UNIVERSITY OF CALGARY

Mixed Objective LQ / Sliding Mode Control

by

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Abstract

The now classic optimization technique of Linear Quadratic (LQ) controller design is, in theory, one of the best design strategies possible. Practically, however, it has limitations. Specifically, it is a simple demonstration to show that LQ control techniques have no robustness margins when applied as an output feedback strategy to a noise-corrupted system.

Alternatively, Sliding Mode Control (SLMC) is an extremely robust control strategy, provided the disturbances are bounded and matched. The primary drawback is that the resulting system performance is never optimal in an LQ sense of the word.

In a novel approach, this thesis blends these two design objectives. The resulting controller exhibits near-LQ performance, while adding the additional feature of robustness to bounded, matched disturbances. The controller is developed for both the state and output feedback cases. The results are then applied to the problem of maintaining an inverted pendulum in the vertical position.

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Notation and Symbols

$\mathbb{F}$	Arbitrary Field
$\mathbb{R}$	Field of Real Numbers
$\mathbb{C}$	Field of Complex Numbers
$\mathbb{Z}$	Field of Integer Numbers
$\mathcal{X}$,$\mathcal{Y}$,...	Vector Space Over $\mathbb{F}$
x,y,...	Element of a Vector Space Over $\mathbb{F}$
α,β,...	Scalar Element of $\mathbb{F}$
$\{x_i\}_{i=1}^n$	Set of n Elements x_i
$\emptyset$	Empty Set
$\in$	Element of
$\subset$	Subset / Subspace
$\cap$	Intersection
$\cup$	Union
$\operatorname{span}\{x\}$	Linear Span of the Set $\{x\}$
$\dim(\mathcal{X})$	Dimension of the Space $\mathcal{X}$
Im	Image
ker	Kernel
$\varphi : \mathcal{X} \to \mathcal{Y}$	Map φ From $\mathcal{X}$ to $\mathcal{Y}$
$\mathcal{L}(\mathcal{X},\mathcal{Y})$	Set of all Linear Maps From $\mathcal{X}$ to $\mathcal{Y}$
A,B,...	Matrix / Matrix Representation of a Map over $\mathbb{F}$
$[\alpha_{ij}]_{i,j=i}^{n,m}$	Array Representation of the n by m Matrix A
A^T	Transpose of the Matrix A
A^*	Complex Conjugate Transpose of the Matrix A

$\det(A)$	Determinant of the Matrix A
$\langle , \rangle$	Inner Product
$\|x\|_p$	p-Norm of x
$\sum$	Summation
$=$	Equality
$\neq$	Negated Equality
$\equiv$	Equality by Identity
$:=$	Equality by Definition
$\simeq$	Isomorphic
$\approx$	Similar
$>, <$	Strictly Greater Than, Strictly Less Than
$\geqslant, \leqslant$	Greater Than, Less Than
$+$	Sum
$\dot{+}$	Direct Sum
$\oplus$	Orthogonal Direct Sum
$\times$	Cartesian Product
$\otimes$	Direct / Kronecker Product
vec	Vector Function
I_n	n by n Identity Matrix
$0_{n \times m}$	n by m Zero Matrix
$\sigma(A)$	Spectrum / Eigenvalues of the Matrix A
$\Rightarrow$	If / Implies / Necessary
$\Leftarrow$	Only If / Implied by / Sufficient
$\Leftrightarrow$	If and Only If / Necessary and Sufficient

List of Acronyms

LQ	Linear Quadratic
LQG	Linear Quadratic Gaussian
LTR	Loop Tranfer Recovery
SLMC	Sliding Mode Control
LMI	Linear Matrix Inequality
VSC	Variable Structure Control
ARE	Algebraic Ricatti Equation

Chapter 1

Introduction

It is a basic fact of life that there is rarely a unique 'best' way of doing things. Of course, it is rarely the case that an optimal solution does not exist to a given situation. Frequently, this lack of a clear solution arises as a direct consequence of an unclear, or perhaps undecided objective.

To illustrate this point in some very broad terms, let us consider the classic tale of the grasshopper and the ant. We are all familiar with the story of the grasshopper who spends his days carelessly playing in the sun, only to meet a bitter demise come winter. The ant, however, carries on in comfort despite the cold, because he spent his summer fastidiously preparing for the approaching winter.

The moral of the story? Most would have us believe that this is a quaint allegory aimed at instilling a strong work ethic and ability to plan ahead in the reader. But is this in fact the case? Perhaps what we have here is a clear example of a situation in which there were multiple objectives, and the characters chose between one or the other. Consider first the grasshopper. Sure, he dies. But so what? His objective was to spend his days in the sun, living his life to fullest, and I dare say he achieved it.

The ant, on the other hand, lives. His days are spent living a joyless life, toiling in the dirt, planning for the encroaching winter. His goal was a long life, and he achieved it. Was one objective more correct than the other? That is not really a topic for discussion here, but it does set the stage for motivating this particular work.

Consider a third option. The grasshopper didn't have to play *every* day, and the ant probably didn't need *all* the supplies he built up. Thus, it would seem that the 'best' option would have been to work on some days, and play on others. There is obviously some sort of merit to this plan, as it would seem that this is how a large portion of the human population lives. In this case, the 'best' solution is achieved by mixing objectives, and following the solution that allows one to enjoy the best of both worlds, as the saying goes.

1.1 Motivation

It is this basic philosophy that has motivated the following work. Of course, the topic at hand is a little more specific. Here, motivation for this thesis arises from the following **Control Problem.**

Control Problem: A linear quadratic (LQ) optimal control strategy is, in theory, one of the best control laws available. However, it is a known fact that this strategy is not robust to any class of disturbances in the output feedback case [10], when operating in a noisy environment. Conversely, sliding mode control (SLMC) is extremely robust to bounded matched disturbances [11], yet typically leads to system performance that is not at all optimal. The control problem at hand, then, is to design a control law that will incorporate the positive aspects of both of these strategies. That is, design a controller that realizes LQ optimal (or near optimal) closed loop performance, while simultaneously achieving the SLMC result of an extremely robust controller.

It turns out that this is by no means a trivial problem, the primary reason being that each control strategy utilizes decidedly different objectives for determining the closed loop structure of the system. Specifically, when designing an LQ controller, the feedback law applied leads to a set of closed loop eigenvalues that are placed anywhere on the open left hand complex plane. Conversely, application of SLMC leads to a situation where a fixed number of the closed loop eigenvalues *must* be located at the origin, and only the remaining eigenvalues are free for placement (see Theorem 3.1).

Thus, the problem at hand is not one of not deciding between two different objectives, but rather one of determining a strategy that allows us to mix them in a manner that is, in some sense, optimal. It is demonstrated in §4.3 that this mixing of objectives can be accomplished by modifying the classic LQ strategy in such a manner that m closed loop eigenvalues are pre-selected, and the remaining ones are free to be optimized. In this way, the two strategies can be mixed, leading to a closed loop system that maintains near-optimal performance, with the added feature of robustness to matched disturbances.

1.2 Previous Work and Related Literature

1.2.1 Motivating Works

By the 1970s, LQ control theory had started to reach maturity, and the method had become a practical design tool. At this point in time, the focus shifted from that of determining the viability of implementing an LQ strategy to that of determining the theoretical performance limits of LQ control. In particular, the question of robustness was of great interest. That is, given a closed loop, output feedback LQ system that is corrupted by noise, how well will the system perform? Or, what guaranteed robustness margins does this system have? It is now a standard result [2] to demonstrate these results for the state-feedback case (i.e., $u = Kx$), but the question remained open for the output-feedback case (i.e., $u = Ky$). It was shown by Doyle [10] in 1978 via counterexample that there are, in fact, no guaranteed robustness margins. A rather demoralizing result!

Over the next few years, interest then grew in development of output feedback control laws that could exhibit some guaranteed robustness margins, even if only to a specific class of disturbances. LQG / LTR results attempted to asymptotically recover the nominal (uncorrupted) state feedback results, but lost physical significance of the Kalman filter weights. Recently, output feedback sliding mode control has received a great deal of attention ([9], [12], [13], [14], [15], [19], [42], [46], [48]). The problem with these types of control strategies is that they suffer from the same drawbacks as conventional state-feedback sliding mode control. That is, the resulting system performance is robust, but in no sense optimal.

Various schemes have, in fact, been introduced to address this problem. Notably, in [47] and [32], additional dynamics were introduced into the control term, with the objective of achieving a smoother control law than possible with conventional SLMC. In this way, the resulting system performance has the potential of achieving closed loop performance that is closer to optimal.

Further, in a recent work by Tang and Misawa [39], a result is presented that has essentially the same goal as this thesis. That is, to design an LQ controller with a preset (real) eigenvalue, allowing the controller to exhibit the benefits of LQ performance as well as SLMC robustness. For reasons explained in §4.1 however, the method presented in [39] has been passed up in this work. In fact, the method employed in this work is drastically different from the one presented in [39].

1.3 Organization of Thesis

A brief outline of the thesis is as follows.

Chapter 2 is purely composed of the preliminary mathematical tools needed for this work.

Chapter 3 presents a comprehensive overview of sliding mode control, and gives insight into why SLMC and LQ control are conflicting design objectives.

Chapter 4 presents the main result of the thesis. Here, conditions are derived that are necessary for the controller to work. That is, it is shown that the open loop system is required to be of the type

- (A, B) controllable, and (A, B, C) complete in the output feedback case.
- $\text{rank}\,(B) = m$

The robustness properties of the proposed controller are then demonstrated for both the state and output feedback situations.

Chapter 5 gives a sample application of the proposed controller. Since the focus of this work was a theoretical development of the control law rather than the implementation of it, all results are simulated, rather than physically realized.

Chapter 6 contains a summary of the main results and some direction for future work.

1.4 Contributions of Thesis

This thesis provides a method for designing a controller that closely mimics nominal, closed loop LQ system results in both the state and output feedback situations, while adding the result of a guaranteed robustness margin in the presence of bounded, matched disturbances. Specifically, the main contributions are:

- The explicit development of the transformation matrix T needed for the result of (4.11). This result transforms the closed loop system matrix to an upper block triangular form. In turn, this allows a K to be solved that simultaneously meets the objective of arbitrary eigenvalue placement for $m$closed loop eigenvalues, while the remaining eigenvalues may be placed optimally.

- The derivation of the controller (4.24). Here, a controller is now created that induces a near optimal, closed loop system response, while adding the feature of invariance to bounded, matched disturbances.

- The derivation of a robust, output feedback controller in §4.4

- A demonstration of the robustness properties of the proposed controller in both the state and output feedback cases.

In addition to these points, the thesis also provides a comprehensive overview of conventional SLMC control theory as it pertains to linear systems, as well as an overview of all the important mathematical tools needed for the analysis.

Chapter 2

Mathematical Preliminaries

In this chapter, some basic facts that are essential to this work will be reviewed. No new concepts will be developed here. Rather, this section will serve as an introduction to the notation that will be employed throughout the work.

2.1 Linear Algebra and Geometry

In the following, the basic concepts that are of particular importance to this work are developed. The material presented in §2.1 and §2.2 is based almost exclusively on the two excellent works of Lancaster and Tismenetsky [27] and Wonham [45].

2.1.1 Linear Spaces and Subspaces

Let $\mathbb{F}$ denote a scalar field, and $\mathbb{F}^n$ denote an n-dimensional vector space over $\mathbb{F}$. That is, $\mathbb{F}^n$ is isomorphic to the n-fold Cartesian product of $\mathbb{F}$, where n is a finite integer. i.e.

$$\mathbb{F}^n = \underbrace{\mathbb{F} \times \mathbb{F} \times \cdots \times \mathbb{F}}_{n - times} \tag{2.1}$$

Further, an additive element $x \in \mathbb{F}^n$ denotes a vector over the underlying field $\mathbb{F}$. In this work, the attention will be limited to finite dimensional *linear* spaces over $\mathbb{R}$ and $\mathbb{C}$, the real and complex numbers.

Since it is the nature of control problems to deal with elements in numerous different spaces at the same time, a notation will be adopted that will clarify the operations. Calligraphic capitals $\mathcal{X}$,$\mathcal{Y}$,... will be used to denote spaces, lower case roman characters $x, y, \ldots$ will be used to denote their elements, and scalars will be denoted with lower case

Greek or Roman characters as appropriate.

Of great importance when dealing with vectors are the operations of addition and scalar multiplication, or the fact that these operations are both *associative* and *distributive*. For example, consider $x_1, x_2 \in \mathcal{X}$ where $\mathcal{X}$ is defined over $\mathbb{F}$, and $\alpha, \beta \in \mathbb{F}$. Then

$$\alpha x_1 \in \mathcal{X} \tag{2.2}$$

$$x_1 + x_2 \in \mathcal{X} \tag{2.3}$$

and

$$(\alpha\beta)\, x_1 = \alpha\,(\beta x_1) \tag{2.4}$$

$$\alpha\,(x_1 + x_2) = \alpha x_1 + \alpha x_2 \tag{2.5}$$

Now, consider a subset $\mathcal{X}_0 \subset \mathcal{X}$. Since the operations of addition and scalar multiplication are defined for all elements of $\mathcal{X}$, they are also defined for all elements of $\mathcal{X}_0$. If these operations are closed in $\mathcal{X}_0$, in that

$$\alpha x_1 \in \mathcal{X}_0 \tag{2.6}$$

$$x_1 + x_2 \in \mathcal{X}_0 \tag{2.7}$$

for all $\alpha \in \mathbb{F}$ and $x_1, x_2 \in \mathcal{X}_0$, then $\mathcal{X}_0$ is called a *subspace* of $\mathcal{X}$, and shall henceforth be denoted as $\mathcal{X}_0 \subset \mathcal{X}$, where $\mathcal{X}_0 \neq \emptyset$.

As per the axioms of a linear space, $\mathcal{X}$ must contain the zero element, 0, typically as an origin. This implies then that a subspace $\mathcal{X}_0 \subset \mathcal{X}$ must also contain the same zero element, and if there is another subspace $\mathcal{X}_1 \subset \mathcal{X}$, then it is always true that $0 \subset \mathcal{X}_1 \cap \mathcal{X}_0 \neq \emptyset$, or two subspaces are never disjoint. Geometrically, this means that one can view a subspace as a hyperplane passing through the origin of the original space.

With the notion of a space in place, one can now define some properties of these spaces. First, define the *span* of a set of elements $\{x_i\}_{i=1}^{n} \in \mathcal{X}$ as the minimal subspace $\mathcal{X}_0$ generated by all linear combinations of the x_i over $\mathbb{F}$. Thus

$$\operatorname{span}\{x_1, x_2, \ldots, x_n\} := \left\{ x \in \mathcal{X} : x = \sum_{i=1}^{n} \alpha_i x_i,\, \alpha_i \in \mathbb{F} \right\} \tag{2.8}$$

This allows $\mathcal{X}$ to be termed finite dimensional it there exists a finite integer n and a set $\{x_i\}_{i=1}^{n}$ whose span is the whole of $\mathcal{X}$. The minimal value of n, in the sense of linearly independent x_i's that span $\mathcal{X}$, is termed the *dimension* of $\mathcal{X}$, denoted $\dim(\mathcal{X})$. When $n \neq 0$, this minimal spanning set forms a *basis* for $\mathcal{X}$.

A property that will be exploited to some degree in this work is the notion of a *sum* and *direct sum* of subspaces. Let $\mathcal{X}_1, \mathcal{X}_2 \subset \mathcal{X}$. Define the sum $\mathcal{X}_1 + \mathcal{X}_2$ as the set of all sums

$$\mathcal{X}_1 + \mathcal{X}_2 := \{x_1 + x_2 : x_1 \in \mathcal{X}_1, x_2 \in \mathcal{X}_2\} \tag{2.9}$$

Note that $\mathcal{X}_1 + \mathcal{X}_2 \subset \mathcal{X}$. Further, define the *intersection* of these subspaces as

$$\mathcal{X}_1 \cap \mathcal{X}_2 := \{x : x \in \mathcal{X}_1 \text{ and } x \in \mathcal{X}_2\} \tag{2.10}$$

As before, $\mathcal{X}_1 \cap \mathcal{X}_2 \subset \mathcal{X}$. If $\mathcal{X}_1 \cap \mathcal{X}_2 = 0$, then any element $x \in \mathcal{X}_1 + \mathcal{X}_2$ admits a unique decomposition

$$x = x_1 + x_2 \tag{2.11}$$

where $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$. Since this property can be quite useful, the space $\mathcal{X}_0 = \mathcal{X}_1 + \mathcal{X}_2$ generated is referred to as the direct sum of $\mathcal{X}_1$ and $\mathcal{X}_2$, and is denoted

$$\mathcal{X}_0 = \mathcal{X}_1 \dot{+} \mathcal{X}_2 \tag{2.12}$$

or, if $\mathcal{X}_0$ can be decomposed into k linearly independent subspaces, in that

$$\bigcap_{i=1}^{k} \mathcal{X}_i = 0 \tag{2.13}$$

then the direct sum of these spaces is denoted

$$\begin{aligned} \mathcal{X}_0 &= \mathcal{X}_1 \dot{+} \mathcal{X}_2 \dot{+} \cdots \dot{+} \mathcal{X}_k \\ &= \sum_{i=1}^{k} \cdot \mathcal{X}_i \end{aligned} \tag{2.14}$$

Also of great importance in the *orthogonal direct sum*. This operation works in the same manner as the direct sum, with the additional feature that

$$\langle x_1, x_2 \rangle = 0 \;\text{ for all }\; x_1 \in \mathcal{X}_1,\; x_2 \in \mathcal{X}_2 \tag{2.15}$$

where $\langle , \rangle$ denotes the inner product. In other words, as well as the spaces being linearly independent, they are also orthogonal. The orthogonal direct sum is denoted

$$\begin{aligned} \mathcal{X}_0 &= \mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \cdots \oplus \mathcal{X}_k \\ &= \bigoplus_{i=1}^{k} \mathcal{X}_i \end{aligned} \tag{2.16}$$

2.1.2 Maps and Matrices

In the manner previously defined, let $\mathcal{X}$ and $\mathcal{Y}$ denote linear spaces over $\mathbb{F}$. A *linear transformation*, or *map*, is a function $\varphi : \mathcal{X} \rightarrow \mathcal{Y}$ such that

$$\varphi(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \varphi(x_1) + \alpha_2 \varphi(x_2) \tag{2.17}$$

for all $x_1, x_2 \in \mathcal{X}$, $\alpha_1, \alpha_2 \in \mathbb{F}$. Maps will herein be denoted with capital roman characters $A, B, \ldots$

Denote $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ as the set of all linear maps $A : \mathcal{X} \rightarrow \mathcal{Y}$. It follows that $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ forms a linear space as well, and thus follows the rules of addition and multiplication by a scalar, i.e.

$$(A_1 + A_2)\, x \quad : \quad = A_1 x + A_2 x \tag{2.18}$$

$$(\alpha A_1)\, x \quad : \quad = \alpha (A_1 x) \tag{2.19}$$

for all $x \in \mathcal{X}$, $A_1, A_2 \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. As per the definition, $A_1 \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $A_1 : \mathcal{X} \rightarrow \mathcal{Y}$ are equivalent statements, and will hence be interchanged freely from this point on.

Let $\mathcal{X}$ be a linear space over $\mathbb{F}$, such that $\dim(\mathcal{X}) = n$. Further, let $\{x_i\}_{i=1}^{n}$ form a basis for $\mathcal{X}$. Thus, if $z \in \mathcal{X}$ then from the definition of a basis set, z may be represented by

$$z = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n \tag{2.20}$$

Forming a vector

$$a = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \tag{2.21}$$

where a is then referred to as the *representation of z with respect to* $\{x_i\}_{i=1}^{n}$. If this notion is expanded to a set of elements $\{z_j\}_{j=1}^{m} \in \mathcal{X}$, the corresponding set of $\{a_j\}_{j=1}^{m}$ can be formed together to give the array

$$A_{\{x\}} = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nm} \end{bmatrix} = [\alpha_{ij}]_{i,j=1}^{n,m} \in \mathbb{F}^{n \times m} \tag{2.22}$$

where

$$z_j = \sum_{i=1}^{n} \alpha_{ij} x_i \tag{2.23}$$

for all $1 \leqslant j \leqslant n$. The array A is known as the *matrix* representation of the (ordered) set $\{z_j\}_{j=1}^{m}$ with respect to the basis $\{x_i\}_{i=1}^{n}$.

