Introduction

A main difficulty in computing a complete list of all irreducible graphs that are not embeddable into the projective plane P is the fact that there are so many of these graphs - in [1] there are 103 different ones listed - and that in a straight forward approach to exhaustively list all these graphs, the same graph gets constructed many times in different ways.

Four techniques turned out to be useful to overcome these difficulties:

Firstly, we consider minimal graphs, that are not embeddable into P. We define a graph G to be minimal non embeddable into P, if G is not embeddable into P, but each proper minor of G is. This definition differs from the standard definition for irreducible graphs by substituting "minor" for "subgraph". It is rather obvious that each minimal graph is also irreducible. But the converse is not true. However, each irreducible graph can be obtained from some minimal graph by several vertex splits and edge deletions. Since each of these operations reduces the valency of the vertices, it follows that from each minimal graph only finitely many irreducible graphs can be obtained. Given a complete list of minimal graphs, it is a finite but tedious task, best done computer assisted, to construct a complete list of all irreducible graphs. Clearly, the set of all minimal graphs is a subset of the set of all irreducible graphs and constructing this subset I turns out to be manageable.

Secondly, we used - whenever possible - symmetries of the graphs to be constructed in order to avoid multiple constructions of the same graph.

Thirdly, some graphs that occur frequently in an exhaustive enumeration of all constructions that could lead to minimal graphs are proved to be forbidden subgraphs for minimal graphs. This serves as a powerful guide to avoid early on constructions that would eventually lead to graphs that are not minimal.

Finally, we partition the set I of all minimal graphs that are not embeddable into P into three disjoint subsets, according to properties that simplify the enumeration of each of these subsets. I_1, is the subset of all graphs in I that are not 3-connected. $I_{3,4}$ is the subset of all graphs in I that are not in I_1, and contain the complete bipartite graph $K_{3,4}$ as a minor.

I_2 is the complement of $I_2 \cup I_{3,4}$ in I. I_1 is relatively easy to enumerate and we will not address this problem here. I_2 has to be computed in [2], [3], [4]. In this paper we will prove that $I_{3,4}$ is contained in the following set of graphs: $\{A_2, B_1, B_7, C_3, C_4, C_7, D_2, D_3, D_9, D_{12}, E_2, E_3, E_5, E_{11}, E_{18}, E_{19}, E_{27}, G\}$, where the notation is the same

as in [1].

The emphasis of this paper is on proving the completeness of this list and not on checking if all the graphs listed are minimal. It is a finite but tedious task, best done computer assisted, to eliminate from this list all graphs that are not minimal.

Definition:

An embedding of a graph G into a 2-dimensional mannifold M is a topological mapping of G, considered as 1-dimensional mannifold, into M.

Definition:

Two embeddings ϕ_1 and ϕ_2 of G into M are equivalent if there exists a homeomorphism h from M to M such that ϕ_2 is the composition of ϕ_1 and h.

Remark:

It is well known that a class of equivalent embeddings of G into M is uniquely described by listing all those circuits in G that are boundaries of faces of M.

Definition:

A graph G is minimal non embeddable into the projective plane P if G cannot be embedded into P but each proper minor of G is embeddable into P.

Clearly, a minimal graph that is not embeddable into P is strict and finite.

Notation:

The main vertices of a graph G are the vertices of valency ≥ 3. The generalized links of G are those arcs of G whose inner vertices are of valency 2 and whose endvertices are main vertices. Clearly, G is homeomorphic to a graph G' whose vertices are the main vertices of G and whose edges connect two vertices iff these vertices are connected by a generalized link in G.

Let H be a subgraph of a graph G and a and b two vertices of H. We say that a and b are bridged by a bridge B of H if a or b are vertices of attachment of B or if a or b are vertices on two generalized links l_1 and l_2 of H with the property that an inner vertex of l_1 and an inner vertex of l_2 are vertices of attachment of B.

Clearly, a and b are bridged iff H can be contracted to a graph $\overline{H}$ homeomorphic to H in which the two vertices corresponding to a and b are vertices of attachment of B.

Definition:

$I_{3,4}$ is the set of all 3-connected minimal graphs that are not embeddable into P and contain the complete bipartite graph $K_{3,4}$ as a minor.

We will prove the

Main Theorem

$I_{3,4}$ is contained in the following list of graphs: $\{A_2, B_1, B_7, C_3, C_4, C_7, D_2, D_3, D_9, D_{12}, E_2, E_3, E_5, E_{11}, E_{18}, E_{19}, E_{27}, G\}$, where the notation is the same as in [1].

Remark:

In this paper we do not want to prove that each graph in that list is minimal, which is a finite but tedious task, best done computer assisted. We will prove that this list is complete, however.

Proof:

The Main Theorem is an immediate consequence of theorem 0-3, stated and proved in the remainder of this paper. □

Before we state theorems 0 - 3, we introduce some more notations.

$K_{3,4}$ has three vertices of valency 4. Each of these can be split into two vertices of valency 3, giving rise to graphs that contain $K_{3,4}$ as a minor.

We partition $I_{3,4}$ into four disjoint sets $I^i_{3,4}$, $0 \le i \le 3$. A graph I belongs to $I^i_{3,4}$ iff I contains a minor that can be obtained from $K_{3,4}$ by i vertex splits, and I contains no minor that can be obtained from I by $i + 1$ vertex splits.

Now we can state the four theorems:

<u>Theorem 0:</u> $I^0_{3,4} = \{A_2, B_1, E_3, E_{18}\}$.

<u>Theorem 1:</u> $I^1_{3,4} \subseteq \{B_7, C_3, C_7, D_3, D_9, D_{12}, E_5, E_{11}, E_{27}\}$.

Theorem 2: $I^2_{3,4} \subseteq \{C_4, D_2, E_2, E_{19}\}$.

Theorem 3: $I^3_{3,4} = \{G\}$.

Theorems 0 and 1 will be proved in the first part of this paper, Theorems 2 and 3 in the second part. Before proving these theorems, we state a few simple conbinatorial facts without proof:

$K_{3,4}$ has exactly six inequivalent embeddings into P, given by Figure 1:

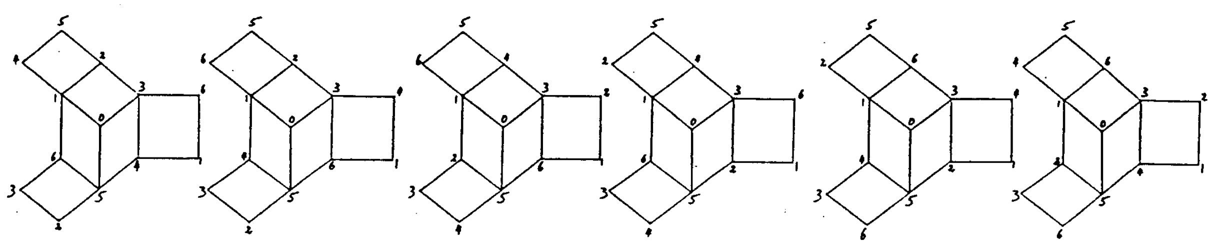


Fig. 1

All graphs that are obtained from $K_{3,4}$ by one vertex split are isomorphic and denoted by $K^1_{3,4}$. $K^1_{3,4}$ has four inequivalent embeddings into P, given by Figure 2:

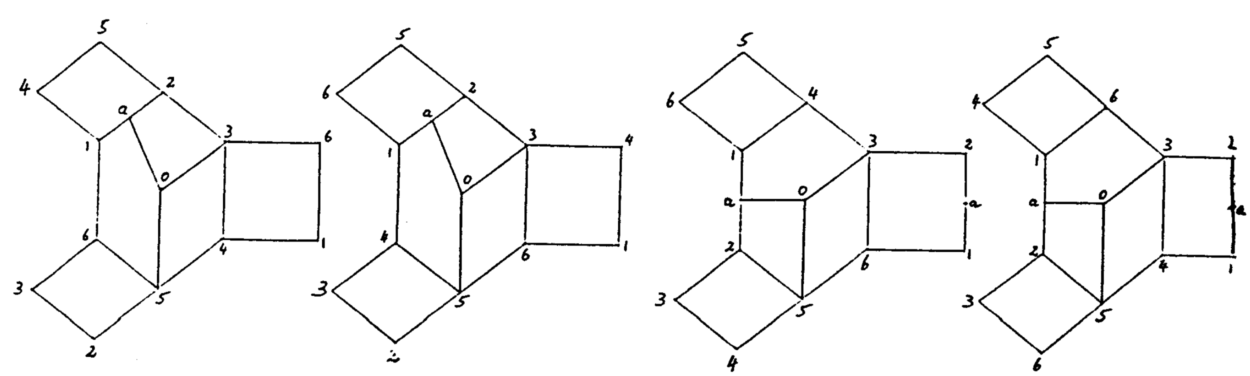


Fig. 2

In the sequel, we will always denote the vertices of $K_{3,4}$ and $K^1_{3,4}$ as in Figure 1 and Figure 2. It is easy to see that with this notation the following four permutations describe automorphisms of $K^1_{3,4}$:

$$(0, 2);\ (4, 6);\ (3, 5);\ (1, a)(0, 4)(2, 6).$$

The automorphism group generated by these four automorphisms partitions the main vertices and generalized links of K into three equivalence classes each:

$\{0, 2, 4, 6\}$	$\{(0, a), (2, a), (1, 4), (1, 6)\}$
$\{a, 1\}$	$\{(0, 3), (0, 5), (2, 3), (2, 5), (3, 4), (5, 4), (3, 6), (5, 6)\}$
$\{3, 5\}$	$\{(1, a)\}$

Now we come to the proof of Theorem 0:

Let I be a graph in $I^0_{3,4}$. Then I contains a subgraph K homeomorphic to $K_{3,4}$. We number the main vertices of K with the numbers 0, 1, 2, 3, 4, 5, 6 in such a way that each generalized link of K connects an even numbered to an odd numbered vertex of K, see Figure 1.

Since $K_{3,4}$ can be embedded into P, K is a proper subgraph of I. We consider the bridges of K. If all the bridges of K bridge only pairs of main vertices of K, then all these bridges are degenerate, since I is 3-connected, and I is a subgraph of the complete graph on seven vertices, K_7. It is a finite but tedious task process to show that in that case $I = A_2$ or $I = B_1$.

