THE UNIVERSITY OF CALGARY

THE WINNER TAKES IT ALL -
A DAY AT THE RACES

by

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ABSTRACT

This thesis deals with a maximization procedure to determine optimum betting strategies for racetrack wagering. The gambler is assumed to have sufficient knowledge to make estimates of the probability of each particular horse winning the race.

The first problem considered is that of dependent, non-mutually exclusive wagers, such as place and show wagers on the same horse. This strategy will then be used in Chapter III in a test of the theories presented. The next problem of concern is the case where several horses are deemed to have positive expected gains. A general analysis designed to maximize the long term growth of the gambler's fortune is presented.

Procedures for profiting from both strategies are presented even for those that have limited knowledge of the subject. The main thrust, however, is to use the results in concert with a good education in horse racing, for maximum effectiveness of the theory.

ACKNOWLEDGEMENTS

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Also, a sincere note of thanks to Mrs. Maria Dourado for her excellent typing job.

Dedicated to my parents

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CHAPTER I

INTRODUCTION

1.1 GAMES OF CHANCE

This thesis deals with maximizing a gambler's bankroll by determining optimal wagers for different gambling situations. Secondly, a statistical method of pinpointing gambling situations which have a positive expected return will be presented.

The model used for this procedure will be racetrack wagering. Horse racing was chosen because it offers one of the few gambling opportunities which can theoretically be beaten. To consider whether any type of gamble has prospects to be profitable, two factors must be considered.

Firstly, the probability of winning the wager must be estimated. In the majority of gambling situations, this is known. Games such as craps, bingo, and keno fall into this category. Other games such as blackjack, have slightly varying probabilities because of the constant changes in the deck.

The other factor which must be considered is that of odds. What profit will the gambler make if his wager is successful? Almost all gambles have pre-set odds. In games such as craps, where both the odds and probability are known and constant, the house is guaranteed an edge in the long run because these factors have been set to their advantage. In order to have any possibility of making a long-run profit, the gambler must find some variability in at least one of these factors. In most casino games, however, both factors are usually constant, allowing no possibility of long term profit.

Blackjack is one casino game where there are prospects for profit. The probability of success, in certain situations, can be high enough to allow the gambler a small edge. This occurs when the number of cards remaining in the deck is relatively small and there are disproportionate numbers of certain valued cards remaining. For example, if the 5's were removed from the deck, the game of blackjack becomes a profitable bet when playing intelligently against house rules.

Another game which offers opportunity for profit is poker. As poker is a game where the gambler bets against other gamblers and not against the house, a player who is superior in skill over the other players will undoubtedly come out ahead in the long run.

Horse racing has characteristics very similar to poker. The racetrack deducts a fixed percentage from all wagers and then divides the remaining money among all gamblers. In this way the house has no interest in what the outcome of the race may be, as they receive the same amount in any event. Therefore, the gambler is betting against other gamblers and not the house.

In horse racing, the odds on a particular horse are determined by the relative amount wagered on him. In this way the odds are determined by the betting public, allowing for a great deal of variability. Also, the probability that any particular horse wins a race is a subjective quantity. Hence, with the uncertainness of the two factors, if a good estimate of probabilities can be determined there are betting opportunities available which offer positive returns.

Once a profitable betting situation has been found, the problem then becomes one of determining the proportion of the gambler's wealth to risk on the opportunity. Wealth is defined as a sum set aside by the gambler for the sole purpose of betting. The amount will then fluctuate according to the success or failure of each particular wager.

Kelly [5] first proposed a procedure to determine optimal betting fractions when considering a single wager. For our purposes, optimal betting fractions will be defined as the strategy which maximizes the long-term growth of the gambler's fortune. This thesis extends that analysis to include cases where a gambler has several profitable bets on a single race. Other adaptions of Kelly's work have been found, notably Thorpe [9], who considered fractions of an investor's wealth to be divided up among different stocks in a portfolio. The main difference in our work is that unlike the stock market, where all stocks can increase in value, only one horse can win a race.

The way the betting public distributes its' money on various horses is also of great interest. Griffith [3] noted the pyschological aspects of this subject, showing that racegoers tend to look for the "big kill" rather than wagering on more likely horses at lower odds. This tendency, echoed by Asch, Malkiel, and Quandt [2] and Snyder [8], has become known as the favorite-longshot bias. This phenomenon concerns itself with the respective underbetting of low-odds horses (favorites) versus the overbetting of longshots. Although the differences are not significant enough to allow for profit, by using this theory in concert with other procedures, wagers which have expected profit can be found.

Since gamblers are by nature risk-seeking individuals, their judgement in wagering situations can tend to be very poor. As horse racing lets the gamblers set their own odds between themselves, this poor judgement can be turned into profit by an intelligent investor.

1.2 PARI-MUTUEL SYSTEM

The pari-mutuel system of wagering is used for gambling games such as horse racing and jai alai. The basic concept is for the house (racetrack) to deduct a certain percentage from all wagers made and divide the remaining money among winning bettors. In order to fully understand the concepts involved in horse race betting, the following definitions are necessary:

POOL	-	This is the total amount of money wagered on one particular kind of bet. e.g. The win pool is the total amount wagered on all horses to win in a particular race.
TRACK-TAKE	-	the percentage of a pool deducted by the racetrack to pay for expenses (purses, salaries, security, etc.).
WIN BET	-	in order to collect, the gambler will need his horse to finish first.
PLACE BET	-	in order to collect, the gambler's horse must finish first or second.
SHOW BET	-	in order to collect, the gambler's horse must finish first, second, or third.
EXOTIC BET	-	any type of bet other than win, place, and show. The following are three examples.
DAILY DOUBLE BET	-	the object of this wager is to pick the winners of two races in a row. (These races are designated by the racetrack).
EXACTOR BET	-	to win this wager, the gambler must select the first and second place horses in exact order.
QUINELLA BET	-	the object of this wager is to select the first two horses to cross the finish line without regard to order.

CALCULATION OF ODDS USING THE PARI-MUTUEL SYSTEM

(a) WIN ODDS

Q_i = win odds (profit per dollar bet given horse i wins the race).

W_i = amount bet to win on horse i, $i = 1,2,\ldots,n$, where n is the number of horses in the race.

W = total amount bet to win on all horses = $\sum_{i=1}^{n} W_i$.

t = track take.

If horse i wins, then the net win pool (after track take) would be paid to bettors who have bet a cumulative amount, W_i. Hence

$$W_i(1+Q_i) = (1-t)W$$

or

$$Q_i = \frac{(1-t)W - W_i}{W_i} \quad \begin{bmatrix}\text{rounded down to nearest} \\ \text{.05, minimum .05}\end{bmatrix} \tag{1.1}$$

(b) PLACE ODDS

R_{ij} = place odds on horse i given he finishes first or second along with horse j

L_i = amount bet to place on horse i

L = total amount bet to place on all horses = $\sum_{i=1}^{n} L_i$.

If horse i finishes first or second along with horse j, the net place pool has the amounts wagered to place, on these horses, deducted from it. The remaining amount is divided in two, with one part divided among horse i bettors and the other part paid to horse j bettors. Hence

$$(1.2) \qquad R_{ij} = \frac{(1-t)L-L_i-L_j}{2L_i} \qquad \begin{bmatrix}\text{rounded down to nearest} \\ \text{.05, minimum .05}\end{bmatrix}.$$

(c) **SHOW ODDS**

T_{ijk} = show odds on horse i given he finishes first, second, or third along with horses j and k

S_i = amount bet to show on horse i

S = total amount bet to show on all horses = $\sum_{i=1}^{n} S_i$.

The payoff on horse i, given he finishes first, second or third along with j and k is calculated similar to the place odds, only once S_i, S_j, and S_k are deducted from the net pool, the remainder is divided into three parts. Therefore the odds on horse i would be:

$$(1.3) \qquad T_{ijk} = \frac{(1-t)S-S_i-S_j-S_k}{3S_i} \qquad \begin{bmatrix}\text{rounded down to nearest} \\ \text{.05, minimum .05}\end{bmatrix}.$$

To fully comprehend the procedure, an example is in order:

		HORSE #								
		1	2	3	4	5	6	7	8	POOL
	WIN	1500	2000	2500	500	1500	3000	200	800	12000
BET TYPE	PLACE	1000	2000	1000	500	2000	2000	200	300	9000
	SHOW	500	800	500	300	1400	1500	200	300	5500

$$W = 12000$$

$$L = 9000$$

$$S = 5500$$

$$\text{Let } t = .15$$

Now, consider the odds with the following race outcome:

1ST PLACE - horse #1

2ND PLACE - horse #6

3RD PLACE - horse #4.

Calculation of win odds for #1 using (1.1) yields:

$$Q_1 = \frac{(1-.15)\ 12000 - 1500}{1500} = 5.80.$$

Hence, a \$2 win wager on #1 would return \$2 + 5.80(2) = \$13.60.

Using (1.2) to calculate place odds on #1 and #6 gives:

$$R_{16} = \frac{(1-.15)\ 9000 - 1000 - 2000}{2(1000)} = 2.325 \text{ (rounded to 2.30)}$$

$$R_{61} = \frac{(1-.15)\ 9000 - 2000 - 1000}{2(2000)} = 1.1625 \text{ (rounded to 1.15).}$$

Therefore, a \$2 wager to place on #1 returns \$6.60, while on \$2 place wager on #6 returns \$4.30.

Using (1.3) to calculate show odds on #1, #6, and #4 yields

$$T_{164} = \frac{(1-.15)\ 5500 - 500 - 1500 - 300}{3(500)} = 1.583 \text{ (round to 1.55)}$$

$$T_{614} = \frac{(1-.15)\ 5500 - 1500 - 500 - 300}{3(1500)} = 0.527 \text{ (round to 0.50)}$$

$$T_{416} = \frac{(1-.15)\ 5500 - 300 - 500 - 1500}{3(300)} = 2.639 \text{ (round to 2.60).}$$

Hence, two dollar show wagers on #1, #6, and #4 would return (respectively) \$5.10, \$3.00, and \$7.20.

It is important to note that place and show odds for a particular horse are dependent on the amount of money bet on the other horses that finish in the top two or top three with him. Because of this dependence, any odds calculated before the outcome of the race becomes known are estimates. The win odds, however, can be calculated before the race with good accuracy as there is no dependence on the performance of other horses.

1.3 HANDICAPPING

In order to isolate wagers that possess a positive expected return, an estimate of each horse's probability of winning must be ascertained. Once this has been accomplished, each probability can be compared to the respective odds to determine whether a bet is indicated.

The set of probabilities, however, can be extremely difficult to estimate. As an example, the 1986 Kentucky Derby offered wagering at more than forty racetracks across North America. Each racetrack had separate wagering on the race, therefore the odds on the horses varied depending on the different crowds' opinions of their probabilities of winning.

The Derby was won by Ferdinand, with the win pays at various tracks having great desparity.

TABLE 1.1

WIN PAY PER $2 WAGER ON FERDINAND AT VARIOUS TRACKS

TRACK	PAY
Churchill Downs	$37.20
Hollywood Park	$16.80
Fairplex Park	$13.20
Evangeline Downs	$90.00

Ferdinand's probability of winning is unknown, but if we assume that the crowd at Churchill Downs (the site of the race) were best informed, and hence their estimate reasonably accurate, certainly a bet at Evangeline Downs (as well as many other tracks) offered a positive expected return.

Large variances in payoffs on the same race are very commonplace today with the advent of simulcasting of major races across North America. A possibility for profit certainly exists, but a punter can only be at one track at a time, making exploitation of these events unlikely.

However, the differences in the payoffs do show that racetrack crowds have widely varying opinions and the odds they set can indeed be wrong. It is these disparities a professional horseplayer looks for to achieve profit.

