

THE UNIVERSITY OF CALGARY

The Brill Gravitational Wave Initial Value Problem

by

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A THESIS

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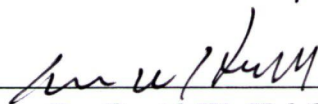
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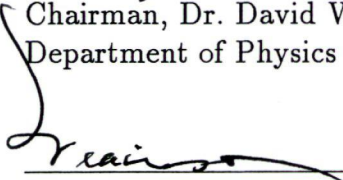
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FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "The Brill Gravitational Wave Initial Value Problem" submitted by Andrew Masterson in partial fulfillment of the requirements for the degree of M.Sc. in Physics.



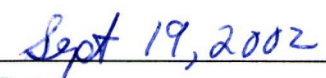
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Abstract

The goal of this thesis is to numerically construct, in its entirety, the spatial 3-volume that arises as the initial slice of an axi-symmetric, vacuum space-time with an initial Brill wave present. This formulation can then be used in future work on the time-evolution of the Brill gravitational wave system.

We were able to construct numerically stable solutions for all of the dynamical variables that are present in the general relativistic Einstein equations, and thus a complete description of the initial slice of the space-time was achieved.

Acknowledgments

No thesis is possible without the dedication and support of one's supervisor, and this case was no different. I would like to extend my gratitude to my supervisor, David Hobill, for taking the time to share his vast knowledge of General and Numerical Relativity with me. Without his guidance and insights, this project would not have been possible.

I would also like to acknowledge all of my friends and loved ones who have provided me with sanity and understanding during this time. Without their encouragement (and prodding), I would be no further than when I first set out on this path. I have a ways to go yet, but thanks to all of those who helped me get to where I am now.

Also, a thanks is in order to all of those teachers and professors who have inspired me throughout the years with their love of physics and mathematics. It is contagious.

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Chapter 1

The Initial Value Problem

In this thesis, our goal is to analyse and numerically construct the Initial Value Problem (IVP) for an axisymmetric gravitational wave system in a vacuum spacetime. To this end, we shall devote this chapter to developing the basic framework necessary to study this problem, the most important of which is the 3+1 formalism of general relativity. Chapter 2 will concentrate on the coordinate and gauge choices that we make to simplify Einstein's equations in order to solve them. Chapter 3 will present the initial conditions and boundary values that are necessary to solve the spacetime on the initial slice, and Chapter 4 will cover the solutions that were generated using this formalism. An overview of the numerical methods used in this thesis is presented in Appendix A.

1.1 A Brief Introduction to General Relativity

Einstein's equation for the relativistic gravitational field is given by:

$$G_{\alpha\beta} = \frac{8\pi G}{c^2} T_{\alpha\beta} \tag{1.1}$$

where we use the convention that Greek indices (α, β) run across all 4 space-time tensor dimensions, whereas the Latin indices (a, b) are 3-dimensional spatial tensor indices. Due to the nature of most problems that are studied in General Relativity, we usually adopt the convention that $G = c = 1$ (where G and c are the universal

gravitational constant and the speed of light in a vacuum, respectively). This can be done without loss of generality, as we are just scaling unit lengths differently. i.e. this implies that

$$1s \equiv 299\,792\,458m = 1.8016 \times 10^{15} kg$$

which arises from setting:

$$299\,792\,458m/s = 1 = 6.67259 \times 10^{-11} \frac{m^3}{kg \cdot s^2}$$

Adopting this convention, our general 4-dimensional equation becomes

$$G_{\alpha\beta} \equiv R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi T_{\alpha\beta} \quad (1.2)$$

Where $G_{\alpha\beta}$ is the Einstein Tensor - a 2nd rank, four dimensional tensor that describes the geometry of the spacetime; $T_{\alpha\beta}$ is the energy-momentum tensor - a 2nd rank, four dimensional tensor that provides information about the distribution of matter and non-gravitational fields; $R_{\alpha\beta}$ is the Ricci Tensor and R is the Ricci scalar (both of which are contractions on the general Riemann tensor¹ $R^\alpha_{\beta\gamma\delta}$); and $g_{\alpha\beta}$ is the metric, which defines lengths and inner products on the spacetime manifold.

No known, general, closed-form solution exists to (1.2), and the large majority of the research that is done in the field of General Relativity consists of attempting to find or verify particular solutions to (1.2). In general, there are two ways of approaching the problem that can lead to a solution.

The first approach consists of a “trial and error” method, wherein the metric $g_{\alpha\beta}$

¹The Riemann tensor is strictly a geometric tensor, and it is linear in the second derivatives of $g_{\alpha\beta}$ and quadratic in the first derivatives of $g_{\alpha\beta}$

is specified as a function of spacetime coordinates and the energy-momentum tensor is constructed from the solution, using

$$T_{\alpha\beta} = \frac{1}{8\pi} G_{\alpha\beta} \quad (1.3)$$

The advantage of this method lies in the the ability to choose a solution that is easy to analyse analytically or numerically, but the obvious (and crippling) disadvantage is that only a very small number of the metrics and energy-momentum tensors constructed in this manner have any physical relevance. In essence, this method is like shooting in the dark, and is about as successful.

The physically relevant method is to specify the energy-momentum tensor, and then proceed to try to solve for the dynamic variables of the system (i.e. the metric and extrinsic curvature components). The obvious advantage of this method is that we can specify the (hopefully) physically relevant spacetime that we wish to study. The (sometimes crippling) disadvantage is that the resulting equations to be solved for the dynamic variables are often highly non-linear and unsolvable short of a foray into the numerical world. And that is where we stand presently with most relativity problems - the "simple" cases of Schwarzschild, Kerr, Reissner-Nordström, etc. have been solved analytically, and any other system that mimics a physically relevant situation requires numerical work due to the excessive complexity and the inability to construct closed-form analytical solutions to (1.2).

1.2 Developing the 3+1 Formalism

In order to simplify the general 4-dimensional tensor problem for numerical work, the “3+1” formalism was introduced by Arnowitt, Deser and Misner [2] (also called the Cauchy formalism), in which the spatial information about the space-time (3) is “split-off” from the temporal information (+1). In this form, we evolve the spacetime from the initial data set by foliating the spacetime along the “time” coordinate, where temporal level surfaces consist of instantaneous spatial 3-volumes. In principle, there is nothing exclusive about our choice of the time variable as our level-surface coordinate, but due to the desire to analyse observable results and make comparisons to Newtonian Gravitational problems, we are driven to choose this splitting. We could thusly choose x , y and t as our variables to “evolve” along level surfaces of z , but the physical meaning becomes much more abstracted and difficult to comprehend, leading to the natural choice of time as our level-surface coordinate.

Thus our goal is to reduce the 4-dimensional tensor equations to a related group of 3-dimensional tensor equations that evolve through time. To accomplish this, we can describe the 4-metric, $g_{\mu\nu}$, as

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta^a \beta_a & \beta_a \\ \beta_a & \gamma_{ab} \end{pmatrix} \quad (1.4)$$

where α represents the lapse function, β_a represents the shift vector and γ_{ab} is the three dimensional (spatial) metric tensor. Figure (1.1) gives a schematic view of how the 3+1 formalism describes the foliated 4-dimensional spacetime with the lapse function and shift vector.

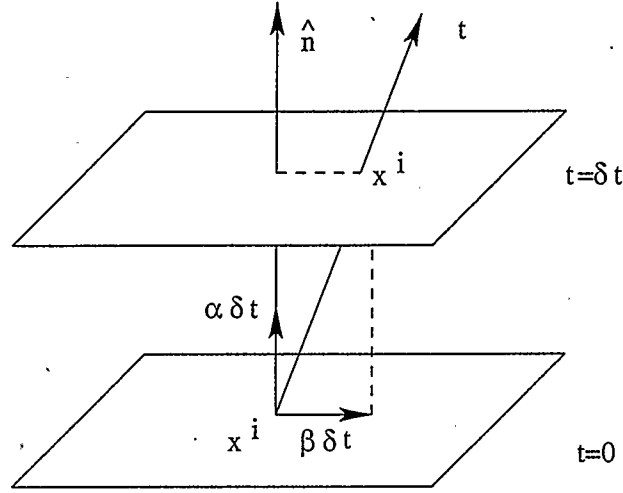


Figure 1.1: A schematic diagram of the splitting of the space-time using the 3+1 formalism. $\hat{n}$ is the normal vector to the surface at each point, α gives the distance, in time, between points on successive sheets and β represents the spatial shift of the coordinates from one sheet to the next. $\delta\tau = \alpha \delta t$, where τ is the proper time

The lapse, α , is a scalar function of time and space over the entire grid that represents orthogonal proper time progression (τ) at each point within the space time (i.e. $\alpha = \alpha(x^a)$). Thus α provides us with a tool to allow the evolution to progress at different rates at different points in the spacetime. This is desirable in regions of large curvature or near spacetime singularities to provide for a longer, more stable evolution. The lapse function, α , can also be used to calculate the interval of proper time from an interval of coordinate time, due to the fact that

$$d\tau = \alpha dt \quad (1.5)$$

The shift vector, β_a , represents the progression of the coordinate or “grid” points

along the spatial direction from one time slice to the next, i.e.

$$dx^i = \beta^i dr \quad (1.6)$$

It can be employed to provide optimal grid resolution in areas of numerically unstable curvature, while minimising grid point calculations in low-curvature areas.

Because we are only interested in the initial value problem, and α and β are variables that affect the time dynamics/evolution of the code, they will require minimal consideration for the remainder of this thesis (although they play a vital role in the evolution equations to be studied at some point in the future).

Thus using (1.4) as our guideline, we can split (1.1) into spatial and temporal equations. The goal of this project is to specify the necessary information on the initial slice (the Initial Value Problem, or IVP), and then use the resulting evolution equations to evolve the spatial information from the initial slice as far into the future as we desire (or as far as numerical stability will allow). Thus we can reduce the general 4-dimensional problem to an initial-value problem and an evolution problem (that is subject to differential constraints at each point during the evolution), which is easier to solve and analyse, numerically.

The 3+1 formulation of the Einstein equations requires one other quantity, however, to describe the geometry on each time slice - the extrinsic curvature (K_{ij}). The extrinsic curvature is a measure of how much the 3-dimensional spacetime on a $t = \text{constant}$ slice is curved relative to a flat, Cartesian, 4 dimensional embedding manifold, and is a well-defined differential geometric quantity². By defining the

²It is also called the "normal" curvature, and the Ricci Tensor (R_{ij}) is the intrinsic or "geodesic" curvature. The metric and extrinsic curvature are also referred to as the first (γ_{ij}) and second (K_{ij})

extrinsic curvature of the spacetime as

$$K_{\alpha\beta} = -\frac{1}{2}\mathcal{L}_0\gamma_{\alpha\beta} = -\frac{1}{2}(\partial_t - \beta^i\nabla_i)\gamma_{\alpha\beta} \quad (1.7)$$

where $\mathcal{L}_0$ is the Lie derivative of the metric along the “time” direction, we get a spatial, symmetric quantity that carries information about the curvature of the spacetime with respect to a flat embedding manifold.

Because the eventual goal is to use this IVP formulation to study the subsequent dynamics of the system, we include the general vacuum evolution equations for K_j^i and γ_{ij} ³:

$$\partial_t\gamma_{ij} = -2\alpha K_{ij} + \partial_i\beta_j - \Gamma_{ji}^l\beta_l + \partial_j\beta_i - \Gamma_{ij}^l\beta_l \quad (1.8)$$

$$\begin{aligned} \partial_t K_j^i = & -g^{id} [\partial_d\partial_j\alpha - \Gamma_{jd}^e\partial_e\alpha] + \alpha [R_j^i + K_j^i K] + \beta^c [\partial_c K_j^i - \Gamma_{jc}^d K_d^i + \Gamma_{dc}^i K_j^d] \\ & + K_c^i [\partial_j\beta^c + \Gamma_{ej}^c\beta^e] - K_j^c [\partial_c\beta^i + \Gamma_{ec}^i\beta^e] \end{aligned} \quad (1.9)$$

Where we have expanded the covariant derivatives out in terms of partial derivatives and connection coefficients, Γ_{jk}^i (which are all defined in terms of the metric γ_{ij}), and standard Einstein summation notation is used (repeated indices indicate a summation over the range of the index). We direct the interested reader to [16] or [17] for a more comprehensive development of these equations.

fundamental forms.

³The notation γ_{ij} is often used to represent the spatial part of the metric, $g_{\alpha\beta}$

1.3 The Constraint Equations

By studying various contractions of the four-dimensional Riemann tensor, we can arrive at a series of four equations in γ^{ab} , K^{ab} and $T^{0\beta}$ that must be satisfied at any given time in the spacetime. These equations play an analogous role to $\nabla \cdot E = 4\pi\rho$ and $\nabla \cdot B = 0$ in Maxwellian electrodynamics, and act as constraints that must be satisfied by the IVP and at each point during the evolution.

In other words, the constraints are a set of spatial equations that must be satisfied on each spatial slice in the spacetime. They can be thought of as differential equations in the spatial metric γ_{ij} and the extrinsic curvature K_{ij} , that can be solved for on the initial slice and each subsequent slice, effectively allowing us to choose an arbitrary slice in the spacetime as our starting point. So the idea is to specify whatever free data we can on the chosen slice, and then solve the constraint equations for the constrained data that remains⁴.

