THE UNIVERSITY OF CALGARY

ZENO'S PARADOXES, THE INFINITE, AND WITTGENSTEIN

by

LEO GROARKE

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled, "Zeno's Paradoxes, The Infinite, and Wittgenstein" submitted by Leo A. Groarke in partial fulfillment of the requirements for the degree of Master of Arts.

John Heintz

Supervisor, John Heintz,
Department of Philosophy.

Brian J. Chellas

Brian Chellas,
Department of Philosophy.

Robert X. Ware

Robert X. Ware,
Department of Philosophy.

July 28, 1977.

ABSTRACT

One might ask why anyone should be concerned with paradoxes Zeno of Elea put forth 2,400 years ago. The conclusions of the paradoxes (for example, that motion is impossible) are so obviously false that it may seem the paradoxes are not worth consideration. Yet they are of enormous interest. For since the paradoxes were first put forth no one has clearly shown what is wrong with the arguments which lead to Zeno's paradoxical conclusions.

Such a situation invites a very sceptical attitude towards reason. For if it cannot be shown where arguments as obviously mistaken as Zeno's go wrong how can we hope to evaluate more important (and less obviously mistaken) arguments in philosophy, ethics, politics or science? The solution of Zeno's paradoxes gives us a valuable indication of how arguments may (and often do) go astray. Such information should interest anyone concerned with the business of reason.

This thesis deals with Zeno's five most important paradoxes (the paradoxes of motion and the paradox of extension). Standard solutions to the paradoxes (with the exception of Aristotle) have been shown inadequate. A different solution to the paradoxes is suggested. Revisions in modern day set theory have resulted from a consideration of the proposed set theoretic solution to the Achilles paradox. An understanding of the paradoxes has also been used as a basis for insight into Wittgenstein's view of philosophy.

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"The comparison with alchemy suggests itself. We might speak of a kind of alchemy in mathematics...

What is typical of the phenomena I am talking about is that a mysteriousness about some mathematical concept is not straightaway interpreted as an erroneous conception, as a mistake of ideas; but rather as something that is at any rate not to be despised, is perhaps even rather to be respected.

All that I can do, is to shew an easy escape from this obscurity and this glitter of concepts.

Strangely, it can be said that there is so to speak a solid core to all these glistening concept-formations. And I should like to say that that is what makes them into mathematical productions.

It might be said: what you see does of course look more like a gleaming Fata Morgana; but look at if from another quarter and you can see the solid body, which only looks like a gleam without a corporeal substrate when seen from that other direction...

'Mathematical logic' has completely deformed the thinking of mathematicians and of philosophers, by setting up a superficial interpretation of the forms of our everyday language as an analysis of the structures of facts...

The philosopher is the man who has to cure himself of many sicknesses of the understanding before he can arrive at the notions of the sound human understanding.

If in the midst of life we are in death, so in sanity we are surrounded by madnesses."

LUDWIG WITTGENSTEIN, Remarks On The Foundations Of Mathematics, IV: 16, 53

PART I

THE PARADOXES OF MOTION

Many years ago Zeno of Elea raised some questions concerning the possibility of motion. He presented arguments designed to show that motion was impossible: that any claim that motion had really taken place was self-contradictory. I don't believe that anyone holds this view today - which proves that some things eventually become evident, even to philosophers...

However, the difficulties Zeno raised were far from silly. They were grounded in legitimate problems concerning space and time, and, although what he claimed to have shown seems to be false, there is far from universal agreement on just what was wrong with his arguments. The debate has lasted these several thousand years. Most likely, it will last several thousand more - which proves that some things don't eventually become evident, even to philosophers.

PAUL BENACERRAF, _Tasks, Super-Tasks, and the Modern Eleatics_[1]

Chapter 1

The Dichotomy Paradox

1. Zeno's paradoxes of motion are significant to science, mathematics, and philosophy. Though they have convinced few that motion is paradoxical, they have been important to the development of modern theories of space, time, and the infinite. Since Zeno's arguments were first advanced, philosophers and mathematicians have continually tried to undermine them. At least in modern times, all have failed miserably.

2. The first paradox of motion, the Dichotomy, is built upon infinity. An infinite collection (set) or series contains an infinite number of elements, an endless number of elements. As Aristotle remarks, "generally, the infinite has this mode of existence: one thing is always taken after another".[2] In an infinite series things can be taken one after the other endlessly. The Dictionary of Philosophy defines infinity as:

> An endless extent of space, time, or any series. Is usually conceived negatively, as having no termination; may be conceived positively... as actually extending without end.[3]

The infinite is endless. It is on this that Zeno's arguments depend.

3. The Dichotomy paradox attempts to establish the impossibility of motion. For consider any motion. In order to be completed its first-half must first be completed. For the remaining half to be completed one-half of it (one-quarter the overall motion) must first be completed. For the remaining quarter of the motion to be completed, one-half of it (one-eighth the overall motion) must first be completed. As this halving procedure can be carried out endlessly an infinite series of smaller and smaller 'half motions' must be completed before the overall motion is finished. First the first half motion must be completed, then the second, then the third, and so on _ad infinitum_. Yet there are an infinite number of these half motions so they must be completed one after the other endlessly. The completion of any half motion in the series _always_ brings one to a new one which must be completed, and never to the completion of the entire motion. Hence the motion is incompletable. (So much for the transportation industry.)

In the Dichotomy we define an infinite series of successive parts of any motion _m_. The first element of the series is the first half of the motion, 1/2m. Every subsequent part of the motion is produced by halving the size of the part that proceeds it. The series 1/2m, 1/4m, 1/8m, 1/16m,... results. The series is endless. No matter how small a member is, it can always be halved to produce a new element in the series. It seems to follow that the motion _m_ is incompletable. For the completion of _m_ does require the completion of all motion segments within the series (first the first half of the motion, then the next quarter, and so on). And the completion of any segment in the series always brings one to a new part of the motion within the series. As there is no last motion segment in the series, the completion of any part of the motion within it never brings one to the completion of all the parts in the series and the consequent completion of the overall motion. In a similar way the counting of all natural numbers could never be completed. For no matter how many numbers in the series "1,2,3,..." one enumerates, further numbers always remain, for the series is endless. The series is infinite.

4. The Dichotomy paradox can be formulated in different ways, and in regard to any continuum. Consider for example line ab in figure (i). 1,2,3, and 4 mark the first four points of division designated by Zeno's halving procedure when it is applied to ab.

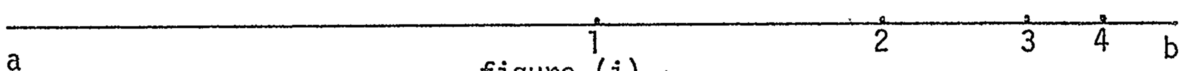

figure (i)

Again this method of division can be applied _ad infinitum_, and therefore individuates an infinite series of ab parts. A series in which the parts could be numbered 1,2,3,... . Paradox results. Because the line contains an infinite number of parts it seems incompletable. Thus, suppose someone wanted to draw the line from left to right. First part 1 would have to be drawn, then part 2, then part 3, and so on _ad infinitum_. New parts of ab (those in the indicated series) would have to be drawn one after the other endlessly. The drawing of any part within the series would always bring one to a next part which must be drawn and never to the completion of the line. We arrive at the paradoxical conclusion that lines like ab cannot be drawn. In an analagous way any continuum may be 'proved' incompletable (we could, for example, consider ab as a time line, and thus prove time intervals incompletable).

In the Physics Aristotle reports two forms of the Dichotomy, both of them involving distances. He talks of

> those who ask, in the terms of Zeno's argument, whether we admit that before any distance can be traversed half the distance must be traversed, and these half-distances are infinite in number, and that it is impossible to traverse distances infinite in number - or some on the lines of this same argument put the questions in another form, and would have us grant that in the time during which a motion is in progress it should be possible to reckon [count] a half-motion before the whole for every half-distance that we get, so that we have the result that when the whole distance is traversed we have reckoned an infinite number, which is admittedly impossible.[4]

Another form of the paradox attempts to show that no motion can ever be started.

> This [dichotomy] comes in two forms. According to the first, Achilles cannot get to the end of any racecourse...indeed, he cannot finish covering any finite distance...
>
> The second form of "The Dichotomy" attempts to show, worse yet, that the runner cannot even get started... Before he can complete the full distance, he must run half of it. But before he can complete the first half, he must run half of that, namely, the first quarter. Before he can complete the first quarter, he must run the first eighth. And so on. In order to cover any distance no matter how short, Zeno concludes, the runner must already have completed an infinite number of runs. Since the sequence of runs he must already have completed as the form of a regression,
>
> ...1/16, 1/8, 1/4, 1/2,
>
> it has no first member, and hence, the runner cannot even get started.[5]

We shall refer to this form of the paradox as the "regressive Dichotomy".

5. The most widely discussed version of the Dichotomy paradox is weaker than the version presented here. It is usually suggested that Zeno construes motion as impossible because it requires that an infinite series of parts of a motion (or an infinite series of distances) be completed in a finite time. Simplicius reports the paradox in a manner that might suggest such an interpretation.

> The argument of Zeno...was as follows. If there is motion there will be something which has traversed an infinite series of distances in a finite time. For since the process of dichotomy has no limit,

> in any continuum there will be an infinite number of halves, since every part of it has a half. A body, therefore, which has traversed a finite distance will have traversed an infinite number of halves in a finite time, i.e. in the time which it actually took to traverse the finite distance in question. He assumes...that it is impossible to traverse an infinite distance in a finite time (because it is impossible to complete an infinite series), and thus does away with the existence of motion.[6]

Unfortunately, many serious treatments of the dichotomy deal with this version of it. Such treatments (e.g., Grünbaum and Salmon[7]) conveniently elude the much stronger form of the paradox already presented. Though they may undermine the form of the paradox they consider, none are successful against the stronger version (and Aristotle's remarks suggest that it is this version which Zeno presented).

6. The given version of the Dichotomy holds that an infinite series of motions is incompletable because there is no last motion within such a series. Against such a paradox it has been argued that it is possible to complete a series of motions without completing a last motion in the series (see for example, Russell and Grünbaum[8]). Such a position is completely untenable. For consider any series of motions. The last motion in such a series is that motion which leaves no motions in the series uncompleted. If there is no last motion in such a series (when it is infinite for example) then there cannot be any individual motion which results in the completion of all motions in the series. But then doing the individual motions in the series one by one (no matter how fast) cannot result in the completion of all motions in the series.

Those who argue that a series of motions (or tasks or acts) may be completed without the completion of a last motion (or task or act) argue that the series can instead be completed by reaching a point where no motions remain to be completed. Such a position (which will be considered more fully later) makes no sense for to reach a point where no motions remain to be completed is _just_ to reach a point where a last motion has been completed (that motion is the one last finished before the series came to an end). One might equally well deny that a particular person controlled some machine because he could instead be said to operate it.

Time is irrelevant to the presented form of the Dichotomy. The impossibility of completing a motion stems from the impossibility of completing an infinite series of submotions one by one, regardless of time. The _Oxford English Dictionary_ defines "to finish" as to "bring to and end, come to the end of,...reach the end".[9] And one cannot reach the end of an infinite series of tasks. Because the series is infinite, it is endless. Considerations about time do not change this fact.

A motion divided according to Zeno's method seems impossible because it requires the one by one completion of the infinite number of submotions Zeno defines. The completion of any part of the motion in the series always brings one to new parts which require completion. One is never brought to the completion of the overall series, no matter how much time is available. Even if an infinite amount of time is at hand, the completion of any part of the motion in the series still brings one to other parts that require completion and not to the end of the motion. It is for _this_ reason that motion seems incompletable, and _not_ because it can be alleged to require an infinite amount of time.

Even if Zeno did not present the form of the Dichotomy which we shall consider, it is the form that most merits consideration. The strength of this version of the paradox will become evident.

6. Though the prevalent form of the dichotomy is not as strong as

it might be, it must be admitted that it has been presented in entertaining ways. Abner Shimony suggests one possible outcome of the paradox.

> _Zeno_ [on being warned of an approaching lion]: In order to run from the zoological garden to the Eleatic school, the lion would first have to traverse half the distance.
>
> _The lion traverses half the distance._
>
> _Zeno_: But there is a first half of that half, and a first half of _that_ half, and yet again a first half of _that_ half to be traversed. And so the halves, would of necessity regress to the first syllable of recorded time - nay, they would recede yet earlier than the first syllable. To have travelled but a minute part of the interval from the zoological garden to the school, the lion would have been obliged to embark upon his travels _infinitely long_ ago.
>
> _The lion bursts into the schoolyard._
>
> _Pupil_: O Master, run, run! He is upon us!
>
> _Zeno_: And thus, by _reductio ad absurdum_, we have proved that the lion could never have _begun_ the course, the mere fantasy of which has so unworthily filled you with panic.
>
> _The pupil climbs an Ionic column, while the lion devours Zeno._
>
> _Pupil_: My mind is in a daze. Could there be a flaw in the Master's argument?[10]

Chapter 2

The Achilles Paradox (Achilles and the Tortoise)

1. Consider a race between Achilles (a very fast runner) and a tortoise. Suppose that the tortoise is given a slight headstart. In reality Achilles quickly catches the tortoise, overtakes him, and wins the race. In his second paradox of motion, Zeno claims to show that Achilles cannot catch the tortoise, no matter how long the race.

When the race begins Achilles proceeds to the point where the tortoise began. In doing so he does not catch the tortoise, for some time elapses and in it the tortoise moves to a new point on the racecourse. Achilles does move closer to the tortoise (for in the time it takes him to cover the distance of the headstart the slower moving tortoise cannot move an equal distance) but he does not catch and overtake it. Next Achilles moves to the new point occupied by the tortoise. But again some time elapses and the slow but continuously moving tortoise moves to another point, ahead of Achilles. Next Achilles moves to this point. But again some time (however minimal) elapses and in it the tortoise moves to another new point, still in front of Achilles. Such reasoning can be expanded indefinitely. Whenever Achilles moves to a point previously occupied by the tortoise some time elapses and in it the tortoise (because he is moving steadily) moves to a point further along the racetrack. The tortoise therefore remains ahead of Achilles and is never overtaken (though Achilles does continually gain on his rival).

According to the Achilles paradox, Achilles must finish an infinite (endless) number of segments of catching before he catches the tortoise. Yet an endless series of parts of the catching cannot be completed one by one. The completion of any part in the series always brings one to a new part that needs completion, and never to the completion of the overall catching. It is concluded that Achilles cannot win his race. One form of the paradox argues that Achilles must occupy an infinite number of positions before he catches the tortoise (those designated by Zeno) and this is impossible. Again it is concluded that Achilles cannot win his race.

Simplicius constructs the Achilles paradox as we have.

> This argument too is based on infinite divisibility, but is set up differently. It would run as follows. If there is motion, the slowest will never be overtaken by the swiftest. But this is impossible, therefore there is no motion...
>
> The argument is called the "Achilles" because of the introduction into it of Achilles who, the argument says, cannot overtake the tortoise he is chasing. For the pursuer, before he overtakes the pursued, must first arrive at the point from which the latter started. But, during the time which it takes the pursuer to get to this point, the pursued has advanced some distance. Even though the pursued, being the slower of the two, covers less ground, he stills advances, for he is not at rest.... Thus, assuming the distance to be successively less without limit, on the principle of the infinite divisibility of magnitudes, it turns out that Achilles fails not only to overtake Hector but even the tortoise.[11]

Aristotle also discusses Zeno's second paradox.

> Zeno's arguments about motion, which cause so much disquietude to those who try to solve the problems they present, are four in

number...

The second is the so-called 'Achilles', and it amounts to this, that in a race the quickest runner can never overtake the slowest, since the pursuer must first reach the point whence the pursued started, so that the slower must always hold a lead. This argument is the same in principle as that which depends on bisection, though it differs from it in that the spaces with which we successively have to deal are not divided into halves. The result of the argument is that the slower is not overtaken: but it proceeds along the same lines as the bisection-argument (for in both a division of the space in a certain way leads to the result that the goal is not reached, though the 'Achilles' goes further in that it affirms that even the quickest runner in legendary tradition must fail in his pursuit of the slowest),...[12]

2. Some have construed the Achilles as resting on the premise that Achilles cannot catch the tortoise <u>in a finite time</u>. It is best not to portray the paradox in this manner, for it is more forceful when seen as a consequence of a runner's inability to complete an infinite number of parts of the catching of the tortoise, regardless of time. Such completion is impossible because new parts of the catching need to be finished one after the other endlessly. The Achilles paradox need <u>not</u> allege that the catching of the tortoise is impossible because it requires an infinite amount of time.

Chapter 3

The Arrow Paradox

1. In order to undermine Zeno's first two paradoxes we might assert that space and time are finitely divisible (and not infinitely divisible). If this is the case it makes no sense to talk of dividing a motion ad infinitum and hence Zeno's arguments make no sense. The third and fourth paradoxes of motion try to show that this move against the first two paradoxes leads to contradiction. Though the Arrow and Stadium as usually presented are <u>not</u> real paradoxes they can be constructed in a way that creates genuine logical paradox.

2. Most often the Arrow paradox is constructed with regard to time. If time is finitely divisible it cannot be divided ad infinitum. It can be divided a definite amount, and no further. If time is finitely divisible the smallest intervals it can be divided into are the fundamental "moments" of time. On such a view time is a conjunction of moments, the passage of time is a succession of moments. Anything that happens within a moment

happens at the same time (for it is impossible to distinguish smaller intervals of time within a moment). In this context consider the flight of an arrow through space. At any particular moment it occupies one position in space. To occupy two positions during one moment the arrow would have to occupy two positions at the same time (for everything during a moment happens at the same time). Because the arrow occupies only one position during a moment it does not move during a moment. It seems to follow that the arrow never moves during the time it is in the air - that time interval is merely a conjunction of moments, a conjunction of intervals of no motion. The arrow cannot move during its flight because it is stationary during all moments it is in the air. Once again we reach apparent contradiction.

This standard version of the arrow does not withstand close scrutiny. It appeals to common sense because motion appears continuous and not constituted of a series of 'moments' of no motion. This appearance however, is easily explained by one who holds that time is finitely divisible. Such a person may claim that the moments of time are so small that we cannot perceive them and cannot perceive the fact that no motion occurs during them. Similarly, though a motion picture is a series of still pictures in quick succession it appears that it encompasses continuous motion. What appears to be a continuous motion may not be, and hence all motion may be discrete (though it appears otherwise). The apparent continuity of motion might be a result of our limited perceptions.

> If we adopt the view that motion merely signifies change of position in the sense that a body is at different positions at different times, then, however strange it may seem, there is nothing illogical in asserting that, since at each instant the body is in a unique position, at that instant it is indistinguishable from a stationary body in the same place. A sequence of photographs of an arrow in flight when viewed separately show it in a succession of quasi-stationary states.

> Owing to the phenomenon of retention of vision, when these pictures are run sufficiently rapidly through a cinema projector the arrow appears to be moving. The difference between the two interpretations depends entirely on how rapidly the photographs succeed each other in our vision, that is, merely on the temporal relations of one photograph to another. If we look upon this as an exact analogy and regard motion as a phenomenon necessarily referring to different instants, Zeno's paradox collapses,[13]

An analogous form of the paradox, based on the supposition that space is finitely divisible, can be similarly undermined.

3. A genuine paradox can be built upon the hypothesis that space or time is finitely divisible. If this hypothesis was correct it would not be possible (in principle) to divide spatial and temporal intervals ad infinitum. There would be smallest possible intervals of time ("moments") and smallest possible intervals of space ("points"). Time would be a conjunction of successive moments, and space a conjunction of successive points.

On the finite divisibility view moments and points must be extensionless. For suppose moments did have some positive extension, say one millisecond. It would follow that there cannot be intervals of time shorter than one moment (1 millisecond). Such time can be represented by the time line in figure (ii), where the distance between each interval represents 1 millisecond.

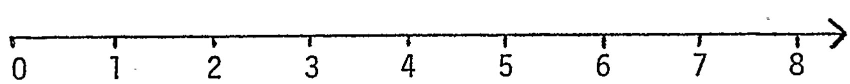

figure (ii)

But then consider an event a situated on the time line as in figure (iii).

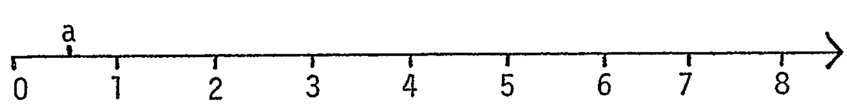

figure (iii)

We would normally say that the start of the first millisecond and event a are separated by a time interval smaller than 1 millisecond. But on the finite divisibility view there can be no such time intervals (for there can be no time intervals shorter than 1 moment). So the start of the first millisecond and event a must occur at the same time (they cannot be separated by an interval equal to or greater than one moment). Similarly, event a must occur at the same time as the end of the first millisecond. Event a is simultaneous with the start of the first millisecond and the end so the start and end of the first millisecond are simultaneous. But this is contradictory, for one millisecond (one moment) is supposed to elapse in this time. The supposition that moments are extended leads to contradiction, so they must be extensionless. Similar reasoning can be applied to spatial points to show that (on the finite divisibility view) they must also be extensionless.

The view that moments and points are extensionless (and hence the finite divisibility view) leads to paradox. If moments and points are extensionless then all space and time intervals, which are nothing but conjunctions of points and moments, must also be extensionless. A conjunction of (a finite number of) time intervals of zero duration cannot produce an extended interval of time. A conjunction of (a finite number of) spatial intervals of zero distance cannot produce distance. This consequence of the finite divisibility view is contradictory, for time and space clearly are extended. So though the standard Arrow paradox is not genuine, the finite divisibility hypothesis does lead to contradiction as perplexing as that of Zeno's first two paradoxes.

Commentators on the Arrow paradox (e.g., Whitrow and Swartz[14]) have dismissed the Arrow without realizing that it can easily be replaced by a stronger paradox. Whitrow makes this mistake when he argues (on the basis of his consideration of the 'old' Arrow and Stadium paradoxes) that:

> We may conclude that the hypothesis of temporal instants, that is, that there is some definite limit to the divisibility of time, is logically preferable to the alternative hypothesis that time is truly continuous, that is, infinitely divisible.[15]

4. Like the dichotomy, many versions of the arrow (some not involving arrows) have been presented.

> An amusing variant of this paradox is given in the article on Zeno in Bayle's famous *Dictionnaire*, published in 1696. He recalls a story told by Sextus Empiricus of the sophist Diodorus who lectured against the existence of motion. Having dislocated his shoulder, he went to have it set. "How?" said the doctor. "Your shoulder dislocated! That cannot be; for, if it moved, it did so either in the place where it was, or in the place where it was not. But it did not move, either in the place where it was, or in the place where it was not, for it could neither act nor suffer in the place where it was not!"[16]

Chapter 4

The Stadium Paradox

1. The most widely discussed version of the Stadium does not involve real contradictions. Like the Arrow however, the Stadium can be replaced by genuine paradox.

2. We may start with the standard form of the Stadium. Suppose time is finitely divisible. It follows that there are smallest possible intervals of time (moments). In this context consider the situation portrayed in figures (ii) and (iii).

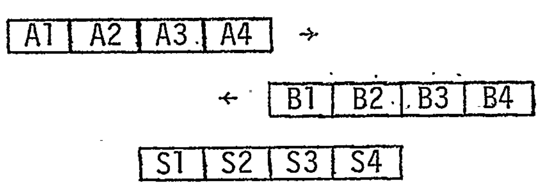

figure (ii)

A, S, and B are each constituted of four objects of one unite length. S is stationary, A moves to the right at the speed of one unit length <u>per moment</u>, and B moves to the left at the same speed. (In the original paradox S was a stadium while A and B were chariots racing in front of it.) Figure (iii) represents the situation one moment after the situation in figure (ii).

A1	A2	A3	A4	→	

←	B1	B2	B3	B4

S1	S2	S3	S4

figure (iii)

In the time that has elapsed B1 has progressed from being adjacent to no part of A to being adjacent to A3. In order for this to be the case there must have been some time when B1 was adjacent to A4. But there is no such time. For time is alleged to be nothing but a conjunction of moments, and at one moment the situation in (ii) holds, and at the next, that in (iii). There is no time between moments, so no time when B1 is adjacent to A4. To indicate a time when B1 is adjacent to A4 we need to be able to distinguish smaller intervals of time than moments, and on the finite divisibility hypothesis, this is not possible. By definition moments are the smallest possible intervals of time.

If one seriously accepts the finite divisibility hypothesis the Stadium paradox is not problematic. It can be said that in the outlined situation there is no time when B1 is adjacent to A4. Such a position is counter-intuitive but this can be explained by saying that situations like that outlined are never perceived because the spatial and temporal intervals they concern are beyond our perception. It appears that B1 must become adjacent to A4 before A3 because we cannot perceive the small intervals where this is not the case.

Whitrow's comments on the standard version of the Stadium are valid.