There is a multitude of information to help the horseplayer determine on which horse(s) he should bet. A bettor must be familiar with trainers, owners, and drivers (or jockeys). He must be able to compare the abilities of the horses based on their past performances. The major factors involved in this comparison are speed, quality of the horse (class), post position, pace, and consistency.

Like any profession, the more you learn about the topic, the better your chance to succeed. So if a gambler has enough experience, he should be able to make fairly accurate estimates regarding the probabilities of each horse winning.

With these estimates, the punter then can compare the odds offered on his horse(s) and determine which bets, if any, offer a positive expected return.

An increasingly popular method of estimating probabilities has arisen in the past few years. With only a basic knowledge of the sport, a horseplayer can use multiple regression software, with the factors previously mentioned being used, to estimate a horse's finish position. These estimates are then regressed to form a set of probabilities.

Although some of these models have had some success, a major problem is that a model is not interchangeable between racetracks. If this procedure is used, it must be developed using data from the racetrack in which you are interested. The main problems in using one of the other models is that all racetracks have different configurations, biases, weather conditions, and quality of horses.

This thesis will not concern itself with handicapping, as this is highly subjective. Those interested in handicapping and the relative statistical importance of various factors are advised to refer to Quirin [7], the definitive work on the subject. Hence, a mathematical model will be presented to keep the study objective.

The betting strategies presented, however, can be utilized by someone who prefers to handicap his own selections. By combining knowledge, through handicapping, with the methods presented here, the gambler will have a powerful tool at his disposal.

1.4 FAVORITE - LONGSHOT BIAS

Although the betting public does make frequent errors in estimating probabilities, on average they tend to do a fairly accurate estimate. Ali [1] performed an extensive study of 20,247 harness races to examine just how accurate they were.

By taking the proportion of the win pool wagered on a particular horse, we have the public's estimate of his probability of winning. Then after ranking the horses according to their favoritism, the actual probability of winning was observed and compared. The results follow.

TABLE 1.2

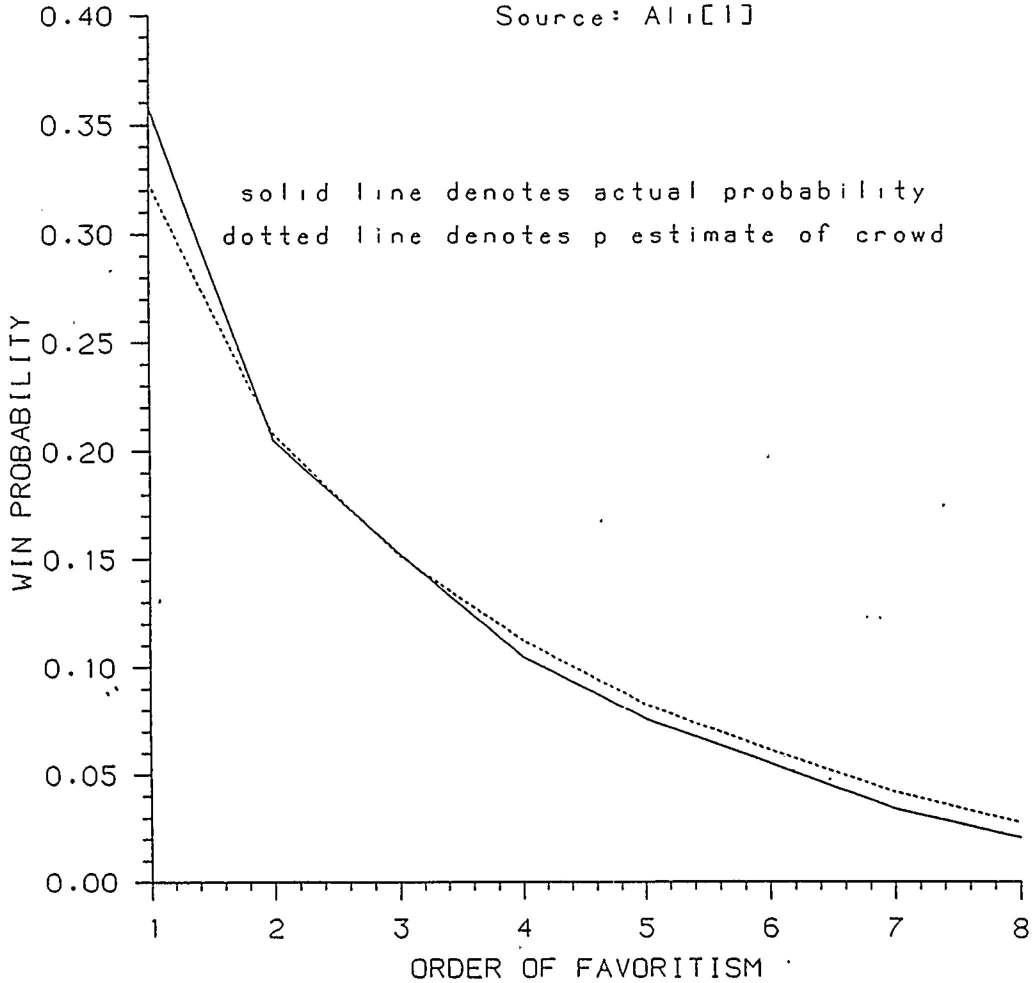
ILLUSTRATION OF FAVORITE/LONGSHOT BIAS

FAVORITES[a]	NUMBER OF RACES	ACTUAL PROBABILITY OF WINNING	CROWD'S ESTIMATED PROBABILITY OF WINNING	Z STATISTIC
FIRST	20,247	0.3583	0.3237	- 10.29*
SECOND	20,247	0.2049	0.2077	- 0.99
THIRD	20,247	0.1526	0.1513	- 0.52
FOURTH	20,247	0.1047	0.1121	3.45*
FIFTH	20,231	0.0762	0.0827	3.49*
SIXTH	20,088	0.0552	0.0611	3.01*
SEVENTH	19,281	0.0341	0.0417	5.80*
EIGHTH	15,749	0.0206	0.0276	6.20*

a Lowest odds horses (e.g. the horse with the lowest odds is the 1st favorite).

* Significantly different at 5% level. SOURCE: ALI [1]

FIGURE 1.1 - PROBABILITY OF 1ST TO 8TH FAVORITES
winning a given race in 20,247 races, 1970-1974
Source: Ali[1]



The results of the study show that the public's estimate of horse i's probability of winning:

$$(1.4) \quad \text{public } P_i = \frac{W_i}{W} \quad i = 1, \ldots, n, \quad n \text{ is number of horses in race.}$$

where W = win pool

W_i = amount bet to win on horse i

is a fairly good estimate of the true probabilities. The bias involved, however, is that the favored horses (those at lower odds) are actually underbet relative to their true probabilities and therefore, the longshots tend to be overbet. It is this observation which will be used as a vital assumption in the following section.

This tendency is not unexpected as we are dealing with risk-seeking individuals with lottery-type mentalities. Although the differences are significant, they are not enough to allow for profit. However, with this knowledge, procedures will be shown which take advantage of the bias.

1.5 HARVILLE FORMULAE

The Harville formulae [4] allow for simple estimation of place and show probabilities, assuming that win probabilities are available.

Under the assumption that horse i wins the race, the probability that j comes second is his probability of winning a race versus the remaining contenders. Hence the probability of an ij finish is:

$$(1.5) \qquad P_{ij} = P_i \cdot \frac{P_j}{1-P_i}$$

where P_k is the probability of horse k winning, $1 \leq k \leq n$, n being the number of horses in the race.

P_{ij} is the probability of horse i finishing first and horse j finishing second ($1 \leq i \leq n$, $1 \leq j \leq n$, $i \neq j$).

Clearly, the probability of horse j finishing second can be approximated at:

$$(1.6) \qquad P(\text{horse } j \text{ second}) = \sum_{\substack{k=1 \\ k \neq j}}^{n} P_k \cdot \frac{P_j}{1-P_k} = P_j \sum_{\substack{k=1 \\ k \neq j}}^{n} \frac{P_k}{1-P_k} .$$

Similarly, the probability of an ijk finish is calculated to be:

$$(1.7) \qquad P_{ijk} = P_i \cdot \frac{P_j}{1-P_i} \cdot \frac{P_k}{1-P_i-P_j}$$

while the probability of horse k finishing third is approximated at:

$$P(\text{horse k finishing 3rd}) = \sum_{\substack{\ell=1 \\ \ell\neq k}}^{n} \sum_{\substack{m=1 \\ m\neq k,\ell}}^{n} P_\ell \cdot \frac{P_m}{1-P_\ell} \cdot \frac{P_k}{1-P_\ell-P_m}$$

$$(1.8) \qquad = P_k \sum_{\substack{\ell=1 \\ \ell\neq k}}^{n} \sum_{\substack{m=1 \\ m\neq k,\ell}}^{n} \frac{P_\ell}{1-P_\ell} \cdot \frac{P_m}{1-P_\ell-P_m}$$

Although these formulae [(1.6), (1.8)] could be found using a hand-held computer for use at the racetrack, the time involved would likely be of great concern to a gambler trying to make betting decisions. Therefore, a simplification of the formulae is required.

Consider the situation where a gambler has an estimate of a horse's probability of winning. By minimizing (1.6), he can determine the absolute lowest probability of his horse finishing second without regard to the other horses' probabilities.

Without loss of generality, let our gambler's horse be #1, with his probability of winning known to be P_1. His probability of finishing second is

$$Z = P_1 \sum_{k=2}^{n} \frac{P_k}{1-P_k} \quad \text{where} \quad \sum_{k=1}^{n} P_k = 1.$$

Since the P_k's are considered unknown, minimize Z with respect to P_k. For $k = 2$ the equation is:

$$\frac{\partial Z}{\partial P_2} = \frac{1}{1-P_2} + \frac{P_2}{(1-P_2)^2} - \frac{1}{\sum_{j=1}^{n-1} P_j} - \frac{1 - \sum_{j=1}^{n-1} P_j}{\left[\sum_{j=1}^{n-1} P_j\right]^2} = 0.$$

Solving for P_2 gives $P_2 = P_n$ and the other minimizations give us to the general statement that $P_i = P_j$, $2 \le i \le n$, $2 \le j \le n$. Therefore, Z is minimized by setting $P_k = \frac{1-P_1}{n-1}$, $k = 2,3,\ldots,n.$

Substituting this value into (1.6) gives:

$$\text{(1.9)} \qquad \min \ P(\text{horse } j \text{ second}) = \frac{P_j(1-P_j)(n-1)}{n-2+P_j} .$$

Similar minimizations give the same result of $P_i = P_j$, $\begin{matrix} i \neq k \\ j \neq k \end{matrix}$, when considering (1.8). This gives rise to:

$$\text{(1.10)} \quad \min \ P(\text{horse } k \text{ finishing 3rd}) = \frac{(n-2)(n-1)\ P_k(1-P_k)^2}{(n-2+P_k)(n-3+2P_k)} .$$

These results give worst case probabilities to the gambler. They are easily found without taking a great deal of time.

An interesting note is that by letting the size of the field, n, go to infinity, (1.9) is simply $P_j(1-P_j)$ and (1.10) becomes $P_k(1-P_k)^2$. This result, although slightly inaccurate, gives the gambler an almost instant worst case estimate of the probabilities without the need of considering the size of the field.

1.6 DR. Z SYSTEM

The Dr. Z system was developed by Dr. William Ziemba and Donald Hausch to determine expected returns on place and show wagers. The favorite - longshot bias is an integral part to the assumptions of this method. Each horse is assumed to have a probability of winning equal to the proportion of the win pool wagered on them.