The development of the constraint equations from (1.2) using the 3+1 formalism is covered in detail elsewhere, so we refer the reader to [15], [9], or to any of the texts on the subject. The end result is that we obtain four spatial equations in the extrinsic curvature, the spatial metric and the energy-momentum densities (all of which are spatial quantities). They are labeled the Hamiltonian, or scalar, constraint and the momentum, or vector, constraints, and they are

$$R + (\text{Tr}K)^2 - K_{ab}K^{ab} = 2\kappa\rho \tag{1.10}$$

⁴This is analogous to specifying two components of the electric and magnetic fields in Maxwellian electrodynamics, and then solving for the remaining components of the two fields using $\nabla \cdot E = 4\pi\rho$ and $\nabla \cdot B = 0$

and

$$\nabla_b(K^{ab} - \gamma^{ab}K) = \kappa j^a \quad (1.11)$$

respectively. Here ∇_b is the covariant derivative, defined with respect to the spatial metric, γ_{ab} ; and in this formalism $\kappa = 8\pi$, $\rho = n^\alpha n^\beta T_{\alpha\beta}$ is the energy density and $j^a = -n_\alpha T^{\alpha a}$ is the momentum density, both of which are spatial quantities (and the n^α are the unit normals).

For this thesis we are studying a vacuum spacetime, and as such $T_{ab} = 0$, meaning that the right hand side of equations (1.10) and (1.11) are 0.

1.4 The Bianchi Identities

One of the basic properties of the Riemann tensor is that it satisfies the Bianchi Identities, which are differential identities that take the form:

$$\nabla_\epsilon R^\alpha_{\beta\gamma\delta} + \nabla_\delta R^\alpha_{\beta\epsilon\gamma} + \nabla_\gamma R^\alpha_{\beta\delta\epsilon} = 0 \quad (1.12)$$

These identities arise simply from a consideration of the underlying geometry that defines the Riemann tensor, and as such they are independent of any other conditions we put upon the spacetime.

Equation (1.12) can also be contracted into the contracted Bianchi identities, which are

$$\nabla_\alpha G^\alpha_\beta = 0 \quad (1.13)$$

which is equivalent to saying, using (1.2), that

$$\nabla_\alpha T^\alpha_\beta = 0 \quad (1.14)$$

These identities show that if the constraints are satisfied on the initial slice, then they are consistent with the evolution equations at all future times. So we can use the constraints to construct our initial value formulation, and then evolve the data off of the initial slice via the evolution equations.

1.5 General Degrees of Freedom

In general, $G_{\alpha\beta}$ has 10 degrees of freedom associated with it because it is a symmetric, 2nd rank, 4 dimensional tensor;

$$G_{\alpha\beta} = \begin{pmatrix} G_{00} & G_{01} & G_{02} & G_{03} \\ G_{01} & G_{11} & G_{12} & G_{13} \\ G_{02} & G_{12} & G_{22} & G_{23} \\ G_{03} & G_{13} & G_{23} & G_{33} \end{pmatrix} \quad (1.15)$$

Four of those degrees of freedom are specified by the constraint equations (1.10) and (1.11); 4 are specified by the coordinate, or gauge choices that we are free to make, leaving us with 2 real degrees of freedom for our system that are manifested in our initial choices for the spacetime geometry.

Just as in Maxwellian electrodynamics, these two degrees of freedom are associated with the two polarisation states of the gravitational field (i.e. gravitational

radiation/waves).

Chapter 2

Coordinate Systems and Gauge Determination

2.1 Coordinate Choice and the Form of the Metric

Once we have derived the general form of the 3+1 equations, it is necessary to start imposing simplifying conditions on the spacetime in order to solve it. The first simplification to make is that we specify an axi-symmetric spacetime, i.e. one in which there is rotational symmetry about the z-axis and reflective symmetry about the equatorial plane ($\theta = \frac{\pi}{2}$)¹. We use the convention that the angle θ is measured from the positive z-axis, the radial coordinate is measured from the origin and the angle ϕ becomes irrelevant due to the symmetry of the coordinate system.

The general line element for an axi-symmetric system is

$$dl^2 = A(t, r, \theta)dr^2 + B(t, r, \theta)d\theta^2 + C(t, r, \theta)drd\theta + D(t, r, \theta)\sin^2\theta d\phi^2 \quad (2.1)$$

¹See figure (2.1)

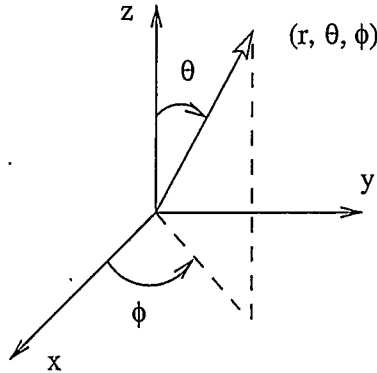


Figure 2.1: Our axi-symmetric coordinate system

as all of the cross-terms involving ϕ vanish due to the symmetries of the spacetime. Following the work of York ([16],[17]), we re-write the metric in terms of a conformally related metric ($\hat{\gamma}_{ab}$) with a conformal factor, ψ , such that

$$\gamma_{ab} = \psi^4 \hat{\gamma}_{ab}$$

in order to create a formulation that is easier to solve analytically and more numerically stable. It can be shown [11] that the Ricci scalar is related to the conformal Ricci scalar ($\hat{R}$) by

$$R = \hat{R}\psi^{-4} - 8\psi^{-5}\hat{\gamma}^{ij}\hat{\nabla}_i\hat{\nabla}_j\psi \quad (2.2)$$

and if $\hat{\gamma}_{ab}$ is a *flat* metric, then $\hat{R} = 0$, simplifying the analysis of the spacetime substantially. We do not, however, choose a conformally *flat* metric for this analysis, but a conformal decomposition is useful nonetheless as a method of determining various aspects of the spacetime, including the mass in Section (4.3.1). This particular choice of ψ also ensures that the Hamiltonian constraint on the initial slice is a second order linear differential equation in ψ , as opposed to most other choices which yield difficult-to-solve, non-linear PDEs [3].

We define the radial coordinate η in terms of r by

$$r = f(\eta)$$

where

$$f(0) = 0$$

to set the origin of our coordinate system at $\eta = 0$. This rescaling of the radial coordinate allows us to experiment with various radial grid spacings when we start working with the physical problem, and allows for easy manipulation of the equations². By defining and deriving all of our equations in terms of this general radial coordinate function, f , the analytics are vastly simplified if we wish to try a different radial function. In the case of a gravitational wave close to the origin, it is preferable to have a large number of grid points closer to the origin to allow for better resolution of the wave and its dynamics, whereas further out along the radial direction where the spacetime approximates flat space and the dynamics are mostly linear we should need less grid points for proper resolution of the physical features of the spacetime. Thus we can designate various radial functions and experiment with these radial functions to provide better stability with fewer grid points (and thus less computation time and storage space).

Using these two definitions, we can re-write the line element in terms of the conformal factor (ψ) and the radial coordinate function (f), giving us

$$dl^2 = \psi^4 (\hat{A} f_{,\eta}^2 d\eta^2 + \hat{B} f^2 d\theta^2 + \hat{C} f f_{,\eta} d\eta d\theta + \hat{D} f^2 \sin^2 \theta d\phi^2) \quad (2.3)$$

Thus we can define our metric using the above conditions, and it becomes

$$g_{ab} = \psi^4 \begin{pmatrix} a f_{,\eta}^2 & c f f_{,\eta} & 0 \\ c f f_{,\eta} & b f^2 & 0 \\ 0 & 0 & d f^2 \sin^2 \theta \end{pmatrix} \quad (2.4)$$

²The general coordinate invariance of General Relativity ensures that this is a valid coordinate choice

where $a = \hat{A}$, $b = \hat{B}$, $c = \hat{C}$ and $d = \hat{D}$

2.2 The Extrinsic Curvature Tensor

In formulating the extrinsic curvature (3)-tensor, we first use the axi-symmetric conditions to eliminate all of the ϕ cross-terms, i.e. $K_{13} = 0$ and $K_{23} = 0$. The symmetric nature of the tensor means that there are only 4 components left to specify, giving the general form for the extrinsic curvature tensor as³:

$$K_{ab} = \psi^4 \begin{pmatrix} H_a & H_c & 0 \\ H_c & H_b & 0 \\ 0 & 0 & H_d \end{pmatrix} \quad (2.5)$$

Where we have used a similar conformal decomposition of the extrinsic curvature to that used for the metric, i.e.

$$K_{ab} = \psi^4 \hat{K}_{ab} \quad (2.6)$$

2.3 Temporal Gauge Conditions

Since we are free to choose our temporal coordinates as we wish, we will investigate some particular choices of the temporal gauge that allow us to explore the portions of the spacetime that we are interested in.

³The actual form used in deriving the equations was slightly different and more complex, because the eventual goal is to use this formalism to perform the evolution of the space time. In that formalism the mixed curvature quantities are used (i.e. K_b^a) which necessitates a different projection formalism, but results in an equivalent covariant curvature tensor, K_{ab}

2.3.1 Maximal Slicing

As the name of this particular gauge indicates, we can specify that

$$\text{Tr}(K) = K = K_a^a = 0, \quad (2.7)$$

which equates to maximising the volume of each time slice. One advantage of this gauge is that it will avoid areas of high curvature in order to maximise these volumes (high curvature means larger surface area of a slice). Another advantage is that the time-evolution equations simplify quite nicely with the use of this gauge, and using (1.9) we can obtain an equation for the lapse that becomes:

$$\nabla^a \nabla_a \alpha = \alpha \left[R + \frac{1}{2}(S - 3\rho) \right] \quad (2.8)$$

Because we are dealing with a vacuum spacetime, we can drop the source terms to obtain

$$\nabla^a \nabla_a \alpha = \alpha R \quad (2.9)$$

It can be noted that as $R \rightarrow \infty$, $\alpha \rightarrow 0$, giving the required large curvature-avoiding property.

Furthermore, we can re-write our Hamiltonian constraint (1.10) as

$$R - K^{ij}K_{ij} = 0 \quad (2.10)$$

where we have substituted (2.7) into (1.10).

2.3.2 Other Slicing Methods

Our choice of Maximal Slicing is not exclusive, and we can choose other methods of slicing our spacetime.

Geodesic slicing is defined by choosing $\alpha = 1$, and if we choose $\beta_a = 0$, then the coordinates of our spacetime will follow geodesics (i.e. the coordinates are in freefall). This method of slicing is not very useful for studying the dynamical properties of the spacetime, but it can be used as a check on the code given some easily calculable analytic properties that it possesses.

Polar slicing imposes the condition that

$$Tr K = K_1^1 \quad (2.11)$$

i.e.

$$K_2^2 + K_3^3 = 0 \quad (2.12)$$

and it gives a parabolic equation for α instead of the elliptic equation in (2.8), thus making it a faster method to solve for α . It does not slice inside the event horizon however, which is arguably a small flaw due to the fact that the spacetime inside an event horizon is causally disconnected from the universe outside the event horizon. Maximal slicing, however, will slice inside the event horizon and can therefore be used to study the dynamics of the interior of a black hole. It can be shown that maximal slicing is also singularity-avoiding in nature, and will stop the evolution within a fixed “distance” of the singularity. Therefore a different slicing method must be used if the entire interior of the black hole is to be studied, and an analysis

of various slicing methods and their properties can be found in [10].

2.4 Spatial Gauge Conditions

After we have chosen our temporal gauge/slicing condition, we have 3 spatial gauge choices that are left from the general coordinate equivalence of General Relativity.

In choosing our metric function, we can specify that $a = b$ in (2.4), and this is called the Isothermal gauge. It is useful in decreasing the number of dynamical metric quantities that must be solved for on each time slice, and it also allows us to develop the Robin boundary condition on the outer boundary of our spacetime. It also happens that an Isothermal gauge is consistent with our definition of an initial Brill wave (see Section (3.3)), and is our first spatial gauge choice.

We also impose the diagonal metric condition so that $c = 0$, which once again will simplify the resulting evolution equations and has been used successfully in other codes [3] [15], and is also consistent with the Brill wave formalism. Its numerical success is partially due to the fact that it decreases the coordinate shear in the spacetime by making the metric diagonal in nature.

Our last gauge choice is to set $d = 1$, such that dynamical information is carried in ψ and as such ψ must be solved for on each time slice of the evolution. This choice is also consistent with the Brill formalism and completes our gauge choices.

Thus the final form of our 3-metric is:

$$\gamma_{ab} = \psi^4 \begin{pmatrix} af_{,n}^2 & 0 & 0 \\ 0 & af^2 & 0 \\ 0 & 0 & f^2 \sin^2 \theta \end{pmatrix} \quad (2.13)$$

Other temporal/spatial gauge choices are discussed in [14].

Chapter 3

Setting Up the Initial Value Problem

3.1 Time-Symmetric Initial Data

Now that we have the proper forms of the metric and extrinsic curvature tensors ((2.13) and (2.5)) for the problem at hand, we can go about solving for the necessary variables to complete our picture of the initial spacetime slice (using our constraint equations, 1.10 and 1.11). We will choose to study a time-symmetric initial slice, meaning that our initial slice is a local isometry surface and has $\frac{\partial}{\partial t}$ as a local Killing vector. This also means that the time derivatives of our metric and extrinsic curvature quantities momentarily vanish.