If one bridge of K bridges the three odd-numbered vertices of K, then I obviously contains E_3 as minor and therefore $I = E_3$. If one bridge of K bridges three even-numbered vertices of K, then I obviously contains E_{18} as minor and therefore $I = E_{18}$.

No bridge of K bridges two even-numbered and one odd-numbered vertex of K, because otherwise I would contain $K^1_{3,4}$ as minor, contrary to the assumption that $I \in I^0_{3,4}$.

To show that no bridge of K bridges two odd-numbered and one even-numbered vertex of K, is somewhat more involved: Assume the contrary and let B be a bridge of K that bridges w.l.o.g. the three vertices 0, 1 and 3. Then I contains a subgraph G homeomorphic to G_0, whose six inequivalent embeddings are given by Figure 3:

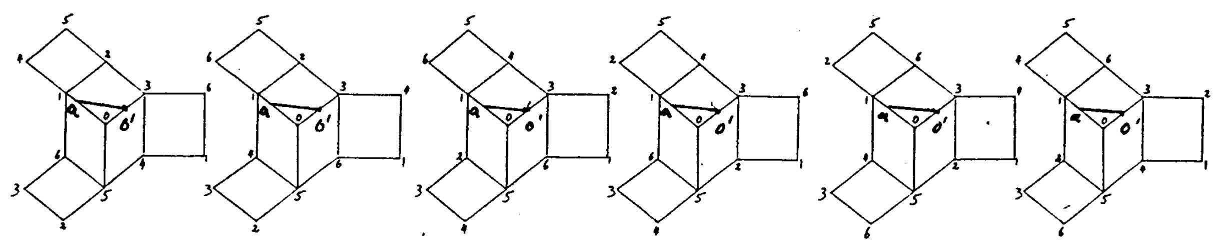


Fig. 3

Without loss of generality we can assume that $0'$ is an inner vertex of the generalized link $(0, 3)$ of K. Furthermore, we can choose $0'$ to be the closest vertex to 0 on $(0, 3)$ such that no inner vertex of the generalized link $(0, 0')$ is bridged to 3 by a bridge of G. (For if there is an inner vertex $0''$ of $(0, 0')$, bridged to an inner vertex b on $(0', 3)$ we redefine G such that $0''$ becomes $0'$, a generalized link connecting $0''$ and b outside of the old G becomes part of G and the link $(0'', 0', b)$ in G becomes part of B.) Similarly, we define the vertex a to be the closest vertex a on $(0, 1)$ at which B is attached to $(1, 0)$. (a may be equal to 1). I therefore contains a subgraph G homeomorphic to G_0 with the property that no inner vertex of (0′, 0) is bridged to an inner vertex of $(0, a)$ or to a and no inner vertex of $(0, a)$ is bridged to an inner vertex of (0, 0′) or to 0.

Obviously, each embedding of $G - (0, a)$ into P can be extended to an embedding of G into P and each embedding of $G - (0', 0)$ into P can be extended to an embedding of G into P. In order for the links (0′, 0) and $(a, 0)$ not to be redundant in I, an inner vertex of (0′, 0) has to be bridged to some main vertex x on G or a vertex of $(a, 0)$ has to be bridged to a main vertex y of G. Since 0 must not be bridged to an even numbered vertex of G (otherwise I would contain $K^1_{3,4}$ as a minor) x can only be 1 if $a \neq 1$ or $x = 5$. Similarly, y can only be 3 or 5. Each of the resulting cases leads to the conclusion that I contains a proper subgraph containing E_3 as minor, contradicting the minimality of I. □

The proof of Theorem 1 follows directly from Lemmas 1-6:

Let I be a graph in $I^1_{3,4}$. Then I contains a subgraph K homeomorphic to $K^1_{3,4}$.

Since $K^1_{3,4}$ is embeddable into P, K is a proper subgraph of I. We now consider the bridges of K. Lemma 1 deals with the possibility that all bridges of K are degenerate and only attached to main vertices of K. Lemma 2 shows that if one bridge is not degenerate, then K can be suitably chosen so that a bridge is attached to an inner vertex of a generalized link of K. As stated earlier, there are only three inequivalent generalized links of K, represented by $(0, a)$, (0, 3), $(1, a)$. Lemma 4 shows that no bridge can be attached to $(0, a)$. Lemma 5 lists all graphs resulting from a bridge attached to an inner vertex of (0, 3) and Lemma 6 lists all graphs resulting from a bridge attached to an inner vertex of $(1, a)$. Lemmas 3 and 4 show that certain graphs are forbidden subgraphs of I and help to shorten the proof considerably. □

In the remainder of this paper, we state and prove Lemmas 1-6.

Lemma 1:

If I is a graph in $I^1_{3,4}$ and if for some subgraph of K of I that is homeomorphic to $K^1_{3,4}$, all the bridges of K are degenerate and are attached only to main vertices of K, then I is contained in the following list of three graphs:

$\{B_7, C_7, D_3\}$, given by the diagrams in Figure 4:

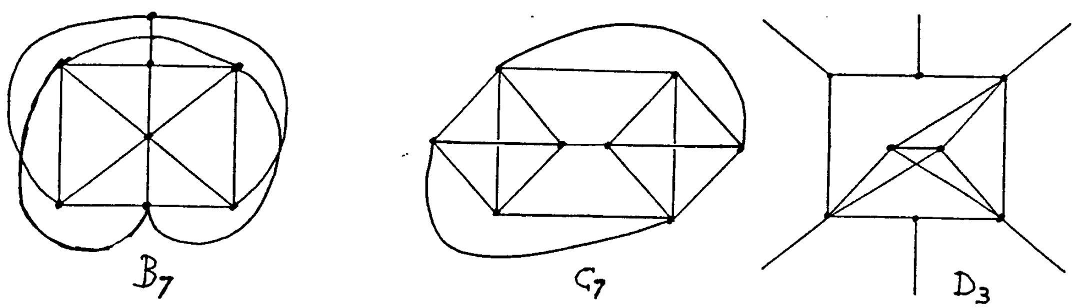


Fig. 4

Proof:

To prove this lemma, we first note that I is a subgraph of the complete graph on eight vertices, K_8. It is a finite but tedious task, best done computer assisted, to show that the only subgraphs of K_8 that belong to I are D_{17}, E_3, E_{18}, and the three graphs listed above. D_{17}, E_3 and E_{18} do not contain $K^1_{3,4}$ as minor. □

Lemma 2:

If I is a graph in $I_{3,4}^{1}$, and if for some subgraph K of I that is homeomorphic to $K_{3,4}^{1}$, a bridge of K is not degenerate, then I contains a subgraph K' homeomorphic to $K_{3,4}^{1}$ with the property that one bridge of K' is attached to an inner vertex of a generalized link of K'.

Proof:

Let B be a non-degenerate bridge of a subgraph K of I, that is homeomorphic to $K_{3,4}^{1}$. Since I is 3-connected, B bridges at least three of the main vertices of K. If any two of these, say a and b, are connected by a generalized link l in K, we obtain K' from K by replacing l by a cycle free arc in B that connects a and b, and clearly one bridge of K' is attached to an inner vertex of a generalized link of K'.

To finish the proof, we show that no bridge of K can bridge three main vertices of K with the property that no two of them are connected by a generalized link of K. Using the symmetries of K, we conclude that the following three triples form a complete set of representatives for the different choices of three such vertices: $\{0, 1, 2\}$, $\{0, 2, 4\}$ and $\{1, 3, 5\}$. If one of these were bridged by a bridge of k, I would obviously contain F_2, E_{18} or E_3 respectively as a proper minor, contradicting the minimality of I. □

Let H_0 be the graph whose two inequivalent embeddings into P are given in Figure 5:

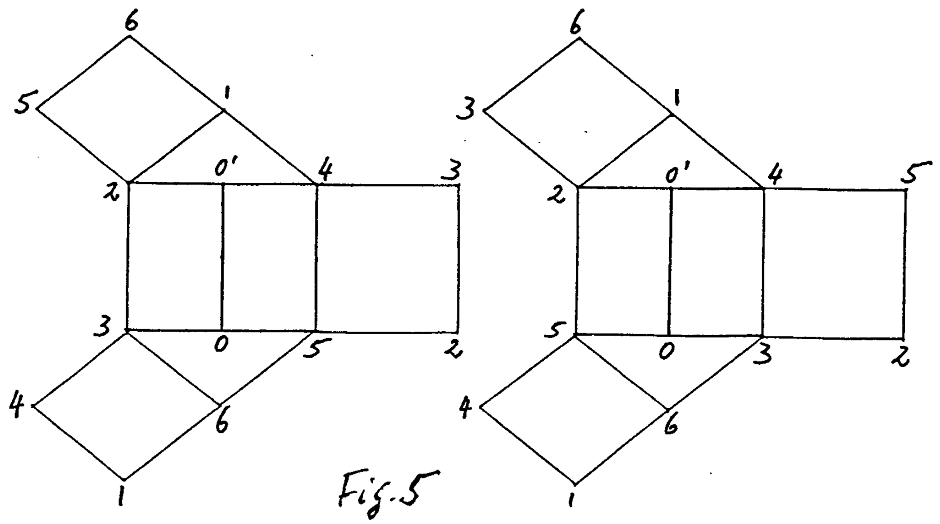


Fig.5

Let $H_{2,2'}$ be the graph obtained from H_0 by splitting the vertex 2 as shown in Figure 6:

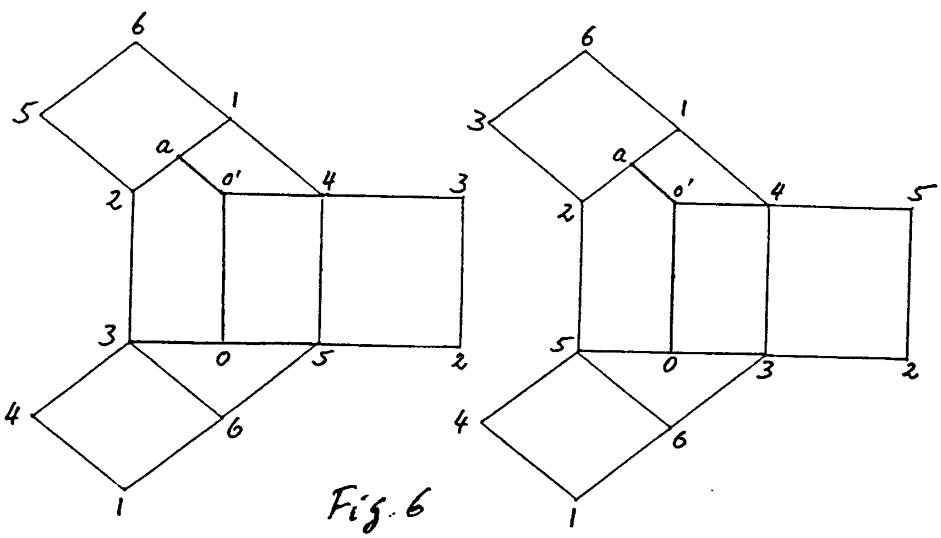


Fig.6

It is easy to check that H_0 and $H_{2,2'}$ have only the two inequivalent embeddings into P given above. It is obvious that the two embeddings of $H_{2,2'}$ are extensions of the two embeddings of H_0 respectively. Furthermore, $H_{2,2'}$ has the two automorphisms

(3, 5) and (0, 6)(0′, 1).