> Despite its ingenuity, this is one of the easiest of Zeno's arguments to answer. For, if space and time are composed of discrete units, then relative motions must be such that the situations typified by the diagrams of Figure 4 can occur at

> successive instants. Zeno's rejection of this possibility is not based on any logical rule but simply on a fallacious appeal to 'common sense'. Indeed, Zeno is in fact guilty of a logical error himself when he makes this appeal, for in fact he is tacitly invoking a postulate of continuity which is incompatible with the hypotheses adopted at the beginning of the argument. Strange as it may seem, if we adopt such hypotheses then motion will be a discontinuous succession of distinct configurations, as in a roll of cinema film, and at no time will intermediate configurations occur.[17]

An analagous version of the Stadium presented by Max Black[18] concerning space can be handled in a similar way.

3. It is not clear how Zeno presented the Stadium's arguments. Aristotle's comments suggest that his version of the paradox was weaker than the modern version.[19] Both forms can be replaced by a real paradox that does arise on the supposition that time or space is finitely divisible. If this supposition is correct there are smallest possible time intervals or smallest possible space intervals (moments or points). Everything that happens at a particular moment happens at the same time, and anything which occupies a particular point occupies the same place in space.

Suppose that moments are of n duration, and consider the situation portrayed in figures (iv) and (v).

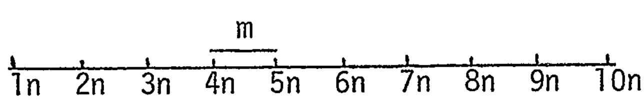

figure (iv)

The lines in the figures are time lines which represent moments of time, the length of time n (the duration of an instant) is represented by a particular length of line. The interval m is of n length so it seemingly takes one moment. When we divide time into moments as in figure (iv) this is the case. However we might divide time into moments as in figure (v).

m

1n 2n 3n 4n 5n 6n 7n 8n 9n

figure (v)

If this is done the time interval m extends within two moments (i.e. within that from 3n-4n and that from 4n-5n). And because it makes no sense to say that a time interval takes up part of a moment it must be said that the time interval m takes two moments to elapse. As each moment is of n duration, the time 2n passes when the interval m (1n in length) elapses. We are left with the contradiction that a time interval n in length requires the time 2n (i.e. twice its time) to elapse.

Suppose points are of n length. If we let the lines in figures (iv) and (v) represent spatial intervals we again meet paradox. For if we divide space as in (iv) the spatial interval a (1n in length) requires 1 point (as it should). But there is no reason why space cannot be divided as in (v). When this is done the spatial interval m extends into two points and therefore occupies two points (for it makes no sense to say it occupies part of a point). Because each point is one n long the spatial interval m (1n long) occupies a spatial interval 2n in length! A given distance is therefore equal to double that distance. Whoever holds space to be finitely divisible is left with paradox.

4. One way around this new paradox is the assertion that moments and points must have zero extension (I have already argued that one who accepts the finite divisibility hypothesis is committed to such a view). Such a move is unsatisfactory because it leads directly to the paradox we have constructed from the arrow. The Stadium is relevant because there are those who hold the position on which it is built. Swartz, for example, holds that the smallest intervals of space are hodons and the smallest

possible intervals of time are (approximately) chronons. He writes:

> It... appears that distances smaller than 10^{-15}m and periods of time smaller than about 10^{-22}s simply do not exist. This means also, of course, that a distance of about 10^{-15}m and a period of time of about 10^{-22} sec cannot be divided anymore; for the parts they would be divided into would be non-existent.[20]

Swartz fails to realize that the presented paradox is a result of his position (for the moments of time, $n = 10^{-22}$ sec, and for the points of space, $n = 10^{-15}$m). We shall see that his treatment of Zeno's paradoxes is inconsistent in other ways.

5. Henceforth we shall call the newly introduced paradoxes the "Arrow" and "Stadium" paradoxes (the new Arrow replaces the old Arrow paradox and the new Stadium replaces its previous counterpart). The Dichotomy, Achilles, Arrow Stadium paradoxes leave us with dilemma. It seems clear that space, time (and motion) can be divided ad infinitum. Yet such a view leads to contradiction. So does the position that time and space are not infinitely divisible but finitely divisible. We are met with paradox whichever way we turn.[21] Zeno might smile.

PART II

SOLVING THE PARADOXES OF MOTION

The results of philosophy are the uncovering of one or another piece of plain nonsense and of bumps that the understanding has got by running its head up against the limits of language...

Philosophy may in no way interfere with the actual use of language; it can in the end only describe it.
For it cannot give it any foundation either.
It leaves everything as it is...

What is your aim in philosophy? - To shew the fly the way out of the fly-bottle.

LUDWIG WITTGENSTEIN, Philosophical Investigations[1]

Chapter 5

The Dichotomy Paradox

1. The solution to Zeno's paradoxes of motion is much simpler than most commentators have imagined. Contrary to standard opinion, the correct solution to the paradoxes does not involve subtle mathematical or scientific reasoning. A clear understanding of the concepts involved in the paradoxes is all that is required. Linguistic confusion has thwarted such understanding.

2. In the Dichotomy paradox motion has <u>not</u> been construed as incompletable because it requires the successive completion of an infinite number of motion segments <u>in a finite time</u>. The problem is not time, for any motion seems incompletable independent of time considerations. Just because (given the infinite divisibility of space or time) it requires a one by one completion of the infinite series of motion segments indicated by Zeno. New parts of the motion must be completed one after the other endlessly. The completion of a motion segment in the series always

brings one to new segments that must be undertaken, and never to the completion of the motion. The Dichotomy paradox depends on the claim that any motion has an infinite number of parts. To undermine the paradox we need consider how a motion may be constituted of parts.

3. The parts of physical objects may be separate or conjoined. If someone says of a broken vase that its parts lie scattered on the ground, they refer to separate parts of the vase. On the other hand, if one says of a particular rope that "Its first half is frayed and worn but its second half is in good shape." one does not imply that the rope has two separate halves. The remark "The thumb is part of the hand." does not imply that the thumb is separate from the other parts of the hand. The parts of an object may be separate or conjoined (continuous). Continuous parts in an object are parts that are not separate, though they could be. An uncut apple has two halves even though the halves are not separate.

Like physical objects, motions may be constituted of separate or continuous parts. Every completed motion has a first, second and final third, but not all have separate thirds. For this to be the case the thirds must be separated by something other than parts of the motion itself (by periods of no motion for example). Obviously, not all motions have separate parts.

4. Because of the difference between separate and continuous parts in a motion, there are two ways in which the parts of a motion may be completed one by one. If the parts are separate, they are completed

(obviously) separately, one by one. One part is completed, then the next, and so on. If the parts of a motion are continuous they are done one after the other but are not done _separately_, one by one. In such a motion a part is finished before the next but not separately from it. The finish of one part and the start of the next are continuous, so a series of disjoint submotions are not required. One overall motion does the job. The parts of a continuous motion are completed one by one _not_ in the sense that they are done separately one by one, but in the sense that they are successive parts of one continuous motion.

5. The Dichotomy is based on a confusion between separate and continuous parts within a motion. Finite continuous motions can (in ordinary circumstances) be completed. Yet a motion which contained an infinite number of separate parts could not be completed. After the completion of _any_ motion segment in the series new motion segments would have to be undertaken separately. The last motion in the series and the end of the series could never be reached.

Zeno does not show that any motion contains an infinite number of separate parts and _therefore_ is incompletable. Rather, he shows that any completed motion contains an infinite number of continuous (conjoined) parts and then argues that an infinite number of _these_ motion segments cannot be completed one by one. This simply is not the case. For though the segments could not be completed _separately_ one by one, they need not be. Rather, they may be completed as continuous parts of one overall motion. The completion of one part is contiguous with the start of the next, and hence different motions are not required for each part. One overall motion suffices. As the submotions are continuous they do not have to be done individually one by one, but can be done as one continuous motion.

Zeno's arguments make it appear as though a motion cannot be completed because they invite us to confuse separate and continuous parts within the motion. Thus, if we imagine the parts Zeno indicates continuously, we can think of the motion as completed (we simply imagine it as a whole). But Zeno forces us to think of the motion in a different way by designating an infinite series of parts which it contains. In the series 1/2 m, 1/4 m, 1/8 m,... he designates each part of the motion separately and asks us to consider separately the completion of each part of the motion. We consider each of the parts individually and therefore cannot imagine all the parts being completed (for in order to do so we would have to think of an infinite number of parts separately, one by one). Yet it does not follow that the motion cannot be completed. It does not follow from the fact that we consider parts of a motion separately that the parts are separate. When we think of the motion in a way more faithful to its nature (that is, as one continuous motion) we can imagine it as completed.

If a hand is moved in a continuous motion the motion does contain the infinite number of parts Zeno indicates. If these parts were separate they would require (among other things) an infinite number of muscle contractions, and that would be impossible. In the actual motion however, the parts are not undertaken individually. Rather, one overall motion which encompasses them all is undertaken, and completed. Because the motion segments are connected they can be completed one by one.

Finite continuous motions can be completed though an infinite series of disjoint motions cannot. Zeno makes it appear as though no motion can be completed because he engenders a confusion between separate and continuous parts.

6. The argument of the Dichotomy is persuasive because it plays on the linguistic similarity of talk about continuous and separate parts. The Dichotomy argument could be given as follows.

> An infinite number of motion segments cannot be completed one by one.
> Any motion contains an infinite number of motion segments.
> Any motion cannot be completed.

Within the argument the first premise is ambiguous. Because it is not clear whether separate or continuous parts are referred to, and because an infinite number of separate motion segments cannot be completed, the premise appears true. The premise must refer to continuous parts however, or the argument is invalid (for the second premise refers to such parts). And in regard to continuous parts an infinite number of successive parts can be completed one by one, though (obviously) the parts are not completed separately one by one. The parts are completed together in one continuous motion. One overall motion is required, and not a series of disjoint submotions.

When we consider an infinite number of continuous parts which are encompassed by a motion we separate the parts for our consideration. In Zeno's series 1/2 m, 1/4 m, 1/8 m, ... each part is designated separately, and thus our designation of the parts is endless and necessarily incomplete. But in the motion itself, the parts are not separate and therefore need not be done separately one by one. We may deal with successive continuous parts separately, but it does not follow that the parts are actually separate. The divisions we impose between the parts are just that, imposed. The divisions do not exist in the actual motion, and therefore its completion does not require the separate completion of an infinite number of separate parts. In reality the motion is <u>not</u> divided into an infinite number of separate parts. It is one continuous motion.

7. Paradoxes analagous to the Dichotomy can be constructed on a confusion of separate and continuous parts within a motion. Suppose for example, that we wish to push a heavy piece of furniture across a room. The furniture can be pushed in a variety of ways. We might push it a foot at a time (that is, we might push it a foot, pause, push it the next foot, pause, and so on). But suppose that proceeding in this way results in damage to the room's floor. If the furniture is pushed a foot at a time parts of the floor cannot withstand the pressure. Imagine then, that we instead push the furniture with one steady continuous push. Despite the fact that we proceed in this way it can still be said that the furniture is moved a foot at a time. The first foot of the moving is completed first, then the second, then the third, and so on. These parts of the motion are completed one by one, and so it might be concluded that the floor is damaged. Here we have a paradox, for it has already been said that the floor is <u>not</u> damaged. But it does <u>not</u> follow that the floor is damaged. The floor is damaged only when the moving is constituted of <u>separate</u> parts one foot long. And of course this is not the case. Though the moving does contain successive parts one foot long (and though these parts are completed one by one) it does <u>not</u> contain separate parts one foot long. Hence the floor is not damaged. To think the floor is damaged is to confuse separate and continuous parts. When it is said that the floor is damaged if the furniture is moved a foot at a time, separate parts of the motion are

referred to. The motion does not contain such parts and therefore the floor is not damaged. Even though the motion does contain continuous parts one foot long.

Like the Dichotomy, the conclusion of the 'furniture paradox' is based on the linguistic similarity of talk of separate and nonseparate parts. The argument for the paradox's conclusion could be given as follows.

> If one moves the furniture a foot at a time the floor is damaged.
> The furniture is moved a foot at a time.
> The floor is damaged.

Such an argument is convincing only when it misassimilates the two ways of moving the furniture a foot at a time. The first premise is true only in regard to separate movements one foot long, whereas the second is true only in regard to continuous submotions. Hence the conclusion does not follow. It appears to only because of the superficial similarityity of talk about separate and continuous parts of the motion. The furniture paradox displays the confusion which can arise from a misassimilation of such parts. Similar confusion seems to underlie the the Dichotomy arguments.

8. A motion may be divided into separate parts by things other than periods of no motion. Thus, a motion which encompasses an infinite number of disjoint accelerations, or one that requires an infinite number of disjoint changes of direction, does have an infinite number of separate parts, and is incompletable because an infinite number of separate parts cannot be completed one by one.

Of course, not all continuous motions are completable. Just because the parts of a motion are continuous it does not follow that a motion is completable. Motions may be incompletable for other reasons (for example, because they require an infinite amount of time or space).

9. A clear distinction between separate and continuous parts undermines all forms of the Dichotomy paradox. The paradox always results from the division of some continuum into an infinite number of successive parts. It is then argued that the continuum is necessarily incomplete because it contains an endless number of parts. The parts in question however, are always continuous rather than separate parts (that is, they are part of an unbroken continuum). It is because of this that the continuum need not be endless. All the parts in question exist together (conjoined) as a unit. That is what it means to say they form a continuum. Though we can consider (designate, etc.) parts of the continuum separately, it does not follow that the parts are themselves separate.

10. The Dichotomy stems from a confusion between separate and continuous parts. A confusion which arises very easily because of the similarity of talk about the two kinds of parts. The paradox begins with the assertion that some motion is completable. Zeno then attempts to perform reductio. An infinite series of separate motion segments cannot be completed and from this Zeno draws his mistaken conclusion, that an infinite number of continuous motion segments are not completable. The conclusion simply does not follow.

Aristotle[2] (and Quan[3]) seem to be the only philosophers who have approached the proper solution to the paradox.

Chapter 6

The Achilles Paradox

1. Zeno's second paradox of motion is based on the same confusion as his first. Zeno shows that Achilles's catching of the tortoise contains an infinite number of parts. It appears to follow that the tortoise cannot be caught, for to do so an infinite number of parts of the catching would have to be completed one by one. To undermine the paradox we must again consider the difference between separate and continuous parts. If the parts of the catching were separate then the catching could not be completed. The completion of any part of the catching would always bring Achilles to new parts which would have to be completed, and never to the end of the catching. Yet the parts of the catching may be continuous parts. They may be completed one by one without being completed <u>individually</u> one by one. One or more continuous bits of catching do the job. Achilles need not complete the parts of the catching separately one by one. The finishing of one continuous part and the start of the next are contiguous and therefore one overall motion

(containing continuous catching) can encompass both. When Zeno discusses the catching he designates its parts separately and in seems to follow that the catching cannot be completed. When the parts are thought of individually, we cannot imagine the catching being completed. But in the race itself the catching need not be constituted of an infinite number of separate parts. Instead it may be encompassed by one or more motions in which the catching of the tortoise is continuous.

2. The Achilles paradox is convincing because in it (as in the Dichotomy) Zeno designates an infinite number of parts of the catching <u>separately</u>, one by one. He has us consider <u>those</u> parts separately and hence we cannot think of them being completed one by one (because we cannot imagine an infinite number of parts <u>separately</u> one by one). This inability arises however, not because the catching is impossible but because we have decided to <u>separate</u> the parts and then consider them one by one. The parts in the actual catching are not separate and hence can be completed.

3. The Achilles paradox can be compared to analogous confusions. Consider two runners, Shorter and Viren, who run in a particular race. Shorter starts first, and Viren begins five minutes later, intent on catching his rival. For most of the distance Viren might run the same speed as Shorter, except for short intervals where he increases his speed and catches Shorter five yards at a time (during the first interval he gains five yards, during the second five more, and so on). Given enough five yards gains, Viren will catch Shorter before the end of the race. But suppose that running in this way is beyond Viren (it requires too much energy) and he proceeds differently. He takes up a <u>steady</u> pace faster than Shorter's,

gains constantly, and catches Shorter before the end of the race. Though Viren does adopt this second method of running, we can still divide his catching into parts where he catches Shorter five yards at a time, and can still describe his run in terms of such parts. Thus Viren first gains five yards on Shorter, then another five, then another five, and so on. One might paradoxically conclude that Viren cannot have caught Shorter, for it was earlier said that such catching is impossible if he attempts to catch Shorter five yards at a time. Such a conclusion confuses separate and continuous parts of the catching. The catching that takes place contains continuous parts in which Viren gains five yards on his rival. But it is not such parts which are alleged to make the catching impossible. Rather, it is only if Viren separates his catching into the disjoint parts suggested that he cannot succeed. These parts are disjoint not because they are separated by periods of no motion, but rather, because they are separated by periods of no catching (0 catching). These periods break the continuity of the catching, and make it impossible (because of Viren's limitations). His catching is not so constituted, and so it does succeed.

In a similar way, Achilles's catching of the tortoise would be incompletable if it was constituted of an infinite series of separate parts. As this is not the case, the second paradox of motion does not establish its alleged conclusion.

Chapter 7

The Arrow and Stadium Paradoxes

1. The second two paradoxes of motion prevent the dismissal of the first two paradoxes on the grounds that space and time are finitely divisible. We have found a different way around those paradoxes. The second two paradoxes simply show that time, space, and motion are infinitely divisible (for the opposite view leads to contradiction).

2. The 'new' Arrow paradox, as described in Chapter 3, shows that the supposition that time or space is finitely divisible leads to absurdity. For if space or time is finitely divisible then space or time must be extensionless (for the smallest parts of space or time would be extensionless and time or space is just the conjunction of such parts). Space and time obviously are extended, so such a consequence refutes (by reductio) the finite divisibility view.

3. Because time and space are not finitely divisible they are infinitely divisible, and can be divided into smaller and smaller parts endlessly.

It follows that there are <u>not</u> smallest time or space intervals (moments or points). The time or space encompassed by any interval can be endlessly broken into smaller and smaller segments. And though this makes it possible to reach 'instants' and 'points' encompassing less and less time and space, none can be reached that contain no time or no space. No matter how small any particular interval is, smaller intervals are always possible.

The idea of time or space intervals so small that they contain no time or space may result from the (true) view that any interval <u>can</u> be divided into intervals so small that the time and space in them is imperceptible. This does not show however, that time and space is not contained within the intervals. We can talk of intervals so small that for <u>practical</u> purposes no time or space occurs within them, but this only means that an insignificant amount of time or space occurs in them and not that absolutely no time or space is within their span.

4. The view that the arrow paradox shows that time and space are infinitely divisible does <u>not</u> involve (as some suggest) the claim that it is logically necessary that space and time are infinitely divisible. Similar situations do not require analagous claims. Suppose for example, that I claim a particular parcel weighs 3 pounds. The claim might legitimately be rejected if it leads to contradiction. Suppose for example that a 2 pound parcel outweighs it on a valid balance scale. If the parcel did weigh three pounds it would outweigh the two pound parcel; it does not do so, and hence the original assumption is rejected. Such rejection does not imply that it is logically necessary that the parcel

does not weigh three pounds. In a similar way, the rejection of the view that time or space are finitely divisible (because it leads to conclusions that contradict the empirical fact that time and space are extended) does not imply that the infinite divisibility of time and space is logically necessary.

5. Like the Arrow, the 'new' Stadium paradox described in Chapter 4 shows that time and space are infinitely divisible (for the opposite view does lead to paradoxical conclusions). As time is infinitely divisible then there are no moments that are the smallest possible intervals of time. Hence it does not make sense to talk of dividing a particular time span into smallest possible parts. Nor does it make sense to talk of dividing spatial and temporal intervals into smallest possible parts in alternate ways. Hence the basis of the Stadium paradox cannot be constructed. As in the case of the Arrow paradox this does <u>not</u> commit one to the assertion that it is logically necessary that space and time are infinitely divisible (for the same reasons).

6. Once the confusion in Zeno's first two paradoxes is uncovered, the second two paradoxes are easily overcome. They simply show that space and time are not finitely divisible. There is a way out of Zeno's dilemmas.

PART III

RUSSELL, SET THEORY, AND THE ACHILLES PARADOX

I shall claim the privilege of a Free-thinker; and take the liberty to inquire into the object, principles, and method of demonstration admitted by the mathematicians of the present age...

I shall subjoin the following questions...

Whether mathematicians do not engage themselves in disputes and paradoxes concerning what they neither do nor can conceive?...

Whether mathematicians...are strictly scrupulous in their own science? Whether they do not submit to authority, take things upon trust, and believe things inconceivable? Whether they have not their mysteries, and what is more their repugnancies and contradictions?

GEORGE BERKELEY, The Analyst (A Discourse Addressed to an Infidel Mathematician)[1]

Chapter 8

Euclid's Fifth Axiom

1. The standard mathematical treatment of the Achilles paradox was first suggested by Bertrand Russell. According to him, the proper solution to the paradox is based on the rejection of 'common sense' principles of logic and mathematics. Russell's 'solution' to the Achilles paradox (contrary to his suggestions) is unsatisfactory. It does not solve Zeno's paradox and leads to <u>completely unnecessary</u> contradictions in modern mathematics.

2. Euclid's fifth axiom is the principle that a set (collection, class[2]) contains more members than any of its proper subsets. A proper subset of a set is a part of the set which does not contain all the set's members. As Euclid puts it, "the whole is greater than the part".[3]

Certainly Euclid's fifth axiom is intuitively appealing. It seems obviously and trivially true. Yet the standard mathematical treatment of the Achilles paradox depends upon a rejection of Euclid's principle.

That is, on the claim that a set and one of its proper subsets may have the same number of members.

> The root cause of the trouble lies in the fact that the laws of ordinary logic, such as we derive from an intuitive appreciation of our experience, inevitably confined to only finite classes, do not apply to infinite collections. For instance, we know that a whole is necessarily bigger than any genuine part itself. Thus, the class of all Asiatics is necessarily smaller than the class of all Homo Sapiens, for the former is only a subclass or a part of the latter. But when we apply this law to an infinite class we fall into error; for as we shall presently see, the infinite class of <u>all</u> integers is <u>exactly</u> equal to any infinite part of itself, as, for example, the infinite class of only <u>even</u> integers.
> The failure to understand that an infinite class can be equal to a part of itself led Zeno to his paradox of Achilles and the tortoise...[4]

3. If one adopts the standard set theoretic view and its rejection of Euclid's fifth axiom, then one must also reject other common sense principles (for example, the principle that a set containing the members of two (nonempty) sets A and B has more members than A or B). Such principles are ingrained in our basic mathematical reasoning. They constitute 'everyday laws of logic' because they are central to our everyday view of the world. The principles are not as central as others (the law of noncontradiction, for example) but are firmly established nonetheless.

Euclid's fifth axiom appears <u>analytically</u> true (that is, true in virtue of meaning alone). The statement "A married man has more wives than a bachelor." is true because of the meaning of the terms "bachelor" and "married man". A married man is a man with at least one wife, and a bachelor is a man without a wife. To understand the meaning of the statement is to understand it as true. The statement "A set has more members than any of its proper subsets." also seems true in virtue of meaning alone. For a proper subset of some set does not (by definition) contain all the members of the whole set. The whole set contains all the members of the subset, plus other elements as well. It has more members than the subset, and hence a greater number of members. So the truth of Euclid's fifth axiom seems analytic. And though it <u>may</u> be the case that some apparent analytic truths are false (for instance, the parallel postulate or certain distributivity laws in quantum mechanics[5]) the rejection of such analytic 'truths' cannot be compared to the rejection of Euclid's fifth axiom. Contrary to standard opinion, there are <u>no</u> good reasons to reject the fifth axiom in set theory. The reasons usually given collapse under rigorous analysis.

Chapter 9

Set Theory and Equivalence Paradoxes

1. The Achilles paradox is not the only reason given for rejecting Euclid's fifth axiom in modern mathematics. Before turning to the paradox we need consider other reasons given for the axiom's rejection.

2. The standard dismissal of the fifth axiom stems from the notion of set equivalence (equipollence) employed within set theory. Two sets are equivalent (if and only if their elements can be put in one-to-one correspondence. Equivalent sets are said to have the same number of members (they share cardinality).

> For a hundred years it has been generally agreed that two classes are equal in number if and only if there is a one-to-one correspondence between their members. A one-one correspondence is said to obtain between the members of classes A and B if and only if there is a correlation such that each member of A is correlated with one and only one member of B, and conversely. For example, in a strictly monogamous community the class of all husbands is equal in number to, or has the same cardinality as, the class of all wives. Without knowing how many husbands or wives there are, we know the classes to be equal because marriage provides a one-one correlation of husbands to wives in such communities.[6]

This notion of equivalence is to be found in all standard treatments of set theory (see Cantor[7], Kamke[8], Frege[9], Dinkines[10], Hunter[11], Russell[12], or any accepted treatment of the subject). The notion of one-one correspondence is easily understood.