With this information, and using the Harville formulae, a regression analysis was performed to determine estimates of expected returns to place and show. The basis of this theory is to pinpoint horses who are underbet to place and/or show when compared with the amount bet on them to win. The results of the regression follow from Ziemba and Hausch [10]

Expected Return to Place

$$\text{on horse } i = 0.319 + 0.559\,\frac{W_i/W}{L_i/L} + \left[2.22 - 1.29\,\frac{W_i}{W}\right](.171-t) - 1$$

Expected Return to Show

$$\text{on horse } i = 0.543 + 0.369\,\frac{W_i/W}{S_i/S} + \left[3.60 - 2.13\,\frac{W_i}{W}\right](.171-t) - 1$$

where W_i = amount bet on horse i to win

W = win pool

L_i = amount bet on horse i to place

L = place pool

S_i = amount bet on horse i to show

S = show pool

t = track take.

These formulae were calculated from information at Exhibition Park in Vancouver, British Columbia where the track take is 17.1%. Hence, the final portion of each formula is an adjustment to account for differences in track takes between Exhibition Park and the track the gambler is using it at.

Although the system has been shown to be reasonably successful, there are some problems with the method which need to be addressed.

These formulae tend to exaggerate the expected returns. Even some of the results in Ziemba and Hausch's book, "Beat the Racetrack", tend to support this. In fact, the authors advise gamblers only to wager on opportunities which offer a minimum of a 14-18 percent edge. This view is also presented by Mitchell [6], while supporting the theories behind the method.

The Dr. Z method is a lazy method for estimating returns as there is little consideration for the fact that the odds on a horse to place and show are dependent on which other horses finish in the money with him. Although considering all the possible outcomes is prohibitive, some weight must be placed on this fact. The information required is available to the horseplayer, so it is important to take advantage of it.

Observance of several Stampede Park races noted that the expected return calculated using regression often varied significantly from the true expected return. As the standard errors for the coefficients of the $\left[\frac{W_i/W}{L_i/L}\right]$ and $\left[\frac{W_i/W}{S_i/S}\right]$ terms are not given, we have little idea of the accuracy of the estimates. Also, the coefficients were found from data involving racetracks other than Stampede Park, making their use here inadvisable.

The correction factor for track take can also cause errant results for racetracks which have takes significantly different from 17.1%. Therefore, an alternative method will be proposed in Chapter III.

CHAPTER II

OPTIMAL BETTING STRATEGIES

2.1 KELLY CRITERION

As an engineer with Bell Telephone, John Kelly Jr. was faced with a problem concerning the efficient transmission of electrical signals. Kelly used the model of a gambler who has access to information which gives him an advantage in a betting opportunity. The problem was to determine an optimal fraction of the gambler's wealth to wager on the event. The solution became known as the Kelly Criterion.

The concept of the Kelly Criterion is to maximize $E \log X$, where X is the random variable representing wealth and E denotes expected value. By considering a simple gambling opportunity with probability p and odds of 1, then the log of the wealth , K, of the gambler would be (from Kelly [5]):

$$K = \lim_{N \to \infty} \left[\frac{W}{N} \log (1+F) + \frac{L}{N} \log (1-F) \right]$$

where W = number of wins in N bets

L = number of losses in N bets

F = fraction of wealth wagered on each event (constant)

$$K = p \log (1+F) + (1-p) \log (1-F)$$

Maximizing K with respect to F gives:

$$F = p - (1-p) = 2p-1.$$

The resulting fraction is simply the edge the gambler has in the wager. Expanding this example to the case where the odds can be different from 1 gives:

$$K = \lim_{N \to \infty} \left[\frac{W}{N} \log (1+QF) + \frac{L}{N} \log (1-F) \right]$$

where Q = odds (profit per dollar wagered).

The maximization of K with respect to F yields:

(2.1) $$F = \frac{p\,Q - (1-p)}{Q}$$

The denominator of the fraction is simply the odds, while the numerator is the expected profit from making the wager. Hence if the betting opportunity does not offer an expected profit, the resulting fraction would be negative. A negative fraction tells the gambler not to wager on the event. A fair bet would result in $F = 0$, as is intuitively correct.

Example

$p = .4 \qquad Q = 3$

$$F = \frac{.4(3) - .6}{3} = .2.$$

Therefore, 20 percent of the gambler's fortune should be wagered on this betting opportunity.

The bets suggested by Kelly can tend to be quite large as the method is based on growth, making it riskier than some methods. The size of these wagers may be too large when considering the utility of the gambler, so an alternative is to use a fractional Kelly strategy. As an example, if the optimal Kelly wager was \$200 and the gambler felt uncomfortable with the size of this wager, he could bet 50% of the Kelly (\$100) as a rule, or any other percentage he is more comfortable with.

By betting a fixed fraction of the Kelly, the gambler is giving up growth potential in exchange for a lessening of the risk involved. Under no circumstances should a gambler wager more than the Kelly criterion, as this would mean a loss in growth as well as an increase in risk.

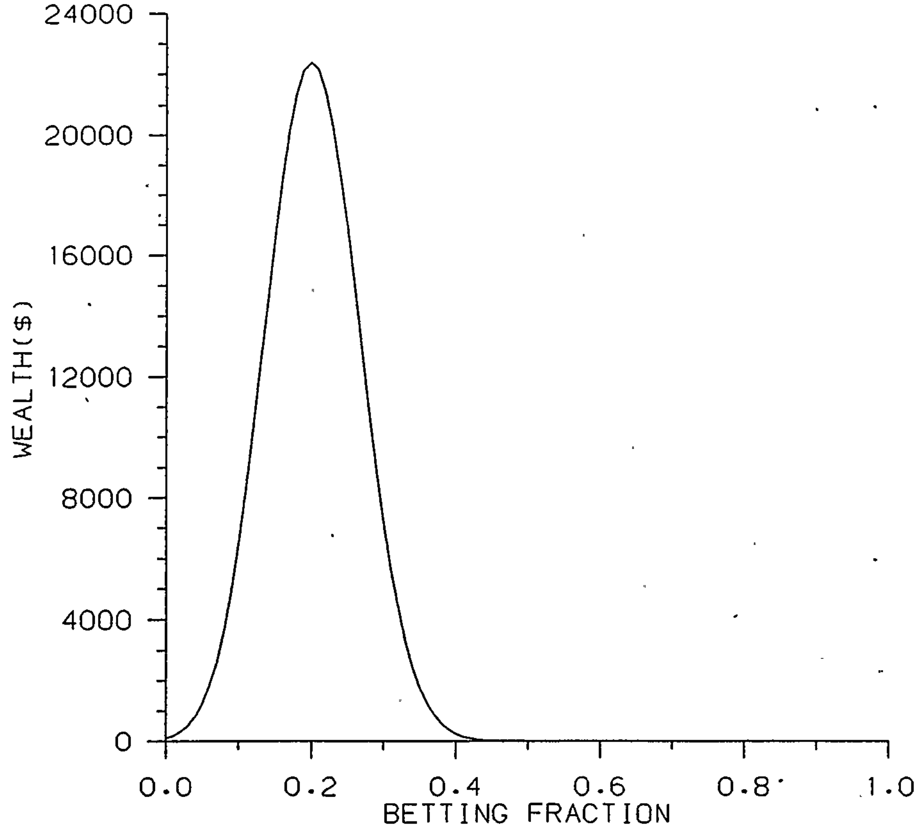
The Kelly criterion maximizes the gambler's bankroll, allowing long-term growth to be maximized. More specifically, the following two properties have been shown by Thorpe [9]:

Property 1: If two gamblers have access to the same betting opportunities, and one uses strategy γ^* maximizing $E \log X$ while the other uses an essentially different strategy γ then the Kelly Bettor (γ^*) will have infinitely more wealth almost surely.

Property 2: The expected time to reach a fixed preassigned goal x is, asymptotically as x increases, least with the Kelly strategy.

The following graph demonstrates how the Kelly fraction maximizes wealth when the expected number of successes is observed. ($N = 100$, $W = 40$, $p = .4$, $Q = 3$). The Kelly fraction F is therefore .2.

FIGURE 2.1 - ILLUSTRATION OF KELLY CRITERION
N=100,W=40,P=.4,Q=3,STARTING WEALTH=$100



2.2 WIN/PLACE PROBLEM

Using the Kelly criterion as a model, we will now consider the situation where the gambler wishes to bet win and place on the same horse. It will be assumed that the probabilities and odds for both wagers are known. Consider the following outcomes:

TABLE 2.1

WIN AND PLACE BET OUTCOMES

	FINISH POSITION OF HORSE		
	1ST	2ND	3RD OR WORSE
WIN BET	WON	LOST	LOST
PLACE BET	WON	WON	LOST

Let P_1 = Probability of horse finishing 1st

P_2 = Probability of horse finishing 2nd

Q_1 = win odds (profit per dollar wagered)

Q_2 = place odds (profit per dollar wagered)

F_1 = Fraction of bankroll bet to win

F_2 = Fraction of bankroll bet to place

B_i = Bankroll after i races

x_1 = number of races where horse won

x_2 = number of races where horse finished second

n = number of races.

Consider the gambler's wealth after n races:

$$B_n = B_0\left[1 + F_1Q_1 + F_2Q_2\right]^{x_1} \left[1 - F_1 + F_2Q_2\right]^{x_2} \left[1 - F_1 - F_2\right]^{n-x_1-x_2}.$$

Maximizing $\ell n\left[\frac{B_n}{B_0}\right]$ with respect to F_1 and F_2, and letting $n \to \infty$ in which case $\frac{x_i}{n} \longrightarrow P_i$, by the weak law of large numbers, one obtains:

$$\frac{P_1Q_1}{1+F_1Q_1+F_2Q_2} - \frac{P_2}{1-F_1+F_2Q_2} - \frac{1-P_1-P_2}{1-F_1-F_2} = 0 \tag{2.2}$$

$$\frac{P_1Q_2}{1+F_1Q_1+F_2Q_2} - \frac{P_2Q_2}{1-F_1+F_2Q_2} - \frac{1-P_1-P_2}{1-F_1-F_2} = 0 \tag{2.3}$$

With (2.2) and (2.3) serving as two equations with two unknowns, the solutions for F_1 and F_2 are:

$$F_1 = \frac{P_1(Q_1-Q_2) - P_2(Q_2+1)}{(Q_1-Q_2)} \tag{2.4}$$

$$F_2 = \frac{P_2Q_1(Q_2+1) - (1-P_1)(Q_1-Q_2)}{Q_2(Q_1-Q_2)} . \tag{2.5}$$

<u>Example</u>

$P_1 = .4$ $\qquad$ $P_2 = .25$

$Q_1 = 2.5$ $\qquad$ $Q_2 = 0.9$

$$F_1 = \frac{(.4)(1.6) - (.25)(1.9)}{1.6} = .103$$

$$F_2 = \frac{(.25)(2.5)(1.9) - (.6)(1.6)}{(.9)(1.6)} = .158.$$

Therefore for a bankroll of \$1000, the optimal wager would be \$103 win and \$158 place. It is important to stress that in order to use these formulae, both wagers should have an expected profit when considered separately.

A common occurrence, when using (2.4) and (2.5), is that one will give a negative value and the other a positive one. This tells one that the wager with the positive value has better growth potential than wagering both together. In this circumstance, the preferred bet should be calculated using (2.1) as only one type of wager will be made.

