, \dots, d_2\right\} \cup \left\{z_0^{\mathbf{k}},\ \mathbf{k} \in \mathbf{Q}_0\right\}$ is a basis for $\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^2$. Then we have $(\lambda_i I - \Gamma_i) z_1^l = \sum_{\mathbf{k}\in\mathbf{Q}_0} a_i^{\mathbf{k}l} z_0^{\mathbf{k}}$ and the relations (2.7) imply

$$W_i(\boldsymbol{\lambda})^\dagger z_1^l = \sum_{j=1}^n V_{ij}^\dagger (\lambda_j I - \Gamma_j) z_1^l = \sum_{j=1}^n V_{ij}^\dagger \sum_{\mathbf{k}\in\mathbf{Q}_0} a_i^{\mathbf{k}l} z_0^{\mathbf{k}} =$$

$$= \sum_{\mathbf{k}\in\mathbf{Q}_0} x_{10}^{k_1} \otimes \cdots \otimes x_{i-1,0}^{k_{i-1}} \otimes \sum_{j=1}^n a_i^{\mathbf{k}l} V_{ij} x_{i0}^{k_i} \otimes x_{i+1,0}^{k_{i+1}} \otimes \cdots \otimes x_{n0}^{k_n}.$$

After we multiply the above expression with $Y_{i0}^{*\dagger}$ on the left-hand side it follows that

$$\sum_{\mathbf{k}\in\mathbf{Q}_0} x_{10}^{k_1} \otimes \cdots \otimes x_{i-1,0}^{k_{i-1}} \otimes \sum_{j=1}^n a_i^{\mathbf{k}l} Y_{i0}^* V_{ij} x_{i0}^{k_i} \otimes x_{i+1,0}^{k_{i+1}} \otimes \cdots \otimes x_{n0}^{k_n} = 0.$$

Since $x_{i0}^{k_i}$ are linearly independent it follows by Lemma 2.19 that

$$\sum_{k_i=1}^{q_i} \sum_{j=1}^n a_i^{\mathbf{k}' \cup_i k_i, l} Y_{i0}^* V_{ij} x_{i0}^{k_i} = 0$$

for all $\mathbf{k}' \in \mathbf{Q}_i$ and all l. This can be written as

$$\sum_{j=1}^n V_{ij}^{\lambda\dagger} \mathbf{a}_j^l = 0.$$

Hence it follows that $\mathbf{a}^l \in \ker \mathcal{D}_0^\lambda$ for all l. Proposition 1.2 implies that $\mathbf{a}^l$ are linearly independent and so it follows that

$$d_1 \leq \dim \ker \mathcal{D}_0^\lambda. \tag{2.28}$$

Next we will show that to every $\mathbf{a} \in \ker \mathcal{D}_0^\lambda$ we can associate a vector $z_1 \in \ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^2 \setminus \ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})$ such that

$$(\lambda_i I - \Gamma_i) z_1 = \sum_{\mathbf{k}\in\mathbf{Q}_0} a_i^{\mathbf{k}} z_0^{\mathbf{k}}. \tag{2.29}$$

Because $\mathbf{a}$ is in the kernel of $\mathcal{D}_0^\lambda$ it follows that $\sum_{j=1}^n V_{ij}^{\lambda\dagger}\mathbf{a}_j = 0$ for all i. This is equivalent to

$$0 = Y_{i0}^* \sum_{k_i=1}^{q_i} \sum_{j=1}^{n} V_{ij} a_i^{\mathbf{k}'\cup_i k_i} x_{i0}^{k_i} = Y_{i0}^* \sum_{k_i=1}^{q_i} U_i \left(\mathbf{a}^{\mathbf{k}'\cup_i k_i}\right) x_{i0}^{k_i}$$

for all $\mathbf{k}' = (k_1, \ldots, k_{i-1}, k_{i+1}, \ldots, k_n) \in \mathbf{Q}_i$ and all i. From the above equations it follows that $\sum_{k_i=1}^{q_i} U_i \left(\mathbf{a}^{\mathbf{k}'\cup_i k_i}\right) x_{i0}^{k_i} \in (\ker W_i(\boldsymbol{\lambda})^*)^\perp$. Lemma 2.12 implies that there exist vectors $x_{i1}^{\mathbf{k}'} \in H_i'$ such that

$$\sum_{k_i=1}^{q_i} U_i \left(\mathbf{a}^{\mathbf{k}'\cup_i k_i}\right) x_{i0}^{k_i} = W_i(\boldsymbol{\lambda})\, x_{i1}^{\mathbf{k}'}.$$

As in the proof of Theorem 2.20 it follows that the vector

$$z_1 = \sum_{j=1}^{n} \sum_{\mathbf{k}'\in\mathbf{Q}_j} x_{10}^{k_1} \otimes \cdots \otimes x_{j-1,0}^{k_{j-1}} \otimes x_{j1}^{\mathbf{k}'} \otimes x_{j+1,0}^{k_{j+1}} \otimes \cdots \otimes x_{n0}^{j_n}$$

is such that relations (2.29) hold. Then, if $\left\{\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^d\right\}$ is a basis for $\ker \mathcal{D}_0^\lambda$, we can associate with every $\mathbf{a}^l$ a vector $z_1 = z_1^l$ as above. The vectors z_1^l, $l = 1, 2, \ldots, d$ are linearly independent because

$$(\lambda_i I - \Gamma_i)\, z_1^l = \sum_{\mathbf{k}\in\mathbf{Q}_0} a_i^{\mathbf{k}l} z_0^{\mathbf{k}}$$

and $\mathbf{a}^l$ are linearly independent. Thus it follows $d_1 \geq \dim\ker \mathcal{D}_0^\lambda$ and together with (2.28) we obtain $d_1 = \dim\ker \mathcal{D}_0^\lambda$. The proof is complete. □

We illustrate the theorem with an example.

Example 2.22 Consider the two-parameter system

$$W_1(\boldsymbol{\lambda}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \lambda_1 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix} \lambda_2 - \begin{bmatrix} 0 & -1 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$W_2(\boldsymbol{\lambda}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda_1 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \lambda_2 - \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

Evidently the matrices V_{10} and V_{20} are singular. So $\boldsymbol{\lambda}_0 = (0,0) \in \sigma(\mathbf{W})$ and we have $\dim\ker V_{10} = 1$ and $\dim\ker V_{20} = 2$. Hence $d_0 = 2$. We choose

$$x_{10} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ Y_{10} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ x_{20}^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ x_{20}^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } Y_{20} = \begin{bmatrix} 2 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then it follows that vectors

$$z_0^{11} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } z_0^{12} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

form a basis for $\ker \boldsymbol{\Gamma}$ and we have

$$V_{11}^{\lambda_0} = V_{12}^{\lambda_0} = [0], \ V_{21}^{\lambda_0} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \text{ and } V_{22}^{\lambda_0} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

The space $H_{\lambda_0} = \mathbb{C} \otimes \mathbb{C}^2 \equiv \mathbb{C}^2$ and we identify the direct sum $H_{\lambda_0} \oplus H_{\lambda_0}$ with $\mathbb{C}^4$. Then

$$\mathcal{D}_0^{\lambda_0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

Because the matrix $\mathcal{D}_0^{\lambda_0}$ has rank 1 it follows that $d_1 = 3$ and we choose $\mathbf{a}^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -2 \end{bmatrix}$, $\mathbf{a}^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{a}^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ to form a basis for $\ker \mathcal{D}_0^{\lambda}$. To construct a vector z_1^1 corresponding to $\mathbf{a}^1$ we need to find vectors x_{11}^{11}, x_{11}^{21} and x_{21}^{11} such that

$$V_{11}x_{10} = W_1(\boldsymbol{\lambda}_0)\, x_{11}^{11},\ -2V_{12}x_{10} = W_1(\boldsymbol{\lambda}_0)\, x_{11}^{21} \text{ and } V_{21}x_{20}^1 - 2V_{22}x_{20}^2 = W_2(\boldsymbol{\lambda}_0)\, x_{21}^{11}.$$

A possible choice is

$$x_{11}^{11} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},\ x_{11}^{21} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} \text{ and } x_{21}^{11} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Similarly we find vectors

$$x_{11}^{12} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},\ x_{11}^{22} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } x_{21}^{12} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

that correspond to $\mathbf{a}^2$, and vectors

$$x_{11}^{13} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ x_{11}^{23} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \text{ and } x_{21}^{13} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

that correspond to $\mathbf{a}^3$. Then

$$z_1^1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -2 \\ 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \quad z_1^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and } z_1^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix},$$

and $\{z_0^1, z_0^2;\ z_1^1, z_1^2, z_1^3\}$ is a basis for $\ker \boldsymbol{\Gamma}^2$. □

2.6 Comments

Multiparameter Spectral Theory has its origins in the work of Klein [117], Bôcher [32, 33, 34], Dixon [59] and Hilb [104, 105] late in the nineteenth century. Also Hilbert [106] and his students considered problems in Multiparameter Spectral Theory. When solving certain boundary value problems by the separation of variables technique we are led to a system of differential equations that are linked only by spectral parameters and this linkage is linear. This is the underlying motivation for many of the developments in Multiparameter Spectral Theory. Some examples of such boundary value problems are the classical ones of acoustic or electromagnetic vibrations and different linearised parts of various bifurcation models involving several parameters as in rotation, convection or explosion. The problem of oscillation of an elliptic membrane is an example that yields two separate differential equations that both contain two spectral variables in a nontrivial fashion, while for example the problem of oscillations for a rectangular membrane leads to a diagonal case and for a circular membrane leads to an upper-triangular (also called mildly coupled) case. The last two cases can be solved using only techniques from the one-parameter case.

The example of the elliptic membrane and similar situations led to studies of special functions at the beginning of this century. Erdélyi gathered such results in [64]. (See also the book by Arscott [6].)

Carmichael [50, 51, 52] was the first to consider multiparameter eigenvalue problems in an abstract setting. He studied, for example in [50], a finite-dimensional multiparameter system generated by a difference equation approximating a system of integral equations. It was Atkinson [10, 8] who laid the foundations of modern Abstract Multiparameter Spectral Theory which led to a revival of the theory in the last 30 years. In the 70s Multiparameter Spectral Theory in an abstract Hilbert space was developed by Binding, Browne, Faierman, Källström, Roach, and Sleeman, to mention a few, in a number of contributions (see Browne's review article [42] and also the enclosed list of references for details). Many of these were brought together in the book by Sleeman [153]. Work on extending multiparameter eigenfunction expansion theorems in a number of directions and under various "definiteness conditions" has been done recently by Binding, Faierman, Gadzhiev, Isaev, Roach, Volkmer and others. Also the recent books of Volkmer [168], Gadzhiev [84] and Faierman [69] present several results on eigenfunction expansion. We also remark that most of research so far involved self-adjoint multiparameter eigenvalue problems. As an exception we mention the paper of Allakhverdiev and Dzhabarzade [2] where they considered a normal multiparameter system, i.e., a system where all the operators V_{ij} are normal operators.

The fundamental tensor space construction that we introduce in this chapter was given by Atkinson in [10, Chapter 6]. In our discussion we partly follow also the presentation of Isaev [112, Lecture 1]. For instance, the idea to use the decomposability set to prove commutativity of the associated system and relation (2.7), is found in [112] (cf. also [4]) where it is used in the infinite-dimensional setting. The notions of spectra, eigenvectors and root vectors for multiparameter systems are defined to correspond to the equivalent notions for the associated system. The corresponding notion of Taylor's spectrum, introduced by Taylor in [161] for an n-tuple of commuting operators, was defined for multiparameter systems by Isaev and Fainstein [111] and studied by Rynne [147]. See also Isaev's Lecture 5 in [112]. In the finite-dimensional setting the notion of Taylor's spectrum for a multiparameter system coincides with the spectrum as defined in Section 2.4.

The linear transformations associated with the square arrays of operators, for instance our transformations $\mathcal{D}$ and $\mathcal{D}_0^\lambda$, are an important tool in the presentation. They were studied already by Atkinson in [10, Chapter 8]. He proved that if there is a nonzero element in the kernel of such a transformation then there is a decomposable element in that kernel (cf. [10, Theorem 8.5.1]). This enabled him to weaken the regularity condition and still prove the expansion result ([10, Theorem 10.6.1]). An interesting related investigation is found in paper of Allakhverdiev and Dzhabarzade [1]. They discuss relations between vectors $V_{ij}x_i$, $j = 1, 2, \ldots, k$, where $i = 1, 2, \ldots, n$ and $k, n \geq 2$, for which $\sum_{j=1}^{k} V_{1j}x_1 \otimes V_{2j}x_2 \otimes \cdots \otimes V_{nj}x_n = 0$.

The structure of the second root vectors in the simple case (cf. Subsection 2.5.1) follows the one of root vectors in Binding's paper [23]. In the general case the transformation $\mathcal{D}_0^\lambda$ carries information about the second root subspace. This will be examined in detail for two-parameter systems in the next chapter.

Chapter 3

Two-parameter Systems

3.1 Introduction

In this chapter we use a matrix equation of Sylvester type to study two-parameter systems

$$W_i(\boldsymbol{\lambda}) = V_{i1}\lambda_1 + V_{i2}\lambda_2 - V_{i0}, \ i = 1, 2. \tag{3.1}$$

First we briefly describe our main ideas.

We identify the tensor product space $\mathbb{C}^{q_1} \otimes \mathbb{C}^{q_2}$ with the space of $q_1 \times q_2$ complex matrices via the isomorphism $\Xi : \mathbb{C}^{q_1} \otimes \mathbb{C}^{q_2} \longrightarrow \mathbb{C}^{q_1 \times q_2}$ defined by

$$\Xi : \begin{bmatrix} a_{11} \\ \vdots \\ a_{q_1 1} \\ a_{12} \\ \vdots \\ a_{q_1 2} \\ \vdots \\ a_{1q_2} \\ \vdots \\ a_{q_1 q_2} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1q_2} \\ a_{21} & a_{22} & \cdots & a_{2q_2} \\ \vdots & \vdots & & \vdots \\ a_{q_1 1} & a_{q_1 2} & \cdots & a_{q_1 q_2} \end{bmatrix}. \tag{3.2}$$

It was shown in Section 2.5, page 80 that we can associate a transformation $\mathcal{D}_0^\lambda$ with an eigenvalue $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$. In the two-parameter case we view

$$\mathcal{D}_0^\lambda = \begin{bmatrix} V_{11}^{\lambda\dagger} & V_{12}^{\lambda\dagger} \\ V_{21}^{\lambda\dagger} & V_{22}^{\lambda\dagger} \end{bmatrix},$$

via the isomorphism Ξ, as a transformation acting on the vector space of pairs of $q_1 \times q_2$ matrices. We also consider the determinantal transformation

$$\Delta_0^\lambda = V_{11}^\lambda \otimes V_{22}^\lambda - V_{12}^\lambda \otimes V_{21}^\lambda$$

as a transformation on the vector space of complex $q_1 \times q_2$ matrices. There is a close relation between the kernels of $\mathcal{D}_0^\lambda$ and of Δ_0^λ as shown in Lemma 2.4. We restate it here in the above setting.

Corollary 3.1 *Suppose that* $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, $X_1, X_2 \in \mathbb{C}^{q_1 \times q_2}$, *is an element of the kernel* $\ker \mathcal{D}_0^\lambda$. *Then both* X_1 *and* X_2 *are in the kernel* $\ker \Delta_0^\lambda$.

In Theorem 2.21 we showed how to associate a basis for the second root subspace with a basis for the kernel of $\mathcal{D}_0^\lambda$, the above result relates this kernel to the kernel of Δ_0^λ, and a matrix $X \in \ker \Delta_0^\lambda$ if and only if $V_{11}^\lambda X V_{22}^\lambda - V_{12}^\lambda X V_{21}^\lambda = 0$. This is our motivation to study the matrix equation

$$AXD^T - BXC^T = 0. \tag{3.3}$$

To do so we use the Kronecker canonical forms for pairs of matrices (A, B) and (C, D). We describe this special block diagonal form for a pair of matrices in the next section. With every block in the Kronecker canonical form of a pair (A, B) we associate an invariant and a chain of vectors called a Kronecker chain. The invariants are of three different types. So, when we study equation (3.3) we would have to consider nine different cases, but because of symmetry with respect to the pairs (A, B) and (C, D) we only need to study six different cases. For any of these cases where there are nontrivial solutions of equation (3.3) we give a basis for the subspace of solutions in terms of underlying Kronecker chains in Subsection 3.2.2. Similarly we study the space of solutions of a pair of equations

$$AX_1 + BX_2 = 0 \text{ and } X_1C^T + X_2D^T = 0 \tag{3.4}$$

in Subsection 3.2.3. Corollary 3.1 is used to relate this system to equation (3.3). With every pair of invariants of (A, B) and (C, D) for which there is a nontrivial solution of the system (3.4) we associate another invariant. We show in Section 3.3 that when a set of invariants is associated this way to pairs of matrices $\left(V_{11}^{\lambda}, V_{12}^{\lambda}\right)$ and $\left(V_{21}^{\lambda}, V_{22}^{\lambda}\right)$ it is equal to the set of invariants of the pair of matrices $\left(\widehat{A}_1, \widehat{A}_2\right)$. Here a pair of commuting nilpotent matrices $A_i = (\lambda_i I - \Gamma_i)\,|_{\ker(\boldsymbol{\lambda}\mathbf{I}-\boldsymbol{\Gamma})^2}$, $i = 1, 2$ is brought to the form (1.2) and the matrices $\widehat{A}_1$ and $\widehat{A}_2$ form a subarray $\widehat{\mathbf{A}}^{01}$ of the array $\mathbf{A}^{01}$ as described in Example 3.3. We also construct a basis for $\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^2$ such that the pair of matrices (A_1^{01}, A_2^{01}) is in Kronecker canonical form and we illustrate the construction with two examples.

3.2 Kronecker Canonical Form and a Special Basis for the Space of Solutions of the Matrix Equation $AXD^T - BXC^T = 0$

3.2.1 Kronecker Canonical Form

We refer to [85, Chapter XII] or [92, Appendix] for recent presentations of the Kronecker canonical form. Our presentation is based on a disposition by Professor H.K. Farahat in a private conversation.

Definition. A pair of complex $m \times n$ matrices (A, B) is *equivalent* to a pair of matrices (C, D) if there exist invertible matrices $P \in \mathbb{C}^{n\times n}$ and $Q \in \mathbb{C}^{m\times m}$ such that

$$C = PAQ \text{ and } D = PBQ. \tag{3.5}$$

First we introduce some special matrices needed in the construction of the

Kronecker canonical form. The $p \times p$ identity matrix is denoted by I_p. The $q \times q$ *Jordan matrix with eigenvalue* α is

$$J_q(\alpha) = \begin{bmatrix} \alpha & 1 & 0 & \cdots & 0 \\ 0 & \alpha & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \alpha \end{bmatrix},$$

where we shall omit α if $\alpha = 0$, and the matrices

$$F_p = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \text{ and } G_p = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

are $p \times (p+1)$ matrices. Here $p, q \geq 1$. Later in the discussion we also use the $p \times p$ matrix

$$H_p = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & \cdot & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}$$

and $p \times q$ matrices $I_{p,q}$ and $H_{p,q}$ that are defined by

$$I_{p,q} = \begin{bmatrix} 0 & I_p \end{bmatrix} \text{ and } H_{p,q} = \begin{bmatrix} 0 & H_p \end{bmatrix}$$

if $p < q$ and

$$I_{p,q} = \begin{bmatrix} I_p \\ 0 \end{bmatrix} \text{ and } H_{p,q} = \begin{bmatrix} H_p \\ 0 \end{bmatrix}$$

if $p > q$. We write $I_{p,p} = I_p$ and $H_{p,p} = H_p$.

The pairs of building blocks of the *Kronecker canonical form* for a pair of matrices are of three different types : (L, p), (M, p) and $(J(\alpha), q)$ where $p \geq 0$, $q \geq 1$ and $\alpha \in \mathbb{C} \cup \{\infty\}$. The building blocks of type (L, p) are of sizes $p \times (p+1)$, the building blocks of type (M, p) are of sizes $(p+1) \times p$ and the building blocks of type

$(J(\alpha), q)$ are of sizes $q \times q$. Here the blocks of types $(L, 0)$ and $(M, 0)$ which are of 'sizes' 0×1 and 1×0, respectively, correspond to a column of 0's and a row of 0's, respectively, in the Kronecker canonical form. Suppose that $p \geq 1$. Then the pairs of building blocks and the corresponding types are :

$$(F_p,\ G_p)\,,\ \text{type}\ (L, p)\,,$$

$$\left(G_p^T,\ F_p^T\right),\ \text{type}\ (M, p)\,,$$

$$(I,\ J_p(\alpha))\,,\ \text{type}\ (J(\alpha), p)$$

if $\alpha \in \mathbb{C}$, and

$$(J_p,\ I)\,,\ \text{type}\ (J(\infty), p)\,.$$

The theorem of Kronecker (cf. [85, p. 37] or [92, Theorem A.7.3]) states that every pair of $m \times n$ complex matrices (A, B) is equivalent to a pair of matrices in block diagonal form with diagonal blocks of types (L, p), (M, p) and $(J(\alpha), q)$. We call this block diagonal form the *Kronecker canonical form of a pair* (A, B). We call the collection

$$\mathcal{I} = \{(L, l_1), \ldots, (L, l_{p_L}); (M, m_1), \ldots, (M, m_{p_M}); (J(\alpha_1), j_1), \ldots, (J(\alpha_{p_J}), j_{p_J});\}$$

of the types of the diagonal blocks the *set of invariants* of a pair (A, B). The elements of the set $\mathcal{I}$ are called the *invariants.* It is a consequence of the theorem of Kronecker that two pairs of $m \times n$ matrices (A, B) and (C, D) are equivalent if and only if they have the same sets of invariants. See [85, Theorem 5, p. 40] or [92, Corollary A.7.4]. Note that in our discussion we view the initial $u \times v$ block of zeros in [92, Theorem A.7.3] (in [85, expression (34), p.39] this is the initial $h \times g$ block of zeros) as a collection of u blocks of type $(L, 0)$ and v blocks of type $(M, 0)$. This enables us to absorb the initial block of zeros into the blocks of types (L, p) and (M, p).

Example 3.2 The pair of matrices

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \end{bmatrix}\right)$$

has the set of invariants $\mathcal{I} = \{(L,0), (L,1), (M,1), (J(2), 2)\}$ and the pair of matrices

$$\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right)$$

has the set of invariants $\mathcal{I} = \{(L,0), (M,0), (M,0), (J(0), 2), (J(\infty), 1)\}$. □

Example 3.3 Suppose that A_1 and A_2 are commuting nilpotent $N \times N$ matrices, that they are brought to the form (1.2) and furthermore, we have $\ker \mathbf{A}^2 = \mathbb{C}^N$. (Here we use the notion introduced in Section 1.2.) We can further assume that the row-cross sections R_j^{01}, $j \in \underline{r_0}$ are linearly independent and $R_j^{01} = 0$, $j = r_0 + 1, r_0 + 2, \ldots, d_0$, where r_0 is the dimension of the subspace of $2 \times d_1$ matrices spanned by R_j^{01}, $j \in \underline{d_0}$. Then we write

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \widehat{\mathbf{A}}^{01} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

where $\widehat{\mathbf{A}}^{01}$ has dimensions $r_0 \times d_1 \times 2$, and $\widehat{\mathbf{A}}^{01} = \left(\widehat{A}_1^{01}, \widehat{A}_2^{01}\right)$. Suppose that $\left(\widehat{B}_1^{01}, \widehat{B}_2^{01}\right)$ is the Kronecker canonical form of the pair $\left(\widehat{A}_1^{01}, \widehat{A}_2^{01}\right)$ and that matrices P and Q are such that

$$\widehat{B}_1^{01} = P\widehat{A}_1^{01}Q \text{ and } \widehat{B}_2^{01} = P\widehat{A}_2^{01}Q.$$

Then the array $UAU^{-1} = \begin{bmatrix} \mathbf{0} & \widehat{\mathbf{B}}^{01} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, where $U = \begin{bmatrix} P & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q^{-1} \end{bmatrix}$ and $\widehat{\mathbf{B}}^{01} =$

$\left(\widehat{B}_1^{01}, \widehat{B}_2^{01}\right)$, is a canonical form for the pair of commuting matrices A_1 and A_2. We remark that for similar, and also for more general, pairs of commuting matrices a canonical form is given by Gel'fand and Ponomarev in [86, Chapter II]. □

Next we introduce the notion of Kronecker basis for a pair of matrices (A, B). With every invariant in $\iota \in \mathcal{I}$ we associate a *Kronecker chain* $\mathcal{C}_\iota$ of linearly independent vectors as follows :
If $\iota = (L, p)$ then $\mathcal{C}_\iota = \left\{u_i,\ i \in \underline{p+1}\right\}$ and

$$\begin{aligned} Bu_1 &= 0, \\ Bu_i &= Au_{i-1}, \quad i = 2, 3, \ldots, p+1, \\ 0 &= Au_{p+1}. \end{aligned}$$

If $\iota = (M, 0)$ then $\mathcal{C}_\iota = \emptyset$ and if $\iota = (M, p)$, $p \geq 1$ then $\mathcal{C}_\iota = \left\{u_i,\ i \in \underline{p}\right\}$ and

$$Bu_i = Au_{i-1}, \quad i = 2, 3, \ldots, p.$$

If $\iota = (J(\alpha), p)$, $\alpha \in \mathbb{C}$, then $\mathcal{C}_\iota = \left\{u_i,\ i \in \underline{p}\right\}$ and

$$\begin{aligned} (\alpha A - B)\, u_1 &= 0, \\ (\alpha A - B)\, u_i &= Au_{i-1}, \quad i = 2, 3, \ldots, p. \end{aligned}$$

And finally, if $\iota = (J(\infty), p)$ then $\mathcal{C}_\iota = \left\{u_i,\ i \in \underline{p}\right\}$ and

$$\begin{aligned} Au_1 &= 0, \\ Au_i &= Bu_{i-1}, \quad i = 2, 3, \ldots, p. \end{aligned}$$

The union of all Kronecker chains of a pair of matrices (A, B) is called a *Kronecker basis* of (A, B).

Remark. Note that if (C, D) is the Kronecker canonical form of a pair of matrices (A, B), and the matrices P and Q are such that relation (3.5) holds, then the columns of the matrix Q^{-1} form a Kronecker basis of (A, B). They are partitioned into Kronecker chains according to the sizes of diagonal blocks of the canonical form (C, D). Note also that if $m = n$ and $A = I$ then the notions of Kronecker canonical form and Kronecker basis coincide with the usual definition of Jordan canonical form of a matrix B and its Jordan basis. □

Suppose that $\mathcal{C}_\iota = \left\{u_i,\ i \in \underline{p+1}\right\}$ where $\iota = (L, p)$ and $\alpha \in \mathbb{C} \cup \{\infty\}$. Then we define vectors $u_i(\alpha)$ so that

$$\sum_{i=0}^{p} \lambda^i u_{i+1}(\alpha) = \sum_{i=0}^{p} (\lambda - \alpha)^i u_{i+1},$$

if $\alpha \in \mathbb{C}$, and

$$\sum_{i=0}^{p} \lambda^i u_{i+1}(\infty) = \sum_{i=0}^{p} \lambda^{p-i} u_{i+1},$$

where λ is an indeterminate. Then we call a chain $\mathcal{C}_\iota(\alpha) = \left\{u_i(\alpha),\ i \in \underline{p+1}\right\}$ the *α-shift of a Kronecker chain $\mathcal{C}_\iota$*. Note that the chains $\mathcal{C}_\iota$ and $\mathcal{C}_\iota(\alpha)$ span the same subspace.

3.2.2 The Matrix Equation $AXD^T - BXC^T = 0$

Next we consider the homogeneous matrix equation (3.3) where A and B are $m_1 \times m_2$ matrices, C and D are $n_1 \times n_2$ matrices and X is the unknown $m_2 \times n_2$ matrix. We define the transformation Λ: $\mathbb{C}^{m_2 \times n_2} \longrightarrow \mathbb{C}^{m_1 \times n_1}$ by

$$\Lambda(X) = AXD^T - BXC^T. \tag{3.6}$$

Then the kernel of Λ is the space of solutions of (3.3). Suppose that $\mathcal{I}_1$ and $\mathcal{I}_2$ are the sets of invariants of the pairs (A, B) and (C, D), respectively, and $\mathcal{C}_1$ and $\mathcal{C}_2$ their corresponding Kronecker bases.

An approach using the Kronecker canonical form to study the matrix equation

$$AXD^T - BXC^T = E \tag{3.7}$$

was outlined by Rózsa in [144]. We include the following detailed discussion on the matrix equation (3.3) because we later need precise expressions for the solutions of the homogeneous equation (3.3). We sketch the proofs using our setting and following [144].

Suppose that $\iota_1 \in \mathcal{I}_1$ and $\iota_2 \in \mathcal{I}_2$ and that $\mathcal{C}_{1\iota_1}$ and $\mathcal{C}_{2\iota_2}$ are the corresponding Kronecker chains. Now we define a set $\mathcal{J}$ of pairs of invariants $(\iota_1, \iota_2) \in \mathcal{I}_1 \times \mathcal{I}_2$. A pair (ι_1, ι_2) is in the set $\mathcal{J}$ if one of the following holds :

(i) $\iota_1 = (L, p_1)$ and $\iota_2 = (L, p_2)$,

(iia) $\iota_1 = (L, p_1)$, $\iota_2 = (M, p_2)$ and $p_1 < p_2$,

(iib) $\iota_1 = (M, p_1)$, $\iota_2 = (L, p_2)$ and $p_1 > p_2$,

(iiia) $\iota_1 = (L, p_1)$ and $\iota_2 = (J(\alpha), p_2)$,

(iiib) $\iota_1 = (J(\alpha), p_1)$ and $\iota_2 = (L, p_2)$ and

(iiic) $\iota_1 = (J(\alpha), p_1)$ and $\iota_2 = (J(\alpha), p_2)$.

Then we associate with a pair of invariants $(\iota_1, \iota_2) \in \mathcal{J}$ a set $\mathcal{A}_{(\iota_1,\iota_2)}$ of matrices as follows :

(i) if $\iota_1 = (L, p_1)$, $\iota_2 = (L, p_2)$, $\mathcal{C}_{1\iota_1} = \left\{u_{1i},\ i \in \underline{p_1+1}\right\}$ and $\mathcal{C}_{2\iota_2} = \left\{u_{2i},\ i \in \underline{p_2+1}\right\}$ then

$$\mathcal{A}_{(\iota_1,\iota_2)} = \left\{A_l;\ A_l = \sum_{i_1+i_2=l+1} u_{1i_1} u_{2i_2}^T,\ l \in \underline{p_1+p_2+1}\right\};$$

(iia) if $\iota_1 = (L, p_1)$, $\iota_2 = (M, p_2)$, where $p_1 < p_2$, $\mathcal{C}_{1\iota_1} = \left\{u_{1i},\ i \in \underline{p_1+1}\right\}$ and $\mathcal{C}_{2\iota_2} = \left\{u_{2i},\ i \in \underline{p_2}\right\}$ then

$$\mathcal{A}_{(\iota_1,\iota_2)} = \left\{A_l;\ A_l = \sum_{i=0}^{p_1} u_{1,p_1+1-i} u_{2,i+l}^T,\ l \in \underline{p_2-p_1}\right\};$$

(iib) if $\iota_1 = (M, p_1)$, $\iota_2 = (L, p_2)$, where $p_1 > p_2$, $\mathcal{C}_{1\iota_1} = \left\{u_{1i},\ i \in \underline{p_1}\right\}$ and $\mathcal{C}_{2\iota_2} = \left\{u_{2i},\ i \in \underline{p_2+1}\right\}$ then

$$\mathcal{A}_{(\iota_1,\iota_2)} = \left\{A_l;\ A_l = \sum_{i=0}^{p_2} u_{1,i+l} u_{2,p_2+1-i}^T,\ l \in \underline{p_1-p_2}\right\};$$

(iiia) if $\iota_1 = (L, p_1)$, $\iota_2 = (J(\alpha), p_2)$, where $\alpha \in \mathbb{C} \cup \{\infty\}$, $\mathcal{C}_{1\iota_1}(\alpha) = \{u_{1i}(\alpha), \ i \in \underline{p_1 + 1}\}$ is the α-shift of the Kronecker chain $\mathcal{C}_{1\iota_1}$and $\mathcal{C}_{2\iota_2} = \{u_{2i}, \ i \in \underline{p_2}\}$ then

$$\mathcal{A}_{(\iota_1,\iota_2)} = \left\{ A_l;\ A_l = \sum_{i_1+i_2=l+1} u_{1i_1}(\alpha)\, u_{2i_2}^T,\ l \in \underline{p_2} \right\};$$

(iiib) if $\iota_1 = (J(\alpha), p_1)$, where $\alpha \in \mathbb{C} \cup \{\infty\}$, $\iota_2 = (L, p_2)$, $\mathcal{C}_{1\iota_1} = \{u_{1i}, \ i \in \underline{p_1}\}$ and $\mathcal{C}_{2\iota_2}(\alpha) = \{u_{2i}(\alpha), \ i \in \underline{p_2 + 1}\}$ is the α-shift of the Kronecker chain $\mathcal{C}_{2\iota_2}$ then

$$\mathcal{A}_{(\iota_1,\iota_2)} = \left\{ A_l;\ A_l = \sum_{i_1+i_2=l+1} u_{1i_1} u_{2i_2}(\alpha)^T,\ l \in \underline{p_1} \right\};$$

(iiic) if $\iota_1 = (J(\alpha), p_1)$, $\iota_2 = (J(\alpha), p_2)$, $\mathcal{C}_{1\iota_1} = \{u_{1i}, \ i \in \underline{p_1}\}$ and $\mathcal{C}_{2\iota_2} = \{u_{2i}, \ i \in \underline{p_2}\}$ then

$$\mathcal{A}_{(\iota_1,\iota_2)} = \left\{ A_l;\ A_l = \sum_{i=1}^{l} u_{1i} u_{2,p_2-i+1}^T,\ l \in \underline{\min\{p_1, p_2\}} \right\}.$$

Using the above setting we have the next important result.