3.1.1 The Lapse and Shift Functions (α and β^a) on the Initial Slice

Due to the freedom in General Relativity to choose coordinates however we wish, we are allowed to specify the lapse and shift functions freely on the initial and subsequent slices. They can be used as controls on the evolution rate and grid stretching at each point in the spacetime during the evolution to help ensure numerical stability and regularity. But because we are only interested in the Initial Value Problem here, and because these two functions are completely arbitrary¹, we can specify them as we wish on the initial slice. Our first simplifying choice is to set $\alpha = 1$, meaning that each point in the spacetime is progressing at the same rate, and that the coordinate

¹In this formulation they must satisfy certain differential relationships (due to the metric evolution equations and gauge choices) on future time slices

time progression is momentarily that of proper time at each point on the manifold. We then choose $\beta_a = 0$, making the coordinate “velocity” disappear on the initial time slice. These are the easiest choices we can make for these variables on the initial slice to simplify the solution of the IVP.

It is important to note that this choice of variables provides us with only one specific initial slice, and other values of α and β_a are permitted. These values, however, provide the simplest framework for future work on the time-evolution.

3.1.2 The Extrinsic Curvature Variables (H_a, H_b, H_c and H_d) on the Initial Slice

In this analysis, because we are restricting our investigation to time-symmetric data, our initial slice is a local isometry surface (with $\frac{\partial}{\partial t}$ as the local Killing vector). Due to the fact that the metric time derivatives momentarily vanish, and because the shift vector’s components are all defined to be zero on the initial slice, the evolution equations for the metric change from:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \partial_i \beta_j - \Gamma_{ji}^l \beta_l + \partial_j \beta_i - \Gamma_{ij}^l \beta_l \quad (3.1)$$

to

$$K_{ij} = 0 \quad (3.2)$$

Thus all of our extrinsic curvature components vanish on the initial slice (i.e. $H_a = H_b = H_c = H_d = 0$). Because Equation (1.11) is trivially satisfied by these conditions in a vacuum spacetime (i.e. the momentum constraints); the only remaining constraint equation that needs to be satisfied on the initial slice is the Hamiltonian

constraint.

3.2 Calculating ψ from the Hamiltonian Constraint

Using the definition for the metric that arises from (2.13) and the extrinsic curvature definition from (2.5), we can calculate the Hamiltonian Constraint from (1.10). Because we are using a time-symmetric set of initial data,

$$K_{ij} = 0$$

Therefore we can rewrite Equation (1.10) as

$$R = 0 \tag{3.3}$$

using the fact that $\rho = 0$ in a vacuum.

Using the definition of R as a contraction on R^a_{bcd} , we can calculate the form of the Hamiltonian constraint that is a differential equation in terms of only metric quantities, giving us the following:

$$\begin{aligned} 0 = & -\frac{8\frac{\partial\psi}{\partial\theta}\cos\theta}{af^2\psi^5\sin\theta} - \frac{8\frac{\partial^2\psi}{\partial\theta^2}}{af^2\psi^5} - \frac{8\frac{\partial^2\psi}{\partial\eta^2}}{af_{,\eta}^2\psi^5} + \frac{8f_{,\eta\eta}\frac{\partial\psi}{\partial\eta}}{af_{,\eta}^3\psi^5} - \frac{16\frac{\partial\psi}{\partial\eta}}{aff_{,\eta}\psi^5} + \frac{\frac{\partial a}{\partial\eta}f_{,\eta\eta}}{a^2f_{,\eta}^3\psi^4} \\ & - \frac{\frac{\partial a}{\partial\eta}}{a^2ff_{,\eta}\psi^4} - \frac{\frac{\partial^2 a}{\partial\eta^2}}{a^2f_{,\eta}^2\psi^4} + \frac{\left(\frac{\partial a}{\partial\eta}\right)^2}{a^3f_{,\eta}^2\psi^4} - \frac{\frac{\partial^2 a}{\partial\theta^2}}{a^2f^2\psi^4} + \frac{\left(\frac{\partial a}{\partial\theta}\right)^2}{a^3f^2\psi^4} \end{aligned} \tag{3.4}$$

Assuming that we choose a and f as our freely specified data on the initial slice²,

²To set up a Brill wave (see Section 3.3) and for coordinate optimisation. The function ψ thusly becomes our constrained data

we can simplify this equation into a second order, linear differential equation in ψ :

$$\begin{aligned} & \psi_{,\eta\eta} + \frac{f_{,\eta}^2}{f^2} \psi_{,\theta\theta} + \left(-\frac{f_{,\eta\eta}}{f_{,\eta}} + \frac{2f_{,\eta}}{f} \right) \psi_{,\eta} + \left(\frac{f_{,\eta}^2}{f^2} \right) \cot \theta \psi_{,\theta} \\ & - \frac{\psi}{8} \left[-\frac{a_{,\eta\eta}}{a} + \frac{a_{,\eta}}{a} \left[\frac{f_{,\eta\eta}}{f_{,\eta}} - \frac{f_{,\eta}}{f} \right] + \left(\frac{a_{,\eta}}{a} \right)^2 - \frac{a_{,\theta\theta}}{a} \frac{f_{,\eta}^2}{f^2} + \left(\frac{a_{,\theta}}{a} \right)^2 \frac{f_{,\eta}^2}{f^2} \right] = 0 \quad (3.5) \end{aligned}$$

So once we have the functions f and a specified, as well as their derivatives, we can proceed to solve (3.5) for ψ . The stability of the method used to solve ψ is highly dependent on the boundary conditions, the function $h(\eta)$ (the initial radial Brill wave profile)³ and the accuracy of the derivatives of f and a . Due to the importance of an accurate and precise solution for the evolution of the spacetime (the eventual aim of this work), we will devote an entire chapter to a study of these solutions, after first developing the form of a .

3.3 Brill Wave Construction

In order to determine our remaining metric variable, a , we shall construct our initial spacetime with a gravitational wave that follows the formulation of Brill. Brill showed [4] that in an asymptotically flat, axi-symmetric spacetime where the metric is specified in spherical polar coordinates the following are equivalent:

- The mass of the hyper-surface is non-negative (i.e. well-defined and physical)
- The line element describing the space time has the form

$$dl^2 = \psi^4 \left[e^{2q} (dr^2 + r^2 d\theta^2) + r^2 \sin^2 \theta d\phi^2 \right] \quad (3.6)$$

³See Section (3.3)

where $q = q(r, \theta)$ and

1. $q(r, 0) = 0$
2. $\lim_{r \rightarrow \infty} q(r, \theta) = O(r^2)$

Therefore the goal is to choose a coordinate system and initial conditions to fulfill these criteria. Because of our desire to scale the radial coordinate for more optimal grid resolution, we introduce once again the radial coordinate function, f , giving our line element the form of

$$dl^2 = \psi^4 \left[e^{2q} (f_{,\eta}^2 d\eta^2 + f^2 d\theta^2) + \sin^2 \theta f^2 d\phi^2 \right] \quad (3.7)$$

In order to ensure that the two properties of $q(r, \theta)$ are satisfied, we choose a form⁴ for q such that

$$q(\eta, \theta) = \xi g(\theta) h(\eta) \quad (3.8)$$

where ξ is the “amplitude” of the Brill wave, and the functions g and h carry the angular and radial information respectively. Thus we can easily satisfy Brill’s first condition by setting

$$g(\theta) = \sin^n \theta \quad (3.9)$$

where n is some even integer to preserve the equatorial reflection symmetry properties that we required earlier⁵. The second condition can be satisfied by using any number of radial functions, and as we will show in Chapter 4 some choices are more numerically stable than others. The choice of our function for $h(\eta)$ is not trivial, but

⁴This choice is in no way exclusive

⁵For the remainder of this thesis, $n = 2$ is used

an investigation into its form shall be left until later.

So comparing (2.3) and (3.7), we can make the identification that $\hat{A} = \hat{B} = e^{2q}$, $\hat{C} = 0$, and $\hat{D} = 1$, and our metric (2.13) becomes:

$$\gamma_{ab} = \psi^4 \begin{pmatrix} e^{2q} f_{,\eta}^2 & 0 & 0 \\ 0 & e^{2q} f^2 & 0 \\ 0 & 0 & f^2 \sin^2 \theta \end{pmatrix} \quad (3.10)$$

which is consistent with our gauge choices from Section (2.4).

3.4 Determining the Metric Variables a and ψ

As was discussed we will be using the time-symmetric Brill formalism to define our metric variables on the initial slice. The final form of the metric comes from (3.10) where

$$q = \xi \sin^n \theta h(\eta) \quad (3.11)$$

and requires at least 3 input parameters, but usually more.

Firstly, we must specify ξ , the “amplitude” of the Brill wave, which determines the relative scaling of the wave for a given choice of h . Secondly, we must specify the integer power, n , of the sine function to be some even number to preserve the reflective symmetry of the spacetime. And the last variables are inputs for the function $h(\eta)$, that usually determine the “starting position” of the wave (i.e. the global maximum of the metric function) and its spatial extent (although this is easily deconvolvable only in the case of a Gaussian wave). It is also possible to require a wavenumber, a frequency, and various other parameters that determine the initial

shape and radial dependence of the wave.

And once we have used these parameters to completely specify a , we can solve for our last metric variable, ψ using equation (3.5).

3.5 Boundary Values and Numerical Methods

In solving the IVP, we really only have to specify the input parameters for the conformal metric function a to determine the shape of the wave, and then solve for the conformal factor, ψ , in order to have the complete description of the spacetime on the initial slice. The next chapter will deal with various input parameters for the function $h(\eta)$ and the solutions they produce, but a note about solving (3.5) should be made first. Due to the fact that it is a linear, second-order differential equation (with non-constant coefficients), we have many numerical methods available to solve this particular equation. As an analytic solution to the equations is not, in general, available, the only way to determine the conformal factor is via some numerical method. Appendix 1 gives some more details of the numerical methods that are used, but the two main schemas that were tried were a Relaxation scheme and a BiConjugate Gradient (BiCG) scheme. The Relaxation scheme tends, in general, to be slower and computationally more expensive, but is more stable and reliable than the BiCG scheme. In this case, however, the results for ψ that were produced by both schemes turned out to be the same to within the numerical accuracy of the machines, leaving us with the obvious choice of the BiCG method for its speed⁶.

In order to properly solve the elliptic differential equation in (3.5), we must

⁶Although some regularity problems occur near the origin

specify boundary conditions that are consistent with our symmetries, thus we set the metric function and the conformal factor to be symmetric along the axis ($\theta = 0$) to preserve the rotational symmetry, symmetric along the equator ($\theta = \frac{\pi}{2}$) to preserve the reflective symmetry and symmetric at the origin ($r = 0$) to preserve regularity. i.e.

$$\left. \frac{\partial g_{11}}{\partial \theta} \right|_{\theta=0} = \left. \frac{\partial g_{11}}{\partial \theta} \right|_{\theta=\frac{\pi}{2}} = \left. \frac{\partial g_{11}}{\partial \eta} \right|_{\eta=0} = 0$$

and

$$\left. \frac{\partial \psi}{\partial \theta} \right|_{\theta=0} = \left. \frac{\partial \psi}{\partial \theta} \right|_{\theta=\frac{\pi}{2}} = \left. \frac{\partial \psi}{\partial \eta} \right|_{\eta=0} = 0$$

Furthermore, we require asymptotic flatness at the outer edge of the grid (so that the wave has “died away” sufficiently) giving

$$g_{11}|_{\eta=\eta_{max}} \simeq 1$$

by equating our metric to the Minkowski flat-space metric there.

Our outer boundary condition for ψ is not quite as easy, but we require that the solution be Schwarzschild - like at the outer boundary, and we impose the Robin boundary condition:

$$\psi_{,\eta} \frac{f}{f_{,\eta}} + \psi - 1 = 0 \quad (3.12)$$

3.5.1 The Robin Boundary Condition

The Robin boundary condition arises from requiring the spacetime to be Schwarzschild-like at the outer boundary, and the condition can be found by writing the

spatial part of the Schwarzschild metric in conformally flat, isotropic coordinates:

$$\gamma_{ab} = \left(1 + \frac{m}{2\rho}\right)^4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2 \sin^2 \theta \end{pmatrix} \quad (3.13)$$

Because we require our solution to be asymptotically flat, we can match Equation (2.13) to Equation (3.13) at the outer boundary. Making the identification that $\rho = f(\eta)$, we find that

$$\psi = 1 + \frac{m}{2f} \quad (3.14)$$

along the outer boundary. We can then take the η derivatives of (3.14) and eliminate the mass term m (which is otherwise arbitrary), to obtain the Robin boundary condition, (3.12).

We will discuss the solutions that were generated using this formulation in the next chapter, and a further discussion of the numerical methods that were employed in this thesis are presented in Appendix A.