> A one-to-one correspondence between two classes is said to be established when some rule is given whereby each element of one class is paired with one and only one element of the other class, and reciprocally...
>
> For example, the class of soldiers in an army can be put into one-to-one correspondence with the class of rifles which they carry, since (as we suppose) each soldier is the owner of one and only one rifle, and each rifle is the property of one and only one soldier...
>
> An example of a relation between two classes which is not a one-to-one correspondence, is furnished by the relation of ownership between the class of soldiers and the class of shoes which they wear; we have here what might be called a two-to-one correspondence between these classes, since each shoe is worn by one and only one soldier, while each soldier wears two and only two shoes...[13]

It was Frege who first suggested the modern notion of equivalence.

> Before tackling the question, What is a number?, i.e. what set-theoretic definitions of numbers are available, we ask the question: What is it for two classes to have the <u>same number</u> of members? Frege remarks that a waiter may lay a group of tables by placing at each position exactly one knife and one fork. He may, on completion of his task, not know <u>how</u> many knives and forks he has put out, but he <u>will</u> know that he has put out the <u>same number</u> of knives and forks. And he knows this because he knows that there is a one-one correspondence between the class of knives and the class of forks. Taking our cue from this story, we say that two classes x and y are <u>equipollent</u> (equinumerous, have the same number of members), in symbols $x \approx y$, if there exists a one-one correspondence between them.[14]

From this notion of one-one correspondence numbers and counting seem to be derived.

> What gives this matching process its great power is that it can be applied universally to all kinds of aggregates - from collections of ewes and pebbles to those of belles and braces, apples and angels, or virtues and vipers. Any two aggregates whatever can be matched so long as the mind is able to distinguish their constituent members from one another.
>
> Gradually men formed the notion of having a series of standard collections for matching the members of any given group or aggregate. One such series consisted of the ten different collection formed by including one or more fingers of their two hands. All collections,

which could, for example, be matched were 'similar' in at least one respect however they might otherwise differ among themselves. They were, as we now say, all equal. These collections were then given names -- One, Two, Three, ... etc. This is the social origin of the practice of counting. Thus, when we now say that the number of petals in a rose is five, all that we mean is that if we start matching the petals one by one with the fingers of one hand, the members of both collections are exhausted simultaneously. By long practice in handling the abstract symbols 1, 2, 3,... we are liable to forget that they are only a shorthand way of describing the result of an operation, viz., that of matching the items of an aggregate with those of some set of standard collections that are presumed to be known. The process is so habitual that it usually escapes notice.[15]

3. The standard mathematical rejection of Euclid's fifth axiom is based on the extension of set (class) equivalence to infinite sets.

Thus we have the following definitions. Two sets are called equivalent if between their respective members a one-to-one correspondence is possible; and the characteristic that one set has in common with all equivalent sets, and by which it distinguishes itself from all other sets not equivalent to itself is called the (cardinal) number of that set. And now we make the fundamental assertion that in these definitions the finiteness of the sets considered is in no sense involved, the definitions can be applied as readily to infinite sets as to finite sets. The concepts "equivalent" and "cardinal number" are thereby transferred to sets of infinitely many objects. The cardinal numbers of finite sets, i.e. the numbers 1, 2, 3,... are called natural numbers; the cardinal numbers of infinite sets Cantor calls "transfinite cardinal numbers" or "transfinite powers".[16]

Frege observed that a necessary and sufficient condition for, say, the number of F's (which we shall write as NxFx) to be the same as the number of G's is that there should be a one-to-one correspondence of the F's and G's. (In that case we say they are <u>numerically equivalent</u>.) This criterion, which is quite general - that is, not restricted to the case where there are only finitely many F's or G's - had already been exploited by Cantor to generalize the notion of cardinal number to infinite classes.[17]

When the notion of equivalence is extended to infinite sets in this way it does appear that a set and one of its proper subsets can have the same number of members. Consider for example the set of all natural numbers and the subset of it that contains all even natural numbers. Both groups are alleged to be equivalent because their members can be paired in one-one correspondence. To do so we can pair any natural number n with the even natural 2n as follows.

$$\begin{array}{l l} 1, 2, 3, 4, 5, 6, 7, 8, \ldots & \text{(the set of natural numbers)} \\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow\ \updownarrow & \\ 2, 4, 6, 8, 10, 12, 14, 16, \ldots & \text{(the set of even natural numbers)} \end{array}$$

figure (i)

Because the two sets are infinite there is a unique number 2n for any number n in the set. One-one correspondence is thereby established. It is usually concluded that the two sets share cardinality (have the same number of members).

> Euclid regarded it as axiomatic that the whole is greater than its part, which is surely true of finite quantities. But the Euclidean axiom seems to fail in the case of infinite collections. In 1638 Galileo argued that although squares are only some among numbers, yet 'there are as many squares as there are numbers because they are just as numerous as their roots, and all the numbers are their roots'. In 1697 Leibniz observed that since every number can be doubled, the number of even numbers is the same as the number of numbers altogether, and this is a clear case of a whole being no greater than a (proper) part of itself, though Bolzano did not choose to regard it in that light.[18]

4. The rejection of Euclid's fifth axiom in set theory has resulted in a series of equivalence paradoxes in modern mathematics. Russell's "Tristram Shandy" paradox is one such antinomy.

> Tristram Shandy, as we know, employed two years in chronicling the first two days of his life, and lamented that, at this rate material would accumulate faster than he could deal with it, so that, as years went by he would be farther and farther from the end of his history. Now I maintain that, if he had lived forever, and had not wearied of his task, then, even if his life had continued as eventfully as it began, no part of his biography would have remained unwritten.[19]

Russell concludes from his reasoning that there are as many years as days in Shandy's life. Such a conclusion is intuitively unacceptable (and this Russell freely admits). Yet the extension of the concept of number to infinite sets and the modern notion of equivalence do entail Russell's conclusion. They have as a consequence the view that there are as many years in Shandy's life as days, simply because the set of days and the set of years in his life can be put in one-one correspondence (as in figure (ii) for example).

days in Shandy's life: 1, 2, 3, 4, 5, 6, 7, 8, ...
$\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$ $\updownarrow$
years in Shandy's life: 1, 2, 3, 4, 5, 6, 7, 8, ...

figure (ii)

Needless to say, it seems paradoxical to suggest that there are the same number of days and years in Shandy's life, given that there are over 300 days in each year. This does follow in standard theory however and set theorists are left (surprisingly not bothered by) this contradiction (it is a contradiction for to live a certain number of days entails <u>not</u> living that number of years). A set theory which does not reject Euclid's fifth axiom need not be saddled with such fantastic claims.

5. Another well known equivalence paradox, "the innkeeper paradox" is often attributed to Hilbert.

> If an innkeeper has a hotel with a finite number of rooms (however large) and every room is occupied, he must turn away new arrivals. However, this is not true in the case of an innkeeper who has a "fully occupied" hotel with an "infinite" number of rooms. Suppose you were the innkeeper of such a hotel and a new guest arrived. How could you accommodate him without putting guests who were formerly in different rooms in one room? The new guest, of course, must have a room to himself and everyone else simultaneously has a room to himself.
> [Answer:]
> Require all guests to move simultaneously to the next room, that is, if a guest is in room n he must move into room n plus 1. Then put the new guest in room 1.[20]

The conclusion clearly does follow on standard set theory. For imagine that there are a (denumerably) infinite number of rooms in the hotel in question, and a guest in every one. The guests and rooms can be numbered 1, 2, 3,...

Suppose a new guest is placed in room 1. On standard theory it is still possible to place the remaining guests in the remaining rooms, for there are as many guests as remaining rooms. This must be the case because the remaining guests (i.e. those in the hotel before the new guest arrived) can be put in one-one correspondence with the remaining rooms (the rooms 2, 3, 4,...). The first guest is paired with the second room, the second guest with the third room, the third guest with the fourth room and so on. Guest n is paired with room n+1, as in figure (iii).

guests:	1, 2, 3, 4, 5, 6, 7, 8, ... n, ...
remaining rooms:	2, 3, 4, 5, 6, 7, 8, 9, ... n+1, ...

figure (iii)

Because the two sets can be put in one-one correspondence it is concluded that there are as many remaining room as guests. So on standard theory, the innkeeper can allow more guests into his "fully occupied" hotel. Yet to be fully occupied is to have no room for more guests. So again we meet paradox (and again paradox that is nonchalantly accepted by set theorists).

Some may suggest that Hilbert's paradox simply shows that a hotel may be fully occupied and still have room for more guests. Such a position is at best ad hoc. It contradicts ordinary language (as any dictionary will testify) and indulges in dogmatism (dogmatism which holds so strongly to a theory that it pronounces consequences of the theory consistent without reasonable consideration).

6. We could continue to devise new equivalence paradoxes indefinitely. Imagine for example, that there are two clocks (the same size) that keep time ad infinitum (they continue to keep time forever). The minute hand

of the first clock makes one revolution once an hour and the minute hand of the second clock makes two revolutions an hour. The minute hand of the second clock moves twice as fast as the minute hand of the first. On standard theory however, it must be concluded that the two minute hands make the same number of revolutions. For though one of the hands makes twice as many revolutions an hour as the other, the revolutions of both clocks can be put in one-one correspondence (as in figure (iv)). In modern set theory it follows that the two minute hands make the same

first clock's revolutions:	1, 2, 3, 4, 5, 6, ...
	↕ ↕ ↕ ↕ ↕ ↕
second clock's revolutions:	1, 2, 3, 4, 5, 6, ...

figure (iv)

number of revolutions. Even though the second minute hand revolves twice as quickly as the first, and engages two revolutions for every one engaged by the first. The second minute hand moves more quickly than the first at all times, but does not make any extra revolutions. So we have another paradox for set theory. More paradoxes could <u>easily</u> be constructed.

7. The unintuitiveness of set theory in regard to equivalence paradoxes is widely recognized, but it is generally claimed that intuition must simply be rejected.

> Two sets are said to have <u>the same cardinal number</u> when all the things in the sets can be <u>paired off</u> one-to-one. After the pairing there are to be no unpaired things in either set... In a Christian community practising technical monogamy, if twenty married couples sit down to dinner the set of husbands will have the same cardinal number as the set of wives.
>
> As another instance of this 'obvious' sameness, we recall Galileo's example of the set of all squares of positive integers and the set of all positive integers:
>
> 1, 4, 9, 16, 25, ...n^2...
> 1, 2, 3, 4, 5, ...n ...
>
> The 'paradoxical' distinction between this and the preceding examples is apparent. If all the wives retire to the drawing room, leaving

> their spouses to sip port and tell stories, there will be precisely twenty human beings sitting at the table, just half as many as there were before. But if all the squares desert the natural numbers, there are just as many left as there were before. Dislike it or not as we may (we should not, if we are rational animals), the crude miracle stares us in the face that <u>a part of a set may have the same cardinal number as the entire set</u>. If anyone dislikes the 'pairing' definition of 'same cardinal number', he may be challenged to produce a comelier. Intuition (male, female, or mathematical) has been greatly overrated. Intuition is the root of all superstition.[21]

Though such remarks are typical, set theorists are wrong about the 'necessity' of rejecting common sense in this part of set theory. It is a simple matter to construct a set theory which retains Euclid's fifth axiom and rejects equivalence paradoxes. Such a theory need not postulate "crude miracles" within its bounds.

Chapter 10

Revising Set Theory

1. To undermine equivalence paradoxes we should first consider ordinary ways of designating the sizes of sets (that is, the number of elements they contain). Though the size of a set may be assessed in terms of natural numbers, weaker means of assessment are possible. We might say that a particular set has "more than three", "less than a million", or "twenty or twenty-one" members. In such instances we indicate the size of a set to some extent, but not as strongly as if we used natural numbers. We shall say that values (for example "more than three") which group together sets that are similar in respect to size but not necessarily the same size, are "weak cardinal values". Values which group together sets that are <u>exactly</u> the same size (have <u>exactly</u> the same number of members) will be called "strong cardinal values". When used to designate set sizes, natural numbers are strong (not weak) cardinalities, for all sets with the same cardinal natural number have exactly the same number of members. On the other hand, the weak cardinal value "more than

three" subsumes all sets with more than three members. Though all such sets are similar in having more than three elements, they are not <u>exactly</u> the same size (consider a set with six members, and one with ten). Sets with the same weak cardinality may have a different number of members, though sets with the same strong cardinality do not.

2. Intuitively, transfinite cardinals are <u>weak</u> cardinalities, and <u>not</u> strong cardinalities. To say the elements of a set are denumerably infinite for example, is to say that there are more elements than can be accounted for by any natural number, and fewer than can be accounted for by higher order infinities. It is not at all clear that two sets with the same transfinite cardinal have the same number of members (for the assumption that they do leads to problematic consequences - the rejection of Euclid's fifth axiom and equivalence paradoxes). The set of all even natural numbers does seem to have fewer members than the set of all naturals. Simply because the set of all natural numbers encompasses all even naturals <u>and</u> all odd naturals. This intuitive view can easily be accommodated by viewing transfinite cardinals as <u>weak</u> cardinalities. As weak cardinalities can subsume groups with different numbers of members, so too may transfinite cardinalities if they are weak cardinal values. The set of all even naturals and the set of all naturals may have the same transfinite cardinal (because they can be put in one-one correspondence) but not the same number of members.

3. There are a number of reasons why transfinite cardinals should be seen as weak cardinalities. First off, such a view allows us to retain Euclid's fifth axiom in regard to abstract (infinite) sets. If transfinite

cardinals are weak cardinalities then a group and one of its subgroups may have the same transfinite cardinal without it following that they have the same number of members (for groups with different numbers of members may take the same weak cardinality). The weak cardinality less than ten may encompass a group with nine members and one of its subgroups with six members, without it being the case that they have the same number of members. Weak cardinalities and weak equivalence cannot be used to refute Euclid's fifth axiom. Accepting transfinite cardinals as weak cardinals allows us to retain the axiom and such retention is compelling, for we are not at liberty to reject 'standard' laws of logic where they need not be rejected. Keeping the fifth axiom also allows us to retain intuitive views of set size. Though it can still be granted that the set of all natural numbers and the set of all even naturals have the same cardinality (the same cardinal 'number') they share the same weak cardinality, and therefore may differ in size. The set of all naturals is the larger set because it is more inclusive; it contains all elements of the other set, plus an infinite number of odd naturals as well.

In a set theory that retains Euclid's fifth axiom it can be said that a set A which is more inclusive than some set B is larger than B. A is more inclusive than B if it contains all the members of B and other members as well. It follows that a set is always larger than any of its proper subsets (for it is more inclusive). This form of set theory need not dismiss Euclid's fifth axiom.

4. Adopting the view that transfinite cardinals are weak cardinalities allows us to eliminate all equivalence paradoxes from set theory. Consider for example, the Tristram Shandy paradox. It is the case that

the days and years in Shandy's life can be put in one-one correspondence and it does follow that the two sets have the same cardinal number. Yet this only shows that the two sets share the same weak cardinality (they are weakly equivalent). It does not follow that there are (exactly) the same number of years and days in Shandy's life. Because there are over 300 days in every year, Shandy's life includes more days than years.

The innkeeper paradox can be handled in a similar way. In the hotel in question it is the case that the number of hotel rooms available when a new guest is admitted is weakly equivalent to the number of guests in the full hotel. That the set of remaining rooms and the set of guests have the same cardinal number is established by correspondence given earlier. But it does not follow that there are the same number of guests and remaining rooms. Because the total number of guests to be accommodated includes one more guest than those earlier accommodated there are not enough rooms. The set of guests now to be accommodated is larger than the set previously accommodated (for it contains all those guests plus one more) and therefore cannot be accommodated. Though standard theory must accept contradiction, it is not necessary to claim that a hotel might be fully occupied and still have room for new guests.

The 'clock' paradox can also be undermined. Though the number of revolutions made by the two minute hands in question are not the same the two sets of revolution do have the same cardinal number. Because the set of revolutions for the second clock includes two revolutions for every revolution in the other set, it contains more revolutions. So the minute hand which moves faster does make a greater number of revolutions. Another paradox is thereby eradicated. The other equivalence paradoxes that could be constructed can be dismissed in similar fashion.

5. On the given set theory, one-one correspondence in regard to infinite sets does not establish 'strong' equivalence (equivalence in regard to strong cardinality). It does not at all follow that all infinite sets have the same (weak) cardinality. Just as there are distinct finite weak cardinal values (for example, "more than four", "less than two hundred and forty two", and "ten or eleven"), we can postulate a variety of transfinite weak cardinal values.

Though an infinite number of weak cardinalities can be applied to infinite sets (for example, "more than one", "more than two", "more than three",...) we shall single out the group of such cardinalities currently thought of as strong cardinalities. For convenience it is only these cardinalities (i.e. those established by one-one correspondence) which we shall refer to as "transfinite cardinals" and "transfinite cardinal numbers". In a similar way we might distinguish a series of finite weak cardinalities like "less than three", "less than six", "less than nine"... (though one-one correspondence would not directly establish these cardinalities).

The size of different transfinite cardinals can be established on the same grounds currently alleged to distinguish the sizes of distinct strong transfinite cardinals. Because some infinite sets can be put in one-one correspondence with the natural numbers and others cannot, the latter have a larger transfinite cardinal than the former. The set of prime numbers and the set of even numbers have the same transfinite cardinal whereas the set of irrational numbers has a larger one. There is nothing inherently strange about talk of a series of successively larger weak cardinalities (consider the series "more than three", "more than six", "more than nine",...).

Because our revised set theory distinguishes transfinite cardinals it allows for transfinite arithmetic. We may, for example, retain the usual definition of the addition of cardinal numbers.

> If a and b are cardinal numbers, and if A and B are disjoint sets with card A = a and card B = b, we write by definition, a+b = card (A∪B).[22]

In doing transfinite arithmetic things are done as usual, except that they are done with the understanding that the transfinite cardinals in question are weak (not strong) cardinalities. It may seem strange to talk of the addition of weak cardinalities but such strangeness explains the surprising results of such addition (for example the result that $\aleph_0 + \aleph_0 = \aleph_0$). These results make the assimilation of finite and transfinite cardinal numbers as strong cardinalities untenable.

Still more needs to be said about the suggested revision in set theory. But before further discussion we should review what that revision is. In natural language we clearly do talk of weak cardinalities. There is no reason why such a notion cannot be carried to the realm of set theory. If it is, and if it is accepted that transfinite cardinalities are weak cardinalities then mathematics may retain Euclid's fifth axiom and equivalence paradoxes are completely undermined. Such a move retains Frege's notion of "same number" only in regard to finite sets. The resulting theory is far more intuitively appealing than one that insists on the extension of Frege's principle to infinite sets (those who belittle intuitive appeal forget that it is just this which established Frege's principle in the first place). Because the new theory distinguishes different transfinite cardinalities along standard lines (except that it postulates them as weak cardinalities) transfinite cardinal arithmetic is retained.

Chapter 11

More Considerations

1. It may be suggested that transfinite cardinals are established by one-one correspondence and therefore cannot be weak cardinalities. Because one-one correspondence establishes strong cardinality for finite sets, it may be suggested that it cannot establish weak cardinality for infinite sets. Such a view begs the question, for it _assumes_ that one-one correspondence is analogous for finite and infinite sets when this is just what is at issue. There are numerous differences between finite and infinite sets which show finite and infinite one-one correspondence _not_ to be analogous.

2. First off, one-one correspondence establishes that a particular finite set has a particular _natural_ number for its cardinal number. If one-one correspondence acted analogously in regard to infinite sets it would have to be the case that it established for a particular infinite set that it had a particular _natural_ number as its cardinal number. Obviously this is not the case. The simple fact that finite sets have natural numbers

for cardinal values and infinite sets do not shows that one-one correspondence does is not analogous in regard to finite and infinite sets. Other obvious differences are the ways in which transfinite cardinals and finite cardinals are related to sets (for example the fact that a set and one of its proper subsets can have the same transfinite cardinal but not the same finite cardinal). Given these differences it seems an open question whether or not transfinite cardinals are strong cardinalities (given that finite cardinals are). As the view that transfinite cardinalities are strong cardinalities leads to contradiction and the rejection of Euclid's fifth axiom, the most reasonable move is to reject the view. One cannot reasonably accept contradiction and reject 'standard' laws of logic when there is <u>no need</u> to.

3. Much more can be said for the view that transfinite cardinals are weak cardinal values. The one-one correspondence that exists with regard to infinite sets is <u>not</u> the same as that which exists in regard to finite sets. The difference can best be brought out by an example. Consider for the moment finite sets only. In regard to them we might adopt the principle that a set A has <u>twice as many</u> members as a set B if and only if its elements can be put in two-to-one correspondence with the elements of B (each element is paired once). The set {1,2,3,4} has twice as many members as the set {1,2} for the elements of the sets can be paired as in figure (v). We can call this principle the 'doubling' principle. Such a princi-

1,2	3,4
1	2

figure (v)

ple has the same intuitive appeal as Frege's principle for deciding whether or not two sets are equinumerous. Recall again the remarks of

Hans Hahn on the use of Frege's principle to define set equivalence.

> Thus we have the following definition. Two sets are called equivalent if between their respective members a one-to-one correspondence is possible; and the characteristic that one set has in common with all equivalent sets, and by which it distinguishes itself from all other sets not equivalent to itself is called the (cardinal) number of that set. And now we make the fundamental assertion that in these definitions the finiteness of the sets considered is in no sense involved, the definitions can be applied as readily to infinite sets as to finite sets. The concepts "equivalent" and "cardinal number" are thereby transferred to sets of infinitely many objects.[23]

<u>Exactly</u> the same argument can be used to argue for extension of the doubling principle to the realm of abstract sets. We make the fundamental assertion that in the definition of the doubling principle the finiteness of the sets considered is in no sense involved. The definitions can be applied as readily to infinite sets as to finite sets. The concept of "twice the number of elements" could therefore be transferred to sets of infinitely many objects. Of course, if we extend the concept in this way havoc breaks loose. For example, the set of natural numbers can be put in two-one correspondence with itself (as in figure (vi) and so it must have twice as many members as itself!

1,2	3,4	5,6	7,8	...
1	2	3	4	...

figure (vi)

The obvious move to make here is to suggest that the principle simply cannot be extended in the suggested way. Yet when we reach contradiction by the extension of Frege's concept of same number standard, theorists simply accept the contradiction. It can with equal reason be suggested in terms of the doubling principle that there are twice as many elements in the set of natural numbers as in the set of natural numbers (one can imagine someone arguing that here intuition must be discarded). The same moves are appropriate in both cases (and both situations warrant the conclusion that the concepts in question cannot be extended to infinite sets without comment).

The doubling principle demonstrates a crucial difference between one-one correspondence for finite sets and one-one correspondence for infinite ones. Finite sets that may be put in one-one correspondence can only be put in one-one correspondence (and not two-one correspondence, three-one correspondence, etc.). Finite sets that have the same number of members can be put in one-one correspondence, whereas finite sets which have a ratio of m/n members can be put in m-to-n correspondence (the doubling principle is an instance of this ratio). In regard to infinite sets (as shown) this kind of correspondence breaks down (the set of natural numbers can be put in one-one, two-one, three-one,... correspondence with itself). Finitely 'equivalent' sets can only be put in one-one correspondence, whereas infinite 'equivalent' sets may have a variety of correspondences. One-one correspondence does not establish uniqueness for infinite sets the way it does for finite sets. This distinction between the two kinds of one-one correspondence makes plausible the suggestion that such correspondence determines different things in regard to finite and infinite sets (i.e. strong cardinality for finite sets and weak cardinality for infinite sets).

4. Those who suggest that transfinite cardinalities are strong cardinalities may argue that this is shown by the fact that addition and multiplication can be extended in regard to such cardinalities. Such a view ignores the obvious dissimilarity of transfinite and finite addition and multiplication, and particularly the fact that subtraction and division are not defined for such cardinals. One cannot get very far

arguing that the extension of arithmetic to transfinite cardinals establishes them as strong cardinalities when only half of it is extended, and with disanalogous results. Furthermore, we can give other series of values which clearly are not strong cardinalities and yet are closed under addition and multiplication. Thus consider the series f, $\aleph_0$, $\aleph_1$, $\aleph_2$, ... A set has f members if and only if it has a finite number of members. f clearly is a weak cardinal value, and the indicated series is not a series of strong cardinalities. Yet it is closed under cardinal addition and multiplication. So it cannot be argued that closure under such operations show that transfinite cardinals must be strong cardinals.

5. One final point should be made for the view that transfinite cardinals are weak cardinal values. If one mistakes obvious weak cardinalities for strong cardinal values then equivalence paradoxes do result. This suggests that the equivalence paradoxes in set theory do result from such misassimilation.

Consider for example, an innkeeper whose hotel has more than seven rooms. By confusing weak and strong cardinal values we can reason along the lines imbedded in set theory to the conclusion that the innkeeper is in a very enviable position. If the hotel is full, it has more than seven guests. If a new guest arrives he can be given the first room and there still remain more than seven rooms to be occupied. So there are more than seven remaining guests and still more than seven empty rooms. If weak and strong cardinalities are confused it may be concluded that there are enough rooms remaining for all previous guests (for the set of remaining rooms and the set of guests share weak cardinality). Does it follow that the manager of a hotel with more than seven rooms need not turn away new guests when his inn is fully occupied? Obviously not. The reasoning rests on considering the grouping value "more than seven" as a strong cardinality. Just because one has more than seven guests to accommodate and more than seven available rooms, it does not follow that each guest can have a separate room. In a similar way, just because an innkeeper has an infinite number of rooms available and an infinite number of guests, it does not follow that they can be accommodated one to a room.