Theorem 3.4 *A basis $\mathcal{A}$ of the kernel* $\ker \Lambda$, *i.e., a basis for the space of solutions of the matrix equation $AXD^T - BXC^T = 0$, consists of the union of all the sets $\mathcal{A}_{(\iota_1,\iota_2)}$ for pairs of invariants (ι_1, ι_2) in the set $\mathcal{J}$.*

Proof. Suppose that (A', B') and (C', D') are the Kronecker canonical forms of the pairs (A, B) and (C, D), respectively, and that there are invertible matrices P, Q, R and S such that

$$A = PA'Q,\ B = PB'Q,\ C = RC'S \text{ and } D = RD'S. \tag{3.8}$$

Then equation (3.3) is equivalent to the equation

$$A'X'(D')^T - B'X'(C')^T = 0, \tag{3.9}$$

where $X' = QXS^T$. Because the matrices A', B', C' and D' are block diagonal, equation (3.9) becomes a collection of equations, one for each pair of invariants $\iota_1 \in \mathcal{I}_1$ and $\iota_2 \in \mathcal{I}_2$. The invariants ι_1 and ι_2 are of three different types. So we would have to consider nine different cases but because of the symmetry we only need to consider six different situations. We write Y for a block of the unknown matrix X in each of the cases considered :

(a) If $\iota_1 = (L, p_1)$ and $\iota_2 = (L, p_2)$ then we have the equation

$$F_{p_1} Y G_{p_2}^T - G_{p_1} Y F_{p_2}^T = 0. \tag{3.10}$$

Then a direct calculation shows that the matrices in the set

$$\mathcal{A}'_{(\iota_1,\iota_2)} = \left\{ J_{p_1+1}^{l-1} H_{p_1+1,p_2+1},\ l \in \underline{p_1+1};\ H_{p_1+1,p_2+1} \left(J_{p_2}^{l}\right)^T,\ l \in \underline{p_2} \right\} \tag{3.11}$$

solve (3.10). They are linearly independent. By a dimension argument it follows that the set (3.11) is a basis for the space of solutions of (3.10).

(b) If $\iota_1 = (L, p_1)$ and $\iota_2 = (M, p_2)$ then we have

$$F_{p_1} Y F_{p_2} - G_{p_1} Y G_{p_2} = 0. \tag{3.12}$$

If $p_1 \geq p_2$ then this equation has the only solution $Y = 0$. If $p_1 < p_2$ then the set

$$\mathcal{A}'_{(\iota_1,\iota_2)} = \left\{ H_{p_1+1,p_2} J_{p_2}^{l-1},\ l \in \underline{p_2 - p_1} \right\}$$

is a basis for the space of solutions of (3.12).

(c) Suppose now that $\iota_1 = (L, p_1)$ and $\iota_2 = (J(\alpha), p_2)$. Using the α-shift $\mathcal{C}_{1\iota_1}(\alpha)$ instead of $\mathcal{C}_{1\iota_1}$ we may assume without loss that $\alpha = 0$. So we suppose that $\iota_1 = (L, p_1)$ and $\iota_2 = (J(0), p_2)$. Then we have

$$F_{p_1} Y J_{p_2}^T - G_{p_1} Y = 0 \tag{3.13}$$

and the set

$$\mathcal{A}'_{(\iota_1,\iota_2)} = \left\{ H_{p_1+1,p_2} \left(J_{p_2}^{l-1}\right)^T,\ l \in \underline{p_2} \right\}$$

is a basis for the space of solutions of (3.13).

(d) If $\iota_1 = (M, p_1)$ and $\iota_2 = (M, p_2)$ we have

$$F_{p_1}^T Y G_{p_2} - G_{p_1}^T Y F_{p_2} = 0.$$

This equation has no nonzero solutions.

(e) If $\iota_1 = (M, p_1)$ and $\iota_2 = (J(\alpha), p_2)$ then we have

$$F_{p_1}^T Y J_{p_2}(\alpha)^T - G_{p_1}^T Y = 0.$$

This equation also does not have a nonzero solution.

(f) Finally we consider the case $\iota_1 = (J(\alpha), p_1)$ and $\iota_2 = (J(\beta), p_2)$. First suppose that $\alpha, \beta \neq \infty$. Then we have

$$Y J_{p_2}(\beta)^T - J_{p_1}(\alpha) Y = 0. \tag{3.14}$$

If $\alpha \neq \beta$ then this equation has the only solution $Y = 0$. If $\alpha = \beta$ then the set

$$\mathcal{A}'_{(\iota_1,\iota_2)} = \left\{ J_{p_1}^{l-1} I_{p_1,p_2},\ l \in \underline{\min\{p_1, p_2\}} \right\} \tag{3.15}$$

is a basis for the space of solutions of (3.14). Similarly it follows that there is no nonzero solution when $\alpha \neq \beta$ and either of α, β is ∞. If $\alpha = \beta = \infty$ then the set $\mathcal{A}'_{(\iota_1,\iota_2)}$ is as above in (3.15).

By definition we have that $X = Q^{-1} X' (S^{-1})^T$. The columns of the matrices Q^{-1} and S^{-1} form Kronecker bases for (A, B) and (C, D), respectively. Then it follows from (a) that the set $\mathcal{A}_{(\iota_1,\iota_2)}$ defined in (*i*) is a basis for the subspace of solutions of the equation (3.3) associated with a pair of invariants $\iota_1 \in \mathcal{I}_1$ and $\iota_2 \in \mathcal{I}_2$ of types $\iota_1 = (L, p_1)$ and $\iota_2 = (L, p_2)$. Similarly, it follows from (b) that the sets $\mathcal{A}_{(\iota_1,\iota_2)}$ of (*iia*) and, by symmetry, also of (*iib*), span the subspaces of the equations associated with the corresponding pairs of invariants. Case (c) implies a similar conclusion for the sets of (*iiia*) and (*iiib*), and case (f) implies a similar conclusion for the set of (*iiic*). Then it follows that the union $\mathcal{A}$ of all the sets $\mathcal{A}_{(\iota_1,\iota_2)}$ corresponding to pairs of invariants $(\iota_1, \iota_2) \in \mathcal{J}$ form a basis for the space of solutions of the equation (3.3) and therefore also a basis of the kernel $\ker \Lambda$. $\square$

Corollary 3.5 *The matrix equation $AXD^T - BXC^T = 0$ has only the trivial solution $X = 0$ if and only if either*

(i) *there are no invariants of type (L, p) in the sets $\mathcal{I}_1$ and $\mathcal{I}_2$ and there is no pair of invariants $(J(\alpha), p_1) \in \mathcal{I}_1$ and $(J(\beta), p_2) \in \mathcal{I}_2$ with $\alpha = \beta$, or*

(ii) *one of the sets of invariants $\mathcal{I}_i$, where i is either 1 or 2, consists only of invariants of the type (M, p_1), while there are invariants of the type (L, p_2) in the other set of invariants but any of them is such that $p_2 \geq p$, where $p = \min\{p_1, \ (M, p_1) \in \mathcal{I}_i\}$.*

3.2.3 The System of Matrix Equations $AX_1 + BX_2 = 0$ and $X_1C^T + X_2D^T = 0$

Next we consider the system of matrix equations (3.4). We define the transformation $\mathcal{L}$ on the space $\mathbb{C}^{n_2 \times m_2} \oplus \mathbb{C}^{n_2 \times m_2}$ by

$$\mathcal{L}\left(\begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\right) = \begin{bmatrix} AX_1 + BX_2 \\ X_1C^T + X_2D^T \end{bmatrix}. \tag{3.16}$$

Suppose that $\mathcal{A} = \cup_{(\iota_1,\iota_2)\in\mathcal{J}} \mathcal{A}_{(\iota_1,\iota_2)}$ is a basis for the kernel of Λ as described in Theorem 3.4. The transformation Λ is defined by (3.6) and the set of invariants $\mathcal{J}$ is defined in the discussion thereafter. Then we write $\mathcal{J}'$ for the set of all the pairs $(\iota_1, \iota_2) \in \mathcal{J}$ that are different from the cases $(\iota_1, \iota_2) = ((L, p), (M, p+1))$ and $(\iota_1, \iota_2) = ((M, p+1), (L, p))$. Now we associate with every pair of invariants $(\iota_1, \iota_2) \in \mathcal{J}'$ a set of pairs of matrices $\mathcal{A}^2_{(\iota_1,\iota_2)}$ as follows. Here the matrices A_l are defined in (*i*)-(*iiic*) on pp. 98-99 for different cases of pairs (ι_1, ι_2) :

(i) If $\iota_1 = (L, p_1)$ and $\iota_2 = (L, p_2)$ then

$$\mathcal{A}^2_{(\iota_1,\iota_2)} = \left\{\begin{bmatrix} -A_l \\ A_{l-1} \end{bmatrix}, \ l \in \underline{p_1 + p_2 + 2}\right\}.$$

where $A_0 = A_{p_1+p_2+2} = 0$,

(ii) If $\iota_1 = (L, p_1)$, $\iota_2 = (M, p_2)$ and $p_1 + 2 \leq p_2$ or $\iota_1 = (M, p_1)$, $\iota_2 = (L, p_2)$ and $p_1 \geq p_2 + 2$ then

$$\mathcal{A}^2_{(\iota_1,\iota_2)} = \left\{ \begin{bmatrix} -A_{l+1} \\ A_l \end{bmatrix}, \ l \in \underline{|p_1 - p_2| - 1} \right\},$$

(iii) In cases (iiia), (iiib) or (iiic), if $\alpha \in \mathbb{C}$, then

$$\mathcal{A}^2_{(\iota_1,\iota_2)} = \left\{ \begin{bmatrix} -A_l \\ \alpha A_l + A_{l-1} \end{bmatrix}, \ l \in \underline{p} \right\},$$

where $p = p_2$ if $\iota_1 = (L, p_1)$ and $\iota_2 = (J(\alpha), p_2)$, $p = p_1$ if $\iota_1 = (J(\alpha), p_1)$ and $\iota_2 = (L, p_2)$ and $p = \min\{p_1, p_2\}$ if $\iota_1 = (J(\alpha), p_1)$ and $\iota_2 = (J(\alpha), p_2)$. Here we write $A_0 = 0$. If $\alpha = \infty$ then

$$\mathcal{C}_{(\iota_1,\iota_2)} = \left\{ \begin{bmatrix} -A_{l-1} \\ A_l \end{bmatrix}, \ l \in \underline{p} \right\}$$

where p is defined as above and $A_0 = 0$.

In the above setting we have the following result :

Theorem 3.6 *The kernel of $\mathcal{L}$, i.e., the space of solutions of the pair of matrix equations $AX_1 + BX_2 = 0$ and $X_1C^T + X_2D^T = 0$, has a basis $\mathcal{A}^2 = \cup_{(\iota_1,\iota_2)\in\mathcal{J}'}\mathcal{A}^2_{(\iota_1,\iota_2)}$, where the sets $\mathcal{A}^2_{(\iota_1,\iota_2)}$ are given above.*

Proof. Let the matrices A', B', C', D', P, Q, R and S be as in the proof of Theorem 3.4. We write $X_i' = QX_iS^T$ for $i = 1, 2$. Then $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \ker \mathcal{L}$ if and only if

$$A'X_1' + B'X_2' = 0 \text{ and } X_1'(C')^T + X_2'(D')^T = 0. \tag{3.17}$$

It is enough to consider the equations (3.17) blockwise because the matrices A', B', C' and D' are block diagonal. It follows from Corollary 3.1 that there might exist nontrivial solutions of a particular block of the equations (3.17) if and only if the corresponding pair (ι_1, ι_2) belongs to $\mathcal{J}$. Suppose that sets $\mathcal{A}'_{(\iota_1,\iota_2)}$ are defined as in the proof of Theorem 3.4. We write Y_1 and Y_2 for the blocks of X_1 and X_2, respectively, considered in each of the cases. First we have the case $\iota_1 = (L, p_1)$ and $\iota_2 = (L, p_2)$. Then we write $Y_1' = \sum_{l=1}^{p_1+p_2+1} \gamma_l A_l'$ and $Y_2' = \sum_{l=1}^{p_1+p_2+1} \delta_q A_q'$, where $A_l' = J_{p_1+1}^{l-1} H_{p_1+1,p_2+1}$, $l \in \underline{p_1+1}$ and $A_l' = H_{p_1+1,p_2+1} \left(J_{p_2}^{l-p_1-1}\right)^T$, $l = p_1+2, p_1+3, \ldots, p_1+p_2+1$ are the elements of the set $\mathcal{A}'_{(\iota_1,\iota_2)}$. It follows from the equations (3.17) that $\gamma_l + \delta_{l+1} = 0$ for $l \in \underline{p_1+p_2}$. Thus the set

$$\mathcal{A}^{2\prime}_{(\iota_1,\iota_2)} = \left\{ \begin{bmatrix} -A_l' \\ A_{l-1}' \end{bmatrix}, \; l \in \underline{p_1+p_2+2} \right\},$$

where $A_0' = A_{p_1+p_2+2}' = 0$, is the basis of the block of the equations (3.17) corresponding to the pair of invariants (ι_1, ι_2). Because the columns of the matrices Q^{-1} and S^{-1} form Kronecker bases for the pairs (A, B) and (C, D), respectively, it follows that $\mathcal{A}^2_{(\iota_1,\iota_2)}$, as defined in (i) above, form the basis of the subspace of $\ker \mathcal{L}$ corresponding to the pair of invariants (ι_1, ι_2).

Using the same method as for the case $\iota_1 = (L, p_1)$ and $\iota_2 = (L, p_2)$ above, we prove that the sets $\mathcal{A}^2_{(\iota_1,\iota_2)}$ in cases (ii) and (iii) form bases for the corresponding subspaces of $\ker \mathcal{L}$. Then the proof is complete. $\square$

Now we define a mapping η on the set of pairs of invariants $\mathcal{J}'$ by

$$\eta(\iota_1, \iota_2) = \begin{cases} (L, p_1+p_2+1), & \text{if } (\iota_1,\iota_2) \text{ is as in } (i), \\ (M, p_2-p_1), & \text{if } (\iota_1,\iota_2) \text{ is as in } (iia), \\ (M, p_1-p_2), & \text{if } (\iota_1,\iota_2) \text{ is as in } (iib), \\ (J(\alpha), p_2), & \text{if } (\iota_1,\iota_2) \text{ is as in } (iiia), \\ (J(\alpha), p_1), & \text{if } (\iota_1,\iota_2) \text{ is as in } (iiib), \\ (J(\alpha), \min\{p_1, p_2\}), & \text{if } (\iota_1,\iota_2) \text{ is as in } (iiic), \end{cases} \tag{3.18}$$

where the cases (i)-$(iiic)$ are defined on page 98. Then the set of invariants $\mathcal{I} = \{\eta(\iota_1,\iota_2), \; (\iota_1,\iota_2) \in \mathcal{J}'\}$ is called the *set of invariants of the kernel of* $\mathcal{L}$. We write $\mathcal{A}^2_\iota = \mathcal{A}^2_{(\iota_1,\iota_2)}$ if $\iota = \eta(\iota_1,\iota_2)$. We will use the set of invariants $\mathcal{I}$ and the corresponding basis $\mathcal{A}^2 = \cup_{\iota\in\mathcal{I}} \mathcal{A}^2_\iota$ to describe a special basis of the second root subspace of the two-parameter system.

3.2.4 Remark on the Matrix Equation $AXD^T - BXC^T = E$ and Root Subspace for Two-parameter Systems

Let us now consider a particular eigenvalue $\boldsymbol{\lambda} = (\lambda_1, \lambda_2) \in \sigma(\mathbf{W})$. Then we write

$$T_i = W_i(\boldsymbol{\lambda}) = -V_{i0} + \lambda_1 V_{i1} + \lambda_2 V_{i2} \qquad \text{for } i = 1, 2.$$

From the properties of determinantal operators in Lemma 2.1 it follows that

$$\lambda_1 \Delta_0 - \Delta_1 = T_1 \otimes V_{22} - V_{12} \otimes T_2$$

and

$$\lambda_2 \Delta_0 - \Delta_2 = V_{11} \otimes T_2 - T_1 \otimes V_{21}.$$

Again we view the above transformations as acting on the space of $n_1 \times n_2$ complex matrices. Suppose that $X_0, X_1, \ldots, X_p \in \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ (identified with $\mathbb{C}^{n_1 \times n_2}$ via the isomorphism Ξ defined by (3.2), only replacing n_i for q_i) form a Jordan chain for Γ_1, i.e.,

$$(\lambda_1 I - \Gamma_1) X_j = X_{j-1}$$

or equivalently

$$(\lambda_1 \Delta_0 - \Delta_1) X_j = \Delta_0 X_{j-1} \tag{3.19}$$

for $j = 0, 1, \ldots, p$ and $X_{-1} = 0$. These are equivalent, via the isomorphism Ξ, to the recursive system of matrix equations

$$T_1 X_j V_{22}^T - V_{12} X_j T_2^T = V_{11} X_{j-1} V_{22}^T - V_{12} X_{j-1} V_{21}^T. \tag{3.20}$$

We have a similar system for the second associated operator Γ_2. If $X_0, X_1, \ldots, X_p$ is its Jordan chain then

$$V_{11} X_j T_2^T - T_1 X_j V_{21} = V_{11} X_{j-1} V_{22}^T - V_{12} X_{j-1} V_{21}^T \tag{3.21}$$

for $j = 0, 1, \cdots, p$ and $X_{-1} = 0$. At every stage of this recursive system of matrix equations we have to solve a matrix equation of the type

$$AXD^T - BXC^T = E. \tag{3.22}$$

This equation can be studied using the Kronecker canonical forms for pairs (A, B) and (C, D) similarly as for the homogeneous equation (3.3). For every type of a pair of invariants we could give solvability conditions and a particular solution when it exists. Then we could apply this result to the recursive system (3.20) (or (3.21)). We would have to solve simultaneously the recursive system (3.20) and a similar system corresponding to $\overline{\lambda_1} I - \Gamma_1^*$. The weakness of the eventual procedure is that it would only give a basis for the root subspace for one of Γ_1 (or Γ_2) and would not necessarily give a basis for the root subspace $\ker(\boldsymbol{\lambda}\mathbf{I} - \mathbf{\Gamma})^N$. In the procedure we would use ideas developed in the theory of marked invariant subspaces in order to prove completeness (see [72] and [73, Section 4.4]). We state the definition of a marked invariant subspace below, but we do not develop the procedure in further detail.

Definition. A subspace $\mathcal{N} \subset \mathbb{C}^N$ is called a *marked invariant subspace* for a transformation $A : \mathbb{C}^N \longrightarrow \mathbb{C}^N$ if $\mathcal{N}$ is invariant for A and if there exists a Jordan basis $\mathcal{B}$ for A on $\mathbb{C}^N$ such that a subset of $\mathcal{B}$ spans $\mathcal{N}$.

Marked invariant subspaces were introduced in [92, Section 2.9]. For further developments see [46, 47]. An earlier related disposition was given by Cater in [53, Lecture 4–3] where he proves a finite-dimensional version of results of Vilenkin [165, pp. 102-106] and Kaplansky [116, Chapter 18].

3.3 A Special Basis for the Second Root Subspace of Two-parameter Systems

We saw in the previous section that we can build a natural basis for the kernel of $\mathcal{L}$ from Kronecker bases of the pairs of matrices (A, B) and (C, D). This result can be applied to the kernel of the transformation $\mathcal{D}_0^\lambda$ in the two-parameter case. We use the setting of Subsection 2.5.2 with $n = 2$. A special basis for $\ker \mathcal{D}_0^\lambda$ can be given using Kronecker bases for the pairs of matrices $\left(V_{i1}^\lambda, V_{i2}^\lambda\right)$, $i = 1, 2$. Suppose that $\mathcal{I}_i$ is the set of invariants of the pair $\left(V_{i1}^\lambda, V_{i2}^\lambda\right)$ and $\mathcal{C}_i = \cup_{\iota \in \mathcal{I}_i} \mathcal{C}_{i\iota}$ is the corresponding

Kronecker basis where $C_{i\iota}$ is a Kronecker chain associated with the invariant $\iota \in \mathcal{I}_i$. We write $\mathcal{I}$ for the set of invariants of the kernel $\ker \mathcal{D}_0^\lambda$ and $\mathcal{A}_\iota^2$ for the subset of the basis $\mathcal{A}^2$ of $\ker \mathcal{D}_0^\lambda$ corresponding to the invariant $\iota \in \mathcal{I}$. We described in Theorem 2.21 a correspondence between elements of a basis for $\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^2 / \ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})$ and elements of a basis for the kernel of $\mathcal{D}_0^\lambda$. Therefore a special basis for $\ker \mathcal{D}_0^\lambda$ induces a special basis for $\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^2$. The exact correspondence is described later in this section. We have three different types of invariants in the set of invariants $\mathcal{I}$ for the kernel $\ker \mathcal{D}_0^\lambda$. It will turn out that they correspond to the three different types of invariants of a pair of matrices $\mathbf{A}^{01} = (A_1^{01}, A_2^{01})$, as defined on page 92.

In the rest of this section we describe the construction of a special basis for $\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^2$ using the basis $\mathcal{A}^2$ for $\ker \mathcal{D}_0^\lambda$. We discuss each of three different types of invariants $\iota \in \mathcal{I}$ separately.

3.3.1 Basis Corresponding to an Invariant $\iota = (L, p)$

Theorem 3.7 *Suppose $\iota = (L,p) = \eta(\iota_1, \iota_2)$ where $\iota_1 = (L, p_1) \in \mathcal{I}_1$ and $\iota_2 = (L, p_2) \in \mathcal{I}_2$ and that $C_{1\iota_1}$ and $C_{2\iota_2}$ are the associated Kronecker chains. Then there exist vectors $x_{il_i} \in H_i$, $i = 1, 2$ and $l_i = 1, 2, \ldots, p_i + 1$ such that*

$$V_{i1} x_{i0}^1 = W_i(\boldsymbol{\lambda})\, x_{i1}^1, \tag{3.23}$$

$$V_{i1} x_{i0}^{l_i} + V_{i2} x_{i0}^{l_i - 1} = W_i(\boldsymbol{\lambda})\, x_{i1}^{l_i} \qquad for\ l_i = 2, 3, \ldots, p_i \tag{3.24}$$

and

$$V_{i2} x_{i0}^{p_i} = W_i(\boldsymbol{\lambda})\, x_{i1}^{p_i + 1}. \tag{3.25}$$

The the vectors

$$z_0^k = \sum_{l=1}^{k} x_{10}^l \otimes x_{20}^{k+1-l}, \qquad k \in \underline{p} \tag{3.26}$$

and

$$z_1^k = \sum_{l=1}^{k} \left(x_{11}^l \otimes x_{20}^{k+1-l} + x_{10}^l \otimes x_{21}^{k+1-l} \right), \qquad k \in \underline{p+1} \tag{3.27}$$

are linearly independent. It also follows that

$$(\lambda_1 I - \Gamma_1)\, z_1^k = z_0^k \tag{3.28}$$

and

$$(\lambda_2 I - \Gamma_2)\, z_1^k = z_0^{k-1} \tag{3.29}$$

for $k \in \underline{p}$, where $z_0^0 = z_0^p = 0$ and $x_{i1}^k = 0$ for $k \leq 0$ and $k > p_i + 1$.

Proof. Suppose that $\mathcal{C}_{i\iota_i} = \left\{u_{ik};\ k \in \underline{p_i+1}\right\}$, $i = 1, 2$. Because $\iota_i = (L, p_i)$ it follows from the definition of a Kronecker chain that

$$\begin{aligned} V_{i2}^{\lambda} u_{i1} &= 0, \\ V_{i2}^{\lambda} u_{il} &= V_{i1}^{\lambda} u_{i,l-1}, \qquad l = 2, 3, \ldots, p_i + 1, \\ 0 &= V_{i1}^{\lambda} u_{i,p_i+1}\,. \end{aligned}$$

Then we have by a standard argument involving Lemma 2.12 that there exist vectors $x_{i1}^k \in H_i'$ such that (3.23), (3.24) and (3.25) hold. We can construct vectors (3.26) and (3.27), and then it follows that

$$(\lambda_1 \Delta_0 - \Delta_1)\, z_1^k = \sum_{l=1}^{k} \begin{vmatrix} W_1(\boldsymbol{\lambda})\, x_{11}^l & V_{12} x_{10}^l \\ W_2(\boldsymbol{\lambda})\, x_{21}^{k+1-l} & V_{22} x_{20}^{k+1-l} \end{vmatrix}^{\otimes} =$$

$$= \sum_{l=1}^{k} \begin{vmatrix} V_{11} x_{10}^l & V_{12} x_{10}^l \\ V_{21} x_{20}^{k+1-l} & V_{22} x_{20}^{k+1-l} \end{vmatrix}^{\otimes} + \sum_{l=1}^{k-1} \begin{vmatrix} V_{12} x_{10}^l & V_{12} x_{10}^l \\ V_{22} x_{20}^{k-l} & V_{22} x_{20}^{k-l} \end{vmatrix}^{\otimes} = \Delta_0 z_0^k$$

and

$$(\lambda_2 \Delta_0 - \Delta_2)\, z_1^k = \sum_{l=1}^{k} \begin{vmatrix} V_{11} x_{10}^l & W_1(\lambda)\, x_{11}^l \\ V_{21} x_{20}^{k+1-l} & W_2(\lambda)\, x_{21}^{k+1-l} \end{vmatrix}^{\otimes} =$$

$$= \sum_{l=1}^{k} \begin{vmatrix} V_{11} x_{10}^l & V_{11} x_{10}^l \\ V_{21} x_{20}^{k+1-l} & V_{21} x_{20}^{k+1-l} \end{vmatrix}^{\otimes} + \sum_{l=1}^{k-1} \begin{vmatrix} V_{11} x_{10}^l & V_{12} x_{10}^l \\ V_{21} x_{20}^{k-l} & V_{22} x_{20}^{k-l} \end{vmatrix}^{\otimes} = \Delta_0 z_0^{k-1}$$

for $k = 1, 2, \ldots, p$. Hence (3.28) and (3.29) hold. Here we assume $z_0^0 = z_0^p = 0$. The vectors z_0^k, $k = 1, 2, \ldots, p-1$ are linearly independent because the vectors x_{i0}^k, $k = 1, 2, \ldots, p_i$ are linearly independent. Then it follows from (3.28) and (3.29) that the vectors $\left\{z_0^k;\ k \in \underline{p}\right\} \cup \left\{z_1^k;\ k \in \underline{p+1}\right\}$ are linearly independent. □

If we restrict the transformations $\lambda_1 I - \Gamma_1$ and $\lambda_2 I - \Gamma_2$ to the joint invariant subspace $\mathcal{N}$ spanned by the vectors $\left\{z_0^k;\ k \in \underline{p}\right\} \cup \left\{z_1^k;\ k \in \underline{p+1}\right\}$ then they commute and we have

$$(\lambda_1 I - \Gamma_1)\mid_{\mathcal{N}}= \begin{bmatrix} 0 & F_p \\ 0 & 0 \end{bmatrix} \text{ and } (\lambda_2 I - \Gamma_2)\mid_{\mathcal{N}}= \begin{bmatrix} 0 & G_p \\ 0 & 0 \end{bmatrix}.$$

Note that the invariant of the pair of matrices (F_p, G_p) is (L,p).

3.3.2 Basis Corresponding to an Invariant $\iota = (M,p)$

Suppose that $\iota = (M,p) = \eta(\iota_1,\iota_2)$ where $\iota_1 = (L,p_1) \in \mathcal{I}_1$, $\iota_2 = (M,p_2) \in \mathcal{I}_2$ and $p = p_2 - p_1 > 1$, and that $\mathcal{C}_{1\iota 1}$ and $\mathcal{C}_{2\iota_2}$ are the associated Kronecker chains. The basis for the case $\iota_1 = (M,p_1) \in \mathcal{I}_1$, $\iota_2 = (L,p_2) \in \mathcal{I}_2$ and $p = p_1 - p_2 > 1$ is obtained symmetrically, interchanging $i = 1$ and $i = 2$.

Theorem 3.8 *If $\iota = (M,p) \in \mathcal{I}$ is as above then there exist vectors $x_{i1}^k \in H_i$, $k = 1,2,\ldots,p_i+1$ $\left(x_{21}^1 = x_{21}^{p_2+1} = 0\right)$ such that*

$$V_{11}x_{10}^1 = W_1(\boldsymbol{\lambda})\, x_{11}^1,$$

$$V_{i1}x_{i0}^k + V_{i2}x_{i0}^{k-1} = W_i(\boldsymbol{\lambda})\, x_{i1}^k \qquad k = 2,3,\ldots,p_i$$

and

$$V_{12}x_{10}^{p_1} = W_1(\boldsymbol{\lambda})\, x_{11}^{p_1+1}.$$

Then the vectors

$$z_0^k = \sum_{l=1}^{p_1} x_{10}^l \otimes x_{20}^{p_1+k-l},\ k \in \underline{p+1},\ i = 1,2,$$

and

$$z_1^k = \sum_{l=1}^{p_1+1} x_{11}^l \otimes x_{20}^{u_1+k-l+1} + \sum_{l=1}^{p_1} x_{10}^l \otimes x_{21}^{u_1+k-l+1},\ k \in \underline{p}$$

are linearly independent. Furthermore, we have

$$(\lambda_1 I - \Gamma_1)\, z_1^k = z_0^{k+1} \tag{3.30}$$

and

$$(\lambda_2 I - \Gamma_2)\, z_1^k = z_0^k \tag{3.31}$$

for $k \in \underline{p}$.

Proof. The vectors x_{i1}^k exist similarly as in the proof of Theorem 3.7 using the definition of the Kronecker chains $\mathcal{C}_{i\iota_i}$ and Lemma 2.12. To prove the theorem we need to establish relations (3.30) and (3.31). These follow by a straightforward calculation :

$$(\lambda_1\Delta_0 - \Delta_1)\, z_1^k = \sum_{l=1}^{p_1+1} \begin{vmatrix} W_1(\boldsymbol{\lambda})\, x_{11}^l & V_{12}x_{10}^l \\ W_2(\boldsymbol{\lambda})\, x_{21}^{p_1+k-l+1} & V_{22}x_{20}^{p_1+k-l+1} \end{vmatrix}^{\otimes} =$$

$$= \sum_{l=1}^{p_1} \begin{vmatrix} V_{11}x_{10}^l & V_{12}x_{10}^l \\ V_{21}x_{20}^{p_1+k-l+1} & V_{22}x_{20}^{p_1+k-l+1} \end{vmatrix}^{\otimes} + \sum_{l=1}^{p_1} \begin{vmatrix} V_{12}x_{10}^l & V_{12}x_{10}^l \\ V_{22}x_{20}^{p_1+k-l} & V_{22}x_{20}^{p_1+k-l} \end{vmatrix}^{\otimes} =$$

$$= \Delta_0 z_0^{k+1}$$

and

$$(\lambda_2\Delta_0 - \Delta_2)\, z_1^k = \sum_{l=1}^{p_1+1} \begin{vmatrix} V_{11}x_{10}^l & W_1(\boldsymbol{\lambda})\, x_{11}^l \\ V_{21}x_{20}^{p_1+k-l+1} & W_2(\boldsymbol{\lambda})\, x_{21}^{p_1+k-l+1} \end{vmatrix}^{\otimes} =$$

$$= \sum_{l=1}^{p_1} \begin{vmatrix} V_{11}x_{10}^l & V_{11}x_{10}^l \\ V_{21}x_{20}^{p_1+k-l+1} & V_{21}x_{20}^{p_1+k-l+1} \end{vmatrix}^{\otimes} + \sum_{l=1}^{p_1} \begin{vmatrix} V_{11}x_{10}^l & V_{12}x_{10}^l \\ V_{21}x_{20}^{p_1+k-l} & V_{22}x_{20}^{p_1+k-l} \end{vmatrix}^{\otimes} =$$

$$= \Delta_0 z_0^k.$$

Then we argue as in the proof of Theorem 3.7 to complete the proof. □

If we restrict the transformations $\lambda_1 I - \Gamma_1$ and $\lambda_2 I - \Gamma_2$ to the joint invariant subspace $\mathcal{N}$ spanned by the vectors $\left\{z_0^k;\ k \in \underline{p+1}\right\} \cup \left\{z_1^k;\ k \in \underline{p}\right\}$, described in the above proof, then we have

$$(\lambda_1 I - \Gamma_1)\,|_{\mathcal{N}} = \begin{bmatrix} 0 & (F_p)^T \\ 0 & 0 \end{bmatrix} \text{ and } (\lambda_2 I - \Gamma_2)\,|_{\mathcal{N}} = \begin{bmatrix} 0 & (G_p)^T \\ 0 & 0 \end{bmatrix}.$$

Note that the invariant of the pair of matrices $\left((F_p)^T, (G_p)^T\right)$ is (M, p).