Lemma 3:

No graph in I contains $H_{2,2'}$ as a minor.

Proof:

Assume I is a graph in I and contains $H_{2,2'}$ as a minor. Let H be a minimal subgraph of I that contains $H_{2,2'}$ as a minor. Clearly, H can be obtained from $H_{2,2'}$ by splitting some of the vertices of valency 4 in $H_{2,2'}$ and by subdividing some of the links of the resulting graph by vertices of valency 2.

If two vertices of H are obtained by splitting the vertex i of $H_{2,2'}$ (which is possible for $i = 3, 4, 5$) we name these two vertices i and i', in such a way that i' is the vertex that is connected by a generalized link to 0 or 0′ respectively. The remaining main vertices of H we named in the same way as the corresponding vertices of $H_{2,2'}$.

Let a be that vertex on the generalized link (2, 2′) of H that is connected to the vertex 2 by a single edge e in I (a may be equal to 2′), and let $\overline{H}$ be the minor of H obtained by contracting the link e. Because of the minimality of I, $\overline{H}$ is embeddable into P. Since $\overline{H}$ contains H_0 as a minor, each embedding of $\overline{H}$ into P induces an embedding of H_0 into P. Since the two inequivalent embeddings of H_0 into P can be extended to embeddings of $H_{2,2'}$ into P, it follows that each of the (one or two) inequivalent embeddings of $\overline{H}$ into P can be extended to embeddings of H into P.

Now consider the graph $\overline{I}$, obtained from I by contracting the link e with endpoints 2 and a. Because of the minimality of I, there exists an embedding of $\overline{I}$ into P, inducing an embedding of $\bar{H}$ into P. This embedding can obviously be extended to an embedding of H into P. For that embedding not to be extendable to an embedding of I into P, H has to have bridges that interfere with such an extension. The bridges of $\bar{H}$ in $\bar{I}$ are the same as the bridges of H in I, except for the fact that the vertices of attachment 2 and a are identified. In any embedding of $\bar{I}$ into P, the bridges of $\bar{H}$ are embedded into faces whose boundaries are circuits in $\bar{H}$. The faces of an embedding of H into P that do not contain the vertex 2 in their boundaries are the same as those of the corresponding embedding of $\bar{H}$ into P. Therefore, for an embedding of H into P not to be extendable to an embedding of I into P, there has to exist a bridge B of H in I that in an embedding of $\bar{I}$ into P is embeddable into a face whose boundary contains the vertex 2. As Figure 6 shows, these boundaries are the following: (2, 1, 4, 0′, 2), (2, 3, 4, 5, 2), (2, 0′, 0, 3, 2), (2, 0′, 0, 5, 2), (2, 1, 6, 3), (2, 1, 6, 5). The last four are obviously equivalent under the automorphisms of $H_{2,2'}$, and we only consider the first one of them:

Let us then first assume that there exists a bridge B of H whose vertices of attachment are all on the circuit (2, a, 2′, 0′, 0, 3′, 3, 2). In an embedding of $\bar{I}$ into P, let $\bar{F}$ be the face into which B is embedded. Clearly, $\bar{F}$ has boundary (2, 0′, 0, 3, 2) or (2, 0′, 4, 1, 2) or (2, 3, 4, 5, 2) if B is only attached to vertices on the generalized link (2, 0′) or (2, 3) of H. This embedding of $\bar{I}$ into P induces an embedding of $\bar{H}$ into P, which in turn can be extended to an embedding of H into P. In this embedding, let F be the face with boundary (2, a, 2′, 0′,0, 3′, 3, 2). Clearly, this embedding can be extended to an embedding of $H \cup B$ into P, by embedding B into F. (If B could not be embedded into F, I would not be minimal since the generalized link (0′, 4) of H would be redundant.) For B not to be redundant, there has to exist another bridge B' of H whose embedding into F interferes with an embedding of B into F, but if B is not only attached to vertices on the link $(2, 0')$ or $(2, 3)$ of H, B and B' can both be embedded into the face $\bar{F}$ with boundary $(2, 0', 0, 3, 2)$. Therefore, one of these two bridges, say B, has to be attached to a vertex b on the generalized link $(2, 3)$ of H, $b \neq 3$ and B' has to be attached to an inner vertex c of the generalized link $(2, a, 2', 0')$ of H. Furthermore, there have to exist two vertices x and y on the boundary of F occurring with b and c in the cyclical order b, c, x, y, such that B bridges b and x and B' bridges c and y.

We can assume w.l.o.g. that B and B' are degenerate and that B does not only bridge 2 and 3 or 2, a and $2'$ and that B' does not only bridge $2'$ and $0'$. (Some simple modification of the graph H into another minimal graph H' of I that contains $H_{2,2'}$ as a minor may be necessary.)

Therefore, we now have to consider the following 5 cases:

Case 1:

$x = 0', y = 0'$:

I contains the minor M which is given by Figure 7:

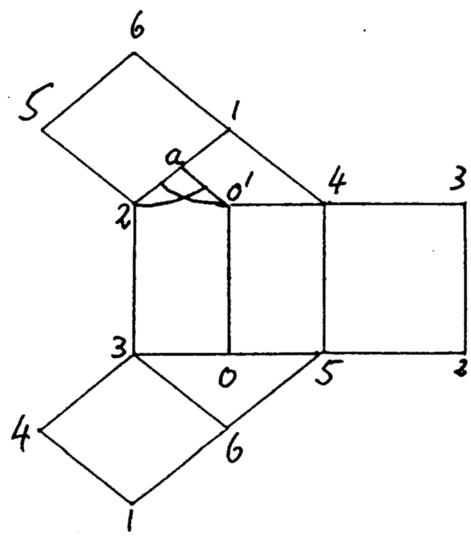


Fig. 7

Clearly, M is not embeddable into P and the links (0, 3), (0, 5), (0, 0′) in M are redundant. This contradicts the minimality of I.

Case 2:

$x = 0', y = 0$:

I contains the minor M given by Figure 8.

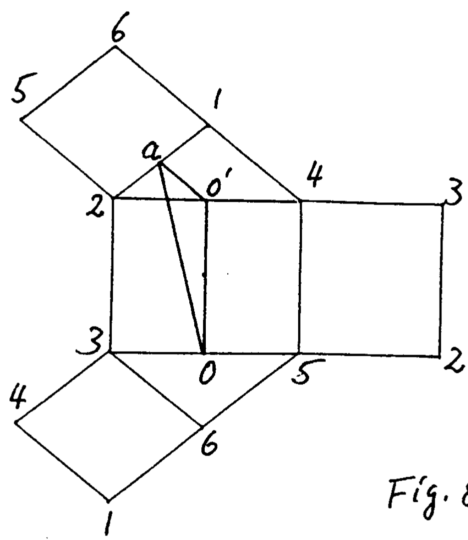


Fig. 8

Clearly, M is not embeddable into P and the link $(0', a)$ is redundant. This contradicts the minimality of I.

Case 3:

$x = 0', y = 3$ or $3'$:

I contains the minor M_0, whose only embedding into P is given by Figure 9:

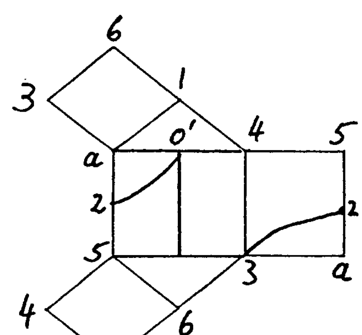


Fig. 9

or the minor M_1, given by Figure 10.

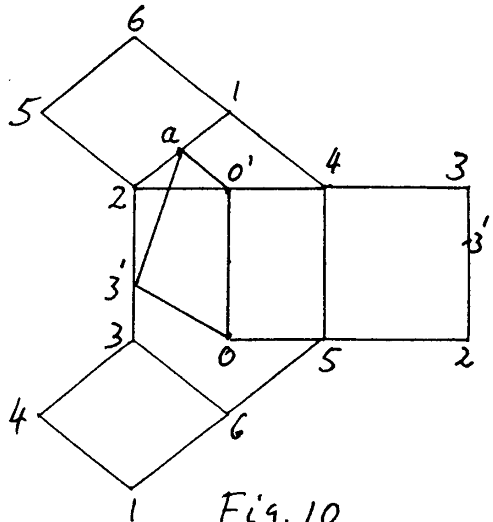


Fig. 10

Clearly, M_1 is not embeddable into P and the link $(0', 4)$ is redundant. This contradicts the minimality of I.

