THE UNIVERSITY OF CALGARY

Existence, Uniqueness and Numerical Realization of Solutions for Thermistor Equations

by

Shijun Zhou

A DISSERTATION

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF

MATHEMATICS AND STATISTICS

CALGARY, ALBERTA

October, 1993

ISBN 0-315-94020-4

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- Entomologie 0353
- Génétique 0369
- Limnologie 0793
- Microbiologie 0410
- Neurologie 0317
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- Chimie organique 0490
- Chimie pharmaceutique 0491
- Physique 0494
- PolymÇres 0495
- Radiation 0754

Mathématiques 0405
Physique
- Généralités 0605
- Acoustique 0986
- Astronomie et astrophysique 0606
- Electronique et électricité 0607
- Fluides et plasma 0759
- Météorologie 0608
- Optique 0752
- Particules (Physique nucléaire) 0798
- Physique atomique 0748
- Physique de l'état solide 0611
- Physique moléculaire 0609
- Physique nucléaire 0610
- Radiation 0756

Statistiques 0463

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a dissertation entitled "Existence, Uniqueness and Numerical Realization of Solutions for Thermistor Equations" submitted by Shijun Zhou in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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ABSTRACT

A coupled pair of nonlinear parabolic partial differential equations, which describe two dimensional heat and current distribution within a thermistor, is reviewed and discussed. For certain boundary conditions two nonlinear parabolic one (space) dimensional differential equations and their corresponding steady state differential equations with different properties are derived from these equations. As the two kinds of partial differential equations have different properties, which are called PTC and NTC respectively, they are discussed separately. Several different methods are used for the theoretical study of the equations. One method is to transform the differential equations into integral equations, through the proof of existence and uniqueness of integral equations, hence existence and uniqueness are obtained. Another method is to change the variables and to change the original boundary value problems to initial value problems. By proving existence and uniqueness for the initial value problems, existence and uniqueness are obtained again. The third one is a monotone method. By using the concept of upper and lower solution, existence and uniqueness (if applicable) are proved. Since the property of the Joule heating function for NTC problems is quite different from that for PTC problems, only existence is obtained and under some special meaning a uniqueness is also obtained. Many numerical experiments have been done. Numerical results are listed in tables and demonstrated by figures. The common property for time dependent problems with an external circuit is that the solutions have a surge. For some unstable solutions, a brief stability analysis is given.

Acknowledgements

I would like to express deepest appreciation to my supervisor Dr. D.R. Westbrook: for his encouraging me to research on the present direction; for his keeping me on the right path and then seriously considering my results and checking every detail, including my Fortran programs and English; for his unfailing support all the time and help in every small aspect; for his giving me the push needed to finish a dissertation; for his financial assistance from an NSERC operating grant and his confidence in me. Without his support, the dissertation would not have been possible.

Thanks to D.R. Westbrook, R.S. Dhaliwal, B.D. Aggarwala, L.B. Bos and C. Nasim. When I attended their lectures, I learnt lots of new materials from them. Thanks to B.D. Aggarwala for his partial financial support from an NSERC operating grant. Thanks to those who let me talk to them about this work, and to all those who gave other help or encouragement.

I also would like to thank the Department of Mathematics and Statistics and the Faculty of Graduate Studies for both the financial and emotional support which enabled me to thrive as a graduate student.

Thanks to my wife for her support during my survival as a graduate student in the University of Calgary.

CONTENTS

List of Tables

List of Figures

Chapter 1

Introduction

In this dissertation, attention is focused on a system of nonlinear parabolic (heat diffusion) differential equations which model the thermistor, an electrical device which is widely used for surge protection, etc. A review of results for the problem is given in chapter two. By considering different boundary conditions two different one (space) dimensional problems are derived. One problem is called PTC problem and the other the NTC problem. This dissertation is devoted to both numerical solutions and existence of theoretical solutions of these problems as they are related to thermistors.

In recent years an active branch of numerical solutions for nonlinear partial differential equations is in the solution of problems with moving boundaries. To solve such problems, a moving finite element method [23, 24, 25, 26, 27, 28] may be used. According to M.J. Baines [27, 28], this method was invented by K. Miller [25] who used it to approximate the solution of diffusion problems with steep moving fronts. The main idea is that the meshpoints are allowed to move according to the conditions at the interface points (or curve). The governing equations at the moving meshpoints are adjusted and the meshsize changes correspondingly. One possible application of moving boundary problems is to the problem of the thermistor.

A typical mathematical description of heat distribution within the thermistor is given by nonlinear parabolic equations. Its corresponding steady state equation is a nonlinear elliptic equation. The boundary conditions considered here are Robin boundary conditions.

X. Chen and A. Friedman [37, 38, 39], by using the concept of weak solution, proved the existence of a solution for this problem. They also use a conformal mapping method to prove the existence and uniqueness. H. Xie and W. Allegretto [40, 41] discussed the existence of solutions under certain assumptions in which $\sigma(u)$ and $k(u)$ take some special forms. D.R. Westbrook [1] gave some numerical methods for obtaining approximate solutions of some steady state problems. More recently A.C. Fowler, I. Frigaard and S.D. Howison[3] have used perturbation and numerical methods to examine a one dimensional time dependent problem which is a special case of the general problem (this is the problem which is here called the NTC problem).

Here we also concentrate on one dimensional equations. Numerical experiments are described and some theoretical proofs of existence and uniqueness of solutions have been studied.

In § 3.1, § 3.2 and § 3.3, steady state problems have been considered. They are divided into two cases, NTC (negative temperature characteristic) and PTC (positive temperature characteristic, both to be defined in § 2.1), which are quite different. The PTC problem has a unique solution for given parameters and boundary conditions but the NTC problem may have more than one solution for some given parameters and boundary conditions. Therefore, different methods are needed to prove the existence (also uniqueness if applicable). We define an operator to transform the differential equations into integral equations and then prove the equivalence of the differential and integral equations. Though the operator itself is not a contraction operator, it can be proved that a power of that operator is a contraction operator. Therefore, by using the fixed point principle, there is a fixed point for that operator. As has been mentioned, there are more than one solution for NTC, so that uniqueness is only proved for the PTC problem. Although the uniqueness is lost for NTC problems, the uniqueness is still true in some other sense (refer § 3.3). The existence, uniqueness and nonuniqueness have also been studied both for NTC and PTC problems when an external circuit is connected.

In chapter 4, many numerical experiments have been done. For the steady state situation, numerical results coincide very well with the theoretical ones. Approximate relations of parameters are also given for one, two and three solutions. For time dependent problems, two methods are used. One is a moving mesh finite element method, the other is for fixed meshpoints. Numerical solutions by both methods converge to the numerical solutions of the steady state problems. If external circuitry is connected, for some parameters, there are more than one steady state solution, one of which is numerically unstable. The details and numerical experiments are given in chapter 4. The comparison of numerical results obtained by both methods has been made. Several figures have been given to demonstrate the results.

In chapter 5, a brief review of stability and instability is given. The analysis of stability and instability is based on perturbation theory. It was found that all the steady state solutions could be obtained numerically, thus theoretically, the corresponding solutions of time dependent problems should converge to one of the solutions of the steady state problems. However, the numerical experiments show that for some steady state solutions there are no corresponding solutions of time dependent problems which converge to these steady state solutions. Hence, there is a need to discuss the stability of these solutions. The method is to investigate the solutions for time dependent problems close to the corresponding solutions for steady state problems. If some increase rapidly in the neighborhood of solutions of the steady state problems, the corresponding solutions for time dependent problems are said to be unstable, otherwise they are stable. As the exact solutions are not explicit expressions, numerical estimates of the tendency in the neighborhood of the solutions are obtained. The numerical results coincide with and demonstrate the conclusion about the stability and instability.

Chapter 2

Preliminaries and Historical Remarks

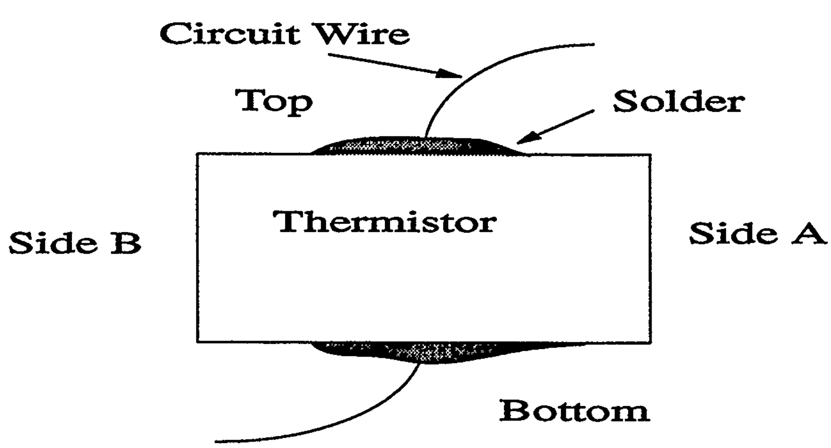
In this chapter, we review the basic idea of the thermistor problem. In § 2.1, the basic definitions and conventions are given. From the same basic equation, two different kinds of one dimensional equation are derived under certain boundary conditions. One has the PTC property and the other has NTC property. Hence thereafter, one is called the PTC problem and the other is called the NTC problem. Most of the recent results have been listed in § 2.2. It can be seen that the most of them require special conditions. For time dependent problems, the nonlinear Joule heating terms are always assumed to be monotone, smooth with bounded derivatives for existence and uniqueness of solutions. A schematic representation of the thermistor is given in § 2.3.

2.1 Basic Definitions and Conventions

The equations here describe heat and current distribution within a thermistor. The thermistor is an electrical device made of ceramic material that can be used as a current surge regulator[1]. In appearance [3], this is a cylinder of typical radius 5mm and typical thickness 2mm, connected into its circuit via wires soldered to the top and bottom; these surfaces are covered with a thin conducting sheet of metal acting as contact. (Figure 2.2.1 is a schematic view of the thermistor). The basis for its performance is its temperature-dependent electrical resistivity, which varies strongly with temperature, increasing by about five orders of magnitude over a temperature range of 100-200$^{\circ}C$ (see Figure 2.2.2).

There are two kinds of thermistors:[3] negative temperature characteristic (NTC) thermistors, whose electrical conductivity σ increases with temperature, and positive temperature characteristic (PTC) thermistors, for which σ decreases with temperature. (Here positive and negative refer to materials whose resistivity is an increasing or nonincreasing function of temperature respectively; conductivity is the inverse of resistivity.)

Figure 2.2.1



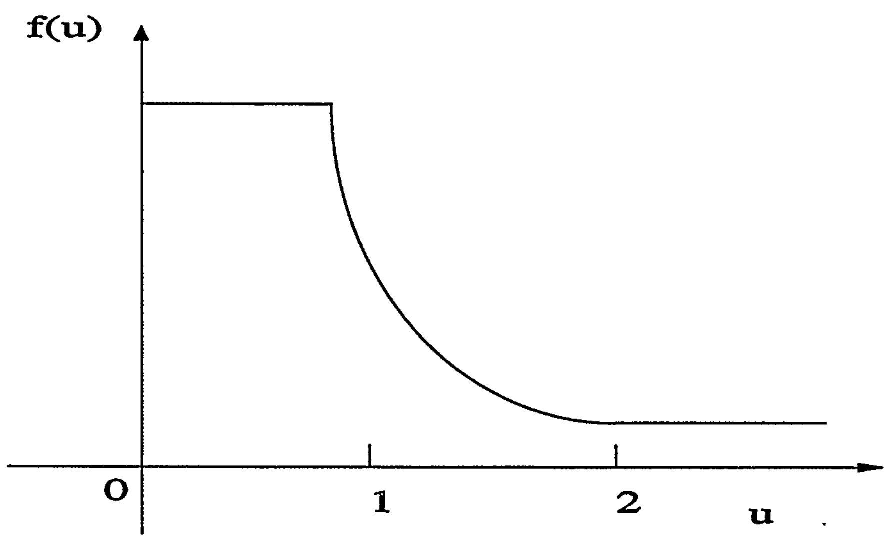
A Thermistor

The general equations to describe the heat distribution within the thermistor, after some scaling of variables, are as follows (in two dimensions):

$$\begin{cases} \nabla(\sigma(u)\nabla\phi) = 0 & ,-a < x < a, \quad -b < y < b \\ \frac{\partial u}{\partial t} = \nabla(k(u)\nabla u) + \gamma\sigma(u)|\nabla\phi|^2 & , \frac{\partial u}{\partial n} + \beta u|_{\partial\Omega} = 0 \end{cases} \tag{2.1}$$

This is a time dependent problem with the two dimensional geometry taken as Cartesian rather than axisymmetric (It is felt that this will not lead to any qualitative

Figure 2.2.2



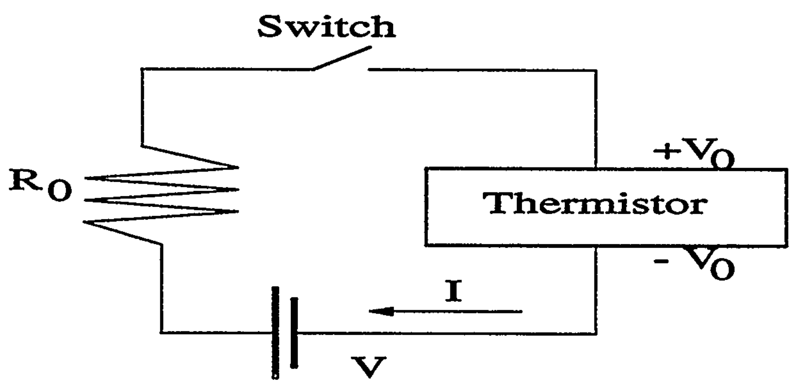
Conductivity as A Function of Temperature

differences in the results). The corresponding steady state problem is as following:

$$\begin{cases} \nabla(\sigma(u)\nabla\phi) = 0 & ,-a < x < a, \ -b < y < b \\ \nabla(k(u)\nabla u) + \gamma\sigma(u)|\nabla\phi|^2 = 0 & , \frac{\partial u}{\partial n} + \beta u|_{\partial\Omega} = 0 \end{cases} \tag{2.2}$$

where ∇ is a gradient operator, Δ is Laplacian operator, $\Omega = [-a, a] \times [-b, b]$, u is a scaled temperature, ϕ is a scaled electrical potential, γ and β are dimensionless

Figure 2.2.3



Thermistor With External Circuit

parameters, $k(u)$ the thermal conductivity is assumed to be constant and the electrical conductivity $\sigma(u)$ is a function of u which changes rapidly between $u = 1$ and $u = 2$ and is a constant when $u < 1$ or $u > 2$. The Figure 2.2.2 is a graph of such a function. Without loss of generality, it can be assumed that b and $k(u)$ are equal to 1. The boundary conditions for the potential ϕ are

$$\phi = \pm v_0 \quad \text{on} \quad y = \pm 1 \tag{2.3}$$

and

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x} = 0 \quad \text{on} \quad x = \pm a \tag{2.4}$$

where v_0 is an unknown constant.

Now consider the case where $\beta = 0$ on $x = a$ (i.e., on side A of Figure 2.2.1). Since $\frac{\partial u}{\partial x} = 0$ on $x = a$, it is possible for u to be a function of y only. Similarly, $\frac{\partial \phi}{\partial x} = 0$ on $x = a$ and ϕ may also be a function of y only. Thus from the first equation of Eq.(2.1) (or (2.2)),

$$\frac{\partial}{\partial y}(\sigma(u)\frac{\partial \phi}{\partial y}) = 0$$

Since $\sigma(u) \not\equiv 0$

$$\frac{\partial \phi}{\partial y} = \frac{C}{\sigma(u)}$$

where C is a constant. Since the average current I is defined by

$$I = \frac{1}{a}\int_0^a \sigma(u)\frac{\partial \phi}{\partial y}|_{y=1} dx = \frac{1}{a}\int_0^a C dx = C$$

Thus $\frac{\partial \phi}{\partial y} = \frac{I}{\sigma(u)}$. By symmetry, $\frac{\partial u}{\partial y} = 0$ on $y = 0$. So Eq.(2.1) and Eq.(2.2) become

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + \frac{\gamma I^2}{\sigma(u)} & ,0 < y < 1 \\ u_y(0) = 0 & ,u_y(1) + \beta u(1) = 0 \end{cases} \tag{2.5}$$

and for the steady state

$$\begin{cases} \frac{d^2 u}{dy^2} + \frac{\gamma I^2}{\sigma(u)} = 0 & ,0 < y < 1 \\ u_y(0) = 0 & ,u_y(1) + \beta u(1) = 0 \end{cases} \tag{2.6}$$

This problem is hereafter labelled NTC.

Now consider the case where $\beta = 0$ on $y = 1$ (i.e., on the top of Figure 2.2.1). Since $\frac{\partial u}{\partial y} = 0$ on $y = 1$, the possible solution is that u is only dependent on x, therefore, $\sigma(u)$ is only dependent upon x while ϕ is still a function of y only. From the first equation of Eq.(2.1) (or Eq.(2.2))

$$\frac{\partial}{\partial y}(\sigma(u)\frac{\partial \phi}{\partial y}) = 0$$

i.e.,

$$\sigma(u)\frac{d^2\phi}{dy^2} = 0$$

Since $\sigma(u) \not\equiv 0$

$$\frac{d\phi}{dy} = C$$

where C is constant. From (2.3), it is easy to see that $\phi = v_0 y$, i.e., $C = v_0$. By symmetry, $\frac{\partial u}{\partial x} = 0$ on $x = 0$. So Eq.(2.1) and Eq.(2.2) become

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \gamma v_0^2 \sigma(u) & ,0 < x < a \\ u_x(0) = 0 & ,u_x(a) + \beta u(a) = 0 \end{cases} \tag{2.7}$$

and

$$\begin{cases} \frac{d^2 u}{dx^2} + \gamma v_0^2 \sigma(u) = 0 & ,0 < x < a \\ u_x(0) = 0 & ,u_x(a) + \beta u(a) = 0 \end{cases} \tag{2.8}$$

Having separated the two cases there is no loss in taking $a = 1$. This case will be labelled the PTC problem.

For Eq.(2.5), Eq.(2.6), Eq.(2.7) and Eq.(2.8), the external circuit is not considered. If the external circuit is connected as in Figure 2.2.3, then

$$V = 2V_0 + IR_0$$

Rescale it as

$$1 = v_0 + I\mu \tag{2.9}$$

where $v_0 = \frac{2V_0}{V}$ (unknown) and $\mu = \frac{R_0}{V}$ (parameter). For Eq.(2.5) and Eq.(2.6), there is

$$v_0 = \phi(1) = \int_0^1 \frac{\partial \phi}{\partial y} dy = \int_0^1 \frac{I}{\sigma(u)} dy = I \int_0^1 \frac{1}{\sigma(u)} dy$$

From (2.9), there is

$$1 = I \int_0^1 \frac{1}{\sigma(u)} dy + I\mu$$

thus

$$I = \frac{1}{\mu + \int_0^1 \frac{1}{\sigma(u)} dy}$$

Therefore, Eq.(2.5) and Eq.(2.6) become

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial y^2} + \frac{\gamma}{(\mu + \int_0^1 \frac{1}{\sigma(u)} dy)^2 \sigma(u)} & ,0 < y < 1 \\ u_y(0) = 0 & ,u_y(1) + \beta u(1) = 0 \end{cases} \tag{2.10}$$

and

$$\begin{cases} \frac{d^2 u}{dy^2} + \frac{\gamma}{(\mu + \int_0^1 \frac{1}{\sigma(u)} dy)^2 \sigma(u)} = 0 & ,0 < y < 1 \\ u_y(0) = 0 & ,u_y(1) + \beta u(1) = 0 \end{cases} \tag{2.11}$$

For Eq.(2.7) and Eq.(2.8), there is $\phi = v_0 y$, hence the current is

$$I = \int_0^1 \sigma(u) \frac{\partial \phi}{\partial y} |_{y=1} dx = v_0 \int_0^1 \sigma(u) dx$$

Similarly, from (2.9), there is

$$1 = v_0 + \mu v_0 \int_0^1 \sigma(u) dx$$

hence

$$v_0 = \frac{1}{1 + \mu \int_0^1 \sigma(u) dx}$$

Therefore, Eq.(2.7) and Eq.(2.8) become

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\gamma\sigma(u)}{(1+\mu\int_0^1 \sigma(u)dx)^2} & ,0 < x < 1 \\ u_x(0) = 0 & ,u_x(1) + \beta u(1) = 0 \end{cases} \tag{2.12}$$

and

$$\begin{cases} \frac{d^2 u}{dx^2} + \frac{\gamma\sigma(u)}{(1+\mu\int_0^1 \sigma(u)dx)^2} = 0 & ,0 < x < 1 \\ u_x(0) = 0 & ,u_x(1) + \beta u(1) = 0 \end{cases} \tag{2.13}$$

In most of this dissertation, $\sigma(u)$ is defined as

$$\sigma(u) = \begin{cases} 1 & ,u < 1 \\ e^{-10(u-1)} & ,1 \leq u \leq 2 \\ e^{-10} & ,2 < u \end{cases} \tag{2.14}$$

This follows the definition used by Fowler et al [3] and has the appropriate behavior. Thus by the definition of $\sigma(u)$ Eq.(2.5), Eq.(2.6), Eq.(2.10) and Eq.(2.11) have NTC property, Eq.(2.7), Eq.(2.8), Eq.(2.12) and Eq.(2.13) have the PTC property. Hereafter they are referred as PTC or NTC problems respectively. The difference for these two problems is that the nonlinear term in one is the reciprocal of that in the other. Therefore, the nonlinear function is monotonically nonincreasing for the PTC problem and the nonlinear function is monotonically increasing for the NTC problem. This difference leads to quite different properties for the solution of these problems.

2.2 Development and Results

Several papers have been written in this area. Most considered the steady state equations (in two dimensions) as follows

$$\begin{cases} \nabla(\sigma(u)\nabla\phi) = 0 & ,-a < x < a,\ -b < y < b \\ \nabla(k(u)\nabla u) + \alpha\sigma(u)|\nabla\phi|^2 = 0 & ,\frac{\partial u}{\partial n} + \beta u|_{\partial\Omega} = 0, \phi|_{y=\pm b} = \pm\phi_0, \frac{\partial\phi}{\partial n}|_{x=\pm a} = 0 \end{cases} \tag{2.15}$$

where Δ is Laplacian operator, $\Delta = \nabla^2$, $\Omega = [-a, a] \times [-b, b]$, u is temperature, ϕ is electrical potential, α and β are dimensionless parameters, $k(u)$ is thermal conductivity and electrical conductivity $\sigma(u)$ is defined by

$$\sigma(u) = \begin{cases} 1 & ,u \leq 1 \\ \delta & ,u > 1 \end{cases} \tag{2.16}$$

where δ is very small real positive number. For time dependent problems,

$$\begin{cases} \nabla(\sigma(u)\nabla\phi) = 0 & ,-a < x < a & , -b < y < b \\ \frac{\partial u}{\partial t} = \nabla(k(u)\nabla u) + \alpha\sigma(u)|\nabla\phi|^2 & ,\frac{\partial u}{\partial n} + \beta u|_{\partial\Omega} = 0 & ,\phi|_{y=\pm b} = \pm\phi_0, \frac{\partial\phi}{\partial n}|_{x=\pm a} = 0 \end{cases} \tag{2.17}$$

where $t > 0$.

For the steady state, X. Chen and A. Friedman [37, 38, 39], G. Cimatti [35, 36], H. Xie and W. Allegretto [41], etc., use the transformation

$$\theta = \frac{1}{2}\phi^2 + \int_{u_0}^{u} \frac{k(s)}{\sigma(s)} ds \tag{2.18}$$

(where ϕ is electrical potential, u is temperature, k and σ are thermal and electrical conductivities, u_0 is a constant) and give existence and uniqueness proofs for the problem.

If we denote

$$F(u) = \int_{u_0}^{u} \frac{k(s)}{\sigma(s)} ds$$

G. Cimatti, in [35, 36], proved that, if $F(u)$ is bounded as $u \longrightarrow \infty$, there are two cases, i.e., either there is only one solution for some boundary condition or there is no solution at all for other boundary conditions. Also, in [35], G. Cimatti proved that, if some more restrictions (e.g., bounded, continuous) on $k(s)$ and $\sigma(s)$ are given, the solution for Eq.(2.15) is bounded.

In [37], X. Chen and A. Friedman discussed the case in which

$$\sigma(u) \begin{cases} > 0 & , u \leq u^* \\ = 0 & , u > u^* \end{cases} \tag{2.19}$$

where u^* is critical temperature and $\sigma(u)$ is continuous at u^*. They construct a continuous and infinitely differentiable function $\sigma_\epsilon(u)$ to approximate $\sigma(u)$. Then using a transformation similar to (2.18), they review the existence proof of a solution $(\phi_\epsilon,\ u_\epsilon)$. After introducing the concept of a weak solution, it is proved that $(\phi_\epsilon,\ u_\epsilon)$ converges to a weak solution $(\phi,\ u)$ of Eq.(2.15). and $u \leq u^*$. Also, in [38], a more specific situation, i.e.,

$$\sigma(u) = \begin{cases} 1 & , u \leq u^* \\ 0 & , u > u^* \end{cases}$$

is considered. A transformation similar to (2.18) and a conformal mapping are used to prove the existence and uniqueness.

H. Xie and W. Allegretto [41], instead of discussing the Robin boundary condition in Eq.(2.15), discuss the Dirichlet boundary value problem. Under the hypothesis of that

$$\sigma(u) = C_1 u^\iota \exp(-C_2/C_3 u)$$

and

$$k(u) = (C_4 + C_5 u + C_6 u^2)^{-1}$$

where C_1, C_2, C_3, C_4, C_5 and C_6 are physical positive constants and ι is a small positive number or non positive number, they proved that there exists at least one solution to Eq.(2.15).

For the time dependent problem (2.17), there is not as much literature as there is for steady state problem. H. Xie and W. Allegretto [40], F.B. Weissler [33], H. Fujita [34] have done some research in the area of classic nonlinear parabolic equations similar to Eq.(2.17). However, F.B. Weissler and H. Fujita mainly contributed to the blow up problem, though the form of equations is the same, the properties of the nonlinear part are quite different. Here we put the emphasis on the case where the equations describe the heat distribution within thermistors. In [40], H. Xie and W. Allegretto assumed that

$$\sigma(u) = C_1 u^{-\iota} \exp(-C_2/u), \quad 0 < \iota < 1, \quad C_1, C_2 > 0$$

and

$$k(u) = (C_3 + C_4 u + C_5 u^2)^{-1}, \quad C_3, C_4, C_5 \geq 0, \quad C_3 + C_4 + C_5 > 0$$

Additionally, they replaced the first equation of Eq.(2.17) by the simple equation $(\sigma(u)\phi_{x_1})_{x_1} = 0$, i.e., the derivative is taken with respect to only one spatial variable and assumed $\sigma(s)$ and $k(s)$ locally Lipschitz continuous and bounded. Then they integrate the first equation of Eq.(2.17), change Eq.(2.17) into an integro-differential equation. After introducing several functional spaces and a variational form, an operator is determined by the variational form. By proving that there exists at least one fixed point for that operator, then the existence of a solution has been proved. For Eq.(2.17), it seems that it is quite difficult to obtain a general conclusion.

All of the above results are theoretical ones. For numerical methods to get numerical solutions for Eq.(2.15) and Eq.(2.17), not many papers can be found. D.R. Westbrook [1] gave some good ideas for the numerical methods. A new function is constructed and then an iterative method is used. The method overcomes the difficulty to determine the point at which the temperature $u = 1$ (after scaling) and lots of numerical experiments are done for different parameters α and β. Also, A.C. Fowler, I. Frigaard and S.D. Howison [3] gave a perturbation analysis and a numerical method for the one dimensional time dependent NTC problem. They use equal stepsize for spatial direction and different time stepsize for time direction. From the numerical results, temperature surges appeared at some time point. They also noticed that there exist possible multiple solutions.

In order to obtain numerical solutions for equations similar to Eq.(2.17), the possibility of a moving finite element method [20, 21, 23, 24, 25, 26, 27, 28] is also raised. In this method, the nodes are allowed to change positions with time. The spatial stepsizes are adjusted at every time step. The method is more accurate in some cases.

2.3 Application

There are many applications of thermistors. One of them is as a fuse. In the circuit of Figure 2.2.3, a short circuit is represented by closing the switch, causing a current surge driven by the external voltage V to pass through the circuit resistance R_0 and the thermistor, thereby heating it. The consequent decrease in the electrical conductivity causes the current to fall until equilibrium is reached, with all the heat generated within the thermistor being lost to its surroundings. In a well-designed thermistor, the final current should be a small fraction of the initial surge.

Chapter 3

Existence and Uniqueness

In this chapter existence and uniqueness are discussed. For convenience the steady state problems are discussed first and then the time dependent problems. In § 3.1, the steady state problem for PTC thermistor has been studied. Before the proof of the existence and uniqueness, the properties of the solutions are thoroughly discussed. An integral operator is introduced so that the original differential equation is transformed into an equivalent operator equation. Since the operator equation has a solution, thus the existence of solution for the differential equation is proved. Using the nonincreasing property of $f(u)$, uniqueness is obtained. In § 3.2, the NTC problem has been discussed. The method is similar to that used in § 3.1. Since in this case, $g(u)$ is not a nonincreasing function, uniqueness is lost. In fact, there are three solutions for some parameters α and β. In § 3.3, a different method is used to prove the existence of solutions. The original boundary problems have been changed into initial value problems. Since the solutions for the initial value problem exist, then the existence of solutions for the boundary value problems is obtained. Here only the monotonic property is used, thus the method is good for both PTC and NTC problems. Though the uniqueness of the original boundary value problem is lost, it can be still proved that if initial value u_0 and β are given, then there is a unique α such that $u(x, u_0, \beta, \alpha)$, a solution for initial value problem, is a unique solution for the original PTC or NTC problem. Since only the monotonic property is used in all proofs in § 3.3, the conclusion can be generalized to all monotonic functions. At the end of § 3.3, the situation when external circuitry is connected is also discussed. The relations of parameters for which there are one, two or three solutions are given. In § 3.5, a more general monotone method is reviewed. As in § 3.3, this method can be used to prove the existence for both NTC and PTC problems. In § 3.6, the last section of this chapter, existence for time dependent problems has been proved.

3.1 Steady State Problem for PTC

In this section the existence and uniqueness of solutions of the following problem are discussed.

$$\begin{cases} \frac{d^2u}{dx^2} + \alpha f(u) = 0 & ,0 < x < 1 \\ u_x(0) = 0 & ,u_x(1) + \beta u(1) = 0 \end{cases} \tag{3.1}$$

and

$$f(u) = \begin{cases} 1 & ,u < 1 \\ e^{-10(u-1)} & ,1 \leq u \leq 2 \\ e^{-10} & ,2 < u \end{cases} \tag{3.2}$$

where α and β are parameters.

Theorem 3.1.1 *For Eq.(3.1) and Eq.(3.2), if $u(x) \leq 1.0$ for $0 \leq x \leq 1$, then*

$$u(x) = \frac{\alpha}{\beta} + \frac{\alpha}{2}(1 - x^2)$$

is the unique solution. Similarly, if $u(x) \geq 2$ for $0 \leq x \leq 1$, then

$$u(x) = \frac{\alpha e^{-10}}{\beta} + \frac{\alpha e^{-10}}{2}(1 - x^2)$$

is the unique solution.

Proof: For $u(x) \leq 1$, $f(u(x)) = 1$, from the first equation of Eq.(3.1) then

$$u(x) = a + bx + cx^2$$

Using the boundary conditions, hence

$$u(x) = \frac{\alpha}{\beta} + \frac{\alpha}{2}(1 - x^2)$$

The uniqueness is trivial since when $u(x) \leq 1$, Eq.(3.1) is a linear differential equation.

For $u(x) \geq 2$, the proof is similar. #

Property 3.1.1 *Suppose the solution of Eq.(3.1) exists and has second derivative, then $u(x)$ is monotonically nonincreasing within the interval $[0, 1]$, and also $u(1) \geq 0$.*

Proof: Integrate the first equation of Eq.(3.1) from 0 to x, we have

$$u_x(x) + \alpha \int_0^x f(u)dx = 0 \tag{3.3}$$

Since $f(u) > 0$, the integral must be positive. That means $u_x(x) < 0$, hence $u(x)$ is strictly monotonically nonincreasing.

As $u_x(1) \leq 0$ and $\beta > 0$, $u(1)$ can not be negative because of the boundary condition of Eq.(3.1). #

Corollary 3.1.1 *If $u(1)$ is zero, then $u(x) \equiv 0$.* #

This conclusion is trivial. Since $u_{xx}(x)$ is negative, $u_x(x)$ is monotonically nonincreasing. As $u_x(0) = 0$ and $u_x(0) \geq u_x(x) \geq u_x(1) = 0$, hence $u_x(x) \equiv 0$, thus $u(x) \equiv u(1) = 0$ (only possible if $\alpha = 0$).