This definition of a matrix then applies directly to linear maps. To see this, consider a map $M \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, where $\mathcal{X}$ is as before, and $\mathcal{Y}$ is a space over $\mathbb{F}$ such that $\dim(\mathcal{Y}) = m$, and $\{y_j\}_{j=1}^{m}$ is a basis set for $\mathcal{Y}$. Then, evaluating the images of $\{x_i\}_{i=1}^{n}$ yields

$$M(x_i) = \sum_{j=1}^{m} \alpha_{ij} y_j \tag{2.24}$$

for all $i = 1, 2, \ldots, n$. Thus, expanding this notation to $\{T(x_i)\}_{i=1}^{n}$ yields the matrix

$$A_{\{x\},\{y\}} = \begin{bmatrix} \alpha_{11} & \cdots & \alpha_{1m} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nm} \end{bmatrix} = [\alpha_{ij}]_{i,j=1}^{n,m} \in \mathbb{F}^{n \times m} \tag{2.25}$$

as before. The matrix $A_{\{x\},\{y\}}$ is then referred to as the matrix representation of the map $M \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ with respect to the basis pair $(\{x\}, \{y\})$. This definition, coupled with the fact that the focus of this work is concerned only with linear operations on linear spaces, allows one to easily drift between the notion of a map and a matrix. Hence, the two terms will be interchanged freely throughout this work. Thus, matrices will henceforth be denoted with capital roman letters $A, B, \ldots$ and the subscripts indicating bases will be dropped in most instances.

The properties and operations of matrices are defined in the usual way. The symbol $\mathbb{F}^{n \times p}$ denotes the class of all $n \times p$ matrices over $\mathbb{F}$. These matrices form a linear space of dimension np over $\mathbb{F}$ by the operations of addition and multiplication by a scalar.

As well, in the usual way, let A^T denote the *transpose* of A via

$$A = [\alpha_{ij}]_{i,j=1}^{n,m} \Leftrightarrow A^T = [\alpha_{ji}]_{i,j=1}^{n,m} \tag{2.26}$$

and let A^* denote the *complex conjugate transpose* of A, so that

$$A = [\alpha_{ij}]_{i,j=1}^{n,m} \Leftrightarrow A^* = [\bar{\alpha}_{ji}]_{i,j=1}^{n,m} \tag{2.27}$$

It is now useful to look at some terms associated with linear maps / matrices.

Let $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. The space $\mathcal{X}$ is referred to as the *domain* of A, the space $\mathcal{Y}$ is the *co-domain*. The *kernel* of A is the subspace

$$\ker A := \{x : x \in \mathcal{X} \text{ and } Ax = 0\} \subset \mathcal{X} \tag{2.28}$$

Similarly, the *image* of A is the subspace

$$\text{Im } A := \{Ax : x \in \mathcal{X}\} \subset \mathcal{Y} \tag{2.29}$$

Consistent with the terminology used thus far, we can define the *rank* of a linear mapping $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ as

$$\text{rank}(A) := \dim(\text{Im } A) \tag{2.30}$$

Or, in terms of a matrix, the row (column) rank of a matrix is equal to the number of linearly independent rows (columns) in the matrix.

Now, let $\mathcal{S} \subset \mathcal{X}$, so that

$$A\mathcal{S} := \{Ax : x \in \mathcal{S}\} \subset \mathcal{Y} \tag{2.31}$$

further, if $\mathcal{R} \subset \mathcal{Y}$, then

$$A^{-1}\mathcal{R} := \{x : x \in \mathcal{X} \text{ and } Ax \in \mathcal{R}\} \subset \mathcal{X} \tag{2.32}$$

By definition, A^{-1} is known as the *inverse image function* of the mapping A, and is thus (generally) an *immersion* mapping, $A^{-1} : \mathcal{Y} \rightarrow \mathcal{X}$. To avoid confusion, the following notation will now be adopted.

If $\dim(\mathcal{X}) = \dim(\mathcal{Y})$, and $\mathrm{Im}(A) = \mathcal{Y}$, $\mathrm{Im}(A^{-1}) = \mathcal{X}$, then the mapping is one to one and onto, and thus linear, since A is linear. In this case, the mapping A^{-1} will be unique, and will simply be referred to as the *inverse* of A. Further, the mapping is said to be *nonsingular*.

2.1.3 Similar Matrices, Equivalence Classes

The notion of similar matrices arises from the notion that a mapping $M \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is, typically, not unique. Consider the fact that the matrix representation of M was derived from the set $\{x_i\}_{i=1}^n$ which formed a basis for $\mathcal{X}$. Now, while the span of a basis set is unique, it is clear that the are many different bases available that have the same span. Thus, it is important to examine how the different basis sets can be transformed.

Consider first the operation of a change of basis in $\mathcal{X}$. Let $\{x_i\}_{i=1}^n$ be a basis set for $\mathcal{X}$, and $\{r_i\}_{i=1}^n$ be another. As per the definition of a matrix

$$T_{\{x\}} = \begin{bmatrix} \tau_{11} & \cdots & \tau_{1n} \\ \vdots & & \vdots \\ \tau_{n1} & \cdots & \tau_{nn} \end{bmatrix} \in \mathbb{F}^{n \times n} \tag{2.33}$$

is the matrix representation of $\{r_i\}_{i=1}^n$ with respect to $\{x_i\}_{i=1}^n$, so that

$$r = T_{\{x\}}x \Leftrightarrow x = T_{\{x\}}^{-1}r \tag{2.34}$$

Similarly, if $\dim(\mathcal{Y}) = m$, and $\{y_j\}_{j=1}^m$ and $\{p_j\}_{j=1}^m$ are (possibly) different bases for $\mathcal{Y}$, then

$$p = T_{\{y\}} y \Leftrightarrow y = T_{\{y\}}^{-1} p \tag{2.35}$$

Now, let $M \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ have a matrix representation

$$y = A_{\{x\},\{y\}} x \tag{2.36}$$

in the bases pair $(\{x\}, \{y\})$. To transform this matrix to a different basis pair, say $(\{r\}, \{p\})$, simple substitution of the above leads to

$$\begin{aligned} T_{\{y\}}^{-1} p &= A_{\{x\},\{y\}} T_{\{x\}}^{-1} r \\ &\Rightarrow p = T_{\{y\}} A_{\{x\},\{y\}} T_{\{x\}}^{-1} r \\ &\Rightarrow A_{\{r\},\{p\}} = T_{\{y\}} A_{\{x\},\{y\}} T_{\{x\}}^{-1} \end{aligned} \tag{2.37}$$

The matrices $A_{\{r\},\{p\}}$ and $A_{\{x\},\{y\}}$ are then said to be *equivalent*, since they are both representations of $M \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$.

A special situation arises when $M \in \mathcal{L}(\mathcal{X})$. Here, one need only look at the basis sets $\{x\}$ and $\{r\}$, so that

$$r = Tx \tag{2.38}$$

Thus, in the bases $\{x\}$, $M \in \mathcal{L}(\mathcal{X})$ has a matrix representation

$$a = A_{\{x\}} b, \;\; a, b \in \mathcal{X} \tag{2.39}$$

and, a change of basis to $\{r\}$ leads to

$$\begin{aligned} T^{-1} a' &= A_{\{x\}} T^{-1} b' \\ &\Rightarrow a' = T A_{\{x\}} T^{-1} b' \\ &\Rightarrow A_{\{r\}} = T A_{\{x\}} T^{-1} \end{aligned} \tag{2.40}$$

where $A_{\{r\}}$ is the matrix representation of $M \in \mathcal{L}(\mathcal{X})$ with respect to $\{r\}$. This relation is quite important in linear algebra, so much so that it has been given a special name. If A and B are square matrices, then A and B are termed *similar* if there exists a nonsingular matrix T such that

$$A = TBT^{-1} \tag{2.41}$$

or, equivalently

$$A \approx B \tag{2.42}$$

Assuming $\dim(\mathcal{X}) = n$, define the set of all matrix representations of $M \in \mathcal{L}(\mathcal{X})$ with respect to different bases in $\mathcal{X}$ as an *equivalence class of similar matrices* over $\mathbb{F}^{n \times n}$. Denote this set as $\mathcal{A}_T^0$. This notion of similarity then gives the value of $A \in \mathcal{A}_T^0$ that is 'simplest' relative to the problem at hand.

2.1.4 Projection Maps, Idempotent Matrices, and Invariant Subspaces

Due to the fact that projectors and idempotent mappings largely define the geometric nature of sliding mode control, it is of great use to explore these maps at this point.

By definition, a map satisfies the condition of *idempotency* if, for some $P \in \mathcal{L}(\mathcal{X})$, $P = P^2$. Such a mapping is also termed a *projector*. This concept will be explored after some preliminary properties are explored.

Theorem 2.1 *If P is idempotent, then*

1. *$(I - P)$ is idempotent.*
2. $\ker(I - P) = \operatorname{Im} P$
3. $\operatorname{Im}(I - P) = \ker P$

Proof. (1) is immediately verified by expansion of $(I - P)^2$, and the definition of an idempotent map. For (2), let $y \in \operatorname{Im} P$ if and only if $y = Px$ for some $x \in \mathcal{X}$. Thus

$$(I - P)y = (I - P)Px$$

$$
\begin{aligned}
&= \left(P - P^2\right) x \\
&= 0 \\
&\Rightarrow y \in \ker\left(I - P\right)
\end{aligned}
$$

Alternatively, if we let $y \in \ker(I - P)$ if and only if $(I - P)\, y = 0$ for some $y \in \mathcal{X}$. Then

$$
\begin{aligned}
(I - P)\, y &= 0 \\
&\Rightarrow y = Py \\
&\Rightarrow y \in \operatorname{Im} P
\end{aligned}
$$

(3) is proved in a similar manner. ■

The above properties are important in that they allow a direct sum decomposition of $\mathcal{X}$, as per the following Theorem

Theorem 2.2 *If P is idempotent, then $\mathcal{X} = \ker P \dot{+} \operatorname{Im} P$.*

Proof. For any $x \in \mathcal{X}$, one may write $x = x_1 + x_2$, with $x_1 = (I - P)\, x \in \ker P$ and $x_2 = Px \in \operatorname{Im} P$. Thus

$$\mathcal{X} = \ker P + \operatorname{Im} P$$

Now examine the subspace generated by $\ker P \cap \operatorname{Im} P$. Here,

$$
\begin{aligned}
\ker P \cap \operatorname{Im} P &= \{x : Px = 0 \text{ and } (I - P)\, x = 0\} \\
&= 0
\end{aligned}
$$

Thus, the sum is direct. ■

It is then possible to form the decomposition

$$\mathcal{X} = \mathcal{X}_1 \dot{+} \mathcal{X}_2 \tag{2.43}$$

where $\mathcal{X}_1 = \ker P$, $\mathcal{X}_2 = \operatorname{Im} P$. This allows the action of P to be visualized as the projection on $\mathcal{X}_1$ along $\mathcal{X}_2$. Conversely, the map $(I - P)$ is the projection on $\mathcal{X}_2$ along $\mathcal{X}_1$.

2.1.5 Invariant Subspaces

Consider the subspace $\mathcal{S} \subset \mathcal{X}$, and let $M \in \mathcal{L}(\mathcal{X})$ have the additional property that $M\mathcal{S} \subset \mathcal{S}$. $\mathcal{S}$ is then termed M-invariant. This concept applies directly to some of the matrix theory developed in this work, and the following may now be stated.

Let $M \in \mathcal{L}(\mathcal{X})$, and let $\mathcal{A}_A^0$ denote the equivalence class of similar matrices associated with M. Further, let $\mathcal{X} = \mathcal{X}_1 \dot{+} \mathcal{X}_2$, such that $M\mathcal{X}_1 \subset \mathcal{X}_1$, on the assumption that this is possible. Thus, the objective is to construct a matrix representation of M that is the most convenient relative to this decomposition of $\mathcal{X}$. To accomplish this, construct a basis

$$\{x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_n\} \tag{2.44}$$

in $\mathcal{X}$, where $k = \dim(\mathcal{X}_1)$ and the first k elements of the set constitute a basis for $\mathcal{X}_1$, and the remaining elements constitute the extension of the basis set to include all of $\mathcal{X}$. Since $M\mathcal{X}_1 \subset \mathcal{X}_1$, $M(x_j) \in \mathcal{X}_1$ for all $j = 1, 2, \ldots, k$. As per §2.1.2, a matrix representation of this mapping would be of the form

$$M(x_j) = \sum_{i=1}^{k} \alpha_{ij} x_i, \quad j = 1, 2, \ldots, k \tag{2.45}$$

Thus, if this matrix is continued for all $j = 1, 2, \ldots, n$, the matrix $A \in \mathcal{A}_A^0$ will be of the form

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \tag{2.46}$$

where $A_1 \in \mathbb{F}^{k \times k}$, $A_3 \in \mathbb{F}^{(n-k)\times(n-k)}$. Thus, A is the matrix representation of $M \in \mathcal{L}(\mathcal{X})$ with respect to the basis set (2.44). So, a decomposition of $\mathcal{X}$ into the direct sum of two subspaces, one of which is M-invariant, admits a matrix representation in block triangular form.

Further, if $\mathcal{X} = \mathcal{X}_1 \dot{+} \mathcal{X}_2$, where both $M\mathcal{X}_1 \subset \mathcal{X}_1$ and $M\mathcal{X}_2 \subset \mathcal{X}_2$, then by the same process as before a matrix $A \in \mathcal{A}_A^0$ will be obtained as

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \tag{2.47}$$

where, again, $A_1 \in \mathbb{F}^{k \times k}$ and $A_2 \in \mathbb{F}^{(n-k)\times(n-k)}$. M itself then admits a decomposition $M = M_1 \dot{+} M_2$ with $M_1 \in \mathcal{L}(\mathcal{X}_1)$, $M_2 \in \mathcal{L}(\mathcal{X}_2)$, so that $A = A_1 \dot{+} A_2$. Accordingly, (2.47) is termed the *direct matrix sum* of A_1 and A_2. Expanding this to a set of p many matrices, the direct matrix sum is defined as

$$\begin{aligned} A &= \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_p \end{bmatrix} \\ &= \sum_{i=1}^{p} \dot{} A_i \end{aligned} \tag{2.48}$$

which implies the following theorem.

Theorem 2.3 *A mapping $M \in \mathcal{L}(\mathcal{X})$ has a representation in $A \in \mathcal{A}_A^0$ in block diagonal form, consisting of p many blocks, if and only if M can be decomposed into the direct sum of p linear maps.*

The idea of a direct matrix sum, block diagonal, and block triangular matrices lead to some nice results, as will be seen in §2.1.6.

2.1.6 Eigenvalues, Eigenvectors, and the Characteristic Polynomial

Consider a mapping $M \in \mathcal{L}(\mathcal{X})$. This map has the action of transforming elements $x \in \mathcal{X}$ into different elements $z \in \mathcal{X}$. That is, the vectors experience a transformation of magnitude and direction under the action of M. Consider, then, an element x such that

$$M(x) = \lambda x \tag{2.49}$$

Here, x constitutes a one dimensional, M-invariant subspace of $\mathcal{X}$, in that x experiences a change only in magnitude, not direction. In terms of a matrix $A \in \mathcal{A}_A^0$, (2.49) takes the form

$$Ax = \lambda x \tag{2.50}$$

Such an element $x \neq 0$ is termed an *eigenvector* of A, and the corresponding scalar λ is termed an *eigenvalue*.

Entire volumes can, and have, been devoted to the study of these entities. This work, however, will only consider the basic properties that are of direct importance.

First, the characteristic polynomial. Consider equation (2.50), so that

$$\begin{aligned} 0 &= (\lambda I - A)\, x \\ &\Rightarrow\; x \in \ker(\lambda I - A) \end{aligned} \tag{2.51}$$

For a non-trivial value of x to occur, $(\lambda I - A)$ must be singular. Or, in other words

$$\det(\lambda I - A) = 0 \tag{2.52}$$

Observe that

$$\begin{aligned} \det(\lambda I - A) &= \gamma_0 + \gamma_1 \lambda + \cdots + \gamma_n \lambda^n, \;\; \gamma_n = 1 \\ &= \sum_{i=0}^{n} \gamma_i \lambda^i && (2.53) \\ &= : \pi(\lambda) && (2.54) \end{aligned}$$

where $\pi(\lambda)$ is referred to as the *characteristic polynomial* of $A \in \mathcal{A}_A^0$. From the construction of $\pi(\lambda)$, the following theorem may be stated

Theorem 2.4 *$\lambda \in \mathbb{F}$ is an eigenvalue of $A \in \mathbb{F}^{n \times n}$ if and only if λ is a zero of the characteristic polynomial of A.*

This allows the determination of the eigenvalues of a matrix without determining the corresponding eigenvectors, which is very useful as far as control theory is concerned.

Denote the set of roots of $\pi(\lambda)$ as $\sigma(A)$, the *spectrum* of A, listed according to multiplicity.

Some properties of the spectrum of a matrix that are pertinent to this work will now be examined. Unless otherwise stated, $\mathcal{X}$ will denote an n-dimensional vector space over $\mathbb{F}$, so that $A \in \mathcal{A}_A^0 \subset \mathbb{F}^{n \times n}$.

Theorem 2.5 *Similar matrices have the same spectrum.*

Proof. See Lancaster and Tismenetsky [27], among others. ∎

Theorem 2.6 *The spectrum of the square matrix*

$$A = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}$$

is given by

$$\sigma(A) = \sigma(A_1) \cup \sigma(A_3) \tag{2.55}$$

Proof. Construction of the characteristic polynomial yields the following:

$$\begin{aligned} \det(\lambda I - A) &= \det \begin{bmatrix} \lambda I - A_1 & -A_2 \\ 0 & \lambda I - A_3 \end{bmatrix} = 0 \\ &\Rightarrow \det(\lambda I - A_1)\det(\lambda I - A_3) = 0 \end{aligned}$$

$\Leftrightarrow \sigma(A) = \sigma(A_1) \cup \sigma(A_3)$. ∎

Note that the above generalizes to any (block) triangular matrix.

Theorem 2.7 *If $A = A^*$ (i.e., A is Hermitian) then $\sigma(A) \subset \mathbb{R}$.*

Proof. See Lancaster and Tismenetsky [27], among others. ∎

Corollary 2.1 *If $A = A^T \in \mathbb{R}^{n \times n}$ (i.e., A is real symmetric) then $\sigma(A) \subset \mathbb{R}$.*

Of course, there is a plethora of other interesting information available regarding the spectrum a matrix. For the purposes of this work, however, the above will constitute a sufficient foundation.

2.1.7 Linear Matrix Equations and the Kronecker Product

In addition to the previously mentioned works by Lancaster and Tismenetsky [27], and Wonham [45], the material in this particular subsection can also be found in [26], [3], and [29].

Of great importance to this thesis are equations of the form

$$A_1XB_1 + A_2XB_2 + \cdots + A_pXB_p = C \tag{2.56}$$

where

$$\begin{aligned} A_1, A_2, \ldots, A_p &\in \mathbb{F}^{n\times n} && (2.57)\\ B_1, B_2, \ldots, B_p &\in \mathbb{F}^{m\times m} && (2.58)\\ C &\in \mathbb{F}^{n\times m} && (2.59) \end{aligned}$$

are known, and the objective is to solve the matrix $X \in \mathbb{F}^{n\times m}$ which is assumed to exist.