3.3.3 Basis Corresponding to an Invariant $\iota = (J(\alpha), p)$

Suppose that $\iota = (J(\alpha), p) = \eta(\iota_1, \iota_2)$ where $\iota_1 = (J(\alpha), p_1) \in \mathcal{I}_1$, $\iota_2 = (J(\alpha), p_2) \in \mathcal{I}_2$ and $p = \min\{p_1, p_2\}$, and that $\mathcal{C}_{1\iota 1}$ and $\mathcal{C}_{2\iota_2}$ are the associated

Kronecker chains. The basis for the case $\iota_1 = (L, p_1)$ and $\iota_2 = (J(\alpha), p_2)$ is obtained using the same arguments as in the case $\iota_1 = (J(\alpha), p_1)$ and $\iota_2 = (J(\alpha), p_2)$ using the α-shift $\mathcal{C}_{1\iota_1}(\alpha) = \left\{u_{1i}(\alpha),\ i \in \underline{p_1+1}\right\}$ and writing $u_{1i}(\alpha) = 0$ for $i \geq p_1 + 1$ if $p_2 > p_1+1$. The case $\iota_1 = (J(\alpha), p_1)$ and $\iota_2 = (L, p_2)$ is analogous, only interchanging $i = 1$ and $i = 2$.

Theorem 3.9 *If $\iota_1 = (J(\alpha), p_1)$ and $\iota_2 = (J(\alpha), p_2)$ then there exist vectors $x_{i1}^k \in H_i$, $k = 1, 2, \ldots, p$ such that*

$$U_i(\alpha)\, x_{i0}^1 = W_i(\boldsymbol{\lambda})\, x_{i1}^1$$

and

$$U_i(\alpha)\, x_{i0}^k + V_{i1} x_{i0}^{k-1} = W_i(\boldsymbol{\lambda})\, x_{i1}^k \ \ k = 2, 3, \ldots, p.$$

Then the vectors

$$z_0^k = \sum_{l=1}^{k} x_{10}^l \otimes x_{20}^{k+1-l}, \quad k \in \underline{p}$$

and

$$z_1^k = \sum_{l=1}^{k} \left(x_{11}^l \otimes x_{20}^{k+1-l} + x_{10}^l \otimes x_{21}^{k+1-l}\right), \quad k \in \underline{p}$$

are linearly independent and furthermore

$$(\lambda_1 I - \Gamma_1)\, z_1^k = \alpha z_0^k + z_0^{k-1} \tag{3.32}$$

and

$$(\lambda_2 I - \Gamma_2)\, z_1^k = z_0^k \tag{3.33}$$

for $k \in \underline{p}$.

Proof. The theorem follows as the previous two given the relations (3.32) and (3.33). These are established using a simple calculation :

$$(\lambda_1 \Delta_0 - \Delta_1)\, z_1^k = \sum_{l=1}^{k} \begin{vmatrix} W_1(\boldsymbol{\lambda})\, x_{11}^l & V_{12} x_{10}^l \\ W_2(\boldsymbol{\lambda})\, x_{21}^{k+1-l} & V_{22} x_{20}^{k+1-l} \end{vmatrix}^{\otimes} =$$

$$= \sum_{l=1}^{k} \begin{vmatrix} U_1(\alpha)\, x_{10}^l & V_{12} x_{10}^l \\ U_2(\alpha)\, x_{20}^{k+1-l} & V_{22} x_{20}^{k+1-l} \end{vmatrix}^{\otimes} + \sum_{l=1}^{k-1} \begin{vmatrix} V_{11} x_{10}^l & V_{12} x_{10}^l \\ V_{21} x_{20}^{k-l} & V_{22} x_{20}^{k-l} \end{vmatrix}^{\otimes} = \alpha \Delta_0 z_0^k + \Delta_0 z_0^{k-1}$$

and

$$(\lambda_2 \Delta_0 - \Delta_2)\, z_1^k = \sum_{l=1}^{k} \begin{vmatrix} V_{11} x_{10}^l & W_1(\boldsymbol{\lambda})\, x_{11}^l \\ V_{21} x_{20}^{k+1-l} & W_2(\boldsymbol{\lambda})\, x_{21}^{k+1-l} \end{vmatrix}^{\otimes} =$$

$$= \sum_{l=1}^{k} \begin{vmatrix} V_{11} x_{10}^l & U_1(\alpha)\, x_{10}^l \\ V_{21} x_{20}^{k+1-l} & U_2(\alpha)\, x_{20}^{k+1-l} \end{vmatrix}^{\otimes} + \sum_{l=1}^{k-1} \begin{vmatrix} V_{11} x_{10}^l & V_{11} x_{10}^l \\ V_{21} x_{20}^{k-l} & V_{21} x_{20}^{k-l} \end{vmatrix}^{\otimes} = \Delta_0 z_0^k.$$

□

If we restrict the transformations $\lambda_1 I - \Gamma_1$ and $\lambda_2 I - \Gamma_2$ to the joint invariant subspace $\mathcal{N}$ spanned by the vectors $\left\{z_0^k;\ k \in \underline{p}\right\} \cup \left\{z_1^k;\ k \in \underline{p}\right\}$ given in the above proof. Then we have

$$(\lambda_1 I - \Gamma_1)\,|_{\mathcal{N}} = \begin{bmatrix} 0 & J_p(\alpha) \\ 0 & 0 \end{bmatrix} \text{ and } (\lambda_2 I - \Gamma_2)\,|_{\mathcal{N}} = \begin{bmatrix} 0 & I_P \\ 0 & 0 \end{bmatrix}.$$

The invariant of the pair of matrices $(J_p(\alpha), I)$ is $(J(\alpha), p)$.

Suppose that for every element in the set of invariants $\mathcal{I}$ of the kernel of $\mathcal{D}_0^\lambda$ we construct vectors z_0^k and z_1^k as explained in the proofs of Theorems 3.7–3.9. Note that they are linearly independent. We denote the set of these vectors by $\mathcal{B}_2'$ and by $\mathcal{N}'$ the subspace they span. The linear transformations $(\lambda_1 I - \Gamma_1)$ and $(\lambda_2 I - \Gamma_2)$ restricted to the subspace $\mathcal{N} = \ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^2$ commute and are nilpotent. Furthermore $\mathcal{N}' \subset \mathcal{N}$. If $\mathcal{N}' \neq \mathcal{N}$ we complete the set $\mathcal{B}_2'$ by a set of vectors, say $\mathcal{B}''$, to the basis $\mathcal{B}_2$ for $\mathcal{N}$. We write $\mathcal{N}'' = \mathcal{L}(\mathcal{B}'')$. Because the vectors z_1^k are as many as $\dim\ker \mathcal{D}_0^\lambda$ and are linearly independent it follows from Theorem 2.21 we can assume that $\mathcal{N}'' \subset \ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})$. We write the pair of restricted transformation $(\lambda_i I - \Gamma_i)\,|_{\mathcal{N}}$, that are nilpotent and commute, in the form (1.2) using the basis $\mathcal{B}_2$. It follows from Theorems 3.7–3.9 that the array $\mathbf{A}^{01}$ has the form

$$\mathbf{A}^{01} = \begin{bmatrix} \widehat{\mathbf{A}}^{01} \\ \mathbf{0} \end{bmatrix}, \tag{3.34}$$

where

$$\widehat{\mathbf{A}}^{01} = \begin{bmatrix} \mathbf{a}_1^{01} & \mathbf{0} & & & \cdots & & & \mathbf{0} \\ & \ddots & & & & & & \\ \mathbf{0} & & \mathbf{a}_{r_1}^{01} & \mathbf{0} & & \cdots & & \mathbf{0} \\ \mathbf{0} & \cdots & \mathbf{0} & \mathbf{a}_{r_1+1}^{01} & & & & \\ & & & & \ddots & & & \vdots \\ \vdots & & \vdots & & & \mathbf{a}_{r_1+r_2}^{01} & & \\ & & & & & & \mathbf{a}_{r_1+r_2+1}^{01} & \mathbf{0} \\ & & & & & & \ddots & \\ \mathbf{0} & \cdots & \mathbf{0} & & \cdots & & \mathbf{0} & \mathbf{a}_{r_1+r_2+r_3}^{01} \end{bmatrix}. \tag{3.35}$$

The first r_1 blocks $\mathbf{a}_j^{01}$ in the array (3.35) correspond to the invariant (L, p_j) in the set $\mathcal{I}$, the next r_2 blocks correspond to the invariants (M, p_j) in the set $\mathcal{I}$ and the last r_3 blocks $\mathbf{a}_j^{01}$ correspond to the invariants $(J(\alpha), p_j)$. The rows of $\mathbf{0}$ at the bottom in (3.34) are as many as there are vectors in the set $\mathcal{N}''$. Note that the array $\mathbf{A}$, where $\mathbf{A}^{01}$ is in the form given by (3.34) and (3.35), is in a canonical form described in Example 3.3. Note also that the set of invariants of the pair of matrices $\left(\widehat{A}_1^{01}, \widehat{A}_2^{01}\right)$ equals $\mathcal{I}$.

To illustrate the preceding construction we discuss two examples.

Example 3.10 Consider again the two-parameter system of Example 2.22. The sets of invariants for the pairs $\left(V_{11}^{\lambda_0}, V_{12}^{\lambda_0}\right)$ and $\left(V_{21}^{\lambda_0}, V_{22}^{\lambda_0}\right)$ that correspond to the eigenvalue $\boldsymbol{\lambda}_0 = (0,0)$ are $\{(L,0),(M,0)\}$ and $\{(L,1),(M,0)\}$, respectively. A Kronecker chain that corresponds to the invariant $(L,1)$ is $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$. The set of invariants for the kernel $\ker \mathcal{D}^{\lambda_0}$ is then $\{(L,2)\}$. We find that vectors

$$x_{11}^1 = x_{11}^2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \text{ and } x_{21}^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, x_{21}^2 = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix}, x_{21}^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

are such that

$$V_{11}x_{10} = W_1(\boldsymbol{\lambda}_0)\, x_{11}^1,\ V_{12}x_{10} = W_1(\boldsymbol{\lambda}_0)\, x_{11}^2$$

and

$$V_{22}x_{20}^1 = W_2(\boldsymbol{\lambda}_0)\,x_{21}^1,\ V_{22}x_{20}^2 - \frac{1}{2}V_{21}x_{20}^1 = W_2(\boldsymbol{\lambda}_0)\,x_{21}^2 \text{ and } -\frac{1}{2}V_{21}x_{20}^2 = W_2(\boldsymbol{\lambda}_0)\,x_{21}^3.$$

Then it follows from Theorems 2.21 and 3.7 that the vectors

$$\begin{bmatrix}1\\0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}; \begin{bmatrix}0\\0\\0\\1\\0\\0\\-1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\-\frac{1}{2}\\1\\1\\0\\-1\\-1\\0\end{bmatrix} \text{ and } \begin{bmatrix}0\\0\\0\\0\\1\\0\\0\\-1\\0\end{bmatrix}$$

form a basis for $\ker(\boldsymbol{\lambda}\mathbf{I}-\boldsymbol{\Gamma})^2$. Note that the above method to construct a basis differs from the method given in the proof of Theorem 2.21 and used in Example 2.22, hence also the bases constructed in the two examples are not the same. □

Example 3.11 Suppose that we are given matrices

$$V_{11}^{\lambda} = \begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},\ V_{12}^{\lambda} = \begin{bmatrix}0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix},$$

$$V_{21}^{\lambda} = \begin{bmatrix}1&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix} \text{ and } V_{22}^{\lambda} = \begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&2&1\\0&0&0&0&2\end{bmatrix},$$

and we write I_k for $k \times k$ identity matrix. Then we form a two-parameter system

$$W_1(\boldsymbol{\lambda}) = \begin{bmatrix} I_6 & V_{11}^{\lambda} \\ 0 & I_6 \end{bmatrix} \lambda_1 + \begin{bmatrix} 0 & V_{12}^{\lambda} \\ 0 & 0 \end{bmatrix} \lambda_2 - \begin{bmatrix} 0 & 0 \\ I_6 & 0 \end{bmatrix}$$

and

$$W_2(\boldsymbol{\lambda}) = \begin{bmatrix} 0 & V_{21}^{\lambda} \\ 0 & 0 \end{bmatrix} \lambda_1 + \begin{bmatrix} I_5 & V_{22}^{\lambda} \\ 0 & I_5 \end{bmatrix} \lambda_2 - \begin{bmatrix} 0 & 0 \\ I_5 & 0 \end{bmatrix}.$$

It follows from the structure of the above two-parameter system that it is nonsingular and that $\boldsymbol{\lambda} = (0,0)$ is an eigenvalue. We also find that the matrices $V_{ij}^{\lambda\dagger}$, $i,j = 1,2$, are the entries of the corresponding matrix $\mathcal{D}_0^{\lambda}$ if we choose $X_{10} = \begin{bmatrix} 0 \\ I_6 \end{bmatrix}$, $Y_{10} = \begin{bmatrix} I_6 \\ 0 \end{bmatrix}$, $X_{20} = \begin{bmatrix} 0 \\ I_5 \end{bmatrix}$ and $Y_{20} = \begin{bmatrix} I_5 \\ 0 \end{bmatrix}$. Then $\{(L,2),(M,3)\}$ is the set of invariants for $\left(V_{11}^{\lambda}, V_{12}^{\lambda}\right)$ and $\{(L,1),(M,1),(J(2),2)\}$ the set of invariants for $\left(V_{21}^{\lambda}, V_{22}^{\lambda}\right)$. The set of pairs of invariants $\mathcal{J}'$ has three elements $\{(L,2),(L,1)\}$, $\{(L,2),(J(2),2)\}$ and $\{(M,3),(L,1)\}$. Applying the mapping η defined by (3.18) we find that the set of invariants of the kernel of $\mathcal{D}_0^{\lambda}$ is $\{(L,4),(M,2),(J(2),2)\}$. □

3.4 Comments

Kronecker in [119] developed his canonical form as the answer to the problem posed by Weierstrass of finding a canonical form for a pair of bilinear forms. The Kronecker canonical form is usually stated in terms of matrix pencils $A\lambda + B$. Because we use the Kronecker canonical form for a pair of matrices $\mathbf{A}^{01} = (A_1^{01}, A_2^{01})$ in a commutative array in the form (1.2), we have chosen to state it in terms of pairs of matrices to keep in tune with our preceding discussion. For an early version of Kronecker's result adapted to matrix pencils see Dieudonné's work [58]. We can also find chapters on the Kronecker canonical form in recent monographs on Linear Algebra, for example [85, 92]. This topic is also of current interest in various applications, e.g. in Control Theory (see [113, 125]), and various further developments :

Van Dooren [61] gave a computational algorithm to find the canonical form, Atkinson [11] extended it to a special class of tensors and Thompson [162] studied it for pairs of self-adjoint matrices. The study of matrix and operator pencils $A\lambda + B$ and also multiparameter pencils $\sum_{i=1}^{n} A_i\lambda_i + B$ motivated by multiparameter eigenvalue problems is found in several papers. For example, Blum [31], Fox, Hayes and Mayers [76] and Hadeler [98], considered numerical methods to find eigenvalues of these pencils, and Binding [22] gave a canonical form for self-adjoint operator pencils $A\lambda + B$ on Hilbert space. Also Bohte in [36] studied numerical methods to calculate eigenvalues of a two-parameter system of pencils.

The matrix equation $AXD^T - BXC^T = E$ has been studied for a long period of time (see [156, 157, 171]). The special cases $XD^T - BX = E$ and also $AXD^T - X = E$ have been thoroughly examined. See [128, Chapter VIII] for early references, some later works being [107, 142, 145, 173]. In [143] Roth gave conditions for existence of a solution of $XD^T - BX = E$. Different proofs of his results were given later in [74, 102]. The authors in [100, 101, 120, 123, 124, 127] suggested different approaches to find explicit solutions of $XD^T - BX = E$. This matrix equation is associated through Roth's results with extensions of block matrices [114, 174] and with the Kronecker sum $I \otimes D - B \otimes I$. The latter was already known to Sylvester (he calls it 'nivellateur') in [156], see also [13, 14, 126]. Eigenvectors and root vectors for the Kronecker sum were given by Trampus in [163]. See also [132, Section 1.2] for a thorough presentation.

The applications of these matrix equations are diverse. Barnett and Storey in [14] discussed problems in stability theory where the equation $XD^T - BX = E$ arises. Epton [63] gives an example of a numerical method for implicit differential equations where solving the equation $AXD^T - BXC^T = E$ is essential. See also [17, 127] for some other applications. The general equation $AXD^T - BXC^T = E$ was studied in [54, 63, 103]. Chu [54, Theorem 1] gave conditions for existence of a unique solution. He also proposes a numerical algorithm to compute this solution. The idea to use the Kronecker canonical forms of two pairs of matrices (A, B) and (C, D), in order to solve this matrix equation, was brought forward by Rózsa in [144].

Two-parameter spectral problems were considered since the early days of Multiparameter Spectral Theory. For example, two-parameter oscillation theorems were proved by Klein [117], Bôcher [32, 33, 34] and Richardson [141]. Dixon [59] studied expansions of functions in terms of eigenfunctions of a pair of coupled two-parameter differential equations of Sturm-Liouville type. Also Camp [48, 49] and Doole [60] proved various two-parameter expansion theorems. Pell [138] studied a two-parameter system of integral equations of Fredholm type. In the 1950s Cordes [55, 56] developed an abstract Hilbert space setting for a special class of two-parameter spectral problems (cf. also [131] for a modern presentation of Cordes's work). Later Arscott considered particular classes of two-parameter spectral problems in [5, 7]. Among recent publications we find work of Binding, Browne, Faierman, Isaev, Seddighi and many others. Most of the early references discuss the right definite case where eigenvectors alone are complete, while Binding and Browne [26] consider the dimensions of root subspaces for general eigenvalues of self-adjoint two-parameter systems.

Chapter 4

Bases for Root Subspaces in Special Cases

4.1 Introduction

In this chapter we study the finite-dimensional completeness problem, i.e., the problem of finding a basis for root subspaces, for special cases of eigenvalues of multiparameter systems.

In the second section of this chapter we consider nonderogatory eigenvalues. Theorem 4.4 is the main result in this case. The method used to prove this result is in part different from the method used to prove the completeness result for simple eigenvalues and can not be directly generalized. When an eigenvalue $\boldsymbol{\lambda}$ is nonderogatory the restricted transformations $A_i = (\lambda_i I - \Gamma_i)\mid_{\ker(\boldsymbol{\lambda}\mathbf{I}-\boldsymbol{\Gamma})^N}$, $i \in \underline{n}$, that are commuting and nilpotent, are assumed to be in upper Toeplitz form. With corresponding root vectors we associate monic matrix polynomials. It turns out that the n-tuples of the first row of the array $\mathbf{A}$, consisting of matrices A_i, $i \in \underline{n}$, form Jordan chains for these matrix polynomials. A chain of vectors $x_0, x_1, \ldots, x_p$ is a Jordan chain for a matrix polynomial $L(\mu)$ at an eigenvalue μ_0 if

$$\sum_{j=0}^{k} \frac{1}{j!} L^{(j)}(\mu_0)\, x_{k-j} = 0, \quad \text{for } k = 0, 1, \ldots, p. \tag{4.1}$$

(For further definitions concerning matrix polynomials see [89, 90].) We also give an algorithm for the construction of a basis for a root subspace. As an application we consider self-adjoint multiparameter systems. We obtain a new result for the real simple eigenvalues of weakly-elliptic multiparameter systems.

Our main results are Theorem 4.18 and Algorithm 4.19 in the third section. We use the same ideas as we did in Section 2.5 when we constructed a basis for the second root subspace. We use essentially one more important fact, that we can separate for all i the kernels of the matrices $W_i(\boldsymbol{\lambda})$ and subspaces spanned by vectors x_{ik}^{g}, for $k \geq 1$, that are used in the construction of root vectors in addition to the vectors $x_{i0} \in \ker W_i(\boldsymbol{\lambda})$. We ensure this by choosing vectors x_{ik}^{g} from a direct complement $H_i' \subset H_i$ of $\ker W_i(\boldsymbol{\lambda})$. This fact is used along with relation (2.7) to obtain equalities of type (4.26). The matrix $\mathcal{S}_m$ plays a role similar to that of the matrices B_0 and $\mathcal{D}_0^{\lambda}$ before. It acts on a space isomorphic to the space spanned by matrices $T_f^{m(1,m-1)}$, $f \in \underline{d_m}$. These were the matrices introduced in Section 1.5. The matrices $T_f^{m(1,m-1)}$, $f \in \underline{d_m}$ are linearly independent and their isomorphic images are elements of the kernel of $\mathcal{S}_m$. Next we can associate with every element in the kernel of $\mathcal{S}_m$ an m-th root vector that is not an $(m-1)$-th root vector. Our proof that this vector is actually a root vector is technically very complicated. We do this in Lemma 4.17. We also prove that we can associate in the same fashion a set of linearly independent root vectors with a basis of the kernel of $\mathcal{S}_m$. We prove by induction that this procedure gives a basis for the root subspace. In the first subsection we establish a basis for the third root subspace and in the second subsection we prove the inductive step. In the third subsection we give Algorithm 4.19 and consider the special case of simple, completely derogatory eigenvalues. Simple eigenvalues in the two-parameter case are always completely derogatory. We also discuss the relation between our expressions for the root vectors for simple eigenvalues of the two-parameter system and the conjecture of Faierman [69, Conjecture 6.1] on the structure of root functions for a class of Sturm-Liouville boundary value problems (0.1).

4.2 Nonderogatory Eigenvalues

In this section we assume that $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ is a nonderogatory eigenvalue. The next result is a well known biorthogonality property between right and left Jordan chains. We write it in the following form for future reference. We assume that index i is fixed.

Definition. A Jordan chain $z_0, z_1, \ldots, z_p$ (as in 0.2) is called *maximal* for a linear transformation V at an eigenvalue λ_0 if $z_p \notin \mathcal{R}(\lambda_0 I - V)$.

Lemma 4.1 *Suppose that $z_0, z_1, \ldots, z_p$ is a maximal Jordan chain for Γ_i at the eigenvalue λ_i and w_0^* is a left eigenvector at the same eigenvalue. Assume also that* $\dim(\ker(\lambda_i I - \Gamma_i)) = 1$. *Then it follows that $w_0^* \Delta_0 z_k = 0$ for $k = 0, 1, \ldots, p-1$ and $w_0^* \Delta_0 z_p \neq 0$.*

Proof. For $k \leq p-1$ we have $0 = w_0^* (\lambda_i \Delta_0 - \Delta_i) z_{k+1} = w_0^* \Delta_0 z_k$. Suppose now that $w_0^* \Delta_0 z_p = 0$. Then it follows that $\Delta_0 z_p \in (\ker(\lambda_i \Delta_0 - \Delta_i)^*)^{\perp} = \mathcal{R}(\lambda_i \Delta_0 - \Delta_i)$ and so there is a vector z_{p+1} such that $(\lambda_i I - \Gamma_i) z_{p+1} = z_p$. This contradicts the assumption that $z_0, z_1, \ldots, z_p$ is maximal Jordan chain. The proof is complete. □

In this section we will denote the family of multiindex sets

$$\left\{(j_1, j_2, \ldots, j_n);\ 0 \leq j_i,\ \sum_{i=1}^{n} j_i = k\right\} \tag{4.2}$$

by Ψ_k for $k = 0, 1, \ldots, p$. Here p is a fixed nonnegative integer.

Lemma 4.2 *Let $\left\{B_k = \left[b_{ij}^k\right]_{i,j=1}^n, k = 0, 1, \ldots, p\right\}$ be a set of matrices and assume that* $\operatorname{rank}(B_0) = n - 1$. *Choose a vector $x_0 \in \ker(B_0) \setminus \{0\}$. Then there exist vectors $x_i, i = 1, 2, \ldots, p$ such that $\sum_{j=0}^{i} B_j x_{i-j} = 0$ for $i = 1, 2, \ldots, p$ if and only if*

$$\sum_{\mathbf{j} \in \Psi_i} \begin{vmatrix} b_{11}^{j_1} & b_{12}^{j_1} & \cdots & b_{1n}^{j_1} \\ b_{21}^{j_2} & b_{22}^{j_2} & \cdots & b_{2n}^{j_2} \\ \vdots & \vdots & & \vdots \\ b_{n1}^{j_n} & b_{n2}^{j_n} & \cdots & b_{nn}^{j_n} \end{vmatrix} = 0 \tag{4.3}$$

for $i = 1, 2, \ldots, p$.

Proof. We construct a matrix polynomial

$$L(\mu) = I\mu^{p+1} + B_p\mu^p + B_{p-1}\mu^{p-1} + \cdots + B_0 = [b_{ij}(\mu)]_{i,j=1}^n .$$

Then

$$L^{(k)}(0) = k! \cdot B_k, \quad k = 0, 1, \ldots, p \tag{4.4}$$

and because $\dim(\ker L(0)) = 1$ the polynomial $L(\mu)$ has only one elementary divisor at $\mu = 0$. Then by [90, Corollary 1.14, p.35] it follows that $\mu = 0$ is a root of degree $p+1$ for the scalar polynomial $d(\mu) = \det L(\mu)$ if and only if matrix polynomial $L(\mu)$ has a Jordan chain $x_0, x_1, \ldots, x_p$ of length $p+1$ at $\mu = 0$. That is, if and only if the vectors $x_0, x_1, \ldots, x_p$, $x_0 \neq 0$, are such that

$$\sum_{j=0}^{k} \frac{1}{j!} L^{(j)}(0)\, x_{k-j} = 0, \qquad \text{for } k = 0, 1, \ldots, p, \tag{4.5}$$

or if we use (4.4), if and only if

$$\sum_{j=0}^{k} B_j x_{k-j} = 0, \qquad \text{for } k = 0, 1, \ldots, p.$$

If the polynomial $d(\mu)$ has $\mu = 0$ as a root of degree $p+1$ then $d(0) = d'(0) = \ldots = d^{(p)}(0) = 0$. Finally the relations

$$d^{(k)}(0) = \sum_{\mathbf{j} \in \Psi_k} \frac{k!}{j_1!\ j_2!\ \cdots\ j_n!} \begin{vmatrix} b_{11}^{(j_1)}(0) & b_{12}^{(j_1)}(0) & \cdots & b_{1n}^{(j_1)}(0) \\ b_{21}^{(j_2)}(0) & b_{22}^{(j_2)}(0) & \cdots & b_{2n}^{(j_2)}(0) \\ \vdots & \vdots & & \vdots \\ b_{n1}^{(j_n)}(0) & b_{n2}^{(j_n)}(0) & \cdots & b_{nn}^{(j_n)}(0) \end{vmatrix} =$$

$$= k! \sum_{\mathbf{j} \in \Psi_k} \begin{vmatrix} b_{11}^{j_1} & b_{12}^{j_1} & \cdots & b_{1n}^{j_1} \\ b_{21}^{j_2} & b_{22}^{j_2} & \cdots & b_{2n}^{j_2} \\ \vdots & \vdots & & \vdots \\ b_{n1}^{j_n} & b_{n2}^{j_n} & \cdots & b_{nn}^{j_n} \end{vmatrix}$$

hold and the result follows. $\square$

The above lemma is used in the proofs of the main results in this section concerning bases for root subspaces for nonderogatory eigenvalues. For the definition see page 78. Let us recall that $H_i' \subset H_i$ is a direct complement of the kernel of $W_i(\boldsymbol{\lambda})$.

Proposition 4.3 *Assume that the eigenvalue $\boldsymbol{\lambda} \in \mathbb{C}^n$ is nonderogatory. Then the following statements are equivalent :*

(i) There exist n-tuples $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p \in \mathbb{C}^n$, $\mathbf{a}_1 \neq 0$ and vectors $x_{i1}, x_{i2}, \ldots, x_{ip} \in H_i'$, $i = 1, 2, \ldots, n$ such that

$$\sum_{j=0}^{k-1} U_i(\mathbf{a}_{k-j})\, x_{ij} = W_i(\boldsymbol{\lambda})\, x_{ik} \qquad \text{for } k = 1, 2, \ldots, p;\ i = 1, 2, \ldots, n. \tag{4.6}$$

(ii) There exist n-tuples $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p \in \mathbb{C}^n$, $\mathbf{a}_1 \neq 0$ and vectors $x_{i1}, x_{i2}, \ldots, x_{i,\,p-1} \in H_i'$, $i = 1, 2, \ldots, n$ such that

$$\sum_{j=0}^{k-1} y_{i0}^* U_i(\mathbf{a}_{k-j})\, x_{ij} = 0 \qquad \text{for } k = 1, 2, \ldots, p;\ i = 1, 2, \ldots, n.$$

(iii) There exists an index h such that $\dim\ker(\lambda_h I - \Gamma_h) = 1$ and Γ_h has a Jordan chain of length $p+1$ at eigenvalue λ_h.

(iv) There exists a set of linearly independent vectors $\{z_0, z_1, \ldots, z_p\} \subset H$ such that $(\lambda_i I - \Gamma_i)\, z_k = \sum_{j=0}^{k-1} a_{i,\,k-j} z_j$ for $k = 0, 1, \ldots, p$; $i = 1, 2, \ldots, n$ and not all $a_{i1} = 0$.

Proof. If we multiply (i) by y_{i0}^* on the left then (ii) follows. Assume now that (ii) holds. In Proposition 2.15 we have already proved that (i) and (ii) are equivalent for $k = 1$. Suppose now that we have already found vectors $x_{i1}, x_{i2}, \ldots, x_{i,k-1}$, where $0 \leq k < p$, such that (4.6) holds. Then $\sum_{j=0}^{k-1} U_i(\mathbf{a}_{k-j})\, x_{ij}$ is orthogonal to the kernel $\ker W_i(\boldsymbol{\lambda})^*$ and so it follows from Lemma 2.12 that there exists a vector $x_{ik} \in H_i'$ such that $\sum_{j=0}^{k-1} U_i(\mathbf{a}_{k-j})\, x_{ij} = W_i(\boldsymbol{\lambda})\, x_{ik}$. We can continue this procedure until $k = p$. Therefore (i) follows.

It is easy to observe that (iv) implies (iii).