Theorem 3.1.2 $u(x) = \frac{\alpha}{\beta} + \frac{\alpha}{2}(1 - x^2)$ *is a solution of Eq.(3.1) iff* $\alpha \leq \frac{2\beta}{\beta+2}$ *(i.e.,*

$u(1) \leq \frac{2}{\beta+2}$ *) and* $u(x) = \frac{\alpha e^{-10}}{\beta} + \frac{\alpha e^{-10}}{2}(1-x^2)$ *is a solution of Eq.(3.1) iff* $\alpha \geq 2\beta e^{10}$ *(i.e.,* $u(1) \geq 2$ *).*

Proof: Here the proof is only given for $\alpha \leq \frac{2\beta}{\beta+2}$. If $\alpha \geq 2\beta e^{10}$, the method is the same.

Necessity. If $u(x) = \frac{\alpha}{\beta} + \frac{\alpha}{2}(1-x^2)$ is a solution, then it must satisfy Eq.(3.1), hence

$$-\alpha + \alpha f(\frac{\alpha}{\beta} + \frac{\alpha}{2}(1-x^2)) = 0$$

i.e.,

$$f(\frac{\alpha}{\beta} + \frac{\alpha}{2}(1-x^2)) = 1$$

so by the property of f,

$$\frac{\alpha}{\beta} + \frac{\alpha}{2}(1-x^2) \leq 1$$

Since $u(x)$ is nonincreasing, $u(x) \leq 1$ iff $\frac{\alpha}{\beta} + \frac{\alpha}{2} \leq 1$ for $0 \leq x \leq 1$, therefore $\alpha \leq \frac{2\beta}{\beta+2}$

Sufficiency. If $\alpha \leq \frac{2\beta}{\beta+2}$, construct

$$u(x) = \frac{\alpha}{\beta} + \frac{\alpha}{2}(1-x^2)$$

hence $u(x) \leq \frac{\alpha}{\beta} + \frac{\alpha}{2} \leq 1$, and $u(x)$ satisfy Eq.(3.1). #

The more interesting problem is when $1 \leq u(x) \leq 2$ for some $x \in [0,1]$. Does the solution exist? Is it unique? To answer those questions, we need to introduce more concepts and results. First, we construct an integral operator from Eq.(3.1),

$$Tu = u_1[1 + \beta(1-x)] - \alpha \int_x^1 f(u(t))(t-x)dt \tag{3.4}$$

where u_1 is an arbitrary real number. Consider the initial value problem

$$\begin{cases} u &= Tu \\ u(1) &= u_1 \\ u'(1) &= -\beta u_1 \end{cases} \tag{3.5}$$

Actually, if $u(x) = Tu(x)$, then $u(1) = u_1$, and since $u'(x) = -\beta u_1 + \alpha \int_x^1 f(u(t))dt$, hence $u'(1) = -\beta u_1$. Therefore the problem is to find u such that

$$u = Tu$$

Before answering that question, the properties of T are discussed. In the following, when it is said $u(x) \leq v(x)$, it means that in the sense of that for every $x \in [0,1]$, $u(x) \leq v(x)$.

Property 3.1.2 *T is a monotonically nondecreasing operator.*

Proof: If $u(x) \leq v(x)$, since f is a nonincreasing function, then

$$f(u(t)) \geq f(v(t))$$

$$(t-x)f(u(t)) \geq (t-x)f(v(t)) \quad \text{if } t \geq x$$

$$\alpha \int_x^1 (t-x)f(u(t))dt \geq \alpha \int_x^1 (t-x)f(v(t))dt$$

$$-\alpha \int_x^1 (t-x)f(u(t))dt \leq -\alpha \int_x^1 (t-x)f(v(t))dt$$

$$\begin{aligned} Tu &= u_1[1+\beta(1-x)] - \alpha \int_x^1 f(u(t))(t-x)dt \\ &\leq u_1[1+\beta(1-x)] - \alpha \int_x^1 f(v(t))(t-x)dt \\ &= Tv \end{aligned}$$

Thus T is a monotonically nondecreasing operator. #

Property 3.1.3 *If $u(x)$ is the cold (then $u_1 = \frac{\alpha}{\beta}$) or hot (then $u_1 = \frac{\alpha e^{-10}}{\beta}$) solution of Eq.(3.1), then*

$$u(x) = Tu(x)$$

Proof: Since $u(x)$ is the cold solution of Eq.(3.1), then $u(x) = \frac{\alpha}{\beta} + \frac{\alpha}{2}(1-x^2)$ and $f(u(t)) = 1$,

$$\begin{aligned} Tu &= \frac{\alpha}{\beta}[1+\beta(1-x)] - \alpha\int_x^1 (t-x)dt \\ &= \frac{\alpha}{\beta}[1+\beta(1-x)] - \alpha\frac{1}{2}(t-x)^2|_x^1 \\ &= \frac{\alpha}{\beta} + \frac{\alpha}{2}(1-x^2) \\ &= u(x) \end{aligned}$$

The result follows. #

Property 3.1.4 *If $u(x)$ is a solution of Eq.(3.1), then $u(x)$ is a solution of Eq.(3.5) where $u_1 = u(1)$.*

Proof: Let $u(x)$ be a solution of Eq.(3.1), change the variable in Eq.(3.1) into t, multiply the first equation with $t - x$ and integrate from x to 1, then

$$\int_x^1 u_{tt}(t)(t-x)dt + \alpha\int_x^1 f(u(t))(t-x)dt = 0$$

Use integration by part for the first term of the above equation, thus

$$u_t(t)(t-x)|_x^1 - \int_x^1 u_t(t)dt + \alpha\int_x^1 f(u(t))(t-x)dt = 0$$

so

$$u_t(1)(1-x) - u(t)|_x^1 + \alpha\int_x^1 f(u(t))(t-x)dt = 0$$

by the boundary conditions of Eq.(3.1), then

$$-\beta u(1)(1-x) - u(1) + u(x) + \alpha \int_x^1 f(u(t))(t-x)dt = 0$$

i.e.,

$$u(x) = u(1)[1+\beta(1-x)] - \alpha \int_x^1 f(u(t))(t-x)dt$$

that is

$$u(x) = Tu(x)$$

The proof is completed. #

Definition 3.1.1 *Let (X,ρ) a metric space, A be an operator from X to X, if $\exists \eta$, $0 \leq \eta < 1$ such that $\forall x, y \in X$, there is*

$$\rho(Ax, Ay) \leq \eta \rho(x,y) \tag{3.6}$$

then A is said to be a contraction operator on X.

Lemma 3.1.1 [32] *(Banach, fixed point principle) If B is a Banach space, A is a contraction operator on B, there exists one and only one $x^* \in B$ such that*

$$x^* = Ax^*$$

#

Lemma 3.1.2 *Let B be a Banach space, A be an operator from B to B. If there exists a natural number n such that A^n is a contraction operator on B, then there must exist one and only one fixed point for A.*

Obviously, if $n = 1$, this lemma is just lemma 3.1.1.

Proof: Let $K = A^n$, then K is a contraction operator on B, hence by lemma 3.1.1 there exists a fixed point $x^* \in B$: $x^* = Kx^*$. Now we say that x^* is a fixed point for A. In fact,

$$AK = AA^n = A^{n+1} = A^n A = KA$$

therefore $K(Ax^*) = A(Kx^*) = Ax^*$, so Ax^* is also a fixed point for K. Since there exists one and only one fixed point for K, thus $Ax^* = x^*$.

If x^1 is any fixed point of A, since $Ax^1 = x^1$, then

$$A^n x^1 = A^{n-1} x^1 = \cdots = x^1$$

Thus x^1 is also a fixed point for $K = A^n$. Since there exists only one fixed point x^* for K, therefore $x^1 = x^*$. Then there exists one and only one fixed point for A. #

Now denote $C[0,1]$ as a space of all continuous function defined on the interval $[0,1]$. Define a norm $||\cdot||$ on $C[0,1]$ as

$$||f|| = \sup_{0\leq x\leq 1} |f(x)| = \max_{0\leq x\leq 1} |f(x)| \tag{3.7}$$

Then $C[0,1]$ is a Banach space.

Theorem 3.1.3 *Let T be defined on $C[0,1]$ by (3.4), and f be defined by (3.2), then for any α and β, there exists one and only one continuous function $u(x) \in C[0,1]$ such that*

$$u = Tu \quad \left(= u_1[1+\beta(1-x)] - \alpha \int_x^1 f(u(t))(t-x)dt \ \right) \tag{3.8}$$

where u_1 is an arbitrary real number.

Proof: Since Tu is defined as

$$Tu = u_1[1+\beta(1-x)] - \alpha \int_x^1 f(u(t))(t-x)dt$$

For any $u(x),\ v(x) \in C[0,1]$, $x \in [0,1]$, by the mean value theorem and $|f'(*)| \leq 10$,

$$\begin{aligned} |Tu(x) - Tv(x)| &= |\alpha \int_x^1 (f(v(t)) - f(u(t)))(t-x)dt| \\ &= |\alpha| \cdot |\int_x^1 f'(*)(v(t) - u(t))(t-x)dt| \\ &\leq 10|\alpha| \cdot ||u - v|| \int_x^1 (t-x)dt \\ &= \frac{10|\alpha|}{2} ||u - v||(1-x)^2 \end{aligned} \tag{3.9}$$

Now use induction to prove for $x \in [0,1]$ that

$$|T^n u - T^n v| \leq \frac{|\alpha|^n 10^n}{(2n)!} ||u - v||(1-x)^{2n} \tag{3.10}$$

For $n = 2$, use (3.9), by the mean value theorem and $|f'(*)| \leq 10$,

$$\begin{aligned} |T^2u(x) - T^2v(x)| &= |\alpha \int_x^1 (f(Tv(t)) - f(Tu(t)))(t-x)dt| \\ &= |\alpha| \cdot |\int_x^1 f'(*)(Tv(t) - Tu(t))(t-x)dt| \\ &\leq \frac{10^2|\alpha|^2}{2} ||u - v|| \int_x^1 (1-t)^2(t-x)dt \\ &= \frac{10^2|\alpha|^2}{1 \cdot 2 \cdot 3 \cdot 4} ||u - v||(1-x)^4 \end{aligned}$$

So for $n = 2$ it is true. Suppose (3.10) is true for n, then

$$\begin{aligned} |T^{n+1}u(x) - T^{n+1}v(x)| &= |\alpha \int_x^1 (f(T^n v(t)) - f(T^n u(t)))(t-x)dt| \\ &= |\alpha| \cdot |\int_x^1 f'(*)(T^n v(t) - T^n u(t))(t-x)dt| \\ &\leq 10|\alpha| \int_x^1 |T^n u - T^n v|(t-x)dt \\ &\leq \frac{10^{n+1}|\alpha|^{n+1}}{(2n)!} ||u - v|| \int_x^1 (1-t)^{2n}(t-x)dt \\ &= \frac{10^{n+1}|\alpha|^{n+1}}{(2n)!} \frac{||u - v||}{(2n+2)(2n+1)} (1-x)^{2n+2} \\ &= \frac{10^{n+1}|\alpha|^{n+1}}{(2(n+1))!} ||u - v||(1-x)^{2n+2} \end{aligned}$$

Hence (3.10) is true. Now take a natural number n such that

$$\eta = \frac{\alpha^n 10^n}{(2n)!} < 1$$

then

$$||T^n u - T^n v|| = \sup_{0 \le x \le 1} |T^n u - T^n v| \le \eta ||u - v||$$

so by lemma 3.1.2, there exists one and only one solution in $C[0,1]$. #

If $u(x) \in C[0,1]$ is a solution of Eq.(3.8), then $u(x)$ is also a solution of (3.5). In fact, the following is true.

Property 3.1.5 *If $u(x) \in C[0,1]$ is a solution of Eq.(3.8), then $u(x)$ is a solution of Eq.(3.5); furthermore it has a second derivative.*

Proof: Since $u(x)$ is a solution of Eq.(3.8), that is

$$u(x) = u_1[1 + \beta(1 - x)] - \alpha \int_x^1 f(u(t))(t - x)dt$$

where f is defined by (3.2). Since f is continuous, thus the derivative of right hand side exists. Differentiate the above equation then

$$u'(x) = -\beta u_1 + \alpha \int_x^1 f(u(t))dt$$

Therefore, $u(1) = u_1$ and $u'(1) = -\beta u_1$, which means that $u(x)$ is a solution of Eq.(3.5). Furthermore, from

$$u'(x) = -\beta u_1 + \alpha \int_x^1 f(u(t))dt$$

it is easy to see that the right hand side is still differentiable by the continuity of f. Differentiate both sides of above equation, we have

$$u''(x) = -\alpha f(u(x))$$

Therefore, $u(x)$ has a second derivative. #

Before we prove that if α and β are fixed there exists a $u(x)$ for Eq.(3.1), a property of $u(x)$ is discussed first. If $u(x)$ is a solution of Eq.(3.5), it can be denoted as $u(x, u_1, \alpha, \beta)$ since u depends upon x, u_1, α, β. From property 3.1.5, $u(x)$ is differentiable, so

$$u_x(x, u_1, \alpha, \beta) = -\beta u_1 + \alpha \int_x^1 f(u(t, u_1, \alpha, \beta))dt \tag{3.11}$$

$$u_{xx}(x, u_1, \alpha, \beta) = -\alpha f(u(x, u_1, \alpha, \beta)) \tag{3.12}$$

Thus from (3.11), if $u(x)$ is a solution of (3.1), then

$$u_x(0, u_1, \alpha, \beta) = -\beta u_1 + \alpha \int_0^1 f(u(t, u_1, \alpha, \beta))dt = 0$$

so

$$u_1 = \frac{\alpha}{\beta} \int_0^1 f(u(t, u_1, \alpha, \beta))dt$$

From the definition of f by (3.2), $e^{-10} \leq f(u) \leq 1$, thus a necessary condition for $u_x(0, u_1, \alpha, \beta) = 0$ is that

$$\frac{\alpha e^{-10}}{\beta} \leq u_1 \leq \frac{\alpha}{\beta}$$

This condition means for fixed α and β that Eq.(3.1) has a solution only if u_1 is chosen in that region. Now we will prove that if u_1, β are fixed nonnegative numbers, there is an α such that $u(x, u_1, \alpha, \beta)$ is a solution of Eq.(3.1). That means the following.

Property 3.1.6 *For any given positive number u_1 and β, there exists a unique α, such that $u(x, u_1, \alpha, \beta)$ satisfies Eq.(3.1).*

Proof: For given positive u_1 and β, for any α, by theorem 3.1.3, there exists $u(x, u_1, \alpha, \beta)$ such that

$$u(x, u_1, \alpha, \beta) = Tu(x, u_1, \alpha, \beta)$$

however in general, $u_x(0, u_1, \alpha, \beta) \neq 0$. From (3.11),

$$u_x(0, u_1, 0, \beta) = -\beta u_1 < 0$$

and also from (3.11)

$$u_x(0, u_1, \alpha, \beta) = -\beta u_1 + \alpha \int_0^1 f(u(t, u_1, \alpha, \beta))dt > -\beta u_1 + \alpha f(u_1)$$

by the nonincreasing property of f and u. Since $f(u_1) \geq e^{-10} > 0$ hence if α is chosen big enough, $u_x(0, u_1, \alpha, \beta) > 0$. By the continuity of $u_x(0, u_1, \alpha, \beta)$ with respect to α, there must exist an α^* such that $u_x(0, u_1, \alpha^*, \beta) = 0$. In order to prove the uniqueness, the strictly monotonic property of $u_x(0, u_1, \alpha, \beta)$ with respect to α is proved, hence uniqueness is obtained. Suppose there is another α^{**} such that $u_x(0, u_1, \alpha^{**}, \beta) = 0$. Without losing generality, suppose $\alpha^{**} > \alpha^*$. Since both $u(x, u_1, \alpha^*, \beta)$ and $u(x, u_1, \alpha^{**}, \beta)$ are solutions of Eq.(3.1), i.e., solutions of Eq.(3.5), hence

$$u(1, u_1, \alpha^*, \beta) = u(1, u_1, \alpha^{**}, \beta) = u_1$$

and

$$u_x(1, u_1, \alpha^*, \beta) = u_x(1, u_1, \alpha^{**}, \beta) = -\beta u_1$$

but

$$u_{xx}(1, u_1, \alpha^*, \beta) = -\alpha^* f(u(1, u_1, \alpha^*, \beta)) = -\alpha^* f(u_1) < 0$$

and

$$u_{xx}(1, u_1, \alpha^{**}, \beta) = -\alpha^{**} f(u(1, u_1, \alpha^{**}, \beta)) = -\alpha^{**} f(u_1) < -\alpha^* f(u_1) < 0$$

therefore

$$u_{xx}(1, u_1, \alpha^{**}, \beta) < u_{xx}(1, u_1, \alpha^*, \beta) < 0$$

which means that in a neighborhood of $x = 1$ the curves are concave down and

$$u(x, u_1, \alpha^{**}, \beta) < u(x, u_1, \alpha^*, \beta) \tag{3.13}$$

Now either the inequality (3.13) is true for all $x \in [0, 1]$, or there is a point $x_0 \in [0, 1]$ such that

$$u(x_0, u_1, \alpha^{**}, \beta) = u(x_0, u_1, \alpha^*, \beta)$$

and

$$u(x, u_1, \alpha^{**}, \beta) < u(x, u_1, \alpha^*, \beta) \quad \text{for} \quad x_0 < x < 1$$

hence by the nonincreasing property of f

$$f(u(x, u_1, \alpha^{**}, \beta)) \geq f(u(x, u_1, \alpha^*, \beta))$$

and

$$\int_x^1 f(u(t, u_1, \alpha^{**}, \beta))dt \geq \int_x^1 f(u(t, u_1, \alpha^*, \beta))dt$$

thus

$$\begin{aligned} & u_x(x_0, u_1, \alpha^{**}, \beta) - u_x(x_0, u_1, \alpha^*, \beta) \\ = \;& \alpha^{**} \int_{x_0}^1 f(u(t, u_1, \alpha^{**}, \beta))dt - \alpha^* \int_{x_0}^1 f(u(t, u_1, \alpha^*, \beta))dt \\ > \;& 0 \end{aligned} \tag{3.14}$$

since $\alpha^{**} > \alpha^*$.

From (3.13) for $x_0 < x < 1$

$$u(x, u_1, \alpha^{**}, \beta) - u(x_0, u_1, \alpha^{**}, \beta) < u(x, u_1, \alpha^*, \beta) - u(x_0, u_1, \alpha^*, \beta)$$

assume x is greater than x_0 and divide the above inequality by $x - x_0$

$$\frac{u(x, u_1, \alpha^{**}, \beta) - u(x_0, u_1, \alpha^{**}, \beta)}{x - x_0} < \frac{u(x, u_1, \alpha^*, \beta) - u(x_0, u_1, \alpha^*, \beta)}{x - x_0}$$

let $x \longrightarrow x_0$ from right side thus

$$u_x(x_0^+, u_1, \alpha^{**}, \beta) \leq u_x(x_0^+, u_1, \alpha^*, \beta)$$

which contradicts (3.14). So there is no such x_0 and (3.13) and hence (3.14) is true for all $x \in [0,1]$. In particular $u_x(0, u_1, \alpha^{**}, \beta) > u_x(0, u_1, \alpha^*, \beta)$, which means $u_x(0, u_1, \alpha, \beta)$ is a strictly increasing function of α. Uniqueness then follows. #

In order to prove that $u(x, u_1, \alpha, \beta)$ is continously dependent upon u_1, Gronwall's inequality is used. Gronwall's inequality is first stated.

Lemma 3.1.3 [30] *If $u(t)$ and $v(t)$ are continuous nonnegative functions on interval $0 \leq t \leq L$ and M is a nonnegative constant, then*

$$u(t) \leq M + \int_0^t v(s)u(s)ds, \qquad 0 \leq t \leq L$$

implies

$$u(t) \leq Mexp(\int_0^t v(s)ds), \qquad 0 \leq t \leq L$$

#

Property 3.1.7 *The solution $u(x, u_1, \alpha, \beta)$ of Eq.(3.5) for fixed α and β is continuously dependent upon u_1. That is $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if*

$$|u_1^* - u_1^{**}| < \delta$$

then

$$|u(x, u_1^*, \alpha, \beta) - u(x, u_1^{**}, \alpha, \beta)| < \varepsilon$$

Proof: Since u is a solution of Eq.(3.5), thus

$$\begin{aligned} u(x, u_1^*, \alpha, \beta) &= T_* u(x, u_1^*, \alpha, \beta) \\ &= u_1^*[1 + \beta(1-x)] - \alpha \int_x^1 f(u(t, u_1^*, \alpha, \beta))(t-x)dt \end{aligned}$$

and

$$\begin{aligned} u(x, u_1^{**}, \alpha, \beta) &= T_{**}u(x, u_1^{**}, \alpha, \beta) \\ &= u_1^{**}[1+\beta(1-x)] - \alpha \int_x^1 f(u(t, u_1^{**}, \alpha, \beta))(t-x)dt \end{aligned}$$

Thus

$$\begin{aligned} & |u(x, u_1^{*}, \alpha, \beta) - u(x, u_1^{**}, \alpha, \beta)| \\ = & \ |(u_1^{*} - u_1^{**})[1+\beta(1-x)] - \alpha \textstyle\int_x^1 (f(u(t, u_1^{*}, \alpha, \beta)) - f(u(t, u_1^{**}, \alpha, \beta)))(t-x)dt \\ \leq & \ |u_1^{*} - u_1^{**}|(1+\beta) + 10\alpha \textstyle\int_x^1 |u(t, u_1^{*}, \alpha, \beta)) - u(t, u_1^{**}, \alpha, \beta)|(t-x)dt \end{aligned}$$

where the Lipschitz condition $|f'(*)| \leq 10$ for f has been used. Then by lemma 3.1.3

$$|u(x, u_1^{*}, \alpha, \beta) - u(x, u_1^{**}, \alpha, \beta)| \leq |u_1^{*} - u_1^{**}|(1+\beta)exp(\int_x^1 10\alpha(t-x)dt)$$

so

$$|u(x, u_1^{*}, \alpha, \beta) - u(x, u_1^{**}, \alpha, \beta)| \leq |u_1^{*} - u_1^{**}|(1+\beta)\exp(\frac{10\alpha}{2}(1-x)^2)$$

hence

$$|u(x, u_1^{*}, \alpha, \beta) - u(x, u_1^{**}, \alpha, \beta)| \leq \exp(\frac{10\alpha}{2})(1+\beta)|u_1^{*} - u_1^{**}|$$

Therefore, $\forall \varepsilon > 0$, take $\delta < \frac{1}{e^{5\alpha}(1+\beta)}\varepsilon$, whenever $|u_1^{*} - u_1^{**}| < \delta$ then

$$|u(x, u_1^{*}, \alpha, \beta) - u(x, u_1^{**}, \alpha, \beta)| < \varepsilon$$

The continuity of $u(x, u_1, \alpha, \beta)$ with respect to u_1 follows. #

Corollary 3.1.2 *$u_x(x, u_1, \alpha, \beta)$ is also continuously dependent upon u_1*

Proof: From (3.11),

$$u_x(x, u_1, \alpha, \beta) = -\beta u_1 + \alpha \int_x^1 f(u(t, u_1, \alpha, \beta))dt$$

Then

$$\begin{aligned}
& |u_x(x,u_1^*,\alpha,\beta) - u_x(x,u_1^{**},\alpha,\beta)| \\
= \; & |\beta(u_1^{**} - u_1^*) + \alpha \int_x^1 (f(u(x,u_1^*,\alpha,\beta)) - f(u(x,u_1^{**},\alpha,\beta)))dt| \\
\leq \; & \beta|u_1^* - u_1^{**}| + \alpha 10 \int_x^1 |u(x,u_1^*,\alpha,\beta) - u(x,u_1^{**},\alpha,\beta)|dt \\
\leq \; & \beta|u_1^* - u_1^{**}| + \alpha 10||u(x,u_1^*,\alpha,\beta) - u(x,u_1^{**},\alpha,\beta)||
\end{aligned}$$

where the Lipschitz condition for f is used. By property 3.1.7, the result is true. #

Theorem 3.1.4 *For any given positive number α and β, there exists one and only one $u_1 \in [\frac{\alpha e^{-10}}{\beta}, \frac{\alpha}{\beta}]$ such that $u(x,u_1,\alpha,\beta)$ obtained through Eq.(3.5) is a solution of Eq.(3.1).*

Proof: As previously discussed, $u_1 \in [\frac{\alpha e^{-10}}{\beta}, \frac{\alpha}{\beta}]$ is a necessary condition. From (3.11),

$$u_x(0,u_1,\alpha,\beta) = -\beta u_1 + \alpha \int_0^1 f(u(t,u_1,\alpha,\beta))dt$$

hence by the property $f(u) \geq e^{-10}$

$$u_x(0,\frac{\alpha e^{-10}}{\beta},\alpha,\beta) = -\alpha e^{-10} + \alpha \int_0^1 f(u(t,\frac{\alpha e^{-10}}{\beta},\alpha,\beta))dt \geq 0$$

and

$$u_x(0,\frac{\alpha}{\beta},\alpha,\beta) = -\alpha + \alpha \int_0^1 f(u(t,\frac{\alpha}{\beta},\alpha,\beta))dt \leq -\alpha + \alpha = 0$$

since $f \leq 1$. By corollary 3.1.2, $u_x(x,u_1,\alpha,\beta)$ is a continuous function of u_1, so there must exist u_1^* in $[\frac{\alpha}{\beta}e^{-10}, \frac{\alpha}{\beta}]$ such that $u_x(0,u_1^*,\alpha,\beta) = 0$.

As for the uniqueness, this may be demonstrated by a method similar to that used for the proof of property 3.1.6. Suppose there exist two values u_1^* and u_1^{**} which make

$$u_x(0,u_1^*,\alpha,\beta) = u_x(0,u_1^{**},\alpha,\beta) = 0$$

Without losing generality, suppose $u_1^* > u_1^{**}$, for any $u(x) \in C[0,1]$ define sequences

$$u_*^n = u^n(x, u_1^*, \alpha, \beta), \quad u_{**}^n = u^n(x, u_1^{**}, \alpha, \beta)$$

by

$$u_*^{n+1} = T_* u_*^n = u_1^*[1 + \beta(1-x)] - \alpha \int_x^1 f(u_*^n)(t-x)dt$$

and

$$u_{**}^{n+1} = T_{**} u_{**}^n = u_1^{**}[1 + \beta(1-x)] - \alpha \int_x^1 f(u_{**}^n)(t-x)dt$$

where $n = 0, 1, \cdots$, and $u_*^0 = u_{**}^0 = u(x)$, then

$$\begin{aligned} u_*^1 &= T_* u_*^0 \\ &= u_1^*[1 + \beta(1-x)] - \alpha \textstyle\int_x^1 f(u(t))(t-x)dt \\ &> u_1^{**}[1 + \beta(1-x)] - \alpha \textstyle\int_x^1 f(u(t))(t-x)dt \\ &= T_{**} u_{**}^0 = u_{**}^1 \end{aligned}$$

by property 3.1.2, T is monotone operator, it is easy to prove

$$u^n(x, u_1^*, \alpha, \beta) = T_*^n u > T_{**}^n u = u^n(x, u_1^{**}, \alpha, \beta) \tag{3.15}$$

In fact, by induction, suppose

$$u_*^n > u_{**}^n$$

hence by the nonincreasing property of f,

$$f(u_*^n) \leq f(u_{**}^n)$$

thus

$$u_*^{n+1} = T_* u_*^n$$

$$\begin{aligned}
&= u_1^*[1+\beta(1-x)] - \alpha \textstyle\int_x^1 f(u_*^n)(t-x)dt \\
&> u_1^{**}[1+\beta(1-x)] - \alpha \textstyle\int_x^1 f(u_{**}^n)(t-x)dt \\
&= T_{**}u_{**}^n = u_{**}^{n+1}
\end{aligned}$$

Therefore the inequality (3.15) is true. By lemma 3.1.2 and theorem 3.1.3, the two function sequences $\{u^n(x, u_1^*, \alpha, \beta)\}_{n=1}^{\infty}$ and $\{u^n(x, u_1^{**}, \alpha, \beta)\}_{n=1}^{\infty}$ are convergent. For convenience, denote the limits of them as $u(x, u_1^*, \alpha, \beta)$ and $u(x, u_1^{**}, \alpha, \beta)$ respectively. Hence

$$u(x, u_1^*, \alpha, \beta) \geq u(x, u_1^{**}, \alpha, \beta)$$

Equality is only possible if $u_1^* = u_1^{**}$ by Eq.(3.5). Suppose the inequality is true. By the nonincreasing property of f,

$$f(u(x, u_1^*, \alpha, \beta)) \leq f(u(x, u_1^{**}, \alpha, \beta))$$

hence

$$\begin{aligned}
u_x(0, u_1^*, \alpha, \beta) &= -\beta u_1^* + \alpha \int_0^1 f(u(t, u_1^*, \alpha, \beta))dt \\
&< -\beta u_1^{**} + \alpha \int_0^1 f(u(t, u_1^*, \alpha, \beta))dt \\
&\leq -\beta u_1^{**} + \alpha \int_0^1 f(u(t, u_1^{**}, \alpha, \beta))dt \\
&= u_x(0, u_1^{**}, \alpha, \beta)
\end{aligned}$$

which means u_1^* and u_1^{**} can not make $u_x(0, u_1, \alpha, \beta) = 0$ simultaneously. Thus the conclusion is obtained. #

3.2 Steady State Problem for NTC

In this section, the equation and boundary condition are the same as that in §3.1 except that here $f(u)$ is replaced by $\frac{1}{f(u)}$. For convenience, denote $g(u) = \frac{1}{f(u)}$, hence

$$g(u) = \begin{cases} 1 & ,u < 1 \\ e^{10(u-1)} & ,1 \leq u \leq 2 \\ e^{10} & ,2 < u \end{cases} \tag{3.16}$$

and Eq.(3.1) can be rewritten as

$$\begin{cases} \frac{d^2u}{dx^2} + \alpha g(u) = 0 & ,0 < x < 1 \\ u_x(0) = 0 & ,u_x(1) + \beta u(1) = 0 \end{cases} \tag{3.17}$$

Since $g(u)$ is defined as the reciprocal of $f(u)$, the properties of $g(u)$ are changed. One of the important changes is that $g(u)$ is a nondecreasing function. However there are still some similar results. For completeness, the conclusions are given in the following theorems and properties.

Theorem 3.2.1 *For Eq.(3.17) and $g(u)$ defined by (3.16), if $u(x) \leq 1.0$ for $0 \leq x \leq 1$, then*

$$u(x) = \frac{\alpha}{\beta} + \frac{\alpha}{2}(1 - x^2)$$

is the unique solution. Similarly, if $u(x) \geq 2$ for $0 \leq x \leq 1$, then

$$u(x) = \frac{\alpha e^{10}}{\beta} + \frac{\alpha e^{10}}{2}(1 - x^2)$$

is the unique solution. #

Property 3.2.1 *Suppose the solution of Eq.(3.17) exists and has second derivative, then $u(x)$ is monotonically nonincreasing for $x \in [0,1]$, and also $u(1) \geq 0$.* #

Corollary 3.2.1 *If $u(1)$ is zero, then $u(x) \equiv 0$.* #

Theorem 3.2.2 *$u(x) = \frac{\alpha}{\beta} + \frac{\alpha}{2}(1-x^2)$ is a solution of Eq.(3.17) iff $\alpha \leq \frac{2\beta}{\beta+2}$ (i.e., $u(1) \leq \frac{2}{\beta+2}$) and $u(x) = \frac{\alpha e^{10}}{\beta} + \frac{\alpha e^{10}}{2}(1-x^2)$ is a solution of Eq.(3.17) iff $\alpha \geq 2\beta e^{-10}$ (i.e., $u(1) \geq 2$).* (Note that for $\beta + 2 < e^{10}$ and $2\beta e^{-10} < \alpha < \frac{2\beta}{\beta+2}$ both solutions exist.) #

As in § 3.1, an operator Q is defined as

$$Qu = u_1[1 + \beta(1-x)] - \alpha \int_x^1 g(u(t))(t-x)dt \tag{3.18}$$

and the corresponding initial value problem is as follows

$$\begin{cases} u &= Qu \\ u(1) &= u_1 \\ u'(1) &= -\beta u_1 \end{cases} \tag{3.19}$$

Instead of being monotonically nondecreasing, the operator Q is monotonically nonincreasing. Thus

Property 3.2.2 *Q is a monotonically nonincreasing operator.* #

Since $g(u)$ is nondecreasing, the proof is obvious.

Property 3.2.3 *If $u(x)$ is the cold (then $u_1 = \frac{\alpha}{\beta}$) or the hot (then $u_1 = \frac{\alpha e^{10}}{\beta}$) solution of Eq.(3.17), then*

$$u(x) = Qu(x)$$

#

Property 3.2.4 *If $u(x)$ is a solution of Eq.(3.17), then $u(x)$ is a solution of Eq.(3.19) where $u_1 = u(1)$.* #

Theorem 3.2.3 *Let Q be defined on $C[0,1]$ by (3.18), and g be defined by (3.16), then for any α and β, there exists one and only one continuous function*

$u(x) \in C[0,1]$ such that

$$u = Qu \quad (= u_1[1 + \beta(1-x)] - \alpha \int_x^1 g(u(t))(t-x)dt \ ) \tag{3.20}$$

where u_1 is an arbitrary real number. #

The proof is as before except that, now n must be chosen large enough that

$$\eta = \frac{\alpha^n (10e^{10})^n}{(2n)!} < 1$$

Property 3.2.5 *If $u(x) \in C[0,1]$ is a solution of Eq.(3.20), then $u(x)$ is a solution of Eq.(3.19); furthermore it has a second derivative.* #

Although the following two results are true, it should be noted that we no longer have uniqueness of the solution of the boundary value problem when α is fixed.