By confusing weak and strong cardinal values, it is possible to construct paradoxes analagous to the equivalence paradoxes of set theories. Here is more evidence that these paradoxes in set theory result from such confusion.

6. The view that transfinite cardinals are weak cardinalities also undermines geometrically based rejections of Euclid's fifth axiom. There are alleged proofs (illegitimate for reasons I will not discuss) that attempt to show that the points of any two lines can be put in one-one correspondence whatever their length (see H. Delong[24], or E. Kramer[25], or Kamke[26]).

If this is the case then the points of a line ab and a part of ab may be put in one-one correspondence, and hence it is argued that the number of points on the whole line is the same as the number of the points on the partial segment. Again it is concluded that the whole of a set may have (exactly) the same number of members as a part of that set: the set of all the points on a line may have the same number of members as a set of all the points on a part of the line).

On the view presented, even if such one-one correspondence could be established, it would not undermine Euclid's fifth axiom. For such correspondence only establishes weak equivalence, and not that the two sets have the same number of members. Because the set of all points on the line includes the members of the other set, and other elements as well, it is the larger set.

7. Consider again the history behind the notion of equivalence employed within modern set theory.

> Galileo noticed that each number has a double and that each even number is the double of just one number [sic, actually Galileo's considerations concerned the one-one correspondence between the natural numbers and their squares[27]]:
>
> | | | | | | |
> |---|---|---|---|---|---|
> | 1 | 2 | 3 | ... | n | ... |
> | 2 | 4 | 6 | ... | 2n | ... |
>
> so that by the criterion of equivalence mentioned before, there are as many numbers as there are even numbers. But, one is inclined to say, there are "more" numbers than even numbers (indeed, "twice as many"). Galileo (and Leibniz after him) concluded that there could not be infinite numbers. The difficulty is real enough. If we (1) adhere to the equivalence criterion for having the same number, and also (2) allow that if A and B are classes, and B contains everything in A and some things not in A, then the number of B is greater than that of A, we cannot assign a number uniquely to the class of inductive numbers...
>
> The solution of the difficulty lies in giving up the assumption that if A is a proper subclass of B, the number of things in B is greater than the number of things in A. This assumption together with the mapping n-2n, the equivalence criterion for sameness of number, and the assumption that every class has just one number, form an inconsistent quadruple. Something had to go and it was the first assumption that went.[28]

The wrong assumption went. A more appealing set theory (one with Euclid's fifth axiom and without equivalence contradictions) is to be gained by rejecting the view that one-one correspondence establishes exact sameness in regard to the number of members in infinite sets. Rather, such correspondence establishes weak cardinality <u>in regard to infinite sets</u>. In rejecting standard set theory we reject startling and untenable views.

> When a one-to-one correspondence exists between the elements of one collection and the elements of another, the number in one set is said to be the same as in the other. Let us emphasize the nature of one-to-one matching by calling to mind some familiar illustrations.

A telephone directory matches subscribers with telephone numbers in one-to-one fashion (if we exclude those with more than one telephone). On a map where all cities of population over 100,000 are indicated by red dots, there will be a one-to-one correspondence between red dots on the map and large cities in the region mapped...

Let us then carry on a piece of one-to-one matching that will have more startling results! Let us write the positive integers in a row, thus- 1,2,3,4,5,6,7,8,9,10,11...- and agree that each integer has a successor, however far out in the series we may go. Now let us pair each integer with its double.

1	2	3	4	5	6	7	8	...	n	...
2	4	6	8	10	12	14	16	...	2n	...

Then, since we have agreed that two sets of things are equal in number when they can be paired in one-to-one fashion, there are as many numbers in our first set as in the second; that is, <u>the number of even integers is the same as the number of all integers</u>. "But surely", you exclaim, "this second set is merely a part of the first! The even integers are only part of the whole set of integers!" Evidently we have, in the set of integers a collection that is <u>numerically equal to a part of itself!</u>[29]

Two sets M and N are called <u>equivalent</u> if they can be put into one-to-one correspondence; that is, if it is possible by some rule to associate each element of M with exactly one element of N, and vice versa...

Observations such as these become somewhat paradoxical if we reflect on the meaning of the statement that two sets have the same number of elements. Cantor suggested the extremely plausible definition that two sets have the same (cardinal) number of elements if they are equivalent. This follows ordinary usage very closely. For example, if in an auditorium every person occupies exactly one set and if there are no empty seats, we say that there are the same number of seats as people. Whatever number we assign to the collection of people. From this it would follow, for example, that the set of even natural numbers contains just as many numbers as the set of natural numbers. This conclusion goes against common sense as well as traditional mathematical conceptions.

...Thus what Cantor did was to show that the axiom <u>the whole is greater than the part</u> is false when applied to infinite sets.[30]

8. Revising the notion of equivalence in set theory does not leave us with an inadequate set theory. Nothing that has been said discounts the standard mathematics concerning set theory. Rather, standard mathematics is seen in a different light.

Perhaps the confusion that has arisen in regard to transfinite cardinals would never have occurred if those concerned had not assimilated finite and infinite cardinals so closely. They would have done well to heed more closely the advice of Cantor.

> All so-called proofs of the impossibility of actually infinite numbers are, as may be shown in every particular case and also on general grounds, false in that they begin by attributing to the numbers in question all the properties of the finite numbers; whereas the infinite numbers, if they are to be thinkable in any form, must constitute quite a new kind of number as opposed to the finite numbers, and the nature of this new kind of number is dependent on the nature of things and is an object of investigation, but not of our arbitrariness or our prejudice.[31]

Chapter 12

Russell and the Achilles Paradox

1. The essence of Russell's treatment of the Achilles paradox is the rejection of Euclid's fifth axiom. He correlates the Tristram Shandy paradox and Zeno's paradox, and argues that the first must be accepted and the second rejected within set theory.

> This paradox, which, I shall show, is strictly correlative to the Achilles, may be called for convenience the Tristram Shandy.
>
> In cases of this kind, no care is superflous in rendering our arguments formal. I shall therefore set forth both the Achilles and the Tristram Shandy in strict logical shape.
>
> I. (1) For every position of the tortoise there is one and only one of Achilles; for every position of Achilles one and only one of the tortoise.
>
> (2) Hence the series of positions occupied by Achilles has the same number of terms as the series of positions occupied by the tortoise.
>
> (3) A part has fewer terms than a whole in which it is contained and with which it is not coextensive.
>
> (4) Hence the series of positions occupied by the tortoise is not a proper part of the series occupied by Achilles.
>
> II. (1) Tristram Shandy writes in a year the events of a day.
> (2) The series of days and years have no last term.
> (3) The events of the nth day are written in the nth year.
> (4) Any assigned day is the nth, for a suitable value of n.
> (5) Hence any assigned day will be written about.

> (6) Hence no parts of the biography will remain unwritten.
> (7) Since there is a one-one correlation between the times of happening and the times of writing, and the former are part of the latter, the whole and the part can have the same number of terms.
>
> ...These two paradoxes are correlative... The Achilles assumes that the whole and part cannot be similar, and deduces a paradox; the other, starting from a platitude, deduces that whole and part may be similar. For common sense, it must be confessed, this is a most unfortunate state of things.
>
> There is no doubt which is the correct course. The Achilles must be rejected, being directly contradicted by Arithmetic. The Tristram Shandy must be accepted, since it does not involve the axiom that the whole cannot be similar to the part. This axiom, as we have seen, is essential to the proof of the Achilles, and it is an axiom doubtless very agreeable to common sense. But there is no evidence for the axiom except supposed self-evidence, and its admission leads to perfectly precise contradictions. The axiom is not only useless but positively destructive, in mathematics, and against its rejection there is nothing to be set except prejudice.[32]

There is no need to comment on Russell's polemic, a hardy prejudice against contradiction does not need defending here. Two things must be said of Russell's remarks. First, it has been shown that we can reject the Achilles paradox without discarding Euclid's fifth axiom, and without accepting equivalence paradoxes like the Tristram Shandy (the given solution to the paradox does not suffer these grave shortcomings). Secondly, Russell is wrong to think his position provides a real solution to the Achilles paradox. Though it does undermine the version of the paradox <u>he</u> gives we can easily create other versions which do not require the premise that "A part has fewer terms than a whole..." Instead we can give the paradox as follows:

(1) In order to catch the Tortoise Achilles must complete an infinite number of parts of the catching, one by one.
(2) An infinite series of parts of the catching cannot be completed one by one.
(3) Hence Achilles cannot catch the tortoise.

This form of the paradox is impervious to Russell's solution. Perhaps some have been convinced by Russell's position because they (mistakenly) accepted the rejection of Euclid's fifth axiom within standard set theory.

PART IV

OTHER SOLUTIONS TO THE PARADOXES

While there may be serious doubt about the subtlety and profundity of the arguments Zeno actually propounded, there can be no doubt that subtle and profound problems have arisen from the consideration of his paradoxes.

WESLEY C. SALMON, Zeno's Paradoxes[1]

Zeno's arguments, in some form, have afforded grounds for almost all theories of space and time and infinity which have been constructed from his time to our own.

BERTRAND RUSSELL, Our Knowledge Of The External World[2]

Chapter 13

A Standard Retort

1. The usual answer to the problems posed by Zeno results from a consideration of a weaker form of the paradoxes than we have considered. It is usually suggested that the Dichotomy paradox alleges motion to be impossible because it requires the completion of an infinite number of segments of motion <u>in a finite time</u>, when an infinite amount of time is required for such completion.

When the paradoxes are portrayed in this way it is often concluded that they can be discounted by the simple assertion that a finite amount of time can contain an infinite number of successive time intervals. Quine, for example, writes that:

> Some of the ancient paradoxes of Zeno belong under the head of falsidical paradoxes. Take the one about Achilles and the tortoise. Generalized beyond those two fictitious characters, what the paradox purports to establish is the absurd conclusion that, so long as a runner keeps running, however slowly, another runner can never overtake him. The argument is that each time the pursuer reaches a spot where the pursued has been, the pursued has moved a bit beyond. When we try to make this argument more explicit, the fallacy that emerges

> is the mistaken notion that any infinite succession of intervals of time has to add up to all eternity. Actually when an infinite succession of intervals of time is so chosen that the succeeding intervals become shorter and shorter, the whole succession may take either a finite or an infinite amount of time. It is the question of a convergent series.[3]

Similarly, Wesley Salmon writes of the Dichotomy and Achilles paradoxes that:

> To whatever extent these paradoxes raised problems about the intelligibility of adding up infinitely many positive terms, the nineteenth-century theory of convergent sequences and series resolved the problem.[4]

Such remarks are not telling against Zeno's paradoxes as we have constructed them. Zeno's conclusions have been construed as following from the incompletability of an infinite series of successive parts one by one, and not from any alleged time requirement. An infinite series seems incompletable irregardless of time, simply because it requires the one by one completion of an infinite series of elements. For though there might be enough time to permit the completion of an infinite number of separate motions, such a series would still be incompletable (simply because the completion of any motion in the series always brings one to a new motion that must be completed, and never to the last motion in the series). Commentators have simply assumed that an infinite series of motions can be completed without a last motion being completed. But to say there is no last motion is just to say there is no motion after which there are no motions left (that is, no motion which completes the series of motions). Given that no motion in the series completes the series, the completion of such motions one by one cannot lead to the completion of the series. To complete the series of motions one must complete a last motion, that motion is the motion last finished before the motions are stopped.

The popularity of the standard solution to Zeno's paradoxes is widespread. Grunbaum's solution (which we will consider later) is a version of it, and it is the solution suggested by such basic sources as the Encyclopedia Britannica.[5] In the Encyclopedia of Philosophy G. Vlastos writes the following of the Dichotomy.

> The argument comes to this:
>
> [F1] To reach G the runner must traverse all Z-intervals (make all the Z-runs).
> [F2] It is impossible to traverse infinitely many intervals (make infinitely many Z-runs).
> [F3] Therefore, the runner cannot reach G.
>
> But why would Zeno assert [F2]? Probably because he made the following further assumption:
>
> [F4] The completion of an infinite sequence of acts in a finite time interval is logically impossible.
>
> This assumption has enormous plausibility. Even in our own time several distinguished thinkers have argued that it is true (Weyl, p. 42; Black, pp. 95 ff.; Thomson, 5 ff.). Has a good case been made for it? An easy way to do so would be to assume that "completing" the sequence here could be defined only as "performing all the acts in the sequence, including the last." If such a definition were mandatory, then, of course, a completed infinite progression such as the Z-sequence (which can have no last member) would be as flat a contradiction as a round square. But "completing" the sequence can be defined, alternatively, as "performing all the acts in the sequence" or "reaching the point when no more acts in the sequence remain to be performed, having omitted none" (see Watling, 39, Owen, 205). Hence, to settle the issue by recourse to the first definition would be to beg the question.[6]

Such comments are at best naive. Within a series of acts "reaching the point when no more acts in the sequence remain to be performed, having omitted none" is to reach a point where one has completed a last act. In undertaking the series all one does is perform its acts one by one. And if none of these acts is a last act then there is no act which brings one to the completion of the series (and hence no completion of the series).

One cannot complain that we beg the question "Can a series of acts be completed without a last act?" when we assert that a last act is required for such completion. Such a view begs the question only in the sense that one begs the question "What is a bachelor?" by saying that "A bachelor is an unmarried man.". "A last act" simply <u>means</u> "an act which completes the series". If there is no such act then one cannot finish the series by completing acts one by one. One does not show that an infinite series of acts can be completed without a last act by showing that such completion can be described without using the locution "last act". All the descriptions used in its place are synonymous. We might on equal grounds deny that a wolfhound is a dog because it can instead be described as "canis familiaris".

Quine and others (for example, Lewis Carroll[7]) who endorse this solution to Zeno's paradoxes simply assume that if there is enough time for an infinite series of motions then such a series is completable. Such an assumption is gratuitous and ignores what must be explained. Such a criticism cannot be made against our solution to the paradoxes of motion.

Chapter 14

Aristotle

1. In essential points, Aristotle's way around Zeno's first two paradoxes is that which we have adopted. As we have distinguished between separate and continuous parts, he distinguishes between "infinite in divisibility" and "infinite in extension".

> "Zeno's argument makes a false assumption in asserting that it is impossible for a thing to traverse or severally come in contact with illimitable things in a limited time. For there are two senses in which a distance or a period of time (or indeed any continuum) may be regarded as illimitable, viz., in respect to its divisibility or in respect to its extension. Now it is not possible to come in contact with quantitively illimitable things in a limited time, but it is possible to traverse what is illimitable in its divisibility; for in this respect time itself is also illimitable. Accordingly, a distance which is (in this sense) illimitable is traversed in a time which is (in this sense) not limited but illimitable; and the contacts with the illimitable (points) are made at 'nows' which are not limited but illimitable in number.[a]

Something 'infinite in divisibility' can (in principle) be divided ad infinitum; it contains an infinite number of continuous parts. Something 'infinite in extension' is actually divided into an infinite number of (separate) parts. According to Aristotle that which is only infinite in divisibility (for example, a finitely long continuous motion) is completable. This agrees with the position already developed. We have said that a motion which has an infinite number of continuous parts is completable (provided the motion does not require an infinite amount of time or space) and this is the essence of Aristotle's claim.

Unlike modern commentators, Aristotle fully realized that we cannot elude Zeno's paradoxes simply by saying that an infinite series of (convergent) parts can be contained within a finite time.

> It is true that in our previous studies concerning movement we solved this puzzle by pointing out that since time, just as much as space, is divisible without limit and with respect to this capacity is illimitable, there is no contradiction in a man passing through an infinite number of points in a time which is 'infinite' in precisely the same sense as the distance to be traversed is. But this solution, though adequate as a reply to the question (which was, whether it is possible in a finite time to go through or to count an infinite number of points), does not really settle the underlying truth or get at realities. For what if a man, dropping the element of distance and the question of the possibility of traversing an infinite number of distances in a finite time, were to confine his question to time only; for this contains an illimitable number of divisions? It would then be no solution to say that there is no limit to the divisibility of time itself, but we should have to fall back upon the truth we have just arrived at.[9]

A position like Quine's explains the possibility of an infinite number of movements in a finite time by appealing to the infinite divisibility of time. But then, as Aristotle rightly points out, we can demand an explanation of how an infinite series of time intervals can be completed (for an infinite series of time intervals would have to be completed one by one). Aristotle continues:

> For whoever divides the continuous into two halves thereby confers a double function upon the point of division, for he makes it both a beginning and an end. And that is just what the counting man, or the dividing man whose half-sections he counts, is doing; and by the very act of division both the line and the movement cease to be continuous; for the movement is not continuous unless the mobile and the time and the track with which it is concerned are continuous. And though it be true that there is no limit to the potential dichotomy of any continuum, it is not true that it is actually dichotomized to infinity. But to make an actual bisection is to effect a motion that is not continuous but interrupted, as is patent

> in the case of one who counts the segments; for he must take the bisecting point twice, once as an end and once as a beginning (which we have seen to involve an interruption of continuity) -- I mean if he does not count the continuous line as one, but the separated halves as two. Accordingly, if we are asked whether it is possible to go through an unlimited number of points, whether in a period of time or in a length, we must answer that in one sense it is possible but in another not. If the points are actual it is impossible, but if they are potential it is possible. For one who moves continuously traverses an illimitable number of points only in an accidental, not in an unqualified, sense; it is an accidental characteristic of the line that it is an illimitable number of half-lengths; its essential nature is something different.[10]

Potential segments of a continuum are segments which the continuum could be divided into (that is, continuous parts of the continuum). Actual segments of something are separate parts of it. A continuum with an infinite number of potential parts is completable (provided it is containable within a finite space and time) and a continuum with an infinite number of separate parts is not completable (though in that case it would not be a continuum). The fact that a continuum does not contain an infinite number of separate parts explains the completability of motion, of Achilles' catching of the tortoise, and of any finite continuum. As Aristotle points out, an attempt to count the potential segments of a motion or to actually divide the motion into such segments is impossible, but then continuity is not preserved. Hence it does not follow that the motion itself (which is continuous) is incompletable.

Though Aristotle did not consider the Arrow and Stadium paradoxes as we have presented them he would probably dismiss them as we have, for he clearly rejects the view that time or space is finitely divisible. Of the ('old') Arrow paradox he writes that:

> The third argument...states that a flying arrow is stationary. This results by granting that time is composed of moments; for if this is not granted, there will be no syllogism leading to that result.[11]

> So there can be no primary part of the time: and the reason is that rest and motion are always in a period of time, and a period of

> time has no primary part any more than a magnitude or in fact anything continuous: for everything continuous is divisible into an infinite number of parts.[12]

It should be pointed out that in recent philosophy Stanislaus Quan has adopted a position similar to Aristotle's in regard to Zeno's first two paradoxes. Because his presentation is not clear I will not consider it here.

Chapter 15

Gilbert Ryle

1. In his book "Dilemmas", Gilbert Ryle offers a contemporary analysis of the Achilles paradox. His treatment is not satisfactory, though some of his general views are tenable.

> Certain sorts of theoretical disputes, such as those we are to consider, are to be settled not by any internal corroboration of those positions, but by an arbitration of a quite different kind -- not, for example, to put my cards on the table, by additional scientific researches, but by philosophical inquiries.[13]

We have seen that some apparently empirical disputes (as to the nature of space and time for example) are solvable by philosophic investigation rather than by scientific research. Ryle believes that the problem he deals with arises from a confusion of different logical categories. The solution to philosophical dilemmas like Zeno's is alleged to be the demonstration of the differences between the categories in question.

> Cryptographers have questions not just of one kind but of multifarious kinds. So have navigators. Yet all or most cryptographic questions differ from all or most navigational questions so widely, not only

> in subject-matter but also in logical style, that we should have no reason for surprise if we found that a man, equally well trained in both disciplines, proved to be able to think powerfully and swiftly in the one field but only slowly and inefficiently in the other. A good High Court Judge might, in the same way, be an inferior thinker in matters of poker, algebra, finance or aerodynamics, however, well coached he might be in its terminology and its techniques. The questions which belong to different domains of thought, differ very often not only in the kinds of subject-matter that they are about, but in the kinds of thinking that they require. So the segregation of questions into their kinds demands some very delicate discriminations of some very unpalpable features...
>
> Sometimes thinkers are at loggerheads with one another, not because their propositions do conflict, but because their authors fancy that they conflict. They suppose themselves to be giving, at least by indirect implication, rival answers to the same questions, when this is not really the case. They are then talking at cross-purposes with one another. It can be convenient to characterize these cross-purposes by saying that the two sides are, at certain points, hinging their arguments upon concepts of different categories, though they suppose themselves to be hinging them upon different concepts of the same category, or vice versa. But it is not more than convenient. It still remains to be shown that the discrepancies are discrepancies of this general kind, and this can be done only by showing in detail how the metiers in ratiocination of the concepts under pressure are more dissimilar from one another or less dissimilar from one another than the contestants had unwittingly supposed.[14]

At times Ryle does approach a proper solution of the Achilles paradox. To accept Zeno's conclusion (that Achilles cannot catch the tortoise) is to confuse a way of dividing the catching with its actual nature.

> Confusedly, he attributes to the...race-track a difference from ordinary... race-tracks, which is really a difference between one division procedure and another division procedure. He ascribes a queer endlessness to Achilles' pursuit of the tortoise, where he should have ascribed an uninteresting non-finality to each of a certain, special way of subdividing two miles.
>
> He is behaving somewhat like the boy who, having learned one card game, namely 'Snap', when he comes to a new card game, like Whist, cannot for awhile help assimilating what he has to do with his cards now to the things he has long since learned to do with those same cards in 'Snap'. He is put out at finding that play which works in 'Snap', does not work in Whist, and vice versa. Yet, in a way, he <u>has</u> learned the rules of Whist -- he has learned enough for some purposes, but not well enough to be safe from relapsing now and again into 'Snap' play and 'Snap' thinking. After all, the cards he is playing with now are the same old cards.[15]

But though some of Ryle's general comments about the Achilles paradox are appropriate, his remarks are usually confused and inadequate. He seems to conclude that the paradox is to be solved by the separation of common sense talk and talk about mathematics.

> We talk about a race in one tone of voice, we talk arithmetic in another tone of voice; but in talking of the arithmetic of a race we have to mix our tones of voice, and in doing this we may easily feel -- and even speak as if -- we were talking out of different sides of our mouths at the same time.
>
> We decide factual questions about the length and duration of a race by one procedure, namely measurement. We decide arithmetical questions by another procedure, namely calculation. But then, given some facts about the race established by measurement, we can decide other questions by calculations applied to those measurements.
>
> The two procedures of settling the different sorts of questions intertwine somehow, into a procedure for establishing by calculation concrete, measureable facts about this particular race... Two separate skills do not, in the beginning, intertwine into one conjoint skill.
>
> ...here we have been talking, so to speak, in one breath with the sporting reporter of a newspaper, and in another breath with our mathematics master, and so find ourselves describing a sprint in terms of numerators and denominators and of relations between fractions in terms of efforts and despairs.[16]

We have already seen that the paradox does not require such a radical solution. And even Ryle himself is doubtful of the arguments he presents (he writes that "In offering a solution of this paradox, I expect to meet the fate of so many who have tried before, namely demonstrable failure."[17]).

In examining the Achilles, Ryle considers a situation that might exist for a mother who cuts a cake for her children. She first cuts the cake in an ordinary way, and then in a way suggested by the Achilles (or dichotomy) paradox.

> She now passes the plate around the children in order of decreasing seniority, and in order that bigger children shall have the bigger portions, she instructs the children always to take not just a bit but exactly half of what is on the plate. The first child begins with half the cake, and leaves a half, the second gets a quarter and leaves a quarter, the third gets an eighth and leaves an eighth, and so on...
>
> ...The size of each piece, if the bisection is exact is a measurable and calculable fraction of the size of the original whole cake; the first slice to be eaten was a half-cake, and so on. The sizes of the slices are fixed in terms of the size of the cake. The partition method employed was from the start a method of operating on the cake as a whole. So, let us say, the second child, playing

> Zeno, were to say, 'What we consume never amounts to the whole cake; so I believe that there never was a whole cake of finite size to consume' he could be refuted by being asked what his own first slice was one-quarter of. There must have been the whole cake, for him to get a quarter of it; and a finite one, for his quarter of it was finite. Or he could be asked what it is, according to him, that the parts consumed never amount to.[18]

Ryle's argument in no way succeeds in establishing the invalidity of Zeno's reasoning. His retort to the child who plays Zeno in regard to the cake is adequate only in a very limited sense. It does point out the falseness of the child's conclusion (for there was a finite cake to begin with, and it was then divided) but it fails to show how his reasoning goes astray, and this is what has to be shown in a proper solution of the paradox. Imagine using Ryle's retort against someone who 'proved' an observed motion impossible by using the reasoning of the dichotomy paradox. When the person finishes considering the observed motion we turn to them and remark "Of course there was a completed motion, for that is what we began by halving". Such a comment fails to unravel the dichotomy paradox. It does point out the falsity of Zeno's conclusion, but it fails to show how his reasoning goes astray. We (or most of us) grant from the start that Zeno' conclusions about motion are false and this is demonstrated every day. But this fact does nothing to unravel the paradoxes, it only establishes them as such (for it makes it clear that their conclusions contradict everyday life). In order to undermine Zeno's paradoxes it is not enough to point out the obvious (that is, that their conclusions are false). Rather, one must point out how the reasoning employed within them is defective.