Suppose now that (i) holds. We define vectors

$$z_k = \sum_{\mathbf{j}\in\Psi_k} x_{1j_1} \otimes x_{2j_2} \otimes \cdots \otimes x_{nj_n}$$

for $k = 0, 1, \ldots, p$. It follows then from (i) and Lemma 2.1

$$(\lambda_i \Delta_0 - \Delta_i)\, z_k = \begin{vmatrix} V_{11}^\dagger & \cdots & V_{1,i-1}^\dagger & W_1(\boldsymbol{\lambda})^\dagger & V_{1,i+1}^\dagger & \cdots & V_{1n}^\dagger \\ V_{21}^\dagger & \cdots & V_{2,i-1}^\dagger & W_2(\boldsymbol{\lambda})^\dagger & V_{2,i+1}^\dagger & \cdots & V_{2n}^\dagger \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1}^\dagger & \cdots & V_{n,i-1}^\dagger & W_n(\boldsymbol{\lambda})^\dagger & V_{n,i+1}^\dagger & \cdots & V_{nn}^\dagger \end{vmatrix} z_k =$$

$$= \sum_{\mathbf{j}\in\Psi_k} \begin{vmatrix} V_{11}x_{1j_1} & \cdots & V_{1,i-1}x_{1j_1} & \sum\limits_{l_1=0}^{j_1-1} U_1\left(\mathbf{a}_{j_1-l_1}\right) x_{1l_1} & V_{1,i+1}x_{1j_1} & \cdots & V_{1n}x_{1j_1} \\ V_{21}x_{2j_2} & \cdots & V_{2,i-1}x_{2j_2} & \sum\limits_{l_2=0}^{j_2-1} U_2\left(\mathbf{a}_{j_2-l_2}\right) x_{2l_2} & V_{2,i+1}x_{2j_2} & \cdots & V_{2n}x_{2j_2} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1}x_{nj_n} & \cdots & V_{n,i-1}x_{nj_n} & \sum\limits_{l_n=0}^{j_n-1} U_n\left(\mathbf{a}_{j_n-l_n}\right) x_{nl_n} & V_{n,i+1}x_{nj_n} & \cdots & V_{nn}x_{nj_n} \end{vmatrix}_{\otimes} \tag{4.7}$$

$$= \sum_{\mathbf{j}\in\Psi_k} \sum_{q,r=1}^{n} \sum_{l_q=0}^{j_q-1} a_{j_q-l_q,r} \begin{vmatrix} \cdots & V_{1,i-1}x_{1j_1} & 0 & V_{1,i+1}x_{1j_1} & \cdots \\ & \vdots & \vdots & \vdots & \\ \cdots & V_{q-1,i-1}x_{q-1,j_{q-1}} & 0 & V_{q-1,i+1}x_{q-1,j_{q-1}} & \cdots \\ \cdots & V_{q,i-1}x_{qj_q} & V_{qr}x_{ql_q} & V_{q,i+1}x_{qj_q} & \cdots \\ \cdots & V_{q+1,i-1}x_{q+1,j_{q+1}} & 0 & V_{q+1,i+1}x_{q+1,j_{q+1}} & \cdots \\ & \vdots & \vdots & \vdots & \\ \cdots & V_{n,i-1}x_{nj_n} & 0 & V_{n,i+1}x_{nj_n} & \cdots \end{vmatrix}_{\otimes} . \tag{4.8}$$

In the displayed determinant (4.8) the first $i-1$ and the last $n-i-1$ columns are the same as in the determinant displayed in (4.7). The vectors $V_{qr}x_{qj_q}$, $r = 1, \ldots, i-1, i+1, \ldots, n$, in (4.8) can be substituted for $V_{qr}x_{ql_q}$ without changing the determinant. The sum (4.8) is then equal to

$$\sum_{\mathbf{j}\in\Psi_k}\sum_{q,r=1}^{n}\sum_{l_q=0}^{j_q-1} a_{j_q-l_q,r}\begin{vmatrix} V_{11}^{\dagger} & \cdots & V_{1,i-1}^{\dagger} & 0 & V_{1,i+1}^{\dagger} & \cdots & V_{1n}^{\dagger} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{q-1,1}^{\dagger} & \cdots & V_{q-1,i-1}^{\dagger} & 0 & V_{q-1,i+1}^{\dagger} & \cdots & V_{q-1,n}^{\dagger} \\ V_{q1}^{\dagger} & \cdots & V_{q,i-1}^{\dagger} & V_{qr}^{\dagger} & V_{q,i+1}^{\dagger} & \cdots & V_{qn}^{\dagger} \\ V_{q+1,1}^{\dagger} & \cdots & V_{q+1,i-1}^{\dagger} & 0 & V_{q+1,i+1}^{\dagger} & \cdots & V_{q+1,n}^{\dagger} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1}^{\dagger} & \cdots & V_{n,i-1}^{\dagger} & 0 & V_{n,i+1}^{\dagger} & \cdots & V_{nn}^{\dagger} \end{vmatrix}.$$

$$\cdot\, x_{1j_1}\otimes\cdots\otimes x_{q-1,j_{q-1}}x_{ql_q}\otimes x_{q+1,j_{q+1}}\otimes\cdots\otimes x_{nj_n}. \tag{4.9}$$

For every multiindex $\mathbf{j}\in\Psi_l$, where $l<k$, the vector $x_{1j_1}\otimes x_{2j_2}\otimes\cdots\otimes x_{nj_n}$ appears exactly n times in the summation (4.9), once for every $q=1,2,\ldots,n$. Then we sum in (4.9) over q and because a determinant with two equal columns is zero it follows that the sum (4.9) equals

$$\sum_{l=0}^{k-1}\sum_{\mathbf{j}\in\Psi_l} a_{k-l,i}\begin{vmatrix} V_{11}x_{1j_1} & V_{12}x_{1j_1} & \cdots & V_{1n}x_{1j_1} \\ V_{21}x_{2j_2} & V_{22}x_{2j_2} & \cdots & V_{2n}x_{2j_2} \\ \vdots & \vdots & & \vdots \\ V_{n1}x_{nj_n} & V_{n2}x_{nj_n} & \cdots & V_{nn}x_{nj_n}\end{vmatrix}^{\otimes} = \sum_{j=0}^{k-1} a_{k-l,i}\Delta_0 z_l.$$

This establishes (*iv*).

To complete the proof we will show that (*iii*) implies (*ii*). This implication was proven in Proposition 2.15 for $k=1$. Assume now that we have already found $\mathbf{a}_1,\mathbf{a}_2,\ldots,\mathbf{a}_{k-1}$ and $x_{i1},x_{i2},\ldots,x_{i,k-2}$; $i=1,2,\ldots,n$ where $k\le p$ such that

$$\sum_{j=0}^{l-1} y_{i0}^{*}U_i\left(\mathbf{a}_{l-j}\right)x_{ij}=0;\qquad l=1,2,\ldots,k-1. \tag{4.10}$$

It remains to show that we can also find $\mathbf{a}_k$ and $x_{i,k-1}$, $i=1,2,\ldots,n$ such that

$$\sum_{j=0}^{k-1} y_{i0}^{*}U_i\left(\mathbf{a}_{k-j}\right)x_{ij}=0.$$

Since (*i*) and (*ii*) are equivalent we can find $x_{i,k-1}$, $i=1,2,\ldots,n$ such that

$$\sum_{j=0}^{k-2} U_i\left(\mathbf{a}_{k-j}\right)x_{ij}=W_i\left(\boldsymbol{\lambda}\right)x_{i,k-1}.$$

Next we build the vectors $z_l = \sum_{\mathbf{j}\in\Psi_l} x_{1j_1} \otimes x_{2j_2} \otimes \cdots \otimes x_{nj_n}$, $l = 0, 1, \ldots, k-1$. In the proof of Proposition 2.15 we showed that

$$z_1 \in \ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^2 / \ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma}).$$

In the Remark following that proof we pointed out that $a_{1h} \neq 0$ for some $h \in \underline{n}$. The same calculation used to show that (i) implies (iv) also proves that $(\lambda_i I - \Gamma_i) z_l = \sum_{j=0}^{l-1} \alpha_{l-j,i} z_j$, $l = 0, 1, \ldots, k-1$. Then the vectors $u_r = \Gamma_h^{k-1-r} z_{k-1}$, $r = 0, 1, \ldots, k-1$ form a Jordan chain for Γ_h of length k $(< p+1)$. Because $\dim\ker(\lambda_h I - \Gamma_h) = 1$, every Jordan chain can be extended to a maximal one (cf.[92, Theorem 2.9.2(b), p. 85]). Lemma 4.1 implies that $w_0^* \Delta_0 u_l = 0, l = 0, 1, \ldots, k-1$ and then, because $\mathcal{L}(\{u_l;\ l = 0, 1, \ldots, k-1\}) = \mathcal{L}(\{z_l;\ l = 0, 1, \ldots, k-1\}) = \ker(\lambda_h I - \Gamma_h)^k$, it also implies that

$$w_0^* \Delta_0 z_l = 0,\ l = 0, 1, \ldots, k-1. \tag{4.11}$$

Next we form the $n \times n$ matrices $B_l = \left[b_{ij}^l\right]_{i,j=1}^n$, $l = 0, 1, \ldots, k-1$ where $b_{ij}^l = y_{i0}^* V_{ij} x_{il}$. The relations (4.10) are equivalent to $\sum_{j=0}^{l-1} B_j \mathbf{a}_{l-j} = 0$ for $l = 1, 2, \ldots, k-1$ and the relations (4.11) are equivalent to

$$\sum_{\mathbf{j}\in\Psi_l} \begin{vmatrix} b_{11}^{j_1} & b_{12}^{j_1} & \cdots & b_{1n}^{j_1} \\ b_{21}^{j_1} & b_{22}^{j_1} & \cdots & b_{2n}^{j_1} \\ \vdots & \vdots & & \vdots \\ b_{n1}^{j_1} & b_{n2}^{j_1} & \cdots & b_{nn}^{j_1} \end{vmatrix} = 0, \qquad \text{for } l = 0, 1, \ldots, k-1.$$

Since $\boldsymbol{\lambda}$ is nonderogatory rank $B_0 = n-1$ and then Lemma 4.2 implies that there exists an n-tuple $\mathbf{a}_k$ such that $\sum_{j=0}^{k-1} B_j \mathbf{a}_{k-j} = 0$ or, equivalently, such that

$$\sum_{j=0}^{k-1} y_{i0}^* U_i(\mathbf{a}_{k-j}) x_{ij} = 0$$

for $i = 1, 2, \ldots, n$. □

Remark. Suppose that the conditions of Proposition 4.3 hold. Then for every index h such that $a_{1h} \neq 0$ (there is always at least one such h because $\mathbf{a}_1 \neq 0$) the vectors $u_l = (\lambda_h I - \Gamma_h)^{p-l} z_p$, $l = 0, 1, \ldots, p$ form a Jordan chain for Γ_h at eigenvalue λ_h.

This implies that the vectors $z_0, z_1, \ldots, z_p$ are linearly independent and they span $\bigcap_{i=1}^n \ker(\lambda_i I - \Gamma_i)^{p+1}$. $\square$

As an immediate consequence of Proposition 4.3 we have

Theorem 4.4 *Suppose $\boldsymbol{\lambda} \in \mathbb{C}^n$ is a nonderogatory eigenvalue for a multiparameter system* $\mathbf{W}$ *such that* $\dim\left(\bigcap_{i=1}^n \ker(\lambda_i I - \Gamma_i)^N\right) = p+1$. *Then there exist* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p \in \mathbb{C}^n, \mathbf{a}_1 \neq 0$, *and* $x_{i1}, x_{i2}, \ldots, x_{ip} \in H_i'$, $i = 1, 2, \ldots, n$ *such that*

$$\sum_{j=0}^{k-1} U_i(\mathbf{a}_{k-j})\, x_{ij} = W_i(\boldsymbol{\lambda})\, x_{ik} \qquad \text{for } k = 1, 2, \ldots, p;\ i = 1, 2, \ldots, n.$$

Moreover the vectors

$$z_k = \sum_{\mathbf{j} \in \Psi_k} x_{1j_1} \otimes x_{2j_2} \otimes \cdots \otimes x_{nj_n},\ k = 0, 1, \ldots, p$$

where Ψ_k *is defined in* (4.2), *are such that*

$$(\lambda_i I - \Gamma_i)\, z_k = \sum_{l=0}^{k-1} a_{k-j,i} z_j$$

and they are a basis for the root subspace $\bigcap_{i=1}^n \ker(\lambda_i I - \Gamma_i)^N$.

Algorithm to Construct a Basis for the Root Subspace of a Nonderogatory Eigenvalue

In the proof of Proposition 4.3 we can also find an algorithm for the construction of vectors z_k, $k = 0, 1, \ldots, p$ that form a basis for the root subspace of a nonderogatory eigenvalue. The construction uses only data from the multiparameter system $\mathbf{W}$.

Algorithm 4.5 *Step I. For* $i = 1, 2, \ldots, n$ *find* $x_{i0} \neq 0$ *and* $y_{i0} \neq 0$ *such that*

$$W_i(\boldsymbol{\lambda})\, x_{i0} = 0 \text{ and } y_{i0}^* W_i(\boldsymbol{\lambda}) = 0.$$

Choose a direct complement H_i' *of* $\ker W_i(\boldsymbol{\lambda})$ *for all* i. *Form* $z_0 = x_{10} \otimes x_{20} \otimes \cdots \otimes x_{n0}$ *and set* $k = 0$.

Step II. Find a matrix polynomial $L_k(\mu) = I\mu^{k+1} + B_k\mu^k + \cdots + B_0$ *and its determinant* $d_k(\mu) = \det L_k(\mu)$. *If*

$$d_k^{(k)}(0) = 0 \tag{4.12}$$

then set $k = k+1$ *and go to Step III, otherwise quit the algorithm.*

Step III. Find $\mathbf{a}_k \in \mathbb{C}^n, \mathbf{a}_1 \neq 0$, *such that*

$$\sum_{l=1}^{k} B_{k-l}\mathbf{a}_l = 0$$

where $B_l = \left[y_{i0}^* V_{ij} x_{il}\right]_{i,j=1}^n$. *For* $i = 1, 2, \ldots, n$ *find vectors* $x_{ik} \in H_i^!$ *such that*

$$\sum_{l=0}^{k-1} U_i(\mathbf{a}_{k-l})\, x_{il} = W_i(\boldsymbol{\lambda})\, x_{ik}.$$

Form $z_k = \sum_{\mathbf{j} \in \Psi_k} x_{1j_1} \otimes x_{2j_2} \otimes \cdots \otimes x_{nj_n}$. *Repeat Step II.*

It follows from Corollary 4.6 that the vectors $z_0, z_1, \ldots, z_k$ obtained in the above algorithm form a basis for the root subspace $\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^N$, i.e., they satisfy the relations

$$(\lambda_i I - \Gamma_i)\, z_l = \sum_{j=0}^{l-1} \mathbf{a}_{l-j,i} z_j \qquad \text{for } l = 0, 1, \ldots, k.$$

Definition. The smallest integer k such that the sum on the left-hand side of the condition (4.12) is not 0 is called the *ascent of* $\boldsymbol{\Gamma}$ *at the eigenvalue* λ.

The next result is the immediate consequence of Lemmas 4.1 and 4.2.

Corollary 4.6 *The ascent of* $\boldsymbol{\Gamma}$ *at the eigenvalue* $\boldsymbol{\lambda}$ *is equal to*

$$\dim\left(\bigcap_{i=1}^{n} \ker(\lambda_i I - \Gamma_i)^N\right).$$

Let us now demonstrate Algorithm 4.5 with an example :

Example 4.7 We consider a multiparameter system

$$W_1(\boldsymbol{\lambda}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}\lambda_1 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix}\lambda_2 - \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$W_2(\boldsymbol{\lambda}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda_1 + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \lambda_2 - \begin{bmatrix} 1 & 0 & -1 \\ -1 & -1 & -1 \\ 0 & -1 & -3 \end{bmatrix}.$$

Because Δ_0 is invertible it is nonsingular. The spectrum is

$$\sigma(\mathbf{W}) = \left\{ (1,-1), \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \right\}.$$

We consider the eigenvalue $\boldsymbol{\lambda}_0 = (1,-1)$. Then we have

$$W_1(\boldsymbol{\lambda}_0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \text{ and } W_2(\boldsymbol{\lambda}_0) = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

We observe that $\dim\ker W_1(\boldsymbol{\lambda}_0) = \dim\ker W_2(\boldsymbol{\lambda}_0) = 1$. To complete *Step I* of Algorithm 4.5 we choose

$$x_{10} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; x_{20} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; y_{10} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } y_{20} = \begin{bmatrix} 4 \\ 0 \\ -1 \end{bmatrix}.$$

We also set $H_1' = \left\{ \begin{bmatrix} 0 \\ a \\ b \end{bmatrix}, \; a, b \in \mathbb{C} \right\}$ and $H_2' = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix}, \; a, b \in \mathbb{C} \right\}$. The matrix $B_0 = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ has rank 1 and therefore $\boldsymbol{\lambda}_0$ is a nonderogatory eigenvalue. We have $z_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Then we go to *Step III*. We choose $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and

$$x_{11} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \text{ and } x_{21} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

since $V_{11}x_{10} = W_1(\boldsymbol{\lambda}_0)\,x_{11}$ and $V_{21}x_{20} = W_2(\boldsymbol{\lambda}_0)\,x_{21}$. Then it follows that $z_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $L_1(\mu) = \begin{bmatrix} \mu^2 & 0 \\ 0 & \mu^2+1 \end{bmatrix}$, so $d_1(\mu) = \mu^2(\mu^2+1)$.

Because $d_1'(0) = 0$ we repeat *Step III*. Now we choose $\mathbf{a}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and vectors

$$x_{12} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \text{ and } x_{22} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so that $V_{11}x_{11} = W_1(\boldsymbol{\lambda}_0)\,x_{12}$ and $V_{21}x_{21} = W_2(\boldsymbol{\lambda}_0)\,x_{22}$. We have $z_2 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $B_2 = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix}$. The matrix polynomial $L_2(\mu) = \begin{bmatrix} \mu^3+2\mu^2 & \mu^2 \\ 0 & \mu^3+1 \end{bmatrix}$ has determinant $d_2(\mu) = \mu^2(\mu+2)(\mu^3+1)$. Because $d_2''(0) \neq 0$ we quit the algorithm. The root subspace at the eigenvalue $\boldsymbol{\lambda}_0$ is three-dimensional and has a basis $\{z_0, z_1, z_2\}$.
□

4.3 Self-adjoint Multiparameter Systems

4.3.1 Elementary Properties

Definition. A multiparameter system $\mathbf{W}$ is called *self-adjoint* if all the transformations V_{ij}, $i = 1, 2, \ldots, n$, $j = 0, 1, \ldots, n$ are self-adjoint, i.e. $V_{ij} = V_{ij}^*$.

In this section we study self-adjoint multiparameter systems. It is an easy consequence of the definition that also all the associated transformations Δ_i, $i = 0, 1, \ldots, n$ are self-adjoint in H with $(\cdot,\cdot)$. The scalar product $(\cdot,\cdot)$ is defined on page 8. For a self-adjoint multiparameter system $\mathbf{W}$ we define a new bilinear form on H by $[x, y] = (\Delta_0 x, y)$ for all $x, y \in H$. The bilinear form $[\cdot,\cdot]$ is an indefinite nondegenerate scalar product because Δ_0 is invertible.

Definition. An operator $T \in \mathcal{L}(H)$ is Δ_0-*self-adjoint* if $[Tx, y] = [x, Ty]$ for all $x, y \in H$.

Lemma 4.8 *The associated transformations* Γ_i, $i = 1, 2, \ldots, n$ *are* Δ_0*-self-adjoint.*

Proof. For any two $x, y \in H$ it follows that

$$[\Gamma_i x, y] = (\Delta_i x, y) = (x, \Delta_i y) = [x, \Gamma_i y] .$$

□

Lemma 4.9 *Suppose that* $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ *and that* $\lambda_i \in \mathbb{R}$ *for an index* i. *Then* $\boldsymbol{\lambda} \in \mathbb{R}^n$.

Proof. Assume that $x \in H$ is an eigenvector for $\boldsymbol{\Gamma}$ at $\boldsymbol{\lambda}$. Consider now a subspace $\mathcal{N} = \bigcap_{i=1}^n \ker(\lambda_i I - \Gamma_i)^N$. Because $\mathcal{N}$ is invariant for Γ_i and Γ_i is Δ_0-self-adjoint it follows from [91, Theorem 3.3] (see also [91, Section 3.4, pp. 37-38]) that $\mathcal{N}$ is nondegenerate for $[\cdot, \cdot]$. Now suppose that $\lambda_j \in \mathbb{C}/\mathbb{R}$ for some $j \neq i$. Then it follows from [91, Corollary 2.6] that $\mathcal{N}$ is neutral in $[\cdot, \cdot]$. This contradicts the assertion that $\mathcal{N}$ is nondegenerate for $[\cdot, \cdot]$. Hence it follows that $\lambda_j \in \mathbb{R}$ for all j. □

Definition. A self-adjoint multiparameter system $\mathbf{W}$ is called *right-definite* if Δ_0 is a positive (or negative) definite matrix in H.

Definition. An eigenvalue $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ is called *semisimple* if

$$\dim\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^2 = \prod_{i=1}^n \dim\ker W_i(\boldsymbol{\lambda}),$$

or equivalently, if the eigenvectors span the root subspace at $\boldsymbol{\lambda}$.

When $\mathbf{W}$ is right-definite the bilinear form $[\cdot, \cdot]$ (or $-[\cdot, \cdot]$) is actually a definite scalar product and therefore $\sigma(\mathbf{W}) \subset \mathbb{R}^n$ and each Γ_i has a basis of eigenvectors. Then we have the following completeness result (see [10, Theorem 10.6.1]). We state it to make this dissertation more complete.

Theorem 4.10 *Assume that* $\mathbf{W}$ *is right-definite. Then the spectrum* $\sigma(\mathbf{W})$ *is real. Furthermore all the eigenvalues are semisimple and there exists a basis for* H *consisting of decomposable eigenvectors for* $\mathbf{\Gamma}$.

Proof. Because Δ_0 is a positive (or negative) definite operator the scalar product $[\cdot,\cdot]$ (or $-[\cdot,\cdot]$) is definite. The transformations Γ_i are self-adjoint in $[\cdot,\cdot]$ and hence $\mathbb{R}^n \supset \sigma(\mathbf{\Gamma}) = \sigma(\mathbf{W})$. Also all the eigenvalues of Γ_i are semisimple. Therefore it follows from Theorem 2.11 that there exists a basis of decomposable tensors for $\bigcap_{i=1}^n \ker(\lambda_i I - \Gamma_i)$ for all $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ and so for

$$H = \bigoplus_{\lambda \in \sigma(\mathbf{W})} \left(\bigcap_{i=1}^{n} \ker(\lambda_i I - \Gamma_i) \right).$$

□

4.3.2 Weakly-elliptic Case

Definition. A self-adjoint multiparameter system is called *weakly-elliptic* if there exists a cofactor Δ_{0ij} of Δ_0 that is a positive definite operator on H.

As an immediate consequence of Theorem 4.4 we have :

Theorem 4.11 *Assume that* $\boldsymbol{\lambda}$ *is a real eigenvalue for a weakly-elliptic multiparameter system* $\mathbf{W}$ *and that* $\dim\ker W_i(\boldsymbol{\lambda}) = 1$ *for all* i. *Then* $\boldsymbol{\lambda}$ *is nonderogatory and there exist* n*-tuples* $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_p$, $\mathbf{a}_1 \neq 0$ *and vectors* $x_{ij} \in H_i'$, $i = 1, 2, \ldots, n$, $j = 0, 1, \ldots, p$ *such that*

$$W_i(\boldsymbol{\lambda}) x_{il} = \sum_{j=1}^{l-1} U_i(\mathbf{a}_j) x_{i,l-j} \qquad \textit{for } l = 0, 1, \ldots, p.$$

The vectors $z_l = \sum_{\mathbf{j} \in \Phi_l} x_{1j_1} \otimes x_{2j_2} \otimes \cdots \otimes x_{nj_n}$ *are such that*

$$(\lambda_i I - \Gamma_i) z_l = \sum_{k=0}^{l-1} a_{ki} z_k.$$

Moreover, if $p+1$ *is the ascent of* $\boldsymbol{\lambda}$ *then the vectors* z_l, $l = 0, 1, \ldots, p$ *form a basis for a root subspace of* $\mathbf{W}$ *at* $\boldsymbol{\lambda}$.

Proof. Suppose that $x_{i0} \in \ker W_i(\boldsymbol{\lambda})$ are nonzero vectors. Then we only need to show that

$$\operatorname{rank}\begin{bmatrix} x_{10}^* V_{11} x_{10} & x_{10}^* V_{12} x_{10} & \cdots & x_{10}^* V_{1n} x_{10} \\ x_{20}^* V_{21} x_{20} & x_{20}^* V_{22} x_{20} & \cdots & x_{20}^* V_{2n} x_{20} \\ \vdots & \vdots & & \vdots \\ x_{n0}^* V_{n1} x_{n0} & x_{n0}^* V_{n2} x_{n0} & \cdots & x_{n0}^* V_{nn} x_{n0} \end{bmatrix} \geq n-1.$$

The result then follows from Theorem 4.4. By definition of a weakly-elliptic multiparameter system it follows that $z_0^* \Delta_{0ij} z_0 \neq 0$ for some i and j. Since

$$z_0^* \Delta_{0ij} z_0 = x_{i0}^* x_{i0} \cdot \left(\widehat{z_0}^i\right)^* \Delta_{0ij} \widehat{z_0}^i,$$

where $\widehat{z_0}^i = x_{10} \otimes \cdots \otimes x_{i-1,0} \otimes x_{i+1,0} \otimes \cdots \otimes x_{n0}$, it follows that the cofactor of $x_{i0}^* V_{ij} x_{i0}$ in the matrix

$$B_0 = \begin{bmatrix} x_{10}^* V_{11} x_{10} & x_{10}^* V_{12} x_{10} & \cdots & x_{10}^* V_{1n} x_{10} \\ x_{20}^* V_{21} x_{20} & x_{20}^* V_{22} x_{20} & \cdots & x_{20}^* V_{2n} x_{20} \\ \vdots & \vdots & & \vdots \\ x_{n0}^* V_{n1} x_{n0} & x_{n0}^* V_{n2} x_{n0} & \cdots & x_{n0}^* V_{nn} x_{n0} \end{bmatrix}$$

is nonzero and so rank $B_0 \geq n-1$. □

Remark. A special case of the weakly-elliptic case is the elliptic case. A multiparameter system is called *elliptic* if Δ_{0ij}, $i = 1, 2, \ldots, n$ are positive definite operators on H for some j. A special case of Theorem 4.11 for the latter case was first proved by Binding [23, Theorem 3.1] in a more general setting with a different method. We remark that we do not generalize his main result [23, Theorem 3.2].

4.4 Simple Eigenvalues

4.4.1 A Basis for the Third Root Subspace

Suppose that $\boldsymbol{\lambda}$ is a simple eigenvalue and that $\mathbf{a}_1^{01}, \mathbf{a}_2^{01}, \ldots, \mathbf{a}_{d_1}^{01}$ form a basis for $\ker B_0$ and $\mathbf{b}_1^{01}, \mathbf{b}_2^{01}, \ldots, \mathbf{b}_{d_1}^{01}$ a basis for $\ker B_0^*$. We write $\mathbf{b}_0 = \left[\mathbf{b}_1^{01}, \mathbf{b}_2^{01}, \ldots, \mathbf{b}_{d_1}^{01}\right]$. We restrict our attention to the root subspace $\mathcal{N} = \ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^N$ and we bring the restricted transformations $(\lambda_i I - \Gamma_i)\,|_{\mathcal{N}}$, that are commuting and nilpotent, to the form (1.2). We refer to (1.3), (1.5) and (1.6) for the definitions of the arrays $\mathbf{A}^{kl}$ and their row and column cross-sections, respectively. It follows from Theorem 1.12 that for every C_k^{12}, $k \in \underline{d_2}$ there exists a unique symmetric matrix T_k such that $R_1^{01}T_k = C_k^{12}$. We choose vectors z_1^k, $k = 1, 2, \ldots, d_1$ such that $\mathcal{B}_1 = \left\{z_0, z_1^1, z_1^2, \ldots, z_1^{d_1}\right\}$ is a basis for $\ker\left(\mathbf{\lambda I} - \mathbf{\Gamma}\right)^2$ and $(\lambda_i I - \Gamma_i)\, z_1^k = a_{ki}^{01} z_0$. Further we have that $z_0 = x_{10} \otimes x_{20} \otimes \cdots \otimes x_{n0}$ and we showed in Proposition 2.17 that there exist vectors $x_{i1}^k \in H_i'$, where $H_i' \subset H_i$ is a direct summand of the kernel $\ker W_i(\boldsymbol{\lambda})$, such that $z_1^k = \sum_{s=1}^n x_{10} \otimes \cdots \otimes x_{s1}^k \otimes \cdots \otimes x_{n0}$ and $U_i\left(\mathrm{a}_k^{01}\right) x_{i0} = W_i(\boldsymbol{\lambda})\, x_{i1}^k$.

Now we define matrices $B_{1k} \in \mathbb{C}^{n \times n}$, $k \in \underline{d_1}$ by

$$B_{1k} = \begin{bmatrix} y_{10}^* V_{11} x_{11}^k & y_{10}^* V_{12} x_{11}^k & \cdots & y_{10}^* V_{1n} x_{11}^k \\ y_{20}^* V_{21} x_{21}^k & y_{20}^* V_{22} x_{21}^k & \cdots & y_{20}^* V_{2n} x_{21}^k \\ \vdots & \vdots & & \vdots \\ y_{n0}^* V_{n1} x_{n1}^k & y_{n0}^* V_{n2} x_{n1}^k & \cdots & y_{n0}^* V_{nn} x_{n1}^k \end{bmatrix},$$

and then we define a matrix $S \in \mathbb{C}^{n \times (d_1+1)d_1/2}$ as follows : for $p \in \underline{\frac{(d_1+1)d_1}{2}}$ we can uniquely choose numbers k and l so that $k \geq l \geq 1$ and $p = \frac{(k-1)k}{2} + l$. Then the p-th column of S is equal to $B_{1,k}\mathrm{a}_l^{01} + B_{1,l}\mathrm{a}_k^{01}$ if $k \neq l$ and to $B_{1,k}\mathrm{a}_k^{01}$ otherwise. The matrix S is called *a symmetrization of the array* $\mathbf{A}^{01}$. We also write $\mathcal{S}_2 = \mathbf{b}_0^* S$.