To attack this topic, we introduce the *Kronecker* (or *tensor*) product of two matrices as

$$A \otimes B = \begin{bmatrix} \alpha_{11}B & \alpha_{12}B & \cdots & \alpha_{1n}B \\ \alpha_{21B} & \alpha_{22}B & \cdots & \alpha_{2n}B \\ \vdots & \vdots & & \vdots \\ \alpha_{n1}B & \alpha_{n2}B & \cdots & \alpha_{nn}B \end{bmatrix} \in \mathbb{F}^{nm\times nm} \tag{2.60}$$

where

$$A = [\alpha_{ij}]_{i,j=1}^{n}$$

It is interesting to note that the definition above does, in fact, follow from the formal definition of $\otimes$ that arises in differential geometry.. To see this, let $A : \mathcal{X} \rightarrow \mathcal{U}$, $B : \mathcal{Y} \rightarrow \mathcal{V}$ so that

$$A \otimes B : \mathcal{X} \otimes \mathcal{Y} \rightarrow \mathcal{U} \otimes \mathcal{V} \tag{2.61}$$

Then, by ordering the basis elements $x_i \otimes y_j$ of $\mathcal{X} \otimes \mathcal{Y}$ as

$$x_1 \otimes y_1, \ldots, x_1 \otimes y_m, x_2 \otimes y_1, \ldots, x_2 \otimes y_m, \ldots, x_n \otimes y_m \tag{2.62}$$

and doing the same for $\mathcal{U} \otimes \mathcal{V}$ leads to the result (2.60) when $A \otimes B$ is represented as a matrix. To see this result in its full splendor, however, requires a more rigorous development of $\otimes$ that is not needed in this work. For this, the interested audience is referred to Wonham [45], among others.

Rather, the objective of this section is to examine some to the properties of $A \otimes B$, where A and B are as previously defined. We note the following properties.

Proposition 2.1 If $A, B, C, \ldots$ are appropriately sized matrices over some field $\mathbb{F}$, then the following relations hold:

1. Let $\alpha \in \mathbb{F}$. Then $(\alpha A) \otimes B = A \otimes (\alpha B) = \alpha (A \otimes B)$

2. $(A + B) \otimes (C + D) = A \otimes C + A \otimes D + B \otimes C + B \otimes D$

3. $(A \otimes B) \otimes D = A \otimes (B \otimes C)$

4. $(A \otimes B)^T = A^T \otimes B^T$

5. $(A \otimes B)(C \otimes D) = AC \otimes BD$

6. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$ (provided A and B are nonsingular)

7. $A \otimes B = (A \otimes I)(I \otimes B) = (I \otimes B)(A \otimes I)$

8. $\det(A \otimes B) = \det(A)^m \det(B)^n$, where $A \in \mathbb{F}^{n \times n}, B \in \mathbb{F}^{m \times m}$

Proof. Refer to [26], [3], and [29]. ∎

Define now a vector valued function of a matrix as

$$\operatorname{vec}(A) = \begin{bmatrix} A_{*1} \\ A_{*2} \\ \vdots \\ A_{*m} \end{bmatrix} \in \mathbb{F}^{nm} \tag{2.63}$$

where $A \in \mathbb{F}^{n\times m}$, and A_{*j} denotes the j^{th} column of A. Thus, $\operatorname{vec}(A)$ is the vector obtained by stacking up the columns of A, and this operation is termed the *vec-function.* It immediately follows that $\operatorname{vec}(\cdot)$ is linear, in that

$$\operatorname{vec}(\alpha A + \beta B) = \alpha \operatorname{vec}(A) + \beta \operatorname{vec}(B) \tag{2.64}$$

for any $A, B \in \mathbb{F}^{n\times m}$, $\alpha, \beta \in \mathbb{F}$. Also, the matrices $A_1, A_2, \ldots, A_p$ are linearly independent in $\mathbb{F}^{n\times m}$ if and only if $\operatorname{vec}(A_1), \operatorname{vec}(A_2), \ldots, \operatorname{vec}(A_p)$ are linearly independent in $\mathbb{F}^{nm}$.

The following theorem indicates the relation between $\otimes$ and $\operatorname{vec}(\cdot)$.

Theorem 2.8 *If $A \in \mathbb{F}^{n\times n}$, $B \in \mathbb{F}^{m\times m}$, $X \in \mathbb{F}^{n\times m}$, then*

$$\operatorname{vec}(AXB) = \left(B^T \otimes A\right) \operatorname{vec}(X) \tag{2.65}$$

Proof. Let $(AXB)_{*j}$ denote the j^{th} column of the matrix (AXB). This element may be expressed as

$$\begin{aligned}
(AXB)_{*j} &= \sum_{k=1}^{m} (AX)_{*k}\, b_{kj} \\
&= \sum_{k=1}^{m} b_{kj}\, (AX)_{*k} \\
&= \sum_{k=1}^{m} (b_{kj}A)\, X_{*k} \\
&= \begin{bmatrix} b_{1j}A & b_{2j}A & \cdots & b_{mj}A \end{bmatrix} \operatorname{vec}(X)
\end{aligned}$$

extending this notation over all $j = 1, 2, \ldots, m$ yields

$$\begin{aligned} \text{vec}(AXB) &= \begin{bmatrix} b_{11}A & \cdots & b_{m1}A \\ \vdots & & \vdots \\ b_{1m}A & \cdots & b_{mm}A \end{bmatrix} \text{vec}(X) \\ &= \left(B^T \otimes A\right) \text{vec}(X) \end{aligned}$$

■

Corollary 2.2 *The following statements are immediate consequences of the above.*

1. $\text{vec}(AX) = (I_m \otimes A)\,\text{vec}(X)$

2. $\text{vec}(XB) = \left(B^T \otimes I_n\right) \text{vec}(X)$

3. $\text{vec}(AX + XB) = \left((I_m \otimes A) + \left(B^T \otimes I_n\right)\right) \text{vec}(X)$

With this background, we are now in a position to examine equation (2.56) with some degree of success. Note that by taking the vec $(\cdot)$ operation on both sides, we obtain

$$Gx = c \tag{2.66}$$

where

$$\begin{aligned} G &= \sum_{i=1}^{p} B_i^T \otimes A_i &(2.67)\\ x &= \text{vec}(X) &(2.68)\\ c &= \text{vec}(C) &(2.69) \end{aligned}$$

which is now a familiar matrix-vector problem. In order to make any statements about existence and / or uniqueness of solutions of (2.66), we generally need some information about $\sigma(G)$. Unfortunately, it is usually difficult to say anything of value about $\sigma(G)$, even when $\sigma(A_i)$ and $\sigma(B_i)$ are known for all i.

However, some special cases do arise that have nice properties. For the purposes of this work, we need only consider the special case where each A_i (B_i) is a scalar polynomial in a fixed $A \in \mathbb{F}^{n\times n}$ ($B \in \mathbb{F}^{m\times m}$). That is

$$C = \sum_{i,j=1}^{p} \alpha_{ij} A^i X B^j \tag{2.70}$$

where $\alpha_{ij} \in \mathbb{F}$. That is, we define a polynomial in two scalar variables, say η and κ, as

$$p(x,y) = \sum_{i,j} \alpha_{ij} \eta^i \kappa^j \tag{2.71}$$

then we may define a polynomial in two matrix variables as

$$p(A;B) = \sum_{i,j} \alpha_{ij} A^i \otimes B^j \tag{2.72}$$

Thus

$$G = p\left(B^T; A\right) \tag{2.73}$$

and the following relation between $\sigma(A)$, $\sigma(B)$ and the eigenvalues of $p(A;B)$ can be stated

Theorem 2.9 *(C. Stephanos, 1900) The spectrum of the matrix $p(A;B)$ are the nm numbers of the form $p(\lambda_r, \mu_s)$, where $\{\lambda_r \mu_s\}_{r=1}^{n} = \sigma(A)$, $\{\mu_s\}_{s=1}^{m} = \sigma(B)$.*

Proof. Refer to [27] and [29], among others. ■

For the purposes of this work, we are concerned with two specializations of this result.

Corollary 2.3 *The eigenvalues of $A \otimes B$ are the nm numbers $\{\lambda_r \mu_s\}_{r,s=1}^{n,m}$.*

Corollary 2.4 *(Sylvester, 1882) The eigenvalues of $(I_m \otimes A) + (B \otimes I_n)$ are the nm numbers $\{\lambda_r + \mu_s\}_{r,s=1}^{n,m}$.*

As an example of an application of the above, consider the case where

$$AX + XB = C \tag{2.74}$$

which appears in one form or another quite frequently in control theory. If $\sigma(A) \cap \sigma(-B) = \emptyset$, then X can be determined uniquely by

$$((I \otimes A) + (B \otimes I)) \operatorname{vec}(X) = \operatorname{vec}(C) \tag{2.75}$$

or

$$\begin{aligned} Gx &= c \\ x &= G^{-1}c \\ &\Rightarrow X = \operatorname{vec}^{-1}(x) \end{aligned} \tag{2.76}$$

where $\operatorname{vec}^{-1}(\cdot)$ denotes an un-stacking operator. Admittedly, this is an abuse of notation, but the operation is clear from the context.

Of course, there is far more that can be said on this topic. The preceding, however, constitutes a sufficient foundation for this particular work.

2.2 Linear Systems

The material in this section can be found in virtually any textbook on control theory that utilizes state space methods. In particular, the material presented here makes use of the notation used in Wonham [45]. Proofs of the Theorems can be found in this reference, among others.

2.2.1 Definition

In this work, only finite dimensional, time invariant, linear systems will be considered. These types of systems constitute a large class of engineering systems, and can be modelled as

$$\dot{x} = Ax + Bu \tag{2.77}$$

$$y = Cx \tag{2.78}$$

for $t \in \mathbb{R}_+$, where $x = x(t)$, $u = u(t)$, $y = y(t)$, and $\dot{x} = \frac{d}{dt}x$. The elements $x \in \mathcal{X}$, $u \in \mathcal{U}$, $y \in \mathcal{Y}$ are referred to as the *state*, *control*, and *output* vectors respectively. Further, assume that

$$\dim(\mathcal{X}) = n \tag{2.79}$$

$$\dim(\mathcal{U}) = m \tag{2.80}$$

$$\dim(\mathcal{Y}) = p \tag{2.81}$$

with $n \geqslant p \geqslant m$. The term *system* will henceforth be understood to mean the set of differential equations (2.77) and (2.78). The majority of this work will primarily be concerned only with the system (2.77), which is equivalent to $y = x$, or $C = I_{\dim(\mathcal{X})}$.

It is useful to then look at some of the general properties associated with (2.77) and (2.78).

2.2.2 Controllability, Feedback, and Eigenvalue Assignment

Denote the linear space of piecewise continuous controls $u(t) \subset \mathcal{U}$ as $\mathcal{U}_c$. Further, the solution $x(t) \subset \mathcal{X}$ of (2.77) is uniquely determined by

$$x(t) = \varphi(t; x_0, u) = e^{At}x_0 + \int_0^t e^{(t-\tau)A}Bu(\tau)\,d\tau \tag{2.82}$$

where $x_0 = x(0)$. A state $x \in \mathcal{X}$ is *reachable* if there exists $t \in [0, \infty]$ and $u \in \mathcal{U}_c$ such that $x = \varphi(t; x_0, u)$. That is, there is a control available that can steer $x(t)$ to x in some finite time. Denote $\mathcal{R}_0 \subset \mathcal{X}$ as the subspace of all states reachable from x_0. The subspace is $\mathcal{R}_0 \subset \mathcal{X}$ termed the *controllable subspace* of the pair (A, B). If $\mathcal{R}_0 = \mathcal{X}$, then this implies that any $x \in \mathcal{X}$ can be reached from x_0 in finite time, and the pair (A, B) is then termed *controllable*. It can be shown that a system is controllable if and only if

$$\mathcal{X} = \mathrm{Im}\begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} \tag{2.83}$$

Feedback is the term used to describe the application of a control $u = Kx$, where

$$K : \mathcal{X} \to \mathcal{U} \tag{2.84}$$

The whole purpose of state feedback is to alter the behavior of the free, uncontrollable system $\dot{x} = Ax$ in some desirable way. Thus, by selecting a feedback map of the type (2.84), the *closed loop* system becomes

$$\dot{x} = (A + BK)\, x \tag{2.85}$$

$$\Rightarrow \quad x(t) = e^{(A+BK)t} x_0 \tag{2.86}$$

indicating that the closed loop system behavior is governed largely by $\sigma(A + BK)$. The main concern here is that $\sigma(A + BK)$ be *stable*, or

$$\sigma(A + BK) \subset \mathbb{C}_- \tag{2.87}$$

where $\mathbb{C}_- := \{\alpha + i\beta : \alpha < 0, \beta \in \mathbb{R}\}$, so that $x(t) \to 0$ as $t \to \infty$. The relation between controllability and spectral assignment can be stated in the following manner.

Theorem 2.10 *The pair (A, B) is controllable if and only if, for every symmetric set $\{\lambda_i\}_{i=1}^n \subset \mathbb{C}$, there exists a map $K : \mathcal{X} \to \mathcal{U}$ such that $\sigma(A + BK) = \{\lambda_i\}_{i=1}^n$.*

Proof. Refer to Wonham, Theorem 2.1 [45]. ■

In this work, the notion of transforming a system into *normal form* will frequently be employed. Specifically, assume the controllable pair (A, B) admits a transformation

$$(A, B) \approx \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \right) \tag{2.88}$$

where $B_2 \in \mathbb{F}^{m \times m}$ is nonsingular, and the blocks A_{ij} are of appropriate size. For some special cases, application of the control $u = Kx$ leads a closed loop system matrix of the form

$$(A + BK) \approx \begin{bmatrix} A_{11} + A_{12}\tilde{K} & A_{12} \\ 0 & M \end{bmatrix} \tag{2.89}$$

Thus, $\sigma(A+BK) = \sigma\left(A_{11} + A_{12}\tilde{K}\right) \cup \sigma(M)$ where M is fixed by some process, and the design objective is to select a control law $\tilde{K}$ such that the performance of the subsystem $\left(A_{11} + A_{12}\tilde{K}\right)$ is in some sense good. The following theorem gives conditions for the controllability of the pair (A_{11}, A_{12}).

Theorem 2.11 *(A_{11}, A_{12}) is controllable if (A, B) is controllable. Further, if B_2 is nonsingular, then (A_{11}, A_{12}) controllable implies (A, B) controllable as well.*

Proof. The Hautus criterion for controllability [18] says that (A, B) is controllable if and only if

$$\operatorname{rank}\left[\begin{array}{cc} \lambda I - A & B \end{array}\right] = n \text{ for all } \lambda \in \mathbb{C}$$

With (A, B) as in (2.88), we have

$$\left[\begin{array}{cc} \lambda I - A & B \end{array}\right] = \left[\begin{array}{ccc} \lambda I_{(n-m)} - A_{11} & -A_{12} & 0 \\ -A_{21} & \lambda I_m - A_{22} & B_2 \end{array}\right] \tag{2.90}$$

If the matrix (2.90) has rank n for all λ, then clearly

$$\operatorname{rank}\left[\begin{array}{cc} \lambda I - A_{11} & -A_{12} \end{array}\right] = n - m \tag{2.91}$$

for all $\lambda \in \mathbb{C}$, which is equivalent to (A_{11}, A_{12}) controllable. On the other hand, if (2.91) holds for all $\lambda \in \mathbb{C}$ and B_2 is nonsingular, then from (2.90), $\left[\begin{array}{cc} \lambda I - A & B \end{array}\right]$ has rank n for all $\lambda \in \mathbb{C}$, so that (A, B) is controllable. ■

2.2.3 Observability and Observers

The results of §2.2.2 are only partially useful, in that it is presumed that x is directly available for measurement at any instant in time. This, however, is often not the case. Consider, once again, the system (2.77), (2.78). The maps A, B, and C are known, as is $u(t)$. If it is possible to compute $x(t)$ from this data, then the system is termed *observable*. More simply, if $u(t) \equiv 0$, then the pair (C, A) is observable if $y(t) = 0$ only when $x(t) = 0$.

Mathematically, this property implies that (C, A) is observable if

$$\ker \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = 0 \tag{2.92}$$

From (2.92), it is immediate that (C, A) is observable if and only if (A^*, C^*) is controllable. Thus, if (A^*, C^*) is controllable then $\sigma(A^* + C^* L^*) = \{\lambda_i\}_{i=1}^n$ as before $\Leftrightarrow \sigma(A + LC) = \{\lambda_i\}_{i=1}^n$, where $L : \mathcal{Y} \to \mathcal{X}$.

Further, if (A, B) is controllable, and (C, A) is observable, then the triple (A, B, C) is termed *complete*.

Assuming (C, A) observable, it is now possible to construct a device that will calculate $x(t)$ from $y(t)$ and $u(t)$. Such a device is termed an *observer*, and takes the form of a differential equation

$$\dot{q} = Wq - Ly + Bu \tag{2.93}$$

where $q \in \mathcal{X}$. The design variables are then $L : \mathcal{Y} \to \mathcal{X}$, and $W : \mathcal{X} \to \mathcal{X}$. Define the observer error as

$$e = x - q \tag{2.94}$$

so that

$$\begin{aligned} \dot{e} &= \dot{x} - \dot{q} \\ &= (A + LC)x - Wq \\ &= (A + LC)e \end{aligned} \tag{2.95}$$

where $W = A + LC$, and L is selected such that $\sigma(A + LC)$ is in some sense good.

The objective at this point is to apply the control law $K : \mathcal{X} \to \mathcal{U}$ as $u = Kq$, where $q(t) \to x(t)$ as $t \to \infty$, and the cascaded closed loop system becomes

$$\begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} A & BK \\ -LC & (A+BC+LC) \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} \tag{2.96}$$

however, recalling (2.94) and (2.95), we see that (2.96) can equivalently be written as

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} (A+BK) & -BK \\ 0 & (A+LC) \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \tag{2.97}$$

and the system performance is determined largely by $\sigma(A+BK) \cup \sigma(A+LC)$. Typically, L is selected such that

$$\operatorname{Re}\sigma(A+LC) \geqslant 4 \operatorname{Re}\sigma(A+BK) \tag{2.98}$$

ensuring that the observer dynamics are fast in comparison to the system response. This structure, when considered with Theorem 2.13, implies that $\sigma(A+BK)$ can be assigned arbitrarily if (A, B, C) is complete.

2.3 Lyapunov Stability

The central concern of any control problem is stability. As it pertains to linear systems, stability implies that a trajectory $x(t) \to 0$ as $t \to \infty$. Thus, a system that satisfies this condition is termed *asymptotically stable*.

Lyapunov (1892) was concerned with general conditions for stability, and numerous conditions for classifying stability without the explicit calculation of $\sigma(A)$. As it applies to this work, this creates an effective design tool when attempting to develop a control law that leads to an asymptotically stable system. In the interest of brevity, we will not delve too deeply into this rather large topic, and only focus on the results that are directly important to this work.

The main result can be summarized as follows. Let $A \in \mathbb{R}^{n \times n}$ and consider the autonomous system

$$\dot{x} = Ax \tag{2.99}$$

Defining a quadratic form $v(x)$ as $v(x) = x^T V x$, where V is symmetric, so that

$$\begin{aligned} \dot{v}(x) &= \dot{x}^T V x + x^T V \dot{x} \\ &= x^T \left(A^T V + VA\right) x \end{aligned} \tag{2.100}$$

and writing

$$A^T V + VA = -W \tag{2.101}$$

Then W is real symmetric as well, so that $\dot{v}(x) = -w(x) = -x^T W x$. Lyapunov noted that, given a positive definite W, then the stability of A can be characterized by the existence of a positive definite solution matrix V. Intuitively, this implies that $v(x)$ is a valid scalar measure of the 'magnitude' x. So, $\dot{v}(x) < 0$ since $W > 0 \Leftrightarrow w(x) > 0$, and this implies that this 'magnitude' is always decreasing, and the system is asymptotically stable.

In this, work, the ability to classify a point, or set of points, $\{x\} \subset \mathcal{X}$ as stable and globally attractive when the system under consideration is nonlinear and autonomous is of importance. That is

$$\dot{x}(t) = f(x; t) \tag{2.102}$$

The following, then, is a restatement of Lyapunov's stability theorem as it pertains to systems of the type (2.102)[24].

Theorem 2.12 *Let $x = 0$ be an equilibrium point for (2.102) and $\mathcal{D} \subset \mathcal{X}$ be a domain containing $x = 0$. Let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function, such that*

$$V(0) = 0 \text{ and } V(x) > 0 \text{ for all } x \in \mathcal{D} - \{0\} \tag{2.103}$$

$$\dot{V}(x) \leqslant 0 \text{ for all } x \in \mathcal{D} \tag{2.104}$$

Then, $x = 0$ is stable. Further, if

$$\dot{V}(x) < 0 \text{ for all } x \in \mathcal{D} - \{0\} \tag{2.105}$$

then $x = 0$ is asymptotically stable.