Property 3.2.6 *For any given positive number u_1 and β, there exists an α, such that $u(x, u_1, \alpha, \beta)$ satisfies Eq.(3.17).* #

Theorem 3.2.4 *For any given real positive numbers α and β, there exists a $u_1 \in [\frac{\alpha}{\beta}, \frac{\alpha e^{10}}{\beta}]$ such that $u(x, u_1, \alpha, \beta)$ obtained through Eq.(3.19) is a solution of Eq.(3.17).* #

In fact, from the numerical solutions obtained in chapter 4 it is easy to see that for given α and β the uniqueness with respect to u_1 is broken.

3.3 Existence for Both PTC and NTC

In §3.1 and §3.2, the existence of solutions for Eq.(3.1) and Eq.(3.17) has been proven. For Eq.(3.1) uniqueness is also obtained. Here another way is used to prove the existence. This method applies for both cases. For convenience, Eq.(3.1) and Eq.(3.17) are written in one form as

$$\begin{cases} \frac{d^2u}{dx^2} + \alpha h(u) = 0 & ,0 < x < 1 \\ u_x(0) = 0 & ,u_x(1) + \beta u(1) = 0 \end{cases} \tag{3.21}$$

where $h(u)$ can be either $f(u)$ or $g(u)$. In order to solve Eq.(3.21), a variable substitution is made, i.e., let $\xi = \sqrt{\alpha}x$, hence $u_x = \sqrt{\alpha}u_\xi$ and $u_{xx} = \alpha u_{\xi\xi}$. So the Eq.(3.21) becomes

$$\begin{cases} \frac{d^2u}{d\xi^2} + h(u) = 0 & ,0 < \xi < \sqrt{\alpha} \\ u_\xi(0) = 0 & ,\sqrt{\alpha}u_\xi(\sqrt{\alpha}) + \beta u(\sqrt{\alpha}) = 0 \end{cases} \tag{3.22}$$

As discussed in §3.1, the solution of Eq.(3.22) is monotonically nonincreasing and $u(x)$ can not be negative. In order to prove the existence of solution for Eq.(3.22), an initial value problem corresponding to Eq.(3.22) is considered first. That is

$$\begin{cases} \frac{d^2u}{d\xi^2} + h(u) = 0 & ,0 < \xi \\ u(0) = u_0 & ,u_\xi(0) = 0 \end{cases} \tag{3.23}$$

where u_0 is an arbitrary positive real number. Formally, the solution of Eq.(3.23) is independent of α.

Lemma 3.3.1 [30] *Let a nonnegative and unbounded function $\psi(x)$ be defined on $[a, b)$ and $\psi(x)$ be integrable over interval $[a, c]$ for any $c < b$. If there is a constant number ν ($0 < \nu < 1$) such that $\lim_{x \to b^-}(b - x)^\nu \psi(x)$ exists, then the* *integral* $\int_a^b \psi(x)dx$ *for unbounded function* $\psi(x)$ *converges. If for* $\nu \geq 1$, $\lim_{x\to b^-}(b-x)^\nu\psi(x) = d > 0$, *or* $\lim_{x\to b^-}(b-x)^\nu\psi(x) = +\infty$, *then* $\int_a^b \psi(x)dx$ *diverges.* #

Theorem 3.3.1 *There is a solution for Eq.(3.23).*

Proof: Multiply first equation of Eq.(3.23) with u_ξ and integrate from 0 to ξ, thus

$$\int_0^\xi u_{\xi\xi}u_\xi d\xi + \int_0^\xi h(u)u_\xi d\xi = 0$$

By property 3.1.1, $u(\xi)$ is strictly monotonically decreasing, i.e., $u(\xi)$ is an invertible function, so denote $u(\xi)$ as u, the second term of the above integral can be written as

$$\int_0^\xi h(u)u_\xi d\xi = \int_{u_0}^u h(u)du = -\int_u^{u_0} h(u)du = -H(u)$$

hence

$$\frac{1}{2}u_\xi^2|_0^\xi - H(u) = 0$$

Using the boundary condition in Eq.(3.23),

$$u_\xi^2 = 2H(u)$$

By property 3.1.1, $u_\xi < 0$, hence

$$u_\xi = -\sqrt{2H(u)}$$

then

$$-\frac{u_\xi}{\sqrt{2H(u)}} = 1$$

Since $\lim_{u\to u_0^-}(u_0-u)^{\frac{1}{2}}/\sqrt{H(u)} = 1/\sqrt{h(u_0)} > 0$, by the lemma 3.3.1 the integral $\int_u^{u_0}\frac{1}{\sqrt{2H(s)}}ds$ exists. Integrate the above equation from 0 to ξ, thus

$$\xi = \int_u^{u_0}\frac{ds}{\sqrt{2H(s)}} = \Phi(u)$$

Since $\xi_u < 0$, ξ is a monotonically decreasing function of u, the inverse function of $\Phi(u)$ exists. Denote the inverse function of $\Phi(u)$ as $u = \phi(\xi)$, $\phi(\xi)$ is a solution of Eq.(3.23). #

Theorem 3.3.2 *For any given nonnegative real number u_0, there is a unique α such that $\phi(\xi)$ (obtained in theorem 3.3.1) is a solution of Eq.(3.22).*

Proof: The problem is now to find an α that makes $\phi(\xi)$ a solution of Eq.(3.22). This means to find an α such that

$$(u_x + \beta u)|_{x=1} = (\sqrt{\alpha}u_\xi + \beta u)|_{\xi=\sqrt{\alpha}} = -\sqrt{2\alpha H(u(\sqrt{\alpha}))} + \beta u(\sqrt{\alpha}) = 0$$

i.e., to find α such that

$$-\sqrt{2\alpha H(\phi(\sqrt{\alpha}))} + \beta\phi(\sqrt{\alpha}) = 0$$

Since $H'(u) = -h(u) < 0$ and $\phi'(\xi) = -\sqrt{2H(u)} < 0$, H and ϕ are monotonically nonincreasing functions. Hence the composition $(H \circ \phi)$ is monotonically nondecreasing and then $-\sqrt{2\alpha(H \circ \phi)(\sqrt{\alpha})} + \beta\phi(\sqrt{\alpha}) = \Psi(\alpha)$ is monotonically nonincreasing. Obviously, $\Psi(0) = \beta u_0 > 0$. By the definition of the function $h(u)$, we can assume $m < h(u) < M$ for $0 < u < u_0$, thus from Eq.(3.23)

$$-M < u_{\xi\xi} < -m$$

Integrate the above inequality from 0 to ξ and use the boundary condition in Eq.(3.23),

$$-M\xi < u_\xi < -m\xi$$

integrate the above inequality from 0 to ξ once more then

$$u_0 - \frac{1}{2}M\xi^2 < u < u_0 - \frac{1}{2}m\xi^2$$

therefore

$$\Psi(\alpha) = (\sqrt{\alpha}u_\xi + \beta u)|_{\xi=\sqrt{\alpha}} < -m\alpha + \beta(u_0 - \frac{1}{2}m\alpha)$$

That is $\Psi(\alpha)$ goes to $-\infty$ as $\alpha \longrightarrow \infty$. Thus there is a unique α such that $\phi(\xi)$ is a solution of Eq.(3.22) for given u_0, β. #

From theorem 3.3.2, for any given u_0 there exists a unique α, thus this defines a function $\alpha = \Theta(u_0)$. As $u(1) = u(\xi = \sqrt{\alpha})$ is also uniquely determined for each u_0, $u(1)$ is also a function of u_0 which can be denoted as $u(1) = U(u_0)$.

3.4 Existence of Solutions for the PTC and NTC Problems with External Circuit

Now consider the solutions of the full problem with the circuit loop. The previous results allow us to draw a curve of " solutions " represented in the α, $u(1)$ plane. This curve may be parameterized by $u(0)$. In the all " cold " portion, by theorem 3.1.2 and theorem 3.2.2, $u(x) = \frac{\alpha}{\beta} + \frac{\alpha}{2}(1-x^2)$, when $u(0) \leq 1$ which means that $\alpha \leq \frac{2\beta}{\beta+2}$, hence $u(1) = \frac{\alpha}{\beta}$. It is preferable to draw the curve in the I, $u(1)$ plane where $\alpha = \gamma I^2$ and $\gamma = 150$ is fixed. Then in the cold portion $u(1) = \frac{\gamma}{\beta}I^2$, which is a parabola.

As for the " hot " portion, the two cases arise. The PTC case is considered first. By theorem 3.1.2, in the hot portion, $u(x) = \frac{\alpha e^{-10}}{\beta} + \frac{\alpha e^{-10}}{2}(1-x^2)$, iff $u(1) \geq 2$ which means $\alpha \geq 2e^{10}\beta$, hence $u(1) = \frac{\alpha e^{-10}}{\beta} = \frac{\gamma e^{-10}}{\beta}I^2$ is a second parabola starting at $I = \sqrt{\frac{2\beta e^{10}}{\gamma}}$. This second parabola may be extended to ∞ in both $u(1)$ and I. These two parabolas are connected by an I, $u(1)$ curve.

The external circuit leads to $I = [1 + \mu \int_0^1 f(u)dx]^{-1}$. From the first equation of

Eq.(3.1), $\alpha \int_0^1 f(u)dx = -u_x|_0^1 = \beta u(1)$, therefore

$$I = [1 + \frac{\mu\beta u(1)}{\gamma I^2}]^{-1}$$

or

$$I + \frac{\mu\beta u(1)}{\gamma I} = 1$$

This gives the third parabola

$$u(1) = \frac{\gamma}{\mu\beta} I(1 - I) \tag{3.24}$$

with vertex at $I = \frac{1}{2}$, $u(1) = \frac{\gamma}{4\mu\beta}$ and passing through $(0,0)$ and $(1,0)$. This curve obviously meets the I, $u(1)$ curve of solution of Eq.(3.1) at least once giving one nontrivial solution to the problem with the external circuit, since the solution curve of Eq.(3.1) begins inside the parabola (3.24) and finishes outside of that parabola. We say that the intersections of those two curves are solutions for PTC problem (2.13). That is because the two curves are both parameterized by I and satisfy Eq.(3.1) and Eq.(3.24) respectively. Thus if there are points satisfying both equations, they will satisfy Eq.(2.13) too. The curves may meet twice or three times giving two or three solutions respectively. There will be three intersections, i.e., three solutions, if the point $(I, u(1)) = (\sqrt{\frac{2\beta}{\gamma(\beta+2)}}, \frac{2}{\beta+2})$ is outside the parabola (3.24). This occurs if

$$\frac{2}{\beta + 2} > \frac{\gamma}{\mu\beta}\sqrt{\frac{2\beta}{\gamma(\beta+2)}}(1 - \sqrt{\frac{2\beta}{\gamma(\beta+2)}})$$

Simplify the above inequality, then

$$\beta > \frac{2\gamma}{2(1+\mu)^2 - \gamma}$$

This gives a condition for the existence of three solutions. From the numerical results and figures in chapter 4, for the exponential functions if the point $(I, u(1)) =$

$(\sqrt{\frac{2\beta}{\gamma(\beta+2)}}, \frac{2}{\beta+2})$ is on the parabola (3.24), that means $\beta = \frac{2\gamma}{2(1+\mu)^2-\gamma}$, there are just two solutions (intersections). In chapter 4, the Figure 4.1.1, Figure 4.1.2 and Figure 4.1.3 demonstrate this.

The case corresponding to Eq.(3.17) is similar and is briefly discussed here. In the hot portion, the solution is $u(x) = \frac{\alpha e^{10}}{\beta} + \frac{\alpha e^{10}}{\beta}(1-x^2)$, iff $u(1) \geq 2$, i.e., $\alpha \geq 2\beta e^{-10}$, hence $u(1) = \frac{\alpha e^{10}}{\beta} = \frac{\gamma e^{10}}{\beta} I^2$. a parabola starting at $I = \sqrt{2\beta e^{-10}\gamma}$. Obviously, this parabola may be extended to ∞ in both I and $u(1)$. Now the external circuit leads to $I = [\mu + \int_0^1 g(u)dx]^{-1}$. From the equation $\alpha \int_0^1 g(u)dx = -u_x|_0^1 = \beta u(1)$, thus

$$I = [\mu + \frac{\beta u(1)}{\gamma I^2}]^{-1}$$

i.e.,

$$\mu I + \frac{\beta u(1)}{\gamma I} = 1$$

which gives a parabola

$$u(1) = \frac{\gamma}{\beta} I(1-\mu I) \tag{3.25}$$

with vertex at $I = \frac{1}{2\mu}$, $u(1) = \frac{\gamma}{4\mu\beta}$ and passing through $(0,0)$ and $(\frac{1}{\mu}, 0)$. Similarly, this curve obviously meets the I, $u(1)$ curve of solution of Eq.(3.17) at least once giving one nontrivial solution, since the curve of solution of Eq.(3.17) begins inside of the parabola (3.25) and finishes outside of that parabola. If they meet twice or three times, there will be two or three solutions respectively. A condition for three possible solutions is that point $(I, u(1)) = (\sqrt{\frac{2\beta}{\gamma(\beta+2)}}, \frac{2}{\beta+2})$ lies outside the parabola (3.25). Hence

$$\frac{2}{\beta+2} > \frac{\gamma}{\beta}\sqrt{\frac{2\beta}{\gamma(\beta+2)}}(1-\mu\sqrt{\frac{2\beta}{\gamma(\beta+2)}})$$

can be simplified as

$$\beta > \frac{2\gamma}{2(1+\mu)^2 - \gamma}$$

This is a condition that there exist three solutions. Similarly, from the numerical results and figures in chapter 4 for the exponential functions, if $\beta = \frac{2\gamma}{2(1+\mu)^2-\gamma}$, there are two solutions. There are three figures in chapter 4 as Figure 4.3.2, Figure 4.3.3 and Figure 4.3.4 which demonstrate these conclusions.

Therefore, for both Eq.(3.1) and Eq.(3.17), when the circuit loop is considered, there is only one solution when $\beta < \frac{2\gamma}{2(1+\mu)^2-\gamma}$, there are three solutions when $\beta > \frac{2\gamma}{2(1+\mu)^2-\gamma}$ and there are possibly two solutions when $\beta = \frac{2\gamma}{2(1+\mu)^2-\gamma}$.

It is easy to notice that the particular exponential functions chosen for $\sigma(u)$ are not necessary for the proof of all conclusions in §3.1, §3.2 and §3.3. The most important things are the monotonic decreasing or increasing, bounded properties and the Lipschitz condition. Thus we have following conclusions.

Corollary 3.4.1 *If f is monotonic nondecreasing or nonincreasing function, which satisfies*

$$0 < m \leq f(u) \leq M < +\infty \qquad u(x) \in C[0,1]$$

and the Lipschitz condition

$$|f(u) - f(v)| \leq M_0 |u - v| \qquad \forall u, v \in C[0,1]$$

where m, M and M_0 are constants, then for any given nonnegative α and β there exists at least one solution for Eq.(3.1). #

In addition, if f is monotonic nonincreasing, a stronger conclusion is achieved, i.e., the uniqueness is obtained.

Corollary 3.4.2 *Assume f satisfies all conditions in corollary 3.3.1, furthermore, f is monotonic nonincreasing function, then for any given nonnegative α and β there exists one and only one solution for Eq.(3.1).* #

3.5 Monotone Method for the PTC and NTC Problems

In §3.1, §3.2 and §3.3, existence and uniqueness under certain conditions have been proved for steady state PTC and NTC problems, i.e., Eq.(2.8) and Eq.(2.6). Here a monotone method[4] is worth a brief review though uniqueness is not obtained. Consider the general equation with the form

$$\begin{cases} Lu + f(x,u) = 0 & ,x \in \Omega \\ Bu = s(x) & ,x \in \partial\Omega \end{cases} \tag{3.26}$$

where $L = \Delta$, $f(x,u)$ is a nonlinear smooth function of x and u, $s(x)$ is a given function and B is a boundary operator defined as

$$Bu = \frac{\partial u}{\partial n} + \beta(x)u, \qquad x \in \partial\Omega$$

Here $\frac{\partial}{\partial n}$ denotes the outward conormal derivative and $\beta(x)$ is assumed nonnegative everywhere on the boundary $\partial\Omega$, i.e., $\beta(x) \geq 0$ for $x \in \partial\Omega$.

Definition 3.5.1 *A smooth function u_0 is said to be an upper solution of Eq.(3.26) if*

$$Lu_0 + f(x,u_0) \leq 0, \qquad Bu_0 \geq s$$

similarly, v_0 is called a lower solution of Eq.(3.26) if

$$Lv_0 + f(x,v_0) \geq 0, \qquad Bv_0 \leq s.$$

The following lemma is then true [4].

Lemma 3.5.1 *Let there exist two smooth functions* $u_0(x) \geq v_0(x)$ *such that*

$$Lu_0 + f(x, u_0) \leq 0, \qquad Bu_0 \geq s$$

and

$$Lv_0 + f(x, v_0) \geq 0, \qquad Bv_0 \leq s.$$

Assume f *is smooth function and* $\frac{\partial f}{\partial u}$ *is bounded on* $\min v_0 \leq u \leq \max u_0$. *Then there exists a regular solution* w *of*

$$Lw + f(x, w) = 0, \qquad Bw = s$$

such that $v_0 \leq w \leq u_0$. #

Therefore we have the following theorems.

Theorem 3.5.1*For Eq.(3.1) and* f *is defined by (3.2)*

$$u_0(x) = K(\frac{1}{\beta} + \frac{1}{2}(1 - x^2)) \qquad \textit{with} \qquad K > \alpha$$

and

$$v_0(x) = 0$$

are upper and lower solutions respectively. Hence there exists a regular solution w *of Eq.(3.1) and Eq.(3.2) such that* $v_0 \leq w \leq u_0$.

Proof: Since $e^{-10} \leq f(u) \leq 1$ by definition of f ,

$$\frac{d^2u_0}{dx^2} + \alpha f(u_0) = -K + \alpha f(u_0) \leq -K + Kf(u_0) = K(f(u_0) - 1) \leq 0$$

and

$$u_{0x}(0) = 0, \qquad u_{0x}(1) + \beta u(1) = 0,$$

hence $u_0(x)$ is a upper solution. Similarly, since

$$\frac{d^2v_0}{dx^2} + \alpha f(v_0) = 0 + \alpha f(v_0) = \alpha f(v_0) \geq 0$$

and

$$v_{0x}(0) = 0, \qquad v_{0x}(1) + \beta v_0(1) = 0,$$

thus $v_0(x)$ is a lower solution. By lemma 3.5.1, there exists a regular w of Eq.(3.1) and Eq.(3.2) such that $v_0 \leq w \leq u_0$. #

Theorem 3.5.2 *For Eq.(3.17) and g is defined by (3.16)*

$$u_0(x) = K(\frac{1}{\beta} + \frac{1}{2}(1 - x^2)) \qquad \textit{with} \qquad K > \alpha e^{10}$$

and

$$v_0(x) = 0$$

are upper and lower solutions respectively. Hence there exists a regular solution w of Eq.(3.17) and Eq.(3.16) such that $v_0 \leq w \leq u_0$. #

The proof is same as that for theorem 3.5.1 except that now $1 \leq g(u) \leq e^{10}$.

3.6 Time Dependent Problem

In this section, existence and uniqueness for time dependent problems are considered. A monotone method the same as that in § 3.5 is used. For convenience, a general form for the time dependent PTC and NTC problems is restated as follows.

$$\begin{cases} \frac{\partial u}{\partial t} = Lu + \alpha h(u) & ,0 < x < 1, 0 < t < t_0 \\ u_x(t,0) = 0, u_x(t,1) + \beta u(t,1) = 0 & ,u(0,x) = s(x) \end{cases} \tag{3.27}$$

where $s(x)$ is in $[0, \frac{\alpha}{\beta} + \frac{\alpha}{2}(1-x^2)]$ when $h(x)$ is defined by (3.2) and $s(x)$ is between 0 and $\frac{\alpha e^{10}}{\beta} + \frac{\alpha e^{10}}{2}(1-x^2)$, when $h(x)$ is defined by (3.16). $\Gamma_t = [0,1] \times (0, t_0)$.

In order to use monotone method, we need

Definition 3.6.1 *A smooth function u_0 is said to be an upper solution of Eq.(3.27) if*

$$Lu_0 + \alpha h(u_0) - \frac{\partial u_0}{\partial t} \leq 0, \qquad Bu_0 \geq s$$

similarly, v_0 is called a lower solution of Eq.(3.27) if

$$Lv_0 + \alpha h(v_0) - \frac{\partial v_0}{\partial t} \geq 0, \qquad Bv_0 \leq s.$$

where operator B is either a boundary operator or an initial value operator and s is 0 when operator B is a boundary operator. Given upper and lower solutions $u_0(t,x)$ and $v_0(t,x)$, with $v_0 \leq u_0$ on Γ_t, we choose C so large that $\alpha h_u + C > 0$ on the region $(x,t) \in \Gamma_t$, $\min_{\Gamma_t} v_0 \leq u \leq \max_{\Gamma_t} u_0$. Then define u_1 by

$$\begin{cases} Lu_1 - Cu_1 - \frac{\partial u_1}{\partial t} = -[\alpha h(u_0) + Cu_0] & , x \in \Gamma_t \\ u_{1x}(t,0) = 0, u_{1x}(t,1) + \beta u_1(t,1) = 0 & , u_1(0,x) = s(x) \end{cases} \tag{3.28}$$

By the maximum principle for parabolic equations it is easily seen that $u_1(t,x) \leq u_0(t,x)$ in Γ_t. The mapping $u_0(t,x) \to u_1(t,x)$ is denoted by $u_1 = Gu_0$. G is a monotone operator[4].

Lemma 3.6.1[4] *Let there exist an upper solution $u_0(t,x)$:*

$$Lu_0 + \alpha h(u_0) - \frac{\partial u_0}{\partial t} \leq 0,$$

$$Bu_0 \geq s \quad on \quad \partial\Gamma_t,$$

and a lower solution $v_0(t,x)$:

$$Lv_0 + \alpha h(v_0) - \frac{\partial v_0}{\partial t} \geq 0,$$

$$Bv_0 \le s \quad on \quad \partial\Gamma_t,$$

with $v_0 \le u_0$. *Define sequences* u_n *and* v_n *inductively by* $u_{n+1} = Gu_n$, $v_{n+1} = Gv_n$. *If* C *is chosen large enough so that*

$$\alpha \frac{\partial h(u)}{\partial u} + C > 0 \quad on \quad \min_{\Gamma_t} v_0 < u < \max_{\Gamma_t} u_0,$$

then the sequences $\{u_n\}$ *and* $\{v_n\}$ *are monotone nonincreasing and nondecreasing respectively. As* n *tends to infinity they both tend to a unique fixed point* $u = Gu$, *which is a strong solution of*

$$Lu + \alpha h(u) - \frac{\partial u}{\partial t} = 0, \quad Bu = s \quad on \quad \partial\Gamma_t.$$

such that $v_0(t,x) \le u(t,x) \le u_0(t,x)$ #

As in §3.5, there are

Theorem 3.6.1 *For Eq.(3.27) and* h *is defined by (3.2)*

$$u_0(t,x) = K(\frac{1}{\beta} + \frac{1}{2}(1-x^2)) \qquad with \qquad K > \alpha$$

and

$$v_0(t,x) = 0$$

are upper and lower solutions respectively. Hence there exists a unique regular solution w *of Eq.(3.27) and Eq.(3.2) such that* $v_0(t,x) \le w(t,x) \le u_0(t,x)$.

Proof: As $h(u)$ is defined by (3.2), $e^{-10} \le h(u) \le 1$. Also $v_0(0,x) \le s(x) \le u_0(0,x)$ is given condition.

$$\frac{\partial^2 u_0}{\partial x^2} + \alpha h(u_0) - \frac{\partial u_0}{\partial t} = -K + \alpha h(u_0) \le -K + Kh(u_0) = K(h(u_0)-1) \le 0$$

and

$$u_{0x}(t,0) = 0, \quad u_{0x}(t,1) + \beta u_0(t,1) = 0$$

hence $u_0(t,x)$ is an upper solution. It is easily seen that

$$\frac{\partial^2 v_0}{\partial x^2} + \alpha h(v_0) - \frac{\partial v_0}{\partial t} = 0 + \alpha h(v_0) \geq 0$$

and

$$v_{0x}(t,0) = 0, \quad v_{0x}(t,1) + \beta v_0(t,1) = 0$$

thus $v_0(t,x)$ is a lower solution. By lemma 3.6.1, there exists a unique regular w of Eq.(3.27) and Eq.(3.2) such that $v_0(t,x) \leq w(t,x) \leq u_0(t,x)$. #

For the same argument, it is easy to get

Theorem 3.6.2 *For Eq.(3.27) and h is defined by (3.16)*

$$u_0(t,x) = K(\frac{1}{\beta} + \frac{1}{2}(1-x^2)) \qquad \textit{with} \qquad K > \alpha e^{10}$$

and

$$v_0(t,x) = 0$$

are upper and lower solutions respectively. Hence there exists a unique regular solution w of Eq.(3.27) and Eq.(3.16) such that $v_0(t,x) \leq w(t,x) \leq u_0(t,x)$. #

Now consider the existence and uniqueness of solution for Eq.(2.12). For consistency, it is rewritten as

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\gamma f(u)}{(1+\mu\int_0^1 f(u)dx)^2} & ,0 < x < 1, 0 < t < t_0 \\ u_x(t,0) = 0, u_x(t,1) + \beta u(t,1) = 0 & ,u(0,x) = s(x) \end{cases} \tag{3.29}$$

where f is defined by (3.2).

It is not difficult to see that during the proof [4] of lemma 3.6.1 the important thing is to construct two monotone sequences, one is nondecreasing and the other is nonincreasing. For Eq.(3.29), though there is an integral as part of denominator, the monotone sequences can be still constructed. Similarly, define an operator G : $u_{n+1} \to u_n$ as

$$\begin{cases} Lu_{n+1} - Cu_{n+1} - \frac{\partial u_{n+1}}{\partial t} = -[\frac{\gamma f(u_n)}{(1+\mu\int_0^1 f(u_n)dx)^2} + Cu_n] & ,0 < x < 1, 0 < t < t_0 \\ u_{n+1,x}(t,0) = 0, u_{n+1,x}(t,1) + \beta u_{n+1}(t,1) = 0 & , u_{n+1}(0,x) = s(x) \end{cases}$$

we say G is monotone operator. In fact, suppose $u \geq v$, then

$$(L - C - \frac{\partial}{\partial t})(Gu - Gv) = -[\frac{\gamma f(u)}{(1+\mu\int_0^1 f(u)dx)^2} - \frac{\gamma f(v)}{(1+\mu\int_0^1 f(v)dx)^2} + C(u-v)]$$

Now it is needed to prove that the right hand side is negative for a large enough constant C. In fact, when $u > v$, $f(u) \leq f(v)$. Hence,

$$(1+\mu e^{-10})^2 \leq (1+\mu\int_0^1 f(u)dx)^2 \leq (1+\mu\int_0^1 f(v)dx)^2 \leq (1+\mu)^2$$

that is

$$\frac{1}{(1+\mu e^{-10})^2} \geq \frac{1}{(1+\mu\int_0^1 f(u)dx)^2} \geq \frac{1}{(1+\mu\int_0^1 f(v)dx)^2} \geq \frac{1}{(1+\mu)^2}$$

therefore

$$\begin{aligned} & \frac{f(u)}{(1+\mu\int_0^1 f(u)dx)^2} - \frac{f(v)}{(1+\mu\int_0^1 f(v)dx)^2} \\ \geq\ & \frac{f(u)}{(1+\mu\int_0^1 f(v)dx)^2} - \frac{f(v)}{(1+\mu\int_0^1 f(v)dx)^2} \\ \geq\ & \frac{f(u)-f(v)}{(1+\mu\int_0^1 f(v)dx)^2} \\ \geq\ & \frac{f(u)-f(v)}{(1+\mu e^{-10})^2} = \frac{f'(u^*)(u-v)}{(1+\mu e^{-10})^2} \end{aligned}$$

where the fact of $f(u) - f(v) \leq 0$ is used, u^* is a function between u and v. As $|f'(u)| \leq 10$, take C large enough such that $C > \frac{10\gamma}{(1+\mu e^{-10})^2}$, then the right hand side is negative. By the maximum principle for parabolic equations, $Gu \geq Gv$, or G is a monotone operator. Therefore, there is following theorem.

Theorem 3.6.3 *For Eq.(3.29) and f is defined by (3.2)*

$$u_0(t,x) = K(\frac{1}{\beta} + \frac{1}{2}(1-x^2)) \qquad \text{with} \qquad K \geq \alpha$$

and

$$v_0(t,x) = 0$$

are upper and lower solutions respectively. Hence there exists a unique regular solution w of Eq.(3.29) and Eq.(3.2) such that $v_0(t,x) \leq w(t,x) \leq u_0(t,x)$. #

Since the nonincreasing property of $f(u)$ is used in the above proof, this method can not be used directly for the NTC problem where the $g(u)$ is nondecreasing. However, it may still be possible to prove the uniqueness if the positive derivative is generalized as a positive operator. As it needs more concepts, such as weak form, weak solution, Banach space, etc., it is not discussed here in details.

Chapter 4

The Numerical Results

In this chapter, the numerical solutions for steady state and time dependent problems are given. There is agreement between the steady state solution calculated in §4.1 and the steady state solution of the time dependent problem as time increases.

4.1 Steady State Problem for PTC

In this part, the steady state problem for PTC is numerically solved for different parameters. The uniqueness and nonuniqueness of the solutions can be numerically obtained and demonstrated by graphs.

4.1.1 Without External Circuit

For convenience, the steady state problem is rewritten as following:

$$\begin{cases} \frac{d^2u}{dx^2} + \alpha f(u) = 0 & , 0 < x < 1 \\ u_x(0) = 0 & , u_x(1) + \beta u(1) = 0 \end{cases} \tag{4.1}$$

and

$$f(u) = \begin{cases} 1 & , u < 1 \\ e^{-10(u-1)} & , 1 < u < 2 \\ e^{-10} & , 2 < u \end{cases} \tag{4.2}$$

where α and β are parameters. Most of the effort is given to the case where $1 \leq u \leq 2$ for some x since the cases for $u < 1$ and $u > 2$ for all x are easy to treat.

Actually, as has been stated in Chapter 3

$$u = \frac{\alpha}{\beta} + \frac{\alpha}{2}(1 - x^2) \tag{4.3}$$

is a solution whenever $\alpha < \frac{1}{(\frac{1}{\beta}+\frac{1}{2})}$. Similarly

$$u = \frac{\alpha e^{-10}}{\beta} + \frac{\alpha e^{-10}}{2}(1 - x^2) \quad , u > 2 \tag{4.4}$$

is a solution whenever $\frac{\alpha e^{-10}}{\beta} > 2$ i.e. $\alpha > 2e^{10}\beta$. For α between these values u will be within [1,2] for some x and an explicit solution is difficult.

In order to solve equation (4.1) numerically we rewrite it as a system of ordinary differential equations

$$\begin{cases} u_x = v & , v(0) = 0 \\ v_x = -\alpha f(u) & , v(1) + \beta u(1) = 0 \end{cases} \tag{4.5}$$

To solve this problem the shooting method[17] is employed. For the details of existence and uniqueness of the method see [17, 18]. Now suppose $u(0) = s$, then we can get $u(x, s)$, $v(x, s)$ as the solution for

$$\begin{cases} u_x = v & , u(0) = s \\ v_x = -\alpha f(u) & , v(0) = 0 \end{cases} \tag{4.6}$$

If $s = s^*$ is a solution of $v(1, s) + \beta u(1, s) = 0$, then $y(x) = u(x, s^*)$ and $z(x) = v(x, s^*)$ is a solution of (4.5), i.e., a solution of (4.1). In order to obtain a solution s^* of $v(1, s) + \beta u(1, s) = 0$, Newton's iterative method is used. We first make an initial guess s^0 for s, by solving (4.6) we get $u(x, s^n)$ and $v(x, s^n)$. If $v(1, s^n) + \beta u(1, s^n) = 0$,

s^n is a solution. Otherwise, consider $s^n + \Delta s$ as new tentative solution, and we try to satisfy $v(1, s^n + \Delta s) + \beta u(1, s^n + \Delta s) = 0$. Since $v(1,s)$ and $u(1,s)$ are not linear functions of s, we can't get an exact solution for Δs directly. Using a Taylor expansion, we get $-\frac{v(1,s^n)+\beta u(1,s^n)}{\frac{dv(1,s^n)}{ds}+\beta\frac{du(1,s^n)}{ds}}$ as the approximation of Δs. Now put $s^{n+1} = s^n - \frac{v(1,s^n)+\beta u(1,s^n)}{\frac{dv(1,s^n)}{ds}+\beta\frac{du(1,s^n)}{ds}}$ as the next guess for s. Repeating the same method, we can get s^{n+2}, thus theoretically a sequence $\{s^m\}_{m=0}^{\infty}$. Also, we can get the solution s^* theoretically. In order to get the approximation of Δs, we assume that $u(x,s)$ and $v(x,s)$ are differentiable functions of s and differentiate the system (4.6) with respect to s, we introduce two new unknowns $\overline{u} = \frac{du}{ds}$, $\overline{v} = \frac{dv}{ds}$, and a new system of differential equations can be rewritten as

$$\begin{cases} u_x = v & ,u(0) = s^n \\ v_x = -\alpha f(u) & ,v(0) = 0 \\ \overline{u}_x = \overline{v} & ,\overline{u}(0) = 1 \\ \overline{v}_x = -\alpha f'(u)\overline{u} & ,\overline{v}(0) = 0 \end{cases} \tag{4.7}$$

and $\Delta s = -\frac{v(1,s^n)+\beta u(1,s^n)}{\overline{v}(1,s^n)+\beta\overline{u}(1,s^n)}$, $n = 0, 1, 2, \cdots$, s^0 arbitrary.