<u>Example</u>

$P_1 = .4$ $\qquad P_2 = .25$

$Q_1 = 2.5$ $\qquad Q_2 = 1.7$

$$F_1 = \frac{.4(.8) - .25(2.7)}{.8} = -.4438$$

$$F_2 = \frac{.25(2.5)(2.7) - .6(.8)}{1.7(.8)} = .8879.$$

The place wager has become so profitable, it is the only type of bet required and has an optimal betting fraction of:

$$F = \frac{.65(1.7) - .35}{1.7} = .4441.$$

Notice this fraction can also be determined by adjusting F_1 upwards to 0 and making a corresponding adjustment downwards to F_2.

$$F = - .4438 + .8879 = .4441.$$

The win-place formulae presented here can be used for any two dependent events which have the following properties:

i) If event A is a success, then event B is a success.

ii) If event B is a success, then event A is not necessarily a success.

Therefore, a person who wishes to bet WIN-SHOW, PLACE-SHOW, WIN-EXACTOR, or various other pairs of wagers, can use (2.4) and (2.5).

2.3 MULTIPLE WIN PROBLEM

The situation which will be considered here is that of betting more than one horse to win in the same race. Several horses may have positive expected gains, so the bettor must determine which one(s) to bet and the respective amounts on each, recalling that only one wager can give a return in any particular race.

Consider the situation where there are m wagers offering a positive expected return:

Let P_i = probability of horse i winning $i = 1,2,\ldots,m$

Q_i = win odds on horse i

F_i = fraction of bankroll bet on horse i

x_i = number of races won by horse i in n races

n = number of races

B_j = bankroll after j races $j = 1,2,\ldots,n$.

The resulting bankroll is:

$$B_n = B_0 \cdot \left[1 - \sum_{i=1}^{m} F_i\right]^{\left[n - \sum_{i=1}^{m} x_i\right]} \prod_{i=1}^{m} \left[1 + F_i Q_i - \sum_{\substack{k=1 \\ k \neq i}}^{m} F_k\right]^{x_i}.$$

Maximizing $\ln\left[\frac{B_n}{B_0}\right]$ with respect to F_i, $i = 1,2,\ldots,m$, the following equations pertain:

$$\frac{x_i Q_i}{1+F_i Q_i - \sum\limits_{\substack{k=1 \\ k \neq i}}^{m} F_k} - \sum_{\substack{j=1 \\ j \neq i}}^{m} \frac{x_j}{1+F_j Q_j - \sum\limits_{\substack{k=1 \\ k \neq j}}^{m} F_k} - \frac{n - \sum\limits_{k=1}^{m} x_k}{1 - \sum\limits_{k=1}^{m} F_k} = 0.$$

Letting $\frac{x_i}{n} \longrightarrow P_i$, by the weak law of large numbers, the m equations are in the form (for $i = 1,2,\ldots,m$)

$$(2.6)_i \qquad \frac{P_iQ_i}{1+F_iQ_i - \sum\limits_{\substack{k=1\\k\neq i}}^{m} F_k} - \sum_{\substack{j=1\\j\neq i}}^{m} \frac{P_j}{1+F_jQ_j - \sum\limits_{\substack{k=1\\k\neq j}}^{m} F_k} - \frac{1-\sum\limits_{k=1}^{m} P_k}{1-\sum\limits_{k=1}^{m} F_k} = 0.$$

Letting $\alpha_i = P_i(Q_i+1)$, $i = 1,2,\ldots,m$, and taking $(2.6)_1 - (2.6)_2$, $(2.6)_1 - (2.6)_3$, ... $(2.6)_1 - (2.6)_m$ yields:

$$\frac{\alpha_1}{1+F_1Q_1-\sum\limits_{j=2}^{m} F_j} = \frac{\alpha_2}{1+F_2Q_2-\sum\limits_{\substack{j=1\\j\neq 2}}^{m} F_j} = \ldots\ldots\ldots = \frac{\alpha_m}{1+F_mQ_m-\sum\limits_{j=1}^{m-1} F_j}.$$

Now letting

$$(2.7)_i \qquad \beta_i = 1 + F_iQ_i - \sum_{\substack{j=1\\j\neq i}}^{m} F_j$$

gives

$$\frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} = \ldots\ldots\ldots = \frac{\alpha_m}{\beta_m}.$$

Let $\underline{F} = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_m \end{bmatrix}$ $\quad \underline{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix}$ $\quad \underline{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

$$\underline{u} = \begin{bmatrix} P_1/\beta_1 \\ P_2/\beta_2 \\ \vdots \\ P_m/\beta_m \end{bmatrix} \qquad Q = \begin{bmatrix} Q_1 & -1 & -1 & -1 & \cdots & -1 \\ -1 & Q_2 & -1 & \cdots & & \vdots \\ \vdots & -1 & \ddots & & & \vdots \\ \vdots & \vdots & & \ddots & & -1 \\ -1 & \cdots & & & -1 & Q_m \end{bmatrix}$$

then the set of equations $(2.7)_i$ $i = 1,\ldots,m$ may be written as:

$$Q\underline{F} = \underline{\beta} - \underline{1}$$

or

(2.8) $\qquad \underline{F} = Q^{-1}(\underline{\beta} - \underline{1})$ as Q is non-singular.

The non-singularity is shown later in the computation of the determinant of the matrix Q. Hence the vector formed from equations $(2.6)_i$, $i = 1, \ldots m$, may be written as

$$Q\underline{u} = \xi\, \underline{1} \text{ or } \underline{u} = Q^{-1}\, \xi\, \underline{1} \text{ where } Q^{-1} = \{Q^{ij}\}.$$

Specifically,

(2.a)
$$u_1 = \xi \sum_{k=1}^{m} Q^{1k}, \; P_1 = \frac{\alpha_1}{1+Q_1}, \quad \text{and}$$

$$\xi = \frac{1 - \frac{\alpha_1}{Q_1+1} - \frac{\alpha_2}{Q_2+1} - \cdots - \frac{\alpha_m}{Q_m+1}}{1 - \sum_{k=1}^{m} F_k} .$$

Now (2.a) may be written as:

$$\frac{P_1}{\beta_1} = u_1 = \xi \sum_{k=1}^{m} Q^{1k} \text{ (this follows for } u_i \text{ by symmetry)}$$

or

(2.b)
$$\frac{\alpha_1}{(Q_1+1)\,\beta_1} = \left[\frac{1 - \sum_{k=1}^{m} \frac{\alpha_k}{Q_k+1}}{1 - \sum_{k=1}^{m} F_k}\right] \sum_{k=1}^{m} Q^{1k}.$$

Now,

$$\begin{aligned} \sum_{k=1}^{m} F_i &= \underline{1}' \, \underline{F} \\ &= \underline{1}' \, Q^{-1} \, (\underline{\beta} - \underline{1}) \\ &= \underline{1}' \, Q^{-1} \, \underline{\beta} - \underline{1}' \, Q^{-1} \, \underline{1}. \end{aligned}$$

Hence (2.b) becomes:

$$\begin{aligned} \alpha_1(1 - \underline{1}' \, Q^{-1} \, \underline{\beta} + \underline{1}' \, Q^{-1} \, \underline{1}) &= \beta_1(1 + Q_1) \left[\sum_{k=1}^{m} Q^{1k}\right] \\ &\quad - (1 + Q_1) \, \alpha_1 \sum_{k=1}^{m} Q^{1k} \cdot \left[\sum_{k=1}^{m} \frac{\beta_k}{1+Q_k}\right]. \end{aligned}$$

Recall $\beta_1\alpha_i = \beta_i\alpha_1$, hence

$$(2.9) \qquad \alpha_1(1 + \underline{1}'\, Q^{-1}\, \underline{1}) - \beta_1(1 + Q_1) \left[\sum_{k=1}^{m} Q^{1k}\right] = \alpha_1\left[\underline{1}'\, Q^{-1}\, \underline{\beta} - (1 + Q_1) \sum_{k=1}^{m} Q^{1k} \sum_{j=1}^{m} \frac{\beta_j}{1+Q_j}\right].$$

If $\alpha_1 = \beta_1$, then

$$1 + \underline{1}'\, Q^{-1}\, \underline{1} - (1+Q_1) \sum_{k=1}^{m} Q^{1k} = \underline{1}'\, Q^{-1}\, \underline{\beta} - (1+Q_1) \sum_{k=1}^{m} Q^{1k} \sum_{j=1}^{m} \frac{\beta_j}{1+Q_j}.$$

As $\underline{\beta}$ depends on factors other than elements of Q, this can only be true if the coefficients of β_i, $i = 1,2,\ldots,m$ are all equal to zero and that the constant term is zero. Hence we must have

$$(2.10) \qquad 1 + \underline{1}'\, Q^{-1}\, \underline{1} = (1+Q_1) \sum_{k=1}^{m} Q^{1k}$$

$$(2.11) \qquad (1+Q_j) \sum_{k=1}^{m} Q^{jk} = (1+Q_1) \sum_{k=1}^{m} Q^{1k}, \text{ for all } j.$$

Conversely if (2.10) and (2.11) hold, then it is easily shown that $\alpha_1 = \beta_1$ from (2.9).

In order to show the truth of these equations, the evaluation of $\sum_{k=1}^{m} Q^{1k}$ and $|Q|$ are necessary. A routine calculation for the determinant of Q yields:

$$|Q| = \prod_{i=1}^{m} (Q_i+1) \left[1 - \sum_{j=1}^{m} \frac{1}{Q_j+1}\right]. \tag{2.12}$$

Note that the matrix Q must be non-singular. This is true due to the fact that the quantity $\sum_{j=1}^{m} \frac{1}{Q_j+1}$ is never equal to 1. As the wagers all offer non-negative returns, clearly

$$P_jQ_j - (1-P_j) \geq 0 \quad \text{for all} \quad j \quad 1 \leq j \leq m$$

$$\text{and} \quad P_jQ_j - (1-P_j) > 0 \quad \text{for some} \quad j \quad 1 \leq j \leq m$$

(to allow for a positive expected gain).

Hence $\sum_{j=1}^{m} P_j > \sum_{j=1}^{m} \frac{1}{Q_j+1}$.

If $\sum_{j=1}^{m} \frac{1}{Q_j+1} = 1$, then $\sum_{j=1}^{m} P_j > 1$, which is clearly impossible. Hence the non-singularity of the matrix Q is shown.

In the evaluation of $\sum_{k=1}^{m} Q^{1k}$, or more generally $\sum_{k=1}^{m} Q^{ik}$ $(1 \leq i \leq m)$, we will consider the elements separately. Also, since $|Q|$ has been evaluated, this factor will be extracted by letting

$$R^{ik} = Q^{ik} \; |Q|$$

and evaluating R^{ik}. The two cases to be considered are:

(a) $i = k$

(b) $i < k$ $\left[\begin{array}{l}\text{recall by symmetry } Q^{ik} = Q^{ki}, \\ \text{therefore } R^{ik} = R^{ki}\end{array}\right]$.