Chapter 4

Solutions to the IVP

4.1 Choosing $h(\eta)$ to determine g_{11}

Initially, a symmetric Gaussian wave was used for the function $h(\eta)$ ¹, making

$$h(\eta) = e^{-\frac{(\eta-\eta_0)^2}{\sigma^2}} + e^{-\frac{(\eta+\eta_0)^2}{\sigma^2}} \quad (4.1)$$

And it is seen that the wave has its peaks at $\pm\eta_0$, and has a Gaussian width of σ . The problem with this type of wave is not in the Initial Value Problem, as it is quite possible to solve for the conformal factor using this type of formulation, but problems crop up in the subsequent evolution because of its properties. Due to the fact that

$$\lim_{\eta \rightarrow 0} h(\eta) \neq 0 \quad (4.2)$$

the Gaussian wave does not die off fast enough at the origin to ensure numerical regularity. Because the Taylor expansions of various metric and curvature quantities have a θ dependence at the origin², and our metric is not going to zero, this can cause numerical shears in the spacetime as the evolution progresses.

So in order to ensure that we are eliminating any angular dependence at the origin, we should choose a different function that identically goes to zero at the

¹In the original code from which this code was developed, a Gaussian Brill wave was used, making this a somewhat historical choice

²See [15]

origin. To this end we choose a Lorentzian-type function, where

$$h(\eta) = \frac{f(\eta)^k}{\left(1 + \frac{f(\eta)}{\eta_0}\right)^l} \quad (4.3)$$

and k , l and η_0 are shape parameters of the wave that we are free to specify. It is important to note that this choice of $h(\eta)$ is identically $= 0$ at the origin, so it can be made to die off fast enough to cancel the angular problems. The effect of each of the three parameters on the shape of the wave is not simple to qualify, but increasing k generally moves the “inside” of the wave (closest to $\eta = 0$) further from the origin, increasing l moves the “outside” of the wave (furthest from $\eta = 0$) further from the origin, and increasing η_0 moves the peak of the wave away from the origin. Furthermore, we must impose the condition that $k + 2 \leq l$, in order that the wave has asymptotic $\frac{1}{r^2}$ -type behaviour. Thus our goal is to choose k and l in such a manner that the wave dies rapidly near the origin (larger k), dies rapidly at the edge of the grid to ensure asymptotic flatness (larger l), but is still wide enough that we can resolve the wave properly with a reasonable grid resolution.

4.2 Choosing the Shape Parameters k , l and η_0

Figures (4.1) and (4.2) contain some graphs of various combinations of (η_0, k, l) that were experimented with to produce a numerically stable Brill Wave. Theoretically, any values of these three shape parameters are allowed, but some will produce more numerically stable codes than others.

From figure (4.1) we can see that using values of k and l that are too small

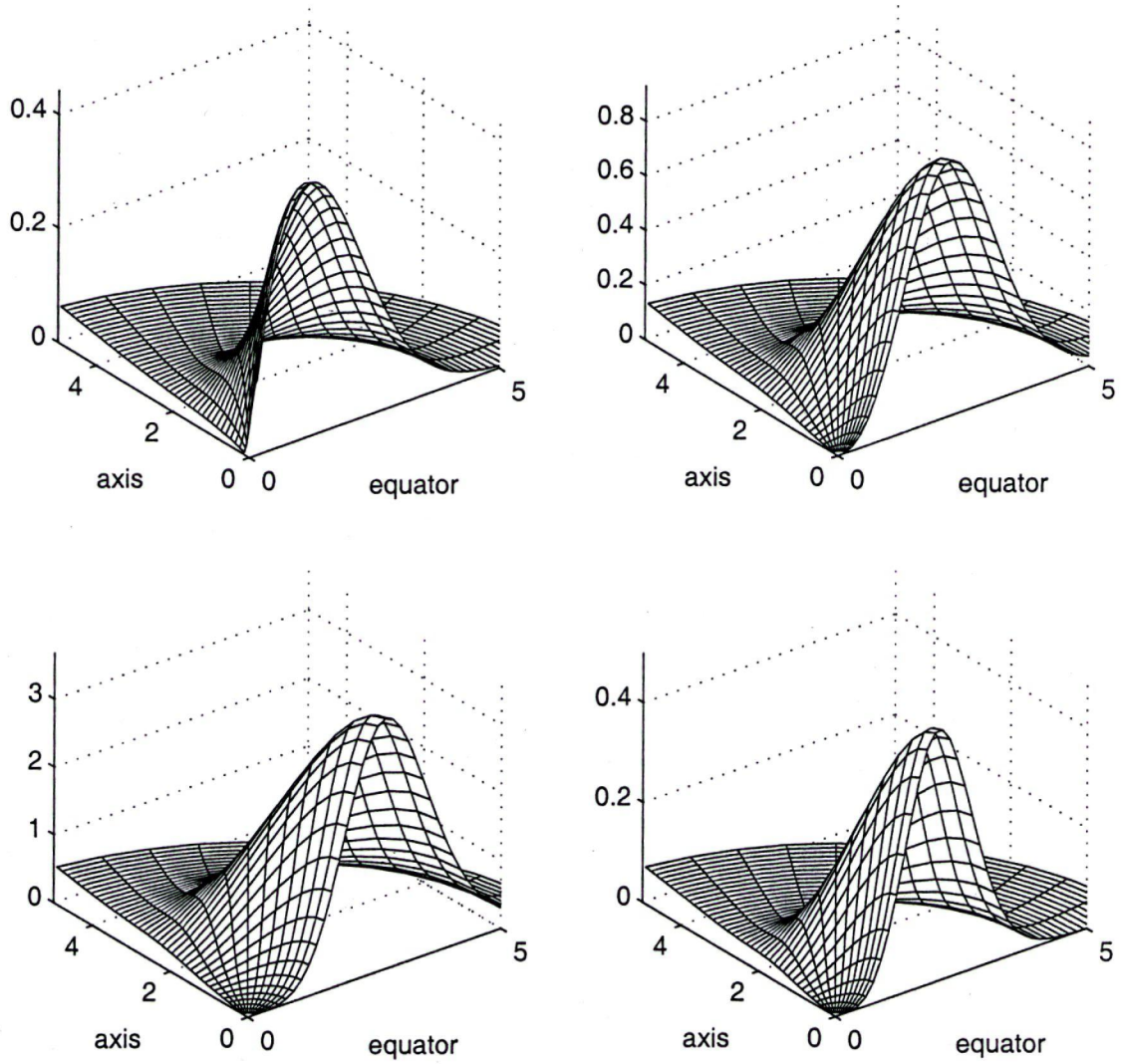


Figure 4.1: All data sets have $\eta_0 = 3$, amplitude $\xi = 1$ and $q(r, \theta)$ on the z -axis. On the top left is $k = 1, l = 3$; the top right is $k = 3$ and $l = 5$; the bottom left is $k = 5, l = 7$ and the bottom right is $k = 5, l = 9$. Radial distances are in η units (arbitrary). Note the difference in vertical scales.

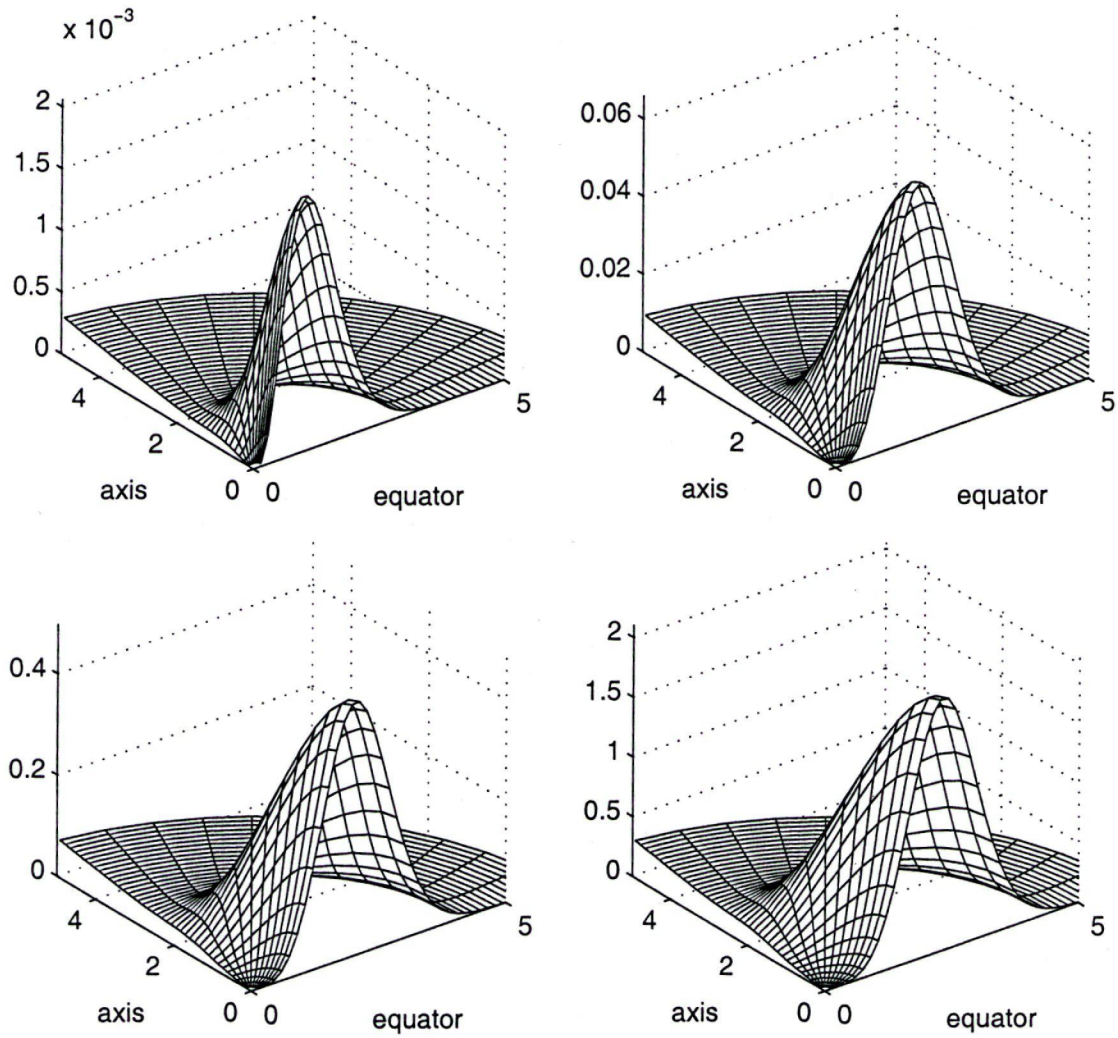


Figure 4.2: All data sets have amplitude $\xi = 1$, $k = 5$, $l = 9$ and $q(r, \theta)$ on the z -axis. On the top left is $\eta_0 = 1$; the top right is $\eta_0 = 2$; the bottom left is $\eta_0 = 3$ and the bottom right is $\eta_0 = 4$. Note the difference in vertical scales.

result in a steep wave near the origin (especially around $\theta = \frac{\pi}{2}$), negating our goal of minimising the angular dependence close to $\eta = 0$. Choosing larger values for k (while still keeping $k + 2 \leq l$) pushes the wave outwards from the origin somewhat, while adding large amounts of energy to it³. Furthermore, it can be seen that increasing k such that $k + 2 = l$ increases the maximum value of the peak of the wave substantially (bottom left), so we increase l further to push the outside of the wave back towards the origin (bottom right), leading to the choices of $k = 5$ and $l = 9$.

Figure (4.2) shows the effects of increasing η_0 , and we can see that around $\eta_0 = 3$ that we have a wave that dies sufficiently at the origin without having a large amplitude. It is important to remember that we are actually using e^{2q} when doing our computations, so keeping $2q_{max} \simeq 1$ is desirable to prevent the wave from getting too distorted. This leads to the choice of $\eta_0 = 3$ for our future calculations.

4.3 Comparing IVP Solutions

Now that the shape parameters have been determined, we are ready to solve the Initial Value Problem for ψ , (3.5). As was mentioned in Section (3.5) and is discussed in Appendix A, two separate numerical algorithms were used to solve the IVP for ψ ; a BiConjugate Gradient method (BICG) and a relaxation method. A comparison of the CPU processing times for the two methods, with variable grid sizes, is given in Table (4.1). It is interesting to note that the maximum value of the Hamiltonian constraint (which theoretically should be zero) actually *increases* as you add more grid points. The reason for this is that when more grid points are added, $\Delta\eta$ becomes

³A wave with the same amplitude that starts further from the origin occupies a larger proper volume and thus has more energy

Number of grid points	BICG		Relaxation	
	CPU time	max Ham constraint	CPU time	max Ham constraint
100×13	.12 s	1.09e-8	2.3 s	3.98e-9
150×18	.30 s	1.22e-7	9.7 s	1.69e-8
200×24	.60 s	5.93e-7	30.6 s	4.32e-9
250×30	1.1 s	1.91e-7	82.0 s	1.07e-7
300×35	1.6 s	5.34e-6	146.2 s	2.27e-7

Table 4.1: CPU processing time required to solve Equation (3.5) and the maximum value of the Hamiltonian constraint as a function of grid resolution for the BICG and Relaxation schemes, using $\eta_0 = 3$, $\xi = 1.5$, $k = 5$, $l = 9$

smaller and our first grid point is closer to the singularity at $\eta = 0$.

And as can be seen in Figure (4.3), the Hamiltonian constraint is small over most of the grid, with the $\frac{1}{r}$ -type errors dominating near the origin of the coordinate system. When we remove that area of the grid, as in Figure (4.4), we see that the errors are “random” errors that appear on the scale of the specified accuracy. So an increase in the maximum value of the Hamiltonian constraint is expected as $\Delta\eta \rightarrow 0$.