Proof. Given $\epsilon > 0$, choose $r \in (0, \epsilon]$ such that

$$B_r = \{x \in \mathcal{X} : \|x\| \leqslant r\} \subset \mathcal{D}$$

Let $\alpha = \min_{\|x\|=r} V(x)$. Then, from (2.103), $\alpha > 0$. Select $\beta \in (0, \alpha)$, and let

$$\Omega_\beta = \{x \in B_r : V(x) \leqslant \beta\}$$

Then, Ω_β is in the interior of B_r. From (2.104), we see that Ω_β has the property that any trajectory starting in Ω_β at time $t = 0$ will remain in Ω_β. As well, since $V(x)$ is continuous and $V(0) = 0$, there is a $\delta > 0$ such that

$$\|x\| \leqslant \delta \Rightarrow V(x) < \beta$$

Then,

$$B_\delta \subset \Omega_\beta \subset B_r$$

and

$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r$$

Thus,

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \leqslant \epsilon \text{ for all } t \geqslant 0$$

which shows that the equilibrium point $x = 0$ is stable. Now, assume (2.105) holds as well. To show asymptotic stability, we need to show that $x(t) \to 0$ as $t \to \infty$. That is, for every $a > 0$, there is $\bar{t} > 0$ such that $\|x(t)\| < a$ for all $t > \bar{t}$. Repeating the previous arguments, we know that for every $a > 0$, we may choose $b > 0$ such that $\Omega_b \subset B_a$. Thus, it is sufficient to show that $V(x(t)) \to 0$ as $t \to \infty$. Since $V(x(t))$ is monotonically decreasing and bounded from below by zero,

$$V(x(t)) \to c \geqslant 0 \text{ as } t \to \infty$$

We show that $c = 0$ via contradiction. Assume $c > 0$. By continuity of $V(x)$, there exists a $d > 0$ such that $B_d \subset \Omega_c$. The limit $V(x(t)) \to c > 0$ implies that the trajectory $x(t)$ lies outside the ball B_d for all $t \geqslant 0$. Let $-\xi = \max_{d \leqslant \|x\| \leqslant r} \dot{V}(x)$, which exists because the continuous function $\dot{V}(x)$ has a maximum over the compact set $\{d \leqslant \|x\| \leqslant r\}$. From (2.105), $-\xi < 0$ and it follows that

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))\, d\tau \leqslant V(x(0)) - \xi t$$

Since the right hand side will eventually become negative, the inequality contradicts the assumption that $c > 0$. ■

Also of importance to this work is the following extension of Lyapunov's Theorem, known as the Barbashin-Krasovskii Theorem.

Theorem 2.13 *Let $x = 0$ be an equilibrium point for* (2.77). *Let $V : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_+$ be a continuously differentiable function such that*

$$\begin{aligned} V(0) &= 0 \textit{ and } V(x) > 0 \textit{ for all } x \neq 0 && (2.106) \\ \|x\| &\rightarrow \infty \Rightarrow V(x) \rightarrow \infty && (2.107) \\ \dot{V}(x) &< 0 \textit{ for all } x \neq 0 && (2.108) \end{aligned}$$

then $x = 0$ is a globally, asymptotically stable point in $\mathcal{X}$.

Proof. See [24], Theorem 3.2. ■

2.4 Chapter Summary

In this chapter, the basic mathematical tools necessary for this work were presented. Notation was introduced, the class of systems to be analyzed was defined, and some basic properties of these systems were covered.

Chapter 3
Sliding Mode Control

3.1 Introduction

As stated in §2.2, it is a well known fact that control of linear systems can be achieved via the application of feedback, $K : \mathcal{X} \to \mathcal{U}$. As will be explained in §4, it is also known that the resulting system

$$\dot{x} = (A + BK)\, x \tag{3.1}$$

can be, in some sense of the word, optimized by minimizing an associated *cost functional*, or *performance index*

$$J = \int_0^\infty (x^* Q x + u^* R u)\, dt \tag{3.2}$$

Theoretically, this type of control law is as good as it gets, and has thus earned its own title and line of study–*linear quadratic* (LQ) control. It does, however, suffer from some practical drawbacks.

Primarily, nonlinear effects in the input signal u form a very realistic limiting factor on the effectiveness of LQ control theory. Typically, these input nonlinearites appear as saturation levels in the actuators. For example, the output torque of a servomotor is subject to a maximum (saturated) value that is in proportion to the size of the motor. Thus, true optimal system performance may not be obtainable due simply to the size constraints on the actuators available. One method of bypassing this problem is to apply 'heavy penalties' to the constrained variables within the cost functional (i.e., alter Q and R) until such time that a practical K is found.

Of course, this idea has some intuitive drawbacks, in that it becomes apparent that the control law is, to some extent, decided at the onset of the design. Then the job almost becomes one of finding an appropriate index that justifies the control law as optimal.

Further, in many realistic systems the only actuator available is a relay-type switch, since these actuators have the practical advantage of reduced complexity in comparison to a continuous valued actuator. Here, the notion of an LQ design scheme has no hope of succeeding.

Thus, the analysis of systems with these particular control constraints within an LQ framework is clearly not appropriate, and something different is required. Here enters the exciting field of *variable structure control* (VSC), so called because it takes advantage of the on-off switching nature of the actuator, rather than ignoring it.

While it would seem apparent that these particular systems do not lend themselves to classic LQ optimization, they are well adapted to optimization with respect to system response (time-optimal control), fuel expenditure (fuel-optimal control), or any number of other quantities.

Optimal relay control was a topic of great interest throughout the period of (roughly) 1945 - 1970. However, not much will be said about this topic here, since it is not of particular interest to the thesis. The interested reader is referred to Ryan [34] for an historical account of VSC.

Two results, pertinent to this work, came out of VSC. First, the somewhat intuitive result was proved that to achieve a time optimal control, the actuators had to operate at their saturation limits. This led to the idea of utilizing relay switching, rather than saturating actuators. Second, it can be shown that the switching sequence required for system optimization could be characterized by a manifold within the state space that acted as the decision mechanism for the switches. That is, if $x \in \mathcal{X}$ is 'above' the manifold, then the controller switches to one structure, and if x is 'below' the manifold, the controller takes a different structure. So, for an m-dimensional controller, each j^{th}

element would have the structure

$$u_j(x;t) = \begin{cases} u_j^+(x;t) & \text{if} \quad \psi(x)_j > 0 \\ u_j^-(x;t) & \text{if} \quad \psi(x)_j < 0 \end{cases} \tag{3.3}$$

where the decision manifold, S, is characterized by the parametric equation

$$S := \{x : \psi(x) = 0\} \tag{3.4}$$

This notion will be clarified in the sequel.

Much interest in the design and analysis of optimal relay control systems occurred in the 1960's, as can be seen in the classic work of Bryson and Ho [4], and still continues to the present (Ailon and Segev, [1]). Of interest to this work is a special case that arose from the study of VSC's. Filippov (1960) is attributed with much of the groundwork for analysis of systems with discontinuous right hand sides, such as a VSC (these results are presented in Ryan [34]). In particular, he was interested in the state behavior, $x(t)$, at the exact moment of the control switch.

Of the various results obtained, the one that is of most interest (as far as this work is concerned) is the situation where both control structures lead to an $x(t)$ that is directed towards the switching manifold S. That is, if $x(t)$ is travelling towards S, and intercepts S at a time τ, then the control structure switches, and an alternate trajectory is followed. However, this trajectory also is directed towards S, so that the net result is that $x(t)$ travels in a direction tangent to S. This type of motion is termed *sliding*, and a new branch of control theory emerged.

From the above heuristic explanation, it is apparent that sliding motion is thus named because $x(t)$ 'slides' along S. This motion was known to occur in optimal VSC simply because at some point, S will coincide with a certain state trajectory. The idea then emerged of creating an arbitrary decision manifold, S, and investigating the possibility of inducing sliding motion on this manifold by means of rapidly varying the structure of the controller. This idea became popular for a number of reasons.

It was shown by Utkin [40], and more generally by Decarlo *et al* [6] that a system could be stabilized by restricting $x(t)$ to S, even if neither of the controls $\{u^+, u^-\}$ alone led to an asymptotically stable system [6]. Instead, it is simply required that each of $\{u^+, u^-\}$ now has the effect of 'steering' $x(t)$ towards S. Then, by rapidly varying the control structure (infinitely fast switching, ideally) the net result will be that the component of $x(t)$ normal to S is zero, so all motion is tangent to S. It becomes apparent, then, that the shape of S determines the closed loop system dynamics, since $x(t)$ will be restricted to S. Thus, with a well designed sliding manifold, SLMC can achieve the same goals of a standard control strategy, such as tracking and regulation.

In addition to the properties of asymptotic stability and relatively simple closed loop dynamics, SLMC is a very robust control strategy. In an important work by Draženović, it was shown that SLMC has guaranteed stability margins in the face of bounded, matched disturbances / model uncertainties [11], [38].

Thus, with the relatively recent advent of high speed digital computers and rapid switching circuitry, SLMC has become a practical reality, and an attractive option in many applications.

The purpose of this chapter will be to provide an overview of some of the fundamental properties of SLMC. The key piece of information that will be omitted is a general discussion / derivation of some of the properties associated with discontinuous feedback control. The purpose of this omission is purely for the sake of brevity, and the interested audience is, again, referred to Ryan [34] and the references contained therein for an in-depth treatment of this topic.

The chapter will be organized as follows. §3.2 gives a brief introduction to manifold theory, with an emphasis on the development of a sliding manifold. §3.3 characterizes SLMC in the ideal situation. In §3.4, the switched control law is developed and shown to be invariant to matched disturbances. A chapter summary is presented in §3.5.

3.2 Sliding Manifolds

In this section, a sliding manifold $\mathcal{S}$ will be defined. State behavior on $\mathcal{S}$ will then be examined when $\mathcal{S}$ is a linear manifold.

3.2.1 General Manifold Theory

The concept of a manifold is an extremely rich mathematical topic. See, for example, Kobayashi and Nomizu [25]. For the purposes of this work, a manifold may be viewed as the subspace generated by solution set of the equation

$$\psi(x) = 0 \tag{3.5}$$

Further, all spaces $\mathcal{X}$,$\mathcal{Y}$,... may also be considered manifolds.

3.2.2 Sliding Manifolds

With this definition of manifold in place, we want to consider the sliding surface as a submanifold $\mathcal{S}$ immersed in $\mathcal{X}$. This type of manifold can be described by

$$\mathcal{S} = \{x : \psi(x) = 0\} \tag{3.6}$$

where $\psi(x)$ denotes some continuous function of x. Equation (3.6) indicates that any general manifold can be chosen. This work, like most literature, only considers a special class of *linear* manifolds that can be described by

$$\mathcal{S} = \{x : Sx = 0\} \tag{3.7}$$

Where $S \in \mathcal{L}(\mathcal{X}, \mathcal{R})$ is, at this point, an arbitrary map. For a more general examination of *nonlinear* sliding manifolds, see Sira-Ramirez [36], [37]. Note that in the case under consideration

$$\operatorname{Im} S = \mathcal{R} \subset \mathcal{X} \tag{3.8}$$

$$\ker S = \mathcal{S} \tag{3.9}$$

The effect of constraining the system (2.77) to S can now be examined

3.3 Ideal Sliding Behavior

In this section the pair (A, B) is assumed controllable, $\operatorname{rank}(B) = m$, and the map S is assumed to have full rank.

The objective now is to design a control such that once $x(t)$ intercepts S, sliding motion will commence and $x(t)$ will remain on S for all subsequent time. If we assume an initial condition of $x_0 \in S$, then

$$\begin{aligned} Sx(t) &\equiv 0 \\ &\Rightarrow S\dot{x}(t) = SAx(t) + SBu(t) \equiv 0 \end{aligned} \tag{3.10}$$

Isolating $u(t)$ in (3.10) results in (dropping the functional dependence on time)

$$u = -(SB)^{-1} SAx \tag{3.11}$$

which is uniquely determined if (SB) is nonsingular. Substitution of (3.11) into (2.77) yields the equivalent system

$$\dot{x} = \left(I - B(SB)^{-1} S\right) Ax \tag{3.12}$$

and the need for an additional switching will now be made clear. Let $P = B(SB)^{-1} S$ and notice that

$$\begin{aligned} P^2 &= B(SB)^{-1} SB(SB)^{-1} S \\ &= B(SB)^{-1} S \\ &= P \end{aligned} \tag{3.13}$$

so that P is idempotent, and thus a projector. Thus, we can find $\{x\} \subset \mathcal{X}$ with $x_0 \in \mathcal{X}$ that will induce sliding motion in (3.12) by observing the following. Begin by transforming (A, B) into normal form as per (2.88), so that

$$(A,B) \approx \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \right)$$

Let $S \in \mathbb{F}^{n \times m}$ take the form

$$S = \begin{bmatrix} S_1 & S_2 \end{bmatrix} \tag{3.14}$$

and assume $S_2 \in \mathbb{F}^{m \times m}$ is nonsingular. Now

$$\begin{aligned} P &= B\,(SB)^{-1}\,S \\ &= \begin{bmatrix} 0_{(n-m)\times(n-m)} & 0 \\ B_2\,(S_2B_2)^{-1}\,S_1 & B_2\,(S_2B_2)^{-1}\,S_2 \end{bmatrix} \\ & \begin{bmatrix} 0 & 0 \\ S_2^{-1}S_1 & I_m \end{bmatrix} \end{aligned} \tag{3.15}$$

so that

$$\begin{aligned} (I-P)\,A &= \begin{bmatrix} I_{n-m} & 0 \\ -S_2^{-1}S_1 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \\ &= \begin{bmatrix} A_{11} & A_{12} \\ -S_2^{-1}S_1A_{11} & -S_2^{-1}S_1A_{12} \end{bmatrix} \end{aligned} \tag{3.16}$$

Define now a transformation T as

$$T = \begin{bmatrix} I_{(n-m)} & 0 \\ S_2^{-1}S_1 & I_m \end{bmatrix} \tag{3.17}$$

$$T^{-1} = \begin{bmatrix} I_{(n-m)} & 0 \\ -S_2^{-1}S_1 & I_m \end{bmatrix} \tag{3.18}$$

so that

$$(I-P)\,A \approx \begin{bmatrix} A_{11} + A_{12}S_2^{-1}S_1 & A_{12} \\ 0 & 0_{m\times m} \end{bmatrix} \tag{3.19}$$

With this construction of $(I-P)\,A$, we may now conveniently examine some of the properties of the closed loop system.

Theorem 3.1 *The closed loop system matrix*

$$A_{cl} = \left(I - B\,(SB)^{-1}\,S\right) A$$

has at least m *zero eigenvalues, or*

$$\sigma\,(A_{cl}) = \{0, \lambda_1, \lambda_2, \ldots, \lambda_p\}\,, \quad p \leqslant n - m \tag{3.20}$$

Proof. Recalling Theorems 2.5 and 2.6, we see in (3.19) that $\sigma\,((I - P)\,A) = \sigma\left(A_{11} + A_{12}S_2^{-1}S_1\right) \cup \sigma\,(0) = \{0, \lambda_1, \lambda_2, \ldots, \lambda_p\}$. ■

Now, we wish to characterize the nature of $x\,(t)$ under this particular feedback law. Begin by making the following assumption

Assumption 3.1 The set

$$\sigma\left(A_{11} + A_{12}S_2^{-1}S_1\right) = \{\lambda_i\}_{i=1}^{n-m}$$

is a symmetric set of distinct, nonzero elements over $\mathbb{C}_-$.

Theorem 3.2 *The symmetric set* $\sigma\left(A_{11} + A_{12}S_2^{-1}S_1\right) \subset \mathbb{C}$ *may be assigned arbitrarily.*

Proof. From Theorem 2.11, the pair (A_{11}, A_{12}) is controllable. Thus, a map $S_2^{-1}S_1$ can always be found such that $\sigma\left(A_{11} + A_{12}S_2^{-1}S_1\right) = \{\lambda_i\}_{i=1}^{n-m} \subset \mathbb{C}$, where $\{\lambda_i\}_{i=1}^{n-m}$ is an arbitrary symmetric set. ■

Combining these facts with the heuristic definition of SLMC given in the introduction of this chapter, namely that the closed loop dynamics are determined by the 'shape' of S, we can visualize the behavior of $x\,(t)$ in a geometric manner.

From Theorem 3.1 and Assumption 3.1, it is apparent that the matrix $(I - P)\,A$ has a nontrivial kernel of dimension m. Any vector in $\ker\,((I - P)\,A)$ is an equilibrium point for the closed loop system, so that the system has an equilibrium subspace rather than an equilibrium point. Further, since S was designed to be an $n - m$ dimensional linear manifold in $\mathcal{X}$, it follows that the columns of the matrix $A_{11} + A_{12}S_2^{-1}S_1$ constitute a (transformed) basis for S. So, for an initial condition of $x_0 \notin S$, it follows that $x\,(t)$ will follow a path parallel to S, and achieve an equilibrium state (i.e., $\dot{x} = 0$) anywhere on the subspace $\ker((I - P)A) \subset \mathcal{X}$.

Thus, asymptotically stable sliding motion will not, in general, occur without the advent of an additional control term such as a switching term, as indicated in §3.1.

To conclude this section, we note the following compatibility condition on S and K, namely that K now has the specific structure

$$K = -(SB)^{-1}SA \tag{3.21}$$

which leads to a singular matrix $(A + BK)$, as per (3.19). This constraint on design objectives is, in fact, one of the main motivations for this work, and will be addressed in §4.3.

3.4 Actual Sliding Behavior

Ideally, once the state trajectory intercepts S, sliding will commence. However, as was shown in §3.3, $x(t)$ will not, in general, intercept S unless $x_0 \in S$. Thus, the task at hand is to now design a control that will force $x(t)$ to intercept S, and ensure that sliding motion will comence at this point. As was was indicated in §3.1, sliding motion may be induced by application of a discontinuous control law. Thus, the task at hand is to design a VSC that will both induce sliding once $x(t)$ intercepts S, as well as ensuring that $x(t)$ intercepts S.

In this section, the structure of the switching control term, u_{sw} will be derived. Various schemes exist to accomplish this objective, see for example [6], [23], [24], [40], and [47]. In this work, the method presented will be based on the works of Decarlo [7], which uses a Lyapunov approach to design a controller which guarantees that S is a globally attractive manifold.

Recall from §3.1 that the controller will utilize a full state feedback law, $u = u(x; t)$, and each j^{th} entry $u_j(x;t)$ of $u(x;t)$ has two possible structures [6], [40].

$$u_j(x;t) = \begin{cases} u_j^+(x;t) & \text{if} \quad \psi(x)_j > 0 \\ u_j^-(x;t) & \text{if} \quad \psi(x)_j < 0 \end{cases} \tag{3.22}$$

Where $\{x : \psi(x) = 0\}$ has previously been defined to be the sliding manifold $S \subset \mathcal{X}$. At this point, $\mathcal{X}$ is necessarily a vector field over $\mathbb{R}$, since the inequalities (3.22) make no sense in $\mathbb{C}$, since $\mathbb{C}$ is not an ordered field. The task at hand is to now give an effective algorithm for designing an appropriate switching control term, u_{sw}.

3.4.1 Switching Control Design

Since no guidelines have been given for selecting a 'good' set of closed loop eigenvalues beyond $\sigma(A_{cl}) \subset \mathbb{C}_- \cup \{0\}$ for asymptotic stability, it would be somewhat inappropriate to examine the problem of selecting the switching matrix, S. Of course, numerous references on this problem exist, and the interested audience is referred to [6], [40], [32], [24], among others. For the remainder of this section, it will be assumed that S has been chosen, and is in some sense good.