Now the problem becomes an initial value problem for the system (4.7). On an IBM-RISC 6000 computer running a Unix operating system, we use the IMSL library routine IVPRK (refer to Appendix A) to solve the initial value problem (4.7). During the progress, we met an interesting phenomena. For some parameters α and β if the initial value s^0 is not close to the solution, then an interesting cycle appears. After 2 or 3 steps, a two cycle appears, so that s^{n+2} is the same as s^n and s^{n+1} is the same as s^{n-1} with s^n different from s^{n-1}. If we make an initial guess very close to the solution, it seems to converge very quickly for any parameters α and β. To overcome the problem of cycling a bisection method is introduced. The bisection method guarantees convergence although the rate of convergence is slower than that obtained by Newton iterations. The technique used is that whenever oscillation appears or the increment of Δs is greater than 0.5 then the bisection subroutine is called and the condition for returning to Newton's iterative scheme is that Δs is less than 0.5. Here we choose the condition that Δs is less than 0.5 by numerical experimental experience. Actually, without the bisection method, oscillation will occur, whenever Δs is greater than 1.0.

The bisection routine is used for choosing the initial value for Newton iteration method. The technique is as follows. From any initial value for s and parameters α, β, the scheme is as following:

(1) Using IMSL library routine IVPRK solve initial value problem (4.7) to get values of $u(1,s^n)$, $v(1,s^n)$, $\overline{u}(1,s^n)$ and $\overline{v}(1,s^n)$. Set $\Delta s = -\frac{v(1,s^n)+\beta u(1,s^n)}{\overline{v}(1,s^n)+\beta\overline{u}(1,s^n)}$, and $T_n = v(1,s^n) + \beta u(1,s^n)$. Goto step (2).

(2) Set $s^{n+1} = s^n + \Delta s$ and solve (4.7) to get values of $u(1,s^{n+1})$, $v(1,s^{n+1})$, $\overline{u}(1,s^{n+1})$ and $\overline{v}(1,s^{n+1})$. Calculate $\Delta s = -\frac{v(1,s^{n+1})+\beta u(1,s^{n+1})}{\overline{v}(1,s^{n+1})+\beta\overline{u}(1,s^{n+1})}$, and $T_{n+1} = v(1,s^{n+1})$ $+\,\beta u(1,s^{n+1})$, goto step (3).

(3) If $\Delta s \leq 0.5$, return to Newton iterations, else goto step (4).

(4) If $T_{n+1} \times T_n \geq 0$, goto step (1), else goto step (5).

(5) Using $s^{n+2} = \frac{s^{n+1}+s^n}{2}$ as initial guess for s to get values of $u(1,s^{n+2})$, $v(1,s^{n+2})$, $\overline{u}(1,s^{n+2})$ and $\overline{v}(1,s^{n+2})$, calculating $\Delta s = -\frac{v(1,s^{n+2})+\beta u(1,s^{n+2})}{\overline{v}(1,s^{n+2})+\beta\overline{u}(1,s^{n+2})}$, and $T = v(1,s^{n+2}) + \beta u(1,s^{n+2})$. If $\Delta s \leq 0.5$, return to Newton iteration, else goto step (6).

(6) If $T \times T_n \geq 0$, then $T_n = T, s^n = s^{n+2}$, goto step (5), else $T_{n+1} = T, s^{n+1} = s^{n+2}$, goto step (5).

The main idea for bisection is as follows: Assume s^* to be the root of $v(1,s) +$

$\beta u(1,s) = 0$, first find two points s^0 and s^1 such that $v(1,s)+\beta u(1,s)$ takes different signs at the two points. Then from these two points, we bisect the interval $[s^0, s^1]$ and get a new point s^2, evaluate the sign of $v(1,s) + \beta u(1,s)$ at point s^2 to choose $[s^0, s^2]$ or $[s^2, s^1]$ as the new interval. Every time there are two points which make $v(1,s) + \beta u(1,s)$ take different signs. Repeating this procedure, a sequence $\{s^k\}$, $k = 1, 2, \ldots$ is obtained. Theoretically, $s^k \to s^*$. As it is known, the convergent rate of this method is linear, so it is used to choose an initial value for Newton's iterative method. In the above bisection scheme, step (1) to step (4) are used to find two points such that $v(1,s) + \beta u(1,s)$ assumes two different signs at these two points; step (5) to step (6) are used to further bisect the interval until the increment $\Delta s \leq 0.5$ and then a return to Newton's iterative method is made.

From the numerical results we can see that for some parameters α and β there are several shifts back and forth between Newton's method and the bisection method, and the method is convergent for any parameters and initial values. During the numerical experimental procedure, an interesting phenomenon is found. If the solutions are between 1 and 2, then for any parameters and initial values, the total number of Newton iterations is almost the same except that the bisection subroutine is used a different number of times. Also, whenever the solutions are bigger than 2 or less than 1, the number of Newton's iterations is always 2 although the number of times the bisection subroutine is used is quite different. Some data are summarized in the Table 4.1.1, where column 6 represents the number of Newton iterations, column 7 represents the number of times that bisection was used, column 8 represents the number of times the bisection routine was called, s represents the initial guess and * means total number of times bisection was used.

Table 4.1.1

Solutions of PTC Problem for Various Parameter α, β

α	β	s	$u(0)$	$u(1)$	6	7	8
999.000	0.005	0.5	9.0936	9.0709	2	19	1
999.000	0.010	0.5	4.5581	4.5355	2	18	1
999.000	0.100	0.5	1.8891	1.8106	11	14	1
750.000	0.100	0.5	1.8615	1.7840	10	15	1
500.000	0.100	0.5	1.8226	1.7465	9	13	1
100.000	0.100	0.5	1.6684	1.5981	13	11	1
75.000	0.100	0.5	1.6410	1.5716	12	12	1
50.000	0.100	0.5	1.6023	1.5344	9	10	1
25.000	0.100	0.5	1.5363	1.4709	8	9	1
15.000	0.100	0.5	1.4878	1.4242	8	10	1
14.000	0.100	0.5	1.4812	1.4179	13	12*	3
10.000	0.100	0.5	1.4493	1.3872	7	8	1
7.500	0.100	0.5	1.4221	1.3611	7	9	1
5.000	0.100	0.5	1.3837	1.3242	8	7	1
1.000	0.100	0.5	1.2323	1.1787	7	5	1
0.850	0.100	0.5	1.2171	1.1641	10	5	1
0.500	0.100	0.5	1.1675	1.1165	6	4	1
0.250	0.100	0.5	1.1030	1.0546	9	3	1

Table 4.1.1(continued)

α	β	s	$u(0)$	$u(1)$	6	7	8
0.100	0.100	0.5	1.0113	0.9642	6	2*	2
0.050	0.100	0.5	0.5250	0.5000	2	1	1
0.005	0.100	0.5	0.0525	0.0500	2	1	1

Col. 6—No. of Newton iterations, Col. 7—No. of times bisection used,

Col. 8—No. of calls to bisection routine.

From Table 4.1.1 it is easy to see that the numerical solutions are exactly the solutions (4.3), (4.4) whenever $u < 1$ or $u > 2$. For cases where u is between 1 and 2, it is possible to make some checks on the numerical solutions. Multiply the first equation by u_x in (4.1) and integrate from 0 to 1 with respect to x, then we get

$$\int_0^1 u_{xx}u_x dx + \alpha \int_0^1 f(u)u_x dx = 0 \tag{4.8}$$

Denote u_0 and u_1 as the values of u at 0 and 1 respectively thus

$$\frac{1}{2}u_x^2|_0^1 + \alpha \int_{u_0}^{u_1} f(u)du = 0 \tag{4.9}$$

and also use the boundary conditions in (4.1). Hence we have

$$\frac{1}{2}\beta^2 u^2(1) - \frac{\alpha}{10}[e^{-10(u_1-1)} - e^{-10(u_0-1)}] = 0 \tag{4.10}$$

If we substitute α, β, and u_0, u_1, obtained by the numerical method, into the left hand side of equation(4.10), it is seen that the error is very small (about 10^{-6}). That suggests that u_0 and u_1 are approximations of the exact solution.

From Table 4.1.1 it is easy to see that for some parameters we can get solution in which critical values of u (i.e., 1 or 2) are obtained, but there do not seem to be values of parameters make u greater than 2 at some points and less than 1 at some other points within the interval [0,1]. Using the same IMSL library routine we can estimate the locations at which u reaches the critical values 1 or 2. The Table 4.1.2 describes this for some cases, where ξ means the value at which $u(\xi) = 1$ or $u(\xi) = 2$. Here only the estimated intervals are given. These results give some indication of the situation for the time dependent problem.

Table 4.1.2

α	β	$u(0)$	$u(1)$	ξ
440.000	0.010	2.0090	1.9990	[0.92,0.96]
0.100	0.100	1.0109	0.9638	[0.48,0.52]
0.096	0.100	1.0029	0.9554	[0.24,0.28]

Location of interface ξ for some α, β.

4.1.2 With External Circuit

Here the steady state one dimensional problem with the extra conditon is considered (An external circuit connected to the thermistor). Numerical solutions are discussed.

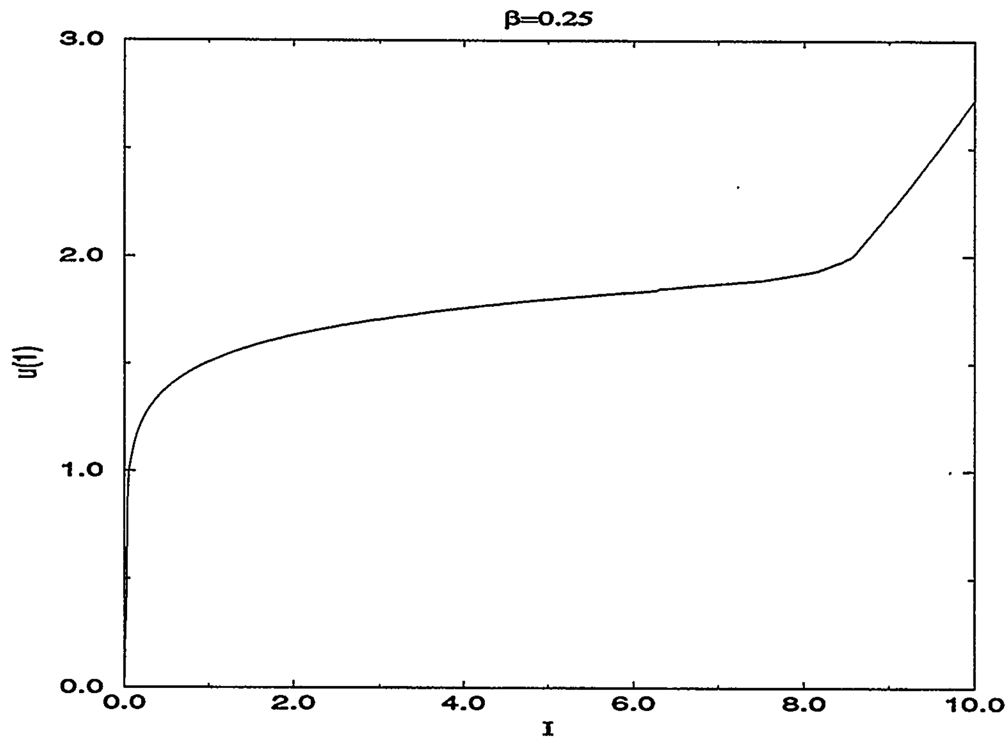
The equations are now as follows

$$\begin{cases} \frac{d2u}{dx^2} + \alpha f(u) = 0 & ,0 < x < 1 \\ u_x(0) = 0 & ,u_x(1) + \beta u(1) = 0 \\ 1 = I + \frac{\mu\beta u(1)}{\gamma I} & ,\alpha = \gamma I^2 \end{cases} \tag{4.11}$$

where γ and β are fixed parameters and $f(u)$ is defined by (4.2). In order to solve Eq.(4.11), it is decomposed into two parts, one as Eq.(4.1) and the other as

$$u(1) = \frac{\gamma I}{\mu\beta}(1 - I) \tag{4.12}$$

Figure 4.1.0



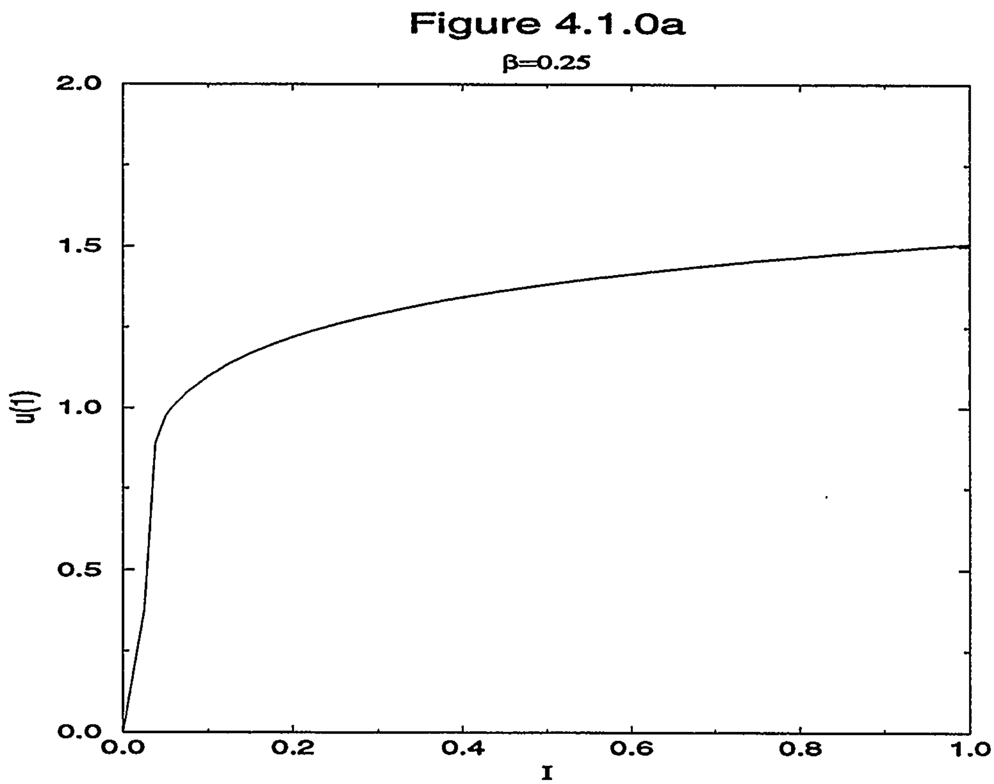
Plot of $u(1)$ *and* I *for solution of (4.1),* $0 \leq I \leq 10$.

For convenience, we rewrite the cold and hot solutions of (4.1) here. In fact, as we know, the cold solution ($u \leq 1$) of (4.1) can be written as

$$u(x) = \frac{\gamma I^2}{\beta} + \frac{\gamma I^2}{2}(1 - x^2)$$

and hot solution ($u \geq 2$) of (4.1) can be written as

$$u(x) = \frac{\gamma I^2 e^{-10}}{\beta} + \frac{\gamma I^2 e^{-10}}{2}(1 - x^2)$$



Plot of $u(1)$ *and* I *for solution of (4.1),* $0 \leq I \leq 1$.

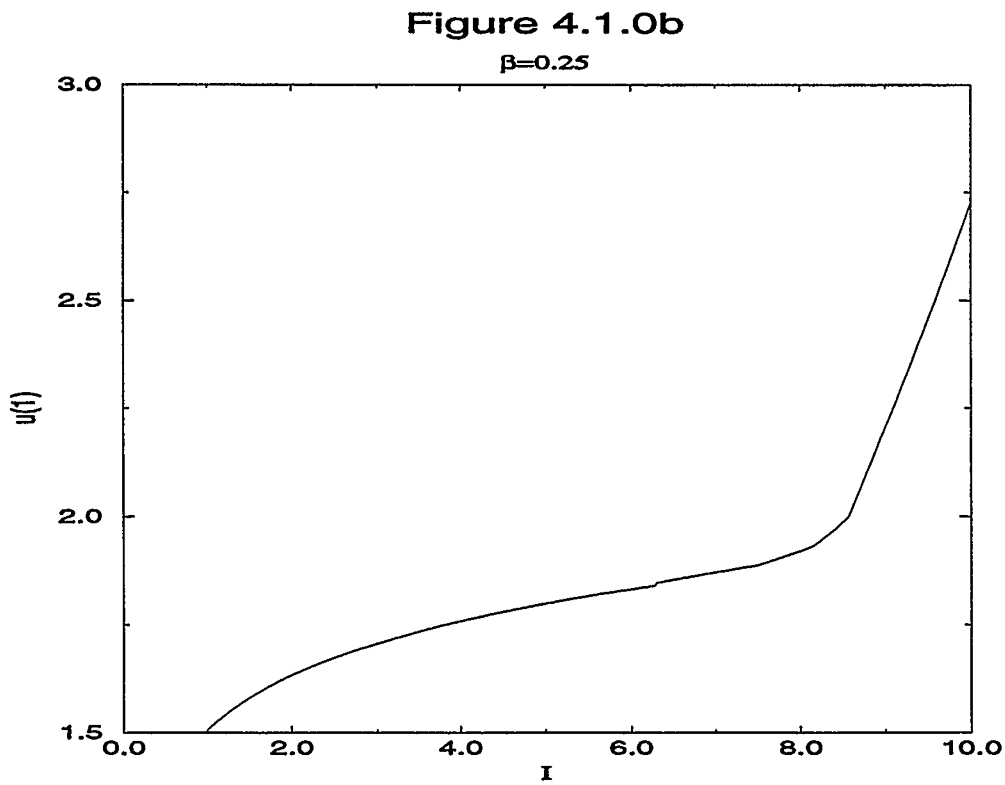
By the property discussed in Chapter 3, $u(x)$ is monotonically nonincreasing function of x, so for the cold solution

$$u(1) = \frac{\gamma I^2}{\beta} \tag{4.13}$$

iff $u(0) = \gamma I^2(\frac{1}{\beta} + \frac{1}{2}) \leq 1$ and for the hot solution

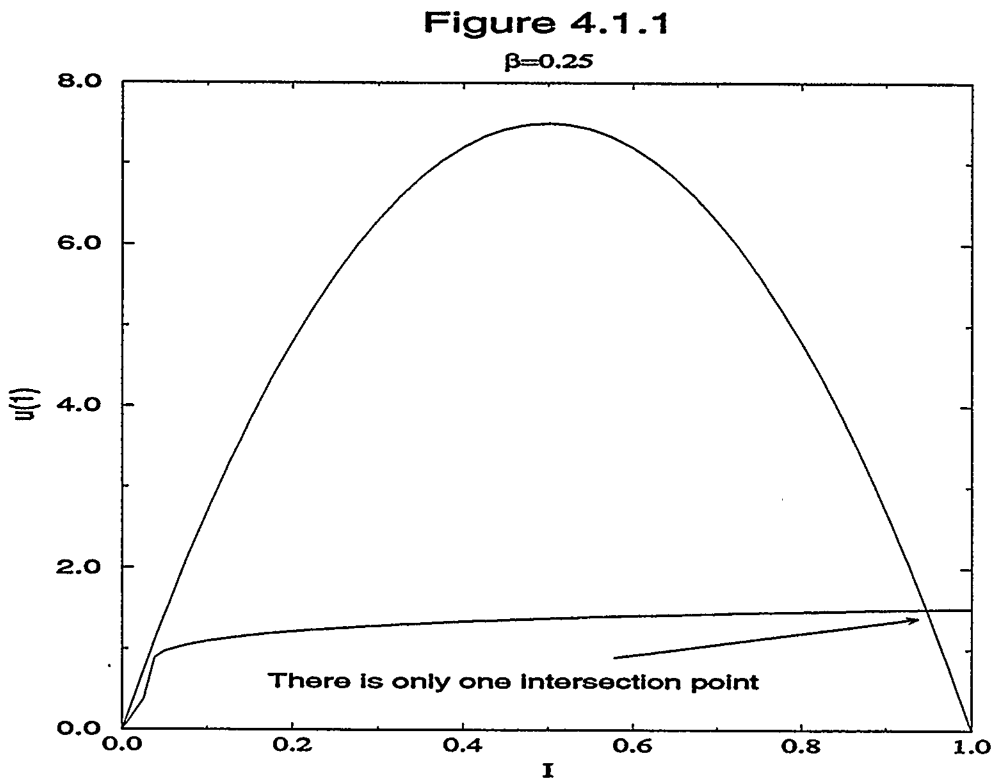
$$u(1) = \frac{\gamma I^2 e^{-10}}{\beta} \tag{4.14}$$

iff $u(1) = \frac{\gamma I^2 e^{-10}}{\beta} \geq 2$.



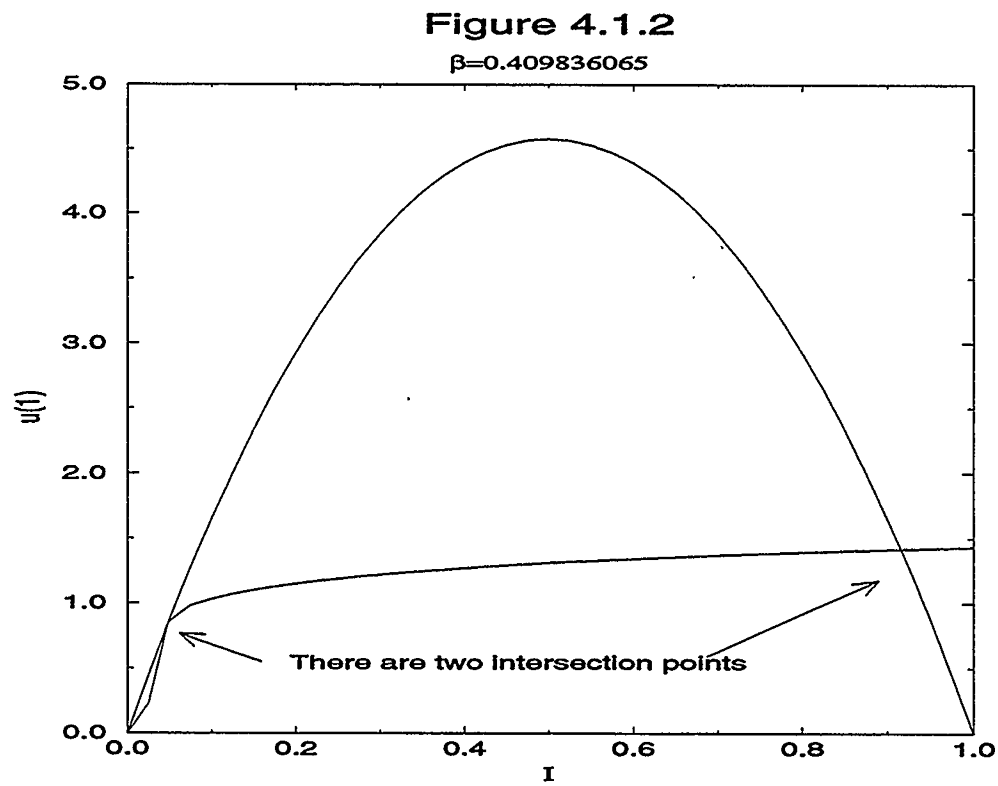
Plot of $u(1)$ *and* I *for solution of (4.1),* $1 \leq I \leq 10$.

In fact, Eq.(4.1) is solved already. Now the problem is how to include (4.12). If γ and β are fixed, the solution of (4.1) is a function of I, so it can be denoted as $U(x, I)$. Let $x = 1$, if $U(1, I)$ is a solution of (4.11), $U(1, I)$ should coincide with (4.12), i.e., $U(1, I) = u(1)$. Problem (4.11) may be solved by iteration. The method is as follows. For a given initial I_0, evaluating $\alpha_0 = \gamma I_0^2$, use the shooting method mentioned in §4.1.1 to solve (4.1) for the fixed α_0, then get $u(1, I_0)$. Now use (4.12) to determine I. Since Eq.(4.12) is quadratic in I two I are obtained, we choose the



Graphs of $u(1)$ and I for solution of (4.1) and solution of (4.12). Intersection point represents solution of coupled problem.

greater one for $\beta = 0.25$ as I_1. Next a $u(1, I_1)$ is obtained as a solution of Eq.(4.1) with $\alpha = \gamma I_1^2$. Repeating this procedure, two sequences $\{I_n\}$ and $\{u(1, I_n)\}$ are obtained. If $|I_{n+1} - I_n|$ and $|u(1, I_{n+1}) - u(1, I_n)|$ are less than a given tolerance, stop otherwise continue until the sequences converge. Therefore the numerical solutions of (4.11) are obtained. The sequences converges because as it is seen in Figure 4.1.1, at the intersection (i.e., solution of Eq.(4.11)) of the two curves, the absolute value of the product of the derivatives one with respect to I and one with respect to $u(1)$ of the two curves is less than 1. Hence the combination of the two curves makes a contraction operator about I, therefore the sequence $u(1, I_n)$ is convergent.

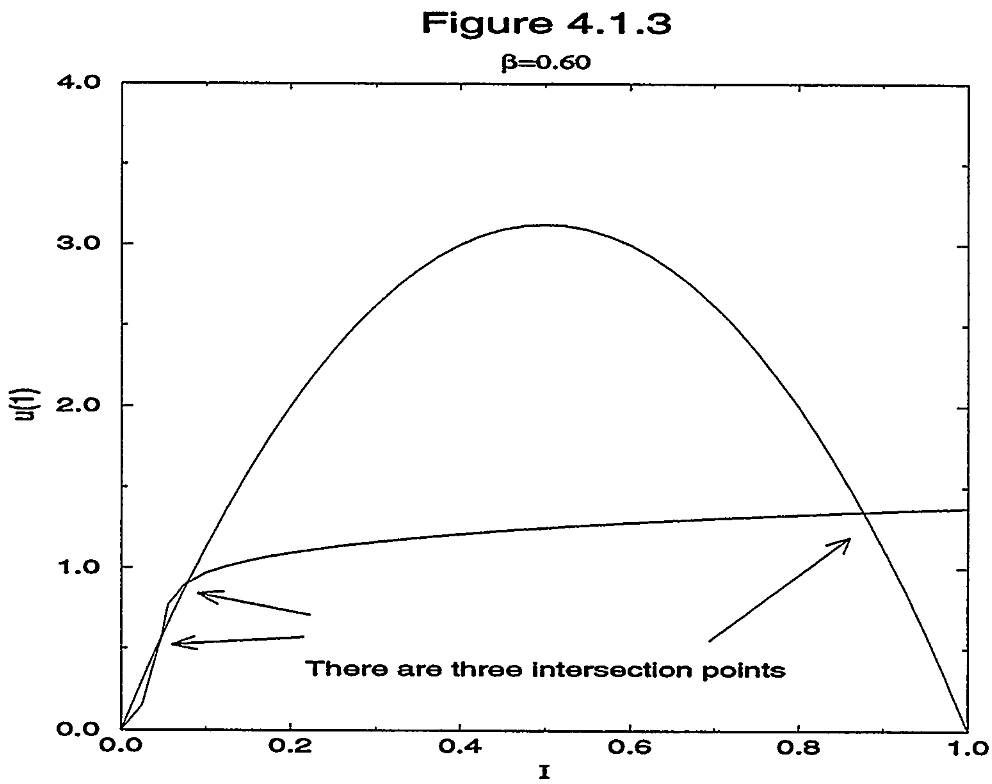


Graphs of $u(1)$ and I for solution of (4.1) and solution of (4.12).

Intersection points are solutions of coupled problem.

The solution to Eq.(4.11) may not be unique. This can be demonstrated graphically. If the solution of (4.1) at $x = 1$ is drawn as in Figure 4.1.0 (for fixed $\beta = 0.25$

for consistency, $\alpha = \gamma I^2, \gamma = 150$, it may be noted that the figure consists of three parts, the two critical points are $I_1 = 0.038490017$ and $I_2 = 8.568637736$. When $I \leq I_1$ and $I \geq I_2$, $u(1)$ is defined by (4.13) and (4.14) respectively; when $I_1 \leq I \leq I_2$, $u(1)$ is obtained by the method discussed in §4.1.1. For ease of reading the Figure 4.1.0, it is decomposed into two figures as Figure 4.1.0a and Figure 4.1.0b),



Graphs of $u(1)$ and I for solution of (4.1) and solution of (4.12).

Intersection points are solutions of coupled problem.

it is easy to see that when condition (4.12) is put into that graph, three cases may occur. One is that (4.12) and (4.1) have one intersection point. The second is that

(4.12) and (4.1) have two intersection points and the third is that (4.12) and (4.1) have three intersection points. The three cases can be obtained by adjusting the parameter β when $\gamma = 150$ and $\mu = 20$ are fixed. The method is as follows.

Consider that (4.13) touches with (4.12), hence,

$$\frac{\gamma I}{\beta\mu}(1-I) = \frac{\gamma I^2}{\beta} \tag{4.15}$$

therefore, $I_0 = \frac{1}{\mu+1}$. That means that (4.13) meets with (4.12) whenever $I = I_0$. However it must be noted that the intersection of (4.12) with (4.13) has meaning only if $u(0) = \gamma I_0^2(\frac{1}{\beta} + \frac{1}{2}) \leq 1$. From $u(0) = \gamma I_0^2(\frac{1}{\beta} + \frac{1}{2}) \leq 1$, there is $\beta \geq \frac{1}{\frac{(\mu+1)^2}{\gamma} - \frac{1}{2}}$. We say that $\beta_0 = \frac{1}{\frac{(\mu+1)^2}{\gamma} - \frac{1}{2}} = 0.409836065$ is a critical value. When $\beta = \beta_0$, (4.13) just touches (4.12) which means that there are two solutions; $\beta > \beta_0$, (4.13) has one intersection point with (4.12) and the end of (4.13) goes above the parabola (4.12) which means that there are three solutions; $\beta < \beta_0$, (4.13) can not meet with (4.12) which means that there is only one solution. The Figure 4.1.1 ($\beta = 0.25$), Figure 4.1.2 ($\beta = 0.409836065$) and Figure 4.1.3 ($\beta = 0.60$) demonstrate this. A similar situation can be discussed when (4.14) intersects, does not intersect and just intersects with (4.12) (here just intersects means that the end point of parabola (4.14) is just on the parabola (4.12)).

4.2 Time Dependent Problem for PTC

Here the time dependent one dimensional PTC problem is considered. Numerical solutions are discussed.

The equation is as follows

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha f(u) & ,0 < x < 1 \\ u_x(0) = 0 & ,u_x(1) + \beta u(1) = 0 \end{cases} \tag{4.16}$$

and

$$f(u) = \begin{cases} 1 & ,u \leq 1 \\ e^{-10(u-1)} & ,1 < u < 2 \\ e^{-10} & ,2 \leq u \end{cases} \tag{4.17}$$

where α and β are parameters.

4.2.1 Moving Meshpoints Method

In order to get the numerical solution of Eq.(4.16), the finite element method is used. At first we expect that the whole solution is within the "cold " region. So the Crank-Nicholson method is used to find the time when the first critical temperature is reached at one point. Whenever the critical temperature is reached, a finite element method is used. Initially, suppose the whole region to be cold, $u_{i,j} \approx u(ih, j\Delta t), i = 0, \ldots, N$, and $h = 1.0/N$, then the Crank-Nicholson difference scheme is as following

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{1}{2}\left[\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right] + \alpha$$

i.e.,

$$\begin{aligned} &-\frac{r}{2}u_{i+1,j+1} + (1+r)u_{i,j+1} - \frac{r}{2}u_{i-1,j+1} \\ &= \frac{r}{2}u_{i+1,j} + (1-r)u_{i,j} + \frac{r}{2}u_{i-1,j} + \alpha\Delta t \end{aligned} \tag{4.18}$$

where $r = \frac{\Delta t}{h^2}, i = 0, \ldots, N$. For the boundary conditions, the central difference is used for good accuracy. In order to use the central difference formula it is necessary to introduce the "fictitious" temperature $u_{-1,j}$ at the external mesh point $(-h, j\Delta t)$, by imagining the thermistor to be extended a distance h at this end. That is

$$\begin{cases} \frac{u_{i+1,j}-u_{i-1,j}}{2h} = 0 & , u_{1,j} = u_{-1,j} & , i = 0 \\ \frac{u_{N+1,j}-u_{N-1,j}}{2h} + \beta u_{N,j} = 0 & , u_{N+1,j} = u_{N-1,j} - 2h\beta u_{N,j} & , i = N \end{cases} \tag{4.19}$$

The temperature $u_{-1,j}$ is unknown and necessitates another equation. This is obtained by assuming that Eq.(4.16) is satisfied at the end point $x = 0$. The $u_{-1,j}$ can be eliminated between these equations. A similar method is used for the other end point $x = 1$.