Thus we can see from Table (4.1) that the BICG method gives larger error values in the Hamiltonian constraint⁴, but it converges on a much quicker time scale (> 10 times faster). In solving the evolution equations, *each time step* will require solving equation (3.5) 2 to 10 times, meaning that the BICG method is certainly preferable, if numerical stability can be achieved⁵.

⁴Both methods were pushed to the limit of their accuracy for these tests

⁵Furthermore, elliptic equations for α , β_η , β_θ , H_a and H_b must be solved for on each time slice - meaning that at least 12 calls to the elliptic equation solver are required per time slice... definitely indicating that the BICG method is preferable

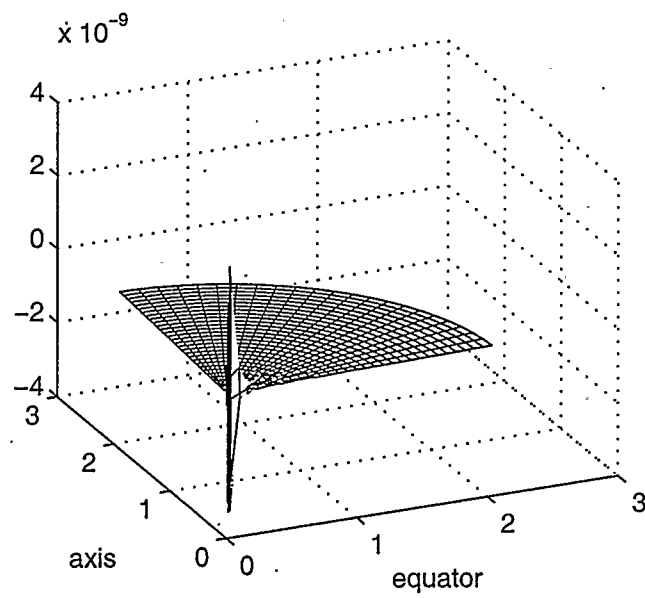


Figure 4.3: The Hamiltonian constraint as calculated using the BICG method. The spike near the origin is due to $\frac{1}{r}$ -type errors.

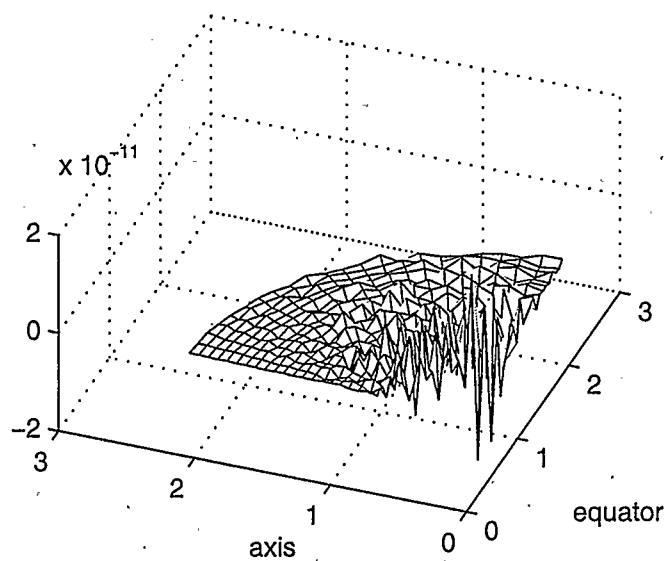


Figure 4.4: The Hamiltonian constraint as calculated using the BICG method, cutting out the area near the origin. Errors are “random” and on the order of the specified accuracy.

4.3.1 Comparative Calculations of the ADM Mass, Mass Aspect and Apparent Horizon Masses

We can define the mass of the spacetime in various ways, but we adopt the usage of two separate mass measurements here, the ADM⁶ mass and the mass aspect⁷. The ADM mass is defined as

$$m_{ADM} = -2f \int_0^\pi \left(\frac{\psi^3}{2 + \frac{\psi_{f,\eta}}{\psi_{,\eta}f}} \right) \sin \theta d\theta \quad (4.4)$$

for our axi-symmetric system, and it is a quasi-local approximation (taken at the outer edge of the grid) to the true ADM mass which should be measured at $r = \infty$. It is a measure of the Hamiltonian “energy” of the system⁸, provided that our metric falls off faster than $\frac{1}{r}$ (which is guaranteed by the Brill criterion), and can be seen to converge to a consistent value as the outer edge of the grid is moved further away.

Our second measure of the “energy” of the system is the mass aspect, which is defined as

$$m = \frac{\psi^2 f}{2} \left(\frac{2\psi_{,\eta}f}{\psi f_{,\eta}} + \frac{a_{,\eta}}{2a} \right) \left(\frac{2\psi_{,\eta}f}{\psi f_{,\eta}} + \frac{a_{,\eta}}{2a} + 2 \right) \quad (4.5)$$

in our axi-symmetric system, and arises from equating g_{11} of our metric to that of the Schwarzschild metric. The mass aspect is measured at the outer edge of the grid, and as we are comparing it to the spherically-symmetric Schwarzschild solution, we require that our angular deviation has died away sufficiently by that point. Theoretically, the ADM mass and the mass aspect should be the same, and

⁶Named after the original formulators: Arnowitt, Deser and Misner

⁷See [15] for more details

⁸Energy is not a well-defined quantity in general relativity

we can check to make sure that our spacetime is sufficiently “flat” at the outer edge by measuring these quantities to see if they agree.

Another physical quantity of interest when studying these spacetimes is apparent horizons. The presence of an apparent horizon⁹ in our spacetime means that an event horizon (i.e. a black hole) is also present in the spacetime, and the event horizon either coincides with or appears outside of the apparent horizon. Event horizons cannot be specifically located with numerical simulations due to the fact that they cannot be locally defined (i.e. they cannot be tested for at any one point during the evolution - the whole spacetime is needed to test for event horizons), but we can use apparent horizons to test for their presence as the presence of an apparent horizon necessitates the presence of an event horizon. Thus if our spacetime has sufficient energy initially, it will have an apparent horizon in it and we can test to see if the apparent horizon solver (from the original code) is working properly.

Table (4.2) shows the measured apparent horizon mass (m_{AH}), ADM mass and mass aspect as a function of the outer radial distance, η_{max} . We can see that an outer distance of $\eta_{max} = 14$ gives very close agreement between the ADM mass and mass aspect, and the apparent horizon mass seems to be converging as well¹⁰. If we move the inner boundary in too far, we do not have an asymptotically flat spacetime at the outer edge of the grid, and as such we still have “energy” in the spacetime that is not accounted for. If we move the outer boundary out too far, we would have to introduce more grid points into the spacetime to cover the inner regions properly¹¹,

⁹The condition to test for an apparent horizon at a point in the spacetime is that outgoing orthogonal null geodesics have zero convergence - a local condition. An event horizon is a global condition arising from the inside of the event horizon being causally disconnected from the outside.

¹⁰Using $r = \sinh(\eta)$, when $\eta = 14$, $r \simeq 6e5$

¹¹As is indicated by the horizon solver’s inability to find an apparent horizon at a larger η_{max}

Maximum Outer Radial Distance (η_{max})	m_{AH}	ADM mass	mass aspect	max Ham constraint
5.0	—	52.92	22.70	8.59e-2
6.0	—	40.38	25.09	4.29e-4
7.0	—	30.34	25.21	4.30e-5
8.0	24.27	27.03	25.24	5.68e-5
9.0	24.21	25.88	25.23	8.56e-6
10.0	24.12	25.47	25.24	1.49e-6
11.0	24.02	25.31	25.24	3.38e-6
12.0	23.90	25.25	25.23	2.97e-6
13.0	23.81	25.22	25.22	1.69e-5
14.0	23.63	25.20	25.21	6.24e-7
15.0	23.62	25.19	25.20	7.97e-7
16.0	—	25.17	25.18	1.26e-7
17.0	—	25.14	25.16	1.84e-7

Table 4.2: apparent horizon mass (m_{AH}), ADM mass, mass aspect and the maximum value of the Hamiltonian constraint as a function of outer radial grid distance (using 200 grid points, $\xi = 14.5$, BICG method, and otherwise the same parameters as Table (4.1).)

ϵ	max Ham constraint	CPU time
1e-10	1.72e-2	.53s
1e-11	2.60e-5	.56s
1e-12	1.59e-5	.58s
1e-13	2.05e-6	.59s
1e-14	1.62e-6	.57s
1e-15	6.45e-7	.53s
1e-16	5.87e-7	.56s
1e-17	5.93e-7	.59s
1e-18	5.94e-7	.61s
1e-19	5.94e-7	.57s
1e-20	5.94e-7	.58s
1e-21	5.94e-7	.62s
1e-22	5.94e-7	.62s

Table 4.3: Maximum Hamiltonian constraint and processing time as a function of the BICG routine error tolerance, ϵ .

and we would be introducing superfluous grid points/calculation time in a region that is almost flat.

It should be noted that the apparent horizon mass is required to be smaller than the mass of the entire space-time (for obvious reasons), and this can be used as a simple check on the solver.

4.3.2 Determining the BICG Solver Tolerance, ϵ

Another variable that affects computational time and numerical stability is the precision to which we require the BICG solver to solve the matrix equation, (3.5). Table (4.3) gives the computational time required for the BICG solver to reach the specified precision in the solution, ϵ . As we are using real numbers that are defined to 16

decimal places in the code¹², we would expect that a tolerance of $\simeq 10^{-16}$ should be the limit of the capability of the solver to solve the problem, and Table (4.3) verifies our prediction.

4.3.3 Calculating Derivatives of g_{11}

Throughout most of this thesis, a second-order finite differencing schema was sufficient to solve the matrix problems that were presented. There is one problem, however, that is most likely linked to the boundary value problem for ψ , that necessitates more accurate methods. As is outlined in Section (3.5), we use Neumann boundary conditions at $r = 0$, $\theta = 0$ and $\theta = \frac{\pi}{2}$. On the outer boundary, however, we use the Robin boundary condition (3.12), which is a mixture of a Dirichlet and a Neumann boundary condition. There are theorems stating the uniqueness of the solution to a general second-order linear differential equation when using strictly Dirichlet boundary conditions, strictly Neumann boundary conditions or strictly Robin boundary conditions. There are not¹³, however, any theorems stating the uniqueness of the solution when mixed boundary conditions are applied as we have here, leading to a potential uniqueness problem.

This led to the discovery of different (non-physical) solutions¹⁴ when second-order derivative terms were used to calculate¹⁵ $g_{11,\eta}$, $g_{11,\theta}$, $g_{11,\eta\eta}$ and $g_{11,\theta\theta}$. This, in turn, necessitated the use of exact derivatives for these quantities, which are (using a

¹²A 64-bit real number

¹³As far as the author is aware

¹⁴i.e. ones with negative ADM masses

¹⁵4th-order correct terms were also tried with similar effects

Lorentzian wave):

$$g_{11,\eta} = \chi e^{2q} \left[\frac{k f_{,\eta}}{f} - \frac{l f_{,\eta}}{\Lambda \eta_0} \right] \quad (4.6)$$

$$\begin{aligned} g_{11,\eta\eta} = \chi e^{2q} & \left[\frac{\chi k^2 f_{,\eta}^2}{f^2} - \frac{2\chi k l f_{,\eta}^2}{f \Lambda \eta_0} + \frac{\chi l^2 f_{,\eta}^2}{\Lambda^2 (\eta_0)^2} + \frac{k f_{,\eta\eta}}{f} - \frac{l f_{,\eta\eta}}{\Lambda \eta_0} \right. \\ & \left. + \frac{k^2 f_{,\eta}^2}{f^2} - \frac{k f_{,\eta}^2}{f^2} - \frac{2k l f^{(-1)} f_{,\eta}^2}{\Lambda \eta_0} + \frac{l^2 f_{,\eta}^2}{\Lambda^2 (\eta_0)^2} + \frac{l f_{,\eta}^2}{\Lambda^2 (\eta_0)^2} \right] \end{aligned} \quad (4.7)$$

$$g_{11,\theta} = \chi e^{2q} n \cot \theta \quad (4.8)$$

and

$$g_{11,\theta\theta} = \chi e^{2q} \left[\cot^2 \theta (\chi n^2 + n^2 - n) - n \right] \quad (4.9)$$

where

$$\Lambda = 1 + \frac{f(\eta)}{\eta_0}$$

and

$$\chi = \frac{2\xi f^k \sin^n \theta}{\Lambda^l}$$

Figures (4.5), (4.6) and (4.7) show the difference between the analytic derivatives and the second order-ones which, in the case of $g_{11,\eta\eta}$ and $g_{11,\theta\theta}$ were quite large.