Further we identify the subspace Θ of symmetric $d_1 \times d_1$ matrices with the space $\mathbb{C}^{(d_1+1)d_1/2}$. Note that Θ is a vector subspace over the complex numbers because $aT^T = (aT)^T$ if $a \in \mathbb{C}$ and T is symmetric. The isomorphism $\psi : \Theta \longrightarrow \mathbb{C}^{(d_1+1)d_1/2}$ is defined by

$$\psi(T) = \begin{bmatrix} t_{11} & t_{12} & t_{22} & t_{13} & t_{23} & t_{33} & \ldots & t_{1d_1} & t_{2d_1} & \cdots & t_{d_1 d_1} \end{bmatrix}^T \tag{4.13}$$

where $T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ t_{12} & t_{22} & \cdots & t_{2n} \\ \vdots & \vdots & & \vdots \\ t_{1n} & t_{2n} & \cdots & t_{nn} \end{bmatrix} \in \Theta.$

Theorem 4.12 *Suppose that* $\mathbf{t} \in \ker \mathcal{S}_2 \backslash \{0\}$ *and* $T = \psi^{-1}(\mathbf{t})$. *Then there exists an* n*-tuple* $\mathbf{a}^{02} \in \mathbb{C}^n$ *such that*

$$\sum_{k,l=1}^{d_1} t_{kl} B_{1k} \mathbf{a}_l^{01} + B_0 \mathbf{a}^{02} = 0. \tag{4.14}$$

Furthermore there exist vectors $x_{i2} \in H_i'$, *for* $i = 1, 2, \dots, n$, *such that*

$$U_i\left(\mathbf{a}^{02}\right) x_{i0} + \sum_{k=1}^{d_1} U_i\left(\mathbf{a}_k^{12}\right) x_{i1}^k = W_i(\boldsymbol{\lambda}) x_{i2} \tag{4.15}$$

where $\mathbf{a}_k^{12} = \sum_{l=1}^{d_1} t_{kl} \mathbf{a}_l^{01}$. *Then the vector*

$$z_2 = \sum_{s=1}^{n} x_{10} \otimes \cdots \otimes x_{s2} \otimes \cdots \otimes x_{n0} + \sum_{k,l=1}^{d_1} t_{kl} \sum_{s=1}^{n-1} \sum_{t=s+1}^{n} x_{10} \otimes \cdots \otimes x_{s1}^k \otimes \cdots \otimes x_{t1}^l \otimes \cdots \otimes x_{n0} \tag{4.16}$$

is in $\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^3 / \ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^2$ *and*

$$(\lambda_i I - \Gamma_i) z_2 = \sum_{k=1}^{d_1} a_{ki}^{12} z_1^k + a_i^{02} z_0. \tag{4.17}$$

Conversely, if $z_2 \in \ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^3 / \ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^2$ *and* (4.17) *holds then if* T *is the unique symmetric matrix such that* $C^{12} = \begin{bmatrix} \mathbf{a}_1^{12} & \mathbf{a}_2^{12} & \cdots & \mathbf{a}_{d_1}^{12} \end{bmatrix} = R_1^{01} T$ *it follows that* $\psi(T) \in \ker \mathcal{S}_2$ *and there exist vectors* $x_{i2} \in H_i$, $i = 1, 2, \dots, n$, *such that* (4.15) *and* (4.16) *hold.*

Proof. Because $\mathbf{t} \in \ker \mathcal{S}_2$ and $T = \psi^{-1}(\mathbf{t})$ it follows that

$$\sum_{k=1}^{d_1} \sum_{l=1}^{d_1} t_{kl} \mathbf{b}_0^* B_{1l} \mathbf{a}_k^{01} = 0.$$

Hence $\sum_{k=1}^{d_1} \sum_{l=1}^{d_1} t_{kl} B_{1l} \mathbf{a}_k^{01} \in (\ker B_0^*)^\perp$ and therefore there exists $\mathbf{a}^{02} \in \mathbb{C}^n$ such that the equality (4.14) holds. By definition of the matrices B_0 and B_{1k} it follows that

$$\sum_{k=1}^{d_1} \sum_{l=1}^{d_1} t_{kl} \sum_{j=1}^{n} y_{i0}^* V_{ij} a_{kj}^{01} x_{i1}^l + \sum_{j=1}^{n} y_{i0}^* V_{ij} x_{i0} a_j^{02} = 0$$

for $i = 1, 2, \dots, n$. Then $U_i(\mathbf{a}^{02}) x_{i0} + \sum_{k=1}^{d_1} U_i(\mathbf{a}_k^{12}) x_{i1}^k \in (\ker W_i(\boldsymbol{\lambda})^*)^\perp$ and so it follows from Lemma 2.12 that there exist vectors $x_{i2} \in H_i'$ such that (4.15) hold for $\mathbf{a}_k^{12} = \sum_{l=1}^{d_1} t_{kl} \mathbf{a}_l^{01}$. Next we form the vector z_2 as in (4.16) and we have

$$(\lambda_i \Delta_0 - \Delta_i) z_2 =$$

$$\sum_{s=1}^{n}\begin{vmatrix} V_{11}x_{10} & \cdots & V_{1,i-1}x_{10} & 0 & V_{1,i+1}x_{10} & \cdots & V_{1n}x_{10} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{s-1,1}x_{s-1,0} & \cdots & V_{s-1,i-1}x_{s-1,0} & 0 & V_{s-1,i+1}x_{s-1,0} & \cdots & V_{s-1,n}x_{s-1,0} \\ V_{s1}x_{s2} & \cdots & V_{s,i-1}x_{s2} & W_s(\boldsymbol{\lambda})\,x_{s2} & V_{s,i+1}x_{s2} & \cdots & V_{sn}x_{s2} \\ V_{s+1,1}x_{s+1,0} & \cdots & V_{s+1,i-1}x_{s+1,0} & 0 & V_{s+1,i+1}x_{s+1,0} & \cdots & V_{s+1,n}x_{s+1,0} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1}x_{n0} & \cdots & V_{n,i-1}x_{n0} & 0 & V_{n,i+1}x_{n0} & \cdots & V_{nn}x_{n0} \end{vmatrix}^{\otimes} +$$

$$+\sum_{k,l=1}^{d_1} t_{kl}\sum_{s=1}^{n-1}\sum_{t=s+1}^{n}\begin{vmatrix} V_{11}x_{10} & \cdots & V_{1,i-1}x_{10} & 0 & V_{1,i+1}x_{10} & \cdots & V_{1n}x_{10} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{s1}x_{s1}^k & \cdots & V_{s,i-1}x_{s1}^k & W_s(\boldsymbol{\lambda})\,x_{s1}^k & V_{s,i+1}x_{s1}^k & \cdots & V_{sn}x_{s1}^k \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{t1}x_{t1}^l & \cdots & V_{t,i-1}x_{t1}^l & W_t(\boldsymbol{\lambda})\,x_{t1}^l & V_{t,i+1}x_{t1}^l & \cdots & V_{tn}x_{tn}^l \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1}x_{n0} & \cdots & V_{n,i-1}x_{n0} & 0 & V_{n,i+1}x_{n0} & \cdots & V_{nn}x_{n0} \end{vmatrix}^{\otimes} =$$

$$= a_i^{02}\begin{vmatrix} V_{11}x_{10} & V_{12}x_{10} & \cdots & V_{1n}x_{10} \\ V_{21}x_{20} & V_{22}x_{20} & \cdots & V_{2n}x_{20} \\ \vdots & \vdots & & \vdots \\ V_{n1}x_{n0} & V_{n2}x_{n0} & \cdots & V_{nn}x_{n0} \end{vmatrix}^{\otimes} +$$

$$+\sum_{k,l=1}^{d_1} t_{kl}\left(\sum_{s=1}^{n}\begin{vmatrix} V_{11}x_{10} & \cdots & 0 & \cdots & V_{1n}x_{10} \\ \vdots & & \vdots & & \vdots \\ V_{s1}x_{s1}^k & \cdots & U_s\left(\mathbf{a}_l^{01}\right)x_{s1}^k & \cdots & V_{sn}x_{s1}^k \\ \vdots & & \vdots & & \vdots \\ V_{n1}x_{n0} & \cdots & 0 & \cdots & V_{nn}x_{n0} \end{vmatrix}^{\otimes} + \right.$$

$$+\sum_{s=1}^{n-1}\sum_{t=s+1}^{n}\left(\begin{vmatrix} V_{11}x_{10} & \cdots & 0 & \cdots & V_{1n}x_{n0} \\ \vdots & & \vdots & & \vdots \\ V_{s1}x_{s1}^{k} & \cdots & 0 & \cdots & V_{sn}x_{s1}^{k} \\ \vdots & & \vdots & & \vdots \\ V_{t1}x_{t0} & \cdots & U_t\left(\mathrm{a}_l^{01}\right)x_{t0} & \cdots & V_{tn}x_{t0} \\ \vdots & & \vdots & & \vdots \\ V_{n1}x_{n0} & \cdots & 0 & \cdots & V_{nn}x_{n0} \end{vmatrix}^{\otimes} +\right.$$

$$\left.+\begin{vmatrix} V_{11}x_{10} & \cdots & 0 & \cdots & V_{1n}x_{n0} \\ \vdots & & \vdots & & \vdots \\ V_{s1}x_{s0} & \cdots & U_s\left(\mathrm{a}_k^{01}\right)x_{s0} & \cdots & V_{sn}x_{s0} \\ \vdots & & \vdots & & \vdots \\ V_{t1}x_{t1}^{l} & \cdots & 0 & \cdots & V_{tn}x_{t1}^{l} \\ \vdots & & \vdots & & \vdots \\ V_{n1}x_{n0} & \cdots & 0 & \cdots & V_{nn}x_{n0} \end{vmatrix}^{\otimes}\right)\Bigg) =$$

$$= a_i^{02}\Delta_0 z_0 + \sum_{k,l=1}^{d_1} t_{kl}\sum_{s=1}^{n}\begin{vmatrix} V_{11}x_{10} & \cdots & U_1\left(\mathrm{a}_l^{01}\right)x_{10} & \cdots & V_{1n}x_{10} \\ \vdots & & \vdots & & \vdots \\ V_{s1}x_{s1}^{k} & \cdots & U_s\left(\mathrm{a}_l^{01}\right)x_{s1}^{k} & \cdots & V_{sn}x_{s1}^{k} \\ \vdots & & \vdots & & \vdots \\ V_{n1}x_{n0} & \cdots & U_n\left(\mathrm{a}_l^{01}\right)x_{n0} & \cdots & V_{nn}x_{n0} \end{vmatrix}^{\otimes} =$$

$$= a_i^{02}\Delta_0 z_0 + \sum_{k=1}^{d_1} a_{ki}^{12}\Delta_0 z_1^k.$$

Conversely, suppose that $z_2 \in \ker\left(\boldsymbol{\lambda}\mathbf{I}-\boldsymbol{\Gamma}\right)^3 / \ker\left(\boldsymbol{\lambda}\mathbf{I}-\boldsymbol{\Gamma}\right)^2$. Then there exist a symmetric matrix $T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1d_1} \\ t_{12} & t_{22} & \cdots & t_{2d_1} \\ \vdots & \vdots & & \vdots \\ t_{1d_1} & t_{2d_1} & \cdots & t_{d_1d_1} \end{bmatrix} \in \mathbb{C}^{d_1\times d_1}$ and a vector $\mathbf{a}^{02} \in \mathbb{C}^n$ such that

$$\left(\lambda_i I - \Gamma_i\right) z_2 = \sum_{k=1}^{d_1} a_{ki}^{12} z_1^k + a_i^{02} z_0 \tag{4.18}$$

for all i and

$$\mathbf{a}_k^{12} = \sum_{l=1}^{d_2} t_{kl} \mathbf{a}_l^{01}. \tag{4.19}$$

Relation (4.19) is a consequence of Theorem 1.12. Next it follows from (2.7) that

$$\sum_{j=1}^{n} V_{ij}^{\dagger} \left(\lambda_j I - \Gamma_j\right) z_2 = W_i \left(\boldsymbol{\lambda}\right)^{\dagger} z_2$$

and from (4.18)

$$\sum_{j=1}^{n} V_{ij}^{\dagger} \left(\sum_{k=1}^{d_1} a_{kj}^{12} z_1^k + a_j^{02} z_0 \right) = W_i \left(\boldsymbol{\lambda}\right)^{\dagger} z_2. \tag{4.20}$$

For $i = 1, 2, \ldots, n$ we choose vectors $v_i \in H_i$ so that $v_i^* x_{i0} = 1$ and $v_i^* x_{i1}^k = 0$ for $k \in \underline{d_1}$. This is possible because $\mathcal{L}\{x_{i0}\} \cap H_i' = \{0\}$. After multiplying (4.20) by $v_1^* \otimes \cdots \otimes v_{i-1}^* \otimes y_{i0}^* \otimes v_{i+1}^* \otimes \cdots \otimes v_n^*$ on the left-hand side we get

$$\sum_{j=1}^{n} y_{i0}^* V_{ij} \sum_{k=1}^{d_1} a_{kj}^{12} x_{i1}^k + \sum_{j=1}^{n} y_{i0}^* V_{ij} a_j^{02} x_{i0} = 0 \tag{4.21}$$

for all i and therefore there exist vectors $x_{i2} \in H_i'$ such that (4.15) hold. Now we form the vector

$$z_2^1 = \sum_{s=1}^{n} x_{10} \otimes \cdots \otimes x_{s2} \otimes \cdots \otimes x_{n0} + \sum_{k,l=1}^{d_1} t_{kl} \sum_{s=1}^{n-1} \sum_{t=s+1}^{n} x_{10} \otimes \cdots \otimes x_{s1}^k \otimes \cdots \otimes x_{t1}^l \otimes \cdots \otimes x_{n0}.$$

The same calculation as above shows that

$$\left(\lambda_i I - \Gamma_i\right) z_2^1 = \sum_{k=1}^{d_1} a_{ki}^{12} z_1^k + a_i^{02} z_0.$$

Hence it follows that $z_2^1 - z_2 \in \ker\left(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma}\right)$ and so there exists a number $\delta \in \mathbb{C}$ such that $z_2 = z_2^1 + \delta z_0$. Without loss we can use the vector $x_{12} + \delta x_{10}$ in place of x_{12}. Then it follows that

$$z_2 = \sum_{s=1}^{n} x_{10} \otimes \cdots \otimes x_{s2} \otimes \cdots \otimes x_{n0} + \sum_{k,l=1}^{d_2} t_{kl} \sum_{s=1}^{n-1} \sum_{t=s+1}^{n} x_{10} \otimes \cdots \otimes x_{s1}^k \otimes \cdots \otimes x_{t1}^l \otimes \cdots \otimes x_{n0}.$$

It remains to be shown that $\psi\left(T\right) \in \ker S_2$. The equalities (4.21) can be written in matrix form as

$$\sum_{k=1}^{d_1} B_{1k} \mathbf{a}_k^{12} + B_0 \mathbf{a}^{02} = 0.$$

Multiplication on the left-hand side by the matrix $\mathbf{b}_0^*$ yields

$$\sum_{k=1}^{d_1} \mathbf{b}_0^* B_{1k} \mathbf{a}_k^{12} = 0$$

and then also

$$\sum_{k=1}^{d_1} \sum_{l=1}^{d_1} \mathbf{b}_0^* B_{1k} \mathbf{a}_l^{01} t_{kl} = 0. \tag{4.22}$$

Finally, we note that the relation (4.22) is equivalent to $\psi(T) \in \ker \mathcal{S}_2$. □

Corollary 4.13 *Suppose that* $\mathcal{T} = \{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_d\}$ *is a basis for* $\ker \mathcal{S}_2$ *and that vectors* $z_2^1, z_2^2, \ldots, z_2^d$ *are associated with* $\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_d$, *respectively, as described in the first paragraph of Theorem* 4.12. *Then*

$$\left\{z_0; z_1^1, z_1^2, \ldots, z_1^{d_1}; z_2^1, z_2^2, \ldots, z_2^d\right\} \tag{4.23}$$

is a basis for $\ker (\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^3$. *We can choose a basis* $\mathcal{T}$ *so that the nonzero* n*-tuples, associated with basis* (4.23), *in the set* $\left\{\mathbf{a}_g^{0k},\ k = 1, 2,\ g \in \underline{d_k}\right\}$ *are linearly independent.*

Conversely, if $z_2^1, z_2^2, \ldots, z_2^{d_2}$ *are such that* $\left\{z_0; z_1^1, z_1^2, \ldots, z_1^{d_1}; z_2^1, z_2^2, \ldots, z_2^{d_2}\right\}$ *is a basis for* $\ker (\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^3$ *and* $T_1, T_2, \ldots, T_{d_2}$ *are symmetric matrices such that* $C_k^{12} = R_1^{01} T_k$, $k \in \underline{d_2}$ *then* $\{\psi(T_1), \psi(T_2), \ldots, \psi(T_{d_2})\}$ *is a basis for* $\ker \mathcal{S}_2$. *In particular, it follows that* $d = d_2$.

Proof. The corollary follows using the correspondence between h and z_2 as described in Theorem 4.12 and the fact that z_2^k are linearly independent if and only if T_k are linearly independent. We only need to show that the n-tuples $\mathbf{a}_g^{02}$ can be chosen so that the nonzero n-tuples in the set $\left\{\mathbf{a}_g^{0k},\ k = 1, 2,\ g \in \underline{d_k}\right\}$ are linearly independent. Suppose that $\mathcal{T}' = \{\mathbf{t}_1', \mathbf{t}_2', \ldots, \mathbf{t}_d'\}$ is a basis for the kernel of $\mathbf{b}_0^* \mathcal{S}_1$. Then there exist n-tuples $\mathbf{a}_g^{02'}$, $g \in \underline{d}$, so that $\mathcal{S}_1 \mathbf{t}_g' + B_0 \mathbf{a}_g^{02'} = 0$. Because $\mathbf{a}_k^{01} \in \ker B_0$, $k \in \underline{d_1}$ we can substitute basis $\mathcal{T}'$ for a basis $\mathcal{T} = \{\mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_d\}$ such that the n-tuples $\mathbf{a}_g^{02}$, satisfying relations $\mathcal{S}_1 \mathbf{t}_g + B_0 \mathbf{a}_g^{02} = 0$, are as required. □

Let us now consider an example.

Example 4.14 Suppose that

$$W_1(\boldsymbol{\lambda}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \lambda_1 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 \end{bmatrix} \lambda_2 - \begin{bmatrix} -1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and

$$W_2(\boldsymbol{\lambda}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \lambda_1 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda_2 - \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}.$$

Because the matrix Δ_0 is invertible $\mathbf{W}$ is nonsingular. We consider the eigenvalue $\boldsymbol{\lambda}_0 = (1, -1)$. The matrices

$$W_1(\boldsymbol{\lambda}_0) = \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \text{ and } W_2(\boldsymbol{\lambda}_0) = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

both have rank 2. We choose vectors

$$x_{10} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \ x_{20} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ y_{10} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } y_{20} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

We also assume that $H_1' = \left\{ \begin{bmatrix} a \\ b \\ b \end{bmatrix}, \ a, b \in \mathbb{C} \right\}$ and $H_2' = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}, \ a, b \in \mathbb{C} \right\}$. Then

we have $z_0 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $B_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So the eigenvalue $\boldsymbol{\lambda}_0$ is simple.

We take $\mathbf{a}_1^{01} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{a}_2^{01} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{b}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Next we choose vectors

$$x_{11}^1 = x_{11}^2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ x_{21}^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } x_{21}^2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

such that $U_i\left(\mathrm{a}_f^{01}\right) x_{i0} = W_i\left(\boldsymbol{\lambda}_0\right) x_{i1}^f$ for $i, f = 1, 2$. Then we have

$$z_1^1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } z_1^2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and also

$$B_{11} = B_{12} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \text{ and } \mathcal{S}_2 = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence $d_2 = \dim \ker \mathcal{S}_2 = 2$. We choose $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$ for a basis of $\ker \mathcal{S}_2$.

Then it follows that $T_1^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $T_2^2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and so

$$\mathbf{A}^{12} = \begin{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ -1 \end{pmatrix} & \begin{pmatrix} -1 \\ 1 \end{pmatrix} \end{bmatrix}.$$

Next we have to find vectors $x_{i2}^f \in H_i'$, for $i, f = 1, 2$, so that

$$V_{i1} x_{i1}^1 - V_{i2} x_{i2}^2 = W_i\left(\boldsymbol{\lambda}_0\right) x_{i2}^1$$

and

$$\left(V_{i1} - V_{i2}\right) x_{i1}^1 + \left(-V_{i1} + V_{i2}\right) x_{i1}^2 = W_i\left(\boldsymbol{\lambda}_0\right) x_{i2}^2.$$

They are

$$x_{12}^1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \; x_{12}^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \; x_{22}^1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \text{ and } x_{22}^2 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}.$$

Then the vectors

$$z_2^1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and

$$z_2^2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

together with z_0, z_1^1 and z_1^2 form a basis for the third root subspace $\ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^3$. □

4.4.2 A Basis for the Root Subspace

By Theorem 1.18 we can conclude that the root subspace at a simple eigenvalue $\boldsymbol{\lambda}$, and the action of the n-tuple of matrices $\boldsymbol{\Gamma}$ on it, is completely described by vectors z_m^f, $m = 0, 1, \ldots, M$ for $f \in \underline{d_m}$, corresponding n-tuples $\mathbf{a}_j^{0m}$, $m = 1, 2, \ldots, M$ for $j \in \underline{r_m}$ and symmetric matrices T_f^m, $m = 2, 3, \ldots, M$, for $f \in \underline{d_m}$ that satisfy the regularity and matching conditions. For $m = 0, 1, 2$ we have already seen in Theorem 4.12 that we can describe the vectors z_m^f using vectors $x_{i0}, x_{i1}^{j_1}, x_{i2}^{j_2}$, $j_1 \in \underline{d_1}$, $j_2 \in \underline{d_2}$, where $i \in \underline{n}$, and matrices $T_{j_1}^2$, $j_1 \in \underline{d_1}$. Our aim is to find an inductive procedure, i.e., an algorithm, to construct the vectors z_m^f for all m and f.

In what follows we again use the sets of multiindices $\Phi_{m,q}$ and $\chi_{\mathbf{l}}$ as defined on page 48. The symbol Ω_q, $1 \le q \le n$, stands for the set of all multiindices $\mathbf{u} = (u_1, u_2, \ldots, u_q)$ such that $1 \le u_1 < u_2 < \cdots < u_q \le n$. Now we state the inductive assumptions. We suppose that we have vectors x_{i0}, $x_{il}^h \in H_i$, $i \in \underline{n}$, $l \in \underline{m-1}$; $h \in \underline{d_l}$, n-tuples $\mathbf{a}_h^{0l}$, $l \in \underline{m-1}$; $h \in \underline{d_l}$ and symmetric matrices $T_h^l \in \mathbb{C}^{\hat{d}_{l-1} \times \hat{d}_{l-1}}$ in the form (1.29), where $\hat{d}_{l-1} = \sum_{i=1}^{l-1} d_i$ and $l = 2, 3, \ldots, m-1$, $h \in \underline{d_l}$, such that :

(i) the matrices $T_h^{l(1,l-1)}$, $h \in \underline{d_l}$ are linearly independent for all l and the matrices T_h^l satisfy the matching conditions (1.25), i.e.,

$$\sum_{k=l_1+l_2}^{l-l_3} \sum_{g=1}^{d_k} t_{g(h_1 h_2)}^{k(l_1 l_2)} t_{h(h_3 g)}^{l(l_3 k)} = \sum_{k=l_2+l_3}^{l-l_1} \sum_{g=1}^{d_k} t_{g(h_1 h_3)}^{k(l_1 l_3)} t_{h(h_2 g)}^{l(l_2 k)} \tag{4.24}$$

where $\mathbf{l} = (l_1, l_2, l_3) \in \Phi_{l,3}$ and $\mathbf{h} = (h_1, h_2, h_3) \in \chi_{\mathbf{l}}$.

(ii) the n-tuples $\mathbf{a}_h^{0l}$, $l = 1, 2, \ldots, m-1$; $h \in \underline{r_l}$ are linearly independent and $\mathbf{a}_h^{0l} = \mathbf{0}$ for $l = 2, 3, \ldots, m-1$ and $h = r_l + 1, r_l + 2, \ldots, d_l$.

(iii) if we define

$$\mathbf{a}_{gh}^{kl} = \sum_{j=1}^{l-k} \sum_{e=1}^{r_j} t_{h(eg)}^{l(jk)} \, \mathbf{a}_e^{0j} \tag{4.25}$$

for $l = 2, 3, \ldots, m-1; h \in \underline{d_l};\ k \in \underline{l-1}$ and $g \in \underline{d_k}$ then the relations

$$\sum_{k=1}^{l-1} \sum_{g=1}^{d_k} U_i \left(\mathbf{a}_{gh}^{kl}\right) x_{ik}^g + U_i \left(\mathbf{a}_h^{0l}\right) x_{i0} = W_i \left(\boldsymbol{\lambda}\right) x_{il}^h \tag{4.26}$$

hold for all i, l and h.

(iv) the vectors $z_0,\ z_1^1, z_1^2, \ldots, z_1^{d_1}$ together with the vectors $z_2^1, z_2^2, \ldots, z_2^{d_2},\ \ldots, z_{m-1}^1,$ $z_{m-1}^2, \ldots, z_{m-1}^{d_{m-1}}$ that are defined by

$$z_k^g = x_{kg}^{\otimes} + \sum_{q=2}^{\min\{k,n\}} \sum_{\mathbf{l} \in \Phi_{kq}} \sum_{\mathbf{h} \in \chi_{\mathbf{l}}} s_{g\mathbf{h}}^{k\mathbf{l}} x_{\mathbf{lh}}^{\otimes}$$

form a basis for the kernel $\ker (\boldsymbol{\lambda} \mathbf{I} - \boldsymbol{\Gamma})^m$. Here the numbers $s_{g\mathbf{h}}^{k\mathbf{l}}$ are defined as in (1.37) for $k = 2, 3, \ldots, m-1;\ q = 2, 3, \ldots, \min\{k, n\};\ g \in \underline{d_k};\ \mathbf{l} \in \Phi_{kq}, \mathbf{h} \in \chi_{\mathbf{l}}$. A vector $x_{\mathbf{ulh}}^{\otimes}$, for $\mathbf{l} \in \Phi_{m-1,q}, \mathbf{h} \in \chi_{\mathbf{l}}, q = 1, 2, \ldots, \min\{m-1, n\}$ (here $\Phi_{k,1} = \underline{k}$ and $\chi_l = \underline{d_l}$) and $\mathbf{u} \in \Omega_q$, is a decomposable tensor $x_{\mathbf{ulh}}^{\otimes} = x_1 \otimes x_2 \otimes \cdots \otimes x_n$ where $x_i = x_{il_j}^{h_j}$ if $i = u_j$ for some j and $x_i = x_{i0}$ otherwise. Then we write

$$x_{\mathbf{lh}}^{\otimes} = \sum_{\mathbf{u} \in \Omega_q} x_{\mathbf{ulh}}^{\otimes} \tag{4.27}$$

for $\mathbf{l} \in \Phi_{m-1,q}$, $\mathbf{h} \in \chi_{\mathbf{l}}$ and $q = 1, 2, \ldots, \min\{m-1, n\}$. We also write

$$\mathcal{B}_{m-1} = \left\{z_0;\ z_1^1, z_1^2, \ldots, z_1^{d_1};\ \ldots\ ; z_{m-1}^1, z_{m-1}^2, \ldots, z_{m-1}^{d_{m-1}}\right\}.$$

By Corollary 4.13 it follows that there exist vectors x_{i0}, x_{il}^h, n-tuples $\mathbf{a}_h^{0l}$ and matrices T_h^l for $l = 1, 2,\ h \in \underline{d_l}$ such that the conditions (i) to (iv) are satisfied. Now we assume that the above conditions are satisfied for $l = 1, 2, \ldots, m-1$ $(m \geq 3)$ and we will prove that we can find vectors x_{im}^f, $i \in \underline{n}$, $f \in \underline{d_m}$, n-tuples $\mathbf{a}_f^{0m}$, $f \in \underline{d_m}$ and symmetric matrices T_f^m, $f \in \underline{d_m}$, so that (i) to (iv) hold. We first introduce some notation.

For $k = 1, 2, \dots, m-1$ and $g \in \underline{d_k}$ we define an $n \times n$ matrix

$$B_{kg} = \begin{bmatrix} y_{10}^* V_{11} x_{1k}^g & y_{10}^* V_{12} x_{1k}^g & \cdots & y_{10}^* V_{1n} x_{1k}^g \\ y_{20}^* V_{21} x_{2k}^g & y_{20}^* V_{22} x_{2k}^g & \cdots & y_{20}^* V_{2n} x_{2k}^g \\ \vdots & \vdots & & \vdots \\ y_{n0}^* V_{n1} x_{nk}^g & y_{n0}^* V_{n2} x_{nk}^g & \cdots & y_{n0}^* V_{nn} x_{nk}^g \end{bmatrix}. \tag{4.28}$$

For the purpose of calculation we write $T_\circ^m$ for unknown symmetric matrices T_f^m in the form (1.29) and $\mathbf{a}_\bullet^{0m}$ for unknown n-tuples $\mathbf{a}_f^{0m}$. The entries of the $\hat{d}_{m-1} \times \hat{d}_{m-1}$ matrix $T_\circ^m$, where $\hat{d}_{m-1} = \sum_{i=1}^{m-1} d_i$, are written $t_{(h_1 h_2)}^{m(l_1 l_2)}$. They must satisfy the matching conditions

$$\sum_{k=l_1+l_2}^{m-l_3} \sum_{g=1}^{d_k} t_{g(h_1 h_2)}^{k(l_1 l_2)} t_{(h_3 g)}^{m(l_3 k)} - \sum_{k=l_1+l_3}^{m-l_2} \sum_{g=1}^{d_k} t_{g(h_1 h_3)}^{k(l_1 l_3)} t_{(h_2 g)}^{m(l_2 k)} = 0 \tag{4.29}$$

for $\mathbf{l} \in \Phi_{m,3}$ and $\mathbf{h} \in \chi_{\mathbf{l}}$. We write the $d_1 \times d_{m-1}$ matrix

$$T_\circ^{m(1,m-1)} = \begin{bmatrix} t_{(11)}^{m(1,m-1)} & t_{(12)}^{m(1,m-1)} & \cdots & t_{(1d_{m-1})}^{m(1,m-1)} \\ t_{(21)}^{m(1,m-1)} & t_{(22)}^{m(1,m-1)} & \cdots & t_{(2d_{m-1})}^{m(1,m-1)} \\ \vdots & \vdots & & \vdots \\ t_{(d_1 1)}^{m(1,m-1)} & t_{(d_1 2)}^{m(1,m-1)} & \cdots & t_{(d_1 d_{m-1})}^{m(1,m-1)} \end{bmatrix}$$

also as a column

$$\tilde{\mathbf{t}}^m = \begin{bmatrix} t_{(11)}^{m(1,m-1)} \\ \vdots \\ t_{(d_1 1)}^{m(1,m-1)} \\ t_{(12)}^{m(1,m-1)} \\ \vdots \\ t_{(d_1 2)}^{m(1,m-1)} \\ \vdots \\ t_{(1 d_{m-1})}^{m(1,m-1)} \\ \vdots \\ t_{(d_1 d_{m-1})}^{m(1,m-1)} \end{bmatrix}. \tag{4.30}$$

The matrix $T_\circ^m$ is in the form (1.29). For every column we define a column vector $\mathbf{t}_{kg}^m$, $k \in \underline{m-1}$, $g \in \underline{d_k}$ of the size $\nu = \min\left\{\sum_{j=1}^{k-1} d_j + g,\ \sum_{j=1}^{m-k} d_j\right\}$ by taking the first

ν entries in the $\left(\sum_{j=1}^{k-1} d_j + g\right)$-th column of $T_\circ^m$. Note that $\mathbf{t}_{kg}^m$ are defined so that they consist of all the entries above and including the main diagonal of $T_\circ^m$ that are not 0 in the form (1.29). We define a column vector $\widehat{\mathbf{t}}^m$ as

$$\widehat{\mathbf{t}}^m = \begin{bmatrix} \mathbf{t}_{11}^m \\ \vdots \\ \mathbf{t}_{1d_1}^m \\ \mathbf{t}_{21}^m \\ \vdots \\ \mathbf{t}_{2d_2}^m \\ \vdots \\ \mathbf{t}_{m-2,1}^m \\ \vdots \\ \mathbf{t}_{m-2,d_{m-2}}^m \end{bmatrix}. \tag{4.31}$$

We split the entries of a matrix T_f^m into two column vectors $\widetilde{\mathbf{t}}_f^m$ and $\widehat{\mathbf{t}}_f^m$. The mapping $T_f^m \xrightarrow{\psi_m} \left(\widetilde{\mathbf{t}}_f^m, \widehat{\mathbf{t}}_f^m\right)$ is a generalization of the transformation ψ defined by (4.13). It is bijective and therefore it has an inverse. The inverse maps two vectors $\widetilde{\mathbf{t}}_f^m$ and $\widehat{\mathbf{t}}_f^m$ into a matrix T_f^m in the form (1.29). We use this inverse mapping in Lemma 4.16.

Now we write the system of equations (4.29) in matrix form as

$$S_m^{21}\widetilde{\mathbf{t}}^m + S_m^{22}\widehat{\mathbf{t}}^m = 0 \tag{4.32}$$

where the entries of the matrices S_m^{21} and S_m^{22} are determined by the system (4.29). These entries are given because we assumed that the matrices T_g^k were given for $k \leq m-1$. Further we want the entries of the matrix $T_\circ^m$ and the n-tuple $\mathbf{a}_\bullet^{0m} = \left[\begin{array}{cccc} a_1^{0m} & a_2^{0m} & \cdots & a_n^{0m} \end{array}\right]^T$ to satisfy the scalar relations

$$\sum_{j=1}^{n} \left(\sum_{k=1}^{m-1} \sum_{g=1}^{d_k} \sum_{l=1}^{m-k} \sum_{h=1}^{r_l} t_{(hg)}^{m(lk)}\, a_{hj}^{0l}\, y_{i0}^* V_{ij} x_{ik}^g + y_{i0}^* V_{ij} x_{i0} a_j^{0m} \right) = 0 \tag{4.33}$$

for all i. These can be written equivalently in matrix form

$$S_m^{11}\widetilde{\mathbf{t}}^m + S_m^{12}\widehat{\mathbf{t}}^m + B_0\mathbf{a}_\bullet^{0m} = 0. \tag{4.34}$$

Again the entries of the matrices S_m^{11} and S_m^{12} are determined by the equations (4.33). We multiply the equation (4.34) by the matrix $\mathbf{b}_0^*$ on the left-hand side and we obtain

$$\mathbf{b}_0^* S_m^{11} \widetilde{\mathbf{t}}^m + \mathbf{b}_0^* S_m^{12} \widehat{\mathbf{t}}^m = 0. \tag{4.35}$$

We choose a matrix $\mathbf{b}_m$ so that its columns form a basis for the kernel

$$\ker \begin{bmatrix} \mathbf{b}_0^* S_m^{12} \\ S_m^{22} \end{bmatrix}^* .$$

Then we define a matrix

$$\mathcal{S}_m = \mathbf{b}_m^* \begin{bmatrix} \mathbf{b}_0^* S_m^{11} \\ S_m^{12} \end{bmatrix} .$$

The equations (4.32) and (4.35) then yield $\mathcal{S}_m \widetilde{\mathbf{t}}^m = 0$.

We now choose vectors $z_m^f \in H,\ f \in \underline{d_m}$ so that $\mathcal{B}_{m-1} \cup \left\{ z_m^f,\ f \in \underline{d_m} \right\}$ is a basis for the space $\ker (\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^{m+1}$. By Corollary 1.18 there exist n-tuples $\mathbf{a}_f^{0m}$ and symmetric matrices T_f^m in the form (1.29) for $f \in \underline{d_m}$ such that (i) holds and for all $i \in \underline{n}$ we have

$$(\lambda_i I - \Gamma_i)\, z_m^f = \sum_{k=1}^{m-1} \sum_{g=1}^{d_k} a_{gfi}^{km}\, z_k^g + a_{fi}^{0m}\, z_0 \tag{4.36}$$

where $\mathbf{a}_{gf}^{km}$ are given by

$$\mathbf{a}_{gf}^{km} = \sum_{l=1}^{m-k} \sum_{h=1}^{r_l} t_{f(gh)}^{m(kl)}\, \mathbf{a}_h^{0l} \tag{4.37}$$

for $k \in \underline{m-1}$ and $g \in \underline{d_k}$. Next we prove three auxiliary results.

Lemma 4.15 *In the above setting it follows that* $\dim \ker \mathcal{S}_m \geq d_m$.