Ryle's answer to the Achilles paradox is analagous to his retort to the child who plays Zeno in regard to the cake. He considers a race between Achilles and the tortoise where Achilles moves at two miles an hour and the tortoise (with a headstart of one mile) moves at one mile an hour. Achilles thus catches the tortoise at the two mile mark.

> Now we spectators of the race might, after the event, go back over this two mile course of his and plant a flag in the ground at the end of each of the sixteen furlongs that Achilles had run. Our last flag would then be planted where the race ended. But now suppose that, when the race is over, we go back over these two miles of the track covered by Achilles, and choose instead to stick one flag into the ground where Achilles started, a second at the half-way point of his total course, a third at the half-way point of the second half of his course, a fourth at the half-way point of the outstanding quarter of his course, and so on. Clearly for every flag we plant, there is always another flag to put in half-way between it and the place where Achilles caught the tortoise. (In fact, of course, we shall soon reach a point where our flags are too bulky for us to continue the operation). We shall never be able to plant a flag just at the place where the race ended, since our principle of flag-planting was that each flag was to be planted half way between the last flag planted and the place where the race ended. In effect our instructions were to plant each flag ahead of the last one but also behind the terminus of the race. If we obey these instructions, it follows that we never plant a flag which is not behind the terminus, and so that we never plant the last flag...
>
> No great mystery seems to confront us here. If we obey the instruction always to leave room for one more flag, we always leave room for one more flag. Nor can the fact that no flag is the last flag convince us that Achille's course was endless, since we knowingly began our flag-planting operations with the datum that this was a two-mile course, the start-line and the terminus of which we knew. The places where we planted our flags were fixed in terms of just this two-mile course, namely one flag at its midpoint, the next at the end of its third half-mile, the next at the end of its fourteenth furlong and so on. We were, all the time, planting flags to makr out determinate portions of the precise two-mile course that Achilles ran. We could, if we had chosen, have worked backwards on the same principle from the terminus of the race; and then we should never get a flag planted on his start-line. Yet this would not persuade us that a race had a finish, but no beginning.
>
> What the distances flagged fail at each stage to amount to is the two mile distance that he had run by the time he caught the tortoise, just because this distance is, according to the instructions, the sum of those flagged distances plus whatever unflagged distance remains outstanding.[19]

Ryle suggests that Zeno's arguments cannot be legitimate because they are based, "though surreptitiously and only by implication", on assumptions that are eventually rejected (i.e. the assumption that Achilles does catch the tortoise). But such a suggestion is completely untenable. For a start, the paradox need not be based on the assumption that Achilles does eventually catch the tortoise. As we have constructed it in Chapter 1 it does not depend on such an assumption for its conclusion. It depends on such an assumption for its paradoxical nature, but not for its arguments. Secondly, even if Zeno's arguments were dependent on the assumption that Achilles catches the tortoise, this would certainly not show them mistaken. Reductio ad absurdum arguments always begin with an assumption (or assumptions) which are eventually rejected. And if such arguments are legitimate there can be nothing wrong with Zeno rejecting an assumption he begins with (for he can be said to be employing reductio). As Ryle does not dismiss reductio arguments outright, and gives no reason why this should be done, his position is unacceptable even if he does correctly construe Zeno's position (and this he does not do). Clearly, Ryle's treatment of the Achilles paradox is unsatisfactory.

Chapter 16

Max Black

1. Though Max Black's general treatment of the four paradoxes of motion might be taken to suggest a position somewhat like the corrected one the details of his consideration differ markedly. Like Ryle, Black fails to distinguish the separate and continuous parts of a thing. His analysis does not solve the paradoxes.

2. Black first considers the Achilles paradox, and denies the paradox's conclusion by denying that there are an infinite number of steps (parts) to Achilles's catching of the tortoise. He argues that an infinite series of parts of the catching cannot be completed, but need not be completed in the actual catching. This suggestion resembles our treatment of the paradox somewhat (for we have said that an infinite series of separate parts can't be completed, but doesn't have to be completed when Achilles catches the tortoise). Black errs by simply stating that the catching does not contain an infinite number of parts (steps). In one sense the

catching <u>is</u> so constituted (that is, of an infinite number of continuous parts), while in another sense it is not so constituted (that is, of an infinite number of separate parts). Instead of distinguishing these two kinds of parts, Black states flatly that the catching does not have an infinite number of parts and thereby denies the infinite divisibility of space and time.

> ...let us return to Achilles. If it really were necessary for him to perform an infinite number of <u>acts</u> ... it would indeed be logically impossible for him to pass the tortoise. But all the things he really does are finite in number; a finite number of steps, heart beats, deep breaths, cries of defiance, and so on. The track on which he runs has a finite number of pebbles, grains of earth, and blades of grass, each of which in turn has a finite though enormous number of atoms. For all of these are things that have a beginning and an end in space or time. But if anybody says we must imagine that the atoms themselves occupy space and so are divisible "in thought", he is no longer talking about spatio-temporal things. To divide a thing "in thought" is merely to halve the numerical interval which we have assigned to it. Or else it is to suppose -- what is in fact physically impossible beyond a certain point -- the actual separation of the physical thing into discrete parts. We can of course choose to say that we shall represent a distance by a numerical interval, and that every part of that numerical interval shall also count as representing a distance; then it will be true a priori that there are infinitely many "distances". But the class of what will then be called "distances" will be a series of pairs of numbers, not an infinite series of spatio-temporal things. The infinity of this series is then a feature of one way in which we find it useful to <u>represent</u> the physical reality; to suppose that therefore Achilles has to <u>do</u> an infinite number of things would be as absurd as to suppose that because I attach two numbers to an egg I must make some special effort to hold its halves together.[20]

What is right about Black's analysis can be captured by distinguishing between the separate and continuous parts in a thing. Black goes too far when he suggests that the series of distances prescribed by Zeno is "a series of pairs of numbers, not an infinite series of spatio-temporal things". To suppose that the numbers do not correspond to actual distances is to suppose that there are no such distances (and hence that distance is only finitely divisible). Black is right in saying that Zeno's series no more shows Achilles to <u>really</u> undertake an infinite number of separate parts of the catching than does the fact that we can attach the fraction 1/2 to an egg show that we must make a special effort to hold the egg together. The egg is not two separate parts. Achilles's catching of the tortoise is not constituted of an infinite number of separate parts. But this does not show that an egg does not consist of two halves, or that Achilles's catching of the tortoise is not constituted of an infinite number of parts. Black goes too far when he concludes that there are not an infinite number of parts to Achilles's catching of the tortoise.

Black mistakenly thinks that there is no problem with rejecting the notion of infinite divisibility.

> ...what reason have we for believing that every extended body must consist of at least two extended parts? Certainly, experience cannot establish it. If true at all, it must be necessarily true, and the strength or weakness of our imaginations has nothing to do with the matter...
>
> Now with regard to the infinite divisibility of space, I must repeat what I have already said about the supposed infinite divisibility of matter. No mental picture that we form of space (a lattice of intangible and transparent cubes?) can have any tendency to support the principle of infinite divisibility; nor can it be established by experience.[21]

We have already seen that there are problems with the view that time and space are only finitely divisible (namely, the new Arrow and Stadium paradoxes). We shall see that Black has no way around these difficulties (in fact, he contradicts himself when he considers the finite divisibility view).

3. Though Black begins by denying the completability of an infinite series of elements, he later relents on his position.

> The argument of the earlier essay took for granted that it was physically impossible for anybody to perform an infinite series of successive "acts" in a finite period of time (or, more generally, to pass through an infinite series of well-bounded changes in such a time). Partly as a result of my critics' comments, I am now no longer confident that any breach of continuity of motion need be

> involved. The following example will help to make the matter clearer.
>
> Suppose Achilles chases the tortoise in a series of leaps, instead of in a smooth motion, and suppose, in order to simplify the discussion, that the tortoise remains at rest while this is going on.
>
> If Achilles's initial distance from the tortoise is d, I thought that an infinite series of jumps 1/2 d, 1/4 d, 1/8 d, ... successively performed in the times 1/2 t, 1/4 t, 1/8 t,... would necessarily involve some breach of continuity at the end point. This is certainly true if Achilles jumps the same height, h, every time. For he is then performing an infinite series of finite vertical oscillations. But suppose the successive heights converge to zero, and let us say for short that the acts performed by him then constitute a convergent series of acts. It is no longer clear that it is physically impossible to perform an infinite convergent series of acts in a finite time. Indeed, if every physical magnitude connected with the hypothetical motion (e.g., the velocities concerned, the forces exerted, etc.) were to define an infinite convergent series of acts, I would have to concede that the motion would not involve abrogation of continuity at the end point. And then I would also have to concede that the illustration of the "infinite machines" did not establish what it was intended to establish, since all of them involved at least one nonconvergent series of acts...[22]

Because he does not distinguish between 'actual' and 'potential' parts of a thing he assumes that an infinite number of separate, but convergent parts can be completed (i.e. because they can be included in a finite time and space). Here he makes the same error as Quine (and Grünbaum). Though it is true that such an infinite series will be containable in a finite time and space it does not follow that such a series is completable. Black's series is incompletable because it involves an infinite series of separate jumps; that is, an endless number of separate jumps. The completion of any jump in the series always brings one to new jumps in the series and never to the completion of the series. The completion of any jump is never the completion of a last jump (that is, the completion of all the jumps in the series). From the fact that the time and space requirements of such a series can be fulfilled, it does not follow that the series can be completed.

In order to retain his claim that Achilles does not perform an infinite series of acts in catching the tortoise, while rejecting the view that an infinite series is incompletable, Black resorts to a new line of reasoning.

> There remains, however, another ground that still leads me to suppose that talk of an infinite series of acts performed in a finite time is illegitimate. I want to argue that it is part of the "grammar" of a word like "jump" that it shall be inadmissible to speak of "jumps" that are <u>indefinitely</u> small or <u>indefinitely</u> brief.
>
> Whenever we are told that Achilles made a certain jump, we are entitled to ask, "How far did he jump?" and "How long did the jump take?" Suppose the answers to the question were to be, respectively, "One-thousandth of an inch" and "One millionth of a second." We should certainly refuse to take such answers seriously. Some people would say the reason for this is simply that what is alleged to have happened is too incredible to be accepted. But I want to urge that there is a <u>logical absurdity</u> in saying that a man jumps a thousandth of an inch, if the word "jump" is understood in any of its ordinary, everyday, uses. When we normally speak or think of a jump made by a man, we have in mind primarily the kind of change of position that can be observed with the unaided sense. The use of the term presupposes the truth of certain generalizations that set limits upon the distances jumped and the times taken for such jumps. When the limits are violated, as in the statement, "Achilles jumped a thousandth of an inch in a millionth of a second", the sentence fails to express a genuine statement...
>
> I believe limitation of scale characterizes the use of a vast number of words applicable to material objects and spatio-temporal events. For example, we can speak sensibly of the color of a tennis ball, but not of the color of an electron. As a special case of limitation of scale, I believe we cannot properly speak of indefinitely brief <u>acts</u>. For this reason I want to reaffirm my original contention (which was central to my discussion of the Achilles paradox) that all the things Achilles "really does are finite in number"... This is to be regarded as a <u>necessary statement</u>, not an empirical affirmation of Achilles' contingent physical limitations. We cannot even imagine what a world would be like in which a person could do infinitely many things in succession in a finite period of time. For that matter, I must now say, we cannot even conceive what it would be like for a man to perform a sufficiently large <u>finite</u> number of acts of a given kind in a finite time. The present objection is to the indefinitely small -- not the infinitely large (though there are other objections to the latter).[23]

Though Black's comments about locutions like "jump" and "act" are convincing, they cannot dismiss Zeno's conclusions. It is not clear, for a start, that his analysis can be carried over to locutions like "distance" and "time interval", and the paradoxes can be translated into such locutions. Even if Black could force us to reject all ordinary language formulations of the paradoxes, he would not thereby undermine Zeno's reasoning. For suppose we did reject the ordinary language formulations of the paradoxes. We could then _introduce_ (by stipulative definition) new jargon and new locutions, and formulate the paradoxes without depending on ordinary language. Instead of using the terms "acts", "jumps", "time intervals" and "distances" to refer to the extremely small phenomena in question we could introduce the terms "macts", "sumps", "fime intervals" and "bistances" to refer to the phenomena in question. We could then reconstruct the paradoxes of motion using this terminology. The paradoxes would sound unfamiliar but they would make the same points, and without violating any of the ordinary language conventions Black depends on. These formulations of the paradoxes could not be refuted on Black's grounds.

4. Black attempts to add strength to his new position by reiterating his denial of the infinite divisibility of space. This denial seems to result from the view that it is the only way out of the Achilles paradox (and of course, we have seen that this is not the case). Such a denial requires answers to Zeno's third and fourth paradoxes of motion, answers which Black does not give. He attempts to undermine the old Arrow paradox by dismissing talk of "motion at an instant" and "rest at an instant" as it _sometimes_ occurs in the construction of the paradox.

> There happen to be good senses in which we _can_ speak of "motion at an instant" or "instantaneous motion" and "rest at an instant" or "instantaneous rest". So it is necessary to show (or, rather to remind ourselves, since everything that needs to be said is familiar) that the sense in which we can speak of a body moving _during_ a period of time is other than the sense in which we can speak of its being in motion _at_ an instant; and the sense in which we can speak of a body being _at_ rest during a period is other than the sense in which we can speak of its being instantaneously at rest. For then it will be clear that the arrow is not at rest at any points of its path except the first and the last, and the argument will collapse. The temptation to say the arrow must be at rest at every instant arises from confusion between the two senses. We are inclined to say, "The arrow is at rest at each instant", when all we are entitled to say is, "It is senseless to speak of a body 'moving _at_ an instant' in the sense in which it is 'moving _during_ an interval of

> time'". Similar confusions arise in connection with the expression "space occupied" that plays an important part in the argument.[24]

But though Black's analysis cuts against some formulations of the arrow paradox (which we need not consider) it clearly does not discount it as it has been put forth here. The fact that Black's comments may be true of our ordinary language use of locutions like "motion at an instant" is beside the point. If one suggests (as Black does) that time is only finitely divisible one must accept that there are smallest possible intervals, beyond which time cannot be divided. For the purposes of discussion we can call these intervals "instants", "moments", or whatever we like. In fact we constructed the paradox without talking of instants at all, but by talking of "moments". If the word "moment" or "instant" is used in the paradox in a way that differs from ordinary language usage, that is beside the point. Black is committed to saying that there are instants in the sense employed in the paradox (for he rejects infinite divisibility) and what we call them is beside the point. Even if we adopt the locution "nonordinary instants" we shall be able to generate Zeno's third conclusion. At most, Black's criticisms can force us to use nonordinary language in formulating the Arrow paradox.

Even worse is Black's handling of the old Stadium paradox, though he fails to realize it. He simply remarks that the paradox does show the untenability of the position that time or space is atomic in nature. He fails to realize that his earlier view (that time and space are not infinitely divisible) commits him to the atomicity view just such a view. He writes:

> ...the supposition that the lengths of the bodies are indivisible leads to insuperable difficulties...
> ...Enough has been said to show that insuperable difficulties follow from any assumption of the atomicity of time, as we already saw them to follow from any hypothesis of the atomicity of space... the hypothesis of the atomicity of space and the hypothesis of atomicity of time produce absurd consequences.[25]

Black thus produces a reductio ad absurdum of his own position. He fails to see that his earlier position concerning the first two paradoxes of motion does not allow him to take this position in regard to the Stadium paradox. He first commits himself to indivisibles (by denying infinite divisibility) and then denies their existence. Obviously such a position is untenable.

Chapter 17

Adolf Grünbaum

1. A recent treatment of Zeno's paradoxes of motion occurs in Adolf Grünbaum's Modern Science and Zeno's Paradoxes. Grünbaum sees the paradoxes as most interesting when construed as arguments against the denseness of spatial and temporal continuums.

> The Dichotomy and Achilles paradoxes are directed against those kinematical theories which assume that between any two instants of time there is at least one other and that between any two space points, there is at least one other. Any class of elements ordered by this kind of betweenness is said to be "dense". It is thus logically possible that diverse classes of elements are each dense with respect to their particular ordering relation of betweenness. And in the case of any one class of elements and a specified betweenness relation, we can ask therefore whether the class constitutes a dense system. Hence, it is significant to ask whether the points of space on a line constitute a dense system no less than to ask whether the instants of time do. By the same token, one can can call the ascription of denseness to physical space into question just as its ascription to time...
>
> By assuming that physical time is a linear mathematical continuum of instants, the modern kinematical theories assert a time interval to be an actually infinite dense set of elements... I shall construe the Dichotomy and the Achilles as offering a *reductio ad absurdum* of the *denseness* of physical time and of motion but *not* as denying their very existence ("reality").[26]

Grünbaum's construction of the paradoxes, and our construction of them can be correlated in <u>some</u> ways.

2. We have construed Zeno's paradoxes of motion as concerning divisibility. If space and time are finitely divisible, they do have smallest segments between which, no segments exist. On such a view space and time are not dense. If space and time are infinitely divisible then they are dense sets of elements. For any two spatial intervals it will be possible to talk of a smaller interval between them. Though Grünbaum builds his account of the paradoxes on a theory of time (of 'becoming') we need not consider that theory here (for such a treatment would be involved and unnecessary).

3. Grünbaum's attack on Zeno's paradoxes does not undermine them as <u>we</u> have constructed them. He sees the paradoxes as arising from the assumption that an infinite number of time intervals cannot be contained within a finite time.

> That our perceptual experience of time poses intellectual obstacles to the denseness postulate for physical time which are not also posed correspondingly in the case of space is patent in the light of the history of Zeno's paradoxes of motion. No mathematically literate person would claim that an avowedly finite space interval is paradoxically infinite on the strength of such infinities as are entailed by its <u>denseness</u>. Specifically, suppose someone were to claim that <u>if there is</u> to be a finite space interval, that interval cannot be the union of either an infinite regression or an infinite progression of subintervals. The reaction which this spatial claim elicits at once is: And why not, pray tell? But the corresponding claim for <u>time</u> intervals constitutes the heart of the conviction which Zeno's Dichotomy and Achilles paradoxes have carried perennially among mathematically literate thinkers.[27]

Grünbaum makes much of the point that an infinite number of segments of space need not constitute an infinite amount of space. Clearly a series of converging segments can be contained in a finite amount of space.

> Within the theoretical framework of standard mathematical physics, consider a linear interval of physical space which is, say, 2 units in length. On that theory, to be such an interval logically involves being a mathematically continuous series of space points. And the denseness property of this kind of continuum assures that, as metricized, the interval includes, for example, a spatial progression of non-overlapping subintervals of lengths
>
> 1, 1/2, 1/4, 1/8,...
>
> If there exists a total interval as postulated by the theory, then *eo ipso* so does such a sequence of subintervals...
>
> Being a progression, our sequence of spatial intervals has no last member. Yet it would be sheer folly mathematically to argue that the absence of a last subinterval in this progression warrants that the total interval, which is known to be spatially finite, must paradoxically also be spatially infinite. I mention this folly because we shall need to see why informed people have been driven to commit the corresponding mathematical folly in the case of time intervals. But why is it a folly to make the allegation of paradox in the spatial case? Because though deneumerably infinite in number, the subintervals are of ever decreasing size such that all of them fit into a total finite interval of length 2 both distributively and collectively; i.e. any one and every one of them fits. That they fit collectively into a finite total interval is evident from the fact that for every n, the notion of the first n of our subintervals in the progression has a length Sn less than 2 given by $Sn = 2 - \frac{1}{2n-1}$ [28]

Grünbaum next suggests (through elaborate argument) that time is dense. We reached the equivalent conclusion (that time is infinitely divisible) simply by using the new Arrow and Stadium paradoxes as reductio arguments against the opposite view. Grünbaum writes:

> It is now clear that just as Galileo recognized that a theoretical cannonical appeal to the sensed attributes of physical objects can be scientifically stultifying, so also the discreteness of perceived coming into being can be misinvoked to encumber the event ontology of theoretical science. And now that we have overcome the objection to the physical intelligibility of temporal denseness by justifying its postulation, we have cleared the way for doing the following in the subsequent parts of this chapter; showing that there is no more reason to infer that a temporally dense set of events must be of infinite duration in virtue of thus being dense than for concluding untutoredly that a spatially dense set of points must be spatially infinite on the strength of its denseness. This demonstration will involve showing in the context of the Dichotomy and the Achilles that the mathematical considerations applied to a finite space interval... can be legitimately carried over to a finite time interval,

> because they are fully as relevant and appropriate in the latter context.[29]

Grünbaum suggests that it is intelligible to talk of an infinite number of time intervals being contained within a finite time interval. But though this seems true, we have already seen (in regard to Quine for example) that it does not undermine Zeno's arguments as they were presented earlier. For the alleged incompletability of a motion (or of Achilles catching of the tortoise) has not been based on the supposition that an infinite number of segments of motion (or an infinite number of time intervals) require an infinite amount of time. The problem is the apparent incompletability of an infinite series of submotions irregardless of time. From the infinite divisibility of time and space it does not follow (as Aristotle realized) that Zeno's conclusions are invalid. It does follow that a finite time interval contains enough <u>time</u> for an infinite series of subintervals, but such a series seems impossible for other reasons. That is, because it requires the one by one completion of an infinite number of successive subintervals. One must agree with Grünbaum when he argues that from the density of a finite temporal or spatial interval it does not follow that the interval is spatially or temporally infinite. Yet on the stronger version of the paradoxes, it still is the case that the interval is incompletable (though not incompletable due to a lack of time). Grünbaum's suggestions do not provide a solution to the strong form of the paradoxes.

4. Grünbaum posits psychological grounds for the forcefulness of Zeno's arguments.

> Since the denseness of time is no longer an issue as such, the refutation which I am about to offer will consist in showing that such infinities as are entailed by the denseness of time do <u>not</u>

> paradoxically allow the deduction that a metrically infinite time interval is required by the runner to traverse the unit space interval. The interrelated false tacit assumptions or fallacies which have served to yield Zeno's paradoxical result of metrical infinity seem to me to arise mainly from the misguided attempt to "reach" the first instant of the motion by the last act of thought of a sequence of thoughts as follows: We begin with the contemplation of the last term of the regression of overlapping temporal subintervals of the motion which the Dichotomy singles out from the total unit time interval, and then we think one by one of the individual members of the regression with a view to thus "reaching" the first instant of the motion "beyond" the regression...
>
> It will be recalled from 2B that human awareness of time exhibits a positive threshold or minimum. This fact can now be seen to have a consequence of fundamental relevance to the appraisal of Zeno's dichotomy argument. For it entails that none of the infinitely many temporal subintervals in the regression whose magnitude is less than the human minimum perceptibilium can be experienced as elapsing in a way that does metrical justice to its actual duration. To succeed, the attempted individual contemplation of all the subintervals would require a deneumerable infinity of mental acts, each of which requires or exceeds a positive minimum duration. Instead of experiencing these subintervals as elapsing in a metrically faithful way, we gain our metrical impression of duration in this context from the time needed by our mental acts of contemplation and not from the respective duration numbers which we associate intellectually with the contemplated subintervals when performing these mental acts.[30]

When we consider the regression (or progression) of temporal subintervals indicated in the dichotomy, every considered subinterval requires a minimal time in our consideration. And because the amount of time so used cannot decrease as do the intervals in Zeno's series, the consideration of all the intervals one by one would require an infinite amount of time (this series of considerations does not converge). Mistakenly, one might conclude that the series of intervals itself requires an infinite amount of time. Such a conclusion is mistaken, but it is not on this conclusion on which the strong form of the dichotomy is based. So Grünbaum's considerations are beside the point.

5. The confusion inherent in not separating the separate parts of a thing and the continuous parts of a thing is shown in Grünbaum's

conclusions about what kind of motion is possible. He compares a continuous 'legato' motion and a 'staccato' motion broken by intervals of no motion.