Using these analytically correct derivatives, it was possible to get a physically meaningful solution for ψ , and an example solution for ψ is shown in Figure (4.8). It is

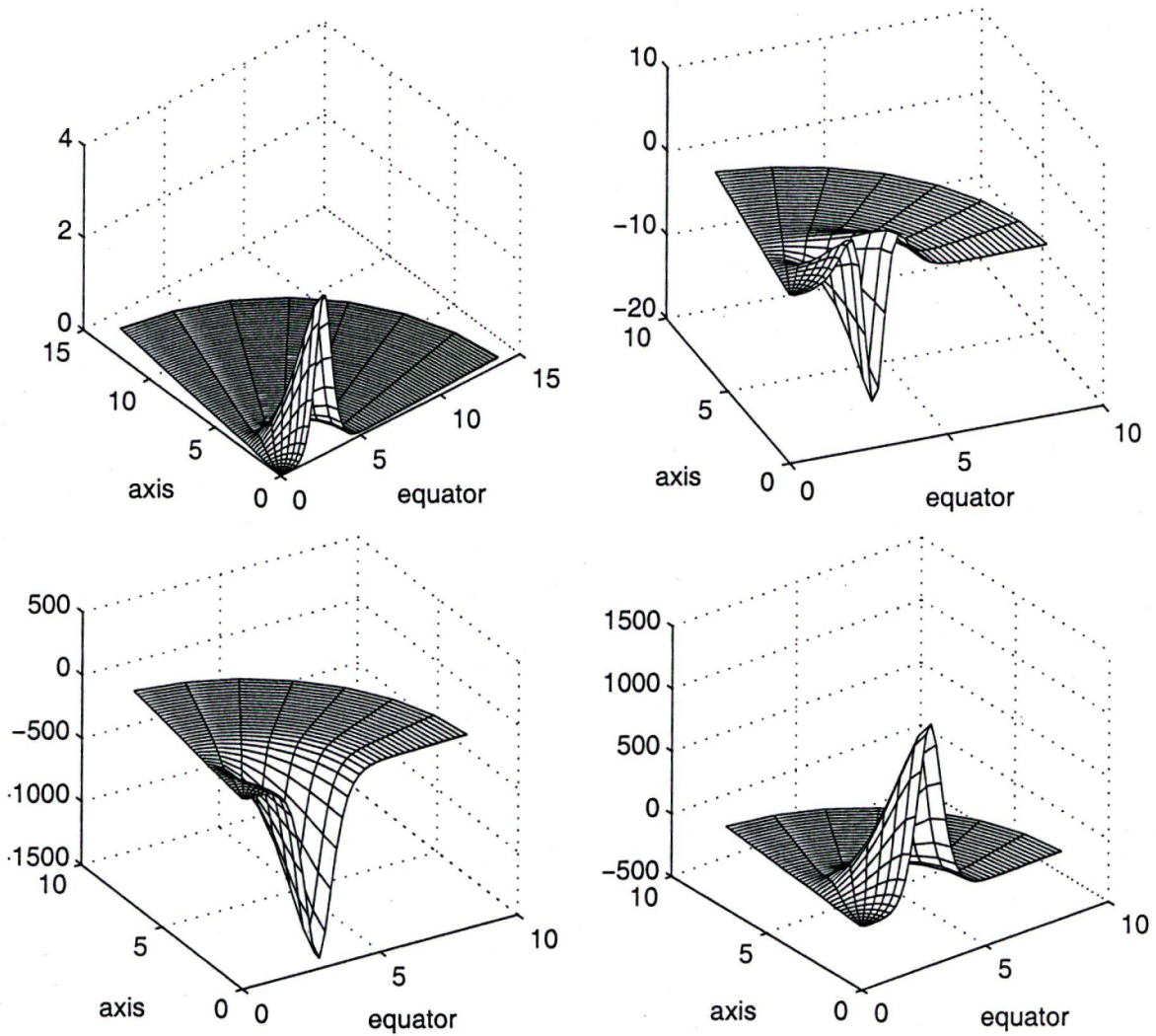


Figure 4.5: Graphs of g_{11} and its derivatives. Top left is g_{11} over the entire grid; top right is $g_{11,\eta\eta}$ calculated using the analytic derivative; bottom left is $g_{11,\eta\eta}$ calculated using a second-order correct method; bottom right is the difference between the two methods of calculating $g_{11,\eta\eta}$ (which is quite large).

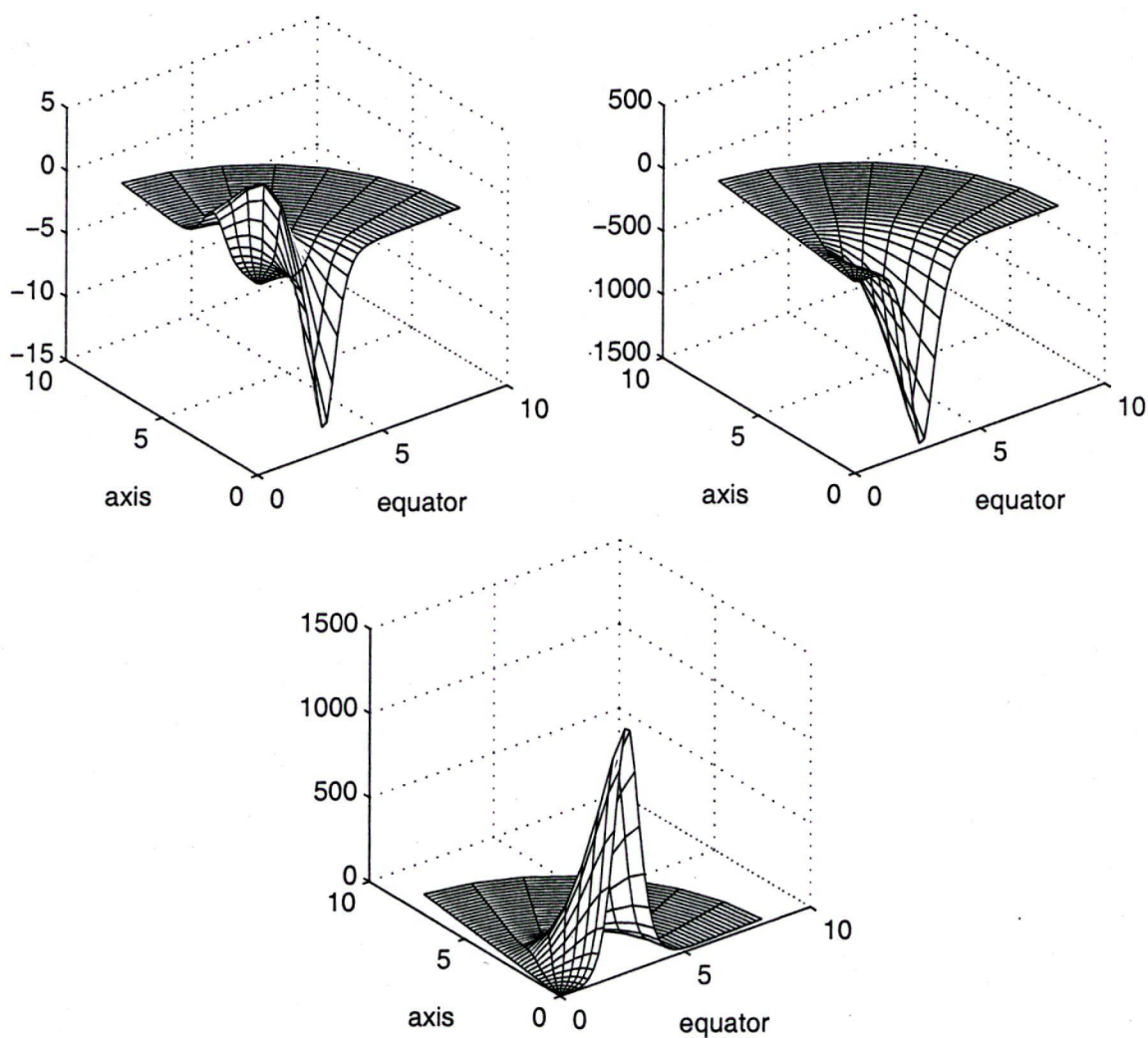


Figure 4.6: Top left is $g_{11,\theta\theta}$ calculated using the analytic derivatives; top right is $g_{11,\theta\theta}$ calculated using a second-order correct method; bottom is the difference between the two top graphs (which is also quite large).

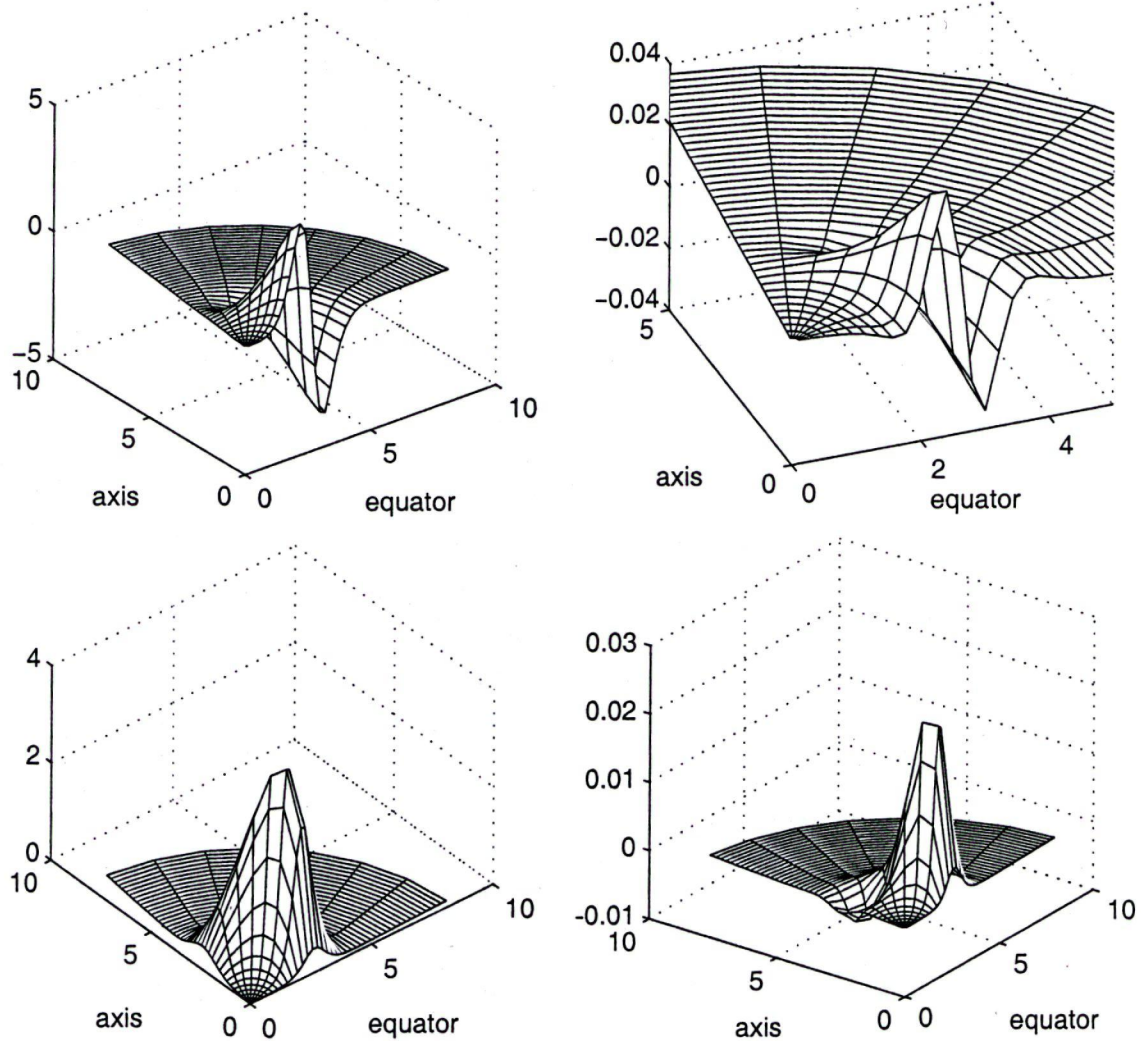


Figure 4.7: Top left is $g_{11,\eta}$ calculated using the analytic derivatives; top right is the difference between the analytic and second-order derivatives for $g_{11,\eta}$ (which is not too large); bottom left is the analytic derivative for $g_{11,\theta}$; bottom right is the difference between the analytic and second order-methods for $g_{11,\theta}$ (also not too large).

uncertain if the non-physical solutions were a result of the incorrect derivative terms or the boundary conditions, but it is an issue that warrants future investigation.