Proof. By Theorem 1.17 it follows that the entries of the matrices T_f^m satisfy the matching conditions (4.29). We put the entries of these matrices into two columns $\widetilde{\mathbf{t}}_f^m$ and $\widehat{\mathbf{t}}_f^m$ as in (4.30) and (4.31) via the isomorphism ψ_m. Then we have $S_m^{21} \widetilde{\mathbf{t}}_f^m - S_m^{22} \widehat{\mathbf{t}}_f^m = 0$. Relation (2.7) implies

$$\sum_{j=1}^{n} V_{ij}^{\dagger} (\lambda_j I - \Gamma_j)\, z_m^f = W_i (\boldsymbol{\lambda})^{\dagger}\, z_m^f$$

for all i and then it follows from relations (4.36) that

$$\sum_{k=1}^{m-1}\sum_{g=1}^{d_k} U_i\left(\mathrm{a}_{gf}^{km}\right)^{\dagger} z_k^g + U_i\left(\mathrm{a}_f^{0m}\right)^{\dagger} z_0 = W_i(\boldsymbol{\lambda})^{\dagger} z_m^f. \tag{4.38}$$

Because we assumed $x_{ik}^g \in H_i'$ and $H_i' \cap \mathcal{L}\{x_{i0}\} = \{0\}$ it follows that there exist vectors $v_i \in H_i$ such that $v_i^* x_{i0} = 1$ and $v_i^* x_{ik}^g = 0$ for $k \in \underline{m-1}$ and $g \in \underline{d_k}$. We multiply the equality (4.38) by a vector $v_1^* \otimes \cdots \otimes v_{i-1}^* \otimes y_{i0}^* \otimes v_{i+1}^* \otimes \cdots \otimes v_n^*$ on the left-hand side. Then it follows, using the structure of vectors z_k^g, $k \leq m-1$, in condition (iv), that

$$\sum_{k=1}^{m-1}\sum_{g=1}^{d_k} y_{i0}^* U_i\left(\mathrm{a}_{gf}^{km}\right) x_{ik}^g + y_{i0}^* U_i\left(\mathrm{a}_f^{0m}\right) x_{i0} = 0 \tag{4.39}$$

for all i and all f. Now we apply the relations (4.37) to obtain

$$\sum_{j=1}^{n}\left(\sum_{k=1}^{m-1}\sum_{g=1}^{d_k}\sum_{l=1}^{m-k}\sum_{h=1}^{r_l} y_{i0}^* V_{ij} x_{ik}^g\, a_{hj}^{0l}\, t_{f(gh)}^{m(kl)} + y_{i0}^* V_{ij} x_{i0}\, a_{fj}^{0m}\right) = 0$$

and then it follows that the vectors $\widetilde{\mathrm{t}}_f^m$, $\widehat{\mathrm{t}}_f^m$ and the n-tuple a_f^{0m} are such that equation (4.34) holds for all f. Therefore the vectors $\widetilde{\mathrm{t}}_f^m$, $f \in \underline{d_m}$ are elements of the kernel of the matrix $\mathcal{S}_m$ and because they are linearly independent we have $d_m \leq \dim\ker \mathcal{S}_m$. □

Lemma 4.16 *Suppose that* $\widetilde{\mathrm{t}}_1^m$ *is an element of the kernel* $\ker \mathcal{S}_m$. *Then there exist a vector* $\widehat{\mathrm{t}}_1^m$ *and an* n*-tuple* a_1^{0m} *such that* (4.32) *and* (4.34) *hold. Furthermore there exist vectors* $x_{im}^1 \in H_i'$, $i \in \underline{n}$ *such that*

$$\sum_{k=1}^{m-1}\sum_{g=1}^{d_k} U_i\left(\mathrm{a}_{g1}^{km}\right) x_{ik}^g + U_i\left(\mathrm{a}_1^{0m}\right) x_{i0} = W_i(\boldsymbol{\lambda})\, x_{im}^1 \tag{4.40}$$

where a_{g1}^{km} *are given by* (4.37) *for* $f=1$ *and* $T_1^m = \psi_m^{-1}\left(\widetilde{\mathrm{t}}_1^m, \widehat{\mathrm{t}}_1^m\right)$.

Proof. From the structure of the matrix $\mathcal{S}_m$ it follows that for an element $\widetilde{\mathrm{t}}_1^m \in \ker \mathcal{S}_m$ there exist a vector $\widehat{\mathrm{t}}_1^m$ and a n-tuple a_1^{0m} such that relations (4.32) and (4.34) hold. We associate with the pair of vectors $\widetilde{\mathrm{t}}_1^m$ and $\widehat{\mathrm{t}}_1^m$, using the inverse of the isomorphism ψ_m^{-1}, a symmetric matrix T_1^m. The relations (4.34) can be written equivalently in the form (4.39) for $f = 1$. Then it follows for every i that

$$\sum_{k=1}^{m-1} \sum_{g=1}^{d_k} U_i \left(\mathrm{a}_{g1}^{km}\right) x_{ik}^g + U_i \left(\mathrm{a}_1^{0m}\right) x_{i0} \in \left(\ker W_i \left(\boldsymbol{\lambda}\right)^*\right)^{\perp}$$

and hence it follows from Lemma 2.12 that there exists a vector x_{im}^1 such that (4.40) holds. □

Lemma 4.17 *Suppose that we have the same setting as in the previous Lemma. We construct a vector*

$$z_m^1 = x_{m1}^{\otimes} + \sum_{q=2}^{\min\{n,m\}} \sum_{\mathbf{l}\in\Phi_{mq}} \sum_{\mathbf{h}\in\chi_{\mathbf{l}}} s_{1\mathbf{h}}^{m\mathbf{l}} x_{\mathbf{lh}}^{\otimes}$$

where the numbers $s_{1\mathbf{h}}^{m\mathbf{l}}$ *are defined in* (1.37) *and the vectors* $x_{\mathbf{lh}}^{\otimes}$ *are defined in* (4.27). *Then it follows that*

$$\left(\lambda_i I - \Gamma_i\right) z_m^1 = \sum_{k=1}^{m-1} \sum_{g=1}^{d_k} a_{g1i}^{km} z_k^g + a_{1i}^{0m} z_0 \tag{4.41}$$

for all i.

Because the proof of this lemma is long and technically complicated we include it in Appendix B. Next we state and prove our main result.

Theorem 4.18 *Suppose that* $\left\{\tilde{\mathbf{t}}_f^m,\ f \in \underline{d}\right\}$ *is a basis for the kernel of* $\mathcal{S}_m$ *where* $d = \dim\ker\mathcal{S}_m$. *Then there exist vectors* x_{im}^f, *matrices* T_f^m *and* n*-tuples* a_f^{0m} *so that conditions* (*i*) *to* (*iv*), *on page* 141, *hold also for* $l = m$. *In particular the union of the set of vectors*

$$\left\{ z_m^f = x_{mf}^{\otimes} + \sum_{q=2}^{\min\{n,m\}} \sum_{\mathbf{l}\in\Phi_{kq}} \sum_{\mathbf{h}\in\chi_{\mathbf{l}}} s_{f\mathbf{h}}^{m\mathbf{l}} x_{\mathbf{lh}}^{\otimes},\ f \in \underline{d_m} \right\}$$

and the set $\mathcal{B}_{m-1}$ *forms a basis for the* $(m+1)$*-th root subspace* $\ker\left(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma}\right)^{m+1}$.

Proof. Suppose that we are given a basis $\mathcal{T} = \left\{\tilde{\mathbf{t}}_f^m,\ f \in \underline{d}\right\}$. Then by Lemma 4.16 it follows that we can find vectors x_{im}^f, symmetric matrices T_f^m and n-tuples a_f^{0m}

such that (i) and (iii) hold for $l = m$. We define r_m as the difference between the number of linearly independent n-tuples in the set $\left\{\mathbf{a}_g^{0k},\ k \in \underline{m},\ g \in \underline{d_k}\right\}$ and the number of linearly independent elements in the set $\left\{\mathbf{a}_g^{0k},\ k \in \underline{m-1},\ g \in \underline{d_k}\right\}$. By a change of a basis $\mathcal{T}$, similar to the one in the proof of Corollary 4.13, we can assume that a basis $\mathcal{T}$ is chosen so that the corresponding n-tuples $\mathbf{a}_f^{0m}$, $f \in \underline{r_m}$ are such that $\mathbf{a}_g^{0k}$, $k \in \underline{m}$, $g \in \underline{r_k}$ are linearly independent and $\mathbf{a}_f^{0m} = \mathbf{0}$ for $f = r_m + 1, r_m + 2, \ldots, d_m$. So condition (ii) holds. From Lemma 4.17 it follows that the vectors $z_m^f \in \ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^{m+1} / \ker(\boldsymbol{\lambda}\mathbf{I} - \boldsymbol{\Gamma})^m$ for $f \in \underline{d}$. They are linearly independent because $\widetilde{\mathbf{t}}_f^m$ are linearly independent. It follows that $d \leq d_m$ and, because $d \geq d_m$ by Lemma 4.15, we have $d = d_m$. Then also (iv) holds for m. □

4.5 Further Discussions

4.5.1 Algorithm to Construct a Basis for the Root Subspace of a Simple Eigenvalue

From the above lengthy discussion we can extract an algorithm which explains how to construct a basis for the root subspace of a simple eigenvalue of a multiparameter system $\mathbf{W}$. It follows from Theorem 4.18 that for every m we have $\dim\ker \mathcal{S}_m = d_m$. When $\dim\ker \mathcal{S}_{m'} = 0$ but $\dim\ker \mathcal{S}_{m'-1} \neq 0$ for some m' it follows that $M = m' - 1$ and the vectors z_m^f constructed for $m = 0, 1, 2, \ldots, M$, $f \in \underline{d_m}$ are a basis for the root subspace. This is used in the following algorithm. We assume that $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ is such that $\dim\ker W_i(\boldsymbol{\lambda}) = 1$ for all i.

Algorithm 4.19 *Step I. For $i \in \underline{n}$ find vectors $x_{i0},\ y_{i0} \in H_i \setminus \{0\}$ such that*

$$W_i(\boldsymbol{\lambda})\, x_{i0} = 0 \text{ and } y_{i0}^* W_i(\boldsymbol{\lambda}) = 0.$$

Find subspaces $H_i' \subset H_i$ so that $H_i = \ker W_i(\boldsymbol{\lambda}) \oplus H_i'$. Form $z_0 = x_{10} \otimes x_{20} \otimes \cdots \otimes x_{n0}$ and

$$B_0 = \begin{bmatrix} y_{10}^* V_{11} x_{10} & y_{10}^* V_{12} x_{10} & \cdots & y_{10}^* V_{1n} x_{10} \\ y_{20}^* V_{21} x_{20} & y_{20}^* V_{22} x_{20} & \cdots & y_{20}^* V_{2n} x_{20} \\ \vdots & \vdots & & \vdots \\ y_{n0}^* V_{n1} x_{n0} & y_{n0}^* V_{n2} x_{n1} & \cdots & y_{n0} V_{nn} x_{n0} \end{bmatrix}.$$

If $\operatorname{rank} B_0 = n$ *then set $M = 0$ and quit the algorithm, if* $\operatorname{rank} B_0 = n - 1$ *then go to Step II,* Algorithm 4.5 *(for nonderogatory eigenvalues), else write* $d_1 = \dim\ker B_0$ *and go to Step II.*

Step II. Find bases $\left\{\mathbf{a}_f^{01},\ f \in \underline{d_1}\right\}$ for $\ker B_0$ *and $\left\{\mathbf{b}_0,\ f \in \underline{d_1}\right\}$ for* $\ker B_0^*$. *For all i find vectors $x_{i1}^f \in H_i'$, $f \in \underline{d_1}$ such that*

$$U_i\left(\mathbf{a}_f^{01}\right) x_{i0} = W_i(\boldsymbol{\lambda})\, x_{i1}^f.$$

Form vectors $z_1^f = \sum_{i=1}^{n} x_{10} \otimes \cdots \otimes x_{i-1,0} \otimes x_{i1}^f \otimes x_{i+1,0} \otimes \cdots \otimes x_{n0}$ and matrices

$$B_{1f} = \begin{bmatrix} y_{10}^* V_{11} x_{11}^f & y_{10}^* V_{12} x_{11}^f & \cdots & y_{10}^* V_{1n} x_{11}^f \\ y_{20}^* V_{21} x_{21}^f & y_{20}^* V_{22} x_{21}^f & \cdots & y_{20}^* V_{n2} x_{21}^f \\ \vdots & \vdots & & \vdots \\ y_{n0}^* V_{n1} x_{n1}^f & y_{n0}^* V_{n2} x_{n1}^f & \cdots & y_{n0}^* V_{nn} x_{n1}^f \end{bmatrix}$$

and $\mathcal{S}_2 = \mathbf{b}_0^* \mathcal{S}$, where $\mathbf{b}_0 = \begin{bmatrix} \mathbf{b}_1^{01} & \mathbf{b}_2^{01} & \cdots & \mathbf{b}_{d_1}^{01} \end{bmatrix}$ and $\mathcal{S}$ is the symmetrization of $\mathbf{A}^{01} = \begin{bmatrix} \mathbf{a}_1^{01} & \mathbf{a}_2^{01} & \cdots & \mathbf{a}_{d_1}^{01} \end{bmatrix}$ as defined on page 133. *Write $d_2 =$* $\dim\ker \mathcal{S}_2$. *If $d_2 = 0$ then set $M = 1$ and quit the algorithm else set $m = 2$ and go to Step III.*

Step III. Find a basis $\left\{\mathbf{t}^1, \mathbf{t}^2, \ldots, \mathbf{t}^{d_2}\right\}$ for $\ker \mathcal{S}_2$, *symmetric matrices T_f^2 (via the isomorphism ψ) and n-tuples $\mathbf{a}_1^{02}, \mathbf{a}_2^{02}, \ldots, \mathbf{a}_{d_2}^{02}$ such that*

(a) $\sum_{g=1}^{d_1} B_{1g} \mathbf{a}_{gf}^{12} + B_0 \mathbf{a}_f^{02} = 0$, *where*

$$\mathbf{a}_{gf}^{12} = \sum_{h=1}^{d_1} t_{f(gj)}^{2(11)} \mathbf{a}_h^{01}, \quad g \in \underline{d_1},\ f \in \underline{d_2}$$

and

(b) *the nonzero n-tuples in the set* $\left\{\mathbf{a}_g^{0k},\ k=1,2,\ g\in \underline{d_k}\right\}$ *are linearly independent.*

Next find vectors $x_{i2}^f \in H_i^!$ *such that*

$$\sum_{g=1}^{d_1} U_i\left(\mathbf{a}_{gf}^{12}\right) x_{i1}^g + U_i\left(\mathbf{a}_f^{02}\right) x_{i0} = W_i(\boldsymbol{\lambda})\, x_{i2}^f.$$

Form vectors

$$z_2^f = \sum_{i=1}^{n} x_{10}\otimes\cdots\otimes x_{i2}^f\otimes\cdots\otimes x_{n0} + \sum_{g,h=1}^{d_2} t_{f(gh)}^{2(11)} \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} x_{10}\otimes\cdots\otimes x_{i1}^g\otimes\cdots\otimes x_{j1}^h\otimes\cdots\otimes x_{n0}$$

and matrices B_{2f}, S_3^{11}, S_3^{12}, S_3^{22} *and* $\mathbf{b}_3$ *as in the displayed formula* (4.28) *and the discussion that follows it. Note that* $S_3^{21}=0$. *Then we find a matrix* $\mathcal{S}_3$ *and write* $d_3 = \dim\ker \mathcal{S}_3$. *If* $d_3 = 0$ *then set* $M=2$ *and quit the algorithm else set* $m=3$ *and go to Step IV.*

Step IV. Find a basis $\left\{\tilde{\mathbf{t}}_1^m, \tilde{\mathbf{t}}_2^m, \ldots, \tilde{\mathbf{t}}_{d_m}^m\right\}$ *for* $\ker \mathcal{S}_m$, *associated symmetric matrices* T_f^m *and n-tuples* $\mathbf{a}_f^{0m}$, *as described in Lemma* 4.16, *so that*

(a) $\sum_{k=1}^{m-1}\sum_{g=1}^{d_k} B_{kg}\mathbf{a}_{gf}^{km} + B_0\mathbf{a}_f^{0m} = 0$, *where*

$$\mathbf{a}_{gf}^{km} = \sum_{l=1}^{m-k}\sum_{h=1}^{d_l} t_{f(gh)}^{m(kl)}\mathbf{a}_h^{0l},$$

and

(b) *the nonzero n-tuples in the set* $\left\{\mathbf{a}_g^{0k},\ k\in \underline{m},\ g\in \underline{d_k}\right\}$ *are linearly independent.*

Then find the numbers $s_{f\mathbf{h}}^{m\mathbf{l}}$, *defined by the recursive relation* (1.38), *for* $\mathbf{l}\in\Phi_{m,q}$, $q=2,3,\ldots,m$ *and* $\mathbf{h}\in\chi_{\mathbf{l}}$ *and vectors* $x_{im}^f\in H_i^!$ *so that*

$$\sum_{k=1}^{m-1}\sum_{g=1}^{d_k} U_i\left(\mathbf{a}_{gf}^{km}\right) x_{ik}^g + U_i\left(\mathbf{a}_f^{0m}\right) x_{i0} = W_i(\boldsymbol{\lambda})\, x_{im}^f\,.$$

Next form vectors z_m^f *as described in (iv), page* 142, *and matrices* B_{mf}, S_{m+1}^{ij}, $i,j=1,2$, $\mathbf{b}_{m+1}$ *and* $\mathcal{S}_{m+1}$ *as in the displayed formula* (4.28) *and the discussion that follows it. Write* $d_{m+1}=\dim\ker\mathcal{S}_{m+1}$. *If* $d_{m+1}=0$ *then set* $M=m$ *and quit the algorithm else set* $m=m+1$ *and repeat Step IV.*

To shed some light on Algorithm 4.19 we consider an example.

Example 4.20 The two-parameter system

$$W_1(\boldsymbol{\lambda}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \lambda_1 + \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \lambda_2 - \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

and

$$W_2(\boldsymbol{\lambda}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \lambda_1 + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \lambda_2 - \begin{bmatrix} 0 & 2 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

is nonsingular because matrix

$$\Delta_0 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

is invertible. Here we identify the tensor space $\mathbb{C}^4 \otimes \mathbb{C}^3$ with the vector space $\mathbb{C}^{12}$ via the Kronecker product. (See page 75.) The spectrum is

$$\sigma(\mathbf{W}) = \left\{(0,-1), \left(\frac{1}{2}, -\frac{3}{2}\right), (1,-1), (1,-2)\right\}.$$

We will find a basis for the root subspace at the eigenvalue $\boldsymbol{\lambda}_0 = (1,-1)$.

We begin with *Step I* of Algorithm 4.19. It follows from above that

$$W_1(\boldsymbol{\lambda}_0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } W_2(\boldsymbol{\lambda}_0) = \begin{bmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Observe that $\dim\ker W_1(\boldsymbol{\lambda}_0) = \dim\ker W_2(\boldsymbol{\lambda}_0) = 1$ and therefore $d_0 = 1$. Then we choose

$$x_{10} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, y_{10} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, x_{20} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ and } y_{20} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and $H_1' = \left\{ \begin{bmatrix} a \\ b \\ b \\ c \end{bmatrix}, \ a,b,c \in \mathbb{C} \right\}$ and $H_2' = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}, \ a,b \in \mathbb{C} \right\}$. So $z_0 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $B_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence it follows that $d_1 = 2$ and the eigenvalue $\boldsymbol{\lambda}_0$ is simple.

We proceed with *Step II*. We choose $\mathbf{a}_1^{01} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{a}_2^{01} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We also choose $\mathbf{b}_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and, because it does not influence further calculations, we will omit it. Vectors $x_{i1}^f \in H_i'$, $i,f = 1,2$, such that $U_i\left(\mathbf{a}_f^{01}\right) x_{i0} = W_i(\boldsymbol{\lambda}_0)\, x_{i1}^f$ are

$$x_{11}^1 = x_{11}^2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \ x_{21}^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } x_{21}^2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.$$

Then

$$z_1^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } z_1^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 1 \\ 1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

and we find $B_{11} = B_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So $\mathcal{S}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and therefore $d_2 = 3$.

We continue with *Step III* choosing matrices

$$T_1^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \; T_2^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } T_3^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$\mathbf{A}^{12} = \begin{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{bmatrix},$$

while $\mathbf{A}^{02} = \mathbf{0}$. Vectors $x_{i2}^f \in H_i'$, $f = 1,2,3$ and $i = 1,2$, such that $U_i\left(\mathbf{a}_{1f}^{12}\right) x_{i1}^1 + U_i\left(\mathbf{a}_{2f}^{12}\right) x_{i1}^2 = W_i\left(\boldsymbol{\lambda}_0\right) x_{i2}^f$ are

$$x_{12}^1 = \begin{bmatrix} 0 \\ 3 \\ 3 \\ -2 \end{bmatrix}, \; x_{12}^2 = \begin{bmatrix} 0 \\ 5 \\ 5 \\ -4 \end{bmatrix}, \; x_{12}^3 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ -2 \end{bmatrix}$$

and

$$x_{22}^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \; x_{22}^2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \text{ and } x_{22}^3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Next we find vectors

$$z_2^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 0 \\ 0 \\ 3 \\ 0 \\ 0 \\ -2 \end{bmatrix}, \; z_2^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \\ 5 \\ 0 \\ 1 \\ 5 \\ 0 \\ 0 \\ -4 \end{bmatrix} \text{ and } z_2^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ -2 \\ 1 \\ 2 \\ 0 \\ 0 \\ -2 \end{bmatrix}$$

and matrices

$$B_{21} = \begin{bmatrix} -2 & -4 \\ 0 & 0 \end{bmatrix}, \; B_{22} = \begin{bmatrix} -4 & -8 \\ -1 & 0 \end{bmatrix} \text{ and } B_{23} = \begin{bmatrix} -2 & -4 \\ 1 & 0 \end{bmatrix}.$$

We now write $T_\circ^{3(12)} = \begin{bmatrix} t_1 & t_2 & t_3 \\ t_4 & t_5 & t_6 \end{bmatrix}$. Because $B_{11} = B_{12} = 0$ it follows that $S_3^{11} = 0$. Then

$$T_{1R}^{1(3)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } T_{2R}^{1(3)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the matrices $T_{1R}^{1(12)} \left(T_\circ^{3(12)}\right)^T$, $g = 1, 2$, are symmetric if $t_2 = t_4$ and $t_3 = t_5$. Next it follows that

$$S_3 = \begin{bmatrix} -2 & -4 & -4 & -8 & -2 & -4 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{bmatrix}$$

and then we have $d_3 = 2$. Therefore we continue with the *Step IV.*

We choose a basis for the kernel of $\mathcal{S}_3$ so that

$$T_1^{3(12)} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \text{ and } T_2^{3(12)} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & -4 \end{bmatrix}.$$

We can also choose $T_1^{3(11)} = T_2^{3(11)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Then the array

$$\mathbf{A}^{23} = \begin{bmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} & \begin{pmatrix} -1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ -1 \end{pmatrix} & \begin{pmatrix} 1 \\ -4 \end{pmatrix} \end{bmatrix}$$

and the arrays $\mathbf{A}^{13}$ and $\mathbf{A}^{03}$ are $\mathbf{0}$. Vectors $x_{i3}^f \in H_i'$, $i, f = 1, 2$, such that

$$U_i\left(\mathrm{a}_{1f}^{23}\right) x_{i2}^1 + U_i\left(\mathrm{a}_{2f}^{23}\right) x_{i2}^2 + U_i\left(\mathrm{a}_{3f}^{23}\right) x_{12}^3 = W_i\left(\boldsymbol{\lambda}_0\right) x_{i3}^f$$

are

$$x_{13}^1 = \begin{bmatrix} 0 \\ 16 \\ 16 \\ -10 \end{bmatrix}, \; x_{13}^2 = \begin{bmatrix} 0 \\ 14 \\ 14 \\ -10 \end{bmatrix}, \; x_{23}^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } x_{23}^2 = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}.$$

Next we find vectors

$$z_3^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ 16 \\ -2 \\ 2 \\ 16 \\ 2 \\ -2 \\ -10 \end{bmatrix} \text{ and } z_3^2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 14 \\ -10 \\ 0 \\ 14 \\ -2 \\ 2 \\ -10 \end{bmatrix}$$

and matrices

$$B_{31} = \begin{bmatrix} -10 & -20 \\ 1 & 0 \end{bmatrix} \text{ and } B_{32} = \begin{bmatrix} -10 & -20 \\ 5 & 0 \end{bmatrix}.$$

Now we will find the matrix S_4. We are looking for a symmetric matrix

$$T_\circ^4 = \begin{bmatrix} v_1 & v_2 & u_1 & u_3 & u_5 & t_1 & t_3 \\ v_2 & v_3 & u_2 & u_4 & u_6 & t_2 & t_4 \\ u_1 & u_2 & w_1 & w_2 & w_4 & 0 & 0 \\ u_3 & u_4 & w_2 & w_3 & w_5 & 0 & 0 \\ u_5 & u_6 & w_4 & w_5 & w_6 & 0 & 0 \\ t_1 & t_2 & 0 & 0 & 0 & 0 & 0 \\ t_3 & t_4 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Because $B_{11} = B_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and there is no symmetry condition on the entries of the matrix $T_\circ^{4(11)} = \begin{bmatrix} v_1 & v_2 \\ v_2 & v_3 \end{bmatrix}$ we can assume that $T_f^{4(11)} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So we omit the entries v_j and write

$$\widetilde{\mathbf{t}}_\circ^4 = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \end{bmatrix} \text{ and } \widehat{\mathbf{t}}_\circ^4 = \begin{bmatrix} u_1 \\ u_2 \\ w_1 \\ u_3 \\ u_4 \\ w_2 \\ w_3 \\ u_5 \\ u_6 \\ w_4 \\ w_5 \\ w_6 \end{bmatrix}.$$

Matrices $T_{gR}^{2(4)}\left(T_\circ^{4(2)}\right)^T$, for $g = 1,2,3$, and $T_{gR}^{1(4)}\left(T_\circ^{4(1)}\right)^T$, for $g = 1,2$, are symmetric if $2t_2 - t_3 - t_4 = 0$, $t_3 - t_4 = 0$, $t_1 + 4t_3 + t_4 = 0$, $u_2 - u_3 = 0$, $2t_1 - t_3 - w_1 = 0$, $t_3 - w_4 = 0$, $2t_2 - t_4 - w_2 = 0$, $t_4 - w_3 = 0$, $t_4 - w_5 = 0$, $u_4 - u_5 = 0$, $t_3 - w_3 = 0$, $t_1 + 4t_3 + w_5 = 0$, $t_4 - w_4 = 0$ and $t_2 + 4t_4 + w_6 = 0$, and so we find matrices S_4^{21} and S_4^{22} such that $S_4^{21}\widetilde{\mathbf{t}}_\circ^4 + S_4^{22}\widehat{\mathbf{t}}_\circ^4 = 0$. Next, the matrices S_4^{11} and S_4^{12} such that $S_4^{11}\widetilde{\mathbf{t}}_\circ^4 + S_4^{12}\widehat{\mathbf{t}}_\circ^4 = 0$ are

$$S_4^{11} = \begin{bmatrix} -10 & -20 & -10 & -20 \\ 1 & 0 & 5 & 0 \end{bmatrix}$$

and

$$S_4^{12} = \begin{bmatrix} -2 & -4 & 0 & -4 & -8 & 0 & 0 & -2 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We find that

$$\mathcal{S}_4 = \begin{bmatrix} 1 & 0 & 4 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 4 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & -2 & 1 & 1 \\ 0 & 2 & -1 & -1 \end{bmatrix}.$$

Because $d_4 = \dim\ker \mathcal{S}_4 = 1$ we have to repeat *Step IV*. First we choose an element in the kernel $\ker \mathcal{S}_4$ so that $T_1^{4(13)} = \begin{bmatrix} -5 & 1 \\ 1 & 1 \end{bmatrix}$, and then we find that the matrix

$$T_1^4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 11 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 5 & 0 & 0 \\ -5 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore we have

$$\mathbf{A}^{34} = \begin{bmatrix} \begin{pmatrix} -5 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{bmatrix}$$

and the arrays $\mathbf{A}^{24}$, $\mathbf{A}^{14}$ and $\mathbf{A}^{04}$ are $\mathbf{0}$. It also follows that

$$T_{1R}^{3(5)} = \begin{bmatrix} -5 \\ 1 \end{bmatrix} \text{ and } T^{3(5)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We write $T_\circ^{5(14)} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$. Then the matrices $T_{gR}^{3(5)} \left(T_\circ^{5(14)}\right)^T$ are symmetric if $t_1 - t_2 = 0$ and $t_1 + 5t_2 = 0$. The only solution of this system of equations is $t_1 = t_2 = 0$. This implies that $\ker \mathcal{S}_5 = \{0\}$ and therefore $d_5 = 0$. So we will quit the algorithm after completing *Step IV* for $m = 4$. It remains to find vectors $x_{i4}^1 \in H_i'$, $i = 1, 2$, such that

$$U_i\left(\mathbf{a}_{11}^{34}\right) x_{i3}^1 + U_i\left(\mathbf{a}_{21}^{34}\right) x_{i3}^2 = W_i\left(\boldsymbol{\lambda}_0\right) x_{i4}^1.$$

They are

$$x_{14}^1 = \begin{bmatrix} 0 \\ -42 \\ -42 \\ 92 \end{bmatrix} \text{ and } x_{24}^1 = \begin{bmatrix} -6 \\ 0 \\ 0 \end{bmatrix}$$

and so

$$z_4^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -18 \\ 30 \\ -42 \\ -6 \\ 30 \\ -42 \\ 8 \\ -20 \\ 92 \end{bmatrix}.$$

The root subspace at $\boldsymbol{\lambda}_0$ has dimension 9 and its basis is

$$\mathcal{B} = \left\{z_0, z_1^1, z_1^2, z_2^1, z_2^2, z_2^3, z_3^1, z_3^2, z_4^1\right\}.$$

□

4.5.2 Completely Derogatory Case

A special case of a simple eigenvalue is a completely derogatory eigenvalue.

Definition. A simple eigenvalue $\boldsymbol{\lambda}$ of a multiparameter system is called *completely derogatory* if $B_0 = 0$. The matrix B_0 is defined by (2.18).

We write $\{\mathbf{e}_k;\ k \in \underline{n}\}$ for the standard basis of $\mathbb{C}^n$. Then the next observation follows :

Proposition 4.21 *Suppose that $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ is completely derogatory. Then $d_1 = n$, $r_m = 0$ for $m \geq 2$ and we can choose $\mathbf{a}_k^{01} = \mathbf{e}_k$ for $k \in \underline{n}$.*

Proof. It follows from Proposition 2.17 that $d_1 = n$ and we can choose $\mathbf{a}_k^{01} = \mathbf{e}_k$. In the previous subsection we established property (ii), page 141. Thus it also follows that $r_m = 0$ for $m \geq 2$. □

Note that it follows from the above proposition that we can assume $\widetilde{R}_M = \begin{bmatrix} I & 0 & \cdots & 0 \end{bmatrix}$ and then

$$T_f^{m(1l)} = \begin{bmatrix} \mathbf{a}_{1f}^{lm} & \mathbf{a}_{2f}^{lm} & \cdots & \mathbf{a}_{d_l f}^{lm} \end{bmatrix}$$

for $l \in \underline{m-1}$.

4.5.3 Two-parameter Simple Case

For $n = 2$ an eigenvalue $\boldsymbol{\lambda} \in \sigma(\mathbf{W})$ such that $\dim\ker W_i(\boldsymbol{\lambda}) = 1$, for $i = 1, 2$, is either nonderogatory if $B_0 \neq 0$ or completely derogatory if $B_0 = 0$, i.e., a simple eigenvalue is completely derogatory. The basis constructed in Algorithm 4.19 has simpler form for $n = 2$. It consists of vectors

$$z_m^f = x_{1m}^f \otimes x_{20} + x_{10} \otimes x_{2m}^f + \sum_{k=1}^{\left[\frac{m}{2}\right]} \sum_{l=k}^{m-k} \sum_{g=1}^{d_k} \sum_{h=1}^{d_l} t_{f(gh)}^{m(kl)} \left(x_{1k}^g \otimes x_{2l}^h + x_{1l}^h \otimes x_{2k}^g \right) \quad (4.42)$$

for $m = 0, 1, \ldots, M$ and $f \in \underline{d_m}$. Faierman conjectured [69, Conjecture 6.1, p. 122] the structure of root functions for nonreal eigenvalues of a two-parameter eigenvalue problem arising from class of Sturm-Liouville boundary value problems (0.1). The setting of these problems implies that all the eigenvalues are such that $\dim\ker W_i(\boldsymbol{\lambda}) = 1$

for both i. Therefore the eigenvalues are either nonderogatory or simple. Our formulae (4.42) for simple case are finite-dimensional simplified versions of his expressions for root functions. Our constants $s_{f(gh)}^{m(lk)}$, that are a counter-part of the constants c_{ijk}^{lqrs} in the expressions for the root functions $u_{jk}^{(i)}$ in [69, Conjecture 6.1], carry further structure that, when used in conjunction with the construction in [69], might lead to a solution of Faierman's conjecture.

4.6 Final Comments

Completeness results form an essential part of Multiparameter Spectral Theory. They were studied since the beginning of the theory. In 1968, in a modern revival of the theory, Atkinson [9] posed a completeness question in terms of the structure of root subspaces. This problem remains unsolved to this day, although partial solutions can be found in the literature cited below, and in this dissertation. (See also Comments to the previous two chapters.)

A basis for the first root subspace (that is the subspace of joint eigenvectors) and a theorem on the decomposition of the space H into a direct sum of root subspaces was given by Atkinson in [10, Chapter 6]. Isaev [109] discussed a general relation that holds for root vectors, similar to the relation (2.7). He also stated the problem of describing root subspaces of the associated system in terms of the underlying multiparameter system [112, Lecture 6, Problem 4].

In [93, Section V.9] Gohberg and Krein proved that a vector function $x(t) = e^{\lambda t}\sum_{k=0}^{p}\frac{t^k}{k!}x_k$ is a solution of a differential equation $L\left(\frac{d}{dt}\right)x(t) = 0$, where L is an operator polynomial, λ its eigenvalue and $x_0, x_1, \ldots, x_p$ is a Jordan chain at λ, Jordan chains for an operator polynomial being defined by (4.1). (One can find a version of this result for matrix polynomials in [122] and a version for holomorphic operator functions in [133].) Gadzhiev in [83] and also [84, Chapter 3] studied a multiparameter generalization of this setting. He found a set of linearly independent vectors in a root subspace, but he did not discuss completeness.