> The staccato motion is the traversal of the Z-sequence [Zeno's series of prescribed parts in the Dichotomy] in unit time by a runner who runs discontinuously as follows: He takes 1/4 of a unit time to traverse the first Z-interval of length 1/2 and rests for an equal amount of time; then he takes 1/8 of a unit time to traverse the second Z-interval of length 1/4 and rest for an equal amount of time, and so on. I shall refer to the... process as the staccato Z-run. And I shall state the conditions for the following conclusion: If one imagines that the two runners run parallel to one another on essentially the same race course, then the two runners depart jointly and arrive jointly at their final destination after a finite time,... And in so doing, each of the two runners traverses ever smaller space intervals in proportionately ever smaller time intervals, whose successive lengths and durations each suitably converge to zero.[31]
>
> The staccato runner at no time lags behind his legato colleague but is either ahead of him or abreast of him. For while running within each of the Z-intervals, the staccato runner's average velocity is twice that of his legato colleague, but his over-all average velocity for the total interval is equal to his colleague's velocity and is less than the velocity of light in vacuo. It follows that if the legato runner reaches his destination in 1 unit of time after traversing the Z-sequence, then so also does the staccato runner. And this conclusion has the following important consequence: Given that the pauses separating the individual traversals carried out by the staccato runner for a geometric progression whose terms converge to zero, it is immaterial to the traversability of the total unit interval in a finite time that the process of traversal consists of no motions separated by pauses of rest... instead of being one uninterrupted motion which can be analyzed into an infinite number of submotions... And if we wish to call the staccato runner's execution of the no separate motions doing "infinitely many things", then his performance shows that infinitely many things can be done in a finite time. What could reasonably have been expected here in the way of a "proof" that the staccato run is possible is the following: A demonstration that, given the kinematical principles of the theory and the boundary conditions, the theory entails the finitude of the total duration of the staccato run. For the allegation of the impossibility of that run was based on its allegedly infinite duration. Thus, I have given a proof of the possibility of the staccato run.[32]

Wesley Salmon[33] also thinks (mistakenly) that Grünbaum proves the possibility of the staccato run. Grünbaum proves only that there is enough time for the staccato run. And though this is the case, there are other reasons why the run is incompletable. It cannot be finished because there is no last motion in the series of motions it encompasses. It is this impossibility which Grünbaum must address if he wishes to 'prove' the completability of the staccato run.

Grünbaum successfully dismisses Zeno's paradoxical conclusions only if they are alleged to follow from the impossibility of completing an infinite number of space or time intervals in a finite time. Since the conclusions can be based on different considerations his account of them is beside the point.

Chapter 18

Paul Swartz

1. The most recent treatment of Zeno's paradoxes occurs in Paul Swartz's book About Time. Though his solution to the paradoxes is not satisfactory his stance is refreshing. Principally because he realizes the inadequacy of standard treatments of the paradox.[34]

> Many people, especially scientists, are under the impression that Zeno's paradoxes have long since been solved, and are now only of historical or anecdotical value. They are generally referring to the so-called mathematical solution, which seems to have originated with Descartes... It is an interesting fact that, while scientists usually think the mathematical solution conclusive, most philosophers do not. In my opinion this touches an essential point of difference between science and philosophy: science as such is primarily interested in the how of things, and only secondarily in the why, but with philosophy just the opposite is true. Therefore the scientist is satisfied with the mathematical solution, which enables him to calculate when and where A overtakes B, but only a bad philosopher, who does not see the difference between a calculation and a solution, can think the mathematical solution adequate. For the philosophical problem is not whether A can really overtake B; we know very well that it can. Nor is it where and when A overtakes B; that precisely is the mathematical problem. The philosophical problem is why Zeno's arguments seem so convincing, while being obviously untrue. Therefore the philosophical solution must show where the error lies in Zeno's reasoning, and why this error is an error. It will be clear that mathematics is quite unable to provide a solution of this kind.

What now does this mathematical solution consist in? It starts by pointing out that both in the first and in the second paradox we have to do with an infinite series. This series is convergent, and consequently has a limit. The series of the first paradox is t, 1/2t, 1/4t..., in which t is the time needed by the body to cover half the distance to its goal. The limit of this infinite series is of course zero. In the second paradox we have to do with the series t, t/n, t/n^2..., in which t is the time which A needs to arrive at B's point of departure and n is the proportion of the velocities of A and B. As n is greater than one this series also has a limit zero. Now the fact that there series each have a limit zero is seen as corresponding to the fact that (in the first paradox) the moving body reaches its destination, or that (in the second paradox) the faster body A overtakes the slower body B. In a way this is undoubtedly correct, but it is not a solution of the paradoxes. If asked what then Zeno's error was it is generally answered that he did not know the concept of limit. But I find it hard to believe that Zeno did not realize that, as the arguments proceed, the distances and intervals of time we have to do with become smaller and smaller, and that they eventually approach to zero. And this is of course just another way of saying that their limit is zero.

What Zeno evidently meant to say is that these distances and intervals of time may approach to zero, but that they never can become zero. The mathematician cannot rebut this argument for the simple reason that he asserts exactly the same thing. For according to mathematics we may go on indefinitely writing down terms in the series I, 1/2, 1/4... without ever reaching zero. (Likewise, according to mathematics we may go on indefinitely halving an interval of space or time without reaching a point respectively instant of zero extension). So if we say that the above series has a limit zero this does not mean that if we keep on writing down terms we shall eventually reach the term zero. We only mean that for any given small number d we can find a term of this series which differs less than d from zero. The term zero is unreachable, although it can be approached as closely as one wishes. But this is in fact precisely the same thing which Zeno asserts, namely that a moving body can never reach its point of destination. The only thing Zeno omits to say is that the body can come unlimitedly close to its goal. The question how it is possible that in practice the point of destination turns out to be not unreachable at all is left unanswered by the mathematician. He disposes of the problem simply by neglecting the difference between infinitesimal and zero. But from a philosophical point of view this is of course inadmissible.[35]

Though he is not as clear as might be desired, Swartz does discount the kind of solution to the paradoxes endorsed by Grünbaum.

...an infinite series of tasks or events or digits or whatever must be open at at least one end; a series which can be completed is necessarily closed, i.e. finite. Grünbaum, in asserting that a π-machine is logically possible (1970, p. 222 ff.),

> unwittingly commits the very same fallacy he reproaches Bernouilli for, viz. the fallacy of thinking that an infinite series has a last or infinitieth member (Grünbaum, 1967, p. 124). For if this series can be completed, no matter in what way, it is closed, i.e. there must be a last member.[35]

2. Despite Swartz's appropriate criticisms of standard solutions to the paradoxes of motion, his solution is no better. He develops a 'relational' theory of time on the basis of it and quantum physics argues that time and space are only finitely divisible.

> According to the relational theory of space and time intervals of space and time which are so small that they are theoretically immeasurable are simply non-existent. In fact, the very concept of 'theoretically immeasurable interval of space or time' is a contradiction in terms, because an interval only exists by the grace of its being perceptible, at least in principle. It thus appears that distances smaller than about 10^{-15} m and periods of time smaller than about 10^{-22} s simply do not exist. This means also, of course, that a distance of about 10^{-15} m and a period of time of about 10^{-22} sec cannot be divided any more; for the parts they would be divided into would be non-existent.[37]

> Once more, only a solution which shows where Zeno's arguments go wrong, and why, can be considered a real solution. For in the beginning Zeno's line of reasoning is undoubtedly correct: it is indisputable that in order to run a mile I must first run half a mile, then a quarter of a mile and so on. It is just as certain, however, that at the end Zeno's line of reasoning is incorrect, for how else could he arrive at the wrong conclusion? I can run a mile, can I not? Consequently the real point at issue is: at what point (or moment) does Zeno's line of reasoning turn from correct into incorrect, and why?
>
> After what was said in the preceding chapter it will be clear what answers, in my opinion, must be given to these questions: the reasoning turns incorrect when the smallest stretches of space and time measurable are reached, and the reason why is that these stretches are not divisible any more. Even if the moving bodies were infinitely small (which they are not, a point which we shall return to in a moment) Zeno's line of reasoning would only be correct up to a distance of about 10^{-15} m and/or an interval of time of about 10^{-22} c. At that moment there is (in case of the first paradox) no distance any more between the body and its destination, which means that the body has reached its destination. The second paradox is solved in the same way: at a certain moment (a quantum of time) both bodies are at the same point (a quantum of space), and that is when one overtakes the other.[38]

3. A number of things must be said about Swartz's position. To begin with, he bases his solution to the paradoxes on a very questionable theory of time. I shall not consider that theory here but will mention that most philosophers and scientists do not think (as Swartz does) that quantum physics establishes that time and space are discrete.[39]

4. Swartz's solution to the first two paradoxes of motion falls prey to the 'new' Arrow paradox introduced earlier. He talks of smallest intervals of time being 10^{-22} seconds long but such talk makes no sense. For if there are smallest intervals, then they must be of 0 duration. Swartz himself writes:

> ...that events that occur with a certain minimum time (about 10^{-22} s) between them cannot be distinguished into earlier and later any more, which means that they occur simultaneously, i.e. at the same moment.[40]

and that

> Strictly speaking it is logically impossible to speak about anything happening at a point or at a moment, and therefore we may neither say that a body is at reast nor that it is in motion at a point or at a moment. We can only speak about motion or rest during a period of time, for we have to know the positions of the body at at least two moments. This holds for all kinds of change; the concept of change implies that the states of a system at two moments are compared and found different.[41]

It follows that there can be no change of time <u>during</u> a moment. Everything in the moment occurs at the same time, including its beginning and end. On Swartz's view a period of time in which no change occurs must be non-existent, of zero extension.

> In a world in which nothing happened, we could also say, time would be standing still; but as the existence and the flow of time are one and the same thing we can just as well say that in a world in which nothing happened there would be no time either. (We can of course not speak of a certain period of standstill, for this would mean that time, while standing still, was flowing on. We must say instead that the first event after this 'period' follows immediately after the last event before this 'period', in other words, that the period is non-existent.[42]

As he also alleges that nothing can change at a moment, moments must be non-existent! Such intervals could not be 10^{-22} sec. long but would have zero extension.

> For an interval of time which is so small that no clock event occurs in it would simply have no duration, i.e. its duration would be zero.[43]

So Swartz is forced into a position where time is constituted of intervals of zero length. It follows that time is not extended (for it is constituted of extensionless intervals) and Swartz has a contradiction (for time obviously is extended). Similar considerations lead to the unacceptable conclusion that space is unextended. We could also construct the new Stadium paradox as a reductio of Swartz's position.

Chapter 19

Whitrow, Kaiser, James and Hegel

1. Excepting Aristotle, most of what has been said about Zeno's paradoxes of motion is inadequate. There are commentaries not examined here. Whitrow and Kaiser accept the first two paradoxes as genuine logical antinomies.

> It is not surprising that the application of the principle of the infinite divisibility of time is found to be associated with logical fictions formed, strictly speaking, in violation of the law of contradiction. For the principle itself involves just such a logical fiction, as is evident when Zeno's Dichotomy paradox -- which he appears to have formulated for a moving body -- is applied to time itself, that is to any clock. In this case, Zeno would assert that before any temporal interval (however small) can elapse, half of that interval must have elapsed, and similarly before this half-interval has elapsed half of it must have elapsed, and so on *ad infinitum*. Therefore, before any interval can elapse a completed infinity of overlapping sub-intervals must have elapsed. One can, therefore, either conclude that the idea of the infinite divisibility of time must be rejected, or else if one wishes to make use of the device, one must recognize that it is, strictly speaking, a logical fiction.
>
> To sum up this somewhat lengthy discussion of Zeno's paradoxes concerning time and motion, it appears that the two based on the concept of indivisible temporal instants are on a different footing from the remaining pair which involve the infinite divisibility of time. The former when correctly analysed are seen not to involve any logical antinomies; although they may seem to conflict with common sense. The latter, however, are true paradoxes involving definite logical antinomies.[44]

> ...the [Achilles] paradox cannot be refuted, but the paradoxical proposal to revise word usage may be refused; we cannot solve the paradox, we can avoid it.[45]

Others (for example, William James) use the paradoxes and their alleged unsolvability as an excuse to discount our ordinary reasoning.

> Either we must stomach logical contradiction, therefore, in these cases; or we must admit that the limit is reached in these successive cases by finite and perceptible units of approach -- drops, buds, steps, or whatever we please to term them, of change, coming wholly when they do come, or coming not at all. Such seems to be the nature of concrete experience, which changes always by sensible amounts, or stays unchanged. The infinite character we find in it is woven into it by our late conception indefinitely repeating the act of subdividing any given amount supposed. The facts do not resist the subsequent conceptual treatment; but we need not believe that the treatment necessarily reproduces the operation by which they were originally brought into existence.
>
> The antinomy of mathematically continuous growth is thus but one more of those many ways in which our conceptual transformation of perceptual experience makes it less comprehensible than ever.[46]

Hegel adopts this sort of view.[47] As Adam Schaff writes

> The Eleatics rejected the contradiction and denied consistently the objectivity of motion, while Hegel accepted objectivity of motion and ... accepted logical contradictions in the sentences descriptive of motion.[48]

The correct solution to the paradoxes shows such positions mistaken. One must look elsewhere for genuine logical contradiction.

PART V

THE PARADOX OF EXTENSION

But a still more general perspective is relevant for clarifying the concept of the infinite. A careful reader will find that the literature of mathematics is glutted with inanities and absurdities which have had their source in the infinite.

HILBERT, <u>On the Infinite</u>[1]

Chapter 20

The Paradox

1. Zeno's paradox of extension (sometimes called "the paradox of plurality") can be constructed in regard to any continuum and the parts it contains. Usually it is constructed in regard to geometric entities.

> Since physical separation of parts is not at issue, we can just as well discuss the composition of the mathematical line. Zeno's argument runs as follows. As we have seen from both the Achilles and Dichotomy paradoxes, any line is infinitely divisible. If we stop short with only a finite number of divisions, it is always possible to carry the division further. The process of halving the line, and then halving the half, is one which has no end. Hence, if the line is made up of parts, as it surely appears to be, then there are infinitely many of them. Now, Zeno poses a simply dilemma. What is the size of the parts? If they have zero magnitude, then no matter how many of them you add together, the result will still be zero. The process of adding zeros never yields any answer but zero. If, however, the parts have a positive non-zero size, then the sum of the infinite collection of them will be infinite. In other words, a line segment must have a length of either zero or infinity; a line segment one inch or one mile long is impossible.[2]

The paradox of extension is based on our concept of a continuum. A continuum is said to consist of points, so Zeno asks what size points are. Any answer appears to produce paradox.

2. Grünbaum's view of the paradox of extension is based on the modern view of space and time. As he remarks;

> It is a commonplace in the analytic geometry of physical space and time that an extended straight-line segment, having positive length, is treated as "consisting of" unextended points, each of which has zero length. Analagously, time intervals of positive duration are postulated to be aggregates of instants, each of which has zero duration.[3]

Such a view seems to produce paradox.

> In the geometric paradox, our philosopher asserts that it is self-contradictory to claim that a line segment consists of points, each having zero length. For he reasons that if a line segment of, say two centimeters, actually does consist of points, then the total length of that segment should be computable by adding the individual lengths of its constituent points. But instead of yielding the required value of two centimeters, this computation, says Zeno, unavoidably yields the paradoxical result of zero centimeters, since a summation of zeros can issue in nothing other than zero. By the same token Zeno argues that it is self-contradictory to maintain that a positive time interval can consist of instants of zero duration.[4]

3. Paul Swartz construes the paradox of extension as an argument against the position (adopted earlier) that space and time are infinitely divisible.

> A second logical argument against the continuum theory is already very old. In fact, it originated with Zeno, and is known as his paradox of plurality. This objection has to do with the fact that according to the continuum theory points and moments have no magnitude. For if they had, they would still be divisible, and consequently they would not be points respectively moments. However, it is inconceivable that out of points and moments without magnitude finite spatial and temporal intervals could be built up. No matter how often one adds zero to zero, the result will never be more than zero. Or conversely, it is clearly impossible that by continually dividing an interval (in space or time) ultimately nothing could be left over.[5]

Other accounts of the paradox are given by William James[6] and P.W. Bridgman.[7]

Chapter 21

Solving The Paradox

1. To solve the paradox of extension we must consider points in continuums. Points are the smallest (fundamental) intervals within a continuum. According to Euclid,

> A point is that which has no parts.[8]

2. The new Arrow and Stadium paradoxes show that a continuum is infinitely divisible. It follows that there are no actual points within a continuum. There can be no smallest intervals within a continuum because any interval can be divided into smaller intervals. Points then, are not actual intervals in a continuum but hypothetical entities. Though there are no actual points (no smallest intervals) within a continuum we postulate such entities to allow a better analysis of continuums. By postulating points within a continuum we can better fix position (place) within the continuum.

As points are hypothesized to be the smallest (fundamental) constituents of a continuum they can be seen as extensionless (of zero dimension). If they had extension there would be smaller intervals. As points are _hypothetical_ entities, the view that they are extensionless does not entail that a continuum actually is a series of unextended points. A continuum is constituted of a series of intervals, but not of any series of _smallest_ intervals (of zero dimension). Such intervals may be hypothesized to facilitate our consideration of continuums, but such points do not actually exist. Because space and time are infinitely divisible, points _must_ be hypothetical.

An understanding of points undermines the paradox of extension. He asks us to consider the fundamental intervals (points) within a continuum and their size. But actually there are _no_ fundamental intervals within a continuum (points are only _hypothetical_ entities). The size of a continuum cannot be calculated from the size of points (for the continuum is not actually a series of such points).

3. The Grünbaum version of the paradox of plurality can be handled in the same way as the general form. Points within time and space are _hypothetical_ entities which are (theoretically) the fundamental parts of time and space. Because time and space are infinitely divisible there are no such entities, though the postulation of them allows us to more easily distinguish position within spatial and temporal intervals. The length of a spatial or temporal interval cannot be considered as resulting from the size of the (hypothesized) points within the interval because the interval actually is not constituted of such points. A space or time interval is a series of intervals but not a series of _smallest_ intervals (of zero extension).

Grünbaum presents only a formal solution to the paradox of extension and not a real solution. Whether or not addition is defined for non-denumerable series, it is impossible that a line or time interval could be extended if it was nothing more than a series of zero-extended points (of unextended union point-sets). If the interval is nothing but the points in question how could extension come into being. All parts of the interval are supposed to be accounted for by points.

Any continuum made up of parts all of zero extension must itself have no extension. The absurdity of Grünbaum's view can be shown by an example. Consider the line in figure (i).

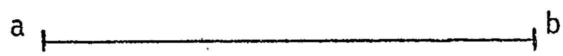

figure (i)

It is the case that there is no money on any of the points within ab, so there is no money on all such points. Yet there are a non-denumerable infinity of occurrences of $0 on the line. On Grünbaum's view I cannot conclude that there is no money on the whole line (for the zero amounts of money cannot be added to get this result)!

Such examples show that the addition of a non-denumerable infinity of zeros _must_ result in zero. This does _not_ imply that addition can in general be defined for a non-denumerable series. Only Swartz seems to have acceptable intuitions on this point.

> Possibly someone or other will object against the above statement that infinite times zero equals zero because in mathematics it is customary to say that infinite times zero equals any arbitrary finite number _a_. However, this is only a convention, made possible by neglecting the difference between zero and infinitesimal. The equation $\underline{a/\infty = 0}$ is, strictly speaking, not correct; the result should not be zero, but an infinitely small number. Now it is generally not necessary to take this difference into account, but this case of points and moments is obviously an exception to this rule.[13]

4. Grünbaum's views are not strong enough as they stand. Even if they bar Zeno's conclusion they are not satisfactory. It is not enough for Grünbaum to claim that the addition of a non-denumerable infinity of zeros does not result in zero. Rather, he must claim that such addition must result in something other than zero. Because lines have positive length and they are constituted of a non-denumerable number of segments of zero length, the addition of such lengths must equal a positive quantity (that is, the length of the line). Furthermore, that quantity must <u>vary</u>, for all lines have a non-denumerable infinity of points though they can have different overall lengths. It follows in the example where no money is put on each point of a line that there must be some amount of money resting on the whole line!

The solution to the paradox of extension which has already been suggested cannot be criticized in this way.

Chapter 22

The Grünbaum Solution

1. Adolf Grünbaum has put forth a solution to the paradox of extension different from the one suggested here. His solution is unacceptable.

2. According to Grünbaum, it is crucial to the solution of the paradox of extension that a finite interval contains a non-denumerably infinite of points.

> ...since each positive interval has a non-denumerable infinity of degenerate subintervals, we see already that the result of determining the length of that interval by "compounding", in some unspecified way, the zero lengths of its degenerate subintervals is far less obvious than it must have seemed to Zeno, who did not distinguish between countably and non-countably infinite sets![10]

According to Grünbaum the paradox of extension could validly be deduced on standard theory if a time or space interval contained a denumerably infinite number of points.

> The length of an interval which is subdivided into an enumerable number of subintervals without common points is equal to the arithmetic sum of the lengths of these subintervals. It follows at once that if the standard mathematical theory containing this result were to assert as well - which it does not! - that an interval

> consists of an enumerable number of points, then Zeno's paradox would be deducible.[11]

Grünbaum goes on to claim that because a line consists of a super-denumerable infinity of points, Zeno's conclusion cannot be deduced.

> Cantor has shown that any collection of positive non-overlapping intervals on a line is at most denumerably infinite. It follows that the degenerate subintervals which are at the focus of our interest are the only kind of non-overlapping subintervals of which there are non-denumerably many in a given interval. Quite naturally, therefore, they create a special situation. The latter is due to the fact that our theory does <u>not</u> assign any meaning to "forming the arithmetic sum", when we are attempting to "sum" a <u>super-denumerable</u> infinity of individual numbers (lengths)! This fact is independent of whether the individual numbers in such a non-denumerable set of numbers are zeros or finite cardinal numbers differing from zero.
>
> Consequently, the theory under discussion cannot be deemed to be <u>ad hoc</u> for precluding the possibility of "adding", in Zenonian fashion, the zero lengths of the continuum of points which "compose" the interval (a,b) to obtain zero as the length of this interval. Though the finite interval (a,b) is the union of a continuum of degenerate subintervals, <u>we cannot meaningfully determine its length in our theory by "adding" the individual zero lengths of the degenerate subintervals</u>.[12]

In mathematics it makes no sense to talk of the sum of a non-denumerable infinity of terms. It therefore makes no sense to talk of the 'sum' of the extensions of points on a line. We cannot conclude that that sum must equal zero because the length of any point within the line is zero.

3. In some ways Grünbaum's solution to the paradox and the solution we have developed are similar. Both views refute the paradox by denying that the length of a continuum is a function of the length of its points. We have denied such a function by arguing that points are hypothetical rather than real entities within a line or time interval. Grünbaum denies such a function by pointing out that there is a non-denumerable infinity of points within a line or time segment. Much can be said against the Grünbaum view.

4. The proper answer to the paradox of extension avoids a criticism of the view that space and time are infinitely divisible forwarded by Swartz.

> ...there are...a number of logical arguments against the continuum theory [the theory that space and time are infinitely divisible]. The first is that the continuum theory contains a contradiction, proclaiming as it does both infinite divisibility and the existence of indivisibles. For surely the concept of infinite divisibility implies that indivisible units are impossible. If time and space were really infinitely divisible it would be impossible that there existed indivisible instants of time or points of space. Nevertheless, the existence of precisely such indivisible points of time and space is one of the tenets of the continuum theory, and also of the exact sciences (especially mathematics). So the continuum theory is inconsistent in that it proclaims the infinite divisibility of space and time, while at the same time maintaining that points of time and space are indivisible. The continuum theorist must choose: either a point of space or time is still further divisible (which would disagree with the accepted meaning of the term 'point'), or there are simply no points of time or space, but only small intervals.[9]

The continuum theory as we have constructed it is not plagued by this inconsistency. We have postulated points as indivisibles, but because we have also postulated them as <u>hypothetical</u> entities we have not contradicted the view that space and time are infinitely divisible. To do so they would have to be postulated as actual entities within space and time (that is, it would have to be claimed that there actually are smallest intervals of time and space). It is not inconsistent to hold to infinite divisibility and indivisibles when the indivisibles are hypothetical entities.

PART VI

WITTGENSTEIN AND PHILOSOPHY

The book deals with the problems of philosophy and shows, as I believe, that the method of formulating these problems rests on a misunderstanding of the logic of our language.

LUDWIG WITTGENSTEIN, _Tractatus Logico-Philosophicus_[1]

Philosophy, as we use the word, is a fight against the fascination which forms of expression exert on us.

LUDWIG WITTGENSTEIN, _The Blue Book_[2]

We want to replace wild conjectures and explanations by quiet weighing of linguistic facts.

LUDWIG WITTGENSTEIN, _Zettel_[3]

Philosophy is a battle against the bewitchment of our intelligence by means of language.

LUDWIG WITTGENSTEIN, _Philosophical Investigations_[4]

Chapter 23

Wittgenstein and Philosophy of Language

1. We have approached Zeno's paradoxes independent of Wittgenstein. Though he does not deal with the paradoxes explicitly he does present the understanding necessary for their solution. To gain an appreciation of <u>some</u> of Wittgenstein's thought and its importance to philosophy we can profitably consider Zeno's paradoxes of motion from a Wittgenstein point of view. Such an inquiry is worthwhile because modern scholars have not understood the details (or the significance) of Wittgenstein's philosophy.