If I contained the minor M_0, let M be a minimal subgraph of I that contains M_0 as a minor. Let l_1 and l_2 be the generalized links of M that correspond to the links $(a, 0')$ and $(3, 2)$ of M_0. Consider the bridges of $M - \{l_1, l_2\}$ in I. Clearly, l_1 and l_2 cannot belong to one and the same of these bridges, because then I would contain as proper minor the non-embeddable graph given by Figure 11:

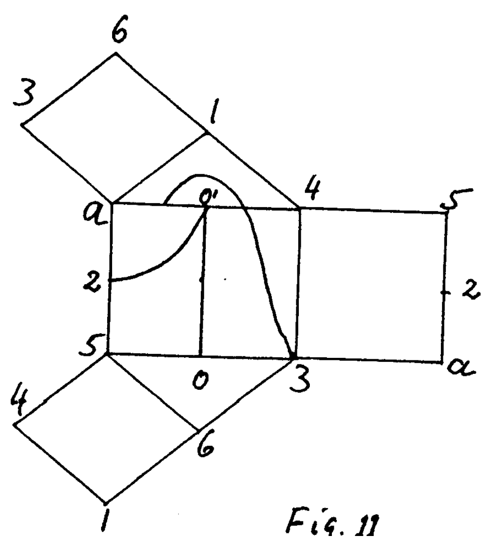


Fig. 11

This contradicts the minimality of I.

Let then B_1 and B_2 be the two bridges of $M - \{l_1, l_2\}$ that contain l_1 and l_2 as subgraphs respectively. Obviously, $\{M_0 - (a, 0'), (2, 3)\}$ and $M - \{l_1, l_2\}$ have only one embedding into P. The embeddings of $I - B_1$, and $I - B_2$ into P are therefore extensions of this embedding. In these extensions, B_1 and B_2 are embedded into different faces, say F_1 and F_2. For B_1 or B_2 not to be redundant, there have to exist bridges C_1 and C_2 of M that interfere with an embedding of B_1 into F_1 and B_2 onto F_2 Clearly, C_1 and C_2 have to be two different bridges, and therefore any one of the four bridges B_1, B_2, C_1, C_2 is redundant. This is again a contradiction to the minimality of I.

Case 4:

$x = 0, y = 3$ or $3'$

In this case, I contains the minor M_0 whose only embedding into P is given by Figure 12:

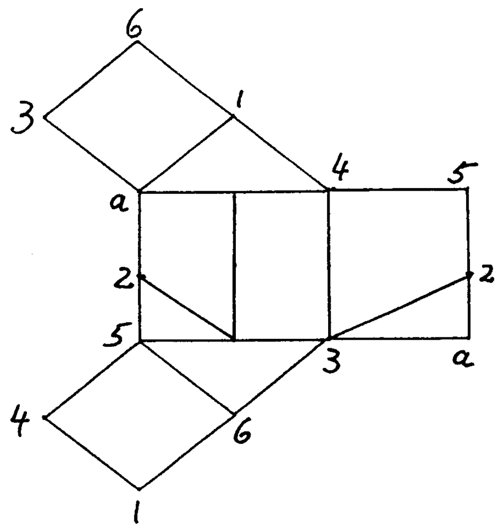


Fig. 12

or the minor M_1, given by Figure 13:

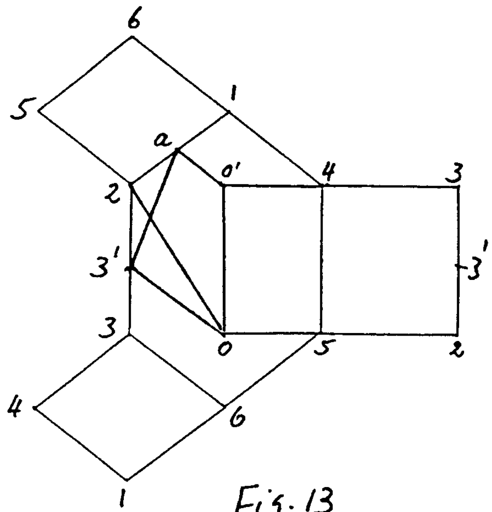


Fig. 13

Analogously to case 3, M_1 is not embeddable into P and we conclude that M_1 is not minimal since $(0', 4)$ is redundant. Also, M_0 is not minimal because of the two links $(2, 3)$ and $(5, 0)$.

This results again in a contradiction to the minimality of I.

Case 5:

$x = 3', y = 3$

I contains the minor M_1 given by Figure 14:

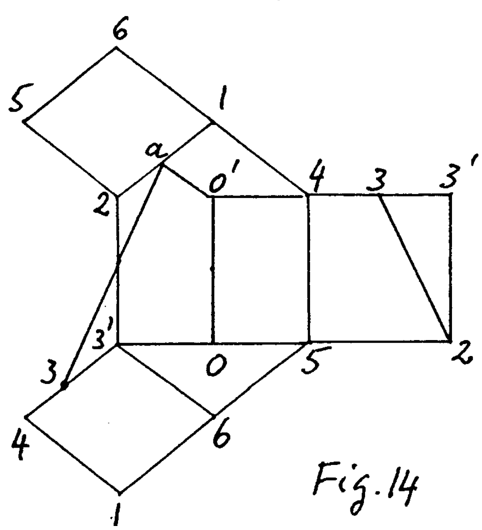


Fig. 14

Clearly, M is not embeddable into P and the link (3, 2) is redundant. This contradicts the minimality of I.

These five cases cover all the essentially different possibilities that could occur if B was attached only to the boundary $(2, a, 2', 0', 0, 3', 3, 2)$.

Assume now that B is not attached only to that boundary or to one equivalent under the automorphisms of $H_{2,2'}$. Then all the vertices of attachment of B have to be on one of the circuits $(2, 1, 4, 0', 2)$ or $(2, 3, 4, 5, 2)$ of $\overline{H}$.

Let us next consider the case that B is attached to the boundary $(2, 1, 4, 0', 2)$ in $\overline{H}$. In an embedding of $\overline{I}$ into P, B is embedded into that face $\overline{F}$ with boundary $(2, 1, 4, 0', 2)$ or $(2, 3, 4, 5, 2)$ if B is only attached to 2 and 4. For B not to be redundant, B must not be embeddable into the face with boundary $(a, 1, 4, 0', a)$, because otherwise each embedding of $\overline{I}$ into P could be extended to an embedding of I into P. Therefore, one vertex of attachement of B in H is the vertex 2.

If B bridges three of the main vertices of H, we have, using the automorphisms of $H_{2,2'}$, the following four choices for vertices to consider: $\{2, a, 0'\}$, $\{2, a, 4\}$, $\{2, 0', 4\}$, $\{2, 0', 1\}$. The first one can be excluded since B must not be attached only to the boundary $(2, a, 2', 0', 0, 3', 3, 2)$. In the remaining three cases it follows easily that $H \cup B$ is not embeddable into P and not minimal, since the links $(2, a)$, $(0', 4)$ and $(0, 3)$ respectively of H are redundant, contradicting the minimality of I.

If B is only attached to the vertices 2 and 4 of H, then any embedding of H can be extended to an embedding of $H \cup B$ by embedding B into the face with boundary $(2, 3, 4, 5, 2)$. For B not to be redundant, there exists a bridge B' of H that bridges 3 and 5.

If B is attached only to 3 and 5, H must have another bridge B'' that bridges 0 and 6, and I therefore contains the minor given by Figure 15,

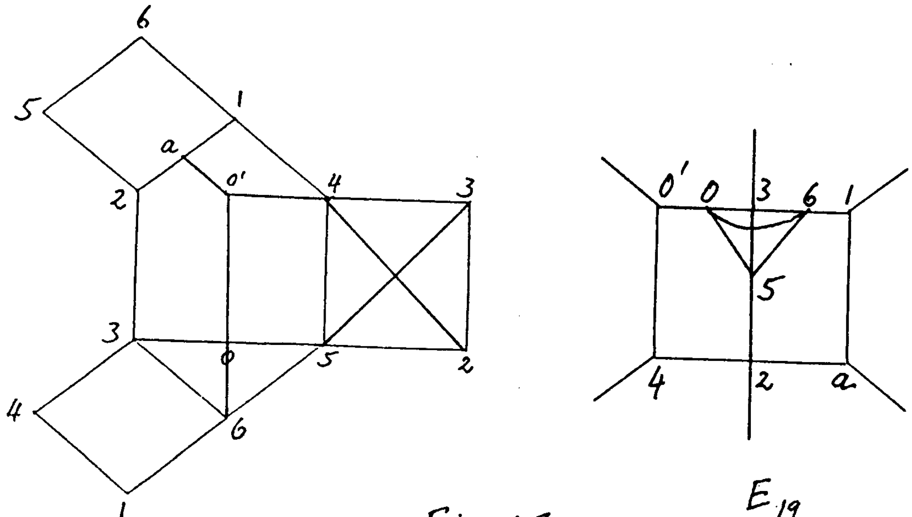


Fig. 15

which obviously contains E_{19} as proper minor, contradicting the minimality of I.

The remaining cases, where B' is attached to a vertex different from 3 and 5, lead again to the same contradiction, following similar reasoning.

Finally, the case that B is attached to the boundary (2, 3, 4, 5, 2) of $\overline{H}$, can be dealt with analogously. □

Lemma 4:

Let I be a graph in $I^1_{3,4}$ and K a minimal subgraph of I homeomorphic to $K^1_{3,4}$. Then no bridge of K is attached to an inner vertex of the generalized link $(0, a)$.

Proof:

Assume the contrary, i.e., let I be a graph in $I^1_{3,4}$, K a minimal subgraph of I homeomorphic to $K^1_{3,4}$ and B a bridge of K attached to an inner vertex $0'$ of the generalized link $(0, a)$ of K. Since I is 3-connected, B has to bridge $0'$ to some main vertex x of K, different from 0 and a.

Because of lemma 3, x cannot be 4 or 6. Therefore, x has to be one of 3, 5, 1 or 2. The first two are equivalent because of the automorphisms of $K^1_{3,4}$, described earlier.

We now consider the three resulting cases:

Case 1:

$x = 3$ or 5.