Case (a) $i = k$

When $i = k$

$$R^{ii} = \begin{vmatrix} Q_1 & & & & -1 \\ & Q_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & -1 & & & Q_m \end{vmatrix}_{(m-1)\times(m-1)}$$

The diagonal elements consist of all the Q_j's except for Q_i making the evaluation possible by substituting into (2.12).

$$R^{ii} = \prod_{\substack{j=1 \\ j\neq i}}^{m} (Q_j+1) \left[1 - \sum_{\substack{j=1 \\ j\neq i}}^{m} \frac{1}{Q_j+1}\right]. \tag{2.13}$$

Case (b) $i < k$

R^{ik} is determined by eliminating row i and column k from the matrix Q and evaluating the determinant of the resulting matrix. Therefore, row k will consist of only -1's as elements. By moving this row upwards one row at a time to the position where row i was, the form of the original matrix is restored.

$$R^{ik} = (-1)^{i+k}(-1)^{k-i-1} \begin{vmatrix} Q_1 & & & & & & & & & \\ & Q_2 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & Q_{i-1} & & & & -1 & & \\ & & & & -1 & & & & & \\ & & & & & Q_{i+1} & & & & \\ & & -1 & & & & \ddots & & & \\ & & & & & & & Q_{k-1} & & \\ & & & & & & & & Q_{k+1} & \\ & & & & & & & & & \ddots \; Q_m \end{vmatrix}_{(m-1)\times(m-1)}$$

A routine evaluation gives:

$$R^{ik} = \prod_{\substack{r=1 \\ r\neq i,k}}^{m} (Q_r+1). \tag{2.14}$$

Looking at $\sum_{k=1}^{m} R^{jk}$ (constant $j = 1, \ldots, m$)

$$\sum_{k=1}^{m} R^{jk} = \left[\prod_{\substack{i=1 \\ i\neq j}}^{m} (Q_i+1)\right] \left[1 - \sum_{\substack{i=1 \\ i\neq j}}^{m} \frac{1}{Q_i+1}\right] + \left[\prod_{\substack{i=1 \\ i\neq j}}^{m} (Q_i+1)\right] \left[\sum_{\substack{i=1 \\ i\neq j}}^{m} \frac{1}{Q_i+1}\right]$$

$$\sum_{k=1}^{m} R^{jk} = \left[\prod_{\substack{i=1 \\ i\neq j}}^{m} (Q_i+1)\right]. \tag{2.15}$$

With these expressions solved, proof of (2.10) and (2.11) becomes possible. Recalling (2.10):

$$1 + \underline{1}' \, Q^{-1} \, \underline{1} = (1+Q_1) \sum_{k=1}^{m} Q^{1k} \quad \text{or}$$

$$|Q| + \underline{1}' \, R^{-1} \, \underline{1} = (1+Q_1) \sum_{k=1}^{m} R^{1k}$$

$$|Q| + \underline{1}' \, R^{-1} \, \underline{1} = \prod_{i=1}^{m} (Q_i+1) \left[1 - \sum_{j=1}^{m} \frac{1}{Q_j+1}\right] + \prod_{i=1}^{m} (Q_i+1) \left[\sum_{j=1}^{m} \frac{1}{Q_j+1}\right]$$

$$= \prod_{i=1}^{m} (Q_i+1)$$

$(Q_1+1) \sum_{k=1}^{m} R^{1k} = \prod_{i=1}^{m} (Q_i+1)$ (from (2.15)). Hence, (2.10) is proved.

Now, recalling (2.11)

$$(1+Q_j) \sum_{k=1}^{m} Q^{jk} = (1+Q_1) \sum_{k=1}^{m} Q^{1k}, \text{ for all } j,$$

$$(1+Q_j) \sum_{k=1}^{m} R^{jk} = (1+Q_1) \sum_{k=1}^{m} R^{1k}, \text{ for all } j,$$

$$(1+Q_j) \sum_{k=1}^{m} R^{jk} = \prod_{i=1}^{m} (Q_i+1) \quad \text{from (2.15) (for all } j),$$

and

$$(1+Q_1) \sum_{k=1}^{m} R^{1k} = \prod_{i=1}^{m} (Q_i+1) \quad \text{from (2.15)}.$$

This proves the relation (2.11).

Therefore, with $\alpha_1 = \beta_1$ and generally $\alpha_i = \beta_i$

$$\underline{F} = Q^{-1} (\underline{\alpha} - \underline{1}).$$

Letting the expected net profit per dollar wagered on horse i to be $\gamma_i = \alpha_i - 1$, where $\alpha_i = P_i(Q_i+1)$, allows the vector $\underline{F}$ to be calculated from

$$\underline{F} = Q^{-1} \underline{\gamma} \tag{2.16}$$

or for a specific wager

$$F_i = \sum_{j=1}^{m} Q^{ij} \gamma_j.$$

<u>Example</u>

Consider a two-bet case ($m = 2$)

$$P_1 = .3 \qquad P_2 = .2$$
$$Q_1 = 4 \qquad Q_2 = 5$$
$$\gamma_1 = .5 \qquad \gamma_2 = .2$$

$$Q = \begin{bmatrix} 4 & -1 \\ -1 & 5 \end{bmatrix} \qquad Q^{-1} = \begin{bmatrix} \frac{5}{19} & \frac{1}{19} \\ \frac{1}{19} & \frac{4}{19} \end{bmatrix}$$

$$\gamma = \begin{bmatrix} .5 \\ .2 \end{bmatrix}$$

$$\underline{F} = Q^{-1}\,\underline{\gamma} = \begin{bmatrix} .142105 \\ .068421 \end{bmatrix}.$$

Hence, approximately 14.2% of the gambler's bankroll should be wagered on bet 1, while 6.8% should be wagered on bet 2.

Example

$m = 5$

$$\begin{array}{lllll} P_1 = .2 & P_2 = .15 & P_3 = .15 & P_4 = .20 & P_5 = .05 \\ Q_1 = 5 & Q_2 = 7 & Q_3 = 10 & Q_4 = 7 & Q_5 = 30 \\ \gamma_1 = .2 & \gamma_2 = .2 & \gamma_3 = .65 & \gamma_4 = .6 & \gamma_5 = .55 \end{array}$$

The resulting F matrix is:

$$F = \begin{bmatrix} .109453 \\ .082090 \\ .100611 \\ .132090 \\ .032475 \end{bmatrix}$$

An interesting result of the maximization process is that if the gambler is faced with several bets on the same race which offer positive expected returns, it is optimal for him to bet all of the horses to win.

Perhaps more interesting is that there can be cases where a horse, whose odds are not high enough to allow for profit, should be bet along with other horses who do have profit potential. The wager on the horse acts as a hedge for the gambler, with the increase in his probability of winning allowing for greater fractions to be wagered on the horses.

Example

$m = 3$

$$P_1 = .25 \qquad P_2 = .10 \qquad P_3 = .05$$
$$Q_1 = 4 \qquad Q_2 = 14 \qquad Q_3 = 24$$
$$\gamma_1 = .25 \qquad \gamma_2 = .5 \qquad \gamma_3 = .25$$

$$F = \begin{bmatrix} .0769231 \\ .0423077 \\ .0153846 \end{bmatrix}.$$

By including a fourth betting opportunity which has $\gamma = 0$, the criterion suggest wagering on all four horses:

$$P_4 = .40$$

$$Q_4 = 1.50$$

$$\gamma_4 = 0$$

$$F = \begin{bmatrix} .113636 \\ .054545 \\ .022727 \\ .127273 \end{bmatrix}.$$

Note that the fourth wager acts as a hedge, giving no loss of it wins, while decreasing substantially the probability of a complete loss. Even slightly poor bets should be considered when their probability of success is high enough to allow for a good hedge. This concept of making wagers on fair and slightly unfair wagers, along with favorable bets, was recognized by Kelly [5].

An interesting observation about these formulae is that the returns are predictable almost instantly. Consider the net profit, R_i, if horse i wins the race, $1 \leq i \leq m$:

$$R_i = K\left[F_i Q_i - \sum_{\substack{j=1 \\ j \neq i}}^{m} F_j\right], \text{ where } K = \text{wealth}.$$

As $\underline{F} = Q^{-1}\,\underline{\gamma}$ or $\underline{\gamma} = \underline{F}Q$, a vector R can be formed as:

$$\underline{R} = KQ\underline{F}$$

$$\text{or} \quad \underline{R} = K\underline{\gamma}\ .$$

Hence if horse i wins, the net profit will be $K\gamma_i$. Consider the previous case of $m = 3$. If horse 1 wins the net profit will be:

$$R_1 = (.0769231(4) - .0423077 - .0153846)K$$
$$= .25K = \alpha_1 K.$$

Similarly,

$$R_2 = (.0423077(14) - .0769231 - .0153846)K = .50K = \alpha_2 K$$

$$R_3 = (.0153846(24) - .0769231 - .0423077)K = .25K = \alpha_3 K.$$

Therefore, with a \$1000 bankroll, a gambler would enjoy a \$500 net profit if horse 2 wins, or a \$250 net profit if either horse 1 or 3 wins, by following the strategy suggested.

Other horse racing applications for the multiple win bet formulae are when a gambler has several mutually exclusive events on which he wishes to wager.

Therefore, if a bettor has found several quinella, exactor, or daily double wagers, which offer betting value, (2.16) can be utilized to determine appropriate betting fractions.

There are possible applications in other forms of gambling for these formulae. Consider a bettor who has knowledge of a crooked roulette wheel such that several numbers are profitable. Another case is that of an unfair die. If we know that 5 and 2 have a one-third chance of occuring each, then the optimal wager at "fair" odds of 5-1 is:

$$F = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$F = \begin{bmatrix} \frac{5}{24} & \frac{1}{24} \\ \frac{1}{24} & \frac{5}{24} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}.$$

Hence, 25% of bankroll should be bet on each of 2 and 5.

2.4 BET EFFECT PROBLEM

A major problem facing a horse player, who uses Kelly fractions, is that by making the "optimal" wager he is in fact causing the bet to become less than optimal. The reason behind this is that the bet he makes is decreasing the odds on his horse, making it slightly less attractive.

Take a sample win bet as an example:

$P_1 = .2 \qquad Q_1 = 7.00 \qquad \text{CAPITAL} = K = 1000$

$t = .15 \qquad W = 10000 \qquad W_1 = 1062.$

From (2.1) the optimal betting fraction is:

$$\frac{.2(7) - .8}{7} = .0857$$

or approximately \$86 on our capital of \$1000. The bet has caused an adjustment in Q equal to (using 1.1):

$$Q' = \frac{.85(10086) - 1148}{1148} \doteq 6.46.$$

Therefore, even though the bet is still attractive, the expected profit has gone down. Our gambler has overbet the opportunity.

A method of alleviating this problem is to consider what the bet, B, does to the odds leaving B unknown. Then by substituting this into (2.1) the gambler will have an optimal bet after the wager has been made. Considering (2.1):

$$B = K\left[\frac{PQ + P - 1}{Q}\right].$$

Now by substituting the revised odds, made by our bet of size B, and utilizing (1.1) the following equation is found:

$$B = K\left[\frac{P\left[\dfrac{(1-t)(W+B) - (W_1+B)}{(W_1+B)}\right] + P - 1}{\left[\dfrac{(1-t)(W+B) - (W_1+B)}{(W_1+B)}\right]}\right].$$

Solving the quadratic for B yields:

(2.17) $$B = \frac{-\,b + \sqrt{b^2-4ac}}{2a}$$

where $a = -t$

$b = (1-t)W-W_1 + K[1-(1-t)P]$

$c = KW_1(1-P) - PK[(1-t)W-W_1]$.

Reconsidering our example,

$a = -\,.15$

$b = 8268$

$c = -63800$

gives $B = 77.2733$

By betting only \$77, Q becomes 6.52. As a check:

$$\frac{.2(6.52) - .8}{6.52} = .077$$

Hence the gambler has made a bet which preserves optimality. Similarly, for place wagering the formula holds with:

$$a = -t$$

$$b = (1-t)\ L - L_1 - L_2 + K[2-(2-t)P]$$

$$c = 2KL_1(1-P) - PK[(1-t)L-L_1-L_2] \tag{2.18}$$

where L = place pool

L_1 = place money on our horse

L_2 = place money on other horse whose places.

The show case gives the following results:

$$a = -t$$

$$b = (1-t)\ S - S_1 - S_2 - S_3 + K[3-(3-t)P]$$

$$c = 3K \cdot S_1(1-P) - P \cdot K[(1-t)S-S_1-S_2-S_3] \tag{2.19}$$

where S = show pool

S_1 = show money on our horse

S_2, S_3 = show money on other horses who show.