Binding [23] proved an important completeness result for real eigenvalues of elliptic multiparameter systems. A special case of his result in finite-dimensions is generalized in Subsection 4.3.2, but we do not recover his full result. The structure of root vectors he gave remains the same as for nonderogatory eigenvalues in our presentation.

In this chapter we prove completeness results for nonderogatory and simple eigenvalues of finite-dimensional multiparameter systems, i.e., eigenvalues $\boldsymbol{\lambda}$ such that

$$\dim\ker W_i(\boldsymbol{\lambda}) = 1$$

for all i. We also give a method for constructing a basis for the second root subspace in Subsection 2.5.2. The general completeness problem, i.e., the problem of finding bases for root subspaces for eigenvalues $\boldsymbol{\lambda}$ when

$$\dim\ker W_i(\boldsymbol{\lambda}) \geq 2 \tag{4.43}$$

for at least one of i, still remains unsolved. We see no immediate obstacle, as far as the overall reasoning is concerned, to generalizing our method to (4.43). On the other hand, it seems technically very complicated and it would require an extensive amount of calculation, as already exhibited in the case of simple eigenvalues. So, we would suggest considering use of other algebraic constructions to model multiparameter eigenvalue problems in order to understand the structure of root vectors better. The paper of Atkinson [9] could be used as the sign-post for the possible directions of research.

There are other immediate questions awaiting to be answered. For example, we already mentioned open problems of representations by tensor products and by multiparameter systems in the Comments section of Chapter 1. Another example is the question of the relation between the root subspaces $\ker(\boldsymbol{\lambda}\mathbf{I}-\boldsymbol{\Gamma})^N$ and $\ker\left(\overline{\boldsymbol{\lambda}}\mathbf{I}-\boldsymbol{\Gamma}^*\right)^N$, where $\overline{\boldsymbol{\lambda}}$ is the n-tuple of complex conjugate numbers $\overline{\lambda}_i$. This question is of special interest in connection with non-real eigenvalues for self-adjoint multiparameter systems. And finally, certainly the most important question for applications is to generalizing finite-dimensional completeness results to infinite-dimensional multiparameter eigenvalue problems.

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Appendix A

Proof of Theorem 1.18

Theorem 1.18 *Suppose that an array* $\mathbf{A}$ *is in the form* (1.2), $d_0 = 1$ *and the nonzero elements in the set* $\left\{\mathrm{a}_f^{0m},\ m \in \underline{M},\ f \in \underline{d_m}\right\}$ *are linearly independent. Then there exist symmetric matrices* $T_f^m,\ m = 2,3,\ldots,M,\ f \in \underline{d_m}$ *in the form* (1.29) *such that the relations*

$$\mathrm{a}_{h_1 f}^{l_1 m} = \sum_{l_2=1}^{m-l_1} \sum_{h_2=1}^{r_{l_2}} t_{f(h_1 h_2)}^{m(l_1 l_2)} \mathrm{a}_{h_2}^{0 l_2} \tag{A.1}$$

hold, where $l_1 \in \underline{m-1}$, $h_1 \in \underline{d_{l_1}}$, *and also*

$$\sum_{k=l_1+l_2}^{m-l_3} \sum_{g=1}^{d_k} t_{g(h_1 h_2)}^{k(l_1 l_2)} t_{f(h_3 g)}^{m(l_3 k)} = \sum_{k=l_1+l_3}^{m-l_2} \sum_{g=1}^{d_k} t_{g(h_1 h_3)}^{k(l_1 l_3)} t_{f(h_2 g)}^{m(l_2 k)}, \tag{A.2}$$

where $\mathbf{l} \in \Phi_m$, $\mathbf{h} \in \chi_{\mathbf{l}}$, $k \in \underline{m-2}$ *and* $g \in \underline{d_k}$. *Moreover the matrices* $T_f^{m(1,m-1)}$, $f \in \underline{d_m}$ *are linearly independent for* $m = 2,3,\ldots,M$.

Proof. The matrices $\widetilde{T}_f^m$ of Proposition 1.17 are such that relations (A.1) hold and matrices $\widehat{T}_f^m$ of (1.27) are symmetric. We construct matrices $\widetilde{T}_f^m$ inductively as described on pages 36-38. To prove the theorem we need to establish three properties of matrices $T_f^{m'}$, $m' = 2,3,\ldots,M$, $f \in \underline{d_{m'}}$. These are :

(i) the matrices $T_f^{m'}$ are symmetric, i.e.

$$T_f^{m'(l_l l_2)} = \left(T_f^{m'(l_2 l_1)}\right)^T$$

for all l_1 and l_2 such that $l_1 + l_2 \leq m'$,

(ii) the products

$$T_{fC}^{m'(l)} \left(T_{gR}^{l(m')}\right)^T, \tag{A.3}$$

where $T_{gR}^{l(m')}$ is defined from (1.33) and

$$T_{fC}^{m'(l)} = \begin{bmatrix} T_f^{m'(1,l+1)} & T_f^{m'(1,l+2)} & \cdots & T_f^{m'(1,m-1)} \\ T_f^{m'(2,l+1)} & T_f^{m'(2,l+2)} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ T_f^{m'(m-l+1,l+1)} & 0 & \cdots & 0 \end{bmatrix},$$

are symmetric for all $l \in \underline{m-2}$ and $g \in \underline{d_l}$, and

(iii) the matrix $T_f^{m'(l_1 l_2)}$ coincides with the matrix

$$\begin{bmatrix} \widehat{T}_f^{m'(l_1 l_2)} & \overline{T}^{m'(l_1 l_2)} \\ \left(\overline{T}_f^{m'(l_2 l_1)}\right)^T & * \end{bmatrix}$$

in the entries other than ones denoted by $*$, for all l_1 and l_2 such that $l_1+l_2 \leq m'$.

First note that condition (*i*) holds for l_1 and l_2 such that $l_1 \neq l_2$ and all m' and f from the definition of matrices $T_f^{m'(l_1 l_2)}$. Condition (*ii*) is equivalent to the condition

$$s_{f(h_1 h_2 h_3)}^{m(l_1 l_2 l_3)} = s_{f(h_2 h_3 h_2)}^{m(l_1 l_3 l_2)}$$

for $\mathbf{l} \in \Phi_m$ and $\mathbf{h} \in \chi_{\mathbf{l}}$. From Proposition 1.17 it follows that the matrices

$$T_{gR}^{1(12)} \left(T_f^{3(12)}\right)^T$$

are symmetric for $g \in \underline{d_1}$ and $f \in \underline{d_3}$. Because we also have that $T_f^{3(11)}$ are symmetric and $T_f^{3(12)} = \widehat{T}_f^{3(12)} = \left(\widehat{T}_f^{3(21)}\right)^T$ the conditions (*i*) – (*iii*) hold for $m' = 2, 3$. We proceed by induction on m'. Suppose that (*i*) through (*iii*) hold for $m' = 2, 3, \ldots, m-1$. We want to prove them also for $m' = m$. By Proposition 1.17 and definition (1.32) we have

$$\sum_{h=1}^{d_{m-1}} t_{h(ig)}^{m-1(1,m-2)} t_{f(jh)}^{m(1,m-1)} = s_{f(igj)}^{m(1,m-2,1)} = s_{f(jgi)}^{m(1,m-2,1)} = \sum_{h=1}^{d_{m-1}} t_{f(ih)}^{m(1,m-1)} t_{h(jg)}^{m-1(1,m-2)}$$

for all $i, j \in \underline{d_1}$ and therefore the matrix $T_{gR}^{m-2(1,m-1)} \left(T_f^{m(1,m-1)}\right)^T$ is symmetric for all $g \in \underline{d_{m-2}}$. Now we proceed with backwards induction on k' to prove that products

$$T_{gR}^{k'(m)} \left(T_{fC}^{m(k')}\right)^T \tag{A.4}$$

are symmetric matrices for all $g \in \underline{d_{k'}}$. We suppose then that matrices (A.4) are symmetric for $k' = m-2, m-3, \ldots, k+1$, $(k \geq 1)$ and we prove that they are symmetric also for $k' = k$. Consider first the product

$$T_{gR}^{k(r,k+r)} \left(T_f^{m(m-k-r,k+r)}\right)^T \tag{A.5}$$

for $r \in \underline{\left[\frac{m-k}{2}\right]}$. It is equal to

$$\sum_{h_1=1}^{d_1} T_{gR}^{k(r,k+r)} T \left({}^{1(\bar{k}+r,k+r+1)}_{h_1 R} T^{m(m-k-r-1,k+r+1)}\right)^T \left(Z_{h_1 R}^{1(m-k-r-1,m-k-r)}\right)^T . \tag{A.6}$$

By the inductive assumption for $m' \leq m-1$ we have

$$\sum_{h_2=1}^{d_{k+r}} t_{h_2(ig)}^{k+r(rk)} t_{j(h_2 h_1)}^{k+r+1(k+r,1)} = s_{j(igh_1)}^{k+r+1(rk1)} = s_{j(h_1 gi)}^{k+r+1(1kr)} = \sum_{h_2=1}^{d_{k+1}} t_{h_2(h_1 g)}^{k+1(1k)} t_{j(h_2 i)}^{k+r+1(k+r,r)}$$

and therefore the product (A.6) is equal to

$$\sum_{h_1=1}^{d_1} \sum_{h_2=1}^{d_{k+1}} t_{h_2(h_1 g)}^{k+1(1k)} T_{h_2 R}^{1(k+r,k+r+1)} \left(T_f^{m(m-k-r-1,k+r+1)}\right)^T \left(Z_{h_1 R}^{1(m-k-r-1,m-k-r)}\right)^T .$$

Next we use the inductive assumption for $k' \geq k+1$ to show that (A.6) is equal to

$$\sum_{h_1=1}^{d_1} \sum_{h_2=1}^{d_{k+1}} t_{h_2(h_1 g)}^{k+1(1k)} T_f^{m(k+r,m-k-r)} \left(T_{h_2 R}^{k+1(m-k-r-1,m-r)}\right)^T \left(Z_{h_1 R}^{1(m-k-r-1,m-k-r)}\right)^T =$$

$$= T_f^{m(k+r,m-k-r)} \sum_{h_1=1}^{d_1} \sum_{h_2=1}^{d_{k+1}} t_{h_2(h_1 g)}^{k+1(1k)} \left(T_{h_2 R}^{k+1(m-k-r-1,m-r)}\right)^T \left(Z_{h_1 R}^{1(m-k-r-1,m-k-r)}\right)^T . \tag{A.7}$$

We use again the inductive assumption on $m' \leq m-1$ to show that

$$\sum_{h_2=1}^{d_{k+1}} t_{h_2(h_1 g)}^{k+1(1k)} t_{i(jh_2)}^{m-r(m-k-r-1,k+1)} = s_{i(h_1 gj)}^{m-r(1,k,m-k-r-1)} =$$

$$= s_{i(h_1jg)}^{m-r(1,m-k-r-1,k)} = \sum_{h_2=1}^{d_{m-k-r}} t_{i(h_2g)}^{m-r(m-k-r,k)} t_{h_2(h_1j)}^{m-k-r(1,m-k-r-1)}.$$

Then it follows that (A.7) equals to

$$T_f^{m(k+r,m-k-r)} \sum_{h_1=1}^{d_1} \left(T_{gR}^{k(m-r-k,m-r)}\right)^T \left(T_{h_1R}^{1(m-k-r-1,m-k-r)}\right)^T \left(Z_{h_1R}^{1(m-k-r-1,m-k-r)}\right)^T =$$

$$= T_f^{m(k+r,m-k-r)} \left(T_{gR}^{k(m-r-k,m-r)}\right)^T. \tag{A.8}$$

So the products (A.5) and (A.8) are equal and therefore the product of the r-th (block) row of the matrix $T_{gR}^{k(m)}$ and q-th ($q = m - r$) column of the matrix $T_f^{m(k)}$ is the transpose of the product of the q-th row of the matrix $T_{gR}^{k(m)}$ and r-th column of the matrix $T_f^{m(k)}$. Now we proceed by backward induction on q to prove that the above is true also for the products of the other rows and columns. Suppose now that for every $r \in \left[\frac{m-k}{2}\right]$ and $q' = m - r, m - r - 1, \ldots, q + 1$ (where $q > \left[\frac{m-k}{2}\right]$) we have

$$\sum_{p=k+r}^{m-q'} T_{gR}^{k(rp)} \left(T_f^{m(q'p)}\right)^T = \sum_{p=q'+k}^{m-r} T_f^{m(rp)} \left(T_{gR}^{k(q'p)}\right)^T. \tag{A.9}$$

We want to prove the relation (A.9) for $q' = q$. By the definition of the matrices $T_f^{m(qp)}$ it follows that

$$\sum_{p=k+r}^{m-q} T_{gR}^{k(rp)} \left(T_f^{m(qp)}\right)^T = \sum_{p=k+r}^{m-q} T_{gR}^{k(rp)} \sum_{h_1=1}^{d_1} \left[\begin{array}{cccc} T_{h_1R}^{1(p,p+1)} & T_{h_1R}^{1(p,p+2)} & \cdots & T_{h_1R}^{1(p,m-q+1)} \end{array}\right] \cdot$$

$$\cdot \left[\begin{array}{cccc} \left(T_f^{m(q-1,p+1)}\right)^T & \left(T_f^{m(q,p+1)}\right)^T & \cdots & \left(T_f^{m(m-p-1,p+1)}\right)^T \\ \left(T_f^{m(q-1,p+2)}\right)^T & \left(T_f^{m(q,p+2)}\right)^T & & 0 \\ \vdots & & & \vdots \\ \left(T_f^{m(q-1,m-q+1)}\right)^T & 0 & \cdots & 0 \end{array}\right] \left[\begin{array}{c} \left(Z_{h_1R}^{1(q-1,q)}\right)^T \\ \left(Z_{h_1R}^{1(q-1,q)}\right)^T \\ \vdots \\ \left(Z_{h_1R}^{1(q-1,m-p)}\right)^T \end{array}\right] =$$

$$= \sum_{h_1=1}^{d_1} \sum_{p=k+r}^{m-q} T_{gR}^{k(rp)} \left[\begin{array}{cccccc} 0 & \cdots & 0 & T_{h_1R}^{1(p,p+1)} & T_{h_1R}^{1(p,p+2)} & \cdots & T_{h_1R}^{1(p,m-q+1)} \end{array}\right] \cdot$$

$$\cdot \left[\begin{array}{cccc} \left(T_f^{m(q-1,k+r+1)}\right)^T & \left(T_f^{m(q,k+r+1)}\right)^T & \cdots & \left(T_f^{m(m-k-r-1,k+r+1)}\right)^T \\ \left(T_f^{m(q-1,k+r+2)}\right)^T & \left(T_f^{m(q,k+r+2)}\right)^T & & 0 \\ \vdots & & & \vdots \\ \left(T_f^{m(q-1,m-q+1)}\right)^T & 0 & \cdots & 0 \end{array}\right].$$

$$\cdot \begin{bmatrix} \left(Z_{h_1R}^{1(q-1,q)}\right)^T \\ \left(Z_{h_1R}^{1(q-1,q)}\right)^T \\ \vdots \\ \left(Z_{h_1R}^{1(q-1,m-k-r)}\right)^T \end{bmatrix}. \tag{A.10}$$

Next we have

$$\sum_{p=k+r}^{m-q} T_{gR}^{k(rp)} \begin{bmatrix} 0 & \cdots & 0 & T_{h_1R}^{1(p,p+1)} & T_{h_1R}^{1(p,p+2)} & \cdots & T_{h_1R}^{1(p,m-q+1)} \end{bmatrix} =$$

$$= \Big[T_{gR}^{k(r,k+r)} T_{h_1R}^{1(k+r,k+r+1)} \quad \sum_{p=k+r}^{k+r+1} T_{gR}^{k(rp)} T_{h_1R}^{1(p,k+r+2)} \quad \sum_{p=k+r}^{k+r+2} T_{gR}^{k(rp)} T_{h_1R}^{1(p,k+r+3)} \quad \cdots$$

$$\cdots \quad \sum_{p=k+r}^{m-q} T_{gR}^{k(rp)} T_{h_1R}^{1(p,m-q+1)} \Big] \tag{A.11}$$

and by the inductive assumption for $m' \leq m-1$ it follows that

$$\sum_{p=k+r}^{k+r+l} \sum_{h_2=1}^{d_p} t_{h_2(ig)}^{p(rk)} \, t_{j(h_2h_1)}^{k+r+l+1(p1)} = s_{j(igh_1)}^{k+r+l+1(rk1)} =$$

$$= s_{j(h_1gi)}^{k+r+l+1(1kr)} = \sum_{p=k+1}^{k+l+1} \sum_{h_2=1}^{d_p} t_{h_2(h_1g)}^{p(1k)} \, t_{j(h_2i)}^{k+r+l+1(pr)}.$$

Therefore the expression (A.11) is equal to

$$\Big[\sum_{h_2=1}^{d_{k+1}} t_{h_2(h_1g)}^{k+1(1k)} T_{h_2R}^{k+1(r,k+r+1)} \quad \sum_{p=k+1}^{k+2} \sum_{h_2=1}^{d_p} t_{h_2(h_1g)}^{p(1k)} T_{h_2R}^{p(r,k+r+2)} \quad \cdots$$

$$\cdots \quad \sum_{p=k+1}^{m-q-r+1} \sum_{h_2=1}^{d_p} t_{h_2(h_1g)}^{p(1k)} T_{h_2R}^{p(r,m-q+1)} \Big] =$$

$$= \sum_{p=k+1}^{m-q-r+1} \sum_{h_2=1}^{d_p} t_{h_2(h_1g)}^{p(1k)} \begin{bmatrix} 0 & \cdots & 0 & T_{h_2R}^{p(r,r+p)} & T_{h_2R}^{p(r,r+p+1)} & \cdots & T_{h_2R}^{p(r,m-q+1)} \end{bmatrix}.$$

Using this last calculation and the inductive assumption for $m' \leq m-1$, respectively, we show that the expression (A.10) is equal to

$$\sum_{h_1=1}^{d_1} \sum_{p=k+1}^{m-q-r+1} \sum_{h_2=1}^{d_p} t_{h_2(h_1g)}^{p(1k)} \begin{bmatrix} 0 & \cdots & 0 & T_{h_2R}^{p(r,r+p)} & T_{h_2R}^{p(r,r+p+1)} & \cdots & T_{h_2R}^{p(r,m-q+1)} \end{bmatrix}.$$

$$\cdot \begin{bmatrix} \left(T_f^{m(q-1,k+r+1)}\right)^T & \left(T_f^{m(q,k+r+1)}\right)^T & \cdots & \left(T_f^{m(m-k-r-1,k+r+1)}\right)^T \\ \left(T_f^{m(q-1,k+r+2)}\right)^T & \left(T_f^{m(q,k+r+2)}\right)^T & & 0 \\ \vdots & & & \vdots \\ \left(T_f^{m(q-1,m-q+1)}\right)^T & 0 & \cdots & 0 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \left(Z_{h_1R}^{1(q-1,q)}\right)^T \\ \left(Z_{h_1R}^{1(q-1,q)}\right)^T \\ \vdots \\ \left(Z_{h_1R}^{1(q-1,m-k-r)}\right)^T \end{bmatrix} =$$

$$= \sum_{h_1=1}^{d_1} \sum_{p=k+1}^{m-q-r+1} \sum_{h_2=1}^{d_p} t_{h_2(h_1g)}^{p(1k)} \begin{bmatrix} T_f^{m(r,q+k)} & T_f^{m(r,q+k+1)} & \cdots & T_f^{m(r,m-r)} \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \left(T_{h_2R}^{p(q-1,q-1+p)}\right)^T & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \left(T_{h_2R}^{p(q-1,q+p)}\right)^T & \left(T_{h_2R}^{p(q,q+p)}\right)^T & & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \left(T_{h_2R}^{p(q-1,m-r)}\right)^T & \left(T_{h_2R}^{p(q,m-r)}\right)^T & \cdots & \left(T_{h_2R}^{p(m-r-p,m-r)}\right)^T & 0 & \cdots & 0 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \left(Z_{h_1R}^{1(q-1,q)}\right)^T \\ \left(Z_{h_1R}^{1(q-1,q)}\right)^T \\ \vdots \\ \left(Z_{h_1R}^{1(q-1,m-k-r)}\right)^T \end{bmatrix} . \tag{A.12}$$

By the assumption for $m' \leq m-1$ it follows that

$$\sum_{u=k+1}^{p} \sum_{h_3=1}^{d_u} t_{h_2(h_1g)}^{u(1k)} \, t_{i(jh_2)}^{q-1+p(q-1,u)} = s_{i(h_1gj)}^{q-1+p(1,k,q-1)} =$$

$$= s_{i(jh_1g)}^{q-1+p(q-1,1,k)} = \sum_{u=q}^{q-1+p-k} \sum_{h_3=1}^{d_u} t_{i(h_2g)}^{q-1+p(uk)} \, t_{h_2(jh_1)}^{u(q-1,1)}$$

and therefore

$$\sum_{p=k+1}^{m-q-r+1} \sum_{h_2=1}^{d_p} t_{h_2(h_1g)}^{p(1k)} .$$

$$\cdot \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \left(T_{h_2R}^{p(q-1,q-1+p)}\right)^T & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \left(T_{h_2R}^{p(q-1,q+p)}\right)^T & \left(T_{h_2R}^{p(q,q+p)}\right)^T & & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \left(T_{h_2R}^{p(q-1,m-r)}\right)^T & \left(T_{h_2R}^{p(q,m-r)}\right)^T & \cdots & \left(T_{h_2R}^{p(m-r-p,m-r)}\right)^T & 0 & \cdots & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} \left(T_{gR}^{k(q,q+k)}\right)^T & 0 & \cdots & 0 \\ \left(T_{gR}^{k(q,q+k+1)}\right)^T & \left(T_{gR}^{k(q+1,q+k+1)}\right)^T & & 0 \\ \vdots & \vdots & & \vdots \\ \left(T_{gR}^{k(q,m-r)}\right)^T & \left(T_{gR}^{k(q+1,m-r)}\right)^T & \cdots & \left(T_{gR}^{k(m-r-k,m-r)}\right)^T \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \left(T_{h_1R}^{1(q-1,q)}\right)^T & 0 & \cdots & 0 \\ \left(T_{h_1R}^{1(q-1,q+1)}\right)^T & \left(T_{h_1R}^{1(q,q+1)}\right)^T & & 0 \\ \vdots & \vdots & & \vdots \\ \left(T_{h_1R}^{1(q-1,m-r-k)}\right)^T & \left(T_{h_1R}^{1(q,m-r-k)}\right)^T & \cdots & T_{h_1R}^{1(m-r-k-1,m-r-k)} \end{bmatrix} . \tag{A.13}$$

Since $\mathbf{Z}_R^{1(m)}$ is a left inverse of $\mathbf{T}_R^{1(m)}$ it follows that

$$\sum_{h_1=1}^{d_1} \begin{bmatrix} \left(T_{h_1R}^{1(q-1,q)}\right)^T & 0 & \cdots & 0 \\ \left(T_{h_1R}^{1(q-1,q+1)}\right)^T & \left(T_{h_1R}^{1(q,q+1)}\right)^T & & 0 \\ \vdots & \vdots & & \vdots \\ \left(T_{h_1R}^{1(q-1,m-r-k)}\right)^T & \left(T_{h_1R}^{1(q,m-r-k)}\right)^T & \cdots & T_{h_1R}^{1(m-r-k-1,m-r-k)} \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \left(Z_{h_1R}^{1(q-1,q)}\right)^T \\ \left(Z_{h_1R}^{1(q-1,q)}\right)^T \\ \vdots \\ \left(Z_{h_1R}^{1(q-1,m-k-r)}\right)^T \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} . \tag{A.14}$$

Now we use the equalities (A.13) and (A.14) to show that the expression (A.12) is equal to

$$\begin{bmatrix} T_f^{m(r,q+k)} & T_f^{m(r,q+k+1)} & \cdots & T_f^{m(r,m-r)} \end{bmatrix} \begin{bmatrix} \left(T_{gR}^{k(q,q+k)}\right)^T \\ \left(T_{gR}^{k(q,q+k+1)}\right)^T \\ \vdots \\ \left(T_{gR}^{k(q,m-r)}\right)^T \end{bmatrix} = \sum_{p=q+k}^{m-r} T_f^{m(r,p)} \left(T_{gR}^{k(qp)}\right)^T.$$

Thus we proved that the relation (A.9) holds also for $q' = q$. We proceed with the induction until $q = \left[\frac{m-k}{2}\right]$. Then it follows that the matrix $T_f^{m(k)} \left(T_{gR}^{k(m)}\right)^T$ is symmetric for all $g \in \underline{d_k}$. Hence we proved the inductive step for $k' = k$. We stop the induction process when $k = 1$. Then it follows that the products (A.3) are symmetric and this proves condition (*ii*) for $m' = m$.

It remains to prove that the matrices $T_f^{m(ll)}$ are symmetric for $l = 2, 3, \ldots, \left[\frac{m}{2}\right]$ and condition (*iii*) for $m' = m$. First we prove by backwards induction on l that the matrices $T_f^{m(ll)}$ are symmetric. Suppose first that m is even and write $l = \frac{m}{2}$. Then we define for every $g, h \in \underline{d_1}$ a matrix

$$U_{gh}^f = T_f^{m(l-1,l+1)} \left(T_{gR}^{1(l,l+1)}\right)^T \left(T_{hR}^{1(l-1,l)}\right)^T.$$

For every $i \in \underline{d_{l+1}}$ and $j \in \underline{d_{l-1}}$ we have

$$\sum_{h_2=1}^{d_l} t_{i(h_2 g)}^{l+1(l1)} t_{h_2(jh)}^{l(l-1,1)} = s_{i(jhg)}^{l+1(l-1,1,1)} = s_{i(jgh)}^{l+1(l-1,1,1)} = \sum_{h_2=1}^{d_l} t_{i(h_2 h)}^{l+1(l1)} t_{h_2(jg)}^{l(l-1,1)}$$

and therefore

$$U_{gh}^f = U_{hg}^f. \tag{A.15}$$

Next it follows from condition (*ii*) for $m' = m$ proven above that

$$\sum_{h_3=1}^{d_{l+1}} \sum_{h}^{d_l} t_{f(ih_3)}^{m(l-1,l+1)} t_{h_3(h_2 g)}^{l+1(l1)} t_{h_2(jh)}^{l(l-1,1)} = \sum_{h_2=1}^{d_l} s_{f(h_2 gi)}^{m(l,1,l-1)} t_{h_2(jh)}^{l(l-1,1)} = \sum_{h_2=1}^{d_l} s_{f(ih_2 g)}^{m(l-1,l,1)} t_{h_2(jh)}^{l(l-1,1)} =$$

$$= \sum_{h_3=1}^{d_{m-1}} \sum_{h_2=1}^{d_l} t_{f(h_3 g)}^{m(m-1,1)} t_{h_3(ih_2)}^{m-1(l-1,l)} t_{h_2(jh)}^{l(l-1,1)} = \sum_{h_3=1}^{d_{m-1}} t_{f(h_3 g)}^{m(m-1,1)} s_{h_3(jhi)}^{m-1(l-1,1,l-1)}.$$

Similarly we show that

$$\sum_{h_3=1}^{d_{l+1}} \sum_{h}^{d_l} t_{f(jh_3)}^{m(l-1,l+1)} t_{h_3(h_2g)}^{l+1(l1)} t_{h_2(ih)}^{l(l-1,1)} = \sum_{h_3=1}^{d_{m-1}} t_{f(h_3g)}^{m(m-1,1)} s_{h_3(ihj)}^{m-1(l-1,1,l-1)}$$

and then because we assumed (*ii*) for $m' = m-1$ it follows that $s_{h_3(ihj)}^{m-1(l-1,1,l-1)} = s_{h_3(jhi)}^{m-1(l-1,1,l-1)}$ and therefore every matrix U_{hg}^f is symmetric. This fact together with (A.15) imply that the sum

$$V^f = \sum_{h=1}^{d_1}\sum_{g=1}^{d_1} Z_{hR}^{1(l-1,l)} U_{hg}^f \left(Z_{hR}^{1(l-1,l)}\right)^T$$

is a symmetric matrix, but

$$V^f = \sum_{h=1}^{d_1} Z_{hR}^{1(l-1,l)} T_f^{m(l-1,l+1)} \left(T_{hR}^{1(l,l+1)}\right)^T \sum_{g=1}^{d_1} \left(Z_{gR}^{1(l-1,l)} T_{gR}^{1(l-1,l)}\right)^T = T_f^{m(ll)}$$

and hence $T_f^{m(ll)}$ is symmetric.

Next we assume that m is odd. For every $g, h \in \underline{d_1}$ we define matrices

$$U_{gh}^f = \begin{bmatrix} T_f^{m(l-1,l+1)} & T_f^{m(l-1,l+2)} \\ T_f^{m(l,l+1)} & 0 \end{bmatrix} \begin{bmatrix} \left(T_{gR}^{1(l,l+1)}\right)^T & 0 \\ \left(T_{gR}^{1(l,l+2)}\right)^T & \left(T_{gR}^{1(l+1,l+2)}\right)^T \end{bmatrix} \cdot$$
$$\cdot \begin{bmatrix} \left(T_{hR}^{1(l-1,l)}\right)^T & 0 \\ \left(T_{hR}^{1(l-1,l+1)}\right)^T & \left(T_{hR}^{1(l,l+1)}\right)^T \end{bmatrix},$$

where $2l+1 = m$. Similarly to the case of even m we show that $U_{gh}^f = U_{hg}^f$ and that the matrices U_{gh}^f are symmetric. Thus also the matrix

$$V^f = \sum_{h=1}^{d_1}\sum_{g=1}^{d_1} \begin{bmatrix} Z_{hR}^{1(l-1,l)} & Z_{hR}^{1(l-1,l+1)} \end{bmatrix} U_{hg}^f \begin{bmatrix} \left(Z_{gR}^{1(l-1,l)}\right)^T \\ \left(Z_{gR}^{1(l-1,l+1)}\right)^T \end{bmatrix}$$

is symmetric and because $V^f = T_f^{m(ll)}$ also the latter matrix is symmetric.

Now we proceed by backward induction on l until $l = 1$. For every $g, h \in \underline{d_1}$ we define matrices

$$U_{gh}^{f(l)} = \begin{bmatrix} T_f^{m(l-1,l+1)} & T_f^{m(l-1,l+2)} & \cdots & T_f^{m(l-1,m-l+1)} \\ T_f^{m(l,l+1)} & T_f^{m(l,l+2)} & & 0 \\ \vdots & & & \vdots \\ T_f^{m(m-l+1,l-1)} & 0 & \cdots & 0 \end{bmatrix}.$$

$$\cdot \begin{bmatrix} \left(T_{gR}^{1(l,l+1)}\right)^T & 0 & \cdots & 0 \\ \left(T_{gR}^{1(l,l+2)}\right)^T & \left(T_{gR}^{1(l+1,l+2)}\right)^T & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \left(T_{gR}^{1(l,m-l)}\right)^T & \left(T_{gR}^{1(l+1,m-l)}\right)^T & \cdots & \left(T_{gR}^{1(m-l,m-l+1)}\right)^t \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \left(T_{hR}^{1(l-1,l)}\right)^T & 0 & \cdots & 0 \\ \left(T_{hR}^{1(l-1,l+1)}\right)^T & \left(T_{hR}^{1(l,l+1)}\right)^T & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \left(T_{hR}^{1(l-1,m-l-1)}\right)^T & \left(T_{hR}^{1(l,m-l-1)}\right)^T & \cdots & \left(T_{hR}^{1(m-l-1,m-l)}\right)^T \end{bmatrix} \cdot$$

As before in the case of even m we show that $U_{gh}^{f(l)} = U_{hg}^{f(l)}$ and by the same method using also the inductive assumption on l we prove that the matrices $U_{gh}^{f(l)}$ are symmetric. Finally the matrix

$$V^{f(l)} = \sum_{h-1}^{d_1} \sum_{g=1}^{d_1} \left[\ Z_{hR}^{1(l-1,l)} \quad Z_{hR}^{1(l-1,l+1)} \quad \cdots \quad Z_{hR}^{1(l-1,m-l)}\ \right] U_{hg}^{f(l)} \begin{bmatrix} \left(Z_{hR}^{1(l-1,l)}\right)^T \\ \left(Z_{hR}^{1(l-1,l+1)}\right)^T \\ \vdots \\ \left(Z_{hR}^{1(l-1,m-l)}\right)^T \end{bmatrix}$$

is symmetric and since $V^{f(l)} = T_f^{m(ll)}$ also the latter matrix is symmetric.