We show that this assumption leads to a contradiction: Since B bridges $0'$ and 3, B is attached to a vertex $3'$ on one of the generalized links of K that contains 3 as an endpoint. Because of the restrictions on x stated above, $3'$ can only be on the generalized link $(0, 3)$ of K, and $3'$ may coincide with 3. Therefore, I contains a minor M_0 whose four inequivalent embeddings into P are given by Figure 15a :

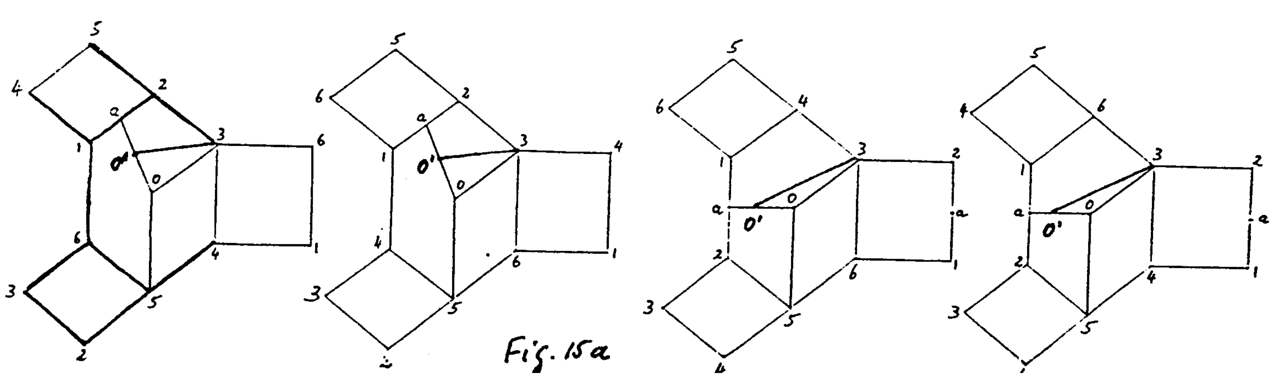


Fig. 15a

Let M be a minimal subgraph of I that contains M_0 as a minor. Clearly, M has four inequivalent embeddings into P which are extensions of the embeddings of M_0 into P. We observe that the generalized circuit (0, 0′, 3′, 0) of M is the boundary of face F in each of these embeddings and also in each of the six inequivalent embeddings of $\overline{M}$ into P, where $\overline{M}$ is obtained from M by contracting the generalized link (1, a) of M. We conclude that no bridge of M can be attached only to that circuit (0, 0′, 3′, 0). Because if C was such a bridge, C would have to be embeddable into F, since otherwise $\overline{M} \cup C$, a proper minor of I, would not be embeddable into P, contradicting the minimality of I. Therefore, for C not to be redundant, there would have to exist another bridge C' of M that interferes with the embedding of C into F. But this again leads to a contradiction, namely that $\overline{M} \cup C \cup C'$ is not embeddable into P.

We also observe that the generalized link (0, 3) of M can be embedded into each of the four inequivalent embeddings of $M - (0, 3)$ into P. Therefore, for (0, 3) not to be redundant, there has to exist a bridge B' of M that either bridges 0′ to one of the vertices 4, 5, 6 or that bridges an inner vertex of (0, 3) to one of the vertices a, 1, 2, 4, 6. However, no inner vertex of (0, 3) can be bridged to one of 2, 4, 6 because of the definition $I^1_{3,4}$, nor can such a vertex be bridged to 1 or a, because in that case I would obviously have E_3 as a proper minor, contradicting the minimality of I. Because of lemma 3, 0′ cannot be bridged to 4 or 6. Therefore, B' bridges 0′ and 5. B' cannot bridge 0′ and 5 to 2 because of the definition of $I^1_{3,4}$, nor to 3 because otherwise I would again contain E_3 as a proper minor. Therefore, B' is only attached to vertices on (5, 0′), at least one of them, say 5′, different from 0′, and to vertices on (0, 0′, a), at least one of them, say 0″, different from 0 and a. From this we draw the conclusion that I contains as minor the graph N_0, whose four inequivalent embeddings into P are given by Figure 16:

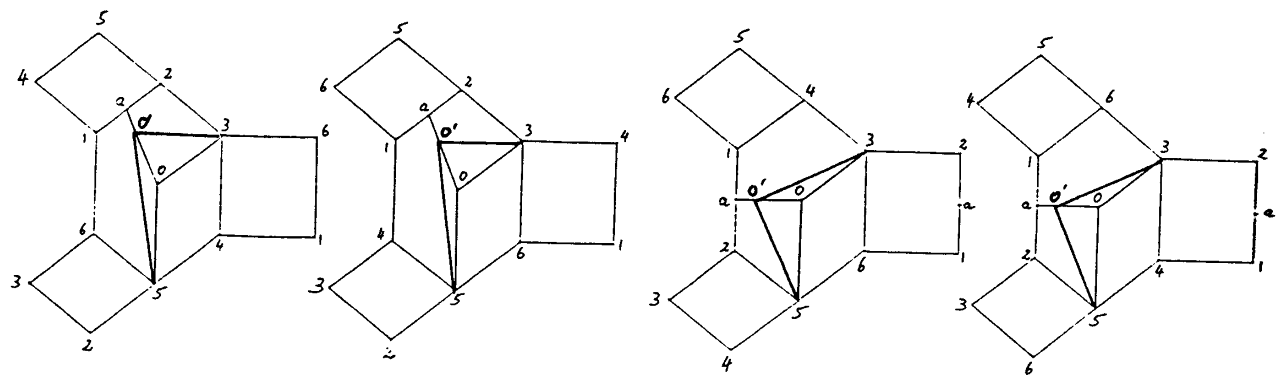


Fig. 16

Let N be a minimal subgraph of I that contains N_0 as a minor. Then N has the four inequivalent embeddings into P, given by Figure 17:

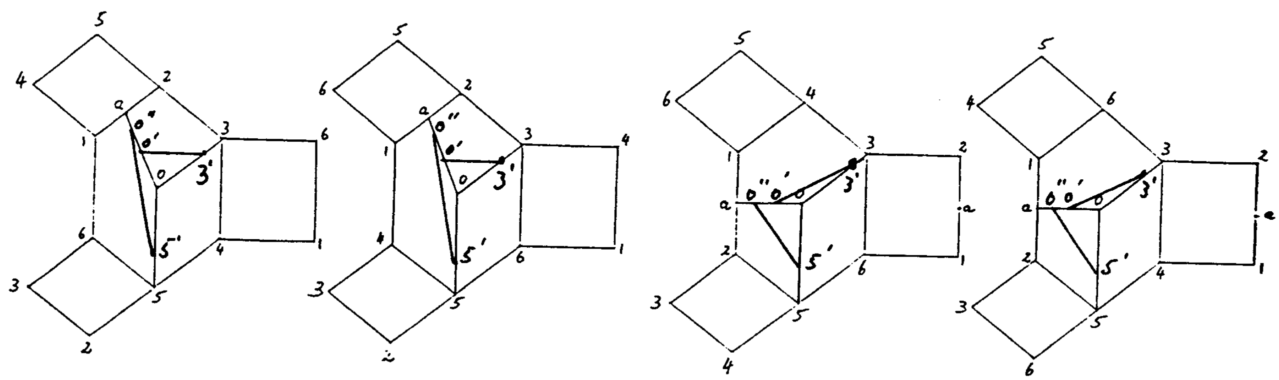


Fig. 17

Here $3'$ and 3, $5'$ and 5, $0''$ and $0'$ are not necessarily different. Because of the automorphisms of $K_{3,4}^{1}$ we can assume w.l.o.g. that $0'$ is between 0 and $0''$. Clearly, the subgraph of M_1 consisting of the three generalized links (0, 0′), (3, 0), (5, 0) of N can be embedded into each of the four inequivalent embeddings of $M - \{(0, 0'), (3, 0), (5, 0)\}$ into P. As before, we conclude that there has to exist a bridge D of M to interfere with at least one of these embeddings and that D has to bridge 0 to one of the vertices a, 1, 2, 4, 6. Again, 2, 4 and 6 can be excluded because of the definition of $I^1_{3,4}$, and 1 or a would lead to the contradiction that I contained E_3 as a proper minor.

This shows that case 1 leads to a contradiction.

Case 2:

$x = 1$

In that case, I contains a minor M_0 whose four inequivalent embeddings into P are given by Figure 18:

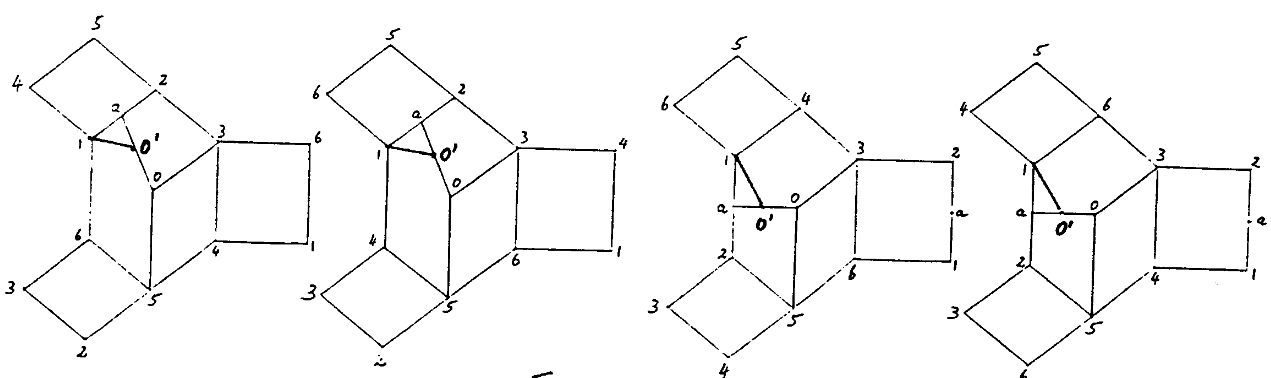


Fig. 18

As in the proof of lemma 3, we note that contracting the generalized link (0, 0′) of M_0 leads to a minor $\overline{M}_0$ of I that has four inequivalent embeddings into P. Clearly, the embeddings of M_0 into P are extensions of those of $\overline{M}_0$ into P. In the same way as in the proof of lemma 3 we conclude that the generalized link (0, 0′) is redundant.

Case 3:

$x = 2$

Finally, this case is dealt with the same way as case 2, again concluding that the generalized link (0, 0′) is redundant. □

Lemma 5:

Let I be a graph in $I^1_{3,4}$ with the property that for some subgraph K of I that is homeomorphic to $K^1_{3,4}$ a bridge of K is attached to an inner vertex of the generalized link (0, 3) of K. Then I is one of the graphs D_{12}, E_{11} or E_{27}.