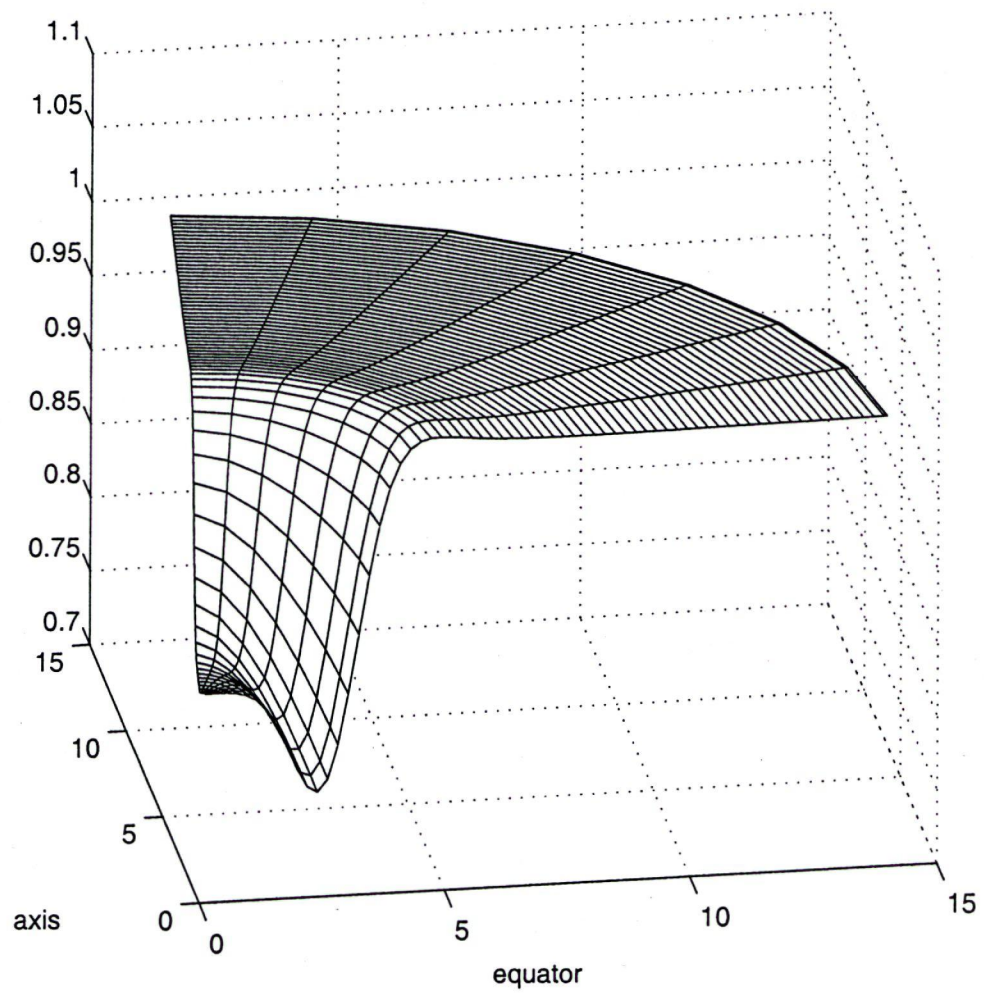


Figure 4.8: An IVP solution for ψ , with $(\eta_0 = 3, k = 5, l = 9, \xi = 1.5)$.

4.4 Conclusion

In the course of this research, we have successfully constructed and implemented a framework for solving the Initial Value Problem in an axi-symmetric, vacuum Brill wave spacetime. Also, a computer code has been created that can be used for future work on the time evolution of the spacetime to look for apparent horizon formation and to study critical phenomenon [8] in axi-symmetric gravitational wave collapse. Some numerical difficulties were encountered during the formulation of the IVP, and the methods outlined in this thesis seem to have been sufficient to solve them; however future work on the evolution equations is the only way to tell if they will stand up under more continuous usage with varying conditions. Due to the highly non-linear, self-interacting and amplifying nature of the evolution equations, it may be necessary to use the more time-consuming Relaxation method to solve (3.5), but the faster BICG method seems sufficient for the time being.

Furthermore, optimal values for the various input parameters for the grid resolution, outer grid location, solver tolerance, wave shape and wave function were determined to provide minimal error balanced with minimal computational time.

Future work on this problem could involve investigation of the uniqueness problem given our mixed boundary conditions, development of a faster relaxation scheme, and further conditioning of (3.5) to help remove the $\frac{1}{r}$ -type errors that appear near the origin, all of which could yield profitable results and insights into the IVP.

Appendix A

Numerical Methods

In solving the Initial Value Problem for ψ , a number of different numerical techniques had to be employed in order to stabilise the solution to the point of solvability. This appendix will cover the major techniques that were used to ensure numerical accuracy and stability.

A.1 Coordinate Choice

Due to the presence of coordinate singularities along $\theta = 0$ (the axis) and $\eta = 0$, we must pay particular attention to those regions of the grid. Our choice of coordinate discretisation can help us to avoid this problem, as we can define our coordinate grid in such a way so as to “straddle” the singularities (and, in fact, this seems sufficient to avoid most numerical problems). This straddling is accomplished by defining our first radial grid point at a $1/2$ step away from the origin, and all subsequent points at integral steps away from each other. Similarly, we define the first θ grid point at a $1/2$ step away from the axis of our system, and continue in integral steps from there. So in the radial direction, if we define our radial grid spacing to be $\Delta\eta$, then our grid has radial nodes located at

$$-\frac{\Delta\eta}{2}, \frac{\Delta\eta}{2}, \frac{3\Delta\eta}{2}, \frac{5\Delta\eta}{2} \dots$$

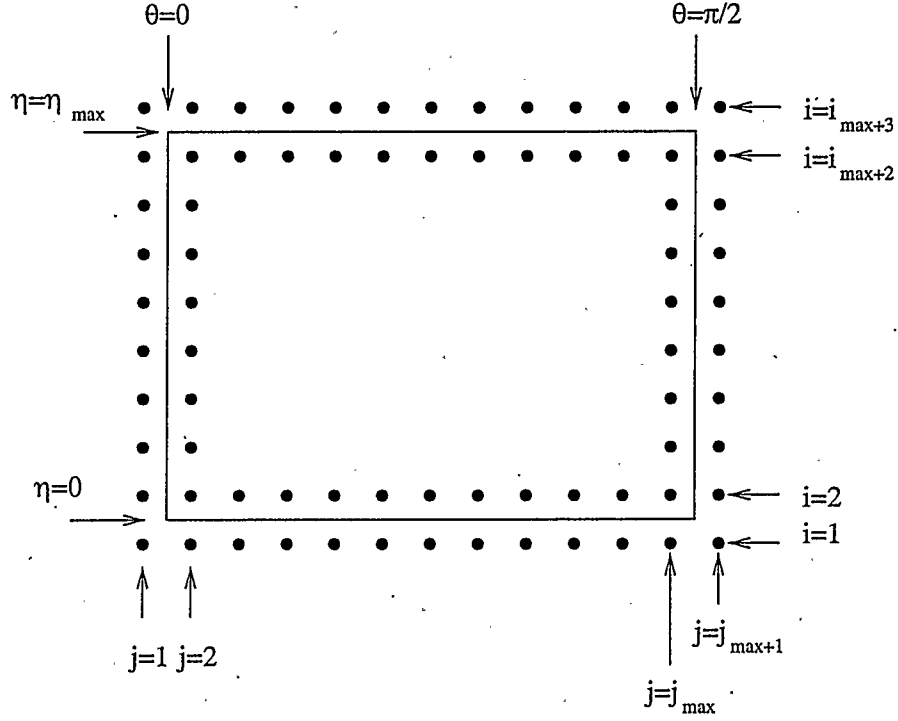


Figure A.1: A schematic of the 2-dimensional discretised grid. Grid points are located at $((i - \frac{3}{2})\Delta\eta, (j - \frac{3}{2})\Delta\theta)$, where $1 \leq i \leq i_{max} + 3$ and $1 \leq j \leq j_{max} + 1$

and defining $\Delta\theta$ as our angular grid spacing, our grid has angular nodes located at

$$-\frac{\Delta\theta}{2}, \frac{\Delta\theta}{2}, \frac{3\Delta\theta}{2}, \frac{5\Delta\theta}{2} \dots$$

in the θ direction. We include the $-\frac{1}{2}$ step as well in order to keep track of our derivative information at $i, j = \frac{1}{2}$, as the second order finite differencing about those points requires the $-\frac{1}{2}$ grid point. Figure (A.1) gives a schematic of what our discretised grid looks like.

Furthermore, $f(\eta)$ is another coordinate option that must be chosen, and we choose

$$f(\eta) = \sinh(\eta) \tag{A.1}$$

throughout this thesis to achieve exponential scaling over the entire grid. This will give a better coordinate resolution near the wave (which is what we are interested in) and the origin (which is numerically problematic), while using fewer grid points in the asymptotically flat region near the outer edge of the grid.

A.2 Discretisation

Throughout this research, we have used a discretisation of the $t = 0$ spatial slice in order to solve for the conformal factor, ψ , numerically. However, in order to take derivatives of the functions that are defined on a discrete (non-continuous) grid, we must make approximations to within some defined tolerance level. For all of the work here, we used methods that are accurate up to second order in whatever quantity we wish to differentiate, so the errors are always on the order of $(\Delta\eta)^2$ in the radial direction or $(\Delta\theta)^2$ in the angular direction. In order to derive the formulae for these derivatives, we shall start with a second order Taylor series approximation about η for a general function, $f(\eta)$:

$$f(\eta + \Delta\eta) \simeq f(\eta) + f_\eta(\eta)\Delta\eta + f_{\eta\eta}(\eta)\frac{\Delta\eta^2}{2!} \quad (\text{A.2})$$

giving also

$$f(\eta - \Delta\eta) \simeq f(\eta) - f_\eta(\eta)\Delta\eta + f_{\eta\eta}(\eta)\frac{\Delta\eta^2}{2!} \quad (\text{A.3})$$

Assuming that we are finite differencing about the point (i, j) , and noting that

$$f(\eta) = f(i, j) \quad (\text{A.4})$$

$$f(\eta + \Delta\eta) = f(i + 1, j) \quad (\text{A.5})$$

$$f(\eta - \Delta\eta) = f(i - 1, j) \quad (\text{A.6})$$

we can subtract (A.2) and (A.3) and solve for the first derivative of f , yielding

$$f_{\eta(i,j)} = \frac{f_{i+1,j} - f_{i-1,j}}{2\Delta\eta} + O[(\Delta\eta)^2] \quad (\text{A.7})$$

while adding (A.2) and (A.3) and solving for the second derivative of f yields

$$f_{\eta\eta(i,j)} = \frac{f_{i+1,j} - 2f_{i,j} + f_{i-1,j}}{(\Delta\eta)^2} + O[(\Delta\eta)^2] \quad (\text{A.8})$$

which are the second order finite differencing approximations for f_{η} and $f_{\eta\eta}$. We can apply these equations, without loss of generality, to derivatives in the θ direction.

A.3 Setting up the Second-Order Differential Equations to Solve Them Numerically

In order to solve the second order equation for ψ (3.5), we must discretise the equation from its continuous form. To do this, we write out the general form of (3.5) as

$$P(\eta, \theta) \psi_{,\eta\eta} + Q(\eta, \theta) \psi_{,\theta\theta} + R(\eta, \theta) \psi_{,\eta} + S(\eta, \theta) \psi_{,\theta} + T(\eta, \theta) \psi = Z(\eta, \theta). \quad (\text{A.9})$$

We then use (A.7) and (A.8) to discretise all of our derivative terms and reduce the differential equation to an algebraic equation for the ψ_i 's. After rearrangement, we

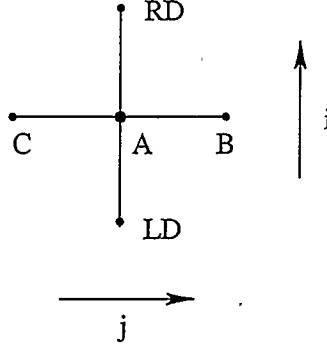


Figure A.2: A schematic of the 2-dimensional stencil from Equation (A.11).

find that

$$\begin{aligned} & \left(\frac{P}{(\Delta\eta)^2} + \frac{R}{2\Delta\eta} \right) \psi_{i+1,j} + \left(\frac{P}{(\Delta\eta)^2} - \frac{R}{2\Delta\eta} \right) \psi_{i-1,j} + \\ & \left(-\frac{2P}{(\Delta\eta)^2} - \frac{2Q}{(\Delta\theta)^2} + T \right) \psi_{i,j} + \left(\frac{Q}{(\Delta\theta)^2} + \frac{S}{2\Delta\theta} \right) \psi_{i,j+1} + \\ & \left(\frac{Q}{(\Delta\theta)^2} - \frac{S}{2\Delta\theta} \right) \psi_{i,j-1} = Z_{i,j}. \end{aligned} \quad (\text{A.10})$$

which can be written schematically as

$$RD_{i,j} \psi_{i+1,j} + LD_{i,j} \psi_{i-1,j} + A_{i,j} \psi_{i,j} + B_{i,j} \psi_{i,j+1} + C_{i,j} \psi_{i,j-1} = Z_{i,j} \quad (\text{A.11})$$

and is shown in Figure (A.2).

This gives us a cross-shaped stencil around each point on the grid, which can then be converted into a matrix equation to be solved for at each grid point. Because the matrix created by this algorithm is very large¹ and it is very sparse², we can adapt

¹The matrix is $nm \times nm$ where n and m are the number of radial and angular grid divisions, usually around 200 and 25 respectively, giving a $5,000 \times 5,000$ matrix to be solved

²Usually only 5 of the mn entries on each row are occupied by non-zero entries, i.e. about 5 out

our numerical algorithm to solve this sort of problem efficiently.

Ideally, we would even like to take advantage of the sparseness of the matrix, which we do by specifying a multiplication routine that requires the least amount of operations possible for the specific form of the matrix in Figure (A.3). Furthermore, because the matrix is so large, direct inversion methods are impractical as the numerical noise would quickly flood out any real information in the matrix. Also, as is mentioned in Chapter 4, similar elliptic equations must be solved at each time step during the evolution, making direct inversion techniques also impractical for future work due to their prohibitive computational time. So our preferred method to solve the matrix in Figure (A.3) is via matrix multiplication and addition only, with no row or column operations necessary. To this end, we used two different algorithms: a “stabilised” Bi-Conjugate Gradient (BiCGStab) routine and a relaxation scheme, both of which provided similar answers to within the specified numerical precision. The major advantage of one method over the other was processing time, as the relaxation method slowly “smooths” out the data over the entire grid in an easy-to-program but inefficient manner, whereas the BiCGStab routine uses information about the derivatives of the matrix quantities to converge more quickly.