When the solution at $x = 0$ reaches 1, the interface point appears. The problem at the next timestep is to choose the new position of the interface point and time stepsize. In order to solve this problem, the following method is employed.

Suppose $x = x_1(t)$ to be the interface at which the temperature is the critical temperature $u_c = 1$ (i.e., $\sigma(u_c) = 1$). Define new variables $\xi = \frac{x}{x_1(t)}, \tau = t$ (ξ is relative coordinate). Then

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - \frac{x}{x_1^2}\dot{x}_1\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \tau} - \xi\frac{\dot{x}_1}{x_1}\frac{\partial}{\partial \xi}$$
$$\frac{\partial}{\partial x} = \frac{1}{x_1}\frac{\partial}{\partial \xi} \tag{4.20}$$

Let $u = u(\xi, \tau)$, note $\frac{\partial u(1,\tau)}{\partial \tau} = 0$ for all τ because $u(1, \tau) = 1$ for all τ. The Eq.(4.16) becomes

$$\frac{\partial u}{\partial \tau} - \xi\frac{\dot{x}_1}{x_1}\frac{\partial u}{\partial \xi} = \frac{1}{x_1^2}\frac{\partial^2 u}{\partial \xi^2} + \alpha\sigma(u(\xi, \tau)) \tag{4.21}$$

At $\xi = 1$, $0 - \frac{\dot{x}_1}{x_1}\frac{\partial u}{\partial \xi}|_{\xi=1} = \frac{1}{x_1^2}\frac{\partial^2 u}{\partial \xi^2}|_{\xi=1} + \alpha$, since $\frac{\partial u}{\partial \xi}|_{\xi=0} = 0, \frac{\partial u}{\partial \xi}|_{\xi=1} = \int_0^1 \frac{\partial^2 u}{\partial \xi^2}d\xi = x_1\int_0^{x_1}\frac{\partial^2 u}{\partial x^2}dx = x_1^2.\frac{1}{x_1}\int_0^{x_1}\frac{\partial^2 u}{\partial x^2}dx$ and $-x_1\dot{x}_1[\frac{1}{x_1}\int_0^{x_1}\frac{\partial^2 u}{\partial x^2}dx] = \frac{\partial^2 u}{\partial x^2}|_{x=x_1} + \alpha$, it is now assumed that $\frac{\partial^2 u}{\partial x^2}$ changes very little for x_1 small (or let $x_1 \to 0, \dot{x}_1 \to \infty$), then

$$(1 + x_1\dot{x}_1)(-\frac{\partial^2 u}{\partial x^2}|_{x=0}) = \alpha \tag{4.22}$$

because $x_1\dot{x}_1(\frac{\partial^2 u}{\partial x^2})|_{x=0} = -(\frac{\partial^2 u}{\partial x^2})|_{x=0} - \alpha, i.e., x_1\dot{x}_1 = -1 - \frac{\alpha}{(\frac{\partial^2 u}{\partial x^2})_{x=0}} = (\frac{\alpha}{k_0} - 1)$. $k_0 = \frac{\partial^2 u}{\partial x^2}|_{x=0}$ is estimated from the previous time step. Thus it is assumed that

$$\frac{d}{dt}(x_1^2) = 2(\frac{\alpha}{k_0} - 1) \tag{4.23}$$

(Note that $x_1\dot{x}_1 = \frac{1}{2}\frac{d}{dt}(x_1^2)$).

Hence at first step if x_1 is determined by the following method. The time step Δt can be chosen as $x_1^2/2(\frac{\alpha}{k_0} - 1)$. In the actual program, x_1 is first assigned a value of $h/8$ where h is the mesh length and Δt chosen as above. Thereafter the problem is solved in two phases.

Denote the interface point as $x = \xi_2$. The interval $[0,1]$ is divided into two segments, $[0,\xi_2]$ and $[\xi_2, 1]$. Since ξ_2 is very small at the beginning, it is not necessary to put any points within $[0,\xi_2]$. $[\xi_2, 1]$ is divided into nc equal segments. Thus the total number of node points is $nt = nc + 2$. Denote $hc = (1 - \xi_2)/nc$, node points by x_i. Hence $x_1 = 0$, $x_2 = \xi_2$, $x_i = \frac{nt-i}{nc}\xi_2 + \frac{i-2}{nc}, i = 3, \ldots, nt$. Since ξ_2 is not fixed, x_i depends upon time t. Define trial and test function as

$$\phi_i(x,t) = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & , [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & , [x_i, x_{i+1}] \end{cases} \tag{4.24}$$

Multiply Eq.(4.16) by $\phi_j(x,t)$ and integrate from 0 to 1, then

$$\int_0^1 (\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \alpha\sigma(u))\phi_j(x,t)dx = 0, \tag{4.25}$$

where $j = 1, \ldots, nt$, i.e.,

$$< u_t, \phi_j > + < u_x, \phi_{j,x} > -\beta u(1,t)\delta_{nt,j} - \alpha < \sigma(u), \phi_j >= 0, \tag{4.26}$$

where $j = 1, \ldots, nt$. Replace u by the approximate form $\sum_{j=1}^{nt} u_j \phi_j$ where u_j are functions of t and ϕ_j are functions of x and t, and integrate by parts, then

$$\sum_{i=1}^{nt}(u_{i,t} < \phi_i, \phi_j > + u_i < \phi_{i,t}, \phi_j > + u_i < \phi_{i,x}, \phi_{j,x} >) -$$
$$\beta u(1, nt)\delta_{nt,j} - \alpha < \sigma(\sum_{i=1}^{nt} u_i \phi_i), \phi_j >= 0, \tag{4.27}$$

where $j = 1, \ldots, nt$ and $< \phi_i, \phi_j >= \int_0^1 \phi_i \phi_j dx$. Hence

$$i = 1, \tag{4.28}$$
$$\frac{1}{3}\xi_2 u_{1,t} + \frac{u_1}{\xi_2} - \frac{u_2}{\xi_2} + \frac{1}{6}\dot{\xi}_2(u_1 - u_2) =$$
$$\frac{\alpha\xi_2}{10(u_2 - u_1)}[e^{-10(u_1-1)} + \frac{e^{-10(u_2-1)} - e^{-10(u_1-1)}}{10(u_2 - u_1)}]$$

$$i = 2, \tag{4.29}$$
$$\frac{1}{6}\xi_2 u_{1,t} + \frac{1}{6}(x_3 - \xi_2)u_{3,t} + \frac{1}{3}\dot{\xi}_2 u_1 + \frac{1}{6}\dot{x}_3 u_2 - (\frac{1}{3}\dot{\xi}_2 + \frac{1}{6}\dot{x}_3)u_3$$
$$-\frac{u_1}{\xi_2} + u_2(\frac{1}{\xi_2} + \frac{1}{x_3 - \xi_2}) - \frac{u_3}{x_3 - \xi_2} = \frac{\alpha hc}{2}$$
$$-\frac{\alpha\xi_2}{10(u_2 - u_1)}[e^{-10(u_2-1)} + \frac{e^{-10(u_2-1)} - e^{-10(u_1-1)}}{10(u_2 - u_1)}]$$

$$i = 3, \tag{4.30}$$
$$\frac{2}{3}hcu_{3,t} + \frac{1}{6}hcu_{4,t} + \frac{1}{hc}(-u_2 + 2u_3 - u_4) + \frac{3nc - 2}{6nc}u_2\dot{\xi}_2$$
$$-\frac{1}{3nc}u_3\dot{\xi}_2 - \frac{3nc - 4}{6nc}\dot{\xi}_2 u_4 = \alpha hc$$

$$i = 4, \ldots, nt - 1, \tag{4.31}$$
$$\frac{hc}{6}(u_{i-1,t} + 4u_{i,t} + u_{i+1,t}) + \dot{\xi}_2(\frac{3nt - 3i + 1}{6nc}u_{i-1} - \frac{2}{6nc}u_i$$
$$-\frac{3nt - 3i - 1}{6nc}u_{i+1}) + \frac{1}{hc}(-u_{i-1} + 2u_i - u_{i+1}) = \alpha hc$$

$$i = nt, \tag{4.32}$$

$$\frac{hc}{6}(u_{nt-1,t} + 2u_{nt,t}) + \frac{1}{6nc}\dot{\xi}_2(u_{nt-1} - u_{nt}) + \frac{1}{hc}(-u_{nt-1} + u_{nt}) \\ +\beta u_{nt} = \frac{\alpha}{2}hc$$

Now denote $\Delta\xi_2$ and Δt as increments of ξ_2 and t respectively. Use

$\int_t^{t+\Delta t} f(t)dt \approx \frac{1}{2}(f(t+\Delta t) + f(t))\Delta t$

and

$\int_t^{t+\Delta t} f_t g dt \approx (f(t+\Delta t) - f(t))(\frac{g(t+\Delta t)+g(t)}{2})$

Integrate the equations $i = 1, \ldots, nt$,using the above approximations. Simplifying the equations we get

$$i = 1, \tag{4.33}$$

$$(\frac{1}{3}\xi_2 + \frac{1}{4}\Delta\xi_2 + \frac{\Delta t}{2(\xi_2 + \Delta\xi_2)})u_1^{n+1} - (\frac{1}{6} + \frac{u_1^n}{12})\Delta\xi_2 \\ -\frac{\Delta t}{2(\xi_2 + \Delta\xi_2)} + (-\frac{1}{3}\xi_2 + \frac{\Delta t}{2\xi_2})u_1^n - \frac{\Delta t}{2\xi_2} \\ -\frac{\alpha}{2}\Delta t[\frac{\xi_2 + \Delta\xi_2}{10(1-u_1^{n+1})}(e^{-10(u_1^n-1)} + \frac{1 - e^{-10(u_1^{n+1}-1)}}{10(1-u_1^{n+1})}) \\ +\frac{\xi_2}{10(1-u_1^n)}(e^{-10(u_1^n-1)} + \frac{1 - e^{-10(u_1^n-1)}}{10(1-u_1^n)})] = 0$$

$$i = 2, \tag{4.34}$$

$$(\frac{1}{6}\xi_2 + \frac{1}{4}\Delta\xi_2 - \frac{\Delta t}{2(\xi_2 + \Delta\xi_2)})u_1^{n+1} + (\frac{u_1^n}{12} - \frac{3nc-2}{12nc}u_3^n \\ +\frac{nc-1}{6nc} + \frac{\alpha\Delta t}{4nc})\Delta\xi_2 + \frac{\Delta t}{2}(\frac{1}{\xi_2 + \Delta\xi_2} + \frac{1}{hc - \frac{\Delta\xi_2}{nc}}) \\ +(\frac{hc}{6} - \frac{1}{4}\Delta\xi_2 - \frac{\Delta t}{2(hc - \frac{\Delta\xi_2}{nc})})u_3^{n+1} - (\frac{1}{6}\xi_2 + \frac{\Delta t}{2\xi_2})u_1^n$$

$$
\begin{aligned}
&-(\frac{hc}{6}+\frac{\Delta t}{2hc})u_3^n+\frac{\Delta t}{2\xi_2}+\frac{\Delta t}{2hc}-\frac{\alpha hc\Delta t}{2}\\
&-\frac{\alpha\Delta t}{2}[\frac{\xi_2+\Delta\xi_2}{10(1-u_1^{n+1})}(1+\frac{1-e^{-10(u_1^{n+1}-1)}}{10(1-u_1^{n+1})})\\
&\frac{\Delta\xi_2}{10(1-u_1^n)}(1+\frac{1-e^{-10(u_1^n-1)}}{10(1-u_1^n)})]=0
\end{aligned}
$$

$$
i=3, \tag{4.35}
$$

$$
\begin{aligned}
&(\frac{2}{3}-\frac{\Delta\xi_2}{2(1-\xi_2)}+\frac{\Delta t}{hc(hc-\frac{\Delta\xi_2}{nc})})u_3^{n+1}+(\frac{1}{6}-\frac{nc-1}{4(1-\xi_2)}\Delta\xi_2\\
&-\frac{\Delta t}{2hc(hc-\frac{\Delta\xi_2}{nc})})u_4^{n+1}+\Delta\xi_2(\frac{u_3^n}{6(1-\xi_2)}-\frac{3nc-5}{12(1-\xi_2)}u_4^n+\frac{3nc-2}{6(1-\xi_2)}\\
&+\frac{\alpha\Delta t}{2(1-\xi_2)})-\frac{\Delta t}{2hc(hc-\frac{\Delta\xi_2}{nc})}-(\frac{2}{3}-\frac{\Delta t}{hc^2})u_3^n\\
&-(\frac{1}{6}+\frac{\Delta t}{2hc^2})u_4^n-\frac{\Delta t}{2hc^2}-\alpha\Delta t=0
\end{aligned}
$$

$$
i=4,\ldots,nt-1, \tag{4.36}
$$

$$
\begin{aligned}
&(\frac{1}{6}+\frac{3nt-3i}{12(1-\xi_2)}\Delta\xi_2-\frac{\Delta t}{2hc(hc-\frac{\Delta\xi_2}{nc})})u_{i-1}^{n+1}+(\frac{2}{3}-\frac{\Delta\xi_2}{2(1-\xi_2)}\\
&+\frac{\Delta t}{hc(hc-\frac{\Delta\xi_2}{nc})})u_i^{n+1}+(\frac{1}{6}-\frac{3nt-3i}{12(1-\xi_2)}\Delta\xi_2-\frac{\Delta t}{2hc(hc-\frac{\Delta\xi_2}{nc})})u_{i+1}^{n+1}\\
&+\Delta\xi_2(\frac{3nt-3i+2}{12(1-\xi_2)}u_{i-1}^n+\frac{u_i^n}{6(1-\xi_2)}-\frac{3nt-3i-2}{12(1-\xi_2)}u_{i+1}^n+\frac{\alpha\Delta t}{2(1-\xi_2)})\\
&-(\frac{1}{6}+\frac{\Delta t}{2hc^2})u_{i-1}^n-(\frac{2}{3}-\frac{\Delta t}{hc^2})u_i^n-(\frac{1}{6}+\frac{\Delta t}{2hc^2})u_{i+1}^n-\alpha\Delta t=0
\end{aligned}
$$

$$
i=nt, \tag{4.37}
$$

$$
\begin{aligned}
&(\frac{1}{6}-\frac{\Delta t}{2hc(hc-\frac{\Delta\xi_2}{nc})})u_{nt-1}^{n+1}+(\frac{1}{3}-\frac{\Delta\xi_2}{4(1-\xi_2)}\\
&+\frac{\Delta t}{2hc(hc-\frac{\Delta\xi_2}{nc})}+\frac{\beta\Delta t}{2hc})u_{nt}^{n+1}+\Delta\xi_2(\frac{u_{nt-1}^n}{6(1-\xi_2)}+\frac{u_{nt}^n}{12(1-\xi_2)}\\
&+\frac{\alpha\Delta t}{4(1-\xi_2)})-(\frac{1}{6}+\frac{\Delta t}{2hc^2})u_{nt-1}^n-(\frac{1}{3}-\frac{\Delta t}{2hc^2}-\frac{\beta\Delta t}{2hc})u_{nt}^n=0
\end{aligned}
$$

In Eq.(4.33), since u_1 is very close to 1 at the beginning, a Taylor series expansion suggest that we use ξ_2 to replace

$$\frac{\xi_2 + \Delta\xi_2}{10(1-u_1^{n+1})}(e^{-10(u_1^n-1)} + \frac{1-e^{-10(u_1^{n+1}-1)}}{10(1-u_1^{n+1})})$$
$$+\frac{\xi_2}{10(1-u_1^n)}(e^{-10(u_1^n-1)} + \frac{1-e^{-10(u_1^n-)}}{10(1-u_1^n)})$$

A similar method was used in Eq.(4.34). For later times the original equations are used. It is very clear that system is nonlinear in $(u_1, \Delta\xi_2, u_3, \ldots, u_{nt})$. Here it must be noted that the second unknown $\Delta\xi_2$ is the increment of ξ_2 since u_2 is always assumed to be 1. In order to solve equations (4.33-4.37), Newton's method is again applied. Since the variable $\Delta\xi_2$ is different from others, some special techniques are used to solve for it. If the system is denoted as $\mathbf{F}(u_1, \Delta\xi_2, u_3, \ldots, u_{nt}) = 0$ where $u = (u_1, \Delta\xi_2, u_3, \ldots, u_{nt})^T$,the Newtonian iterative scheme can be written as $u^{n+1} = u^n - (\nabla\mathbf{F}(u^n))^{-1}\mathbf{F}(u^n)$. Let $\Delta u = u^{n+1} - u^n$, hence $[\nabla\mathbf{F}(u^n)]\Delta u = -\mathbf{F}(u^n)$ must be solved. First let us examine the form of Jacobian $\nabla\mathbf{F}(u^n)$. Obviously most of the entries are zero, and for convenience a_{ij} means that entry is not zero. The second column corresponds to $\Delta\xi_2$.

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 & 0 & 0 & \ldots & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 & 0 & \ldots & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 & 0 & \ldots & 0 \\ 0 & a_{42} & a_{43} & a_{44} & a_{45} & 0 & \ldots & 0 \\ 0 & a_{52} & 0 & a_{54} & a_{55} & a_{56} & \ldots & 0 \\ \vdots & \vdots & \vdots & & & \ddots & \ddots & \\ 0 & a_{nt2} & 0 & 0 & 0 & 0 & \ldots & a_{ntnt} \end{pmatrix} \tag{4.38}$$

The problem then is to solve a system $Ax = b$ where $x = (x_1, x_2, \dots, x_{nt})^T$ and $b = (b_1, \dots, b_{nt})^T$. Denote $a_2 = (a_{32}, \dots, a_{nt2})^T$, $b^1 = (b_3, \dots, b_{nt})^T$,

$$C = \begin{pmatrix} a_{33} & a_{34} & 0 & 0 & \dots & 0 \\ a_{43} & a_{44} & a_{45} & 0 & \dots & 0 \\ 0 & a_{54} & a_{55} & a_{56} & \dots & 0 \\ \vdots & & & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & \dots & a_{ntnt} \end{pmatrix} \tag{4.39}$$

$x^1 = (x_3^1, \dots, x_{nt}^1)^T$ and $x^2 = (x_3^2, \dots, x_{nt}^2)^T$. Solve $Cx^1 = b^1$ and $Cx^2 = a_2$ first (where C is a tridiagonal matrix). Then let $x = (x_1, x_2, x_3^1 - x_2x_3^2, \dots, x_{nt}^1 - x_2x_{nt}^2)^T$ and choose x_1 and x_2 so that the remaining equations of the systems $Ax = b$ are satisfied. This method allows the Δu to be found, and hence u^{n+1}. Repeat this until either $\Delta\xi_2 \leq 0$ or $\xi_2 \geq \frac{1}{nt}$. If $\Delta\xi_2 \leq 0$ for the first time step after the introduction of the interface, this means that the interface point ξ_2 is almost equal to 0, i.e., the temperature reaches its critical value (which here is 1) only at the point $x = 0$. If $\xi_2 > \frac{1}{nt}$, then $[0, \xi_2]$ is partitioned into two pieces. As ξ_2 increases, more node points will be added to the interval $[0, \xi_2]$, i.e., this interval is partitioned into ns equal segments.

In the actual algorithm, when the temperature at the center point becomes "hot", the program uses four subroutines. The first subroutine is devoted to the case in which only one point is a hot point. As time increases, ξ_2 increases. When ξ_2 is greater than h, more nodal points will be added, then the next subroutine is called. The second subroutine deals with the case where there are more than one mesh point on both sides of the interface point. If ξ_2 keeps moving forward, the number of mesh points at the right side of interface point reduce to one. Hence, the third subroutine is engaged. When ξ_2 is very close to the right end point ($x = 1$), it is considered that the whole region is within the transient region. Therefore, the fourth subroutine is applied. Repeat using the fourth subroutine until the absolute value of biggest difference between newly obtained values and old values at all points satisfies a specified tolerance and then stop.

Table 4.2.1

α	β	$u(0)$	$u(1)$	$v(0)$	$v(1)$	error
7.50	0.250	1.362675	1.240080	1.362729	1.240063	0.00005
3.50	0.250	1.289988	1.172719	1.290052	1.172717	0.00006
2.50	0.250	1.257984	1.143097	1.258052	1.143100	0.00006
1.00	0.250	1.171109	1.062799	1.171199	1.062828	0.00009
0.90	0.250	1.161152	1.053608	1.161241	1.053635	0.00008
0.80	0.250	1.150019	1.043332	1.150117	1.043369	0.00009
0.70	0.250	1.137422	1.031710	1.137516	1.031741	0.00009
0.50	0.250	1.105845	1.002560	1.106811	1.002506	0.00005
0.40	0.250	1.084141	0.981651	1.084082	0.981590	0.00005
0.35	0.250	1.069982	0.966818	1.069938	0.966847	0.00004
0.30	0.250	1.051984	0.947115	1.051951	0.947085	0.00003

End point values of solutions for various α, β.

u is the solution of the time dependent problem as $t \longrightarrow \infty$.

v is the numerical solution of the steady state problem.

In the first three subroutines, similar methods are used to solve $[\nabla \mathbf{F}(u^n)]\Delta u =$

$-\mathbf{F}(u^n)$. The criterion for adding and removing nodal points is as follows: when $\xi_2 \le 0.5$ and $\xi_2 \ge \frac{ns \times h}{2}$, the nodal points between 0 and ξ_2 are doubled. In this way, when ξ_2 is very close to the middle point, the number of nodal points between 0 and ξ_2 is the same as the total number which is taken initially when the whole region was considered as a cold region. When $1-\xi_2 \le \frac{nc \times h}{2}$, $\xi_2 \ge 0.75$ and $nc > 1$, half of nodal points are removed. This means when ξ_2 moves towards 1, less and less nodal points between ξ_2 and 1 remain, the number of nodal points are determined by how close ξ_2 is to 1. When the sum of increment $\xi_2 + \Delta\xi_2$ is greater than 1, the whole region is considered as a transient region. The timestep Δt is also adjusted according to the ratio of previous values of $\Delta\xi_2$ and Δt.

The Table 4.2.1 gives the results obtained using the above mentioned program, where $u(0)$ and $u(1)$ represent numerical solutions of Eq.(4.16) at points 0 and 1 respectively after the steady state conditions have been attained and $v(0)$ and $v(1)$ the corresponding numerical solutions of the steady state problem (4.1) corresponding to Eq.(4.16). "Error" means the maximum error of the two numerical solutions corresponding to Eq.(4.16) and the steady state problem (4.1). From the Table 4.2.1 it is easy to see that biggest error is 0.00009 and smallest 0.00003. Also from Table 4.2.1 it can be seen that when α is between 0.4 and 0.3 inclusively, two phases appear. One phase is for the cold region and another one is for the transient region. This means that for those α the whole region is not in the transient state and therefore the interface point ξ_2 needs to be located.

The Table 4.2.2 gives the results obtained by two methods. One is the program mentioned above and the other is the IMSL shooting method. From the results it can be seen that the outcomes match very well. The value of ξ_2 on the fifth column is obtained by the above mentioned program and the approximated interval of ξ_2 on the eighth column is obtained by using the IMSL subroutine for the steady state problem.

Table 4.2.2

α	β	$u(0)$	$u(1)$	ξ_2	$v(0)$	$v(1)$	ξ_2
0.40	0.25	1.084141	0.981651	0.920	1.084082	0.981590	[0.88,0.92]
0.35	0.25	1.069982	0.966818	0.846	1.069938	0.966847	[0.84,0.88]
0.30	0.25	1.051984	0.947115	0.731	1.051951	0.947085	[0.72,0.76]

Position of the " interface " point of ξ_2 from the time dependent solution as $t \longrightarrow \infty$ and from the steady state solution.

Also through the program, ξ_2 can be traced during the whole time period before the solution reaches steady state. For some α and β pictures are given for demonstration. From the pictures it is easy to see that for fixed β, when α increases, the time to become hot decreases and the time period for the temperature to reach a pseudo-steady state is also shortened. When the temperature reaches pseudo-steady state, it still needs some Newton iterations for the temperature to reach steady state. The results from the pictures match very well with the theoretical conclusion, i.e. , when α is bigger, which means the heat source is stronger, the temperature increases rapidly and reaches the steady state very quickly. Otherwise, the temperature increases very slowly and it takes a longer time to reach steady state.

4.2.2 Fixed Meshpoints Method

A fixed meshpoints method is now used to solve (4.16) where $f(u)$ is defined by (4.2). Here the unconditionally stable Crank-Nicholson difference scheme is used throughout. The x direction stepsize is $\Delta x = \frac{1}{32}$ (we choose this number for comparison with the results of §4.2.1) and time direction stepsize Δt is chosen arbitrarily (of course $r = \frac{\Delta t}{(\Delta x)^2}$ is kept reasonably bounded). The scheme is as follows:

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{1}{2}\left[\frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{(\Delta x)^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(\Delta x)^2}\right] + \alpha f(u_{i,j})$$

i.e.,

$$\begin{aligned} & -\frac{r}{2}u_{i+1,j+1} + (1+r)u_{i,j+1} - \frac{r}{2}u_{i-1,j+1} \\ = & \frac{r}{2}u_{i+1,j} + (1-r)u_{i,j} + \frac{r}{2}u_{i-1,j} + \alpha\Delta t f(u_{i,j}) \end{aligned} \tag{4.40}$$

where $r = \frac{\Delta t}{(\Delta x)^2}, i = 0, \ldots, N$. For the boundary conditions, a similar central difference at endpoints $x = 0$ and $x = 1$ is used as in §4.2.1. That is

$$\begin{cases} \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} = 0 & , u_{1,j} = u_{-1,j} & , i = 0 \\ \frac{u_{N+1,j} - u_{N-1,j}}{2\Delta x} + \beta u_{N,j} = 0 & , u_{N+1,j} = u_{N-1,j} - 2\Delta x \beta u_{N,j} & , i = N \end{cases} \tag{4.41}$$

The advantage of the fixed meshpoints method is that the scheme is easy to form and the numerical solutions are easy to obtain. Actually, the system obtained by discretization is a tridiagonal system. So it is very easy to solve. The disadvantage of this method is that there is no indication where the interface point ξ, where

$u(t, \xi) = 1$, is located. The numerical results are summarized in Table 4.2.3 where $u(0)$ and $u(1)$ is the numerical solution of (4.16) at endpoints $x = 0$ and $x = 1$ respectively and $v(0)$ and $v(1)$ is the numerical solution of (4.1) at endpoints $x = 0$ and $x = 1$ respectively.

Table 4.2.3

α	β	$u(0)$	$u(1)$	$v(0)$	$v(1)$	error
7.50	0.250	1.362750	1.240166	1.362729	1.240063	0.00010
3.50	0.250	1.290071	1.172801	1.290052	1.172717	0.00008
2.50	0.250	1.258070	1.143189	1.258052	1.143100	0.00009
1.00	0.250	1.171216	1.062906	1.171199	1.062828	0.00008
0.90	0.250	1.161258	1.053703	1.161241	1.053635	0.00007
0.80	0.250	1.150134	1.043435	1.150117	1.043369	0.00007
0.70	0.250	1.137532	1.031807	1.137516	1.031741	0.00007
0.50	0.250	1.105827	1.002569	1.106811	1.002506	0.00098
0.40	0.250	1.084067	0.981579	1.084082	0.981590	0.00002
0.35	0.250	1.069945	0.966875	1.069938	0.966847	0.00003
0.30	0.250	1.051941	0.947079	1.051951	0.947085	0.00001

End point values of solutions.

u is the solution of the time dependent solution as $t \longrightarrow \infty$.

v is the steady state solution.

Comparing with Table 4.2.1, it is seen that the errors with the fixed meshpoint method are almost ten times bigger than the errors with the moving meshpoint method. The biggest error in Table 4.2.3 (for parameters $\alpha = 0.5$ and $\beta = 0.25$) is 0.00098 while the biggest in Table 4.2.1 is 0.00009. That is to say, though the moving meshpoints method is more complicated than the fixed meshpoints method, the moving meshpoints method gives more accurate numerical solutions. Also the moving meshpoints method gives a more accurate location of interface point than the fixed meshpoints method, which only gives a possible interval in which the interface point is located.

The above numerical solution is only for α fixed. Now the situation for α is function of $u(x,t)$ is considered in the following. For convenience, the time dependent equation is written as follows:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha f(u) & ,0 < x < 1 \\ u_x(0) = 0 & ,u_x(1) + \beta u(1) = 0 \end{cases} \tag{4.42}$$

where β, μ and γ are parameters, $f(u)$ is defined by (4.2) and α is a function of u, i.e., $\alpha = \frac{\gamma}{(1+\mu\int_0^1 f(u)dx)^2}$. This arises when the external circuit is included. A reasonable value for γ is 150 and for μ a value of 20. Comparing with Eq.(4.16), Eq.(4.42) is more complicated since α is dependent on $u(x,t)$. In order to solve Eq.(4.42), a semi-implict difference scheme is used. Here semi-implicit means that the linear part is implicit but the nonlinear part is explicit. This makes it easy to handle the integral term. That is

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta t} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{(\Delta x)^2} + \frac{\gamma f(u_{i,j})}{integral}$$

i.e.,

$$-ru_{i+1,j+1} + (1+2r)u_{i,j+1} - ru_{i-1,j+1} \\ = u_{i,j} + \frac{\gamma \Delta t f(u_{i,j})}{integral} \tag{4.43}$$

where $r = \frac{\Delta t}{(\Delta x)^2}$ and $i = 0, 1, \cdots, N$, $integral = [1 + \mu \Delta x \frac{1}{2} \sum_{i=2}^{N} [f(u_{i-1,j}) + f(u_{i,j})]]^2$ i.e., a trapezoidal rule over the values of u at the gridpoints (time step j) is used. For the boundary condition, the central difference (4.41) is employed. Since in Eq.(4.42) α is no longer constant, the numerical experiments are done for various β. As it is mentioned in Chapter 3, as time increases, Eq.(4.42) reaches its steady state, therefore the steady state solutions of Eq.(4.42) should match the solutions of Eq.(4.11). Thus for different β there should exist one, two or three solutions to Eq.(4.42). The numerical results are summarized in following Table 4.2.4, where $I = \sqrt{\frac{\alpha}{\gamma}}$ and $\alpha = \frac{\gamma}{(1+\mu \int_0^1 f(u)dx)^2}$, $u(0)$ and $u(1)$ represent the numerical solutions (at both endpoints $x = 0$ and $x = 1$) obtained when the initial values $u_{i,0}$ are all zeros; $v(0)$ and $v(1)$ represent the numerical solutions obtained when the initial values $u_{i,0}$ are greater than 1.0. Theoretically, when $\beta > 0.4098$ there should be three solutions, one is that the whole solution is in the cold region, the second one is a solution spanning both cold and transient regions, the third one is the whole solution in the transient region; when $\beta < 0.4098$ there is only one solution in the transient region. From the Table 4.2.4 it is easy to see that when $\beta < 0.4098$ (critical value for β), there is only one numerical solution; when $\beta > 0.4098$, there are two solutions, one is the whole solution in the cold region and another one is the whole solution in the transient region. How about the one which spans both cold and transient region? Why are we unable to get it numerically? The problem is that that steady state solution is unstable. As for the case $\beta = 0.4098$, $v(0)$ reaches 1.0

for the cold branch and it agrees with the numerical results in §4.1.2.