2. The parts of Wittgenstein's philosophy we need consider are found <u>throughout</u> his work (and demonstrate the connection between the 'earlier' and 'later' Wittgenstein). Maurice Drury has made significant comments concerning the continuity of Wittgenstein's thought.

> This would not be the place, nor would I have the ability to discuss the differences and developments which can be found between the <u>Tractatus Logico-Philosophicus</u> and the <u>Philosophical Investigations</u>. But this I must place on record. When Wittgenstein was

> living in Dublin and I was seeing him constantly he was at that time hard at work on the manuscript of the Investigations. One day we discussed the development of his thought and he said to me (I can vouch for the accuracy of the words): "My fundamental ideas came to me very early in life." ...I think perhaps the remark that Wittgenstein, after his conversations with Sraffa felt like a tree with all its branches lopped off, has been misinterpreted. Wittgenstein chose his metaphors with great care, and here he says nothing about the roots or the main trunk of the tree, these - his fundamental ideas - remain I believe unchanged.[5]

Drury's sentiments are enforced by a clear understanding of Wittgenstein's philosophy.

3. In considering Wittgenstein and Zeno's first four paradoxes we must turn to philosophy of language. The paradoxes lend credence to Wittgenstein's view that philosophical problems are to be solved through an understanding of language and the way it works.

In order to deal with the four paradoxes we need emphasize Wittgenstein's realization that the same word or phrase in a language may be used in very different ways. In different contexts the same expression may have very different reference or meaning. On different occasions the word "bolt" may stand for "rifle bolt", "door bolt", "screw bolt", "lightning bolt" or "bolt of cloth". The arrow of a cross-bow is also called a "bolt", and the word may be used as a verb with various meanings (as in "Please bolt the door." and "The horses may bolt.")
On different occasions the expression "bolt" does not have a constant meaning. This is not to say that propositions like "I bought a new bolt today." are always ambiguous. Usually context prevents ambiguity. If in the middle of bolting together two metal sheets a workman unfamiliar with cross-bows remarks, "Give me another 3/4 inch bolt.", he is not (in normal circumstances) talking of cross-bow arrows.

Even if we use words and phrases which do not have multiple meanings, they can be used in very distinct ways. Thus I may use the word "chair" to refer to a kitchen chair or to an arm chair, and may use the word "ball" to refer to a baseball, a basketball, a medicine ball, a rubber ball, a soccer ball or a football. In different situations the word is used differently (to refer to different things).

In the Tractatus Wittgenstein distinguishes different uses of a linguistic expression through the notions of "sign" and "symbol". Symbols, not signs, have a unique use in language.

The sign is the part of the symbol perceptible by the senses.[6] In written discourse a sign may be a particular mark on a piece of paper, in spoken language an uttered sound, and in sign language a particular hand signal. Signs are definitely not symbols because (as noted in regard to "bolt" and "ball") a sign may have a different use on different occasions (when used in alternate ways, the sign "bolt" constitutes different symbols).

> Two different symbols can therefore have the same sign (the written sign or the sound sign) in common - they then signify in different ways.[7]

The sign "bat" may signify in at least two ways; by referring to baseball bats or to a particular species of flying mammal. When used in one of these ways the sign takes a different meaning than when used in the other way, and is a different symbol. The sign may also be used in different ways by taking different reference when meaning is constant. Thus we might use the phrase "the bat" to refer to different baseball bats on different occasions.

> It can never indicate the common characteristic of two objects that we symbolize them with the same signs but by different *methods of symbolizing*. For the sign is arbitrary. We could therefore equally well choose two different signs and where then would be what was common in the symbolization.[8]
>
> (Wittgenstein's emphasis)

Because signs are arbitrary, two things designated by the same sign (but different symbols) need not be the same sort of thing. The same sign ("bat") designates baseball bats and flying bats, but it does not follow that they are the same thing. We might also refer to baseball bats as (baseball) "clubs" and that would not establish that they are intrinsically similar to those clubs that are groups of people (chess clubs, boating clubs,...) or the similarly called suit of playing cards. If all clubs were similar, card games would be a dangerous pastime.

We often use linguistic expressions in a variety of ways.

> In the language of everyday life it very often happens that the same word signifies in two different ways - and therefore belongs to two different symbols - or that two words, which signify in different ways, are apparently applied in the same way in the proposition.
>
> Thus the word "is" appears as the copula, as the expression of existence; "to exist" as an intransitive verb like "to go"; "identical" as an adjective; we speak of <u>something</u> but also of <u>something</u> happening.
>
> (In the proposition "Green is green" - where the first word is a proper name and the last an adjective - these words have not merely different meaning but they are different symbols.)[9]
>
> (Wittgenstein's emphasis)

Different symbols may be encompassed by the same sign and different kinds of symbols may be encompassed by the same kind of sign (we use substantive nouns to refer to physical objects as well as 'abstract objects' like justice and beauty, and we also talk of having "scruples").

Because the same sign can take different meaning or reference in different contexts the meaning or reference of a sign is not always clear when the sign is considered independent of context.

> Colloquial language is a part of the human organism and is not less complicated than it.
>
> From it it is humanly impossible to gather immediately the logic of language.[10]

Ordinary language is not immediately clear because it is not clear from signs alone (from the perceptible part of the symbol). One must also consider how a sign is being used in a particular context.

> Language disguises the thought; so that from the external form of the clothes one cannot infer the form of the thought they clothe, because the external form of the clothes is constructed with quite another object than to let the form of the body be recognized.[11]

Natural language strives for convenience and this is furthered by the use of signs which have different meaning or reference on different occasions (consider for example, personal pronouns like "I", "he", and "they").

> The silent adjustments to understand colloquial language are enormously complicated.[12]

The silent adjustments to understand colloquial language depend on particular occasions of use.

According to Wittgenstein, the difference between signs and symbols is the cause of much misunderstanding.

> Thus there easily arise the most fundamental confusions (of which the whole of philosophy is full).[13]

Simple confusions (and puns) arise in everyday discourse because a word or phrase is used in very different ways. Thus it might mistakenly be concluded that "the Bombay duck" refers to a bird (actually it is a kind of fish) or that "clever horses" are ingenious ones (actually they are good-natured horses). In philosophy such confusion leads to much dilemma (for example, Zeno's paradoxes of motion).

In the Tractatus Wittgenstein wants to avoid symbol-sign confusion by employing a symbolism (a language) which does not assimilate different symbols under one sign.

> In order to avoid these errors, we must employ a symbolism which excludes them, by not applying the same sign in different symbols and by not applying signs in the same way which signify in different ways. A symbolism, that is to say, which obeys the rules of logical grammar - of logical syntax.

> (The logical symbolism of Frege and Russell is such a language, which, however, does still not exclude all errors.[14]
> (Wittgenstein's emphasis)

In philosophy one is therefore concerned with linguistic considerations.

> All philosophy is "Critique of language" (but not at all in Mauthner's sense). Russell's merit is to have shown that the apparent logical form of the proposition need not be its real form.[15]

It was Russell's merit to show (in his theory of descriptions) that the apparent logical form of a proposition (that shown by the signs it employs) is not necessarily the real logical form. What the proposition appears to say in virtue of the signs it employs may not be what it does say (for similarity of meaning and reference does <u>not</u> follow from similarity of sign).

> The point is only that the logical part of what is signified should be completely determined just by the logical part of the sign and the method of symbolizing: sign and method of symbolizing <u>together</u> must be logically identical with what is signified.[16]

4. Any proper theory of language use (and any attempt to understand 'the logic of language') must concede that a linguistic sign may have different meaning and reference in different contexts. Wittgenstein's later philosophy still emphasizes this aspect of language, though he no longer concerns himself with formal languages. Again he argues that the apparent similarity of linguistic expressions confuses us when we attempt to do philosophy.

> Think of the tools in a tool-box: there is a hammer, pliers, a saw, a screw-driver, a rule, a glue-pot, glue, nails, and screws. The functions of words are as diverse as the functions of these objects...
> Of course, what confuses us is the uniform appearance of words when we hear them spoken or meet them in script or print. For their <u>application</u> is not presented to us so clearly. Especially when we are doing philosophy!
> It is like looking into the cabin of a locomotive. We see handles all looking more or less alike. (Naturally, since they

are all supposed to be handled.) But one is the handle of a crank which can be moved continuously (it regulates the opening of a valve); another is the handle of a switch, which has only two effective positons, it is either off or on; a third is the handle of a break; a fourth, the handle of a pump: it has an effect only so long as it is moved to and fro.[17]

A main source of our failure to understand is that we do not command a clear view of the use of our words.--Our grammar is lacking in this sort of perspicuity. A perspicuous representation produces just that understanding which consists in 'seeing connexions'. Hence the importance of finding and inventing intermediate cases.

The concept of a perspicuous representation is of fundamental significance for us. It earmarks the form of account we give, the way we look at things.[18]

In the use of words one might distinguish 'surface grammar' from 'depth grammar'. What immediately impressed itself upon us about the use of a word is the way it is used in the construction of the sentence, the part of its use - one might say - that can be taken in by the ear.--And now compare the depth grammar, say of the verb "to mean", with what its surface grammar would lead us to suspect. No wonder we find it difficult to know our way about.[19]

Surface grammar is the grammar of linguistic signs, and depth grammar is the grammar of linguistic symbols. It is depth grammar, and not surface grammar, which determines the meaning or reference of an expression when it is used.

Wittgenstein's remark in the Investigations that

We remain unconscious of the prodigious diversity of all the everyday language-games because the clothing of our language makes everything look alike.[20]

can be compared to the Tractatus remark that

Language disguises the thought; so that from the external form of the clothes one cannot infer the form of the thought they clothe, because the external form of the clothes is constructed with quite another object than to let the form of the body be recognized.[21]

Elsewhere Wittgenstein still stresses the nonidentity of linguistic signs and meaning or reference, the importance of context to meaning.

How words are understood is not told by words alone.[22]

The words "Gottlob! Noch etwas Weniges hat man geflüchtet--vor den Fingern der Kroaten," and the tone and glance that go with them seem indeed to carry within themselves every last nuance of the meaning

> they have. But only because we know them as part of a particular scene. But it would be possible to construct an entirely different scene around these words so as to shew that the special spirit they have resides in the story in which they come.[23]

> I cannot carry out the order because I don't understand what you mean.--Yes, I understand you now."--What went on when I suddenly understood him? Here there were various possibilities. The order may for example have been given with a wrong emphasis; and the right emphasis suddenly occurred to me. In that case I should say to a third party "Now I understand him, he means..." and should repeat the order with the right emphasis. And now, with the right emphasis, I should understand him;[24]

According to Wittgenstein, philosophical problems often arise because we misunderstand our language, we misassimilate expressions which differ in meaning or reference even though they do share the same sign and the same kind of sign.

> The problems arising through a misinterpretation of our forms of language have the character of <u>depth</u>. They are deep disquietudes; their roots are as deep in us as the forms of our language and their significance is as great as the importance of our language. --Let us ask ourselves: why do we feel a grammatical joke to be <u>deep</u>? (And that is what the depth of philosophy is.)[25]

> Suppose people used always to point to objects in the following way: they describe a circle as it were round the object with their finger in the air; in that case a philosopher could be imagined who said: "All things are circular, for the table looks like <u>this</u>, the stove like <u>this</u>, the lamp like <u>this</u>", etc., drawing a circle around the thing each time.[26]

It is this sort of confusion which leads to Zeno's four paradoxes of motion and the inadequacy of standard attempts to resolve them.

Chapter 24

The Paradoxes of Motion

1. The essence of our solution to the paradoxes of motion is the distinction between separate and continuous parts. The distinction between the two kinds of parts is hidden by the fact that the same words and phrases are used to signify separate and continuous (possibly separate) parts.

Imagine separate parts of a large and elaborate table (the parts must be joined to make the table functional). Suppose Jack has all the parts but does not know how to conjoin them. It follows that he (now) cannot use the table. The table can be put together without Jack's help. Afterwards it can still be said that Jack has all the parts of the table and still does not know how to put them together. It would be a mistake to argue that he cannot use the table. Such a conclusion is not warranted because Jack now has the parts of the table <u>conjoined</u>. He has no extra parts, but the parts have been put together. Nevertheless, this unwarranted argument and the earlier (warranted) one may seem equally valid, for they can both be represented as follows.

> Jack has the parts of the table.
> Jack does not know how to put the parts of the table together.
> Jack cannot use the table.

The difference of course, is that in the first argument the parts in question are separate and in the second argument they are not. In the second argument the parts are parts the table could be divided into, not parts it is divided into. The form of our language hides this fact.

Imagine two pineapples, one cut into sixteen slices and one uncut. Though the second is uncut we can still say it contains sixteen parts (we might, for example, mark off such parts with a pen). Suppose that the sliced pineapple cannot be picked up with one hand because it has sixteen parts and that is too many to be picked up in this way. It does not follow that the uncut pineapple cannot be picked up with one hand because it too has sixteen parts and that is too many to be picked up in this way. In the uncut pineapple the parts are not separate and therefore it may be possible to pick the pineapple up whole. In the case of the uncut pineapple this is not possible, because the sixteen parts are separate parts. Again it is possible to misassimilate two arguments because they look the same. The (sound) argument in regard to the cut pineapple and the unsound one in regard to the uncut pineapple can both be represented as follows.

> The pineapple has sixteen parts.
> That is too many to pick up with one hand.
> The pineapple cannot be picked up with one hand.

The difference in the arguments is that in the first separate parts are referred to, and in the second parts that are not separate.

2. The argument of the Dichotomy succeeds only if we confuse separate

and continuous parts. The argument can be put as:

> Every motion has an infinite number of parts.
> A motion with an infinite number of parts is incompletable.
> Any motion is incompletable.

The second premise is true only with regard to motions constituted of an infinite number of separate parts, and Zeno does not refer to such parts. Similar things can be said of the argument of the Achilles paradox, and our solution of the Arrow and Stadium paradoxes is a direct result of our dealings with these first two paradoxes.

3. It should be clear how our solution to the paradoxes of motion reflects Wittgenstein's understanding of language and philosophy. Central to his thought is the thesis that philosophical problems arise from linguistic misunderstanding - misunderstanding that is the result of superficial linguistic similarity.

> The cases in which particularly we wish to say that someone is misled by a form of expression are those in which we would say "he would not talk as he does if he were aware of this difference in the grammar of such-and-such words, or if he were aware of this other possibility of expression" and so on.[27]

> When words in our ordinary language have prima facie analagous grammars we are inclined to try to interpret them analogously; i.e. we try to make the analogy hold throughout.[28]

In the paradoxes of motion talk of "parts" that are continuous and "parts" that are separate is misassimilated. These two different uses of the expression "parts" are mistakenly thought to be one because they are syntactically similar. Once one sees past this superficial similarity the paradoxes are seen in proper light.

Chapter 25

Philosophy

1. Zeno's paradoxes of motion show the significance of Wittgenstein's thought (for a clear understanding of Wittgenstein does undermine the paradoxes). The solution to the paradoxes can also be used to gain an understanding of Wittgenstein's view of philosophy.

2. In dealing with the paradoxes of motion we found ourselves concerned with a description of the way language works. Significantly, the facets of language with which we were concerned were not obscure or unusual. Rather we pointed out something obvious about language (the difference between reference to continuous parts and reference to separate parts) and thereby solved the paradoxes. Our procedure in considering the paradoxes reflects Wittgenstein's view that in philosophy we are concerned with the description of unexceptional facets of ordinary language.

> The aspects of things that are most important for us are hidden because of their simplicity and familiarity. (One is unable to notice something -- because it is always before one's eyes).

The real foundations of his enquiry do not strike a man at all. Unless that fact has at some time struck him. -- And this means: we fail to be struck by what, once seen, is most striking and most powerful.[29]

The problems arising through a misinterpretation of our forms of language have the character of depth. They are deep disquietudes; their roots are as deep in us as the forms of our language and their significance is as great as the importance of our language. -- Let us ask ourselves: why do we feel a grammatical joke to be deep? (And that is what the depth of philosophy is.)[30]

We want to establish an order in our knowledge of the use of language: an order with a particular end in view; one out of many possible orders; not the order. To this end we shall constantly be giving prominence to distinctions which our ordinary forms of language easily make us overlook. This may make it look as if we saw it as our task to reform language.

Such a reform for particular practical purposes, an improvement in our terminology designed to prevent misunderstandings in practice, is perfectly possible. But these are not the cases we have to do with.[31]

The results of philosophy are the uncovering of one or another piece of plain nonsense and of bumps that the understanding has got by running its head up against the limits of language. These bumps make us see the value of the discovery.[32]

The work of the philosopher consists in assembling reminders for a particular purpose.[33]

It is because philosophy deals with familiar linguistic facts that it is not a theory in the normal sense. It does not require explanation as science does.

Philosophy simply puts everything before us, and neither explains nor deduces anything. -- Since everything lies open to view there is nothing to explain. For what is hidden, for example, is of no interest to us.

One might also give the name "philosophy" to what is possible before all new discoveries and inventions.[34]

The object of philosophy is the logical clarification of thoughts.

Philosophy is not a theory but an activity.

A philosophical work consists essentially of elucidations.

The result of philosophy is not a number of "philosophical propositions", but to make propositions clear.

Philosophy should make clear and delimit sharply the thoughts which otherwise are, as it were, opaque and blurred.[35]

It was true to say that our considerations could not be scientific ones. It was not of any possible interest to us to find out empirically 'that, contrary to our preconceived ideas, it is possible to think such-and-such' -- whatever that may mean. (The conception of thought as a gaseous medium.) And we may not advance any kind of theory. There must not be anything hypothetical in our considerations. We must do away with all explanation, and description alone must take its place. And this description gets its light, that is to say its purpose, from the philosophical problems. These are, of course, not empirical problems; they are solved, rather, by looking into the workings of our language, and that in such a way as to make us recognize those workings: in despite of an urge to misunderstand them. The problems are solved, not by giving new information, but by arranging what we have always known. Philosophy is a battle against the bewitchment of our intelligence by means of language.[36]

it is ... of the essence of our investigation that we do not seek to learn anything new by it. We want to understand something that is already in plain view. For this is what we seem in some sense not to understand.

Augustine says in the Confessions "quid est ergo tempus? si nemo ex me quaerat scio; si quaerenti explicare velim, nescio".[37] -- This could not be said about a question of natural science ("What is the specific gravity of hydrogen?" for instance). Something that we know when no one asks us, but no longer know when we are supposed to give an account of it, is something that we need to remind ourselves of. (And it is obviously something of which for some reason it is difficult to remind oneself.)[38]

Wittgenstein once commented that:

What I give is the morphology of the use of an expression. I show that it has kinds of uses of which you had not dreamed. In philosophy one feels forced to look at a concept in a certain way. What I do is suggest, or even invent, other ways of looking at it. I suggest possibilities of which you had not previously thought. You thought that there was one possibility, or only two at most. But I made you think of others. Furthermore, I made you see that it was absurd to expect the concept to conform to those narrow possibilities. Thus your mental cramp is relieved, and you are free to look around the field of use of the expression and to describe the different kinds of uses of it.[39]

Zeno's paradoxes of motion are the result of a 'mental cramp' surrounding the use of the word "part". Once it is seen that the word has two exceedingly different uses in regard to separate and continuous parts our mental cramp is relieved and the paradox disappears. As Wittgenstein suggests we do not solve this philosophical dilemma discovering new facts about space, time or language. Rather we remind ourselves of the difference between separate and continuous parts. Once we take this reminder to heart Zeno's arguments are no longer compelling. We deal with the problems <u>not</u> by discovering new facts about the world or language, but by reminding ourselves of the obvious.

3. Zeno's paradoxes of motion, and the solutions of them do not establish that Wittgenstein was correct about all philosophy. Nevertheless, our considerations do suggest that there is something to be found in Wittgenstein's thought. I suggest much more waits to be uncovered.

FOOTNOTES TO PART I

1. Paul Benacerraf, "Tasks, Super-Tasks, and the Modern Eleatics", Zeno's Paradoxes, edited by Wesley Salmon, The Bobbs- Merrill Company, Inc., 1970, p. 103.

2. Aristotle, "Physics", The Basic Works of Aristotle, edited by Richard McKeon, translation by R.P. Hardie and R.K. Gage, Random House, 1941, Bk. III, Ch. 6, 206a, 27 (p. 265).

3. Ralph B. Winn, Dictionary of Philosophy, edited by Dagbert Runes, Littlefield, Adams & Co., 1974, p. 146.

4. Aristotle, op. cit., Bk. VIII, Ch. 8, 263a, 4-10 (p. 383).

5. Wesley Salmon, Zeno's Paradoxes, p. 9-10.

6. Simplicius, Physics, 1289 (5), John M. Robinson, An Introduction to Early Greek Philosophy, Houghton-Mifflin, 1968.

7. see Adolf Grünbaum, Modern Science and Zeno's Paradoxes, Wesleyan University Press, 1967 and Wesley Salmon, Space, Time and Motion, Dickenson Publishing Co. Inc. 1975.

8. see Grünbaum, op. cit., and Bertrand Russell, "The Problem of Infinity Considered Historically", Zeno's Paradoxes, edited by Salmon.

9. The Concise Oxford Dictionary, edited by H.W. Fowler and F.G. Fowler, Fifth Edition, Oxford At The Clarendon Press, 1964.

10. Abner Shimony, "Resolution of the Paradox: A Philosophical Puppet Play", Zeno's Paradoxes, edited by Salmon, p. 1-3.

11. Simplicius, op. cit.

12. Aristotle, op. cit., Bk. VI, Ch. 8, 239b, 10-24.

13. G.J. Whitrow, The Natural Philosophy of Time, Nelson, 1966, p. 139.

14. see G.J. Whitrow, The Natural Philosophy of Time, and Paul Swartz, About Time, Van Nostram, 1975.

15. Whitrow, op. cit., p. 152.

16. Whitrow, op. cit., p. 137.

17. Whitrow, op. cit., p. 135.

18. see Max Black, Problems of Analysis, Cornell University Press, 1954.

19. see Aristotle, op. cit.

20. Paul Swartz, <u>About Time</u>, Van Nostram, 1975, p. 201.

21. The position that space and time simply are not divisible (finitely or infinitely) seems patently absurd.

FOOTNOTES TO PART II

1. Ludwig Wittgenstein, Philosophical Investigations, Basil Blackwell, 1958, comments 119, 124 and 309.

2. I have discussed Aristotle's solution to the paradoxes of motion in Chapter 14.

3. Stanislaus Quan has discussed the dichotomy paradox in "The Solution of Zeno's First Paradox", Mind, 1968. I have not discussed his treatment of the paradoxes because (I think) it is not clear on some very important points.

FOOTNOTES TO PART III

1. George Berkeley, "The Analyst", The Works of George Berkeley Volume 4, edited by A.A. Luce and T.E. Jessop, Nelson, 1948, p. 95-102.

2. We shall use the terms "set", "collection" and "class" as synonymous expressions.

3. Euclid, The Thirteen Books of Euclid's Elements, Dover Publications, 1956, Bk. 1, Common Notions #5, (p. 155).

4. Jagjit Singh, Great Ideas of Modern Mathematics, Dover Publications, Inc., 1959, p. 93-94.

5. In this regard see Hilary Putnam, "The Logic of Quantum Mechanics", Mathematics, Matter, and Method, Cambridge University Press, 1975 or Garret Birkhoff and John Von Neumann, "The Logic of Quantum Mechanics", Annals of Mathematics, XXXVII (1936).

6. Irving Copi, The Theory of Logical Types, Routledge and Kegan Paul, p. 3.

7. Georg Cantor, Contributions to the Founding of the Theory of Transfinite Numbers, Dover Publications, 1955, p. 87.

8. Kamke, Theory of Sets, Dover Publications, 1950, p. 14.

9. Frege, The Foundations of Arithmetic, Basil Blackwell, 1974, p. 85.

10. F. Dinkines, Elementary Theory of Sets, Appleton-Century-Crofts, 1964, p. 54-55.

11. Geoffrey Hunter, Metalogic, Macmillan, 1971, p. 16.

12. Bertrand Russell, Introduction to Mathematical Philosophy, Simon and Schuster, p. 15.

13. Edward Huntington, The Continuum and Other Types of Serial Order, Dover Publications, 1971, p. 3, 4, 88.

14. E.J. Lemmon, Introduction to Axiomatic Set Theory, Routledge and Kegan Paul, 1968, p. 93.

15. Jagjit Singh, op. cit., p. 5-6.

16. Hans Hahn, "Infinity", The World of Mathematics, edited by J. Newman, Simon and Schuster, 1956, Vol. 3, p. 1593.

17. Charles Parsons, "Mathematics, Foundations of", Encyclopedia of Philosophy, edited by Paul Edwards, MacMillan, 1967, Vol. 5, p. 195.