CHAPTER III

HOW TO WIN, IF GAMBLE YOU MUST

3.1 FINDING PROFITABLE PLACE AND SHOW WAGERS

In section 1.4 a method for finding worst case place and show probabilities was presented. A procedure for estimating the odds for these wagers is required to determine if the wagers offer betting value.

As place and show payoffs are dependent on which horses come in along with our selection, an exact calculation is impossible. To consider all the possible outcomes, with the proper weightings, can be cumbersome, if not impossible. Hence, a simplification is required. A horse-player does not have the time, even with access to a pocket computer, to enter all the necessary information and wait for the computer to perform the operations.

One method of handling this problem is to find the worst case odds. In the place case, the lowest possible odds for a particular horse is when the horse with the most money wagered on him, among the remaining horses, is assumed to be in the top two. Similarily in the show case, the worst odds would be if the two horses with the most show money wagered on them finish in the top three along with our selection.

Referring to (1.2) and (1.3) and the example in that section, consider the worst case place and show odds for #6. The highest remaining amount bet to place is on horses 2 and 5. If either of these horses finish in the top two along with 6, the worst case odds would occur. In this case, utilizing (1.2) they are:

$$R_{62} = \frac{.85(9000) - 2000 - 2000}{2(2000)} = .9125 \quad \text{rounds to} \quad 0.90.$$

Similarly, the lowest possible show odds for #6 would occur if he came in along with #2 and #5 giving the following odds

$$T_{625} = \frac{(.85)(5500) - 1500 - 800 - 1400}{3(1500)} = .217 \quad \text{rounds to} \quad .20.$$

By using the preceding methods for determining worst case probabilities and odds, it is then possible to determine the worst case expected return.

Since the true expected return is likely to be higher than this figure, the suggested Kelly wager will be smaller than it should be. However, this will not be a problem, as the growth is increased in return for increased safety. A fractional Kelly strategy is therefore automatically being used when utilizing worst-case analysis.

Consider the following pools which occurred on December 8, 1985 for the tenth race at Stampede Park, Calgary:

TABLE 3.1

RACE 10, DECEMBER 8, 1985 POOLS, STAMPEDE PARK

HORSE NUMBER

	1	2	3	4	5	6	7	8	POOLS
WIN	2062	1826	256	1720	2437	895	514	587	10297
PLACE	999	958	118	823	623	576	238	432	4767
SHOW	337	381	118	465	243	253	185	145	2127

The bettors have estimated #5's probability of winning at 2437/10297 = 0.237. Using (1.8) and (1.9), his minimum probability of finishing second and third are:

$$\min P(\#5 \text{ finishes second}) = \frac{(.237)(.763)(7)}{6.237} = .203$$

$$\min P(\#5 \text{ finishes third}) = \frac{6(7)(.237)(.763)^2}{(6.237)(5.474)} = .170.$$

Therefore, the probability of a bettor winning a place bet would be minimized at .440 and the show probability is .610.

Now consider the worst case odds for #5. The payoff would be minimized if #1 came in along with #5 giving: (using (1.2) with $t = .132$)

$$R_{51} = \frac{.868(4767) - 623 - 999}{2(623)} = 2.01 \quad \text{rounded to 2.00.}$$

For the show case, the odds are lowest if our horse comes in with 2 and 4. (It is interesting to notice that the #1 horse appears to be well overbet to place, yet underbet to show. A wagerer, however, must prefer to bet on #5 as his probability of winning is highest while his show odds are higher then #1). The odds are (using (1.3)):

$$T_{524} = \frac{.868(2127) - 243 - 381 - 465}{3(243)} = 1.04 \quad \text{rounded to} \quad 1.00.$$

Therefore, the minimum expected returns are:

$$\text{E(RETURN TO PLACE)} = .440(2.00) - .560 = .320$$

$$\text{E(RETURN TO SHOW)} = .610(1.00) - .390 = .220.$$

The wagers both offer good value to the gambler. Using (2.4) and (2.5), the optimal betting fractions are:

<u>PLACE</u>	<u>SHOW</u>
$P_1 = .440$	$P_2 = .170$
$Q_1 = 2$	$Q_2 = 1$

$$F_1 = \frac{(.440)(1) - .170(2)}{1} = .100$$

$$F_2 = \frac{(.170)(2)(2) - (.560)(1)}{(1)(1)} = .120.$$

The #5 horse won the race with #6 second and #1 third. The actual odds paid were $Q_1 = 2.55$ and $Q_2 = 1.35$ meaning the wager offered more value than expected.

3.2 FINDING PROFITABLE QUINELLA AND EXACTOR WAGERS

Using the theory of the Harville formulae, estimates of exactor and quinella probabilities can be found.

The probability of an exactor combination occurring is clearly (1.5):

$$P_{ij} = P_i \cdot \frac{P_j}{1-P_i}.$$

Similarly the probability of a quinella combination is:

$$P_{ij} + P_{ji} = P_i \cdot P_j \left[\frac{1}{1-P_i} + \frac{1}{1-P_j}\right].$$

At most major racetracks, monitors are available to allow bettors to observe the odds on all combinations. Since there are so many possible combinations, there tends to be some discrepancy in the payoffs when compared to the estimates made by the public in the win pool.

As these odds are not dependent on any other factor, there is no need to estimate these values. Also, exactor and quinella pools tend to be much larger than place and show pools, such that any further bets do not have as much of a negative effect on odds as they did in the place and show cases.

The probabilities of winning, however, are much lower because of the many possible outcomes. This can make losing streaks very long. Another problem is that the track-take at most tracks is larger for quinella and exactor bets than for place and show wagers. This makes finding exactor and quinella bets which offer betting value more difficult to come by. They do occur, however, and the following is an example of an exactor race which offers a positive expected return:

RACE 8 NOVEMBER 22, 1985 STAMPEDE PARK, CALGARY

	WIN MONEY	WIN PROB(EST) (using (1.4))
FAVORITE # 1	2316	.3990
2ND FAVORITE #4	1131	.1948
OTHER HORSES	2358	–
WIN POOL	5805	

EXACTOR POOL	16273	(track take = .192)
AMOUNT BET ON 1-4	1370	
AMOUNT BET ON 4-1	994	
ODDS ON 1-4	8.55	
ODDS ON 4-1	12.20	

$$\text{PROB OF 1-4} = .3990\ \frac{(.1948)}{.6010} = .1293$$

$$\text{PROB OF 4-1} = .1948\ \frac{(.3990)}{.8052} = .0965.$$

$$\text{E(RETURN ON 1-4 WAGER)} = .1293(8.55) - .8707 = 0.2348$$

$$\text{E(RETURN ON 4-1 WAGER)} = .0965(12.20) - .9035 = 0.2738.$$

Both wagers offer positive expected returns. As only one of the bets can win the betting strategy of multiple win betting applies. Utilizing (2.16)

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = Q^{-1}\begin{bmatrix} .2348 \\ .2738 \end{bmatrix}$$

where

$$Q = \begin{bmatrix} 8.55 & -1 \\ -1 & 12.20 \end{bmatrix}$$

$$\therefore\ Q^{-1} = \begin{bmatrix} .118091 & .009680 \\ .009680 & .082761 \end{bmatrix}$$

giving

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} .0304 \\ .0249 \end{bmatrix}.$$

Therefore, approximately 3% of the gambler's bankroll should be placed on the 1-4 wager with 2.5% being placed on 4-1. The outcome of the race was 1-4. Hence with a $1000 bankroll, we would have bet $30 on 1-4 and $25 on 4-1. We would have won $286.50 and hence made a net gain of $231.50.

Quinella wagers can be considered in a similar manner:

RACE 6 DECEMBER 7, 1985 STAMPEDE PARK, CALGARY

	WIN MONEY	WIN PROB(EST) (using (1.4))
FAVORITE # 5	2998	.3879
2ND FAVORITE #4	2746	.3553
OTHER HORSES	1985	–
WIN POOL	7729	

QUINELLA POOL	10403	(track take = .192)
AMOUNT BET ON 4-5	3567	
ODDS ON 4-5	1.35	

$$\text{PROBABILITY OF 4-5 QUINELLA} = .3879\left[\frac{.3553}{.6121}\right] + .3553\left[\frac{.3879}{.6447}\right]$$

$$= .4389$$

$$\text{E(RETURN ON 4-5 QUINELLA WAGER)} = .4389(1.35) - .5611$$

$$= .0314.$$

As only one wager is involved, a simple Kelly is used:

$$F = \frac{.0314}{1.35} = .0233 \ .$$

The optimal wager would be 2.33% of bankroll.

The previous examples concentrated on the top two favorites because their actual probabilities are slightly higher than the crowd's estimates, due to the favorite-longshot bias. This gives an extra measure of security. By considering horses at longer odds, you are losing this extra edge and maybe more.

Also the combinations involving the favorites have the higher probabilities of success, which is very important because Kelly-style wagers are very sensitive to changes in probability.

When wagering on these types of wagers, however, the probability is still fairly low. Hence, the opportunity for long losing streaks exist. For a novice horse player (or even an experienced one) these streaks can be extremely difficult to bear. Therefore, with place and show wagers having very high probabilities, this style of betting will be tested.

It is important to note that exactor and quinella wagering is still a very viable method of making money, but only for a regular player who goes to the races often enough to allow the number of plays required for his winnings to converge to the expected wealth.

3.3 A SUGGESTED METHOD IN ACTION

To test the theories presented in this thesis, races at Stampede Park, Calgary from November 22, 1985 to January 19, 1986 were examined. The final pools were used to apply the methods objectively.

All races were tested with the exception of the daily Maiden contest. Maidens are horses, usually very young, who have never won a race. For these reasons they tend to be very unpredictable. Also since this study is of harness races and maidens have a high tendency to go off-stride, it was considered prudent to ignore such contests for this study.

The remaining 324 races were tested to determine if the favorite was underbet to place and/or show. If a prospective bet to place or show was uncovered, formulae (2.18) or (2.19) were used to determine the optimal wager. If both place and show wagers offered value, formulas (2.4) and (2.5) were utilized. All wagers were considered in a worst-case analysis and all payoffs were adjusted, if our wager was successful, to account for our hypothetical wagers.

With a starting bankroll of $200, the results follow:

DEFINITION OF TERMS FOR TABLE 3.2

MONTH	-	month of year
DATE	-	day of month
RACE	-	race number
NO.	-	horse number
FIELD	-	number of horses in race
W. PROB.	-	probability of winning based on proportion of win pool wagered on horse (formula 1.4)
P. PROB	-	worst case place probability (using (1.9) + win prob.)
S. PROB	-	worst case show probability (using (1.10) + place probability)
P. ODDS	-	worst case place odds
S. ODDS	-	worst case show odds
P. EDGE	-	minimum expected return on place wager (0 denotes negative)
S. EDGE	-	minimum expected return on show wager (0 denotes negative)
P. BET	-	suggested wager to place (using (2.4) and (2.18))
S. BET	-	suggested wager to show (using (2.5) and (2.19))
P. PAY	-	actual place odds adjusted for place bet suggested
S. PAY	-	actual show odds adjusted for show bet suggested
BANK	-	bankroll of gambler (start of $200).