Both methods were used as checks on each other, and Figure (A.4) shows the conformal factor, ψ , calculated using the two different methods on equation (3.5) with a Lorentzian Brill wave.

of 5,000. See figure (A.3) for an idea of what the matrix looks like - empty spaces are 0 entries.

$$\Sigma \Psi = Z$$

where

$$\Sigma = \begin{bmatrix} a_{11} & b_{11} & \dots & rd_{11} & & & \\ c_{12} & a_{12} & b_{12} & & rd_{12} & & \\ & c_{13} & a_{13} & b_{13} & & rd_{13} & \\ & & \ddots & \ddots & \ddots & & \\ ld_{21} & & & c_{21} & a_{21} & b_{21} & rd_{21} \\ & \ddots & & & \ddots & & \\ & & ld_{m-1,n} & & c_{m-1,n} & a_{m-1,n} & b_{m-1,n} & rd_{m-1,n} \\ & & & \ddots & & \ddots & & \\ & & & ld_{m,n-1} & & c_{m,n-1} & a_{m,n-1} & b_{m,n-1} \\ & & & & ld_{m,n} & \dots & c_{m,n} & a_{m,n} \end{bmatrix}$$

$$\Psi = \begin{bmatrix} \psi_{11} \\ \psi_{12} \\ \psi_{13} \\ \vdots \\ \psi_{21} \\ \vdots \\ \psi_{m-1,n} \\ \vdots \\ \psi_{m,n-1} \\ \psi_{m,n} \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} Z_{11} \\ Z_{12} \\ Z_{13} \\ \vdots \\ Z_{21} \\ \vdots \\ Z_{m-1,n} \\ \vdots \\ Z_{m,n-1} \\ Z_{m,n} \end{bmatrix}$$

Figure A.3: A schematic of the matrix equation that needs to be solved for $\psi(\eta, \theta)$ in Equation (A.11)

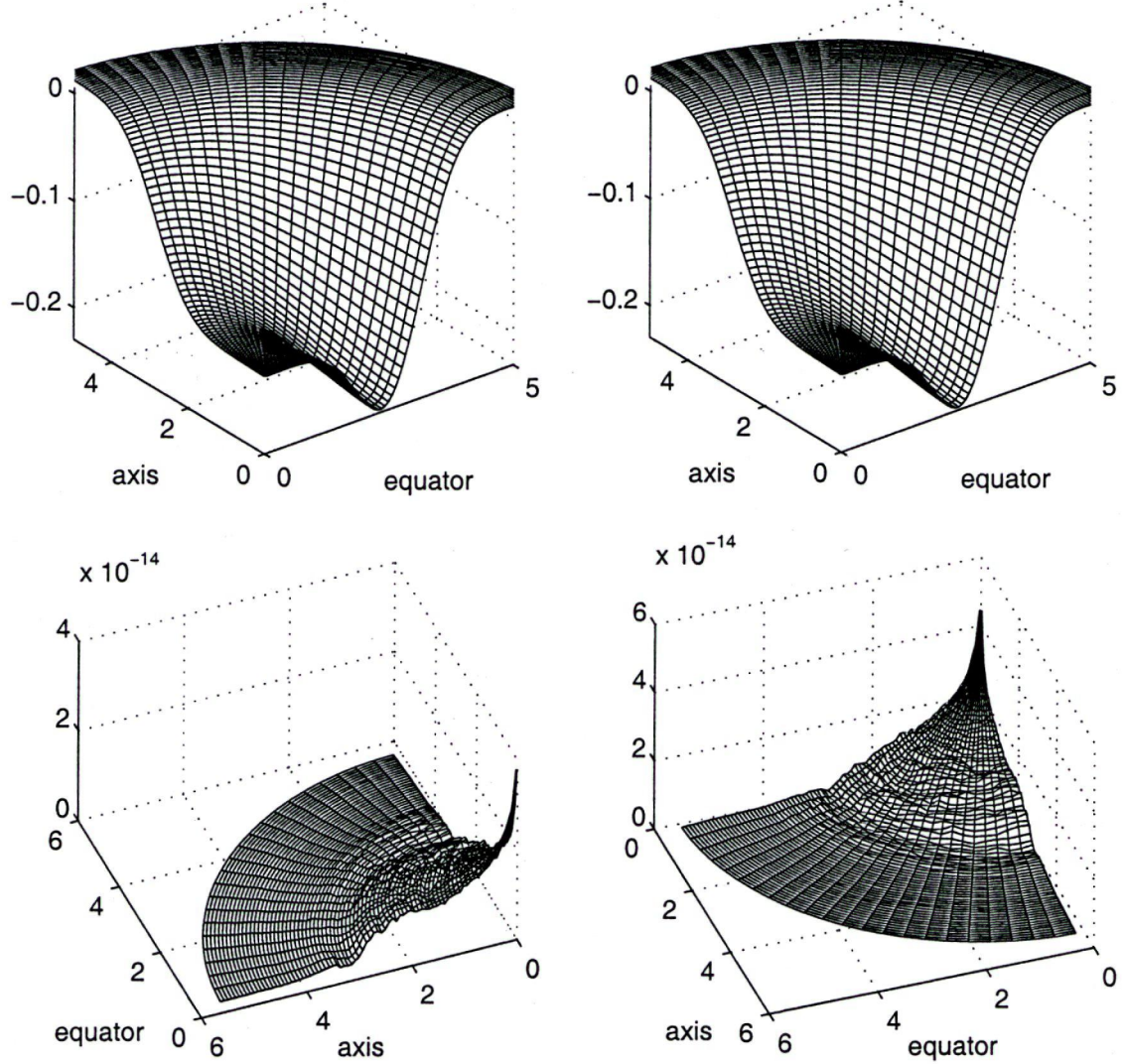


Figure A.4: A comparison of the BICG and Relaxation methods. The top left is Equation (3.5) solved for $\psi - 1$ using the BICG method, and the top right uses the relaxation method. Both were solved using a Lorentzian data set of $(\eta_0 = 3, k = 5, l = 9, \xi = 1.5)$. The bottom left shows the absolute difference between the two solutions from the top, and the bottom right shows the relative difference between the two solutions. $\psi - 1$ was solved for instead of ψ , due to the fact that the solution lies very close to 1, and solving for $\psi - 1$ instead helps to remove subtraction errors.

A.4 Boundary Conditions

When defining the boundary conditions for ψ at $\eta = 0$, we know that³

$$\left. \frac{\partial \psi}{\partial \eta} \right|_{\eta=0} = 0$$

meaning that in our discretised model that we set

$$\psi\left(\frac{-\Delta\eta}{2}, \theta\right) \equiv \psi_{1,j} = \psi_{2,j} \equiv \psi\left(\frac{\Delta\eta}{2}, \theta\right) \quad (\text{A.12})$$

to make the function symmetric about $\eta = 0$. Similarly,

$$\left. \frac{\partial \psi}{\partial \theta} \right|_{\theta=0} = 0$$

meaning that

$$\psi\left(\eta, \frac{-\Delta\theta}{2}\right) \equiv \psi_{i,1} = \psi_{i,2} \equiv \psi\left(\eta, \frac{\Delta\theta}{2}\right) \quad (\text{A.13})$$

and

$$\left. \frac{\partial \psi}{\partial \theta} \right|_{\theta=\frac{\pi}{2}} = 0$$

gives

$$\psi\left(\eta, \frac{\pi}{2} - \frac{\Delta\theta}{2}\right) \equiv \psi_{i,j_{max}} = \psi_{i,j_{max}+1} \equiv \psi\left(\eta, \frac{\pi}{2} + \frac{\Delta\theta}{2}\right). \quad (\text{A.14})$$

Thus we have three of our four boundary conditions defined, and our outer boundary condition ($\eta = \eta_{max}$) comes from the Robin boundary condition (3.12). By

³See Section 3.5

discretising (3.12) for $\tilde{\psi} = \psi - 1$ ⁴, we get

$$\frac{\tilde{\psi}_{i+1,j} - \tilde{\psi}_{i-1,j}}{2\Delta\eta} \left(\frac{f}{f,\eta} \right) + \tilde{\psi}_{i,j} = 0 \quad (\text{A.15})$$

which means that on the outer boundary (assuming $imax + 3$ is the very outer grid point) that

$$\tilde{\psi}_{imax+3,j} = -\frac{2\Delta\eta f,\eta}{f} \tilde{\psi}_{imax+2,j} + \tilde{\psi}_{imax+1,j}. \quad (\text{A.16})$$

So using the boundary conditions in (A.12), (A.13), (A.14) and (A.16) to set up the matrix in Figure (A.3), and using equations (A.9) and (A.11), we have a fully defined matrix problem to be solved.

A.5 The Form of Equation (3.5)

Due to the fact that we are trying to minimise numerical errors in the code, it is to our advantage to re-write any equations we solve so that we are not dividing by small numbers. Recalling (3.5),

$$\begin{aligned} & \psi_{,\eta\eta} + \frac{f_{,\eta}^2}{f^2} \psi_{,\theta\theta} + \left(-\frac{f_{,\eta\eta}}{f,\eta} + \frac{2f_{,\eta}}{f} \right) \psi_{,\eta} + \left(\frac{f_{,\eta}^2}{f^2} \right) \cot \theta \psi_{,\theta} \\ & - \frac{\psi}{8} \left[-\frac{a_{,\eta\eta}}{a} + \frac{a_{,\eta}}{a} \left[\frac{f_{,\eta\eta}}{f,\eta} - \frac{f_{,\eta}}{f} \right] + \left(\frac{a_{,\eta}}{a} \right)^2 - \frac{a_{,\theta\theta}}{a} \frac{f_{,\eta}^2}{f^2} + \left(\frac{a_{,\theta}}{a} \right)^2 \frac{f_{,\eta}^2}{f^2} \right] = 0 \end{aligned} \quad (\text{A.17})$$

In equation (A.17), we have a $\cot \theta$ term, and we are dividing by f and $f_{,\eta}$, so

⁴This was done for numerical reasons; because the solution to ψ is close to 1, we transform to $\tilde{\psi}$ to avoid subtraction errors in the code

we should analyse their behaviour on the grid. By choosing

$$f = \sinh(\eta) \quad (\text{A.18})$$

we find that

$$f_{,\eta} = \cosh(\eta) \quad (\text{A.19})$$

And using Taylor series expansions around $\eta = 0$, we find that

$$f(\epsilon) \simeq \epsilon \quad (\text{A.20})$$

and

$$f_{,\eta}(\epsilon) \simeq 1 + \frac{\epsilon^2}{2!} \quad (\text{A.21})$$

meaning that $f(\eta)$ will be small near the origin - indicating that we should numerically condition our system to get rid of $\frac{1}{f}$ behaviour. The derivative, $f_{,\eta}$ will be close to 1 near the origin, so it is not problematic. As we proceed to the outer edge of the grid, both terms go as $\frac{e^\eta}{2}$, and as such are well-conditioned.

Furthermore,

$$\cot \theta = \frac{\cos \theta}{\sin \theta} \quad (\text{A.22})$$

meaning that near $\theta = 0$

$$\cot \bar{\epsilon} \simeq \frac{1 - \frac{\bar{\epsilon}^2}{2!}}{\bar{\epsilon}} \quad (\text{A.23})$$

and we should numerically condition the equations to remove any $\frac{1}{\sin \theta}$ type behavior near the axis.

Near $\theta = \frac{\pi}{2}$, $\cot \theta \simeq 0$ and numerical stability can be retained as long as we don't

divide by $\cot \theta$ or $\cos \theta$.

This suggests the equivalent, but the more numerically stable form of (A.17);

$$\begin{aligned} & \left(\frac{f^2}{f_{,\eta}^2} \right) \sin \theta \psi_{,\eta\eta} + \sin \theta \psi_{,\theta\theta} + \left(-\frac{f_{,\eta\eta} f^2}{f_{,\eta}^3} + \frac{2f}{f_{,\eta}} \right) \sin \theta \psi_{,\eta} + \cos \theta \psi_{,\theta} \\ & - \frac{\psi}{8} \sin \theta \left[-\frac{a_{,\eta\eta} f^2}{a f_{,\eta}^2} + \frac{a_{,\eta}}{a} \left(\frac{f_{,\eta\eta} f^2}{f_{,\eta}^3} - \frac{f}{f_{,\eta}} \right) + \left(\frac{a_{,\eta} f}{a f_{,\eta}} \right)^2 - \frac{a_{,\theta\theta}}{a} + \left(\frac{a_{,\theta}}{a} \right)^2 \right] \\ & = 0 \text{ (A.24)} \end{aligned}$$

This method of solving the IVP gives a noticeable improvement over (3.5), giving a maximum error in the Hamiltonian constraint of $3.4\text{e-}7$ instead of $5.9\text{e-}7$ for the Lorentzian data set ($\eta_0 = 3, k = 5, l = 9, \xi = 1.5$).

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