Proof:

Let I be a graph in $I^1_{3,4}$ with the property stated in lemma 5 and let B be a bridge of K that is attached to an inner vertex 0′ of (0, 3). Since I is 3-connected, B has to bridge 0′ to some main vertex x of K, different from 0 and 3. Also, x has to be different from 2, 4, 6 and 5, since vertex 5 would lead to a contradiction to lemma 4 and any one of the vertices 2, 4, 6 would lead to a contradiction with the definition of $I^1_{3,4}$. Therefore, x has to be one of a or 1.

Case 1:

$x = 1$

Since B bridges 0′ and 1, B is attached to a vertex on one of the generalized links of K that contain 1 as an endpoint. Because of the restrictions on x stated above, x can only be on the generalized link $(0', 1)$ of K. Because of lemma 4 and the symmetries of $K_{3,4}^{1}$, x has to be equal to 1. Therefore I contains a minor M_0 whose two inequivalent embeddings into P are given by Figure 19:

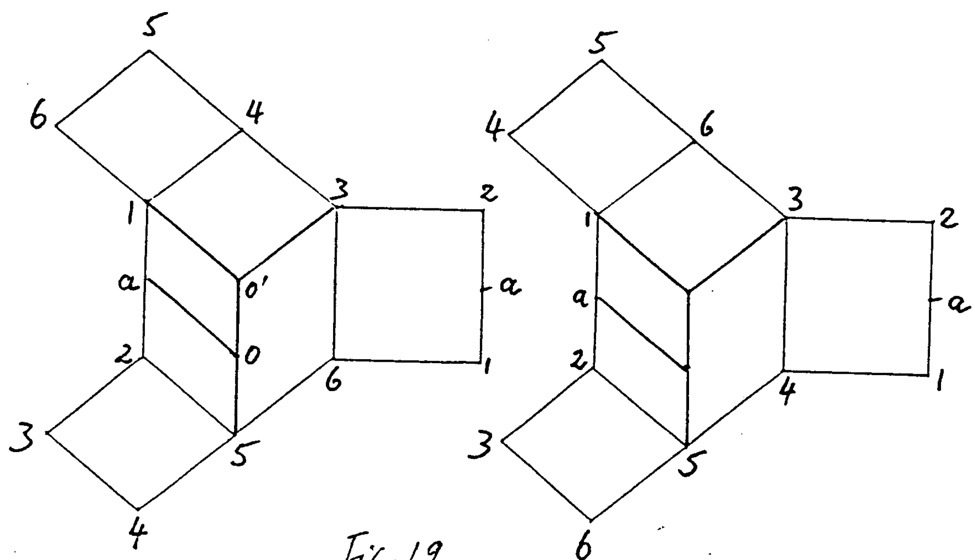


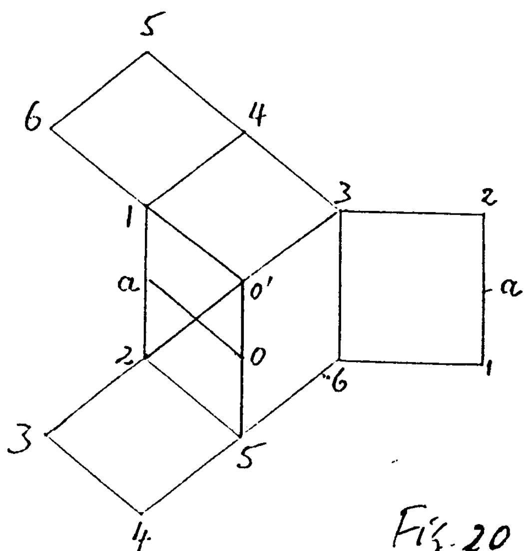
Fig. 19

Let M be a minimal subgraph of I that contains M_0 as a minor. M obviously has the two automorphisms given by $(4, 6)$ and $(0', 2)(0, a)(1, 5)$. Therefore the main vertices of M are partitioned into the five equivalence classes $\{0', 2\}$, $\{1, 5\}$, $\{4, 6\}$, $\{0, a\}$, $\{3\}$ and the generalized links equivalent to $(0, a)$ are the the following: $(2, a)$, $(1, 4)$, $(1, 6)$, $(0', 0)$, $(5, 4)$, $(5, 6)$.

Since I is 3-connected, any bridge of M is attached to M at two different vertices on two different generalized links of M. Among the endpoints of these links there is at least one pair $\{x, y\}$ that is not connected by a generalized link of M (since M has no generalized triangles). Because of the symmetries of M, given above, there are ten different equivalence classes of such pairs, that can be described by the following representatives: $\{0', 2\}$, $\{0', 4\}$, $\{0, 4\}$, $\{0', 5\}$, $\{0', 4\}$, $\{1, 3\}$, $\{4, 6\}$, $\{1, 5\}$, $\{1, 0\}$, $\{0, 3\}$.

We will now investigate the ten resulting cases and show that they lead to contradictions, unless I contains one of the graphs D_{12}, E_{11}or E_{27}.

$\{x, y\}$ is different from $\{0', 2\}$, because otherwise would contain F_1 as a proper minor, Figure 20,



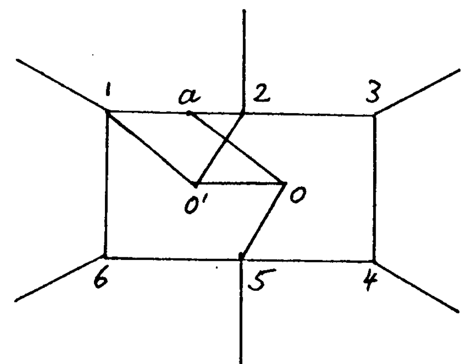


Fig. 20

F_1

contradicting the minimality of I.

$\{x, y\}$ is different from $\{0', 4\}$ because of the definition of $I^{1}_{3,4}$.

$\{x, y\}$ is different from $\{0, 4\}$ because of lemma 3.

$\{x, y\}$ is different from $\{0', 5\}$ because otherwise I would contain a minor that is excluded by lemma 4.

If $\{x, y\} = \{0', a\}$, then B cannot bridge $0'$ and a to 4 or 6 because of the definition of $I^{1}_{3,4}$, nor to 0, 1, 2, 5 because of Lemma 4. Therefore B is either degenerate, attached only to $0'$ and a, or B bridges $0'$, a and 3. In that case, I is the graph E_{27}.

Figure 21:

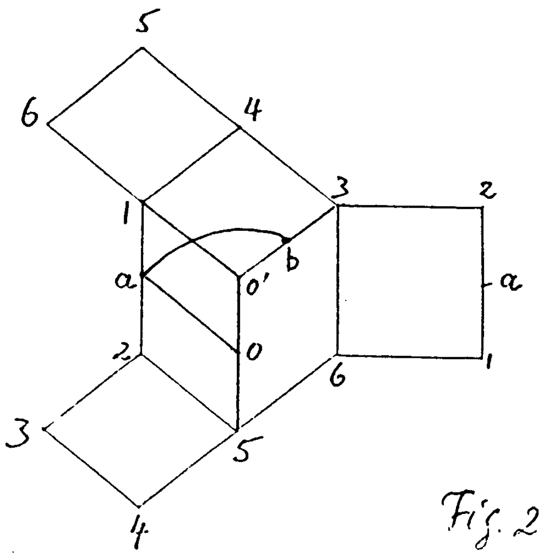
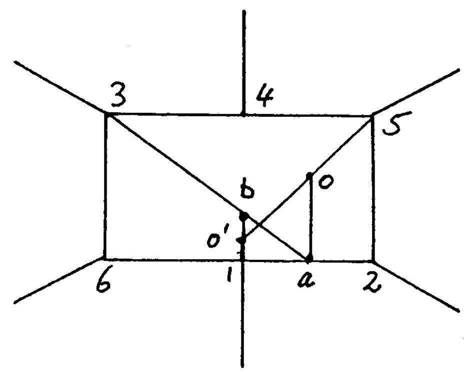


Fig. 21



E_{27}

If $\{x, y\} = \{1, 3\}$, then B cannot bridge 1 and 3 to 0′, 2, 4, 5, 6 or a because of Lemma 4, nor to 0 or 5, because in that case I would contain E_3 as a proper minor. Therefore, B is degenerate, attached only to 1 and 3.

If $\{x, y\} = \{4, 6\}$, then B cannot bridge 4 and 6 to 1 or 5 because of Lemma 4 nor to 3 because of the definition of $I^1_{3,4}$. Because of the symmetries of M and the previously excluded cases $\{0', 4\}$ and $\{0, 4\}$, B cannot bridge 4 and 6 to 0′, 0, 2 or a either. Therefore, B is degenerate, attached only to 4 and 6.

If $\{x, y\} = \{1, 5\}$, then B cannot bridge 1 and 5 to 4 or 6 because of Lemma 4, nor to 0′, 2 or 3 because of the symmetries of M and the cases $\{1, 3\}$ and $\{0', 5\}$ dealt with earlier. Therefore, B is either degenerate, attached only to 1 and 5, or B bridges 1 and 5 to 0 or a. The last two possibilities are equivalent because of the symmetries of M, and in either case I is the graph E_{11}, Figure 22:

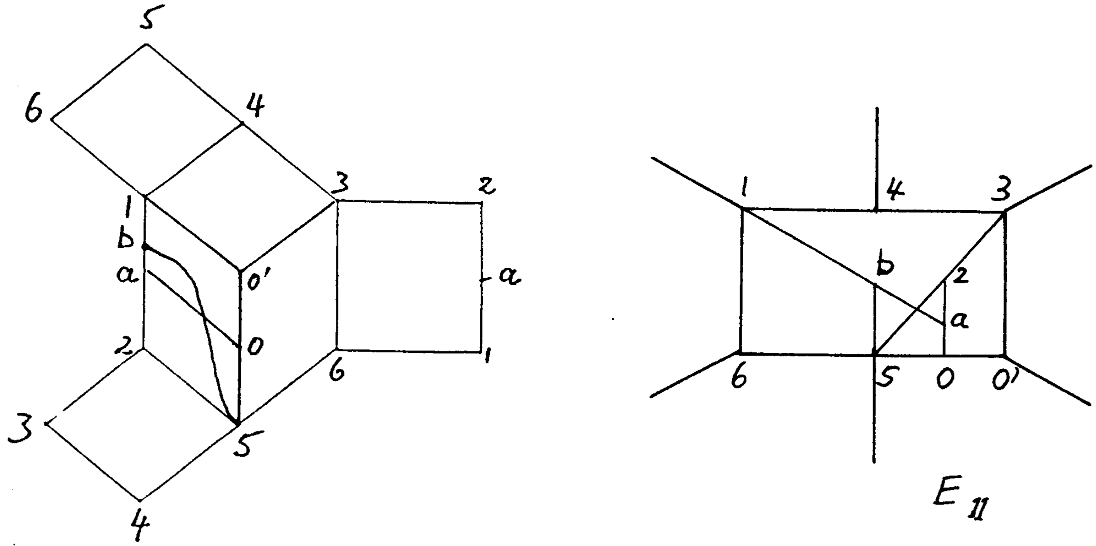


Fig. 22

If $\{x, y\} = \{1, 0\}$, then B cannot bridge 1 and 0 to 0′, 4, 6 or a because of Lemma 4, nor to 2, 3 or 5 because of the symmetries of M and the cases $\{0', 5\}$, $\{1, 3\}$, $\{1, 5\}$ previously dealt with. Therefore, B is degenerate, attached only to 1 and 0.