TABLE 3.2 – PLACE/SHOW WORKOUT (Nov.22/85–Jan.19/86,Stampede Park,Calgary)

MONTH	DATE	RACE	NO.	FIELD	W.PROB	P.PROB	S.PROB	P.ODDS	S.ODDS	P.EDGE	S.EDGE	P.BET	S.BET	P.PAY	S.PAY	BANK
11.	22.	4.	7.	8.	0.28	0.50	0.67	1.15	0.39	0.0907	0.0000	13.	0.	0.00	0.00	187.00
11.	22.	7.	4.	8.	0.29	0.52	0.70	0.48	0.52	0.0000	0.0742	0.	21.	0.80	1.05	209.05
11.	22.	9.	1.	8.	0.26	0.47	0.64	0.89	0.62	0.0000	0.0537	0.	13.	0.00	0.00	196.05
11.	22.	10.	1.	8.	0.21	0.40	0.56	1.20	0.86	0.0000	0.0522	0.	8.	0.00	0.00	188.05
11.	23.	2.	5.	8.	0.49	0.76	0.90	0.35	0.20	0.0340	0.0859	0.	65.	0.35	0.15	197.80
11.	23.	9.	2.	8.	0.36	0.62	0.79	0.58	0.40	0.0000	0.1105	0.	41.	0.75	0.35	212.15
11.	24.	2.	5.	8.	0.33	0.57	0.75	0.94	0.28	0.1205	0.0000	21.	0.	0.00	0.00	191.15
11.	29.	4.	5.	8.	0.56	0.82	0.93	0.24	0.17	0.0212	0.0961	0.	98.	0.20	0.15	205.85
11.	30.	7.	2.	8.	0.26	0.48	0.65	1.13	0.48	0.0357	0.0000	5.	0.	1.25	0.55	212.10
12.	1.	2.	5.	8.	0.49	0.76	0.89	0.28	0.21	0.0000	0.0890	0.	72.	0.40	0.20	226.50
12.	4.	4.	4.	8.	0.44	0.71	0.86	0.43	0.28	0.0167	0.1076	0.	71.	0.60	0.25	244.25
12.	4.	6.	5.	7.	0.53	0.80	0.92	0.21	0.09	0.0000	0.0192	0.	33.	0.35	0.30	254.15
12.	4.	7.	4.	8.	0.33	0.58	0.75	0.77	0.39	0.0393	0.0599	0.	29.	0.00	0.00	225.15
12.	4.	10.	1.	8.	0.26	0.48	0.66	0.97	0.55	0.0000	0.0294	0.	8.	1.05	0.55	229.55
12.	6.	2.	4.	7.	0.46	0.74	0.88	0.43	0.27	0.0593	0.1359	0.	95.	0.40	0.30	258.05
12.	6.	4.	2.	5.	0.62	0.88	0.97	0.21	0.05	0.0749	0.0185	64.	30.	0.00	0.20	200.05
12.	6.	8.	2.	6.	0.34	0.60	0.78	0.41	0.35	0.0000	0.0666	0.	30.	0.70	0.35	210.55
12.	7.	5.	5.	8.	0.25	0.46	0.63	1.02	0.75	0.0000	0.1146	0.	25.	1.30	0.80	230.55
12.	7.	10.	5.	8.	0.38	0.64	0.80	0.52	0.36	0.0000	0.1042	0.	52.	0.60	0.50	256.55
12.	8.	7.	5.	8.	0.34	0.59	0.77	0.75	0.38	0.0515	0.0676	0.	35.	0.00	0.00	221.55
12.	8.	10.	5.	8.	0.23	0.43	0.60	2.01	1.03	0.3266	0.2418	16.	28.	2.25	1.20	291.15
12.	11.	2.	7.	8.	0.30	0.54	0.72	0.86	0.57	0.0168	0.1319	0.	39.	0.00	0.00	252.15
12.	11.	3.	4.	8.	0.35	0.60	0.77	0.94	0.60	0.1729	0.2455	0.	79.	0.95	0.50	291.65
12.	11.	5.	3.	8.	0.32	0.57	0.74	0.66	0.40	0.0000	0.0451	0.	23.	0.00	0.50	303.15
12.	11.	6.	1.	7.	0.35	0.61	0.79	0.76	0.25	0.0910	0.0000	28.	0.	2.55	1.55	374.55
12.	11.	7.	2.	8.	0.29	0.52	0.70	0.86	0.75	0.0000	0.2297	0.	77.	0.00	0.00	297.55
12.	11.	9.	4.	8.	0.37	0.63	0.79	0.64	0.34	0.0375	0.0702	0.	42.	0.60	0.30	310.15
12.	11.	10.	4.	8.	0.31	0.55	0.73	0.82	0.52	0.0170	0.1136	0.	42.	1.40	0.75	341.65
12.	13.	2.	4.	5.	0.42	0.71	0.88	0.24	0.18	0.0000	0.0432	0.	56.	0.30	0.30	358.45
12.	13.	3.	6.	6.	0.30	0.55	0.74	0.92	0.25	0.0700	0.0000	22.	0.	0.00	0.30	336.45
12.	13.	4.	4.	5.	0.47	0.76	0.91	0.36	0.16	0.0461	0.0697	0.	110.	0.00	0.10	347.45
12.	13.	5.	8.	8.	0.46	0.73	0.87	0.39	0.29	0.0222	0.1430	0.	134.	0.50	0.35	394.35
12.	13.	7.	1.	8.	0.23	0.43	0.60	1.02	1.30	0.0000	0.3967	0.	75.	0.00	0.90	461.85
12.	13.	10.	6.	8.	0.29	0.52	0.69	0.97	0.35	0.0376	0.0000	14.	0.	0.00	0.00	447.85
12.	14.	3.	3.	8.	0.28	0.51	0.69	0.52	0.48	0.0000	0.0320	0.	16.	0.90	1.00	463.85
12.	14.	5.	6.	6.	0.35	0.61	0.79	0.60	0.28	0.0000	0.0225	0.	20.	0.00	0.35	470.85
12.	14.	6.	4.	5.	0.42	0.71	0.88	0.34	0.15	0.0000	0.0153	0.	22.	0.50	0.10	473.85
12.	14.	7.	3.	7.	0.27	0.49	0.68	0.83	0.49	0.0000	0.0183	0.	9.	1.05	0.90	481.15
12.	14.	9.	2.	8.	0.44	0.70	0.86	0.79	0.37	0.2690	0.1865	85.	115.	1.00	0.40	612.15
12.	14.	10.	4.	8.	0.30	0.53	0.70	0.82	0.68	0.0000	0.1911	0.	94.	1.25	0.55	663.85
12.	15.	5.	3.	8.	0.41	0.67	0.83	0.36	0.23	0.0000	0.0364	0.	60.	0.35	0.20	675.85
12.	15.	7.	8.	8.	0.29	0.52	0.69	0.63	0.47	0.0000	0.0274	0.	21.	0.60	0.60	688.45
12.	15.	9.	6.	7.	0.34	0.59	0.77	0.83	0.29	0.0840	0.0007	53.	0.	0.75	0.30	728.20
12.	15.	10.	6.	7.	0.29	0.53	0.71	0.63	0.43	0.0000	0.0307	0.	26.	0.00	0.00	702.20

TABLE 3.2 (continued)

MONTH	DATE	RACE	NO.	FIELD	W.PROB	P.PROB	S.PROB	P.ODDS	S.ODDS	P.EDGE	S.EDGE	P.BET	S.BET	P.PAY	S.PAY	BANK
12.	18.	3.	4.	8.	0.44	0.71	0.86	0.34	0.20	0.0000	0.0393	0.	81.	0.30	0.25	722.45
12.	18.	4.	4.	8.	0.49	0.76	0.90	0.30	0.14	0.0031	0.0308	0.	89.	0.40	0.35	753.60
12.	18.	5.	5.	8.	0.40	0.66	0.82	0.67	0.33	0.1167	0.1036	0.	148.	0.00	0.00	605.60
12.	18.	6.	6.	6.	0.49	0.77	0.91	0.33	0.10	0.0325	0.0132	40.	0.	0.50	0.20	625.60
12.	18.	7.	7.	8.	0.35	0.60	0.77	0.56	0.49	0.0000	0.1615	0.	124.	0.55	0.45	681.40
12.	18.	8.	2.	7.	0.42	0.69	0.85	0.73	0.21	0.2123	0.0440	154.	0.	0.65	0.30	781.50
12.	18.	10.	5.	8.	0.28	0.50	0.68	1.13	0.59	0.0810	0.0867	0.	50.	0.00	0.00	731.50
12.	20.	8.	3.	8.	0.53	0.79	0.92	0.34	0.16	0.0747	0.0779	0.	235.	0.45	0.10	755.00
12.	20.	9.	8.	8.	0.26	0.48	0.66	0.68	0.52	0.0000	0.0085	0.	5.	0.00	0.95	759.75
12.	21.	6.	8.	8.	0.33	0.58	0.75	1.10	0.53	0.2273	0.1592	80.	120.	1.30	0.70	947.75
12.	21.	7.	4.	8.	0.41	0.67	0.83	0.50	0.38	0.0186	0.1607	0.	239.	0.50	0.40	1043.35
12.	21.	9.	2.	8.	0.40	0.67	0.83	0.42	0.30	0.0000	0.0869	0.	145.	0.55	0.30	1086.85
12.	22.	2.	1.	8.	0.39	0.66	0.82	0.29	0.31	0.0000	0.0831	0.	126.	0.50	0.50	1149.85
12.	22.	5.	6.	8.	0.25	0.46	0.63	1.17	0.63	0.0050	0.0360	0.	25.	1.20	0.55	1163.60
12.	22.	6.	2.	6.	0.40	0.68	0.85	0.30	0.20	0.0000	0.0250	0.	59.	0.30	0.20	1175.40
12.	22.	7.	2.	7.	0.39	0.66	0.82	0.53	0.39	0.0193	0.1569	0.	212.	0.55	0.40	1260.20
12.	22.	10.	6.	8.	0.23	0.42	0.59	1.77	1.16	0.1928	0.2935	0.	128.	1.95	1.10	1401.00
12.	27.	4.	5.	8.	0.28	0.50	0.68	1.20	0.83	0.1225	0.2519	0.	168.	1.20	0.60	1501.80
12.	27.	8.	6.	8.	0.43	0.69	0.85	0.76	0.36	0.2329	0.1649	135.	215.	0.00	0.40	1452.80
12.	27.	9.	2.	8.	0.27	0.49	0.66	1.17	0.24	0.0788	0.0000	59.	0.	0.00	0.00	1393.80
12.	28.	2.	4.	8.	0.18	0.34	0.50	2.12	1.27	0.0836	0.1414	0.	43.	0.00	1.10	1441.10
12.	28.	5.	6.	8.	0.46	0.73	0.87	0.28	0.19	0.0000	0.0483	0.	166.	0.00	0.00	1275.10
12.	28.	6.	4.	7.	0.41	0.67	0.84	0.53	0.31	0.0385	0.1087	0.	220.	0.65	0.25	1330.10
12.	28.	7.	2.	8.	0.36	0.62	0.79	0.41	0.29	0.0000	0.0271	0.	46.	0.25	0.25	1341.60
12.	28.	9.	2.	8.	0.30	0.53	0.71	0.65	0.49	0.0000	0.0622	0.	69.	0.65	0.40	1369.20
12.	28.	10.	6.	8.	0.24	0.45	0.62	0.77	0.67	0.0000	0.0470	0.	30.	0.90	0.60	1387.20
12.	29.	6.	2.	7.	0.31	0.56	0.74	0.89	0.38	0.0598	0.0229	52.	0.	1.10	0.85	1444.40
12.	29.	7.	5.	8.	0.31	0.56	0.73	0.63	0.39	0.0000	0.0278	0.	41.	0.90	0.45	1462.85
12.	29.	10.	1.	8.	0.40	0.67	0.83	0.55	0.27	0.0440	0.0649	0.	137.	0.00	0.00	1325.85
12.	30.	2.	1.	8.	0.39	0.66	0.82	0.57	0.27	0.0410	0.0498	0.	78.	0.00	0.20	1341.45
12.	30.	4.	6.	8.	0.33	0.57	0.75	0.80	0.63	0.0405	0.2302	0.	228.	0.00	0.55	1466.85
12.	30.	6.	3.	8.	0.34	0.60	0.77	0.78	0.19	0.0735	0.0000	72.	0.	0.00	0.00	1394.85
1.	1.	10.	5.	8.	0.37	0.63	0.79	0.62	0.37	0.0219	0.0963	0.	175.	0.80	0.40	1464.85
1.	3.	6.	5.	8.	0.34	0.59	0.76	0.80	0.30	0.0745	0.0002	78.	0.	0.00	0.00	1386.85
1.	3.	7.	5.	8.	0.41	0.67	0.83	0.65	0.21	0.1201	0.0137	153.	0.	0.00	0.00	1233.85
1.	3.	9.	7.	7.	0.31	0.56	0.74	0.70	0.35	0.0000	0.0056	0.	7.	0.70	0.40	1236.65
1.	3.	10.	4.	8.	0.28	0.51	0.68	1.00	0.68	0.0249	0.1574	0.	99.	0.00	0.00	1137.65
1.	4.	2.	4.	8.	0.45	0.72	0.87	0.27	0.15	0.0000	0.0109	0.	30.	0.30	0.35	1148.15
1.	4.	4.	4.	7.	0.32	0.57	0.75	0.58	0.36	0.0000	0.0261	0.	39.	0.70	0.30	1159.85
1.	4.	5.	1.	8.	0.27	0.49	0.68	1.15	0.50	0.0592	0.0029	36.	0.	0.00	0.00	1123.85
1.	4.	8.	6.	7.	0.34	0.59	0.77	0.80	0.39	0.0719	0.0766	0.	90.	0.85	0.55	1173.35
1.	4.	10.	6.	8.	0.45	0.72	0.87	0.38	0.18	0.0000	0.0349	0.	105.	0.40	0.25	1199.60
1.	5.	4.	4.	8.	0.24	0.44	0.61	0.83	0.65	0.0000	0.0186	0.	13.	0.00	0.00	1186.60
1.	5.	5.	4.	8.	0.19	0.38	0.54	1.41	0.93	0.0000	0.0481	0.	20.	1.75	1.35	1213.60