Let us now examine the slightly more involved problem of ensuring that sliding motion is actually induced on S. To do this, recall §2.3, in which Lyapunov stability was presented. Define a Lyapunov candidate $V : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}_+$, such that

$$V(S) = 0 \tag{3.23}$$

$$V(x) > 0 \text{ for all } x \notin S \tag{3.24}$$

$$\|x\|_2 \rightarrow \infty \text{ if and only if } V(x) \rightarrow \infty \text{ for all } x \notin S \tag{3.25}$$

$$\dot{V}(x) < 0 \text{ for all } x \notin S \tag{3.26}$$

Then, S is a globally attractive manifold in $\mathcal{X}$, in that any trajectory $x(t)$ will tend toward S in finite time. Further, since the dynamics on S have been defined to be 'good' (i.e., asymptotically stable), then the controller will be globally stabilizing.

Defining $z = Sx$, and viewing $\|z\|$ as a measure of the 'distance' from x to the manifold S when $x \notin S$, we may construct

$$V = \frac{1}{2}\langle z, z\rangle = \frac{1}{2}\sum_{j=1}^{m} z_j^2 \tag{3.27}$$

So that

$$\frac{d}{dt}V = \langle z, \dot{z}\rangle = \sum_{j=1}^{m} z_j \dot{z}_j \tag{3.28}$$

For global stability, it is required that

$$\langle z, \dot{z}\rangle < 0 \tag{3.29}$$

for all $t \geqslant 0$ and $z \neq 0$. Sufficient conditions for satisfying (3.29) are to let

$$z_j \dot{z}_j = -\varphi_j |z_j| \tag{3.30}$$

$$\Rightarrow \quad \dot{z}_j = -\varphi_j \operatorname{sign}(z_j) \tag{3.31}$$

where $\varphi_j > 0$ for all $j = 1, 2, \ldots, m$. Thus,

$$\begin{aligned} \dot{z} &= -\begin{bmatrix} \varphi_1 & & \\ & \ddots & \\ & & \varphi_m \end{bmatrix} \operatorname{sign}(z) \\ &= -F \operatorname{sign}(z) \end{aligned} \tag{3.32}$$

such that

$$\dot{z} = S\dot{x} = SAx + SBu \tag{3.33}$$

Thus

$$u = -(SB)^{-1} SAx - (SB)^{-1} F \operatorname{sign}(z) \tag{3.34}$$

$$= u_{eq} + u_{sw} \tag{3.35}$$

And the closed loop system becomes

$$\dot{x} = \left(I - B(SB)^{-1} S\right) Ax - B(SB)^{-1} F \operatorname{sign}(z) \tag{3.36}$$

Now, as per Theorem 2.13, the switching term $B(SB)^{-1} F \operatorname{sign}(z)$ ensures stability of the closed loop system. That is, application of this control will result in $x(t)$ converging to $\mathcal{S}$ for all $x_0 \in \mathcal{X}$, as well as ensuring that $x(t)$ remains on $\mathcal{S}$ once $x(t)$ intercepts $\mathcal{S}$.

It is now appropriate to demonstrate the primary advantage of SLMC–invariance to bounded, matched disturbances / uncertainties.

3.4.2 Disturbance Rejection in Sliding

As mentioned in §3.1, SLMC is invariant to bounded, matched disturbances only. That is, disturbances that enter the system via the same path as the control input. The analysis of Draženović [11], [38] illustrates the property extremely well. Let

$$\dot{x} = Ax + Bu + \delta \tag{3.37}$$

Where δ is an unknown disturbance. If the initial state is on $\mathcal{S}$, then $Sx = 0$ for all $t \geqslant 0$, and

$$\begin{aligned} S\dot{x} &= 0 = SAx + SBu + S\delta \\ &\Rightarrow\ u = -(SB)^{-1} S(Ax + \delta) \end{aligned} \tag{3.38}$$

For the sake of accuracy, note that if δ is unknown in (3.38), then the controller cannot be expected to imitate it. That aside, the closed loop system becomes

$$\begin{aligned} \dot{x} &= Ax - B(SB)^{-1} SAx - B(SB)^{-1} S\delta + \delta \\ &= \left(I - B(SB)^{-1} S\right)(Ax + \delta) \end{aligned} \tag{3.39}$$

and, for δ to have no effect on $\dot{x}(t)$, it is required that

$$\left(I - B(SB)^{-1} S\right)\delta = 0 \tag{3.40}$$

$$\Rightarrow\ \delta = B(SB)^{-1} S\delta$$

$$\Rightarrow\ \delta \in \operatorname{Im} B \tag{3.41}$$

Thus, if the controller is to demonstrate invariance to disturbances, the disturbances must be matched. If the disturbances are in fact matched, then without loss of generality the disturbances can be modelled as

$$\delta = B\xi \tag{3.42}$$

Where ξ is still unknown, but assumed norm-bounded, so that

$$\|\xi\| \leqslant \rho \tag{3.43}$$

Where $\rho \in \mathbb{R}_+$. Now, to show that the controller can be designed to achieve the result of invariance, consider again the Lyapunov candidate $V : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ and

$$\begin{aligned} \frac{d}{dt}V &= \sum_{j=1}^{m} z_j \dot{z}_j \\ &= (Sx)^T (S\dot{x}) < 0 \end{aligned} \tag{3.44}$$

for all $t \geqslant 0$. Substitute (3.34), into (3.44) to achieve

$$(Sx)^T \left(SAx + SB\left(-(SB)^{-1} SAx - (SB)^{-1} F \operatorname{sign}(z)\right) + SB\xi\right) < 0 \tag{3.45}$$

Thus

$$\begin{aligned} (Sx)^T (-F \operatorname{sign}(z) + SB\xi) &= \sum_{j=1}^{m} -z_j \varphi_j \operatorname{sign}(z_j) + \sum_{j,k=1}^{m} z_j (SB)_{jk} \xi_k \\ &= \sum_{j=1}^{m} -\varphi_j |z_j| + \sum_{j,k=1}^{m} z_j (SB)_{jk} \xi_k < 0 \end{aligned} \tag{3.46}$$

Note that

$$\sum_{j,k=1}^{m} z_j (SB)_{jk} \xi_k \leqslant \sum_{j,k=1}^{m} |z_j| \left|(SB)_{jk}\right| |\xi_k| \tag{3.47}$$

And, without loss of generality, let

$$\rho = \sum_{k=1}^{m} |\xi_k| \tag{3.48}$$

That is, we define ρ to be the 1-norm of the disturbance vector ξ. Sufficient conditions for satisfying (3.46) are to then select each φ_j via

$$-\sum_{j=1}^{m} \varphi_j |z_j| \quad < \quad -\sum_{j,k=1}^{m} |z_j| \left|(SB)_{jk}\right| |\xi_k|$$

$$\sum_{j=1}^{m} \varphi_j \quad > \quad \sum_{j,k=1}^{m} \left|(SB)_{jk}\right| |\xi_k| \tag{3.49}$$

$$\Rightarrow \quad \varphi_j > \max_k \left|(SB)_{jk}\right| \rho \tag{3.50}$$

This will ensure that (3.44) is negative definite as required, and the system (3.36) will converge to $\mathcal{S}$.

3.5 Summary

This chapter presented a comprehensive development of SLMC. All the essential design features were developed for a certain class of linear systems, and the resulting controller was shown to have the desired robustness margins.

Chapter 4

Mixed Objective LQ / SLMC Control

4.1 Introduction

The problem of eigenvalue placement within specified regions of $\mathbb{C}$ is by no means a new problem. Descartes [8] is attributed as being the first person to get this study underway, when he showed in 1637 how to reduce geometrical problems to the solution of algebraic equations.

Later, in the 19th century, there was great interest among mathematicians and engineers in the study of stability as it pertained to differential equations. Conditions for the eigenvalues of a matrix to lie in the open left hand plane of $\mathbb{C}$ were first implied by Hermite [20] in 1856, and later explicitly obtained by Routh [33] and Hurwitz [21] in 1877 and 1895, respectively. An alternative, but equivalent, solution to this problem was also developed in 1892 by Lyapunov [28].

In the 20th century, the focus shifted from stable eigenvalue placement to that of *optimal* stable eigenvalue placement. An excellent treatment of this now classic topic can be found in Anderson and Moore [2]. More recently, there has been a great amount of interest in optimal eigenvalue placement within prescribed regions of $\mathbb{C}$. Techniques for optimal placement within various geometric shapes (strips, circles, etc.) have been treated by Gutman and Jury [16], Haddad and Bernstein [17], and Wang and Bernstein [43], [44] among others.

More recently, the techniques applied in these works have been generalized in a paper by Chilali, *et al* [5]. In all of the above mentioned works, the basic technique has been to define an open region of $\mathbb{C}$ via a set of *linear matrix inequalities*, an LMI region.

The Lyapunov function associated with the problem is then modified in such a way that eigenvalues lying outside the prescribed regions are treated as unstable.

In this chapter, a different approach is taken. Linear systems of the type introduced in §2.2 are considered, and the basic objective is to determine an optimal state feedback control law, $K : \mathcal{X} \rightarrow \mathcal{U}$, that places at least m of the closed loop eigenvalues of (2.85) on the negative real axis. By doing this, an SLMC style switching term may then be added to the control law, creating a closed loop system that is in some sense optimal, as well as robust to matched disturbances [11].

Since the previously mentioned LMI techniques require that the entire closed loop spectrum lie entirely within an LMI region or, the intersection of p of these open regions, the design objective of this chapter cannot be met with these techniques, in general, since $\mathbb{R}_-$ is a closed set in $\mathbb{C}$.

It is worth mentioning that this concept has previously been dealt with by Tang and Misawa [39]. In that particular work, the approach was to select a state weighting matrix Q_0 in (3.2) and inspect the resulting closed loop spectrum. If there was at least one real eigenvalue, the problem was solved. If this did not occur, a real eigenvalue was selected, and an alternative Q was then found that was in some sense 'closest' to Q_0. The problem with this method was that the resulting Q could end up being arbitrarily far away, thus negating any physical significance that may be attached with Q_0.

In this work, a much simpler method is used that is a modification of a recent work by Iracleous and Alexandridis [22]. The idea is to find a similarity transform of the closed loop system matrix that allows the system to be viewed as the cascaded sum of two subsystems. The feedback law is then found by using arbitrary pole placement for one subsystem, thus fixing m eigenvalues as real, and then optimizing the location of the remaining $(n - m)$ eigenvalues.

Once this feedback map K is obtained, focus is switched to SLMC. As outlined in §3, SLMC is an attractive option in many applications, due to its robustness properties. The problem is that the nominal closed loop performance of the sliding mode controller is typically nowhere close to optimal, due to the fact that sliding mode control works by artificially reducing the order of the closed loop dynamics by an order of m. This situation has been dealt with to some degree by the author [35] by adding additional dynamics to the closed loop system in the form of a dynamic compensator that seeks to recover the missing dynamics. This work takes a much simpler approach, in that an original result states that for systems with a closed loop spectrum containing at least m distinct, real eigenvalues, an equivalent sliding mode controller can be constructed. Thus, the nominal closed loop dynamics are preserved, and the additional feature of invariance to bounded matched disturbances is added.

The chapter will be organized as follows. §4.2 presents the derivation of an optimal state feedback map $K : \mathcal{X} \rightarrow \mathcal{U}$ that places m of the closed loop eigenvalues on $\mathbb{R}_{-}$. §4.3 presents the main result, that being the development of a control law that preserves nominal LQ system performance while adding a robustness margin to the closed loop system via a switching term on the controller. §4.4 extends these results to an output feedback controller that utilizes a full-order state estimator. Results are summarized in §4.5.

4.2 An LQ Regulator With Preset Eigenvalues

This section covers, in detail, the derivation of a gain map $K : \mathcal{X} \rightarrow \mathcal{U}$ that allows the designer to specify m of the closed loop eigenvalues, and place the remaining $(n - m)$ poles at a location that is optimal with respect to some index. The results are presented here using $\mathbb{C}$ as the underlying field only for the purpose of full generality. All results translate directly to $\mathbb{R}$. This method first appeared in [22], but has been significantly altered for this work.

As a quick aside, we note that the most intuitively simple way to accomplish this objective would be to decouple the open loop system into two controllable subsystems, so that

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4.1}$$

and the problem is solved easily, since the two subsystems can be dealt with separately. However, it was shown by Moorse [31] and Molinari [30] that this result cannot be achieved in general.

Consider, then, the system

$$\dot{x} = \acute{A}x + \acute{B}u \tag{4.2}$$

where $\left(\acute{A}, \acute{B}\right)$ is completely controllable, $x \in \mathcal{X}$, $u \in \mathcal{U}$, and $\operatorname{rank}\left(\acute{B}\right) = m = \dim(\mathcal{U}) \leqslant \dim(\mathcal{X}) = n$. Transforming $\left(\acute{A}, \acute{B}\right)$ to normal form as per (2.88) gives

$$\left(\acute{A}, \acute{B}\right) \approx \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \right) = (A, B) \tag{4.3}$$

where $B_2 \in \mathbb{C}^{m \times m}$ is nonsingular. Further, define two matrices $X_1 \in \mathbb{C}^{(n-m)\times(n-m)}$ and $X_2 \in \mathbb{C}^{m \times (n-m)}$, where X_1 is nonsingular. Thus, we may construct a nonsingular transformation T as

$$T = \begin{bmatrix} X_1 & 0 \\ X_2 & B_2 \end{bmatrix} \tag{4.4}$$

so that

$$T^{-1} = \begin{bmatrix} X_1^{-1} & 0 \\ -B_2^{-1}X_2X_1^{-1} & B_2^{-1} \end{bmatrix} \tag{4.5}$$

Now, consider the closed loop system $(A + BK)$. Here, K will be of the form

$$K = \begin{bmatrix} K_1 \in \mathbb{C}^{(n-m)\times m} & K_2 \in \mathbb{C}^{m \times m} \end{bmatrix} \tag{4.6}$$

and application of the above transformation results in

$$T^{-1}(A + BK)T = \begin{bmatrix} \bar{A}_{11} + \bar{A}_{12}X_2 & \bar{A}_{12}B_2 \\ \left(\bar{A}_{22} + K_2\right)X_2 + \left(\bar{A}_{21} + K_1\right)X_1 & \left(\bar{A}_{22} + K_2\right)B_2 \end{bmatrix} \tag{4.7}$$

where

$$\begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} = \begin{bmatrix} X_1^{-1} A_{11} X_1 & X_1^{-1} A_{12} \\ B_2^{-1}\left(A_{21} - X_2 X_1^{-1} A_{11}\right) & B_2^{-1}\left(A_{22} - X_2 X_1^{-1} A_{12}\right) \end{bmatrix} \tag{4.8}$$

Define a particular M such that $\sigma(M) \subset \mathbb{C}_-$, and let

$$\begin{aligned} \left(\bar{A}_{22} + K_2\right) B_2 &= M \\ K_2 &= M B_2^{-1} - \bar{A}_{22} \\ &= M B_2^{-1} - B_2^{-1}\left(A_{22} - X_2 X_1^{-1} A_{12}\right) \end{aligned} \tag{4.9}$$

Further, constrain the closed loop system by

$$\begin{aligned} 0 &= \left(\bar{A}_{22} + K_2\right) X_2 + \left(\bar{A}_{21} + K_1\right) X_1 \\ 0 &= M B_2^{-1} X_2 + \left(\bar{A}_{21} + K_1\right) X_1 \\ &\Rightarrow K_1 = -M B_2^{-1} X_2 X_1^{-1} - \bar{A}_{21} \\ &\Rightarrow K_1 = -M B_2^{-1} X_2 X_1^{-1} - B_2^{-1}\left(A_{21} - X_2 X_1^{-1} A_{11}\right) \end{aligned} \tag{4.10}$$

and the resulting closed loop system matrix becomes

$$(A + BK) \approx \begin{bmatrix} \bar{A}_{11} + \bar{A}_{12} X_2 & \bar{A}_{12} B_2 \\ 0 & M \end{bmatrix} \tag{4.11}$$

And so, the closed loop spectrum is determined by

$$\sigma(A_{cl}) = \sigma\left(\bar{A}_{11} + \bar{A}_{12} X_2\right) \cup \sigma(M) \tag{4.12}$$

The goal is to then use any method, potentially some LQ optimization technique [2] to assign $\sigma\left(\bar{A}_{11} + \bar{A}_{12} X_2\right)$. That is, the system can be viewed as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} B_2 \\ 0 & M \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \bar{A}_{12} \\ 0 \end{bmatrix} v \tag{4.13}$$

where M may be selected arbitrarily, $v \in \mathcal{V}$, and the objective is to come up with a state feedback law $X_2 : \mathcal{V} \to \mathcal{X}_1$. To accomplish this, the pair $\left(\bar{A}_{11}, \bar{A}_{12}\right)$ must be fully controllable. The following proposition gives conditions for this to occur.

Proposition 4.1 Let (A, B) be given as in (2.88), with B_2 nonsingular and (A, B) controllable. Let $M \in \mathbb{C}^{m \times m}$ be given, and X_1, X_2 be any two matrices such that

$$T = \begin{bmatrix} X_1 & 0 \\ X_2 & B_2 \end{bmatrix}$$

is well defined and nonsingular. Then, there is a $K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ such that

$$T^{-1}(A + BK)T = \begin{bmatrix} \bar{A}_{11} + \bar{A}_{12}X_2 & \bar{A}_{12}B_2 \\ 0 & M \end{bmatrix}, \; T^{-1}B = \begin{bmatrix} 0 \\ I_m \end{bmatrix}$$

where

$$\begin{aligned} \bar{A}_{11} &= X_1^{-1}A_{11}X_1 \\ \bar{A}_{12} &= X_1^{-1}A_{12} \end{aligned}$$

and $(\bar{A}_{11}, \bar{A}_{12})$ is controllable.

Proof. As per (4.9), (4.10), the required K is

$$\begin{aligned} K_1 &= -MB_2^{-1}X_2X_1^{-1} - B_2^{-1}\left(A_{21} - X_2X_1^{-1}A_{11}\right) \\ K_2 &= MB_2^{-1} - B_2^{-1}\left(A_{22} - X_2X_1^{-1}A_{12}\right) \end{aligned}$$

Further, from Theorem 2.11, (A_{11}, A_{12}) is a controllable pair. Thus, since X_1 is nonsingular, it easily follows that $(\bar{A}_{11}, \bar{A}_{12})$ is also a controllable pair. ■

Assuming the condition of Theorem 4.1 is met, the feedback law $X_2 : \mathcal{V} \to \mathcal{X}_1$ can be designed. The design methodology used here will be that of classic LQ regulator design, [2], [45] but any of the specialized methods referenced in the introduction may now be applied as well. That is, if the cost functional, J, is defined as

$$J = \int_0^\infty \left(x_2^* Q x_2 + v^* R v\right) dt \tag{4.14}$$

where $Q \geqslant 0$, $R > 0$, and v is a pseudo-control term, then it is a standard result to show that J is minimized by selecting v as

$$\begin{aligned} v &= R^{-1}\bar{A}_{12}^{*}Px_2 \qquad (4.15) \\ &\Rightarrow X_2 = R^{-1}\bar{A}_{12}^{*}P \qquad (4.16) \end{aligned}$$

where P is the maximal Hermitian solution of the algebraic Riccati equation (ARE)

$$P\bar{A}_{11} + \bar{A}_{11}^{*}P - P\bar{A}_{12}R^{-1}\bar{A}_{12}^{*}P + Q = 0 \qquad (4.17)$$

Note that the dynamics of x in $\mathcal{X}_2$ correspond to an uncontrollable subspace in $\mathcal{X}$, relative to the control problem (4.13). K is now solved as

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix} \qquad (4.18)$$

The resulting structure of $\sigma(A + BK)$ will now be exploited further. Note that for the sequel, M must be defined in the following manner

Assumption 4.1 $M = M^{*} < 0 \Leftrightarrow \sigma(M) \subset \mathbb{R}_{-}$. Further, assume that all elements of $\sigma(M)$ are distinct.

4.3 Mixed LQ / SLMC Design

Consider the fact that Theorem 3.1 implied that conventional SLMC and LQ design strategies were not compatible. Presented in this section is a proposed technique that will bridge the gap between these two powerful techniques.