If $\{x, y\} = \{0, 3\}$ then B cannot bridge 0 and 3 to 0′ or a because of lemma 4, nor to 1, 2, 4, 5, or 6 because of the symmetries of M and the cases $\{0', a\}$, $\{0, 4\}$,$\{1, 3\}$ previously dealt with. Therefore B is degenerate, attached only to 0 and 3.

We are now left with the possibility that M has some degenerate bridge, bridging some of the six pairs of vertices $\{0', a\}$, $\{1, 3\}$, $\{4, 6\}$, $\{1, 5\}$, $\{1, 5\}$, $\{1, 0\}$, $\{0, 3\}$.

Obviously, for the degenerate bridge attached to 0′ and a not to be redundant, the degenerate bridge attached to 1 and 0 is required, and vice versa. In that case, I is the graph D_{12}, Figure 23:

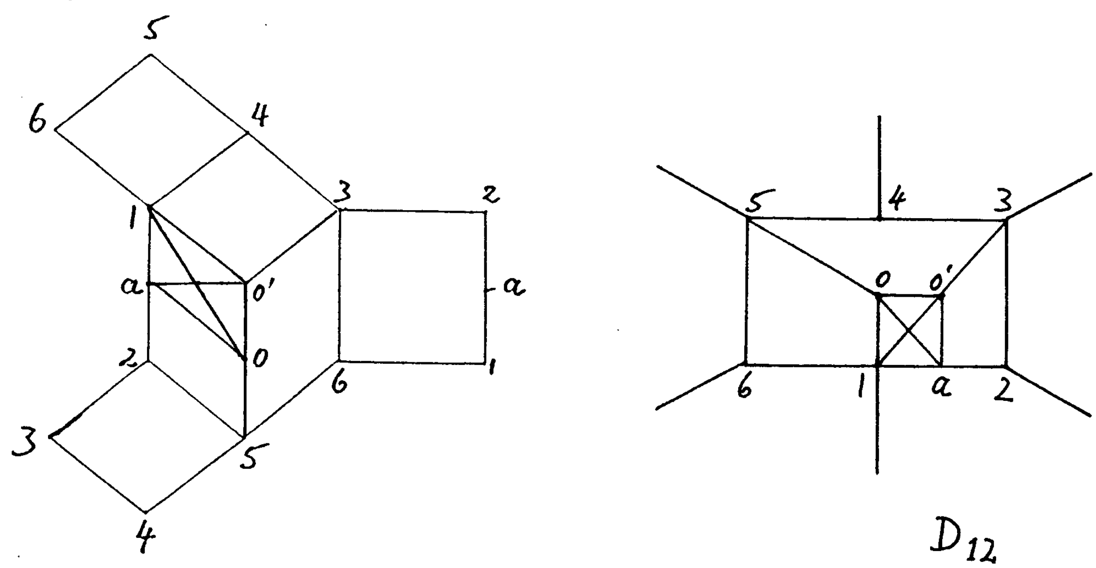
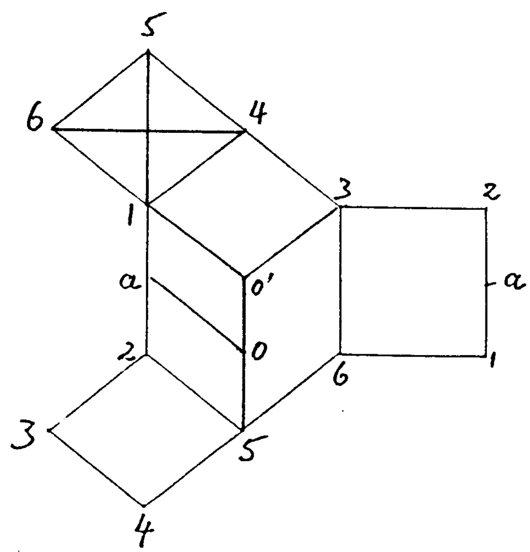


Fig. 23

Similarly, the degenerate bridge attached to 4 and 6 needs the degenerate bridge attached to 1 and 5 and vice versa. It follows that M cannot have either of these bridges, because otherwise I would contain the graph D_3 a proper minor, Figure 24:



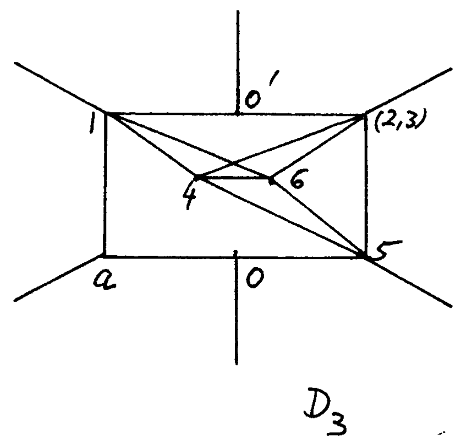


Fig. 24

Finally, M cannot have degenerate bridges attached to 3 and 0 or to 3 and 1, because no other possible bridge of M can interfere with these.

This finishes the discussion of case 1, $x = 1$.

Case 2:

$x = a$

This case can be reduced to case 1: Since now B is assumed to bridge $0'$ and a, B is attached to a vertex a' on one of the generalized links of K that contain a as an endpoint. Because of the restrictions on X stated earlier, a' can only be on the generalized link $(0', a)$ of K, and a' may coincide with a. Therefore I contains a minor N_0 whose four inequivalent embeddings into P are given by Figure 25:

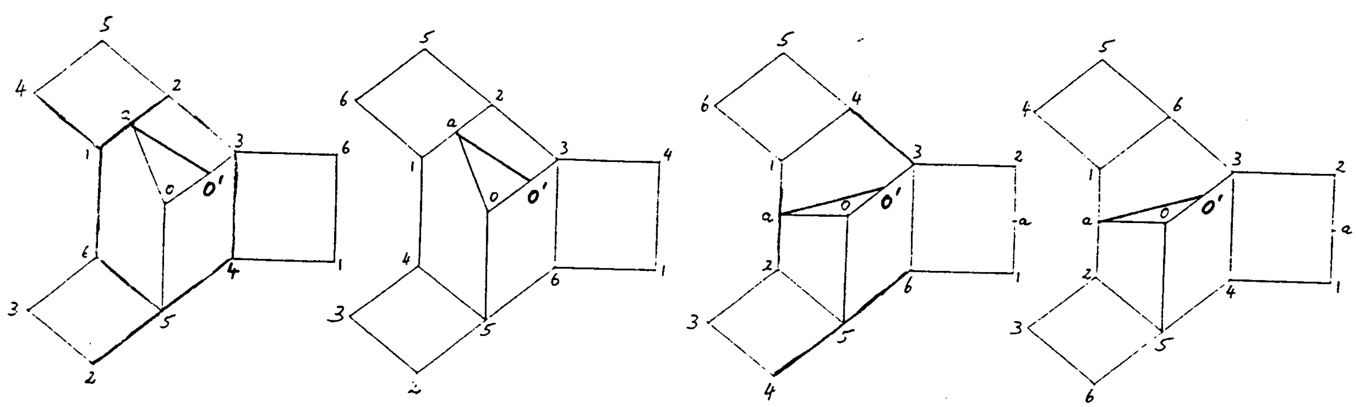


Fig. 25

Let N be a minimal subgraph of I that contains N_0 as a minor. Clearly, the four inequivalent embeddings of N into P are extensions of the four embeddings of N_0 into P. As in the proof of lemma 4, case 1, we conclude that no bridge of N can be attached only to the generalized circuit (0, 0′, a', 0) of N. Furthermore, for the generalized link (0, a) of N not to be redundant, there has to exist a bridge B of N that bridges 0′ to one of the vertices 1, 2, 4, 5 or 6 (since by lemma 4 no bridge can be attached to an inner vertex of (0, a)). Again, because of the definition of $I_{3,4}^{1}$, 2, 4 and 6 are excluded, and because of lemma 4 and the symmetries of $K_{3,4}^{1}$ 5 is excluded. Therefore, B bridges 0′ and 1, and we are back to case 1. □

Lemma 6:

Let I be a graph in $I_{3,4}^{1}$ with the property that for no subgraph K of I that is homeomorphic to $K_{3,4}^{1}$ a bridge of K intersects the generalized link (0, 3) of K in an inner vertex, but for some subgraph K of I that is homeomorphic to $K_{3,4}^{1}$, a bridge of K intersects the generalized link (1, a) in an inner vertex. Then I is one of three graphs C_3, D_9 orE_5.

Proof:

Let I be a graph with the properties stated in lemma 6 and let B be a bridge of K that intersects the generalized link (1, a) in an inner vertex b. Since I is 3-connected, B has to bridge b to one of the vertices 0, 2, 3, 4, 5 or 6 of K. B cannot bridge b to 0, 2, 4 or 6 since that would, because of the symmetries of $K_{3,4}^{1}$, contradict lemma 4 or the properties of I stated in lemma 6. B cannot bridge b to 3 and 5, because otherwise I would have E_3 as a proper minor. Because of the symmetries of $K_{3,4}^{1}$ we assume w.l.o.g. that B bridges b and 3. From the constraints on B, derived earlier, it follows that B is degenerate and only attached to b and 3.