To complete the proof of the inductive step for m it remains to prove condition (*iii*) for $m' = m$. We write

$$T_f^{m(l_1 l_2)} = \begin{bmatrix} T_{fN}^{m(l_1 l_2)} \\ T_{fS}^{m(l_1 l_2)} \end{bmatrix},$$

where $T_{fN}^{m(l_1 l_2)} \in \mathbb{C}^{r_{l_1} \times d_{l_2}}$ and $l_1, l_2 \in \underline{m-1}$, and

$$T_{fN}^{m(l)} = \begin{bmatrix} T_{fN}^{m(1l)} & T_{fN}^{m(1,l+1)} & \cdots & T_{fN}^{m(1,m-1)} \\ T_{fN}^{m(2l)} & T_{fN}^{m(2,l+1)} & & 0 \\ \vdots & & & \vdots \\ T_{fN}^{m(m-l,l)} & 0 & \cdots & 0 \end{bmatrix}$$

for $l \in \underline{m-1}$. Similarly we have

$$T_{gR}^{k(l_1 l_2)} = \begin{bmatrix} T_{gRN}^{k(l_1 l_2)} \\ T_{gRS}^{k(l_1 l_2)} \end{bmatrix}$$

where $T_{gRN}^{k(l_1 l_2)} \in \mathbb{C}^{r_{l_1} \times d_{l_2}}$. By Proposition 1.17 we have

$$\sum_{p=k+1}^{m-q} \widetilde{T}_{gR}^{k(1p)} \left(\widetilde{T}_f^{m(qp)}\right)^T = \sum_{p=k+q}^{m-1} \widetilde{T}_f^{m(1p)} \left(\widetilde{T}_{gR}^{k(qp)}\right)^T$$

for all $k \in \underline{m-2}$, $q \in \underline{m-k-1}$ and $g \in \underline{d_k}$. Let us recall that the matrices $\widetilde{T}_f^{m(l_1 l_2)}$ are given (cf. the beginning of this proof). Because we already showed that condition (ii) holds for $m' = m$ it follows that

$$\sum_{p=k+1}^{m-q} T_{gR}^{k(1p)} \left(T_{Nf}^{m(qp)}\right)^T = \sum_{p=k+q}^{m-1} T_f^{m(1p)} \left(T_{gNR}^{k(qp)}\right)^T .$$

Because $T_f^{m(1p)} = \widetilde{T}_f^{m(1p)}$ and because we assumed that condition (iii) holds for $m' \leq m-1$ it follows that

$$\sum_{p=k+q}^{m-1} T_f^{m(1p)} \left(T_{gNR}^{k(qp)}\right)^T = \sum_{p=k+q}^{m-1} \widetilde{T}_f^{m(1p)} \left(\widetilde{T}_{gR}^{k(qp)}\right)^T$$

and therefore

$$\sum_{p=k+1}^{m-q} \widetilde{T}_{gR}^{k(1p)} \left(\widetilde{T}_f^{m(qp)}\right)^T = \sum_{p=k+1}^{m-q} T_{gR}^{k(1p)} \left(\widetilde{T}_f^{m(qp)}\right)^T = \sum_{p=k+1}^{m-q} T_{gR}^{k(1p)} \left(T_{Nf}^{m(qp)}\right)^T . \tag{A.16}$$

Next we define for every $k \in \underline{m-2}$ a $(d_1 d_k) \times \left(\sum_{l=k+1}^{m-1} d_l\right)$ matrix

$$T_{RA}^{k(m)} = \begin{bmatrix} T_{1R}^{k(1,k+1)} & T_{1R}^{k(1,k+2)} & \cdots & T_{1R}^{k(1,m-1)} \\ T_{2R}^{k(1,k+1)} & T_{2R}^{k(1,k+2)} & \cdots & T_{2R}^{k(1,m-1)} \\ \vdots & \vdots & & \vdots \\ T_{d_k R}^{k(1,k+1)} & T_{d_2 R}^{k(1,k+2)} & \vdots & T_{d_k R}^{k(1,m-1)} \end{bmatrix}$$

and a $\left(d_1 \sum_{l=1}^{m-2} d_l\right) \times \left(\sum_{l=2}^{m-1} d_l\right)$ matrix

$$\mathbf{T}_{RA}^{(m)} = \begin{bmatrix} T_{RA}^{1(m)} \\ 0 \quad T_{RA}^{2(m)} \\ 0 \quad T_{RA}^{3(m)} \\ \vdots \\ 0 \quad T_{RA}^{m-2(m)} \end{bmatrix},$$

where the sizes of the 0 blocks are determined by the sizes of matrices $T_{RA}^{k(m)}$. The matrix $\mathbf{T}_{RA}^{(m)}$ is left invertible because the columns of the matrix

$$\begin{bmatrix} T_{1R}^{k(1,k+1)} \\ T_{2R}^{k(1,k+1)} \\ \vdots \\ T_{d_kR}^{k(1,k+1)} \end{bmatrix}$$

are linearly independent for $k \in \underline{m-2}$. From the equalities (A.16) it follows that

$$\mathbf{T}_{RA}^{(m)} \left(\widetilde{T}_f^{m(1)}\right)^T = \mathbf{T}_{RA}^{(m)} \left(T_{fN}^{m(1)}\right)^T$$

and because $\mathbf{T}_{RA}^{(m)}$ is left invertible it follows that $\widetilde{T}_f^{m(1)} = T_{fN}^{m(1)}$. Because we showed that the matrices T_f^m are symmetric it follows then that the matrix $T_f^{m(l_1l_2)}$ coincides with the matrix

$$\begin{bmatrix} \widehat{T}_f^{m(l_1l_2)} & \overline{T}_f^{m(l_1l_2)} \\ \overline{T}_f^{m(l_2l_1)} & * \end{bmatrix}$$

in the entries other than $*$. Therefore condition (*iii*) holds also for $m' = m$ and so the proof of the inductive step for m' is done. We proceed by induction until $m = M$, and this completes the proof. □

Appendix B

Proof of Lemma 4.17

Lemma 4.17 *Suppose that we have the same setting as in Lemma* 4.16. *We construct a vector*

$$z_m^1 = x_{m1}^{\otimes} + \sum_{q=2}^{\min\{n,m\}} \sum_{\mathbf{l}\in\Phi_{mq}} \sum_{\mathbf{h}\in\chi_{\mathbf{l}}} s_{\mathbf{1h}}^{m\mathbf{l}} x_{\mathbf{lh}}^{\otimes}$$

where the numbers $s_{\mathbf{1h}}^{m\mathbf{l}}$ *are defined in* (1.37) *and the vectors* $x_{\mathbf{lh}}^{\otimes}$ *are defined in* (4.27). *Then it follows that*

$$(\lambda_i I - \Gamma_i)\, z_m^1 = \sum_{k=1}^{m-1} \sum_{g=1}^{d_k} a_{g1i}^{km}\, z_k^g + a_{1i}^{0m} z_0 \tag{B.1}$$

for all i.

Proof. We use a direct calculation to show (B.1). First we have

$$(\lambda_i\Delta_0 - \Delta_i)\, z_m^1 = (\lambda_i\Delta_0 - \Delta_i)\, x_{m1}^{\otimes} + \sum_{q=2}^{\min\{m,n\}} \sum_{\mathbf{u}\in\Omega_q} \sum_{\mathbf{l}\in\Phi_{m.q}} \sum_{\mathbf{h}\in\chi_{\mathbf{l}}} (\lambda_i\Delta_0 - \Delta_i)\, s_{\mathbf{1h}}^{m\mathbf{l}} x_{\mathbf{ulh}}^{\otimes}. \tag{B.2}$$

From the basic properties of the operator determinants it follows that

$$(\lambda_i\Delta_0 - \Delta_i) = \begin{vmatrix} V_{11} & \cdots & V_{1,i-1} & W_1(\boldsymbol{\lambda}) & V_{1,i+1} & \cdots & V_{1n} \\ V_{21} & \cdots & V_{2,i-1} & W_2(\boldsymbol{\lambda}) & V_{2,i+1} & \cdots & V_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1} & \cdots & V_{n,i-1} & W_n(\boldsymbol{\lambda}) & V_{n,i+1} & \cdots & V_{nn} \end{vmatrix}$$

and hence $(\lambda_i\Delta_0 - \Delta_i)\, x_{m1}^{\otimes} =$

$$= \sum_{u=1}^{n} \begin{vmatrix} \cdots & V_{1,i-1}x_{10} & 0 & V_{1,i+1}x_{10} & \cdots \\ & \vdots & \vdots & \vdots & \\ \cdots & V_{u-1,i-1}x_{u-1,0} & 0 & V_{u-1,i+1}x_{u-1,0} & \cdots \\ \cdots & V_{u,i-1}x_{um}^{1} & W_u(\boldsymbol{\lambda})\, x_{um}^{1} & V_{u,i+1}x_{um}^{1} & \cdots \\ \cdots & V_{u+1,i+1}x_{u+1,0} & 0 & V_{u+1,i+1}x_{u+1,0} & \cdots \\ & \vdots & \vdots & \vdots & \\ \cdots & V_{n,i-1}x_{n0} & 0 & V_{n,i+1}x_{n0} & \cdots \end{vmatrix}^{\otimes} =$$

$$= \sum_{u=1}^{n}\sum_{k=1}^{m-1}\sum_{g=1}^{d_k} \begin{vmatrix} V_{11}x_{10} & \cdots & 0 & \cdots & V_{1n}x_{10} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & U_u\left(\mathbf{a}_{g1}^{km}\right) x_{uk}^{g} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ V_{n1}x_{n0} & \cdots & 0 & \cdots & V_{nn}x_{n0} \end{vmatrix}^{\otimes} + \Delta_0 a_{1i}^{0m} z_0 \qquad \text{(B.3)}$$

by virtue of (4.40).

Next we consider the right most summation in (B.2). The relations (4.26) and (4.37) show that

$$(\lambda_i\Delta_0 - \Delta_i)\, s_{\mathbf{1h}}^{ml} x_{\mathbf{ulh}}^{\otimes} = \sum_{p=1}^{q}\sum_{k_1=1}^{l_p-1}\sum_{g_1=1}^{d_{k_1}}\sum_{k_2=1}^{l_p-k_1}\sum_{g_2=1}^{r_{k_2}} s_{\mathbf{1h}}^{ml} t_{h_p(g_1g_2)}^{l_p(k_1k_2)} \cdot$$

$$\cdot \begin{vmatrix} V_{11} & \cdots & V_{1,i-1} & 0 & V_{1,i+1} & \cdots & V_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{u_p-1,1} & \cdots & V_{u_p-1,i-1} & 0 & V_{u_p-1,i+1} & \cdots & V_{u_p-1,n} \\ 0 & & 0 & U_{u_p}\left(\mathbf{a}_{g_2}^{0k_2}\right) & 0 & \cdots & 0 \\ V_{u_p+1,1} & \cdots & V_{u_p+1,i-1} & 0 & V_{u_p+1,i+1} & \cdots & V_{u_p+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1} & \cdots & V_{n,i-1} & 0 & V_{n,i+1} & \cdots & V_{nn} \end{vmatrix} x_{\mathbf{u},\widehat{\mathbf{l}^p}\cup_p k_1,\widehat{\mathbf{h}^p}\cup_p g_1}^{\otimes} + \qquad \text{(B.4)}$$

$$+\sum_{p=1}^{q} s_{1\mathbf{h}}^{m\mathbf{l}} \begin{vmatrix} V_{11} & \cdots & V_{1,i-1} & 0 & V_{1,i+1} & \cdots & V_{1n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{u_p-1,1} & \cdots & V_{u_p-1,i-1} & 0 & V_{u_p-1,i+1} & \cdots & V_{u_p-1,n} \\ 0 & & 0 & U_{u_p}\left(\mathbf{a}_{h_p}^{0l_p}\right) & 0 & \cdots & 0 \\ V_{u_p+1,1} & \cdots & V_{u_p+1,i-1} & 0 & V_{u_p+1,i+1} & \cdots & V_{u_p+1,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1} & \cdots & V_{n,i-1} & 0 & V_{n,i+1} & \cdots & V_{nn} \end{vmatrix} x^{\otimes}_{\widehat{\mathbf{u}^p},\widehat{\mathbf{l}^p},\widehat{\mathbf{h}^p}}. \tag{B.5}$$

Here we use the symbols $\widehat{\mathbf{l}^p}$, $\widehat{\mathbf{h}^p}$ and $\widehat{\mathbf{u}^p}$ to denote omission of the component with the index p in multiindices $\mathbf{l}$, $\mathbf{h}$ and $\mathbf{u}$ respectively. The symbols $\widehat{\mathbf{l}^p} \cup_p k_1$ and $\widehat{\mathbf{h}^p} \cup_p g_1$ indicate the replacement of the component l_p in $\mathbf{l}$ with k_1 and the replacement of the component h_p in $\mathbf{h}$ with g_1 respectively. For $\mathbf{l} \in \Phi_{m,q}$ we write $L = \sum_{i=1}^{q} l_i$ and $L_p = \sum_{i=1, i\neq p}^{q} l_i$. The symbol L_p is well defined also for $\widehat{\mathbf{l}^p}$.

For $q = 2$ the sum of terms (B.5) over $\mathbf{u} \in \Omega_2$, $\mathbf{l} \in \Phi_{m,2}$ and $\mathbf{h} \in \chi_1$ equals

$$\sum_{\mathbf{u}\in\Omega_2} \sum_{\mathbf{l}\in\Phi_{m,2}} \sum_{\mathbf{h}\in\chi_1} \sum_{p=1}^{2} t_{1(h_1h_2)}^{m(l_1l_2)} \begin{vmatrix} V_{11} & \cdots & 0 & \cdots & V_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & & U_{u_p}\left(\mathbf{a}_{h_p}^{0l_p}\right) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ V_{n1} & \cdots & 0 & \cdots & V_{nn} \end{vmatrix} x^{\otimes}_{\widehat{\mathbf{u}^p},\widehat{\mathbf{l}^p},\widehat{\mathbf{h}^p}}.$$

In the following calculation we use the relation (4.37) and the definitions of the sets of multiindices. Note that $\sum_{\mathbf{l}\in\Phi_{m,2}} = \sum_{l_1=1}^{m-1}\sum_{l_2=1}^{m-l_1} = \sum_{l_2=1}^{m-1}\sum_{l_1=1}^{m-l_2}$. Then the above expression is equal to

$$\sum_{\mathbf{u}\in\Omega_2} \sum_{l_1=1}^{m-1} \sum_{h_1=1}^{d_{l_1}} \begin{vmatrix} V_{11} & \cdots & 0 & \cdots & V_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & U_{u_2}\left(\mathbf{a}_{h_1 1}^{l_1 m}\right) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ V_{n1} & \cdots & 0 & \cdots & V_{nn} \end{vmatrix} x^{\otimes}_{u_1,l_1h_1} +$$

$$+\sum_{\mathbf{u}\in\Omega_2}\sum_{l_2=1}^{m-1}\sum_{h_2=1}^{d_{l_2}}\begin{vmatrix} V_{11} & \cdots & 0 & \cdots & V_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & U_{u_1}\left(\mathrm{a}_{h_2 1}^{l_2 m}\right) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ V_{n1} & \cdots & 0 & \cdots & V_{nn} \end{vmatrix} x^{\otimes}_{u_2,l_2h_2} =$$

$$= \sum_{u_1=1}^{n}\sum_{u_2=1,u_2\neq u_1}^{n}\sum_{l=1}^{m-1}\sum_{h=1}^{d_l}\begin{vmatrix} V_{11} & \cdots & 0 & \cdots & V_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & U_{u_2}\left(\mathrm{a}_{h1}^{lm}\right) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ V_{n1} & \cdots & 0 & \cdots & V_{nn} \end{vmatrix} x^{\otimes}_{u_1,l,h}\,.$$

Now we add the expressions (B.3) and the above one. The sum is

$$\sum_{u=1}^{n}\sum_{l=1}^{m-1}\sum_{h=1}^{d_l}\begin{vmatrix} V_{11} & \cdots & V_{1,i-1} & U_1\left(\mathrm{a}_{h1}^{lm}\right) & V_{1,i+1} & \cdots & V_{1n} \\ V_{12} & \cdots & V_{2,i-1} & U_2\left(\mathrm{a}_{h1}^{lm}\right) & V_{2,i+1} & \cdots & V_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1} & \cdots & V_{n,i-1} & U_n\left(\mathrm{a}_{h1}^{lm}\right) & V_{n,i+1} & \cdots & V_{nn} \end{vmatrix} x^{\otimes}_{ulh} + \Delta_0 a_{1i}^{0m} z_0 =$$

$$= \sum_{l=1}^{m-1}\sum_{h=1}^{d_l}\Delta_0 a_{h1i}^{lm} x^{\otimes}_{lh} + \Delta_0 a_{1i}^{0m} z_0. \tag{B.6}$$

From the definition of the sets of multiindices it follows that

$$\sum_{\mathbf{l}\in\Phi_{m,q}}\sum_{\mathbf{h}\in\chi_{\mathbf{l}}} = \sum_{\widehat{\mathbf{l}^p}\in\Phi_{m-1,q-1}}\sum_{\widehat{\mathbf{h}^p}\in\chi_{\widehat{\mathbf{l}^p}}}\sum_{l_p=1}^{m-L_p}\sum_{h_p=1}^{d_{l_p}}. \tag{B.7}$$

In the summation $\sum_{\mathbf{l}\in\Phi_{m-1,q}}\sum_{k=L}^{m-1}$ a value $k=k'$ appears exactly for those $\mathbf{l}\in\Phi_{m-1,q}$ that have $L\le k'$. Thus

$$\sum_{\mathbf{l}\in\Phi_{m-1,q}}\sum_{k=L}^{m-1} = \sum_{k=q}^{m-1}\sum_{\mathbf{l}\in\Phi_{k,q}}. \tag{B.8}$$

Applying relation (B.7), Corollary 1.24, (i) and relation (B.8), respectively, it follows that

$$\sum_{\mathbf{l}\in\Phi_{m,q}}\sum_{\mathbf{h}\in\chi_{\mathbf{l}}}\sum_{k_1=1}^{l_p-1}\sum_{g_1=1}^{d_{k_1}}\sum_{k_2=1}^{l_p-k_1}\sum_{g_2=1}^{r_{k_2}} s_{\mathbf{l}\mathbf{h}}^{m\mathbf{l}}\, t_{h_p(g_1g_2)}^{l_p(k_1k_2)} = \tag{B.9}$$

$$= \sum_{\widehat{\mathbf{l}^p}\in\Phi_{m-1,q-1}} \sum_{\widehat{\mathbf{h}^p}\in\chi_{\widehat{\mathbf{l}^p}}} \sum_{k_1=1}^{m-L_p-1} \sum_{g_1=1}^{d_{k_1}} \sum_{k_2=1}^{m-L_p-k_1} \sum_{g_2=1}^{r_{k_2}} \sum_{l_p=k_1+k_2}^{m-L_p} \sum_{h_p=1}^{d_{l_p}} s_{1\mathbf{h}}^{m\mathbf{l}} t_{h_p(g_1g_2)}^{l_p(k_1k_2)} =$$

$$= \sum_{\mathbf{l}\in\Phi_{m-1,q}} \sum_{\mathbf{h}\in\chi_{\mathbf{l}}} \sum_{k_2=1}^{m-L} \sum_{g_2=1}^{r_{k_2}} \sum_{k_3=L}^{m-k_2} \sum_{g_3=1}^{d_{k_3}} s_{g_3\mathbf{h}}^{k_3\mathbf{l}} t_{1(g_2g_3)}^{m(k_2k_3)} = \tag{B.10}$$

$$= \sum_{k_3=p}^{m-1} \sum_{g_3=1}^{d_{k_3}} \sum_{\mathbf{l}\in\Phi_{k_3,q}} \sum_{\mathbf{h}\in\chi_{\mathbf{l}}} \sum_{k_2=1}^{m-k_3} \sum_{g_2=1}^{r_{k_2}} s_{g_3\mathbf{h}}^{k_3\mathbf{l}} t_{1(g_2g_3)}^{m(k_2k_3)}. \tag{B.11}$$

In the step (B.10) of the above calculation we wrote l_p instead of k_1 and h_p instead of g_1. Next we sum the terms (B.4) over $\mathbf{l}\in\Phi_{m,q}$ and $\mathbf{h}\in\chi_{\mathbf{l}}$. From the result of the previous calculation it follows that the sum is equal to

$$\sum_{k_3=p}^{m-1} \sum_{g_3=1}^{d_{k_3}} \sum_{\mathbf{l}\in\Phi_{k_3,q}} \sum_{\mathbf{h}\in\chi_{\mathbf{l}}} \sum_{k_2=1}^{m-k_3} \sum_{g_2=1}^{r_{k_2}} s_{g_3\mathbf{h}}^{k_3\mathbf{l}} t_{1(g_2g_3)}^{m(k_2k_3)} \begin{vmatrix} V_{11} & \cdots & 0 & \cdots & V_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & & U_{u_p}\left(\mathbf{a}_{g_2}^{0k_2}\right) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ V_{n1} & \cdots & 0 & \cdots & V_{nn} \end{vmatrix} x_{\mathbf{u},\mathbf{l},\mathbf{h}}^{\otimes}.$$

Here we write l_p instead of k_1 and h_p instead of g_1 as in (B.10). Using the relation (4.37) it follows that the above sum is equal to

$$\sum_{k_3=p}^{m-1} \sum_{g_3=1}^{d_{k_3}} \sum_{\mathbf{l}\in\Phi_{k_3,q}} \sum_{\mathbf{h}\in\chi_{\mathbf{l}}} s_{g_3\mathbf{h}}^{k_3\mathbf{l}} \begin{vmatrix} V_{11} & \cdots & 0 & \cdots & V_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & & U_{u_p}\left(\mathbf{a}_{g_3 1}^{k_3 m}\right) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ V_{n1} & \cdots & 0 & \cdots & V_{nn} \end{vmatrix} x_{\mathbf{u},\mathbf{l},\mathbf{h}}^{\otimes}. \tag{B.12}$$

Similarly to when we showed that (B.9) equals (B.11) we now show, using Corollary 1.24 and relation (B.8), that for any index $p\in\underline{q}$ ($q\geq 3$),

$$\sum_{\mathbf{l}\in\Phi_q} \sum_{\mathbf{h}\in\chi_{\mathbf{l}}} s_{1\mathbf{h}}^{m\mathbf{l}} = \sum_{\mathbf{l}\in\Phi_{m,q}} \sum_{\mathbf{h}\in\chi_{\mathbf{l}}} \sum_{k_3=L_p}^{m-l_p} \sum_{g_3=1}^{d_{k_3}} s_{g_3\widehat{\mathbf{h}^p}}^{k_3\widehat{\mathbf{l}^p}} t_{1(g_3h_p)}^{m(k_3l_p)} =$$

$$= \sum_{\widehat{\mathbf{l}^p}\in\Phi_{m-1,q-1}} \sum_{\widehat{\mathbf{h}^p}\in\chi_{\widehat{\mathbf{l}^p}}} \sum_{k_3=L_p}^{m-1} \sum_{g_3=1}^{d_{k_3}} \sum_{l_p=1}^{m-k_3} \sum_{h_p=1}^{r_{l_p}} s_{g_3\widehat{\mathbf{h}^p}}^{k_3\widehat{\mathbf{l}^p}} t_{1(g_3h_p)}^{m(k_3l_p)} =$$

$$= \sum_{k_3=q}^{m-1} \sum_{g_3=1}^{d_{k_3}} \sum_{\widehat{\mathbf{l}^p} \in \Phi_{k_3,q-1}} \sum_{\widehat{\mathbf{h}^p} \in \chi_{\widehat{\mathbf{l}^p}}} s_{g_3 \widehat{\mathbf{h}^p}}^{k_3 \widehat{\mathbf{l}^p}} \sum_{l_p=1}^{k_3} \sum_{h_p=1}^{r_{l_p}} t_{1(g_3 h_p)}^{m(k_3 l_p)}.$$

We use this equality to show that the sum over $\mathbf{l} \in \Phi_{m,q}$ and $\mathbf{h} \in \chi_{\mathbf{l}}$ for $q \geq 3$ of the terms (B.5) is equal to

$$\sum_{k_3=q}^{m-1} \sum_{g_3=1}^{d_{k_3}} \sum_{\widehat{\mathbf{l}^p} \in \Phi_{k_3,q-1}} \sum_{\widehat{\mathbf{h}^p} \in \chi_{\widehat{\mathbf{l}^p}}} s_{g_3 \widehat{\mathbf{h}^p}}^{k_3 \widehat{\mathbf{l}^p}} \sum_{l_p=1}^{k_3} \sum_{h_p=1}^{r_{l_p}} t_{1(g_3 h_p)}^{m(k_3 l_p)}.$$

$$\cdot \begin{vmatrix} V_{11} & \cdots & 0 & \cdots & V_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & & U_{u_p}\left(\mathrm{a}_{h_p}^{0 l_p}\right) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ V_{n1} & \cdots & 0 & \cdots & V_{nn} \end{vmatrix} x_{\widehat{\mathbf{u}^p},\widehat{\mathbf{l}^p},\widehat{\mathbf{h}^p}}^{\otimes} =$$

$$= \sum_{k_3=q}^{m-1} \sum_{g_3=1}^{d_{k_3}} \sum_{\mathbf{l} \in \Phi_{k_3,q-1}} \sum_{\mathbf{h} \in \chi_{\mathbf{l}}} s_{g_3 \mathbf{h}}^{k_3 \mathbf{l}} \begin{vmatrix} V_{11} & \cdots & 0 & \cdots & V_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & & U_{u_p}\left(\mathrm{a}_{g_3 1}^{k_3 m}\right) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ V_{n1} & \cdots & 0 & \cdots & V_{nn} \end{vmatrix} x_{\widehat{\mathbf{u}^p},\mathbf{l},\mathbf{h}}^{\otimes}. \tag{B.13}$$

Next the sum of the expressions (B.13) over $q = 3, 4, \ldots, \min\{m, n\}$ and $\mathbf{u} \in \Omega_q$ is

$$\sum_{q=3}^{\min\{m,n\}} \sum_{\mathbf{u} \in \Omega_q} \sum_{k_3=q}^{m-1} \sum_{g_3=1}^{d_{k_3}} \sum_{\mathbf{l} \in \Phi_{k_3,q-1}} \sum_{\mathbf{h} \in \chi_{\mathbf{l}}} s_{g_3 \mathbf{h}}^{k_3 \mathbf{l}}.$$

$$\cdot \begin{vmatrix} V_{11} & \cdots & 0 & \cdots & V_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & U_{u_p}\left(\mathrm{a}_{g_3 1}^{k_3 m}\right) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ V_{n1} & \cdots & 0 & \cdots & V_{nn} \end{vmatrix} x_{\widehat{\mathbf{u}^p},\mathbf{l},\mathbf{h}}^{\otimes} =$$

$$= \sum_{q=2}^{\min\{m,n\}-1} \sum_{\mathbf{u} \in \Omega_q} \sum_{v=1, v \notin \mathbf{u}}^{n} \sum_{k_3=q}^{m-1} \sum_{g_3=1}^{d_{k_3}} \sum_{\mathbf{l} \in \Phi_{k_3,q-1}} \sum_{\mathbf{h} \in \chi_{\mathbf{l}}} s_{g_3 \mathbf{h}}^{k_3 \mathbf{l}}.$$

$$\cdot \begin{vmatrix} V_{11} & \cdots & 0 & \cdots & V_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & U_v\left(\mathbf{a}_{g_3 1}^{k_3 m}\right) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ V_{n1} & \cdots & 0 & \cdots & V_{nn} \end{vmatrix} x^{\otimes}_{\mathbf{ulh}} \cdot \tag{B.14}$$

If we reverse the order of summation over q and k_3 we have

$$\sum_{q=2}^{\min\{m,n\}} \sum_{k_3=q}^{m-1} = \sum_{k_3=2}^{m-1} \sum_{q=2}^{\min\{k_3,n\}} \quad \text{and} \quad \sum_{q=2}^{\min\{m,n\}-1} \sum_{k_3=q}^{m-1} = \sum_{k_3=2}^{m-1} \sum_{q=2}^{\min\{k_3,n-1\}} .$$

From (B.12), (B.14) and the last relation it follows that the sum of terms (B.4) for $q \geq 2$ and terms (B.5) for $q \geq 3$ over $\mathbf{u} \in \Omega_q$, $\mathbf{l} \in \Phi_{q,m}$ and $\mathbf{h} \in \chi_{\mathbf{l}}$ is

$$\sum_{k_3=2}^{m-1} \sum_{g_3=1}^{d_{k_3}} \sum_{q=2}^{\min\{k_3,n\}} \sum_{\mathbf{u}\in\Omega_q} \sum_{p=1}^{q} \sum_{\mathbf{l}\in\Phi_{k_3,q}} \sum_{\mathbf{h}\in\chi_{\mathbf{l}}} s_{g_3\mathbf{h}}^{k_3\mathbf{l}} \begin{vmatrix} V_{11} & \cdots & 0 & \cdots & V_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & U_{u_p}\left(\mathbf{a}_{g_3 1}^{k_3 m}\right) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ V_{n1} & \cdots & 0 & \cdots & V_{nn} \end{vmatrix} x^{\otimes}_{\mathbf{u},\mathbf{l},\mathbf{h}} +$$

$$+ \sum_{k_3=2}^{m-1} \sum_{g_3=1}^{d_{k_3}} \sum_{q=2}^{\min\{k_3,n-1\}} \sum_{\mathbf{u}\in\Omega_q} \sum_{v=1, v\notin\mathbf{u}}^{n} \sum_{\mathbf{l}\in\Phi_{k_3,q}} \sum_{\mathbf{h}\in\chi_{\mathbf{l}}} s_{g_3\mathbf{h}}^{k_3\mathbf{l}} \cdot$$

$$\cdot \begin{vmatrix} V_{11} & \cdots & 0 & \cdots & V_{1n} \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & U_v\left(\mathbf{a}_{g_3 1}^{k_3 m}\right) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ V_{n1} & \cdots & 0 & \cdots & V_{nn} \end{vmatrix} x^{\otimes}_{\mathbf{ulh}} =$$

$$= \sum_{k_3=2}^{m-1} \sum_{g_3=1}^{d_{k_3}} \sum_{q=2}^{\min\{k_3,n\}} \sum_{\mathbf{u}\in\Omega_q} \sum_{\mathbf{l}\in\Phi_{k_3,q}} \sum_{\mathbf{h}\in\chi_{\mathbf{l}}} s_{g_3\mathbf{h}}^{k_3\mathbf{l}} \cdot$$

$$\cdot \begin{vmatrix} V_{11} & \cdots & V_{1,i-1} & U_1\left(\mathbf{a}_{g_3 1}^{k_3 m}\right) & V_{1,i+1} & \cdots & V_{1n} \\ V_{12} & \cdots & V_{2,i-1} & U_2\left(\mathbf{a}_{g_3 1}^{k_3 m}\right) & V_{2,i+1} & \cdots & V_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ V_{n1} & \cdots & V_{n,i-1} & U_n\left(\mathbf{a}_{g_3 1}^{k_3 m}\right) & V_{n,i+1} & \cdots & V_{nn} \end{vmatrix} x^{\otimes}_{\mathbf{ulh}} =$$

$$= \sum_{k_3=2}^{m-1} \sum_{g_3=1}^{d_{k_3}} \sum_{q=2}^{\min\{k_3,n\}} \sum_{\mathrm{l}\in\Phi_{k_3,q}} \sum_{\mathrm{h}\in\chi_{\mathrm{l}}} s_{g_3\mathrm{h}}^{k_3\mathrm{l}} \Delta_0 a_{g_3 1 i}^{k_3 m} x_{\mathrm{lh}}^{\otimes}. \tag{B.15}$$

Then it follows from the equality (B.2) and the calculations that followed it that $(\lambda_i\Delta_0 - \Delta_i)\, z_m^1$ is equal to the sum of the expressions (B.6) and (B.15), and thus

$$(\lambda_i\Delta_0 - \Delta_i)\, z_m^1 =$$

$$= \sum_{k=1}^{m-1} \sum_{g=1}^{d_k} \Delta_0 a_{g1i}^{km} x_{kg}^{\otimes} + \Delta_0 a_{1i}^{0m} z_0 + \sum_{k=2}^{m-1} \sum_{g=1}^{d_k} \sum_{q=2}^{\min\{k,n\}} \sum_{\mathrm{l}\in\Phi_{k,q}} \sum_{\mathrm{h}\in\chi_{\mathrm{l}}} s_{g\mathrm{h}}^{k\mathrm{l}} \Delta_0 a_{g1i}^{km} x_{\mathrm{lh}}^{\otimes} =$$

$$= \sum_{k=1}^{m-1} \sum_{g=1}^{d_l} \Delta_0 a_{g1i}^{km} \left(x_{kg}^{\otimes} + \sum_{q=2}^{\min\{k,n\}} \sum_{\mathrm{l}\in\Phi_{k,q}} \sum_{\mathrm{h}\in\chi_{\mathrm{l}}} s_{g\mathrm{h}}^{k\mathrm{l}} x_{\mathrm{lh}}^{\otimes} \right) + \Delta_0 a_{1i}^{0m} z_0 =$$

$$= \sum_{k=1}^{m-1} \sum_{g=1}^{d_l} \Delta_0 a_{g1i}^{km} z_k^g + \Delta_0 a_{1i}^{0m} z_0.$$

□

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