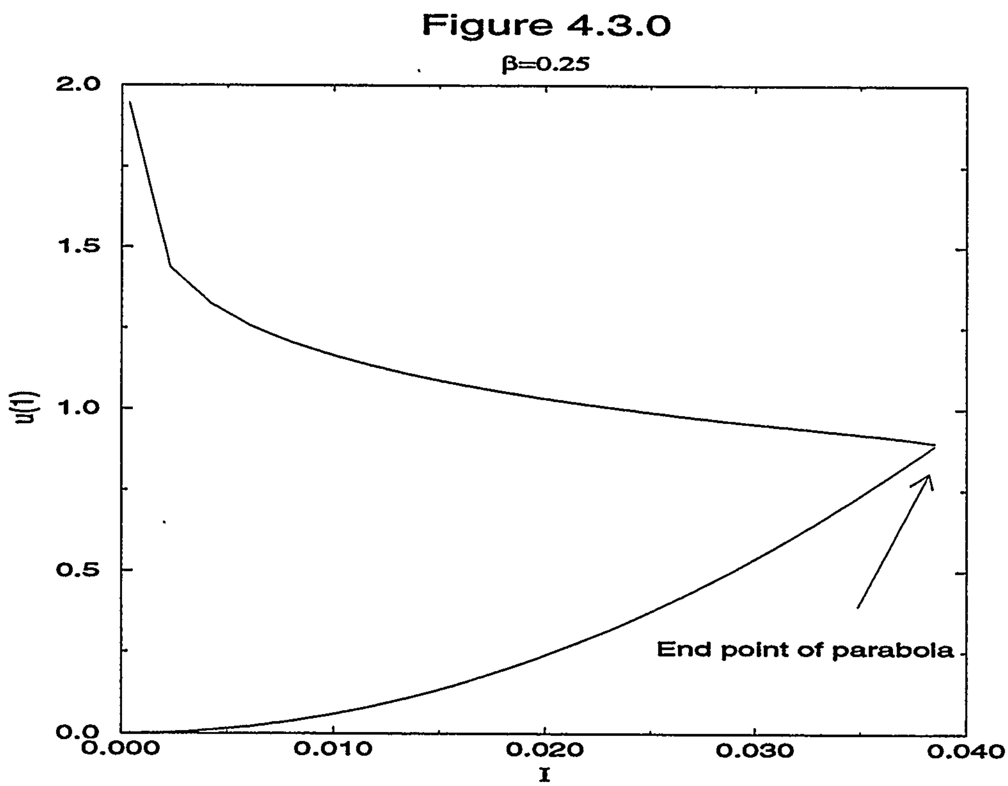
Table 4.2.4

β	I	$u(0)$	$u(1)$	I	$v(0)$	$v(1)$
.6000	.4762E−1	.7369E+0	.5668E+0	.8771E+0	.1588E+1	.1348E+1
.5000	.4762E−1	.8503E+0	.6802E+0	.8974E+0	.1599E+1	.1381E+1
.4500	.4762E−1	.9258E+0	.7558E+0	.9074E+0	.1605E+1	.1400E+1
.4100	.4762E−1	.9996E+0	.8295E+0	.9154E+0	.1611E+1	.1416E+1
.4099	.4762E−1	.9998E+0	.8297E+0	.9155E+0	.1611E+1	.1416E+1
.4098	.4762E−1	.1000E+1	.8299E+0	.9155E+0	.1611E+1	.1416E+1
.4000	.9174E+0	.1612E+1	.1420E+1	.9174E+0	.1612E+1	.1420E+1
.3500	.9274E+0	.1620E+1	.1443E+1	.9274E+0	.1620E+1	.1443E+1
.3000	.9373E+0	.1629E+1	.1469E+1	.9373E+0	.1629E+1	.1469E+1
.2500	.9473E+0	.1640E+1	.1498E+1	.9473E+0	.1640E+1	.1498E+1
.2000	.9573E+0	.1654E+1	.1533E+1	.9573E+0	.1654E+1	.1533E+1

End point values of solutions of time dependent problem with external circuit as $t \longrightarrow \infty$. u and v are solutions obtained from different initial values.

4.3 Numerical Solutions for NTC Problem

In this section, numerical solutions for the NTC problems are given. As in § 4.2, the figures and tables are also included. Numerical results show that the characteristics of solutions of Eq.(3.1) and Eq.(3.17) are mainly determined by the nondecreasing or nonincreasing property of the functions at the right hand side.

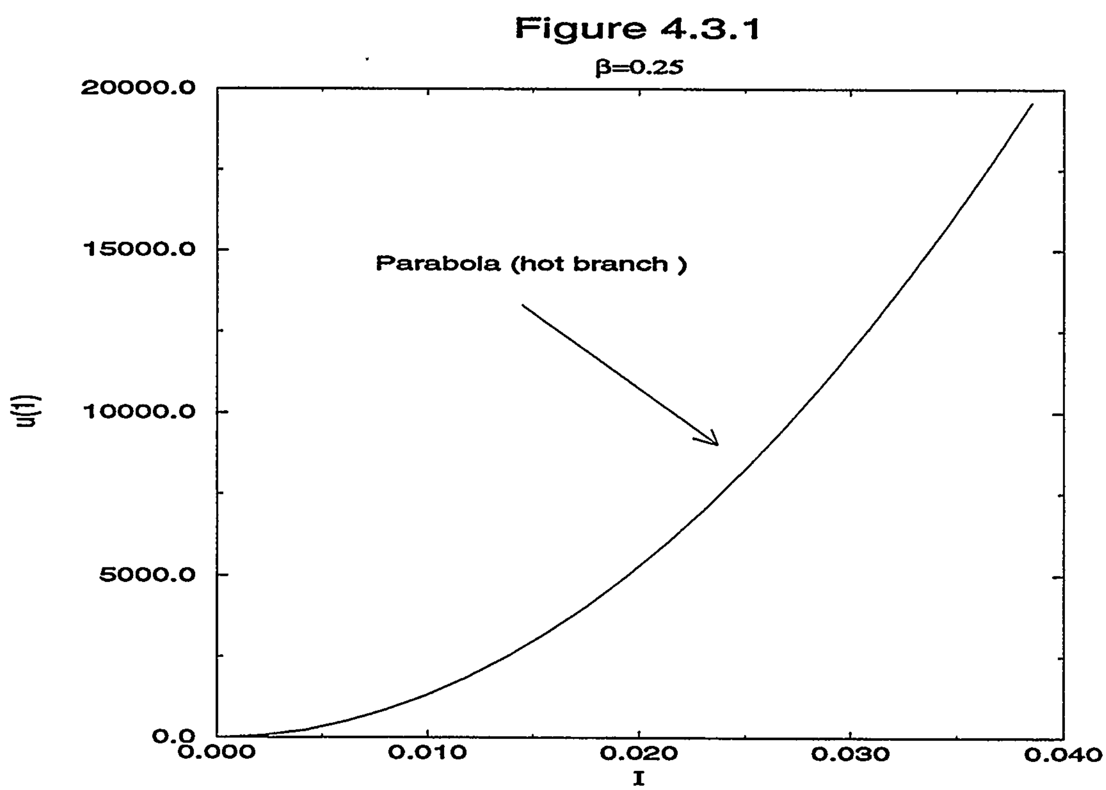


Plot of $u(1)$ and I for solution of (4.45), $0 \le u(1) \le 2$.

4.3.1 Without External Circuit

As for NTC problem, the function for right hand side is defined as (reciprocal of f defined by (4.2))

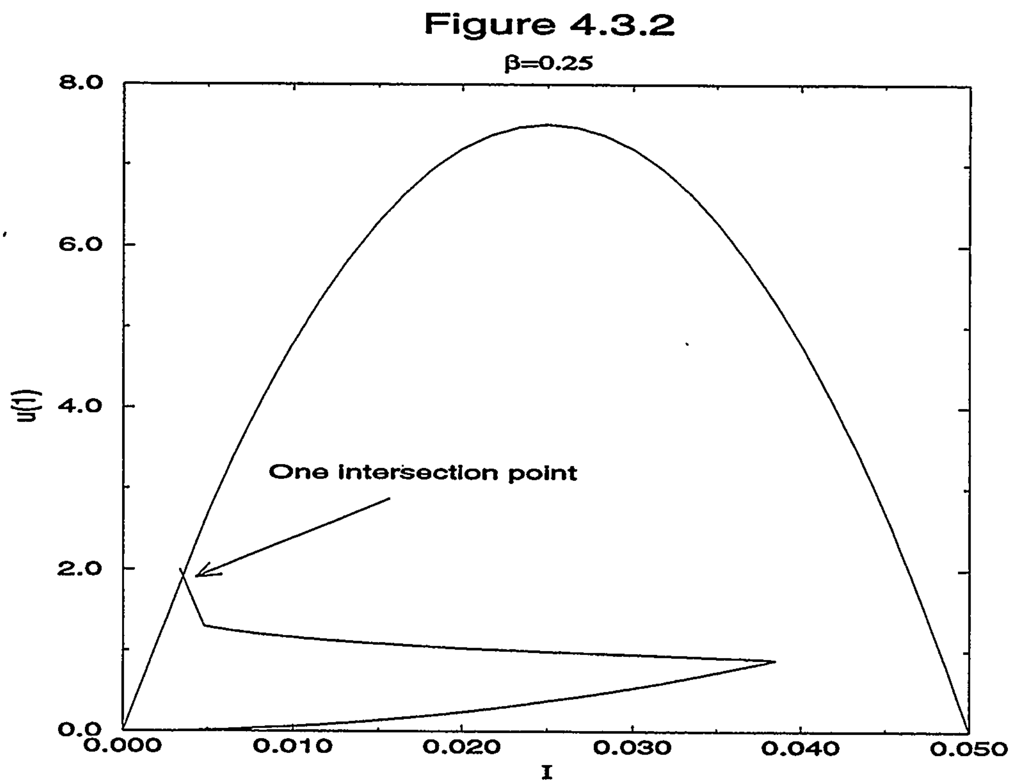
$$g(u) = \begin{cases} 1 & ,u \leq 1 \\ e^{10(u-1)} & ,1 < u < 2 \\ e^{10} & ,2 \leq u \end{cases} \tag{4.44}$$



Plot of $u(1)$ and I for solution of (4.45), $2 \leq u(1) \leq 20000$ (large range).

and the problem is as follows

$$\begin{cases} \frac{d^2u}{dx^2} + \alpha g(u) = 0 & ,0 < x < 1 \\ u_x(0) = 0 & ,u_x(1) + \beta u(1) = 0 \end{cases} \tag{4.45}$$



Graphs of $u(1)$ and I for solution of (4.50) and solutions of (4.52).

Intersection point represents solution of coupled problem.

Comparing to the original problem discussed in §4.1.1, the only difference is the nonlinear function $g(u)$. However this difference of functions makes the solution $u(x)$ very different. That is, for some fixed α and β, the solution $u(x)$ is not unique. Again as in §4.1.1,

$$u(x) = \frac{\alpha}{\beta} + \frac{\alpha}{2}(1 - x^2) \tag{4.46}$$

iff $u \leq 1$ in the interval [0, 1] and

$$u(x) = \frac{\alpha e^{10}}{\beta} + \frac{\alpha e^{10}}{2}(1 - x^2) \tag{4.47}$$

iff $u \geq 2$ in the interval [0, 1]. For (4.46) if $u(0) \leq 1$ then

$$\alpha \leq \frac{1}{\frac{1}{\beta} + \frac{1}{2}} = \frac{2\beta}{2 + \beta} \tag{4.48}$$

and for (4.47) if $u(1) \geq 2$ then

$$\alpha \geq 2\beta e^{-10} \tag{4.49}$$

Hence for $\alpha \in [2\beta e^{-10}, \frac{2\beta}{2+\beta}]$, (4.46) and (4.47) can both be solutions of (4.45). These are the easily found ones. In fact there exists $u(x)$ between 1 and 2. Thus for certain α and β there exist three solutions. For demonstration, $\beta = 0.25$ is fixed, hence for $\alpha \in [0.000022699, 0.222222222]$, there exist three solutions. For convenience, the graphs demonstrate the relations of $u(1)$ and α (for consistency, here $\alpha = \gamma I^2, \gamma = 150$). The graphs are called Figure 4.3.0 and Figure 4.3.1.

4.3.2 With External Circuit

Here the steady state one dimensional problem for the NTC with the external circuit is considered. The numerical solution is discussed. The equation is as follows: to find I and $u(x)$, $0 < x < 1$ such that

$$\begin{cases} \frac{d^2u}{dx^2} + \alpha g(u) = 0 & ,0 < x < 1 \\ u_x(0) = 0 & ,u_x(1) + \beta u(1) = 0 \end{cases} \tag{4.50}$$

$$1 = \mu I + \frac{\beta u(1)}{\gamma I}, \qquad \alpha = \gamma I^2 \tag{4.51}$$

where γ and β are fixed parameters and $g(u)$ is defined by (4.44).

From Eq.(4.50) and (4.51) it can be seen that $\alpha = \gamma I^2$ and $u(1)$ are related. Thus it is more complicated to solve Eq.(4.50) and (4.51) than (4.45). In order to get the numerical solution of Eq.(4.50) and (4.51), the IMSL routine is used as in the PTC problem. So the strategy here is to fix I, therefore α, use the shooting method (usually with more than one iteration) to obtain $u(1)$, then modify α according to $u(1)$ by using the relation (4.51) and continue iterating, until both I and $u(1)$ converge to some values and a numerical solution to Eq.(4.50) and (4.51) is obtained.

From the equation of Eq.(4.51), it is easy to see that

$$u(1) = \frac{\gamma I}{\beta}(1 - \mu I) \tag{4.52}$$

so $u(1)$ is a parabolic function of I. On the other hand, from Eq.(4.50) and the definition of $g(u)$ by (4.44), the following are obvious

$$u(x) = \frac{\alpha}{\beta} + \frac{\alpha}{2}(1 - x^2), \quad provided \ \ u \leq 1 \ \ or \ \ \alpha < \frac{2\beta}{\beta + 2} \tag{4.53}$$

and

$$u(x) = \frac{\alpha e^{10}}{\beta} + \frac{\alpha e^{10}}{2}(1 - x^2), \quad provided \ \ 2 \leq u \ \ or \ \ \alpha > 2\beta e^{-10} \tag{4.54}$$

Now we see that if γ and β are fixed, the solution to Eq.(4.50) and (4.51) for any given I is not unique. Actually, the region of I for nonuniqueness can be determined as follows. From (4.3), if $u(0) = \frac{\alpha}{\beta} + \frac{\alpha}{2} \leq 1$, then $u(x) \leq 1$ for $0 \leq x \leq 1$ because $u(x)$ is nonincreasing function of x. Thus, $\alpha = \gamma I^2 \leq \frac{1}{\frac{1}{\beta}+\frac{1}{2}}$, i.e., $I^2 \leq \frac{2\beta}{\gamma(\beta+2)}$ (e.g., if $\beta = 0.25, \gamma = 150$, then $I \leq 0.03849$). From (4.54), it is easy to see that, if $u(1) = \frac{\gamma I^2 e^{10}}{\beta} \geq 2$, then $u(x) \geq 2$ for $0 \leq x \leq 1$. Hence, $\frac{\gamma I^2 e^{10}}{\beta} \geq 2$, i.e., $I^2 \geq \frac{2\beta}{\gamma e^{10}}$ (e.g., if $\beta = 0.25, \gamma = 150$, then $I \geq 0.0003890$). Therefore, for $I \in [\frac{2\beta}{\gamma e^{10}}, \frac{2\beta}{\gamma(\beta+2)}]$, the number of solutions to Eq.(4.50) and (4.51) is at least two. The Figure 4.3.0 and Figure 4.3.1 are demonstration graphs for the nonuniqueness corresponding to I. The graphs consist of three parts, i.e., Figure 4.3.0 for $u(1) \leq 1, 1 \leq u(1) \leq 2$ and Figure 4.3.1 for $2 \leq u(1)$ within the same region of I. That is to say, the graphs are drawn for the relation of $u(1)$ to I. The numerical results are given in Table 4.3.1. Where $\beta = 0.25$ and $\gamma = 150$ are fixed, I is a parameter changed within the interval mentioned above with stepsize of 20th of the interval length, $u(1)$ and $u(0)$, $v(1)$ and $v(0)$, $w(1)$ and $w(0)$ represent the solutions with respect to $u(x) \geq 2$, $2 \geq u(x) \geq 1$ and $1 \geq u(x)$. From Table 4.3.1, it can be seen that at both ends of the interval the two branches of the solution almost joined together. At $I = 0.3890E-3$, the difference of $u(1)$ and $v(1)$ is 0.052, at $I = 0.3850E-1$, the difference of $v(1)$ and $w(1)$ is 0.0003, much smaller and almost negligible. From Figure 4.3.0, Figure 4.3.2, Figure 4.3.3 and Figure 4.3.4, it is easy to see that the cold and transient solutions joined together at the right end of the interval. Thus the solution $u(x)$ is apparently not differentiable with respect to the parameter α at the joint point. That means there is a wedge at the joint point.

If the values of the solutions are considered at the right end point $x = 1$ and α is replaced by γI^2, then

$$u(1) = \frac{\gamma I^2}{\beta}, u \leq 1 \tag{4.55}$$

Table 4.3.1

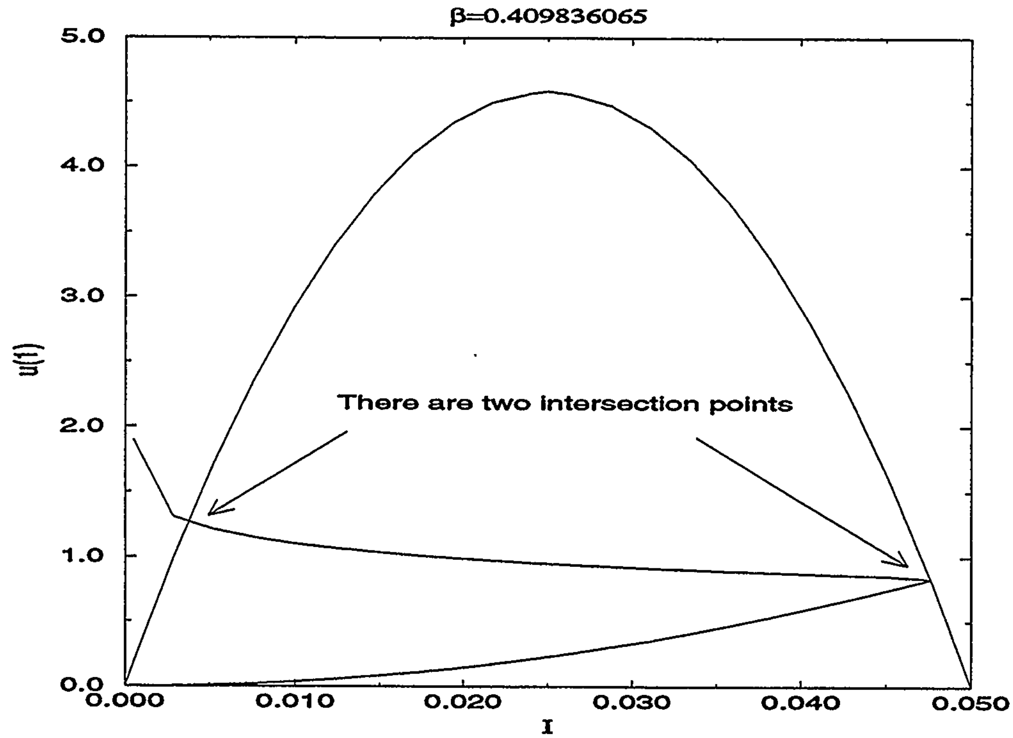
I	$u(0)$	$u(1)$	$v(0)$	$v(1)$	$w(0)$	$w(1)$
.3890E-3	.2250E+1	.2000E+1	.2197E+1	.1948E+1	.9079E−4	.1021E−3
.2295E-2	.7828E+2	.6958E+2	.1680E+1	.1438E+1	.3554E−2	.3159E−2
.4200E-2	.2623E+3	.2331E+3	.1546E+1	.1326E+1	.1191E−1	.1058E−1
.6106E-2	.5543E+3	.4927E+3	.1462E+1	.1257E+1	.2516E−1	.2237E−1
.8011E-2	.9542E+3	.8482E+3	.1401E+1	.1206E+1	.4332E−1	.3851E−1
.9917E-2	.1462E+4	.1300E+4	.1353E+1	.1166E+1	.6638E−1	.5901E−1
.1182E-1	.2078E+4	.1847E+4	.1313E+1	.1133E+1	.9434E−1	.8386E−1
.1373E-1	.2802E+4	.2491E+4	.1279E+1	.1105E+1	.1272E+0	.1131E+0
.1563E-1	.3634E+4	.3230E+4	.1250E+1	.1080E+1	.1650E+0	.1466E+0
.1754E-1	.4574E+4	.4065E+4	.1224E+1	.1058E+1	.2076E+0	.1846E+0
.1944E-1	.5621E+4	.4997E+4	.1200E+1	.1038E+1	.2552E+0	.2269E+0
.2135E-1	.6777E+4	.6024E+4	.1179E+1	.1020E+1	.3077E+0	.2735E+0
.2326E-1	.8041E+4	.7147E+4	.1159E+1	.1004E+1	.3651E+0	.3245E+0
.2516E-1	.9413E+4	.8367E+4	.1141E+1	.9887E+0	.4273E+0	.3799E+0
.2707E-1	.1089E+5	.9682E+4	.1124E+1	.9747E+0	.4945E+0	.4396E+0
.2897E-1	.1248E+5	.1109E+5	.1107E+1	.9616E+0	.5666E+0	.5036E+0
.3088E-1	.1418E+5	.1260E+5	.1090E+1	.9492E+0	.6436E+0	.5721E+0
.3278E-1	.1598E+5	.1420E+5	.1073E+1	.9372E+0	.7255E+0	.6448E+0
.3469E-1	.1789E+5	.1590E+5	.1055E+1	.9253E+0	.8122E+0	.7220E+0
.3659E-1	.1991E+5	.1770E+5	.1035E+1	.9123E+0	.9039E+0	.8035E+0
.3850E-1	.2204E+5	.1959E+5	.1001E+1	.8896E+0	.1001E+1	.8893E+0

End point values for multiple solutions u, v, w of the steady state NTC problem.

and

$$u(1) = \frac{\gamma I^2 e^{10}}{\beta}, 2 \leq u \tag{4.56}$$

Figure 4.3.3



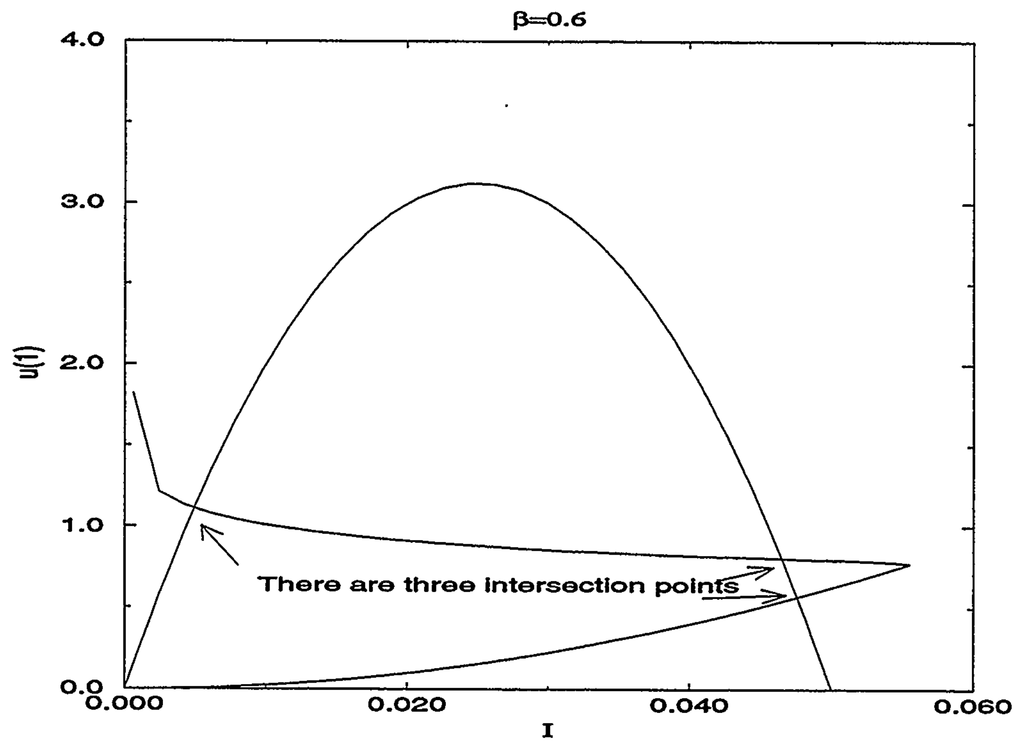
Graphs of $u(1)$ *and* I *for solution of (4.50) and solutions of (4.52).*

Intersection points are solutions of coupled problem.

Now consider the solution of Eq.(4.50) and (4.51), that is, condition (4.52) is in force for Eq.(4.45). As it is seen, (4.52) is a parabola, which has two intersection points, $I = 0$ and $I = \frac{1}{\mu}$, when $u(1) = 0$. The maximum of $u(1)$ is reached at $I = \frac{1}{2\mu}$, and the maximum value of $u(1)$ is $\frac{\gamma}{4\beta\mu}$ which depends on β if γ and μ are fixed. If (4.52) is drawn on Figure 4.3.0, it is obvious that, except for the trivial solution

(i.e., $\alpha = 0$ and $u = 0$), there are three possible cases. Case one, there is only one solution, case two there are two solutions, case three there are three solutions. Since (4.55) is also a parabola about I, it is easy to see that, if (4.55) for some β just reaches (4.52) with $u(0) < 1$ on the $I > \frac{1}{2\mu}$ side, there are exactly two solutions, if (4.55) does not reach (4.52), that means only one solution, if (4.55) intersects with (4.52) and goes outside that parabola, then there are three solutions. In fact, the

Figure 4.3.4



Graphs of $u(1)$ and I for solution of (4.50) and solutions of (4.52). Intersection points are solutions of coupled problem.

region of β can be determined for the three cases. Consider that (4.55) just reaches

(4.52), hence $\frac{\gamma I^2}{\beta} = \frac{\gamma I}{\beta}(1-\mu I)$, then $I = \frac{1}{\mu+1}$. From (4.3), $u(0) = \frac{\alpha}{\beta} + \frac{\alpha}{2}$ and $u(0)$ can be at most 1, thus $\frac{\alpha}{\beta} + \frac{\alpha}{2} = 1$, i.e., $\beta = 0.409836065 = \beta_0$. We say that this value is critical value for β. Obviously, (4.3) is true for all $u(0) \leq 1$, especially for $u(0) = 1$, hence $\alpha = \frac{2\beta}{\beta+2} = 2 - \frac{4}{\beta+2}$, which means α is an nondecreasing function of β. Denote $I_0 = \frac{1}{\mu+1}$ (also $\alpha_0 = \gamma I_0^2$) corresponding to β_0, then if $\beta > \beta_0$, $I = \sqrt{\frac{\alpha}{\gamma}} > I_0$, which means that right end of (4.55) goes outside of (4.52). So the curves shown in Figure 4.3.0 and Figure 4.3.1 should have three intersection points. The Figure 4.3.2, Figure 4.3.3 and Figure 4.3.4 demonstrate this.

In the above only the intersection of the right end of the "all cold" branch meeting with the parabola (4.52) is discussed. For the all hot branch a similar result can be considered.

4.4 Time Dependent Problem for NTC

For completeness, the numerical solution for the time dependent NTC problem is also done. As it is discussed in § 4.3, the NTC problem has three solutions. For the steady state NTC problem all three solutions can be numerically obtained. However, for the time dependent NTC problem, whenever it has three solutions, one of them is unstable. In fact, starting with initial value $u = 0$, the numerical solution always converges to all cold solution for $\alpha \in [2\beta e^{-10}, \frac{2\beta}{2+\beta}]$. Starting with initial value $u > 2$, the numerical solution always converges to the all hot solution for $\alpha \in [2\beta e^{-10}, \frac{2\beta}{2+\beta}]$, in which interval there exist three solutions. Of course, for α is outside that interval there is only one solution either cold or hot. Hence the numerical solution always converges to the corresponding cold or hot solution even if the initial value is between 0 and 2. Thus for $\alpha \in [2\beta e^{-10}, \frac{2\beta}{2+\beta}]$, the third solution is unstable. It can not be numerically obtained. Therefore, the moving meshpoint method is not tried since only the simple all hot or all cold solutions are obtained. For the case with external circuit connected, the results are given in § 4.7. The instability is briefly discussed in Chapter 5.

4.5 Numerical Solutions for Smooth Functions

The given functions for $f(u)$ taken from [3] have discontinuous first derivatives at $u = 1$ and $u = 2$. For comparison, numerical results are also obtained for smooth functions, that is, the functions have continuous first derivatives. In § 4.5.1, a smooth function corresponding to the PTC problem is considered. In § 4.5.2, a smooth function related to the NTC problem is studied.

4.5.1 Smooth Function for PTC Problem

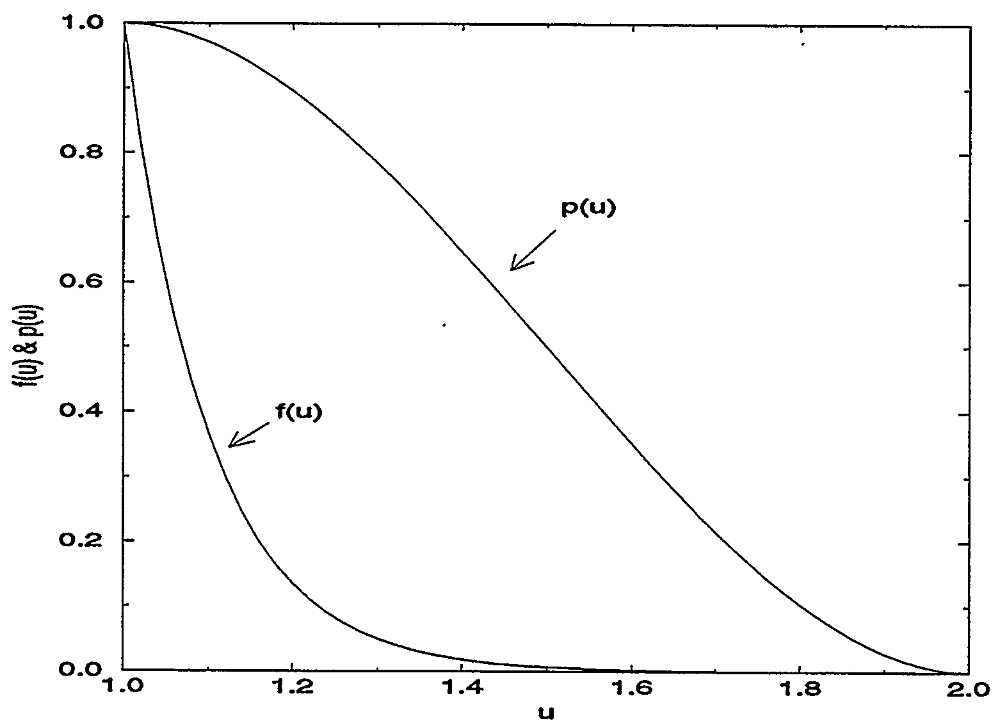
For comparison, the function $\sigma(u)$ and hence $f(u)$ for problem (4.1) is modelled by a smooth function. The choice of smooth function is a cubic Hermite interpolation for the function $\sigma = 1$ for $u \leq 1$ and $\sigma = e^{-10}$ for $u \geq 2$. This is given by

$$p(u) = \begin{cases} 1 & ,u < 1 \\ (2u-1)(u-2)^2 + e^{-10}(5-2u)(u-1)^2 & ,1 \leq u \leq 2 \\ e^{-10} & ,2 < u \end{cases} \tag{4.57}$$

and the derivative of $p(u)$ is as follows

$$p'(u) = \begin{cases} 0 & ,u < 1 \\ 6(1-e^{-10})(u-2)(u-1) & ,1 \le u \le 2 \\ 0 & ,2 < u \end{cases} \tag{4.58}$$

Figure 4.5.1



Graph of nonlinear functions for PTC problems.

The difference of $p(u)$ and $\sigma(u)$ is that $\sigma(u)$ is only continuous but $p(u)$ has continuous first derivative. Now the problem (4.1) can be written as

$$\begin{cases} \frac{d^2u}{dx^2} + \alpha p(u) = 0 & ,0 < x < 1 \\ u_x(0) = 0 & ,u_x(1) + \beta u(1) = 0 \end{cases} \tag{4.59}$$

In order to solve (4.59), the method discussed in §4.1.1 is applied. Similarly, the numerical solutions are summarized in Table 4.5.1. Comparing with Table 4.1.1, it is easy to find that when $1 \leq u \leq 2$, the numerical solutions of (4.59), with α, β the same, are greater than the solutions of (4.1). In Table 4.5.1 column 6 represents the number of Newton iterations, column 7 represents the number of times that bisection was used, column 8 represents the number of times the bisection routine was called, s represents the initial guess. * means total number of bisection times.

Table 4.5.1

α	β	s	$u(0)$	$u(1)$	6	7	8
999.000	0.005	0.5	9.0936	9.0709	2	40	1
999.000	0.010	0.5	4.5581	4.5355	2	34	1
999.000	0.100	0.5	2.0072	1.9752	1	31	1
750.000	0.100	0.5	2.0040	1.9669	1	31	1
500.000	0.100	0.5	1.9985	1.9679	10	94*	4
100.000	0.100	0.5	1.9884	1.9432	5	59*	2
75.000	0.100	0.5	1.9840	1.9363	5	29	1
50.000	0.100	0.5	1.9802	1.9285	1	29	1
25.000	0.100	0.5	1.9667	1.9127	1	25	1
15.000	0.100	0.5	1.9534	1.8950	1	23	1
14.000	0.100	0.5	1.9509	1.8940	1	22	1
10.000	0.100	0.5	1.9412	1.8721	10	18	1
7.500	0.100	0.5	1.9293	1.8582	12	19	1
5.000	0.100	0.5	1.9089	1.8356	7	18	1

Table 4.5.1(continued)

α	β	s	$u(0)$	$u(1)$	6	7	8
1.000	0.100	0.5	1.7644	1.6875	4	12	1
0.850	0.100	0.5	1.7415	1.6649	3	11	1
0.500	0.100	0.5	1.6504	1.5760	3	15	1
0.250	0.100	0.5	1.4794	1.4110	2	8	1
0.100	0.100	0.5	1.0465	0.9968	3	2	2*
0.050	0.100	0.5	0.5250	0.5000	2	1	1
0.005	0.100	0.5	0.0525	0.0500	2	1	1

Solution of PTC problem for various α, β. Col. 6—No. of Newton iterations. Col. 7—No. of times bisection used. Col. 8—No. of calls to bisection routine.