18. Irving Copi, op. cit., p. 2-3.

19. Russell, "Mathematics and the Metaphysicians", The World of Mathematics, edited by J. Newman, Vol. 3, p. 1587.

20. Howard Delong, A Profile of Mathematical Logic, Addison-Wesley Publishing Company, 1970, p. 73-74, 245.

21. E.T. Bell, Men of Mathematics, Simon and Schuster, 1965, p. 358-359.

22. Paul Halmas, Naive Set Theory, P. Van Nostrand Co., 1960, p. 94.

23. Hans Hahn, "Infinity", The World of Mathematics, edited by J. Newman, Vol. 3, p. 1593.

24. Howard Delong, op. cit.

25. Edna Kramer, The Mainstream of Mathematics, Fawcett Publications Inc., 1951.

26. Kamke, op. cit.

27. see Galileo, Dialogues Concerning Two New Sciences, Northwestern University, 1939.

28. J. Thomson, "Infinity in Mathematics and Logic", The Encyclopedia of Philosophy, Vol. 4, p. 185-186.

29. Kramer, op. cit., p. 322-323.

30. Delong, op. cit., p. 72-73.

31. Cantor, op. cit., p. 74.

32. Russell, The Principles of Mathematics, George Allen and Unwin Ltd., 1903, p. 358-359.

FOOTNOTES TO PART IV

1. Wesley Salmon, Introduction to Zeno's Paradoxes, edited by Wesley Salmon, The Bobbs-Merrill Company, Inc., 1970, p. 7-8.

2. Bertrand Russell, Our Knowledge of the External World, W.W. Norton & Company Inc., 1929, p. 193.

3. W.V. Quine, The Ways of Paradox, Harvard University Press, 1976, p. 3.

4. Wesley Salmon, Space, Time & Motion: A Philosophical Introduction, Dickenson Publishing Co., Inc., 1975, p. 38.

5. see Alastair Duman, "Zeno of Elea", Encyclopedia Britannica, Encyclopedia Britannica Inc., 1971, Volume 23, p. 959.

6. G. Vlastas, "Zeno of Elea", Encyclopedia of Philosophy, edited by Paul Edwards, Collier MacMillan Publishers, 1967, Volume 8, p. 373.

7. see Lewis Carroll, "What the Tortoise Said to Achilles", contained in Readings On Logic, second edition, edited by I. Copi and J. Gould, The Macmillan Company, 1972.

8. Aristotle, "Physics" Bk. VIII, Ch. 8, 263a 4-0, from The Basic Works of Aristotle, edited by R. McKeon, Random House Inc., 1941.

9. Ibid., 11-18.

10. Ibid., 263a, 18-263b, 9.

11. Ibid., Bk. VI, Ch. 8, 239b, 30.

12. Ibid., 20-25.

13. Gilbert Ryle, Dilemmas, Cambridge University Press, 1966, p. 5.

14. Ibid., p. 8-11.

15. Ibid., p. 49-50.

16. Ibid., p. 53-54.

17. Ibid., 37.

18. Ibid., p. 40

19. Ibid., p. 41-42.

20. Max Black, Problems of Analysis, Cornell University Press, 1954, p. 109-110.

21. Ibid., p. 120.

22. Ibid., p. 114-115.

23. Ibid., p. 116-118.

24. Ibid., p. 138-139.

25. Ibid., p. 152-154.

26. Adolf Grünbaum, Modern Science and Zeno's Paradoxes, Wesleyan University Press, 1967, p. 37-38.

27. Ibid., p. 52.

28. Ibid., p. 41-42.

29. Ibid., p. 63.

30. Ibid., p. 65.

31. Ibid., p. 79.

32. Ibid., p. 82-83.

33. see Salmon, Space, Time & Motion: A Philosophical Introduction, p. 47-48.

34. In this regard it should also be mentioned that A.J. Ayer dismisses the standard treat of the paradoxes of motion in The Central Questions of Philosophy, Penguin, 1973, p. 17-21. His discussion is not significant enough to be considered here.

35. Paul Swartz, About Time, Van Nostram, 1975, p. 216-219.

36. Ibid., p. 221.

37. Ibid., p. 201.

38. Ibid., p. 231.

39. In this regard, see Grünbaum, Modern Science and Zeno's Paradoxes, p. 109-112.

40. Swartz, op. cit., p. 202-203.

41. Ibid., p. 232-233.

42. Ibid., p. 31.

43. Ibid., p. 146.

44. G.J. Whitrow, The Natural Philosophy of Time, Nelson, 1966, p. 152.

45. D. Kaiser, "Language and the 'Achilles' Paradox", Philosophia Mathematica, June-December, 1968.

46. William James, Some Problems of Philosophy, Longmans, Green, and Co., 1948, p. 185-196.

47. see Georg Hegel, Hegel's Logic, edited by William Wallace, Oxford, 1975, p. 133.

48. Adam Schaff, "Principle of Contradiction", contained in Readings On Logic, edited by Copi and Gould, p. 160.

FOOTNOTES TO PART V

1. David Hilbert, "On the Infinite", Philosophy of Mathematics Selected Readings, edited by Paul Benacerraf and Hilary Putnam, Prentice Hall Inc., 1964, p. 135.

2. Wesley Salmon, Space, Time and Motion, Dickenson Publishing Co. Inc., 1975, p. 52.

3. Adolf Grünbaum, Modern Science and Zeno's Paradoxes, Wesleyan University Press, 1967, p. 115.

4. Adolf Grünbaum, "Modern Science and Refutation of the Paradoxes of Zeno", Zeno's Paradoxes, edited by Wesley Salmon, p. 165.

5. Paul Swartz, About Time, Van Nostram, 1975, p. 198.

6. see William James, Some Problems of Philosophy, Longmans, Green, & Co. Ltd., 1948, p. 155.

7. see P.W. Bridgman, "Some Implications of Recent Points of View in Physics", Revue Internationale de Philosophie, III, No. 10.

8. Euclid, The Thirteen Books of Euclid's Elements, Dover Publications, 1956, Book 1, definition 1 (p. 153).

9. Swartz, op. cit., p. 199.

10. Grünbaum, op. cit., p. 121.

11. Ibid., p. 123.

12. Ibid., p. 130.

13. Swartz, op. cit., p. 219.

FOOTNOTES TO PART VI

1. Ludwig Wittgenstein, Tractatus Logico-Philosophicus, Routledge and Kegan Paul, 1922, p. 27.

2. Ludwig Wittgenstein, The Blue and Brown Books, Basil Blackwell / Harper and Row, 1958, p. 27.

3. Ludwig Wittgenstein, Zettel, edited by G.E.M. Anscombe and G.H. von Wright, University of California Press, 1970, comment 447.

4. Ludwig Wittgenstein, Philosophical Investigations, Basil Blackwell, 1958, comment 109.

5. Maurice Drury, The Danger of Words, Routledge and Kegan Paul, 1973, p. ix.

6. Wittgenstein, Tractatus Logico-Philosophicus, proposition 3.32.

7. Ibid., proposition 3.321.

8. Ibid., proposition 3.322.

9. Ibid., proposition 3.323.

10. Ibid., proposition 4.002.

11. Ibid., proposition 4.002.

12. Ibid., proposition 4.002.

13. Ibid., proposition 3.324.

14. Ibid., proposition 3.325.

15. Ibid., proposition 4.0031.

16. Ludwig Wittgenstein, Notebooks 1914-1916, edited by G.E.M. Anscombe and G.H. von Wright, Basil Blackwell, 1961, p. 19.

17. Wittgenstein, Philosophical Investigations, comments 11-12.

18. Ibid., comment 122.

19. Ibid., comment 664.

20. Ibid., p. 224.

21. Wittgenstein, Tractatus Logico-Philosophicus, proposition 4.002.

22. Wittgenstein, Zettel, comment 144.

23. Ibid., comment 176.

24. Ibid., comment 287.

25. Wittgenstein, Philosophical Investigations, comment 111.

26. Wittgenstein, Zettel, comment 443.

27. Wittgenstein, The Blue and Brown Books, p. 28.

28. Ibid., p. 7.

29. Wittgenstein, Philosophical Investigations, comment 129.

30. Ibid., comment 111.

31. Ibid., comment 132.

32. Ibid., comment 119.

33. Ibid., comment 127.

34. Ibid., comment 126.

35. Wittgenstein, Tractatus Logico-Philosophicus, proposition 4.112.

36. Wittgenstein, Philosophical Investigations, comment 109.

37. "What is time? If no one asks me I know, if I try to explain it I don't know."

38. Wittgenstein, Philosophical Investigations, comment 89.

39. Wittgenstein, quoted in Norman Malcolm, Ludwig Wittgenstein: A Memoir, Oxford University Press, 1958.

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APPENDIX 1

Wittgenstein, Logic and the "Tractatus"[1]

In *Letters From Ludwig Wittgenstein*[2] an unfinished memoir, fifty-four letters, and an informative appendix combine into a worthwhile and readable commentary on Wittgenstein's early life and work.

Wittgenstein's intense despondency at this point in his life is strikingly evident in the letters. In a typical passage he writes:

> I have had a most miserable time lately. Of course only as a result of my own baseness and rotteness. I have continually thought of taking my own life, and the idea still haunts me sometimes. I have sunk to the lowest point. May you never be in that position! Shall I ever be able to raise myself up again? Well, we shall see. -- Reclam will not have my book. I don't care anymore, and that is a good thing.[3]

Some of the correspondence (for example, letter 54) does lend credence to William Bartley's view (expressed in his book *Wittgenstein*) that this depression was partly caused by Wittgenstein's inability to control homosexual urges. Engelmann does not discuss the matter.

Interesting things are said by Engelmann in regard to Wittgenstein's view on ethics, religion and art.

> This [the philosophy of the *Tractatus*] led to an attitude to life that comes nearest perhaps to that sought by Tolstoy: an ethical totalitarianism in all questions, a single-minded and painful preservation of the purity of the uncompromising demands of ethics, in agonizing awareness of one's own permanent failure to measure up to them. This is the demand Wittgenstein makes on himself.[4]

> Was Wittgenstein religious? If we call him an agnostic, this must not be understood in the sense of the familiar polemical agnosticism that concentrates, and prides itself, on the argument that man could never know about these matters.
> The idea of a God in the sense of the Bible, the image of God as the creator of the world, hardly ever engaged Wittgenstein's attention, but the notion of a last judgement was of profound concern to him...

> Above all, he was never a mystic in the sense of occupying his mind with mystic-gnostic fantasies. Nothing was further from his mind than the attempt to paint a picture of the world beyond (either before or after death), about which we cannot speak.[5]

> Again, and again Wittgenstein emphasized the significance of the 'happy ending'. To make a film without a happy ending, he thought, was to misunderstand the fundamentals of the cinema. Carrying the argument further, he said it was the essence of art in general to lead to a positive conclusion...
> The basic idea behind this view, it seems to me, is that art must always, in one sense or another, lead to a <u>solution</u>; the individual work of art, then, is an example demonstrating such a solution.[6]

The (provocative) views on art are not discussed as much as might be hoped.

Above all, the depth of Wittgenstein's feeling in all areas is evident *Letters From Ludwig Wittgenstein*. At one point Engelmann relates how Wittgenstein is carried away with passion at a music rehearsal and consequently interrupts the musicians. Though they greet the outsider with mild contempt we are told that at a later rehearsal:

> Wittgenstein, now completely accepted by the four musicians, did most of the talking, and his objections and advice were heard as defentially as if Gustav Mahler had himself interrupted the rehearsal.[7]

Engelmann continues:

> I have experienced similar scenes in other fields where Wittgenstein acted in the same way. Invariably those concerned realized very soon that they were presented with an opportunity to learn too important to miss. But whenever he was not <u>completely</u> sure of his ground, he would not open his mouth.[8]

A picture emerges of a man extremely sensitive and reflective, and thereby of strong character. It is perhaps not surprising that such a man should become one of this century's most influential philosophers.

II

If anything mars Engelmann's memoir it is his treatment of the *Tractatus Logico-Philosophicus*. His dismissal of the logical theory of the *Tractatus* is uncalled for and prevents his remarks from being anything but introductory. His initial complaint that the book's logical theory has been over emphasized and its mystical conclusions underplayed may be valid, but the suggestion that the logical theory of the *Tractatus* is unimportant to the book cannot stand. Though Wittgenstein was (as Engelmann points out) concerned with ethical and 'mystical' conclusions, he was also a logician. The *Tractatus* stands as an integrated whole and in it mystical conclusions are drawn from (among other things) logical theory.

Much evidence establishes the importance of logic to the *Tractatus*. When attempting to have the book published for example Wittgenstein once remarked that:

> the decimal numbers of my remarks absolutely must be printed alongside them, because they alone make the book perspicuous and clear; without the numbering it would be an incomprehensible jumble.[9]

And anyone familiar with the *Tractatus* knows that the decimal numbers in question progress from a theory of logic and language to mystical and ethical conclusions. This itself denies Engelmann's claim that

> ...it could be said with greater justice that Wittgenstein drew certain logical conclusions from his fundamental mystical attitude to life and the world.[10]

It has also been reported that the logical parts of the *Tractatus Logico-Philosophicus* were completed first, and the mystical parts last.[11]

In the *Tractatus* Wittgenstein writes that:

> My propositions are elucidatory in this way: he who understands me finally recognizes them as senseless, when he has climbed out through them, on them, over them. (He must so to speak throw away the ladder, after he has climbed up on it).
> He must surmount these propositions, then he sees the world rightly.[12]

The early parts of the *Tractatus* are important if only because it is them that must be surmounted in order to see the world rightly. It is these propositions that lead as steps to Wittgenstein's final (mystical) conclusions. A ladder is important to one trying to reach the roof of a particular building even though it loses its importance once that goal is achieved. The vehicle may be discarded after attaining the goal, but it is not to be dismissed beforehand.

Engelmann's claim that

> We do not understand Wittgenstein unless we realize that it was philosophy that mattered to him and not logic, which merely happened to be the only suitable tool for elaborating his world picture.[13]

and his view that

> Wittgenstein's true role and significance is different, indeed opposite to what is generally supposed.[14]

deny the very essence of the *Tractatus*. In it logic and philosophy are not separable, but one (thus the very name *Tractatus* *Logico*-*Philosophicus*). For Wittgenstein, logic and philosophy simply aren't opposites.

> The object of philosophy is the logical clarification of thoughts.[15]

In his 1913 *Notes on Logic* Wittgenstein clearly expresses the view that logic is fundamental to philosophy.

> The word 'philosophy' ought always to designate something over or under, but not beside, the natural sciences. Philosophy gives no pictures of reality, and can neither confirm nor confute scientific investigations. It consists of logic and metaphysics, the former its basis... Distrust of grammar is the first prerequisite for philosophising. Philosophy is the doctrine of the logical form of scientific propositions (not primitive propositions only).[16] (my emphasis)

Bernard Leitner has written of the building who's interior Wittgenstein designed later in his life that:

> Photographs of details may show something of Wittgenstein's thinking as an architect, but they do not give evidence of the particularity, the intellectual oneness, the absolute inter-

> dependence and complementarity of his spatial concept and the design of details.[17]

It is just this intellectual oneness, this absolute interdependence, that characterizes the logical and mystical parts of the *Tractatus*. Engelmann exaggerates when he says that the *Tractatus* "is not a treatise on the nature of human language."[18] Rather; he should say that the *Tractatus* is not just a treatise on the nature of human language. How is he to explain Wittgenstein's *Notebook* remark that "My whole task consists in explaining the nature of propositions."[19] (Wittgenstein's emphasis)?

III

The greatest failing of the Engelmann memoir is its confused idea of what the *Tractatus* considers mystical. It severs the logical from the mystical and then claims that the *Tractatus* has been misunderstood because Wittgenstein's readers:

> have so far been without the master key to its understanding, and because they are looking for its significance in all possible aspects which I know to have been of no or only marginal importance to him.[20]

But this rejection of the logic of the *Tractatus* (allegedly in favour of the mystical) rests on a mistaken idea of what the *Tractatus* considers mystical and 'non-mystical'. Engelmann is impressed by passages like

> 6.41 The sense of the world must lie outside the world.
>
> 6.42 Hence, also there can be no ethical propositions. Propositions cannot express anything higher.
>
> 6.421 It is clear that ethics cannot be expressed. Ethics is transcendental. (Ethics and aesthetics are one.)

but he fails to point out that for Wittgenstein, logic too is part of the mystical.

6.13 Logic is not a theory but a reflexion of the world.
Logic is transcendental.

...6.22 The logic of the world which the propositions of logic show in tautologies, mathematics show in equations.

Logic is mystical, it is beyond what can be said because it can be shown in propositions (tautologies) and

4.1212 What can be shown cannot be said.

For Wittgenstein, logic is not merely a stepping stone towards the mystical, it is a part of the mystical (and this is why logical theory must eventually be discarded before the world is seen rightly). The separation of 'the logical' and 'the mystical' which Engelmann relies on so heavily runs contrary to the *Tractatus*.

IV

That Wittgenstein had an early interest in logic cannot be doubted. His high esteem for Frege and Russell is clear[21] and it is with them he associated his work (thus his consultations with Frege and his study under Russell at Cambridge). In a letter to Russell from Norway he writes:

> My whole day passes between logic, whistling, going for walks and being depressed...
> It's extraordinary, isn't it, what a huge and infinitely strange science logic is? Neither you nor I knew that, I think, a year and a half ago.[22]

Within the *Tractatus* itself it is only Frege and Russell that Wittgenstein credits as being influential to his work.

> I will only mention that to the great works of Frege and the writings of my friend Bertrand Russell I owe in large measure the stimulation of my thoughts.[23]

Engelmann's suggestion that Wittgenstein was philosophically influenced by his Viennese contemporaries is at best superficial (it is not enough, for example, to say that he read Kraus' periodical and admired it). When Wittgenstein looked for philosophical insight he turned to logicians.

Whereas Russell is mentioned 30 times in the *Tractatus* and Frege 18 times, comments concerning Kraus and Loos are never made (in an earlier version of the *Tractatus* now entitled *Prototractatus* further reference is made to Frege and Russell (in 6.211 for example)). Similarly, in Wittgenstein's surviving pre-Tractatus *Notebooks* Frege's name appears eight times and Russell's nine while "Kraus" and "Loos" make no appearance. That Wittgenstein did not always hold admiration for his Viennese contemporaries is clear from some of the Engelmann letters. In letter 21 he writes:

> A few days ago I looked up Loos. I was horrified and nauseated. He has become infected with the most virulent bogus intellectualism! He gave me a phamplet about a proposed 'fine arts office', in which he speaks about a sin against the Holy Ghost. This surely is the limit! I was already depressed when I went to Loos, but that was the last straw![24]

Negative comments are also made of Ludwig Ficker (see letters 41 and 45).

Perhaps the most telling evidence against Engelmann's view about the *Tractatus* is Wittgenstein's own comments in a letter to Russell.

> I also sent my M.S. [Manuscript] to Frege. He wrote me a week ago and I gather that he doesn't understand a word of it all. So my only hope is to see <u>you</u> soon and explain it all to you, for it is VERY hard not to be understood by a single soul![25]

These remarks suggest that it is only Frege and Russell (two logicians) who Wittgenstein saw as capable of understanding his work. The remarks are made <u>after</u> Wittgenstein's discussions with Engelmann and if he thought that Engelmann had understood the book (by understanding its mystical conclusions but not its logic) we would expect him to express different sentiments here.

Engelmann himself seems to admit the importance of logic to the *Tractatus* when he writes that:

> Like all great new systems of philosophy, Wittgenstein introduces only a single, one might say 'small', modification -- but it is a fundamental one... It is meaningless, he says, to talk about the sphere of the transcendental, the metaphysical; and he rests this statement on a strong logical foundation.[26]

Yet Engelmann cannot establish that the book presents a strong logical foundation to any statement -- simply because he fails to examine the logic of the book.

V

One final criticism needs to be made of the Engelmann memoir. His concluding talk (p. 135-136) on "wordless faith" typifies the kind of speculation philosophy can well do without. Because Wittgenstein's philosophy leads to a respect for the mystical which results in silence, Engelmann dubs it "wordless faith". And because wordless faith consists of silence it is not possible, he says, to verbally misconstrue it (for there is no verbal doctrine to misconstrue). But this is wishful thinking. Wittgenstein is not satisfied with <u>any</u> silence, but only that born of a particular understanding. And that understanding is to be furthered by words (in particular, the words of the *Tractatus*). So in the end Wittgenstein's philosophy is tied to words. And that those words can be misconstrued is readily shown by the positivists' interpretation of the *Tractatus* (which Engelmann himself dismisses).

Despite its shortcomings, *Letters From Ludwig Wittgenstein* IS profitable reading. It is unfortunate that Paul Engelmann did not live to complete his memoir; the unwritten chapters would no doubt cast further light on Wittgenstein and his work. Obviously they cannot be thoroughly examined here.

"Whereof one cannot speak, thereof one must be silent."[27]

APPENDIX 2

A Further Note On Set Theory

Besides the significant change in set theory suggested in Part III the remarks that I have made in regard to Zeno's paradoxes of motion also affect standard accounts of set theory. Thus, set theorists mistakenly suggest that an infinite series of acts is completable. Hilary Putnam writes

> The intuitive 'definition' of 'finite' is often thought to be this: A set is finite if one could get through counting it. There are counter-examples (Zeno's paradox) if one does not require that it should take a minimum time (say, 'one minute') to count out each member of the set: for if I take a half-minute to count 'one', a quarter minute to count 'two', an eighth-minute ... etc., then I could count out an infinite collection in a finite time. If the infinite collection has the order $\omega + 1$ (or any other order type, with a last member) then there could even be a last element counted (say, after having counted out the first w elements in one minute, I take one more half-minute and count out the ω + 1st). Thus an infinite set can be 'counted in a finite time', even if we require that there should be a 'last element counted'.[1]

Boolos and Jeffrey make similar remarks.

> If a set is enumerable, Zeno can enumerate it in one second by writing out an infinite list faster and faster. He spends 1/2 second writing the first entry in the list; 1/4 second writing the second entry; 1/8 second writing the third; and in general, he writes each entry in half the time he spent on its predecessor. At no point during the one second interval has he written out the whole list, but when one second has passed, the list is complete![2]

It should be clear from Chapters 1 and 13 that such remarks are not tenable (they are based on the incorrect view that the completion of an infinite series of acts is blocked by an insufficient amount of time).

FOOTNOTES TO APPENDIX 1

1. I have included this book review because it undermines one current interpretation of Wittgenstein. All the comments made could also be applied to the views expressed by Janik and Toulmin in their book Wittgenstein's Vienna (Touchstone, 1973). I do not have time to consider other interpretations of Wittgenstein's work.

2. Paul Engelmann, Letters From Ludwig Wittgenstein, With A Memoir, edited by Brian McGuinness, Basil Blackwell, 1967.

3. Ibid., p. 33.

4. Ibid., p. 109.

5. Ibid., p. 77-79.

6. Ibid., p. 93.

7. Ibid., p. 90.

8. Ibid.

9. Ludwig Wittgenstein, from a letter contained in Prototractatus: An early version of Tractatus Logico-Philosophicus, edited by B.F. McGuinness, T. Nyberg and G.H. von Wright, Routledge and Kegan Paul, 1971, p. 18.

10. Engelmann, op. cit., p. 97.

11. see G.H. von Wright, A Biographical Sketch, contained in Norman Malcolm, Ludwig Wittgenstein: A Memoir, Oxford University Press, 1958, p. 7-8.

12. Ludwig Wittgenstein, Tractatus Logico-Philosophicus, Routledge and Kegan Paul, 1922, proposition 6.54.

13. Engelmann, op. cit., p. 97.

14. Ibid., p. 122.

15. Wittgenstein, Tractatus Logico-Philosophicus, proposition 4.112.

16. Ludwig Wittgenstein, Notes on Logic, contained in Ludwig Wittgenstein, Notebooks 1914-1916, edited by G.E. M. Anscombe and G.H. von Wright, Basil Blackwell, 1961, p. 93.

17. B. Leitner, "Wittgenstein's Architecture", Art Forum, February 1970.

18. Engelmann, op. cit., p. 99.

19. Wittgenstein, Notebooks 1914-1916, p. 39.

20. Engelmann, op. cit., p. 87.

21. In a published review written before the Tractatus ("On Logic And How Not To Do It", Cambridge Review, March 1913), he writes of an author who (he says) "has not taken the slightest notice of the great work of modern mathematical logicians - work which has brought about an advance in Logic comparable only to that which made Astronomy out of Astrology and Chemistry out of Alchemy". There can be no doubt that the logicians he has in mind are Russell and Frege (and perhaps Peano).

22. Ludwig Wittgenstein, Letters to Russell, Keynes and Moore, edited by G.H. von Wright, Basil Blackwell, 1976, p. 45.

23. Wittgenstein, Tractatus Logico-Philosophicus, p. 29.

24. Engelmann, op. cit., p. 17.

25. Wittgenstein, Letters to Russell, Keynes and Moore, p. 37.

26. Engelmann, op. cit., p. 109.

27. Wittgenstein, Tractatus Logico-Philosophicus, proposition 7.

FOOTNOTES TO APPENDIX 2

1. Hilary Putnam, "The Thesis That Mathematics Is Logic", Mathematics, Matter, and Method, Cambridge University Press, 1975, p. 24.

2. George Boolos and Richard Jeffrey, Computability And Logic, Cambridge University Press, 1974, p. 14.