4.3.1 Lyapunov Design

Assume that $\sigma(A_{cl})$ has already been designed via the technique of §4.2, and the goal is to now find the map S and switching term u_{sw} as per §3.2 / §3.4 that will allow the two design objectives to be compatible.

Recall [7] that the switching controller can be designed via a Lyapunov approach by again viewing $\|z\| = \|Sx\|$ as a measure of the 'distance' from x to S, and a Lyapunov candidate $V : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ as

$$V = \frac{1}{2}\langle z, z\rangle \Rightarrow \frac{d}{dt}V = \langle z, \dot{z}\rangle \tag{4.19}$$

For global stability, it is required that

$$\langle z, \dot{z}\rangle < 0 \tag{4.20}$$

for all $t \geqslant 0$. Typically, this problem is solved by the approach outlined in §3.4. Consider, however, that an equally valid sufficient condition for satisfying (4.20) is to let

$$\begin{aligned} z_j \dot{z}_j &= -\varphi_j |z_j| - \gamma_j z_j^2 \\ &\Rightarrow \dot{z}_j = -\varphi_j \operatorname{sign}(z_j) - \gamma_j z_j \end{aligned} \tag{4.21}$$

where $\varphi_j, \gamma_j > 0$ for all $j = 1, 2, \ldots, m$. Thus

$$\begin{aligned} \dot{z} &= -\begin{bmatrix} \varphi_1 & & \\ & \ddots & \\ & & \varphi_m \end{bmatrix} \operatorname{sign}(z) - \begin{bmatrix} \gamma_1 & & \\ & \ddots & \\ & & \gamma_m \end{bmatrix} z \\ &= -F \operatorname{sign}(z) - Gz && (4.22) \\ &= SAx + SBu && (4.23) \end{aligned}$$

Solving (4.22), (4.23) for u and substituting this value into (2.77) yields

$$\begin{aligned} u &= -(SB)^{-1}(SAx + GSx + F \operatorname{sign}(z)) \\ &= -(SB)^{-1}((SA + GS)x + F \operatorname{sign}(z)) && (4.24) \\ &= u_{eq} + u_{sw} && (4.25) \\ &\Rightarrow \dot{x} = \left(A - B(SB)^{-1}(SA + GS)\right)x - B(SB)^{-1}F \operatorname{sign}(z) && (4.26) \end{aligned}$$

Now, comparison of (4.26) with the classic $A + BK$ structure yields

$$A + BK = A - B(SB)^{-1}(SA + GS)$$

$$
\begin{aligned}
K &= -(SB)^{-1}(SA+GS) \\
S(A+BK)+GS &= 0 \\
SA_{cl}+GS &= 0
\end{aligned}
\tag{4.27}
$$

(4.27) can be restated as

$$\left[\left(A_{cl}^T \otimes I_m\right) + \left(I_n \otimes G\right)\right] \operatorname{vec}(S) = 0 \tag{4.28}$$

$$Hs = 0 \tag{4.29}$$

$$\Rightarrow \quad s \in \ker(H) \tag{4.30}$$

where $H = \left[\left(A_{cl}^T \otimes I_m\right) + \left(I_n \otimes G\right)\right]$, $s = \operatorname{vec}(S)$. Since S is full rank by assumption, $\dim(\ker H) \geqslant m$ is required. Recall from §2.1.7 that

$$\sigma(H) = \left\{\lambda_i + \gamma_j\right\}_{i,j=1}^{n,m} \tag{4.31}$$

where

$$\sigma\left(A_{cl}^T\right) = \left\{\lambda_i\right\}_{i=1}^{n} \tag{4.32}$$

$$\sigma(G) = \left\{\gamma_j\right\}_{j=1}^{m} \tag{4.33}$$

so that (4.29) is satisfied non-trivially if each $\gamma_j = -\lambda_p$ for each j and some p. Recall, though, that $\gamma_j \in \mathbb{R}_+$ by construction for all $j = 1, 2, \ldots, m$. To simplify matters, assume that all γ_j are distinct, which implies that $\sigma(A+BK)$ must contain at least m distinct real eigenvalues $\left\{-\gamma_j\right\}_{j=1}^{m}$ to solve (4.29). This, however, is has been accomplished by Assumption 4.1, so that

$$\sigma(M) = \left\{-\gamma_j\right\}_{j=1}^{m} \tag{4.34}$$

With this, S can then be determined non-trivially as

$$S = \operatorname{vec}^{-1}(s) \tag{4.35}$$

giving the closed loop system

$$\dot{x} = (A+BK)x - B(SB)^{-1}F\operatorname{sign}(z) \tag{4.36}$$

Thus, the system is now like an LQ controller with an additional switching gain. It can now be shown that by adding the switching term u_{sw}, the system (4.36) is invariant to a certain class of disturbances.

4.3.2 Robustness Properties of the Proposed Controller

As mentioned earlier, SLMC is invariant to matched disturbances only. The proposed controller is no different. That is, the proposed controller is robust only against disturbances / modelling errors that enter the system via the same path as the control input. Again we will employ the analysis of [11] to illustrate the property. Let

$$\dot{x} = Ax + Bu + \delta \tag{4.37}$$

where δ is an unknown disturbance. Apply the control law

$$u = K(x - \delta) \tag{4.38}$$

which is flawed, in that if δ is unknown, then the controller cannot be expected to imitate it. That aside, the system is now

$$\begin{aligned} \dot{x} &= Ax + BK(x-\delta) + \delta \\ &= (A+BK)x + (I-BK)\delta \end{aligned} \tag{4.39}$$

So, for δ to have no effect on $\dot{x}$, it is required that

$$\begin{aligned} (I-BK)\delta &= 0 \\ &\Rightarrow \delta = BK\delta \\ &\Rightarrow \delta \in \operatorname{Im} B \end{aligned} \tag{4.40}$$

So, if the controller is to demonstrate invariance to disturbances, the disturbances must be matched. Specifically, it will now be demonstrated how to select the elements of the switching gain matrix F to achieve this objective.

If the disturbances are in fact matched, then without loss of generality the disturbances may again be modelled as

$$\delta = B\xi \tag{4.41}$$

where ξ is still unknown, but assumed norm-bounded as in §3.4.2, so that

$$\|\xi\| \leqslant \rho \tag{4.42}$$

where $\rho \in \mathbb{R}_{+}$. To show that the controller can be designed to achieve the result of invariance, consider again the Lyapunov candidate $V : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{+}$ and

$$\begin{aligned} \frac{d}{dt}V &= \sum_{j=1}^{m} z_j \dot{z}_j \\ &= (Sx)^T (S\dot{x}) < 0 \end{aligned} \tag{4.43}$$

for all $t \geqslant 0$. Substitute (4.36) into (4.43) to achieve

$$\begin{aligned} 0 &> (Sx)^T \left(SAx + SB\left(-(SB)^{-1}(SA+GS)\,x - (SB)^{-1} F \operatorname{sign}(z)\right) + SB\xi\right) \\ 0 &> (Sx)^T \left(-GSx - F \operatorname{sign}(z) + SB\xi\right) \end{aligned} \tag{4.44}$$

Now

$$\begin{aligned} (Sx)^T \left(-GSx - F \operatorname{sign}(z) + SB\xi\right) &= \sum_{j=1}^{m} -\gamma_j z_j^2 + \sum_{j=1}^{m} -z_j \varphi_j \operatorname{sign}(z_j) + \sum_{j,k=1}^{m} z_j (SB)_{jk}\,\xi_k \\ &= -\sum_{j=1}^{m} \left(\gamma_j z_j^2 + \varphi_j |z_j|\right) + \sum_{j,k=1}^{m} z_j (SB)_{jk}\,\xi_k \\ &= -\sum_{j=1}^{m} \left(\gamma_j |z_j| + \varphi_j\right) |z_j| + \sum_{j,k=1}^{m} z_j (SB)_{jk}\,\xi_k \end{aligned} \tag{4.45}$$

Note that

$$\sum_{j,k=1}^{m} z_j (SB)_{jk}\,\xi_k \leqslant \sum_{j,k=1}^{m} |z_j| \left|(SB)_{jk}\right| |\xi_k| \tag{4.46}$$

and, once again, let ρ to be the 1-norm of the disturbance vector ξ, so that

$$\rho = \sum_{k=1}^{m} |\xi_k|$$

Sufficient conditions for satisfying (4.45) are to then select each φ_j via

$$\begin{aligned}
-\sum_{j=1}^{m}\left(\gamma_j\left|z_j\right|+\varphi_j\right)\left|z_j\right| &< -\sum_{j,k=1}^{m}\left|z_j\right|\left|(SB)_{jk}\right|\left|\xi_k\right| \\
\sum_{j=1}^{m}\left(\gamma_j\left|z_j\right|+\varphi_j\right) &> \sum_{j,k=1}^{m}\left|(SB)_{jk}\right|\left|\xi_k\right| \\
\sum_{j=1}^{m}\varphi_j &> \sum_{j,k=1}^{m}\left(\left|(SB)_{jk}\right|\left|\xi_k\right|-\gamma_j\left|s_{jk}x_{kj}\right|\right) \qquad (4.47)\\
\Rightarrow &\quad \varphi_j > \max_k\left|(SB)_{jk}\right|\rho \qquad (4.48)
\end{aligned}$$

since $\gamma_j > 0$ for all $j = 1, 2, \ldots, m$, ensuring that the term $\sum_k \gamma_j \left|s_{jk}x_{kj}\right|$ can only serve to decrease $\sum_k \left|(SB)_{jk}\right| \left|\xi_k\right|$ over all values of j. Thus, this choice of φ_j will ensure that (4.43) is negative definite as required, yielding the desired robustness characteristics.

4.4 Extension to the Output Feedback Case

4.4.1 Construction of The Observer

In this section, consider the situation where x is not available for direct measurement. That is, the system is now of the form (2.77), (2.78)

$$\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{aligned}$$

where the triple (A, B, C) is complete. With this assumption, it is now possible, as per §2.2.3, to design a full order observer for the system. Recall from (4.24) that

$$u = -(SB)^{-1}\left((SA+GS)\,x + F\,\mathrm{sign}\,(z)\right)$$

when x was available for direct measurement. Now, the objective is to use q, the estimate of x instead, so that (4.24) becomes

$$u = -(SB)^{-1}((SA+GS)q + F\,\mathrm{sign}(\bar{z})) \tag{4.49}$$

where

$$\bar{z} = Sq \tag{4.50}$$

Substitution of (4.49) into (2.77) then results in

$$\begin{aligned} \dot{x} &= Ax - B(SB)^{-1}(SA+GS)q - B(SB)^{-1}F\,\mathrm{sign}(\bar{z}) && (4.51)\\ &= Ax + BKq - B(SB)^{-1}F\,\mathrm{sign}(\bar{z}) && (4.52) \end{aligned}$$

Applying the observer of §2.2.3, so that

$$\dot{q} = (A+LC)q - Ly + Bu$$

and the estimator dynamics now become

$$\begin{aligned} \dot{q} &= (A+LC)q - LCx - B(SB)^{-1}(SA+GS)q - B(SB)^{-1}F\,\mathrm{sign}(\bar{z})\\ &= (A+LC+BK)q - LCx - B(SB)^{-1}F\,\mathrm{sign}(\bar{z}) && (4.53) \end{aligned}$$

The resulting system is then

$$\begin{bmatrix} \dot{x} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} A & BK \\ -LC & (A+LC+BK) \end{bmatrix} \begin{bmatrix} x \\ q \end{bmatrix} - \begin{bmatrix} B(SB)^{-1}F\,\mathrm{sign}(\bar{z}) \\ B(SB)^{-1}F\,\mathrm{sign}(\bar{z}) \end{bmatrix} \tag{4.54}$$

or, in terms of the error dynamics, $e = x - q$, the system may more conveniently be written as

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} (A+BK) & -BK \\ 0 & (A+LC) \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} - \begin{bmatrix} B(SB)^{-1}F\,\mathrm{sign}(S(x-e)) \\ 0 \end{bmatrix} \tag{4.55}$$

Examination of (4.55) immediately reveals that by selecting $\sigma(A+LC)$ to be suitably fast in comparison to $\sigma(A+BK)$, (4.55) will reduce to (4.36) once the estimator converges, and the closed loop system will behave in the expected manner. That is, (4.55) will reduce to (4.26) when e vanishes.

4.4.2 Robustness of the Closed Loop System

The next point to examine is the robustness of the system (4.55). This analysis will be carried out as in previous sections via a Lyapunov approach.

As before, assume that the disturbances are matched and bounded, so that the estimator dynamics are now governed by

$$\dot{q} = (A + LC)\, q - LCx + B\,(u + \xi) \tag{4.56}$$

Noting (4.50), a Lyapunaov candidate $V : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_+$ may be constructed as

$$V(\tilde{z}) = \langle \tilde{z}, \tilde{z} \rangle > 0 \ \text{ for all } \ \tilde{z} \neq 0 \tag{4.57}$$

and for the desired robustness property, it is required that

$$\frac{d}{dt} V(\tilde{z}) = (Sq)^T (S\dot{q}) < 0 \ \text{ for all } \ \tilde{z} \neq 0 \tag{4.58}$$

Substitution of (4.24) into (4.50) then results in

$$\begin{aligned} S\dot{q} &= S\left[(A + LC)\, q - LCx - B\,(SB)^{-1}(SA + GS)\, q - B\,(SB)^{-1} F \operatorname{sign}(z) + B\xi\right] \\ &= -SLC\,(x - q) - GSq - F \operatorname{sign}(z) + SB\xi \end{aligned} \tag{4.59}$$

which leads to

$$(Sq)^T (S\dot{q}) = -(Sq)^T SLCe - (Sq)^T GSz - (Sq)^T F \operatorname{sign}(Sq) + (Sq)^T SB\xi < 0 \tag{4.60}$$

or

$$\begin{aligned} -\sum_{j,p=1}^{m,n} (z_j)\,(SLC)_{jp}\,(e_p) - \sum_{j=1}^{m} (z_j)^2 \gamma_j - \sum_{j=1}^{m} |z_j|\, \varphi_j + \sum_{j,p=1}^{m} (z_j)\,(SB)_{jk}\,(\xi_k) &< 0 \\ \sum_{j,p=1}^{m,n} (z_j)\,(SLC)_{jp}\,(e_p) + \sum_{j=1}^{m} (z_j)^2 \gamma_j + \sum_{j=1}^{m} |z_j|\, \varphi_j - \sum_{j,p=1}^{m} (z_j)\,(SB)_{jk}\,(\xi_k) &> 0 \\ \sum_{j,p=1}^{m,n} (z_j)\,(SLC)_{jp}\,(e_p) + \sum_{j=1}^{m} (z_j)^2 \gamma_j + \sum_{j=1}^{m} |z_j|\, \varphi_j > \sum_{j,p=1}^{m} (z_j)\,(SB)_{jk}\,(\xi_k) & \end{aligned} \tag{4.61}$$

Note that

$$\sum_{j,p=1}^{m,n} (z_j)\,(SLC)_{jp}\,(e_p) \ \leqslant \ \sum_{j,p=1}^{m,n} |z_j| \left|(SLC)_{jp}\right| |e_p| \tag{4.62}$$

$$\sum_{j,p=1}^{m} (z_j)\,(SB)_{jk}\,(\xi_k) \;\leqslant\; \sum_{j,p=1}^{m} |z_j|\,\left|(SB)_{jk}\right|\,|\xi_k| \tag{4.63}$$

which then immediately leads to

$$\sum_{j,p=1}^{m,n} (z_j)\,(SLC)_{jp}\,(e_p) \leqslant \sum_{j,p=1}^{m} |z_j|\,\left|(SB)_{jk}\right|\,|\xi_k| < \sum_{j,p=1}^{m,n} (z_j)\,(SLC)_{jp}\,(e_p) + \sum_{j=1}^{m} (z_j)^2\,\gamma_j + \sum_{j=1}^{m} |z_j|\,\varphi_j \tag{4.64}$$

and

$$\sum_{j,p=1}^{m} \left|(SB)_{jk}\right|\,|\xi_k| < \sum_{j,p=1}^{m,n} \left|(SLC)_{jp}\right|\,|e_p| + \sum_{j=1}^{m} |z_j|\,\gamma_j + \sum_{j=1}^{m} \varphi_j$$

So that

$$\sum_{j,p=1}^{m} \left|(SB)_{jk}\right|\,|\xi_k| \leqslant \sum_{j=1}^{m} \max_k \left|(SB)_{jk}\right|\,\rho < \sum_{j,p=1}^{m,n} \left|(SLC)_{jp}\right|\,|e_p| + \sum_{j=1}^{m} |z_j|\,\gamma_j + \sum_{j=1}^{m} \varphi_j \tag{4.65}$$

Finally

$$\max_k \left|(SB)_{jk}\right|\,\rho - \sum_p \left|(SLC)_{jp}\right|\,|e_p| - |z_j|\,\gamma_j < \max_k \left|(SB)_{jk}\right|\,\rho < \varphi_j \tag{4.66}$$

And the final result of

$$\max_k \left|(SB)_{jk}\right|\,\rho < \varphi_j \tag{4.67}$$

is obtained.

Thus, if each switching gain φ_j is chosen according to (4.67) for all $j = 1, 2 \ldots, m$, then sufficient conditions for satisfying (4.58) will be achieved. In turn, the resulting closed loop system will exhibit the desired robustness properties.

4.5 Summary

This chapter has presented a practical, constructive algorithm to design a controller that is both near optimal in terms of a cost functional, and robust in terms of its ability to reject matched disturbances in the manner of an SLMC controller.

Chapter 5

Application–Inverted Pendulum

In this chapter, we consider the application of the proposed control law to a cart-mounted inverted pendulum. The model under consideration appears in [41], and all numerical calculations were performed on the MATLAB software package.

5.1 Problem Formulation

Consider the inverted pendulum system illustrated in Fig. 5.1.

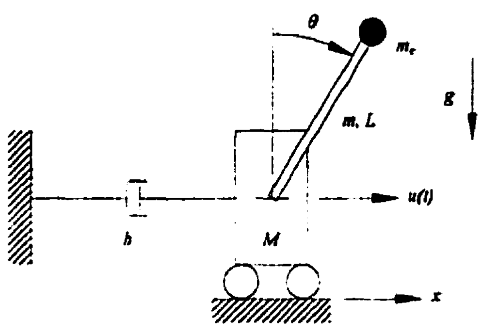

Figure 5.1 Inverted Pendulum System

The system consists of the following

- A cart of mass M.

- A uniform rod of mass m, length L, with an attached mass m_c on the tip.

The rod pivots without friction on the cart in the plane of the page, while the cart rolls without slipping along the x axis. The cart is forced by the applied input $u(t)$, and the coefficient of viscous damping on the cart is b.

Using Newton's Second Law of motion, the system can be modelled as

$$\begin{aligned} I_p\ddot{\theta} + m_oL\ddot{x}\cos\theta - m_ogL\sin\theta = 0 \\ m_t\ddot{x} + m_oL\ddot{\theta}\cos\theta - m_oL\dot{\theta}^2 + b\dot{x} = u(t) \end{aligned} \tag{5.1}$$

where

$$m_t = M + m + m_c \tag{5.2}$$

$$m_o = \frac{m}{2} + m_c \tag{5.3}$$

$$I_p = \left(\frac{m}{3} + m_c\right)L^2 \tag{5.4}$$

A detailed derivation of (5.1) may be found in [41].