4.5.2 Smooth Function for NTC Problem

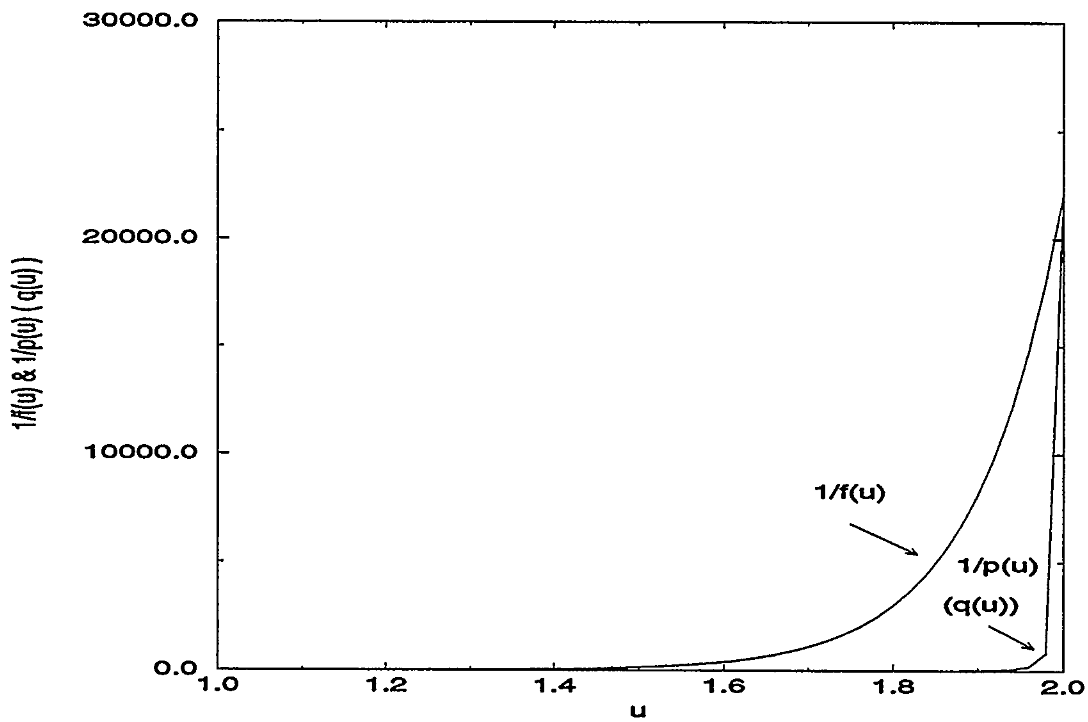
For the NTC problem (4.45) with function $f(u)$ defined by (4.44), the smooth function for (4.44) is not the directly Hermitian interpolation of (4.44) but the reciprocal of (4.57), that is

$$q(u) = \begin{cases} 1 & ,u < 1 \\ \frac{1}{(2u-1)(u-2)^2+e^{-10}(5-2u)(u-1)^2} & ,1 \le u \le 2 \\ e^{10} & ,2 < u \end{cases} \tag{4.60}$$

and the derivative of $q(u)$ is as follows

$$q'(u) = \begin{cases} 0 & ,u < 1 \\ -\frac{6(1-e^{-10})(u-2)(u-1)}{[(2u-1)(u-2)^2+e^{-10}(5-2u)(u-1)^2]^2} & ,1 \leq u \leq 2 \\ 0 & ,2 < u \end{cases} \tag{4.61}$$

Figure 4.5.2



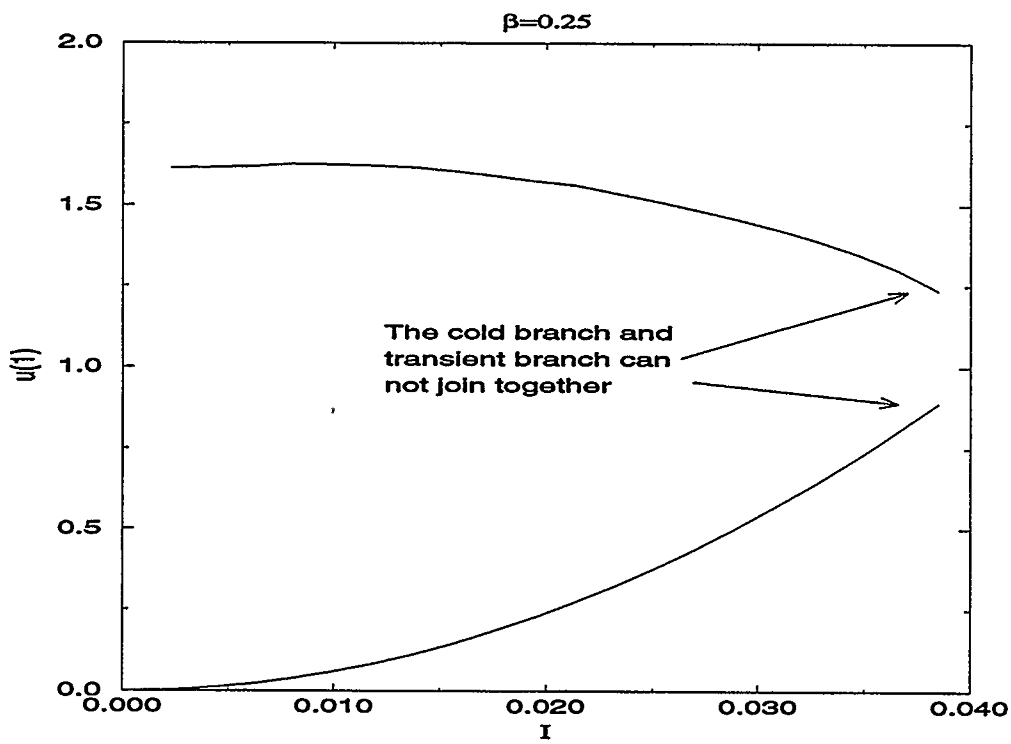
Graph of nonlinear functions for NTC problems.

Thus the problem (4.45) becomes

$$\begin{cases} \frac{d^2u}{dx^2} + \alpha q(u) = 0 & ,0 < x < 1 \\ u_x(0) = 0 & ,u_x(1) + \beta u(1) = 0 \end{cases} \tag{4.62}$$

Similarly, the numerical results are summarized in Table 4.5.2. Comparing with Table 4.3.1, it is easy to see that when $u \leq 1$ and $u \geq 2$, the solutions are the same; but for $1 \leq u \leq 2$, solutions in Table 4.5.2 are greater than those in Table 4.3.1 with the same α and β. However, because of the smoothness of $q(u)$, the properties of the solution have been changed accordingly. From Table 4.3.1, the three branches of the solution can join together at two ends separately. However, from Table 4.5.2,

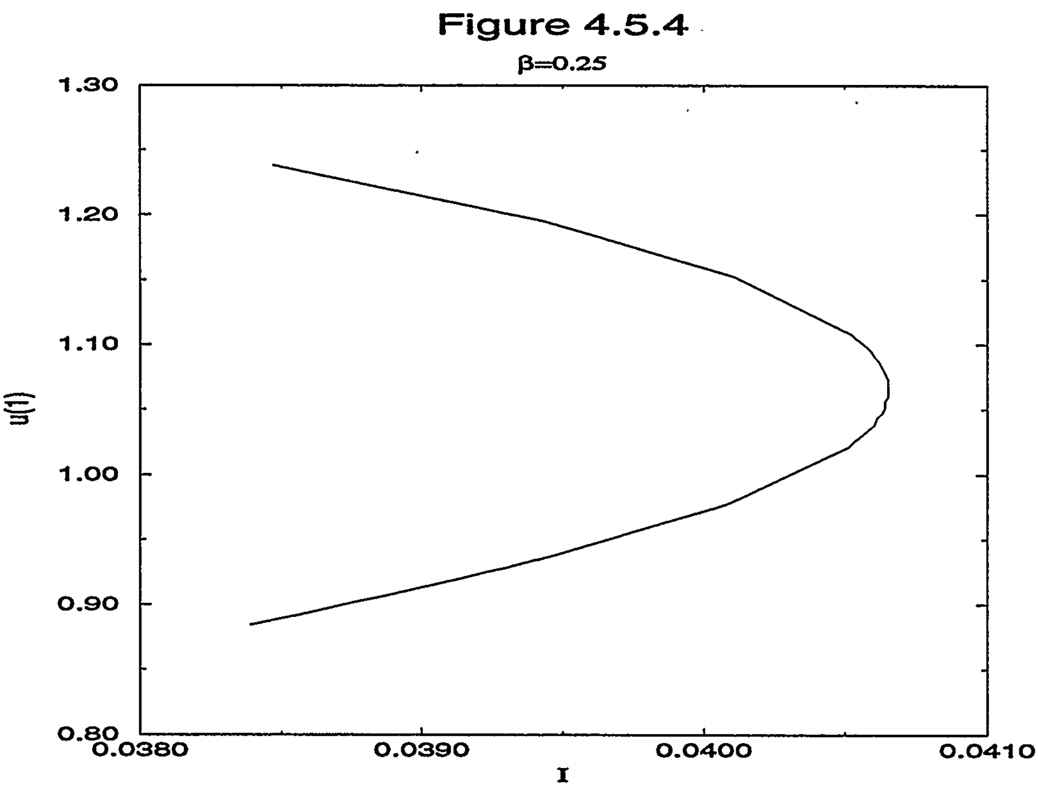
Figure 4.5.3



Plots of $u(1)$ and I for solution of (4.60) and (4.62).

we can see different results. The difference of the values at the right end of the cold and transient solutions is quite different. Also the same method is not convergent for I beyond $I = 0.385E-1$. An alternative method is used, that is, fixing $u(0)$ to find α by the shooting method. Now for fixed β, $u(0)$, denote $\overline{u} = \frac{du}{d\alpha}$, $\overline{v} = \frac{dv}{d\alpha}$ as two new unknowns, use the similar method as in §4.1.1. A new system of differential equations can be obtained from (4.62) as follows:

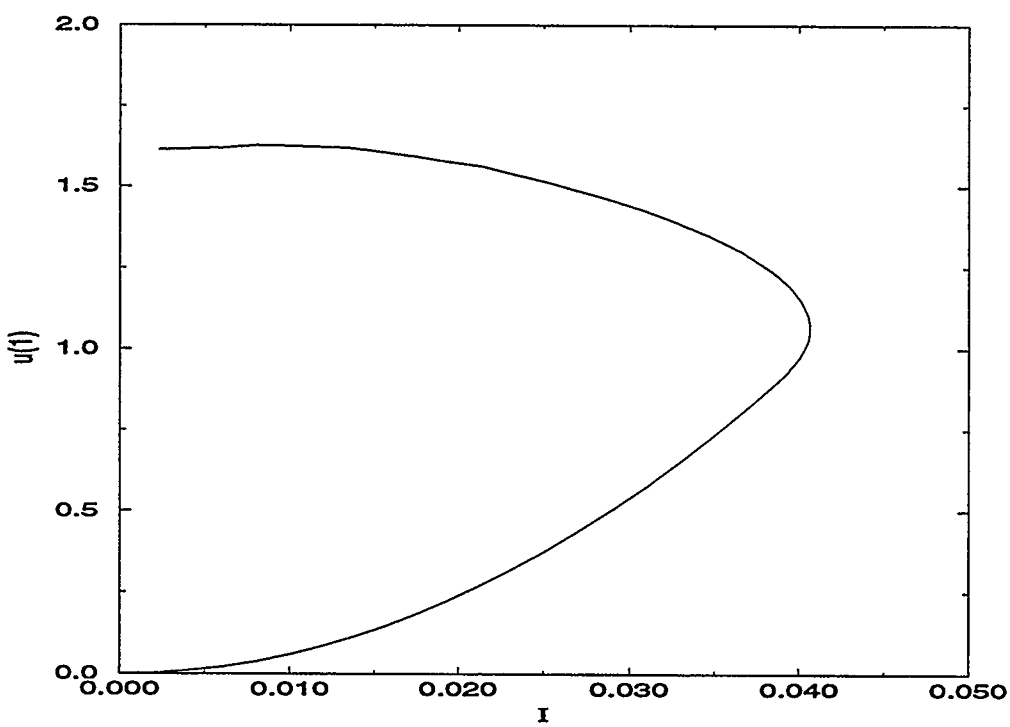
$$\begin{cases} u_x = v & , u(0) = s \\ v_x = -\alpha^n f(u) & , v(0) = 0 \\ \overline{u}_x = \overline{v} & , \overline{u}(0) = 0 \\ \overline{v}_x = -\alpha^n f'(u)\overline{u} - f(u) & , \overline{v}(0) = 0 \end{cases} \tag{4.63}$$



Plots of $u(1)$ and I for solution of the smooth NTC problem using the shooting method (4.63).

where s is fixed, $\alpha^{n+1} = \alpha^n + \Delta\alpha$, $\Delta\alpha = -\frac{v(1,\alpha^n)+\beta u(1,\alpha^n)}{\bar{v}(1,\alpha^n)+\beta\bar{u}(1,\alpha^n)}$, $n = 0, 1, 2, \cdots$, α^0 is arbitrary. Here a numerical method similar to that in § 4.1.1 is used to get α. The results are summarized in Table 4.5.3, where $u(0)$ is changed by small stepsize then different values for α are obtained. Using the data obtained in Table 4.5.3, a graph is drawn as Figure 4.5.4 and also Figure 4.5.3 is drawn according to the data in Table 4.5.2 (note, for consistency, here the x coordinate has been changed to I, where $\alpha = \gamma I^2$, $\gamma = 150$). Comparing these two graphs, it is easy to see that two graphs

Figure 4.5.5



Combined plot of $u(1)$ and I for the smooth NTC problem.

can be connected. The graph in Figure 4.5.4 is just that part missed in Figure 4.5.3.

Figure 4.5.5 is the combination of Figure 4.5.3 and Figure 4.5.4. The two graphs joined perfectly. Also from Figure 4.5.4, it seems that the derivatives of $u(1,\alpha)$ with respect to α is infinite at the end of the interval for I. So that's why we need to use

Table 4.5.2

I	$u(0)$	$u(1)$	$v(0)$	$v(1)$	$w(0)$	$w(1)$
.2295E-2	.7828E+2	.6958E+2	.1998E+1	.1616E+1	.3554E−2	.3159E−2
.4200E-2	.2623E+3	.2331E+3	.1990E+1	.1611E+1	.1191E−1	.1058E−1
.6106E-2	.5543E+3	.4927E+3	.1978E+1	.1619E+1	.2516E−1	.2237E−1
.8011E-2	.9542E+3	.8482E+3	.1958E+1	.1627E+1	.4332E−1	.3851E−1
.9917E-2	.1462E+4	.1300E+4	.1944E+1	.1625E+1	.6638E−1	.5901E−1
.1182E-1	.2078E+4	.1847E+4	.1924E+1	.1622E+1	.9434E−1	.8386E−1
.1373E-1	.2802E+4	.2491E+4	.1897E+1	.1617E+1	.1272E+0	.1131E+0
.1563E-1	.3634E+4	.3230E+4	.1876E+1	.1605E+1	.1650E+0	.1466E+0
.1754E-1	.4574E+4	.4065E+4	.1852E+1	.1592E+1	.2076E+0	.1846E+0
.1944E-1	.5621E+4	.4997E+4	.1825E+1	.1576E+1	.2552E+0	.2269E+0
.2135E-1	.6777E+4	.6024E+4	.1783E+1	.1562E+1	.3077E+0	.2735E+0
.2326E-1	.8041E+4	.7147E+4	.1758E+1	.1538E+1	.3651E+0	.3245E+0
.2516E-1	.9413E+4	.8367E+4	.1730E+1	.1514E+1	.4273E+0	.3799E+0
.2707E-1	.1089E+5	.9682E+4	.1698E+1	.1487E+1	.4945E+0	.4396E+0
.2897E-1	.1248E+5	.1109E+5	.1662E+1	.1458E+1	.5666E+0	.5036E+0
.3088E-1	.1418E+5	.1260E+5	.1623E+1	.1426E+1	.6436E+0	.5721E+0
.3278E-1	.1598E+5	.1420E+5	.1580E+1	.1390E+1	.7255E+0	.6448E+0
.3469E-1	.1789E+5	.1590E+5	.1531E+1	.1349E+1	.8122E+0	.7220E+0

Table 4.5.2 (continued)

I	$u(0)$	$u(1)$	$v(0)$	$v(1)$	$w(0)$	$w(1)$
.3659E-1	.1991E+5	.1770E+5	.1473E+1	.1300E+1	.9039E+0	.8035E+0
.3850E-1	.2204E+5	.1959E+5	.1399E+1	.1237E+1	.1001E+1	.8893E+0

End point values of the multiple solution u, v, w of the steady state NTC problem with (4.60).

Table 4.5.3

$u(0)$	$u(1)$	α	$u(0)$	$u(1)$	α
0.99500000	0.88444442	0.22111109	1.17499995	1.04273701	0.24741493
1.00000000	0.88888890	0.22222222	1.17999995	1.04710269	0.24757470
1.00500000	0.89333266	0.22333126	1.18499994	1.05146742	0.24770039
1.00999999	0.89777416	0.22443224	1.19000006	1.05583107	0.24779195
1.01499999	0.90221232	0.22552381	1.19500005	1.06019342	0.24784940
1.01999998	0.90664661	0.22660190	1.20000005	1.06455481	0.24787290
1.02499998	0.91107804	0.22766449	1.20500004	1.06891501	0.24786241
1.02999997	0.91550541	0.22871037	1.20500004	1.06891501	0.24786241
1.03499997	0.91992903	0.22973785	1.21000004	1.07327390	0.24781790
1.03999996	0.92434883	0.23074605	1.21500003	1.07763171	0.24773957
1.04499996	0.92876470	0.23173484	1.22000003	1.08198822	0.24762738
1.04999995	0.93317693	0.23270294	1.22500002	1.08634341	0.24748141

Table 4.5.3(continued)

$u(0)$	$u(1)$	α	$u(0)$	$u(1)$	α
1.05499995	0.93758607	0.23364586	1.23000002	1.09069741	0.24730173
1.10000002	0.97712988	0.24094184	1.23500001	1.09505010	0.24708843
1.14999998	1.02089286	0.24610418	1.24000001	1.09940147	0.24684161
1.15499997	1.02526367	0.24643457	1.24500000	1.10375130	0.24656127
1.15999997	1.02963352	0.24673088	1.25000000	1.10809994	0.24624757
1.16499996	1.03400230	0.24699304	1.29999995	1.15150189	0.24129973
1.16999996	1.03837013	0.24722104	1.35000002	1.19472814	0.23316069

Values of $u(1)$ and α for given $u(0)$ using shooting method (4.63).

a different method to find the relation of u and α. A similar result can be obtained at the other end of the I interval.

4.6 Time Dependent Problems for Smooth Functions

Now the time dependent problems with smooth functions are considered. Numerical results for the PTC problem are considered first followed by those for the NTC problem.

4.6.1 Smooth Function for PTC Problem

There are two cases considered here. One is that the α and β are fixed. The other is that α depends upon u. Here the smooth function is defined by (4.57). For convenience, the time dependent problem with smooth function is written as

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha p(u) & ,0 < x < 1 \\ u_x(0) = 0 & ,u_x(1) + \beta u(1) = 0 \end{cases} \tag{4.64}$$

Use the Crank-Nicholson difference scheme defined by (4.40) and (4.41). The numerical results are summarized in Table 4.6.1 where $u(0)$ and $u(1)$ is the steady state of the numerical solution of (4.64) at endpoints $x = 0$ and $x = 1$ respectively and $v(0)$ and $v(1)$ is numerical solution of (4.59) at endpoints $x = 0$ and $x = 1$ respectively. The biggest error in Table 4.6.1 is 0.00025 and the smallest error is 0.000004. Comparing with Table 4.2.3, it is easy to see that for the same parameters the values of $u(0)$ and $u(1)$ in Table 4.6.1 are greater than that in Table 4.2.3.

For the case when α depends on $u(x,t)$, the equation is the same as (4.64) except that $\alpha = \frac{\gamma}{(1+\mu\int_0^1 p(u)dx)^2}$. The scheme (4.43) and boundary central difference scheme (4.41) are used. The numerical results are summarized in Table 4.6.2. Similarly as in § 4.2.2, $I = \sqrt{\frac{\alpha}{\gamma}}$ and $\alpha = \frac{\gamma}{(1+\mu\int_0^1 p(u)dx)^2}$, $u(0)$ and $u(1)$ represent the numerical solutions (at both endpoints $x = 0$ and $x = 1$) obtained with initial values $u_{i,0}$ all zeros; $v(0)$ and $v(1)$ represent the numerical solutions obtained with initial values $u_{i,0}$ greater than 1.0.

Table 4.6.1

α	β	$u(0)$	$u(1)$	$v(0)$	$v(1)$	error
3.50	0.250	1.836574	1.674007	1.836824	1.674157	0.000250
2.50	0.250	1.797332	1.631792	1.797487	1.631837	0.000155
1.00	0.250	1.641450	1.476472	1.641456	1.476449	0.000023
0.90	0.250	1.617426	1.453506	1.617425	1.453485	0.000021
0.80	0.250	1.588639	1.426191	1.588637	1.426177	0.000014
0.70	0.250	1.553291	1.392912	1.553296	1.392908	0.000004
0.50	0.250	1.448911	1.295864	1.448927	1.295889	0.000025
0.40	0.250	1.363989	1.217883	1.364019	1.217913	0.000030
0.35	0.250	1.304856	1.163926	1.304904	1.163972	0.000048
0.30	0.250	1.225620	1.091959	1.225690	1.092025	0.000070
0.25	0.250	1.106199	.984013	1.106335	0.984140	0.000136

End point values of the time dependent solution

as $t \longrightarrow \infty$ and of the steady state solution.

From Table 4.6.2, it is easy to see that when $\beta < 0.4098$, only one numerical solution is obtained; when $\beta \geq 0.4098$, there are two numerical solutions obtained. Actually, there are three steady state solutions when $\beta > 0.4098$. However, from

Table 4.6.2, only two solutions are obtained. The reason is the same as in § 4.2.2, i.e., the third steady state solution is unstable so it is not obtained numerically. Also, comparing Table 4.6.2 with Table 4.2.4, it is seen that, when $\beta < 0.4098$, the cold solutions are the same in Table 4.2.4 and Table 4.6.2. They should of course be the same since when $\beta < 0.4098$, $u(0)$ and $u(1)$ are values for the all cold solution. When $\beta < 0.4098$, the one solution in Table 4.6.2 is greater than that in Table 4.2.4.

Table 4.6.2

β	I	$u(0)$	$u(1)$	I	$v(0)$	$v(1)$
.6000	.4762E−1	.7369E+0	.5668E+0	.8237E+0	.1984E+1	.1815E+1
.5000	.4762E−1	.8503E+0	.6802E+0	.8568E+0	.1985E+1	.1840E+1
.4500	.4762E−1	.9258E+0	.7558E+0	.8726E+0	.1986E+1	.1852E+1
.4100	.4762E−1	.9996E+0	.8295E+0	.8850E+0	.1986E+1	.1862E+1
.4099	.4762E−1	.9998E+0	.8297E+0	.8850E+0	.1986E+1	.1862E+1
.4098	.4762E−1	.1000E+1	.8299E+0	.8850E+0	.1986E+1	.1862E+1
.4000	.8706E+0	.1987E+1	.1865E+1	.8880E+0	.1987E+1	.1865E+1
.3000	.9176E+0	.1988E+1	.1890E+1	.9176E+0	.1988E+1	.1890E+1
.2500	.9319E+0	.1989E+1	.1904E+1	.9319E+0	.1989E+1	.1904E+1
.2000	.9459E+0	.1990E+1	.1917E+1	.9459E+0	.1990E+1	.1918E+1

End point values of time dependent solutions as $t \longrightarrow \infty$ for different initial values.

4.6.2 Smooth Function for NTC Problem

As in § 4.6.1, two cases are considered. One case is that α is fixed and the other is that α is a functional of $u(x,t)$. For convenience, the time dependent problem is written as

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha q(u) & ,0 < x < 1 \\ u_x(0) = 0 & ,u_x(1) + \beta u(1) = 0 \end{cases} \tag{4.65}$$

where α and β are fixed constant parameters, $q(u)$ is defined by (4.60). The numerical results are summarized in Table 4.6.3. Similarly as in § 4.3, there should be three solutions for $I \in [0.3890 \times 10^{-3}, 0.3850 \times 10^{-1}]$ (where $I = \sqrt{\frac{\alpha}{150}}$). However, only two solutions are obtained. The solution which is in transient region is unstable, so it is impossible to get it numerically. From Table 4.6.3, it is easy to see that the two solutions are either less than 1 or greater than 2, actually for these two cases the exact solutions are given. Thus it is practical to compare the numerical results and exact solutions. They match very well.

As for α depends upon $u(x,t)$, the problem is same as Eq.(4.65) and except that $\alpha = \frac{\gamma}{(\mu+\int_0^1 q(u)dx)^2}$ which is different from PTC problem. The numerical results are listed in Table 4.6.4, where $u(0)$, $u(1)$ and $v(0)$, $v(1)$ are numerical solutions at the end points $x = 0$ and $x = 1$ respectively, and also the numerical solutions of u and v are obtained by the same program except that the initial values $u(j), j = 1, \cdots, 51$ for u are all zero and initial values $v(j), j = 1, \cdots, 51$ for v are greater than 1. From Table 4.6.4, it is easy to see that, when β is close to or greater than 0.4098, there are two numerical solutions (actually, there should be three solutions, one of them

Table 4.6.3

I	$u(0)$	$u(1)$	$w(0)$	$w(1)$
.3890E−03	.225076E+01	.200426E+01	.102417E−03	.907926E−04
.2295E−02	.783094E+02	.696083E+02	.355357E−02	.315874E−02
.4200E−02	.262269E+03	.233128E+03	.119053E−01	.105825E−01
.6106E−02	.554322E+03	.492731E+03	.251645E−01	.223685E−01
.8011E−02	.954162E+03	.848144E+03	.433173E−01	.385043E−01
.9917E−02	.146221E+04	.129974E+04	.663825E−01	.590067E−01
.1182E−01	.207723E+04	.184642E+04	.943042E−01	.838260E−01
.1373E−01	.280278E+04	.249136E+04	.127245E+00	.113106E+00
.1563E−01	.363217E+04	.322860E+04	.164899E+00	.146577E+00
.1754E−01	.457412E+04	.406589E+04	.207663E+00	.184590E+00
.1944E−01	.561877E+04	.499446E+04	.255090E+00	.226747E+00
.2135E−01	.677711E+04	.602410E+04	.307679E+00	.273492E+00
.2326E−01	.804392E+04	.715016E+04	.365192E+00	.324615E+00
.2516E−01	.941174E+04	.836599E+04	.427291E+00	.379814E+00
.2707E−01	.108949E+05	.968440E+04	.494629E+00	.439670E+00
.2897E−01	.124780E+05	.110916E+05	.566500E+00	.503555E+00
.3088E−01	.141776E+05	.126023E+05	.643662E+00	.572144E+00
.3278E−01	.159759E+05	.142008E+05	.725306E+00	.644716E+00
.3469E−01	.178919E+05	.159039E+05	.812291E+00	.722037E+00
.3659E−01	.199055E+05	.176938E+05	.903708E+00	.803296E+00
.3850E−01	.220379E+05	.195892E+05	.100052E+01	.889349E+00

End point values of solutions as $t \longrightarrow \infty$ of (4.65).

is unstable, so it is difficult to get it numerically). For $\beta < 0.4098$, there is only one solution. It is noticed that both Table 4.6.2 and Table 4.6.4 have same phenomena, that is, the temperatures at center point are almost the same while the temperatures at right end point increase as β decreases.

Table 4.6.4

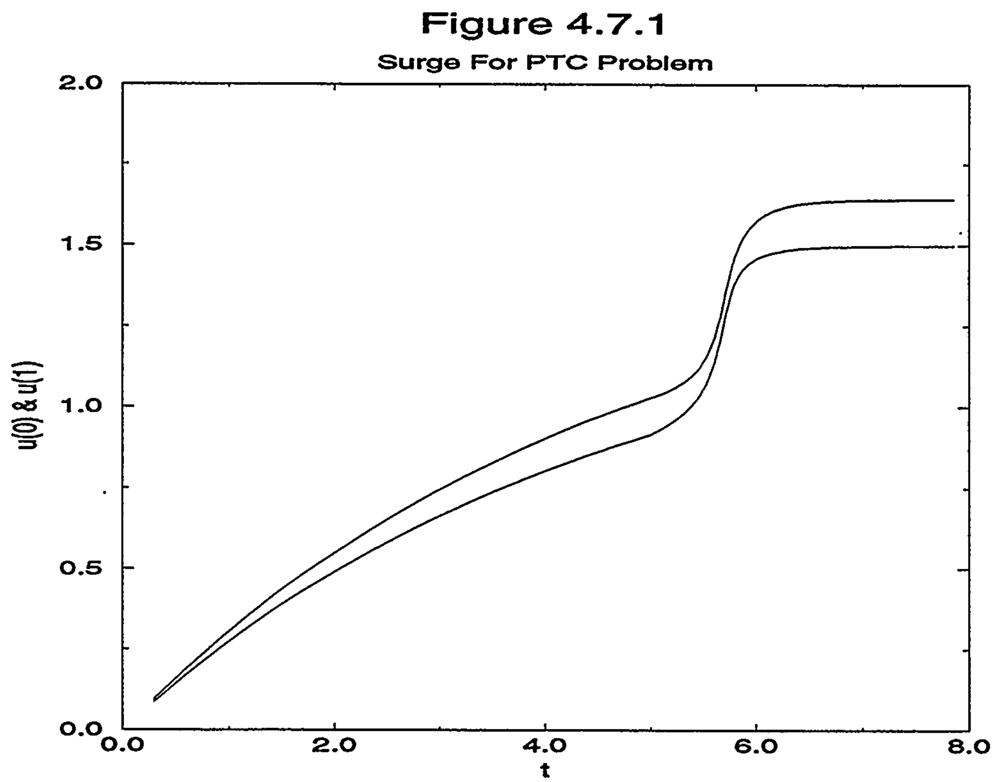
β	I	$u(0)$	$u(1)$	I	$v(0)$	$v(1)$
.6000	.4762E−1	.7369E+0	.5668E+0	.5502E−2	.1996E+1	.1268E+1
.5000	.4762E−1	.8503E+0	.6802E+0	.4794E−2	.1996E+1	.1352E+1
.4500	.4762E−1	.9258E+0	.7558E+0	.4414E−2	.1996E+1	.1399E+1
.4100	.4762E−1	.9996E+0	.8295E+0	.4096E−2	.1996E+1	.1438E+1
.4099	.4762E−1	.9998E+0	.8297E+0	.4095E−2	.1996E+1	.1438E+1
.4098	.4762E−1	.1000E+1	.8299E+0	.4095E−2	.1996E+1	.1438E+1
.4000	.4762E−1	.1020E+1	.8496E+0	.4015E−2	.1996E+1	.1449E+1
.3000	.4015E−2	.1996E+1	.1560E+1	.4015E−2	.1996E+1	.1560E+1
.2500	.2681E−2	.1996E+1	.1622E+1	.2681E−2	.1996E+1	.1622E+1
.2000	.2183E−2	.1996E+1	.1689E+1	.2183E−2	.1996E+1	.1689E+1

End point values for solutions of coupled NTC problem

as $t \longrightarrow \infty$ for different initial values.