Therefore, I contains a subgraph whose four inequivalent embeddings into P are given by Figure 26:

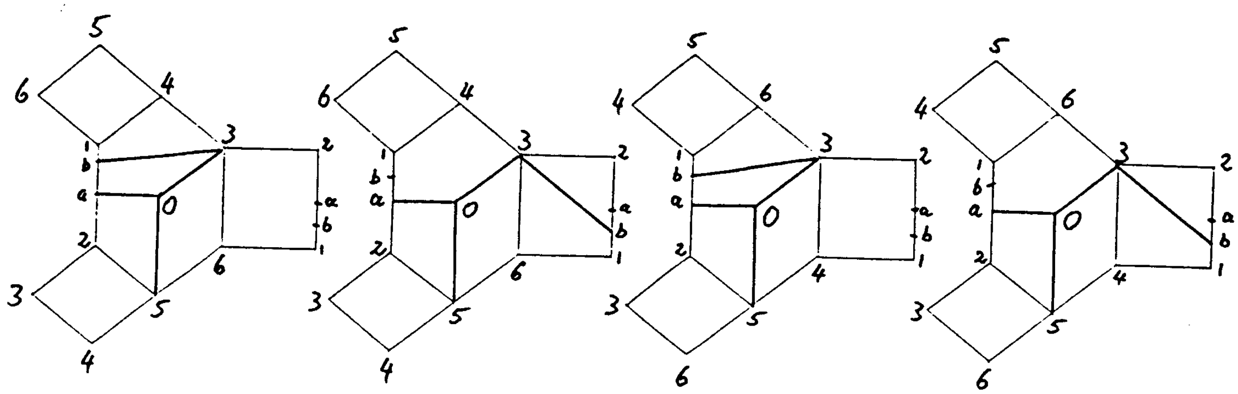


Fig. 26

Obviously, M has the three automorphisms (0,2); (4, 6); (0, 4)(1, a), which partition the vertices of G into the five equivalence classes $\{0, 2, 4, 6\}$,$\{1, a\}$, $\{3\}$, $\{5\}$, $\{b\}$.

As in the proof of lemma 5 we conclude that any bridge of M bridges one pair of main vertices $\{x, y\}$ of M that are not connected by a link of G. Because of the symmetries of G, given above, there are seven different equivalence classes of such pairs that can be described by the representatives $\{5, b\}$, $\{1, a\}$, $\{1, 3\}$, $\{0, 2\}$, $\{a, 5\}$, $\{3, 5\}$, $\{0, 4\}$.

If a bridge B of M bridges 5 and b, then $I = E_5$. Figure 27:

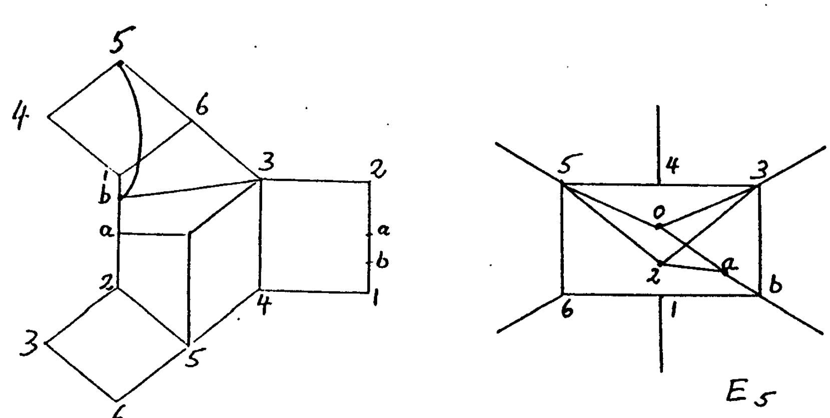
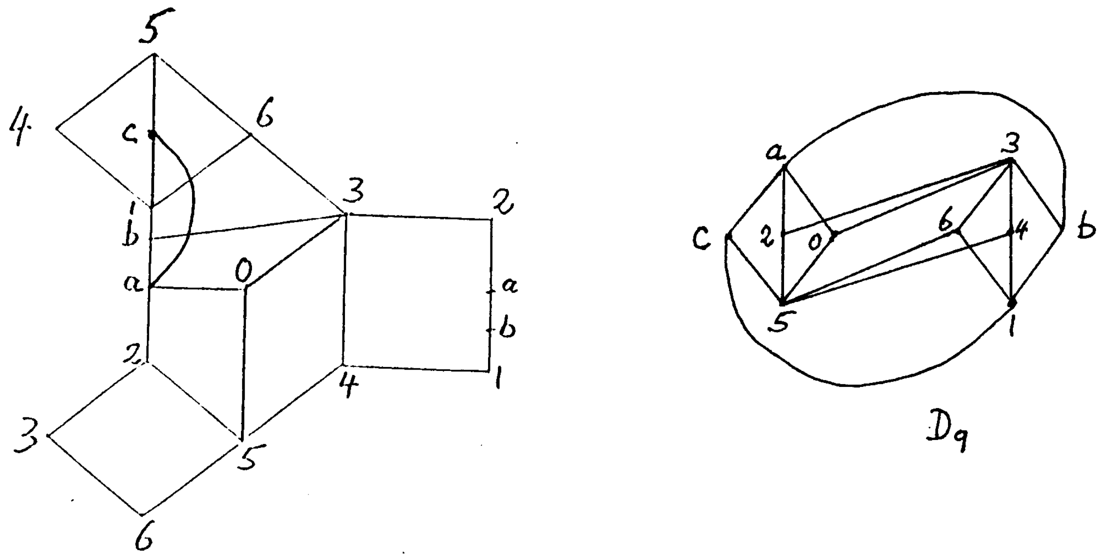


Fig. 27

If a bridge B of G bridges 1 and a, it follows easily that B has to bridge 1 and a to 5, and $I = D_9$, Figure 28:



D_9

From the symmetries of $K_{3,4}^{1}$ and the conditions in Lemma 6 follows that if a bridge B of M bridges any other pair of vertices, $\{x, y\}$, then B is degenerate and only attached to x and y.

A degenerate bridge B of G cannot be attached to 1 and 3, because then B would clearly be redundant.

A degenerate bridge B of M cannot be attached to 0 and 2, because otherwise G would need a bridge B' attached to a and 5, and I would have the proper minor D_3, Figure 29:

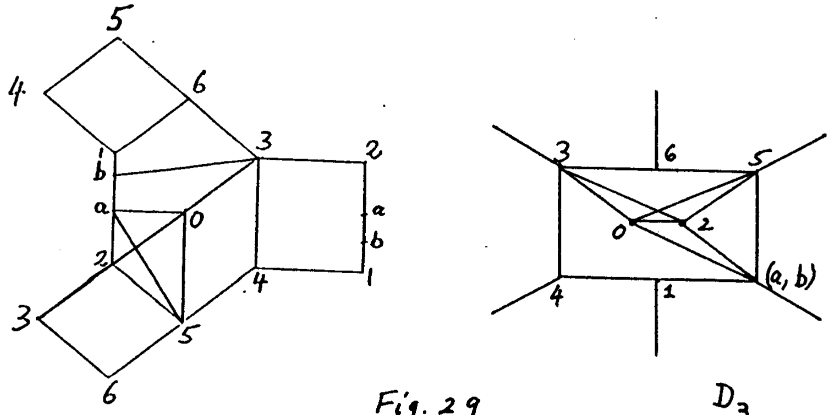


Fig. 29

D_3

and vice versa, M cannot have a dgenerate bridge attached to a and 5.

If M has a degenerate bridge attached to 3 and 5, then M needs two more bridges, which can only be degenerate bridges attached to $\{0,4\}$ and some equivalent pair $\{x', y'\}$. This leads to only one choice for $\{x', y'\}$, namely $\{2, 6\}$. The $I = C_3$, Figure 30:

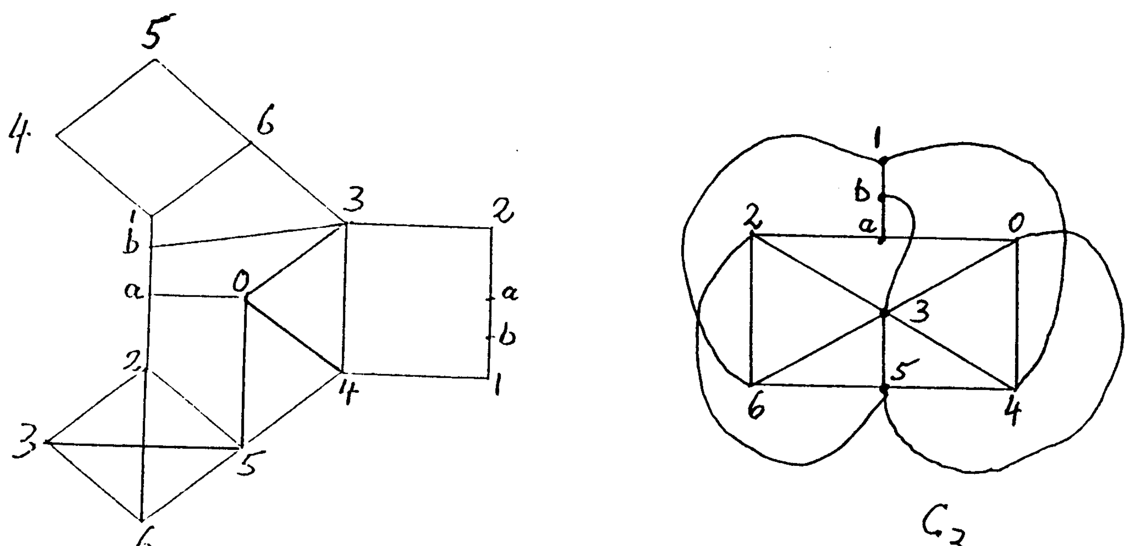


Fig. 30

Finally, if M had four degenerate bridges, all attached to $\{0, 4\}$ and the other three equivalent pairs, we would obtain a graph that is not embeddable into P. However, this graph is not minimal since it has the automorphism $(1, 2)(3, 4)(5, 6)(b, c)$, where c is an inner vertex of the generalized link $(a, 2)$ of G. This shows that a bridge is attached to an inner vertex of a generalized link of G equivalent to $(0, a)$, contradicting Lemma 4. □

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