We wish to now generate a linearized model of (5.1) of the form (2.77), (2.78). To accomplish this, define

$$z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \tag{5.5}$$

and a linear approximation of (5.1) in companion form is found as

$$\frac{d}{dt}\hat{z} = A\hat{z} + Bu \tag{5.6}$$

$$\hat{y} = C\hat{z} \tag{5.7}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \alpha_{23}\alpha_{42} - \alpha_{22}\alpha_{43} & \alpha_{43} & \alpha_{23} \end{bmatrix} \tag{5.8}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{5.9}$$

$$C = \begin{bmatrix} \alpha_{23}\beta_2 - \alpha_{43}\beta_1 & 0 & \beta_1 & 0 \end{bmatrix} \tag{5.10}$$

with

$$\begin{bmatrix} \alpha_{22} & \alpha_{23} \\ \alpha_{42} & \alpha_{43} \end{bmatrix} = \begin{bmatrix} \frac{I_p b}{(m_o L)^2 - I_p m_t} & \frac{(m_o L)^2 g}{(m_o L)^2 - I_p m_t} \\ \frac{-m_o L b}{(m_o L)^2 - I_p m_t} & \frac{m_t m_o L g}{(m_o L)^2 - I_p m_t} \end{bmatrix} \tag{5.11}$$

$$\begin{bmatrix} \beta_2 \\ \beta_4 \end{bmatrix} = \begin{bmatrix} \frac{-I_p}{(m_o L)^2 - I_p m_t} \\ \frac{-m_o L}{(m_o L)^2 - I_p m_t} \end{bmatrix} \tag{5.12}$$

This model is based on the assumption that only the position, x, of the cart is available for direct measurement. The calculations used to achieve this particular (A, B, C) are given in the Appendix.

At this point, it becomes useful to introduce some numerical values in the interest of actually performing the simulation. That said, let

$$I_p = 0.20 \tag{5.13}$$

$$m_o = 0.22 \tag{5.14}$$

$$m_t = 0.50 \tag{5.15}$$

$$L = 1 \tag{5.16}$$

$$b = 1 \tag{5.17}$$

which leads to an open loop unstable system. That is

$$\sigma(A) = \{-4.77, 0.45 \pm i5.00\} \tag{5.18}$$

and the need for a stabilizing control is immediately seen.

In this particular situation, we have (A, B) controllable and $\text{rank}(B) = 1 = m$. Thus, the methodology of §4 may be applied.

5.2 The Control Law

In this section, the controller will be given in three parts. First, the feedback law K will be solved as per §4.2. The switching term of §4.3 will be given next, and finally an observer of the type.§2.2.3 / §4.4 will be constructed.

5.2.1 The State Feedback Law

Select X_1 in (??) as

$$X_1 = I_3 \tag{5.19}$$

so that

$$T = \begin{bmatrix} I_3 & 0 \\ X_2 & 1 \end{bmatrix} \tag{5.20}$$

Recalling (4.8), we may construct $\bar{A}_{11}$ and $\bar{A}_{12}$, so that X_2 may now be solved by (arbitrarily) selecting the weighting matrices Q and R in (4.14) as

$$Q = 100I_3 \tag{5.21}$$

$$R = 1 \tag{5.22}$$

and by solving the associated ARE (4.17). This then allows the construction of $\bar{A}_{21}$ and $\bar{A}_{22}$ as per (4.8).

The next step is to select M as per Assumption 4.1. An acceptable choice is

$$M = -3 \tag{5.23}$$

which then leads to the solution of the feedback law K as per (4.9) / (4.10) as

$$K = \begin{bmatrix} -30.00 & 55.49 & -32.41 & -10.81 \end{bmatrix} \tag{5.24}$$

The closed loop system matrix is now $A + BK$, and

$$\sigma(A + BK) = \{-9.95, -3.00, -0.87 \pm i0.50\} \tag{5.25}$$

so the resulting system is now asymptotically stable.

For the sake of comparison, we can also construct a feedback law K_{lq} based on conventional LQ theory by selecting

$$Q_{LQ} = 100 I_4 \tag{5.26}$$

and defining

$$J_{LQ} = \int_0^\infty \left(x^T Q_{LQ} x + u^T R u\right) dt \tag{5.27}$$

In this way, the two cost functionals J (i.e, (4.14)) and J_{LQ} are as similar as possible, in the sense that all state and control elements are weighted equally relative to the system under consideration. An optimal K_{LQ} is then found as

$$K_{lq} = \begin{bmatrix} -10.00 & 4.84 & -33.51 & -9.62 \end{bmatrix} \tag{5.28}$$

resulting in

$$\sigma(A + BK_{LQ}) = \{-8.96, -2.22 \pm i2.92, -0.08\} \tag{5.29}$$

Thus, the proposed algorithm does not seem to preserve the 'optimal' spectrum. This, however, should be expected since m of the eigenvalues are located arbitrarily in the proposed algorithm, whereas they are free to be located anywhere in $\mathbb{C}_-$ when an LQ strategy is employed. This difference in eigenvalue location is clearly illustrated in Fig. 5.2.

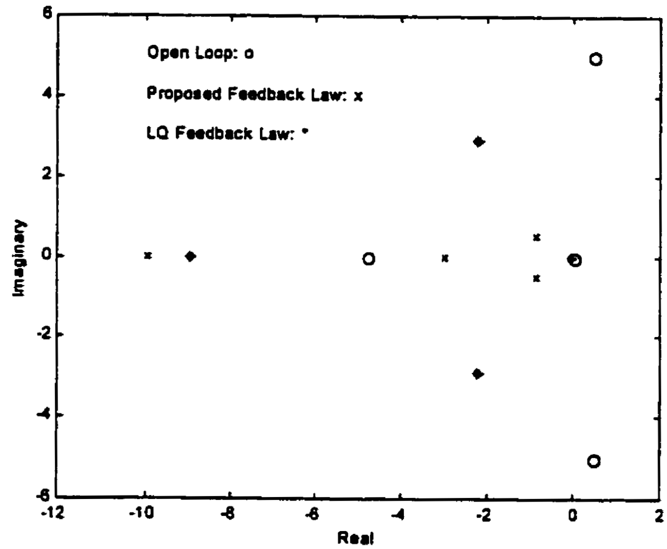

Figure 5.2 Comparison of Eigenvalue Locations in $\mathbb{C}$ for Various Control Strategies

5.2.2 The Switching Term

At this point, the task at hand is to determine the matrix S in (4.27), i.e.

$$S(A+BK)+GS=0 \tag{5.30}$$

so that

$$\left[\left((A+BK)^T \otimes I_m\right)+(I_n \otimes G)\right] \operatorname{vec}(S)=0 \tag{5.31}$$

Recall from (4.34) that a non-trivial S can be found by selecting G as

$$G=-M=3 \tag{5.32}$$

which leads to

$$S=\left[\begin{array}{cccc} 10.00 & 18.27 & 11.68 & 1.00 \end{array}\right] \tag{5.33}$$

All that remains at this point is to determine the switching gain matrix, F, in (4.36). For an appropriate choice of F, define a matched disturbance

$$\xi \in [-0.20, 0, 20] \subset \mathbb{R} \tag{5.34}$$

that is, ξ is any element of the evenly distributed, closed set $[-0.20, 0.20]$. Thus, $\rho = 0.20$, and there results

$$\begin{aligned} F &> \max_k |(SB)_k| \, \rho = 0.20 \\ &\Rightarrow F = 0.250 \end{aligned} \tag{5.35}$$

as per (4.48). The closed loop system (4.36) can now be simulated. This is done in §5.3.

5.2.3 The Observer

We now wish to design an observer of the type described in §2.2.3. Recall from (2.93) that the observer dynamics will take the form

$$\begin{aligned} \dot{q} &= Wq - Ly + Bu \\ &= (A + LC)\, q - LCx + Bu \end{aligned}$$

and that L should be chosen as per (2.98), so that

$$\operatorname{Re} \sigma (A + LC) \geqslant 4 \operatorname{Re} \sigma (A + BK)$$

Thus, an appropriate choice of $\sigma (A + LC)$ could be

$$\sigma (A + LC) = \{-38, -41 \pm i3, -45\} \tag{5.36}$$

so that L may be constructed as

$$L = \begin{bmatrix} -818.3 \\ -223.0 \\ 8789.1 \\ -60.5 \end{bmatrix} \tag{5.37}$$

and the closed loop system (4.55) may now be formed.

We are now in a position to simulate the system.

5.3 Simulation Results

The results will be presented in two parts. For the first part, we introduce the following assumption.

Assumption 5.1 (Temporary) $C = I_4$. Thus, a full state feedback law $u = Kx$ may be employed, and the observer is not needed.

Further, all results are presented in (z, y) coordinates rather than $(\hat{z}, \hat{y})$ coordinates, simply because the latter have no physical meaning. On the other hand, the pair (z, y) corresponds to the tangible, physical properties of the model.

As well, in all simulations the system is given an initial state of

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 0.09 \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \tag{5.38}$$

which corresponds to the situation of the cart and pendulum starting at rest, but not in the equilibrium position. That is, the system is let go with the pendulum about 5° from vertical.

5.3.1 Full State Feedback Results

It is useful to examine the system response on a case-by-case basis. Let us begin with the most basic situation, in which the switching term has been omitted.

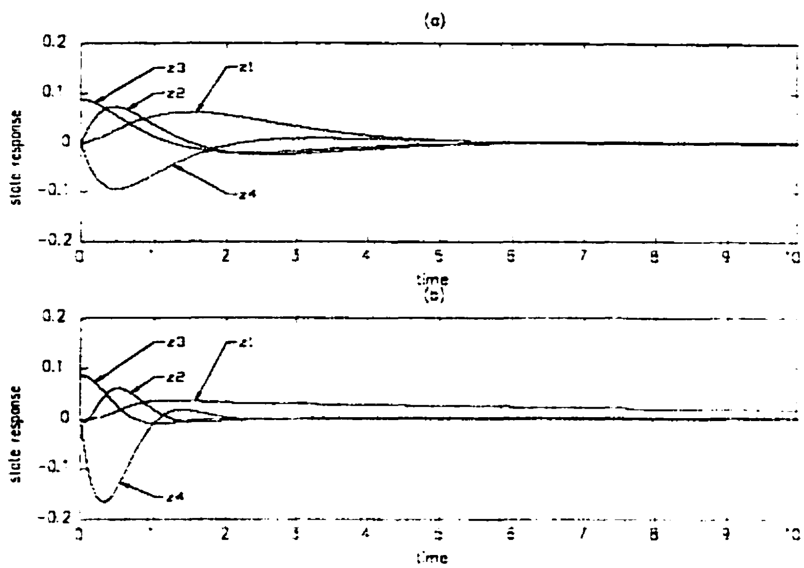

Figure 5.3 Comparison of State Responses for (a) The Proposed Controller Without Switching and (b) Standard LQ Controller

Case One: No Switching Present

The objective here is to give some insight as to how well the proposed K derived in §5.2.1 fares against the standard LQ result. Fig. 5.3 and Fig. 5.4 demonstrate the state responses and the control histories (respectively) for the two situations.

We see that the state responses in Fig. 5.3 are, more or less, the same–disregarding the z_3 response. That is, the response time is similar in both cases, the magnitude of the responses are similar, and Fig. 5.4 indicates that the control histories are also comparable.

In Fig. 5.3 we see the state z_1, the cart position, stabilizing very slowly in the LQ case. This is due to the fact that the LQ designed controller leads to a closed loop system matrix with a very 'slow' eigenvalue. To elaborate, note from Fig. 5.3 that

$$-0.08 \in \sigma(A+BK_{LQ}) \tag{5.39}$$

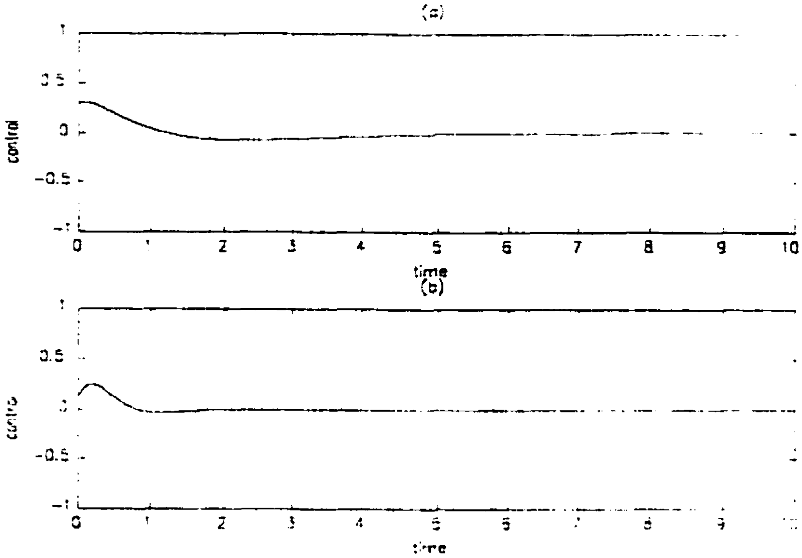

Figure 5.4 Comparison of Control Histories for (a) The Proposed Controller Without Switching and (b) Standard LQ Controller

which is quite close to zero. Since this value is quite small in comparison with the other elements of $\sigma\left(A+BK_{LQ}\right)$, the slow response is expected. Further, since the z_3 value does not fully stabilize during the course of the simulation, the value of the comparison is somewhat questionable on a purely qualitative level.

On a quantitative level, however, the results are quite nice. First, we examine of value of J_{LQ} for each controller over the simulation, i.e.

$$J_{LQ} = 63 \text{ when } u = K\hat{z} \tag{5.40}$$

$$J_{LQ} = 21 \text{ when } u = K_{LQ}\hat{z} \tag{5.41}$$

These numbers on their own are meaningless, except that the whole objective of an LQ strategy is to reduce the value of J_{LQ}, which has clearly been done. In addition, we note the proposed controller gives a result that is reasonably close to the minimal value, in that the values of J_{LQ} are both at least within the same order of magnitude. Thus,

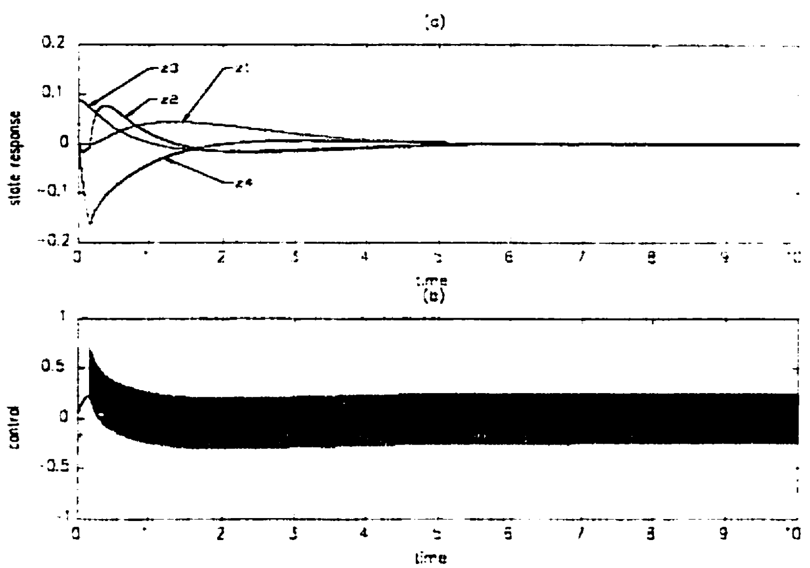

Figure 5.5 Closed Loop Response of the System Under the Proposed Controller, Zero Disturbances Present. (a) State Response and (b) Control History

it is reasonable to assume that the objective of designing a controller that in some way preserves LQ performance characteristics has been achieved.

Case Two: Switching Present, No Disturbances

Fig. 5.5 demonstrates the nominal performance of the system with the switching term activated. The objective here is to examine the effect of the control switching on the state trajectories–the idea being that they should be roughly the same as the results presented in Fig. 5.3, part (a). In fact, this seems to be the case.

In part (a) of Fig. 5.5, we see that the response of $\dot{\theta}$ has been somewhat altered. Rather than a smooth trajectory as in Fig. 5.3, the $\dot{\theta}$ trajectory is now strongly affected by the addition of the switching term, in that it takes a sharp turn about 0.2 seconds into the simulation (the time that the switching is activated) and proceeds to exhibit a first order

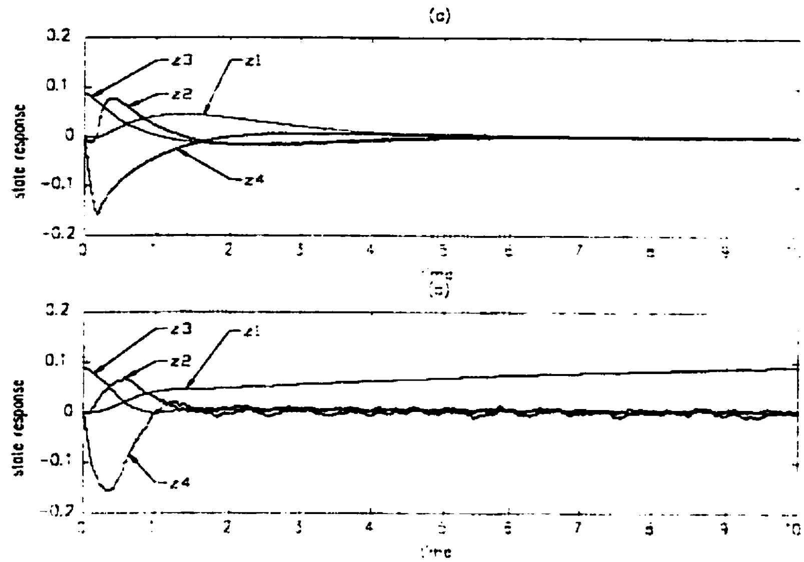

Figure 5.6 Comparison of State Responses with Matched Disturbances Present. (a) Proposed Controller and (b) Standard LQ Controller

response. This would tend to indicate strong coupling between $\dot{\theta}$ and u. Other than that, however, we see in Fig. 5.5 that the remaining three states exhibit similar responses to those shown in Fig. 5.3, part (a).

Thus, it is reasonable to assume that the objective of designing a switching controller that in some way preserves LQ performance characteristics has also been achieved.

Note, as well, the expected control chatter exhibited in Fig. 5.5, part (b).

Case Three: Matched Disturbances Present

Here, the primary advantage of the proposed controller is seen. Fig. 5.6 compares the performance of the proposed controller to that of the standard LQ controller, with noise of the type (5.34) injected. Fig. 5.7 shows the resulting control histories. In part (a) of Fig. 5.6, the expected result of complete disturbance rejection is shown, while part (b)

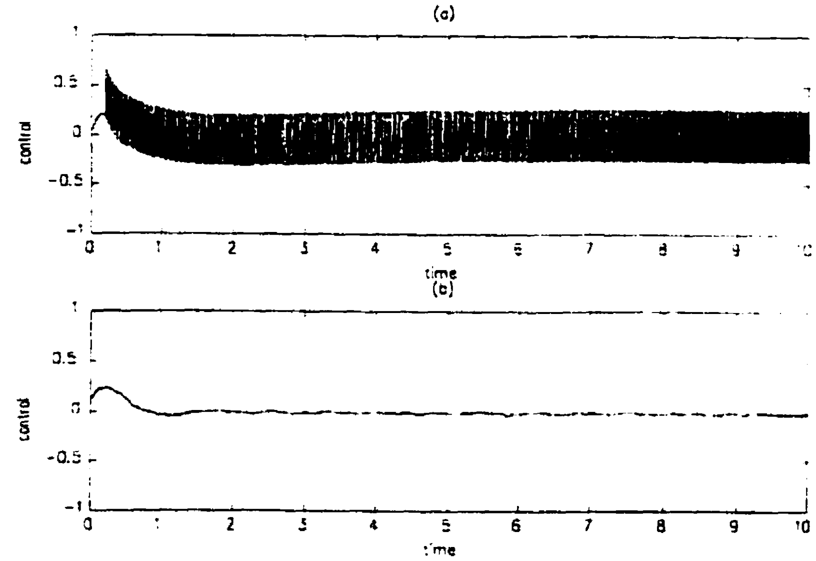

Figure 5.7 Comparison of Control Histories for (a) Proposed Controller and (b) Standard LQ Controller. Matched Disturbances Present.

demonstrates that the LQ controller is unable to completely stabilize the system.

As well, part (a) of Fig. 5.7 demonstrates that the control history, once switching is activated, is identical to the case where no disturbances are present, i.e. Fig. 5.5. Part (b) of Fig. 5.7, however, further demonstrates the inability of the LQ control to fully cope with this particular disturbance.

Note that as in Fig. 5.3, the steadily destabilizing value of x should again be ignored, as the result arises from a numerical problem within the MATLAB software, rather than the system itself.

5.3.2 Output Feedback Results

We now remove Assumption 5.1, and let $C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$, so that only the cart position x is available for measurement.