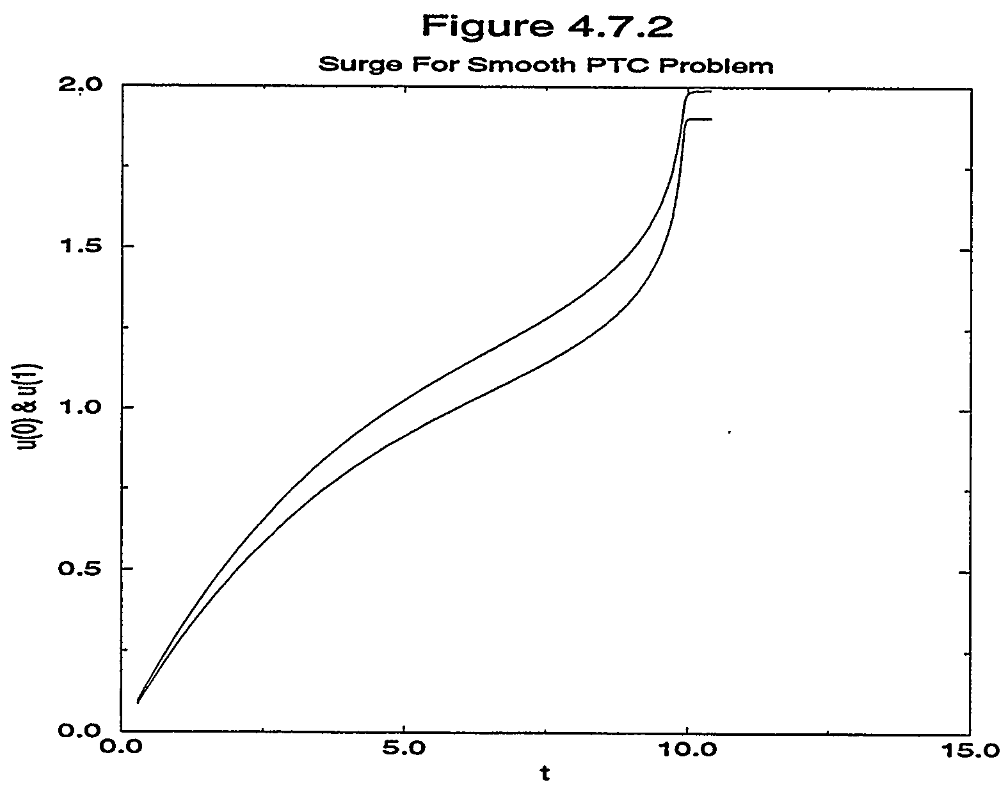
4.7 Conclusion for the Numerical Results

In the above several sections, different numerical methods are used. The numerical experiments for steady state PTC and NTC problems, time dependent PTC and NTC problems, PTC and NTC problems with smooth functions are



Lower curve is $u(1)$ and upper curve is $u(0)$ as functions of t

performed. From the numerical results, the properties of PTC and NTC problems are quite different. One of the important different properties is that if the external circuit is not connected for certain fixed α and β PTC problems have only one solution but NTC problems have three solutions. However, if the external circuit is connected, the situations are the same and they all have one, two or three solutions with respect to different parameters α and β. Another common property is that all time dependent solutions with the external circuit have surges. For Figure 4.7.1, Figure 4.7.2, Figure 4.7.3 and Figure 4.7.4, $\beta = 0.25$ is fixed. The value $\beta = 0.25$ is chosen because for this β there is only one solution which is greater than 1, thus the surges will appear.



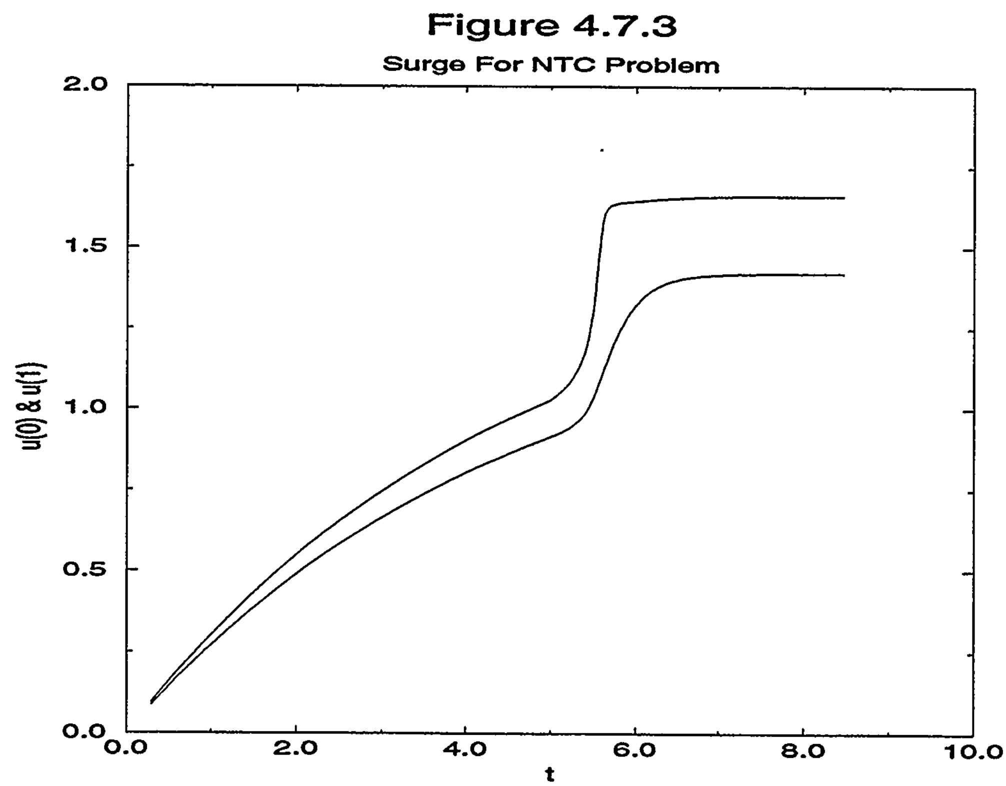
Lower curve is $u(1)$ and upper curve is $u(0)$ as functions of t

In order to get data for the four figures in this section, the Crank-Nicholson difference scheme is used. The interval [0, 1] is divided into 50 equidistant pieces so that there are 51 unknowns. In the figures, $u(0)$ and $u(1)$ are the numerically obtained values at end points $x = 0$ and $x = 1$ respectively. t coordinate is for time.

For Figure 4.7.1 and Figure 4.7.2, the equations are as follows:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\gamma}{(1+\mu\int_0^1 f(u)(\quad or \quad p(u))dx)^2} f(u)(\quad or \quad p(u)) & ,0 < x < 1 \\ u_x(0) = 0 & ,u_x(1) + \beta u(1) = 0 \end{cases}$$

where f is defined by (4.2) and $p(u)$ is defined by (4.57), β is a fixed parameter. As



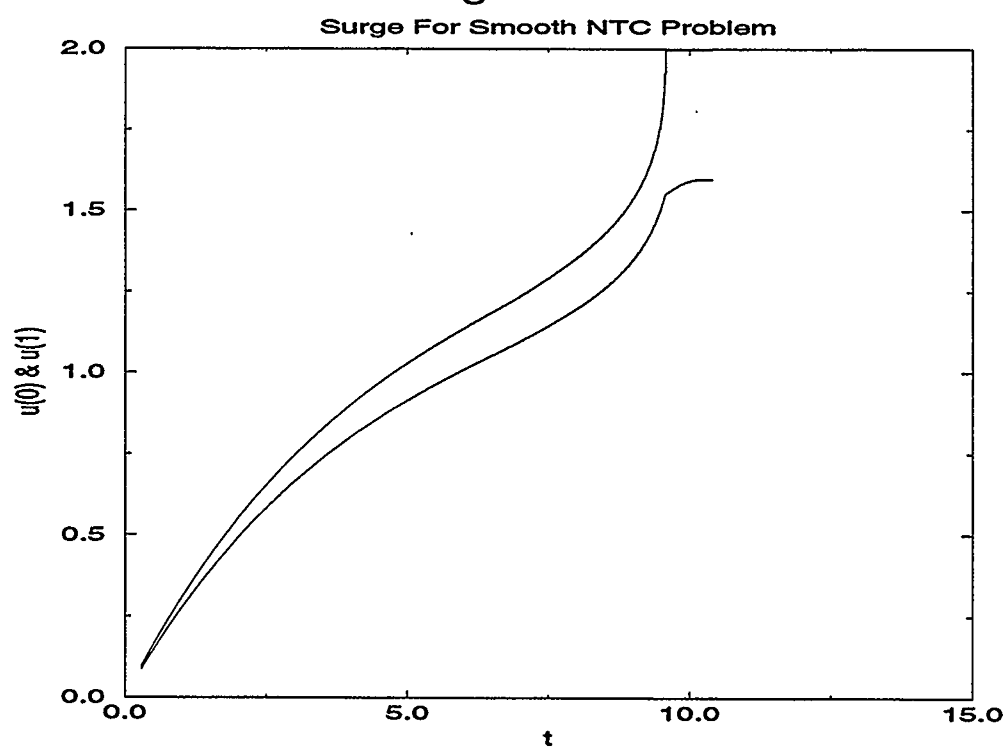
Lower curve is $u(1)$ and upper curve is $u(0)$ as functions of t

for Figure 4.7.3 and Figure 4.7.4, the equations are as follows:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\gamma}{(\mu+\int_0^1 g(u)(\quad or \quad q(u))dx)^2} g(u)(\quad or \quad q(u)) & ,0 < x < 1 \\ u_x(0) = 0 & ,u_x(1) + \beta u(1) = 0 \end{cases}$$

where $g(u)$ is defined by (4.44) and $q(u)$ is defined by (4.60). The difference of the two equations above is in the denominator of the second term on the right hand side. This difference makes the figures look a little bit different. For Figure 4.7.1 and Figure 4.7.2, the graphs change smoothly when surges appear. For Figure 4.7.3 and Figure 4.7.4, the graphs change rapidly and steeply when surges appear. Also the difference between $u(0)$ and $u(1)$ for PTC problems are smaller than that for NTC problems.

Figure 4.7.4



Lower curve is $u(1)$ and upper curve is $u(0)$ as functions of t

From the four figures, it is easy to see that for smooth functions more time is required to reach their steady state. For the smooth functions, whenever the solutions reach their steady states, they will satisfy the stopping criterion (the absolute maximum of difference between two time steps at each nodes is less than a fixed small number) very quickly. Another common property for the problems with smooth functions is that the numerical values are greater than that obtained from the corresponding original ones.

As it is known, when $\beta < 0.4098$, there is only one solution which is greater than 1 and the surges also appear. Thus the numerical experiments are also done for $\beta =$ 0.40, 0.35, 0.30, 0.20, 0.15, 0.10 and 0.05, the results and figures look similar. They all are between 1 and 2. The difference between $u(0)$ and $u(1)$ for NTC problems is greater than that for PTC problems. The graphs for NTC problems change more rapidly and steeply than for PTC problems.

Chapter 5

Convergence and Stability Analysis

Since the steady state solutions for time dependent problems are not unique for some parameters, is it possible to get all solutions numerically? In the actual numerical experiments, it seems difficult to obtain some solutions. As mentioned in §4.2.2 for Eq.(4.42), it is difficult to get the numerical solution, corresponding to the one spanning cold and transient regions, which theoretically exists. Now through discussion of linear stability, though a direct proof is not obtained, the numerical results would give some indication for that situation.

5.1 Linear Stability Analysis

Here a general idea of linear stability analysis [13, 14, 29, 30, 31, 49] is discussed. Generally for

$$\frac{du}{dt} = F(x,u), \qquad t > 0 \quad \& \quad u(0,x) = u_0(x) \tag{5.1}$$

where $F(x,u)$ is a continuously differentiable function of x and u, x is a space coordinate in the interval $[0,l]$ and l is a constant, $u_0(x)$ is a function of x and independent on t. Let $u_1(x)$ satisfy

$$F(x,u) = 0 \tag{5.2}$$

Then $u_1(x)$ is said a steady state solution of (5.1) with initial value $u_1(x)$.

Definition 5.1 *Suppose the solution of Eq.(5.1) exists and is denoted as $u(t,x)$. A steady state solution $u_1(x)$ is said to be stable if for any ε there exists a δ_ε such that*

$$|u(t,x) - u_1(x)| \le \varepsilon, \quad t > 0$$

whenever $|u(0,x) - u_1(x)| < \delta_\varepsilon$. If a steady state solution of (5.1) is not stable, it is said to be unstable.

Obviously, if a steady state solution of (5.1) is unstable, it is hard to get that solution by a numerical method because any error for initial values will grow. However, is it possible for us to know for what kind of function F the corresponding solution is stable?

Let $u_1(x)$ be a solution of (5.2) and $u(t,x)$ be a solution of (5.1) corresponding to the initial value $u_0(x)$ which is close to $u_1(x)$. Let δ be an arbitrary small number and $v(t,x)$ an arbitrary function such that

$$u(t,x) = u_1(x) + \delta v(t,x)$$

Substitute $u(t,x)$ into (5.1), thus

$$\begin{aligned} \delta\frac{dv}{dt} &= F(x, u_1(x) + \delta v(t,x)) - F(x, u_1(x)) \\ &= F_u(x, u_1(x))\delta v(t,x) + R(x, u_1(x), v(t,x), \delta) \end{aligned} \tag{5.3}$$

If δ is very small and R is of higher order in δ, $\lim_{\delta\to 0}\frac{|R|}{\delta} = 0$, it would appear that the behavior of the solution of (5.3) is determined by the linearized equation

$$\frac{dv}{dt} = F_u(x, u_1(x))v(t,x) \tag{5.4}$$

A basic result about stability may be stated as follows. For convenience, denote

$$\kappa \equiv F_u(x, u_1(x)) \tag{5.5}$$

Theorem 5.1 [13, 14, 29, 31, 49] *Let $u_1(x)$ be a steady state solution of (5.1) and assume that*

$$F(x, u_1(x) + \delta v(t,x)) = F_u(x, u_1(x))\delta v(t,x) + R(x, u_1(x), v(t,x), \delta)$$

where the remainder term R is $O(\delta^2)$ for δ sufficiently small. Then $u_1(x)$ is stable if $\kappa < 0$ and unstable if $\kappa > 0$, where κ is given by (5.5). #

The theorem 5.1 is for the case where F_u is a scalar function of x and u, etc.. If it is an operator, the result is true except that instead of using the sign of F_u the sign of the eigenvalues is used. The more general result may be stated as follows:

Theorem 5.2 [14, 29, 31, 49] *Let $u_1(x)$ be a steady state solution of (5.1) and assume that*

$$F(x, u_1(x) + \delta v(t,x)) = F_u(x, u_1(x))\delta v(t,x) + R(x, u_1(x), v(t,x), \delta)$$

where the remainder term R is $O(\delta^2)$ for δ sufficiently small and F_u is a self-adjoint operator. Then $u_1(x)$ is stable if all eigenvalues of F_u are less than zero and unstable if at least one of the eigenvalues of F_u is greater than zero. #

To find an eigenvalue for our operator is not easy. However, for a self-adjoint operator, even if it is not easy to find its eigenvalues, it is still possible to tell whether the operator has a negative eigenvalue or not. This result is based on the Rayleigh quotient. If $A : X \to X$ is a self-adjoint operator, where X is a Hilbert space,

$$R(x) = \frac{(Ax, x)}{(x, x)}$$

is said to be the Rayleigh quotient[14, 31]. If $||x|| = 1$, then $R(x) = (Ax, x)$. It is easy to see that if x is an eigenvector and λ is an eigenvalue, then $R(x) = \lambda$. Define

$$L_A = \inf_{x \in X, x \neq 0} R(x) = \inf_{x \in X, ||x||=1} (Ax, x) \tag{5.6}$$

and

$$U_A = \sup_{x \in X, x \neq 0} R(x) = \sup_{x \in X, ||x||=1} (Ax, x) \tag{5.7}$$

Hence, if A is bounded below, L_A is finite; if A is bounded above, U_A is finite.

Theorem 5.3 [14, 31] *Let A be symmetric and bounded below. If there is an element $x \in X$ for which the infimum in (5.6) is attained, (L_A, x) is an eigenpair and L_A is the lowest eigenvalue of A. Similarly, if there is an element $y \in X$ for which the supremum in (5.7) is attained, (U_A, y) is an eigenpair and U_A is the largest eigenvalue of A.* #

From theorem 5.3, if A is a self-adjoint operator and there is a function which makes the Rayleigh quotient negative, then there is at least one negative eigenvalue for A. Similarly, if there is a function which makes the Rayleigh quotient positive, then A has at least one positive eigenvalue.

5.2 The Numerical Results for Stability and Stability Analysis

Based on the general ideas in §5.1, the NTC problem met in §4.4 and §4.6.2 will be discussed now. Suppose $u_0(x)$, $\alpha_0 = \gamma I_0^2$ is a steady state solution of (2.10). Thus

$$\begin{cases} \frac{d^2 u_0(x)}{dx^2} + \alpha_0 g(u_0(x)) = 0 & ,0 < x < 1 \\ u_{0,x}(t, 0) = 0 & ,u_{0,x}(t, 1) + \beta u_0(t, 1) = 0 \end{cases} \tag{5.8}$$

where g is defined by (3.16) (or $\frac{1}{\sigma(u)}$), $\alpha_0 = \gamma I_0^2$, $I_0 = \frac{1}{\mu+\int_0^1 g(u_0(x))dx}$.

Using the linearized stability analysis method, set

$$u(t,x) = u_0(x) + \delta v(t,x) \tag{5.9}$$

with δ small. Substituting in (2.10), then

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \gamma I^2 g(u) & ,0 < x < 1 \\ u_x(t,0) = 0 & ,u_x(t,1) + \beta u(t,1) = 0 \end{cases}$$

where $I = \frac{1}{\mu+\int_0^1 g(u)dx}$. Since $\frac{\partial u_0(x)}{\partial t} = 0$ and $u_0(x)$ satisfies (5.8), therefore

$$\begin{cases} \delta\frac{\partial v}{\partial t} = \frac{\partial^2 u_0}{\partial x^2} + \delta\frac{\partial^2 v}{\partial x^2} + \gamma I^2 g(u_0 + \delta v) & ,0 < x < 1 \\ v_x(t,0) = 0 & ,v_x(t,1) + \beta v(t,1) = 0 \end{cases} \tag{5.10}$$

So if we can solve Eq.(5.10) and get $v(t,x)$, then according to the decaying or growing of $v(t,x)$ we can say that the steady state solution of (2.10) is stable or unstable. However it is not easy to get the solution to Eq.(5.10). A linearized method is used to obtain an approximate solution to Eq.(5.10). Using a Taylor expansion, thus

$$g(u_0 + \delta v) = g(u_0) + g'(u_0)\delta v + R_1$$

where R_1 is of higher order in δ, and

$$\begin{aligned} I = & \frac{1}{\mu+\int_0^1 g(u)dx} & = \frac{1}{\mu + \int_0^1 (g(u_0) + g'(u_0)\delta v + R_1)dx} \\ = & \frac{1}{\mu+\int_0^1 g(u_0)dx+\delta\int_0^1 g'(u_0)vdx+\cdots} & = \frac{1}{\frac{1}{I_0} + \delta\int_0^1 g'(u_0)vdx + \cdots} \\ = & \frac{I_0}{1+\delta I_0\int_0^1 g'(u_0)vdx+\cdots} & = I_0(1 - \delta I_0 \int_0^1 g'(u_0)vdx + \cdots) \end{aligned}$$

where $\cdots$ means the omitted part contains terms with higher order (at least 2) of δ. Similarly, using the above results

$$\gamma I^2 = \gamma I_0^2(1 - 2\delta I_0 \int_0^1 g'(u_0)vdx + \cdots)$$

and

$$\begin{aligned}\gamma I^2 g(u) &= \gamma I_0^2(1-2\delta I_0 \textstyle\int_0^1 g'(u_0)v dx + \cdots)(g(u_0)+g'(u_0)\delta v + \cdots) \\ &= \gamma I_0^2 g(u_0) + \delta(\gamma I_0^2 g'(u_0)v - 2\gamma I_0^3 g(u_0) \textstyle\int_0^1 g'(u_0)v dx) + \cdots\end{aligned}$$

If we omit all terms which contain higher orders of δ and divide by δ on both sides, then Eq.(5.10) becomes

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \gamma I_0^2 g'(u_0)v - 2\gamma I_0^3 g(u_0) \int_0^1 g'(u_0)v dx & , 0 < x < 1 \\ v_x(t,0) = 0 & , v_x(t,1) + \beta v(t,1) = 0 \end{cases} \tag{5.11}$$

As it is known, the operator on the right hand side of Eq.(5.11) is not a self-adjoint operator. Even if there is a function which makes the right hand side to be negative, it could not be said that there is a negative eigenvalue for that operator. However, in the NTC problem since $g'(u_0) > 0$ the term $-2\gamma I^3 g'(u_0) \int_0^1 g(u_0)v dx$ is a stablizing term (by experience), adding this term to the right hand side makes the right hand side operator self-adjoint. If the modified operator is unstable, the original operator should be more unstable. Thus, instead of considering Eq.(5.11), the following equation is considered.

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \gamma I_0^2 g'(u_0)v - 2\gamma I_0^3 g(u_0) \int_0^1 g'(u_0)v dx & \\ \qquad -2\gamma I_0^3 g'(u_0) \int_0^1 g(u_0)v dx & , 0 < x < 1 \\ v_x(t,0) = 0 & , v_x(t,1) + \beta v(t,1) = 0 \end{cases} \tag{5.12}$$

Therefore, if the operator on the right hand side of Eq.(5.12) has positive eigenvalue, then the steady state solution $u_0(x)$ is unstable. The following method is just one to demonstrate that the right hand side operator of Eq.(5.12) has positive eigenvalue. To form the Rayleigh quotient $(Ax, x)/||x||^2$, now multiply the right hand side of the first equation of (5.12) by v and integrate from 0 to 1 with respect to x, thus

$$R(v)\int_0^1 v^2dx = \int_0^1 v_{xx}vdx + \gamma I_0^2\int_0^1 g'(u_0)v^2dx - 4\gamma I_0^3\int_0^1 g(u_0)vdx\int_0^1 g'(u_0)vdx$$

Simplifying

$$R(v)\int_0^1 v^2dx = vv_x|_0^1 - \int_0^1 v_x^2dx + \gamma I_0^2\int_0^1 g'(u_0)v^2dx - 4\gamma I_0^3\int_0^1 g(u_0)vdx\int_0^1 g'(u_0)vdx$$

Using the boundary conditions

$$\begin{aligned} R(v)\int_0^1 v^2dx &= -\beta v^2(1) - \int_0^1 v_x^2dx + \gamma I_0^2\int_0^1 g'(u_0)v^2dx \\ &\quad -4\gamma I_0^3\int_0^1 g(u_0)vdx\int_0^1 g'(u_0)vdx \end{aligned} \tag{5.13}$$

To examine if there is a function which makes the right hand side of Eq.(5.13) to be positive or not choose $v(x) = \cos(\lambda x)$ then

$$\begin{aligned} R(v)\int_0^1 cos^2(\lambda x)dx &= -\beta cos^2(\lambda) - \int_0^1 \lambda^2 sin^2(\lambda x)dx \\ &\quad +\gamma I_0^2\int_0^1 g'(u_0)cos^2(\lambda x)dx \\ &\quad -4\gamma I_0^3\int_0^1 g(u_0)cos(\lambda x)dx\int_0^1 g'(u_0)cos(\lambda x)dx \end{aligned}$$

hence

$$\begin{aligned} \frac{1}{2}R(v)(1+\frac{1}{2\lambda}sin(2\lambda)) &= -\beta cos^2(\lambda) - \frac{\lambda^2}{2}(1-\frac{sin(2\lambda)}{2\lambda}) \\ &\quad +\gamma I_0^2\int_0^1 g'(u_0)cos^2(\lambda x)dx \\ &\quad -4\gamma I_0^3\int_0^1 g(u_0)cos(\lambda x)dx\int_0^1 g'(u_0)cos(\lambda x)dx \end{aligned} \tag{5.14}$$

For convenience, denote

$$R(v) = A(u_0,\alpha,\mu,\beta,\lambda) = \frac{1}{\frac{1}{2}(1+\frac{1}{2\lambda}sin(2\lambda))}[-\beta cos^2(\lambda) \tag{5.15}$$

$$-\frac{\lambda^2}{2}(1-\frac{sin(2\lambda)}{2\lambda})+\gamma I_0^2\int_0^1 g'(u_0)cos^2(\lambda x)dx$$
$$-4\gamma I_0^3\int_0^1 g(u_0)cos(\lambda x)dx\int_0^1 g'(u_0)cos(\lambda x)dx]$$

Thus (5.15) is the Rayleigh quotient. That's why in the following the sign of $A(u_0,\alpha,\mu,\beta,\lambda)$ is discussed. Considering the boundary conditions, choose λ so that

$$v_x(0)=0, \qquad v_x(1)+\beta v(1)=0$$

Therefore

$$-\lambda sin(\lambda)+\beta cos(\lambda)=0$$

i.e.,

$$\lambda tan(\lambda)=\beta \tag{5.16}$$

As it is known, there are infinitely many solutions to Eq.(5.16). However, to prove that the operator on the right hand side of Eq.(5.12) has a positive eigenvalue, if we can find one λ for which $\cos(\lambda x)$ makes $A(u_0,\alpha,\mu,\beta,\lambda)$ positive, then based on the general discussion in § 5.1, the operator on the right hand side of Eq.(5.12) has positive eigenvalue, so the smallest positive solutions of (5.16) can be taken. Results obtained giving λ for various β are summarized in Table 5.1.1.

Table 5.1.1

β	λ	β	λ	β	λ	β	λ
.100	.31105286	.250	.48009443	.400	.59324193	.550	.68005705
.150	.37787765	.300	.52179116	.450	.62444466	.600	.70506549
.200	.43284073	.350	.55922329	.500	.65327120	.650	.72850811

Smallest solution λ of (5.16) for given β.

Now we have the values of β and λ. The three solutions of I_0 (i.e., α_0) and u_0 for the steady state situation can be obtained numerically. Thus, when $\beta > 0.4098$, there are three solutions, that means there exist three different values of α for a fixed β.

Table 5.1.2

β	λ	α	β	λ	α
0.450	0.62444466	0.34013605e+00	0.500	0.65327120	0.34013605e+00
		0.24069230e−02			0.28149190e−02
		0.33633682e+00			0.33233449e+00
0.550	0.68005705	0.34013605e+00	0.600	0.70506549	0.34013605e+00
		0.32305950e−02			0.36516950e−02
		0.32855827e+00			0.32496199e+00
0.650	0.72850811	0.34013605e+00	0.700	0.75055808	0.34013605e+00
		0.40758720e−02			0.45013170e−02
		0.32153054e+00			0.31825082e+00

Values of α and λ for given β.

As discussed in §4.3.2, there always exists a cold solution with

$$\frac{\gamma I^2}{\beta} = \frac{\gamma I}{\beta}(1 - \mu I)$$

thus $I_0 = \frac{1}{1+\mu}$. Therefore for $\beta > 0.4098$, there always exists a cold solution with $I_0 = \frac{1}{1+\mu}$, i.e., α_0 is fixed though β is different. It is easy to see from Figure 4.3.4, one other α (corresponding to the solution spanning cold and transient regions) should be close to α_0 and the third one corresponding to the whole solution in the

Table 5.1.3

β	λ	α	$u(0)$	$u(1)$	$A(u_0, \alpha, \mu, \beta, \lambda)$
		0.34013605E+0	0.92592603	0.75585800	−0.77986223
0.45	0.62444466	0.33633682E+0	1.03792620	0.83581495	2.38273615
		0.24069231E−2	1.64676213	1.22786081	−33.17415905
		0.34013605E+0	0.85034019	0.68027216	−0.85352647
0.50	0.65327120	0.33233449E+0	1.06160045	0.82755232	3.27942348
		0.28149190E−2	1.64484048	1.18660295	−35.73973656
		0.34013605E+0	0.78849727	0.61842924	−0.92495513
0.55	0.68005705	0.32855827E+0	1.08037066	0.81649649	3.87784278
		0.32305950E−2	1.64317393	1.14788091	−38.15972233
		0.34013605E+0	0.73696148	0.56689346	−0.99423480
0.60	0.70506549	0.32496199E+0	1.09650326	0.80412000	4.42084587
		0.36516951E−2	1.64167249	1.11149311	−40.44283008
		0.34013605E+0	0.69335431	0.52328628	−1.06144810
0.65	0.72850811	0.32153055E+0	1.11053240	0.79098970	4.90334769
		0.40758718E−2	1.64029598	1.07726705	−42.60123539
		0.34013605E+0	0.65597671	0.48590869	−1.12667477
0.70	0.75055808	0.31825081E+0	1.12317467	0.77747375	5.25375581
		0.45013172E−2	1.63901687	1.04503202	−44.64574623

Table to demonstrate the stability of solution of the NTC problem.

transient region is far away from the other two. The different values of α corresponding to the β and λ are listed in Table 5.1.2. The values of α and β are obtained by the methods discussed in § 4.1.2 and §4.2.2.

Now the value of $A(u_0, \alpha, \mu, \beta, \lambda)$ can be numerically evaluated. Using the data given in Table 5.1.2 and the corresponding $u_0(x)$ (actually the values are numerically obtained at meshpoints), the numerical values of $A(u_0, \alpha, \mu, \beta, \lambda)$ and the values of $u_0(0)$ and $u_0(1)$ are listed in Table 5.1.3. Here the integrals in (5.15) are evaluated by a trapezoidal rule over the values of u at the meshpoints. From Table 5.1.3, it is found that the values of $A(u_0, \alpha, \mu, \beta, \lambda)$ assume positive numbers for the solution spanning cold and transient regions, that means that the operator on the right hand side of Eq.(5.12) has positive eigenvalue, thus the corresponding solutions are unstable. So this numerical result gives some indication why we can not get that solution for the time-dependent problem even if we take the initial values very close to the solution. As for the other two solutions, it seems that they are stable (the values of $A(u_0, \alpha, \mu, \beta, \lambda)$ assume negative numbers at those solutions) and the numerical results for steady state problems agree very well with the numerical results obtained for time-dependent problems.

Chapter 6

Conclusion

In this dissertation, attention has been focused on a system of nonlinear parabolic equations which model the thermistor. First, a partial differential equation which describes the heat and current distribution within a thermistor is given, then under some specified boundary conditions two types of nonlinear one (space) dimensional parabolic equations are derived. One type has the PTC property, and the other has NTC property.

Theoretically, existence and uniqueness (whenever applicable) are studied both for steady state and time dependent problems. The properties of the solutions are thoroughly discussed. Three methods are employed to prove the existence and uniqueness for steady state problems. One is to transform the original differential equations into integral equations. By proving the existence and uniqueness of the solutions for integral equations, existence and uniqueness for original problems are obtained.

The other method is to change the variables. Instead of the original boundary value problems being considered, corresponding initial value problems are studied. From the relationship of the original boundary value problems and the initial value problems, the existence and uniqueness are obtained again.

The third method is a monotone method. By using concept of upper and lower solutions, existence and uniqueness for steady state problems are obtained. As for the time dependent problems, if the external circuit is not connected, existence and uniqueness are obtained too. If the external circuit is connected, existence and uniqueness are obtained only for the PTC type problem.

Many numerical experiments are done both for steady state problems and time dependent problems. There are three methods used to obtain numerical solutions. The first one is a shooting method which is used for steady state problems. The second one is moving meshpoint method in which the stepsize, interval length and the number of meshpoints are changing according to some interface conditions. The third one is fixed meshpoints in which the stepsize, interval length and the number of meshpoints are fixed. Obviously, the moving meshpoint method is more complicated than the fixed meshpoint method. Numerical results showed that accuracy of the numerical solutions obtained by the moving meshpoint method is better than that obtained by the fixed meshpoint method. Since the algorithm for the moving meshpoint method is much more complicated than that for the fixed meshpoint method, the advantage of the moving meshpoint method is jeopardized though the precision is improved by using moving meshpoint method. The fixed meshpoint method gives steady state solutions which agree sufficiently with those obtained by other methods. Hence the moving meshpoint method is only used once for PTC problem. For most problems considered the conductivity used is continuous but not differentiable everywhere, for comparison, the same problems with smooth conductivity are numerically solved. Many numerical results are obtained. Numerical results for typical parameters are listed in several tables and some of them are demonstrated in figures. The solutions for the problems with smooth conductivity change more smoothly at interface points than the solutions for original problems with the non-differentiable conductivity.

As one of the three (if there are three) solutions can not be found numerically, the concept of stability and instability is reviewed. Thus the stability and instability of solutions are briefly discussed.

In this dissertation, not only the NTC problem is studied as in most other literature [3, 4, 35, 36, 37, 38, 40, 41], the PTC problem is also studied in detail. The different properties of PTC and NTC problems are also reflected in their solutions. Those properties are studied thoroughly. For PTC problem, if the external circuit is not connected, for the given parameters α and β, there always exists one solution. But for NTC problem, if the external circuit is not connected, for some given parameters α and β, there exist three solutions, for some other given parameters α and β, there exist only one solution. All solutions of the steady state PTC and NTC problems can be numerically obtained. As for time dependent problems without external circuit, the solution of the PTC problem can be numerically obtained since there is only one solution and the numerical solution agrees sufficiently well with the numerical solution of the corresponding steady state PTC problem. The NTC problem has three solutions, but only two of them can be numerically obtained. One of them seems quite unstable and for any initial value the solution converges to one or another of the stable solutions, the third solution can not be obtained numerically even if it exists theoretically. If the external circuit is connected, both for PTC and NTC problems there exist three situations. That is, there exist one, two or three solutions for different parameters α and β. If there exist three solutions for the time dependent external circuit connected problems, the common property of the solutions for PTC and NTC problems is that one of the three solutions is numerically unstable. If there exist one or two solutions, all of them can be numerically obtained.

Though many theoretical and numerical results have been obtained, there are still some uncertainties, such as uniqueness (at most locally) for NTC problems and a direct proof of the instability, etc. Currently, there seems to be little literature contributed to two dimensional problems.

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Appendix A

IMSL Subroutine IVPRK[50]

There are lots of IMSl subroutines installed in AIX machines. Here only the related subroutine is included.

Name: IVPRK (Single Precision)

Purpose: Solve an iniatial-value problem for ordinary differential equations using the Runge-Kutta-Verner fifth-order and sixth-order method.

Usage: CALL IVPRK(IDO,N,FCN,T,TEND,TOL,PARAM,Y)