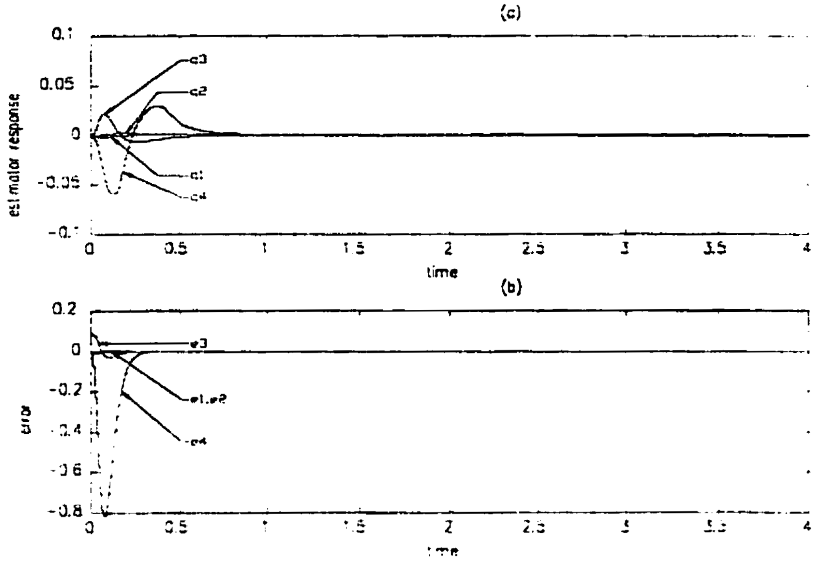

Figure 5.8 Estimator Response (a) and Error Dynamics (b) for the Simulated System–Zero Disturbances Present

Again, the system was simulated using an initial state of

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 0.09 \\ 0 \end{bmatrix}$$

and the estimator was initialized to zero. i.e.

$$q_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{5.42}$$

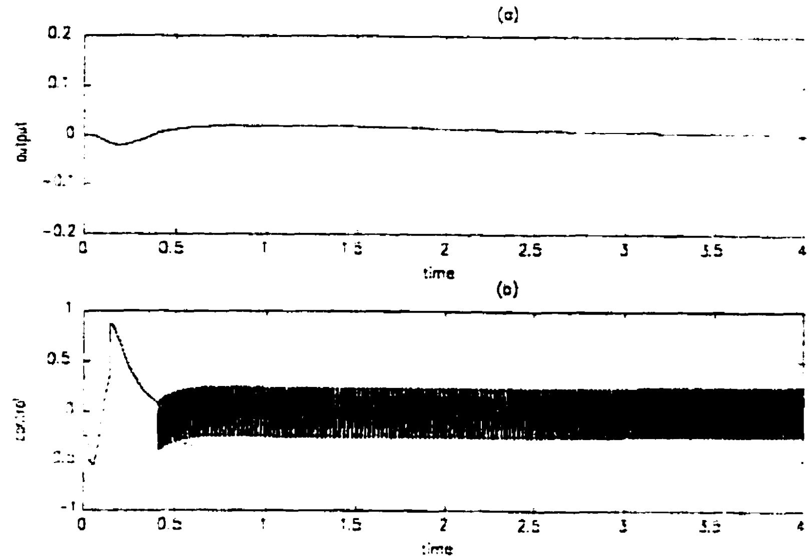

Figure 5.9 System Output (a) and Control Input (b) for the System–Zero Disturbances Present

Note that unlike the state-feedback simulations, the simulations in this section were carried out for only four seconds, as opposed to ten. The reason for this is twofold. First, the primary goal of this simulation was to examine the response of the observer. Due to the relatively fast response of the error dynamics in comparison to those of the state dynamics, it makes it a rather contrived effort to run the simulation for the entire ten seconds. Second, the output variable (i.e. the cart position) is not prone to a 'large' response in this particular situation. Thus, as can easily be seen in Fig. 5.9 and Fig. 5.11, it is hard to say anything of significance by examination of these plots alone.

To elaborate on the first point, Fig. 5.8 demonstrates the fast response of the observer. By examination of Fig. 5.8 (b), we see that the error term drops to zero almost immediately (about 0.3 seconds). As expected, we see that the control chatter has no effect on the error term.

As well, Fig. 5.9 demonstrates the gratifying result that the proposed controller

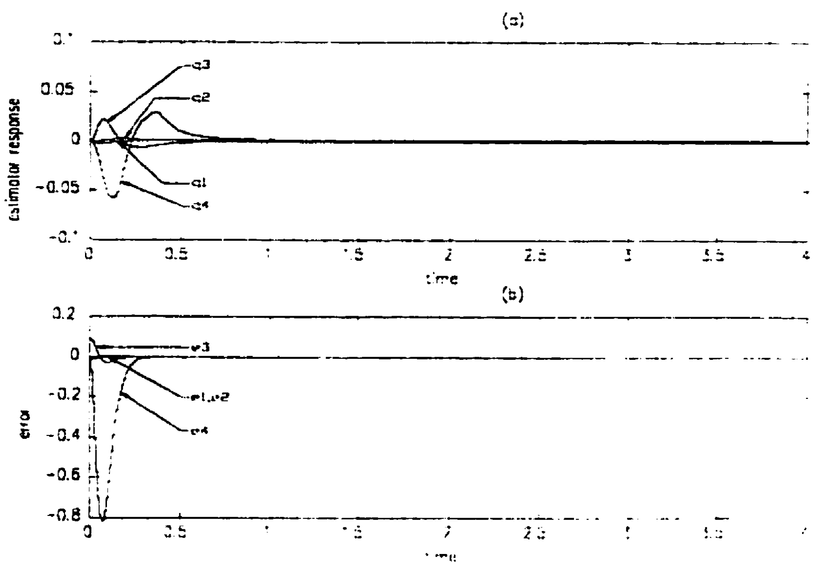

Figure 5.10 Estimator Response (a) and Error Dynamics (b) for the Simulated System–Matched Disturbances Present

not only works, but works in a similar manner to that of the conventional state feedback model. This result is greatly clarified in Fig. 5.12 (a), where the cart position is shown for the state and output feedback controllers. Here we see the expected result, in that the output feedback controller exhibits a lag in response, followed by a period of convergence to the state feedback result. That the lines do not completely intercept is an acceptable result in this situation, as this slow response can be attributed to the relatively slow closed loop eigenvalue of $(A + BK)$.

As well, by comparing the results of Fig. 5.9 (b) and Fig. 5.5 (b), we see that the control histories in both simulations are virtually identical once the error dynamics stabilize. Further, since the control does not 'leap' to some absurd value (for example, up to 100 for a short period, or some other impractical situation) in either case, it is reasonable to assume that the proposed controller constitutes a plausible strategy.

All that remains at this point is to examine the performance of the output feedback

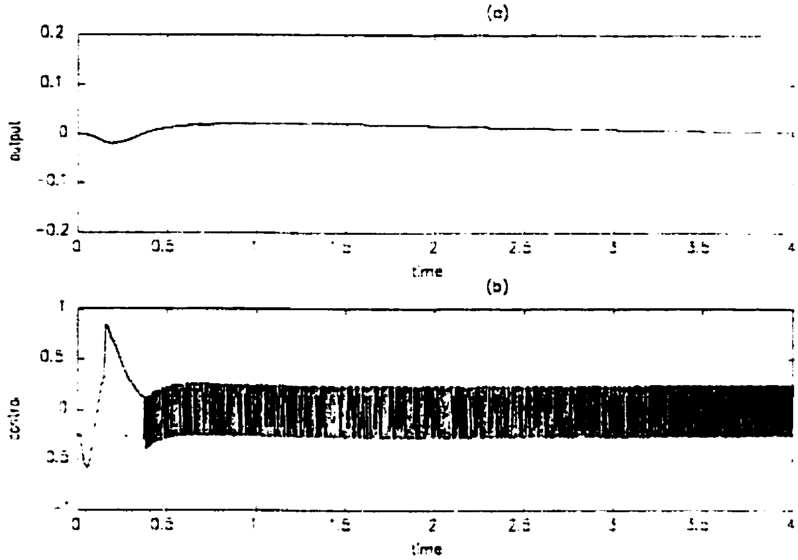

Figure 5.11 System Output (a) and Control Input (b) for the System–Matched Disturbances Present

controller in the presence of noise. By inspection, we see that Fig. 5.11 and Fig. 5.9 are virtually identical, once the control chattering begins. Inspection of Fig. 5.12 (b) further illustrates this result. Thus, the goal of perfect nominal performance in the presence of bounded, matched disturbances is achieved.

Further, comparison of Fig. 5.8 and Fig. 5.10 shows that the estimator and error dynamics are identical in each situation, indicating not only robustness in the closed loop (output feedback) system, but also in the estimator. This robustness is expected, however, due to the switching term as per (4.24).

5.4 Summary

In this chapter, the proposed controller was implemented on the now classic control

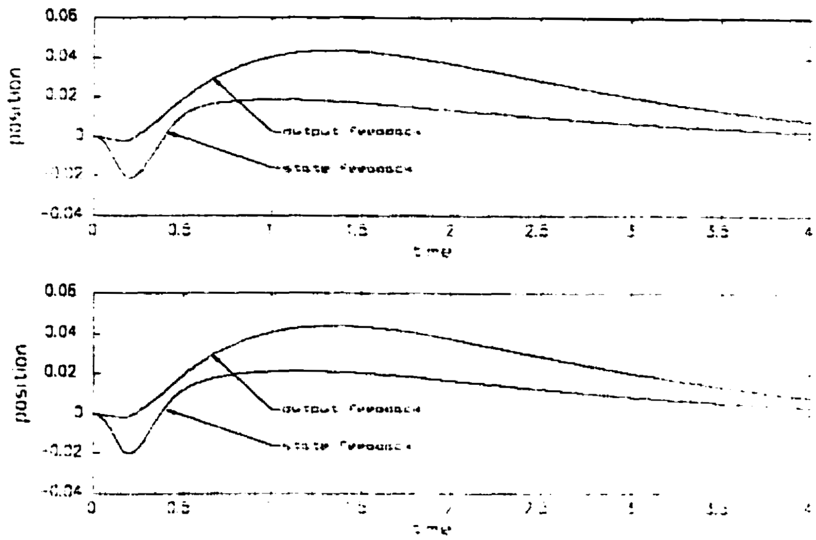

Figure 5.12 Comparison of Cart Positions Using State Feedback and Output Feedback. Zero Disturbances Present (a) and Bounded Matched Disturbances (b).

problem of an inverted pendulum, and the results were promising. That is, the controller was shown to be asymptotically stabilizing from an arbitrary, yet realistic, initial condition, and the resulting system was shown to have the expected robustness margins in both the state and output feedback systems.

Chapter 6

Summary and Recommendations

6.1 Summary

This thesis has provided a constructive algorithm for designing a mixed objective LQ / SLMC controller. The proposed controller was shown to exhibit the desired robustness properties of a conventional SLMC controller, while at the same time preserving near LQ performance. The development of this particular controller was made possible by exploiting a number of results developed within the thesis.

First, a specific similarity transform T was explicitly found in §4.2. Application of this transformation to a generic closed loop, state feedback system matrix $(A + BK)$ yielded the result (4.11). In turn, this allowed the control problem to be split into two parts. Specifically, this transformation allowed m elements of $\sigma(A + BK)$ to be arbitrarily determined by the designer, while the remaining $(n - m)$ elements could be located via some optimization algorithm. For the sake of simplicity, conventional LQ design techniques were employed in §4.2, but it was noted that more specialized methods could also be used.

This result allowed the solution of a feedback law $K : \mathcal{U} \rightarrow \mathcal{X}$ that placed the elements of $\sigma(A + BK)$ in the mixed objective manner outlined above. In turn, the class of systems to which the proposed algorithm could successfully be applied to was determined. As expected, Theorem 4.1 demonstrated that the pair (A, B) needs to be controllable, as well as $\operatorname{rank}(B) = m$.

Next, a Lyapunov design technique was employed in §4.3 that allowed us to solve the problem of mixing LQ and SLMC design objectives. In particular, it was found that the objectives could be mixed if $\sigma(A+BK)$ contained at least m distinct, real elements. With this, a sliding manifold S could then be designed that does not have the usual effect of eliminating m elements of $\sigma(A+BK)$, as per Theorem 3.1.

So, by employing the results of §4.2, the main result (4.24) of the thesis was solved. The resulting system was then shown to have the desired robustness abilities by selecting the gain matrix F in (4.22) appropriately.

The result was then extended to the output feedback case in §4.4. In this section, a full-order observer was employed that was able to recover the results of the full state feedback case. More significantly, the result (4.67) showed that the proposed output feedback controller exhibits robustness to bounded, matched disturbances. Thus, this thesis has provided a significant result in that it has managed to contribute a practical, robust, output feedback controller for a certain class of linear systems.

The algorithm was then demonstrated on a physically motivated example in §5, and the results were generally promising. That is, the resulting obtained by application of the proposed controller were shown to be reasonably similar to those obtained by LQ design methods, with the added feature of robustness to bounded, matched disturbances.

As well, the output feedback controller was also simulated with the proposed full order observer. The results indicated that the objective of designing a robust, output feedback controller was achieved.

In addition to the above mentioned results, this thesis has also provided a reasonable exposition on the development of sliding mode control, as well as the mathematical tools necessary for the analysis.

6.2 Recommendations

One of the main result in this thesis was the selection of the matrix G in (4.22). A feasible controller was constructed by selecting

$$\sigma(G) = -\sigma(M)$$

making it possible to construct a full rank S, which then defined the sliding manifold $\mathcal{S} = \{x : Sx = 0\}$. This was fine.

The problem, however, is the simplifying Assumption 4.1. Here it was required that all m elements of $\sigma(M)$ were chosen to be distinct, making it simple to non-trivially solve (4.29). It is the belief of the author that this assumption should not be necessary. In fact, the same result may possibly be achieved by performing a slight variation on the method proposed in §4.

Explicitly, if the matrix M in (4.11) were made to be a scalar, say $-\gamma$, so that $A_{11} \in \mathbb{C}^{(n-1)\times(n-1)}$, or

$$Acl \approx \begin{bmatrix} A_{11} & A_{12} \\ 0 & -\gamma \end{bmatrix} \tag{6.1}$$

Then, it should be possible to optimize the resulting $(n-1)$ dimensional remaining subsystem, rather than an $(n-m)$ dimensional one. This would likely lead to results that are closer to a fully optimized system than the method presented in this work, while still achieving the result of invariance to matched disturbances. G could then be selected as

$$G = \begin{bmatrix} \gamma & & \\ & \ddots & \\ & & \gamma \end{bmatrix} \in \mathbb{R}^{m\times m}$$

which *should* lead to $\dim(\ker H) \geqslant m$ in (4.29).

Of course, some problems exist with this proposition.

First, a generalized transformation T that will accomplish the result (6.1) is not known at this time. Further, the size of $\dim(\ker H)$ in (4.29) will no longer be obvious, as can be seen in [29]. That is, the result of $\dim(\ker H) \geqslant m$ in (4.29) may not be achieved.

These issues are, however, presented in the hope of possibly kindling future research interest.

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Appendix A
Derivation of The Inverted Pendulum System

Recall again the inverted pendulum system illustrated in Fig. 5.1, whose motion is described by (5.1). To generate a linearized model of (5.1) of the form (2.77), (2.78), recall that

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \\ \theta \\ \dot{\theta} \end{bmatrix} \tag{A.1}$$

so that (5.1) becomes

$$\begin{bmatrix} m_o L\cos z_3 & I_p \\ m_t & m_o L\cos z_3 \end{bmatrix} \begin{bmatrix} \dot{z}_2 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} m_o L\cos z_3 & I_p \\ m_t & m_o L z_4 \sin z_3 \end{bmatrix} \begin{bmatrix} z_2 \\ z_4 \end{bmatrix} + \begin{bmatrix} m_o g L \sin z_3 \\ u(t) \end{bmatrix} \tag{A.2}$$

which leads to

$$\begin{bmatrix} \dot{z}_2 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} f_2(z_1, z_3, z_3, z_4, u; t) \\ f_4(z_1, z_3, z_3, z_4, u; t) \end{bmatrix} \tag{A.3}$$

So that, when viewed with $(A.1)$, the system can be stated as

$$\dot{z} = F(z, u; t) \tag{A.4}$$

where $\dot{z}_1 = z_2$, and $\dot{z}_3 = z_4$. A standard method of generating a linear approximation of $(A.4)$ is to set

$$A = \left[\left. \frac{\partial f_i}{\partial z_j} \right|_{\substack{x=x_{eq} \\ u=u_{eq}}} \right]_{i,j=1}^{n} \tag{A.5}$$

$$B = \left[\left. \frac{\partial f_i}{\partial u_k} \right|_{\substack{x=x_{eq} \\ u=u_{eq}}} \right]_{i,k=1}^{n,m} \tag{A.6}$$

So, via some tedious but straightforward calculations, the application of $(A.5)$, $(A.6)$ to $(A.4)$ leads to

$$\acute{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{I_p b}{Y} & \frac{(m_o L)^2 g}{Y} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-m_o L b}{Y} & \frac{m_t m_o L g}{Y} & 0 \end{bmatrix} \quad (A.7)$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \alpha_{22} & \alpha_{23} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \alpha_{42} & \alpha_{43} & 0 \end{bmatrix} \quad (A.8)$$

and

$$\acute{B} = \begin{bmatrix} 0 \\ \frac{-I_p}{Y} \\ 0 \\ \frac{-m_o L}{Y} \end{bmatrix} \quad (A.9)$$

$$= \begin{bmatrix} 0 \\ \beta_2 \\ 0 \\ \beta_4 \end{bmatrix} \quad (A.10)$$

where

$$Y = (m_o L)^2 - I_p m_t$$

Further, since only x is available for direct measurement, there results

$$\acute{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \quad (A.11)$$

and the system becomes

$$\dot{z} = \acute{A}z + \acute{B}u \quad (A.12)$$

$$y = \acute{C}z \quad (A.13)$$

and the triple $\left(\acute{A}, \acute{B}, \acute{C}\right)$ is complete, as can be verified by application (2.83), (2.92). Applying now a particular nonsingular transformation T_o to $(A.12)$, $(A.13)$ gives the triple (A, B, C) of (5.8), (5.9), (5.10). In particular, T_o is found via the following steps.

1. Calculate the *Transfer Function* of the system in the Laplace domain as

$$\begin{aligned} G(s) &= \acute{C}\left(sI_4 - \acute{A}\right)^{-1}\acute{B} \\ &= \frac{\beta_1 s^2 + (\alpha_{23}\beta_2 - \alpha_{43}\beta_1)}{s^4 - \alpha_{22}s^3 - \alpha_{43}s^2 - (\alpha_{23}\alpha_{42} - \alpha_{22}\alpha_{43})\, s} \end{aligned} \tag{A.14}$$

   allowing immediate construction of the triple (A, B, C), as this triple is in controllable canonical form.

2. Construct

$$W_1 = \left[\begin{array}{cccc} \acute{B} & \acute{A}\acute{B} & \acute{A}^2\acute{B} & \acute{A}^3\acute{B} \end{array}\right] \tag{A.15}$$

   which will be nonsingular, since $\left(\acute{A}, \acute{B}\right)$ is controllable.

3. Construct

$$\begin{aligned} W_2 &= \left[\begin{array}{cccc} B & AB & A^2B & A^3B \end{array}\right] \\ &= T_o\left[\begin{array}{cccc} \acute{B} & \acute{A}\acute{B} & \acute{A}^2\acute{B} & \acute{A}^3\acute{B} \end{array}\right] \\ &= T_oW_1 \end{aligned}$$

   which will also be nonsingular.

4. Solve

$$T_o = W_2W_1^{-1} \tag{A.16}$$

Now, via substitution of the numerical values given in §5.1, the system becomes

$$\frac{d}{dt}\hat{z} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -120.29 & -20.91 & -3.88 \end{bmatrix} \hat{z} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \qquad (A.17)$$

$$\hat{y} = \begin{bmatrix} 0 & 33.05 & 4.26. & 0 \end{bmatrix} \hat{z} \qquad (A.18)$$