TABLE 3.2 (continued)

MONTH	DATE	RACE	NO.	FIELD	W.PROB	P.PROB	S.PROB	P.ODDS	S.ODDS	P.EDGE	S.EDGE	P.BET	S.BET	P.PAY	S.PAY	BANK
1.	5.	8.	5.	5.	0.47	0.76	0.91	0.23	0.13	0.0000	0.0382	0.	97.	0.50	0.15	1228.15
1.	5.	9.	5.	8.	0.43	0.70	0.85	0.33	0.22	0.0000	0.0523	0.	122.	0.35	0.25	1258.65
1.	5.	10.	2.	8.	0.18	0.35	0.51	2.00	0.80	0.0740	0.0000	26.	0.	0.00	0.00	1232.65
1.	6.	4.	1.	7.	0.27	0.50	0.68	0.83	0.76	0.0000	0.2052	0.	107.	0.00	0.00	1125.65
1.	6.	7.	4.	7.	0.33	0.59	0.76	0.63	0.33	0.0000	0.0230	0.	22.	0.00	0.00	1103.65
1.	10.	3.	5.	8.	0.52	0.79	0.91	0.37	0.20	0.0889	0.1080	0.	359.	0.40	0.25	1193.40
1.	10.	5.	5.	8.	0.48	0.75	0.89	0.38	0.24	0.0483	0.1182	0.	336.	0.55	0.25	1277.40
1.	10.	6.	4.	5.	0.42	0.70	0.88	0.44	0.10	0.0188	0.0000	29.	0.	0.45	0.10	1290.45
1.	10.	10.	2.	8.	0.38	0.64	0.81	0.60	0.21	0.0330	0.0000	45.	0.	0.55	0.20	1315.20
1.	11.	2.	3.	8.	0.35	0.61	0.78	0.66	0.16	0.0189	0.0000	18.	0.	0.00	0.20	1297.20
1.	11.	3.	1.	8.	0.46	0.73	0.88	0.38	0.20	0.0248	0.0655	0.	197.	0.55	0.25	1346.45
1.	11.	4.	1.	7.	0.33	0.59	0.76	0.76	0.34	0.0438	0.0326	15.	45.	1.05	0.60	1389.20
1.	11.	10.	4.	8.	0.43	0.69	0.85	0.60	0.38	0.0000	0.1858	0.	336.	0.00	0.00	1053.20
1.	12.	3.	2.	8.	0.49	0.76	0.90	0.25	0.15	0.0000	0.0423	0.	142.	0.25	0.10	1067.40
1.	12.	6.	3.	8.	0.59	0.85	0.95	0.21	0.06	0.0374	0.0129	117.	0.	0.30	0.25	1102.50
1.	12.	8.	3.	7.	0.52	0.79	0.92	0.21	0.11	0.0000	0.0339	0.	165.	0.00	0.15	1127.25
1.	13.	4.	1.	6.	0.35	0.62	0.80	0.76	0.52	0.0936	0.2226	0.	186.	1.10	0.55	1229.55
1.	13.	9.	4.	8.	0.37	0.62	0.79	0.50	0.31	0.0000	0.0448	0.	54.	0.00	0.25	1243.05
1.	13.	10.	7.	8.	0.33	0.57	0.74	0.47	0.53	0.0000	0.1466	0.	110.	0.00	0.00	1133.05
1.	15.	4.	5.	6.	0.42	0.69	0.86	0.32	0.22	0.0000	0.0588	0.	128.	0.45	0.20	1158.65
1.	15.	6.	2.	6.	0.40	0.68	0.85	0.33	0.36	0.0000	0.1638	0.	176.	0.70	0.45	1237.85
1.	15.	7.	6.	8.	0.27	0.49	0.67	1.04	0.50	0.0238	0.0122	13.	0.	0.00	0.80	1224.85
1.	15.	8.	2.	6.	0.48	0.76	0.90	0.14	0.11	0.0000	0.0094	0.	23.	0.00	0.30	1231.75
1.	17.	5.	2.	8.	0.36	0.61	0.78	0.61	0.47	0.0000	0.1573	0.	199.	0.00	0.00	1032.75
1.	17.	8.	4.	7.	0.49	0.76	0.90	0.45	0.20	0.1158	0.0852	71.	229.	0.40	0.15	1095.50
1.	18.	2.	8.	8.	0.36	0.61	0.78	0.64	0.33	0.0189	0.0538	0.	73.	0.95	0.40	1124.70
1.	18.	4.	1.	6.	0.45	0.73	0.89	0.30	0.17	0.0000	0.0444	0.	137.	0.00	0.00	987.70
1.	18.	6.	1.	7.	0.44	0.71	0.86	0.60	0.24	0.1451	0.0801	105.	95.	0.55	0.30	1073.95
1.	18.	9.	6.	6.	0.47	0.75	0.90	0.36	0.08	0.0371	0.0000	71.	0.	0.30	0.05	1095.25
1.	18.	10.	2.	7.	0.23	0.44	0.62	0.87	0.67	0.0000	0.0408	0.	20.	1.15	0.75	1110.25
1.	19.	2.	2.	8.	0.38	0.64	0.80	0.53	0.28	0.0000	0.0377	0.	62.	0.55	0.35	1131.95
1.	19.	4.	5.	5.	0.45	0.73	0.90	0.27	0.22	0.0000	0.1089	0.	245.	0.25	0.10	1156.45
1.	19.	5.	1.	7.	0.33	0.57	0.75	0.60	0.40	0.0000	0.0665	0.	77.	0.80	0.50	1194.95
1.	19.	6.	4.	6.	0.43	0.70	0.87	0.28	0.15	0.0000	0.0051	0.	12.	0.25	0.10	1196.15
1.	19.	8.	1.	8.	0.36	0.61	0.78	0.58	0.28	0.0000	0.0083	0.	13.	0.00	0.00	1183.15

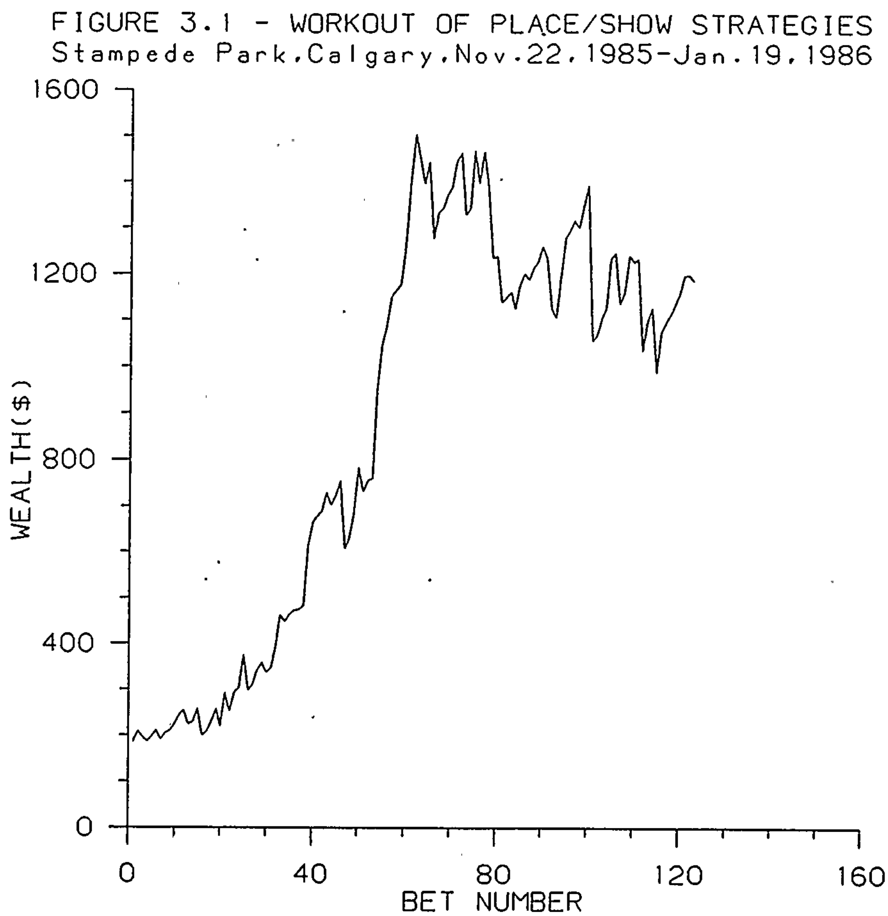


FIGURE 3.1 - WORKOUT OF PLACE/SHOW STRATEGIES
Stampede Park,Calgary,Nov.22,1985-Jan.19,1986
WEALTH($)
1600
1200
800
400
0
0
40
80
120
160
BET NUMBER

Considering the races pinpointed as having positive expected return wagers to place and/or show, the accuracy of the probabilities are tested in detail. Of 324 races considered, 64 races had profitable place wagers. In 39 races the bet would have been won. The expected number of wins based on our probabilities was 39.98.

The actual return based on a $2 wager to place on each of the 64 horses was 12.5%. This compares favorably to the minimum expected return of 7.39%. Since we are underestimating the odds, this bonus is not unexpected.

A similar procedure performed on the 110 show bets gives the following results:

87 races collected

86.25 expected number of races collected

ACTUAL RETURN = 19.73%

MINIMUM EXPECTED RETURN = 8.31%.

Certainly both types of wagers performed as well as expected with the probabilities shown to be very accurate. The actual return to show is much higher than the minimum expected return. This again was expected as the only result which gives the worst case odds is far less likely than even the worst case result for place pays is, allowing for a higher probability of a bonus in the show odds versus the place odds.

The result of the study are very encouraging. After 123 races bet, our gambler has a bankroll six times of that he started with. Both place and show wagers showed solid returns.

An interesting observation is that prospective show wagers occurred far more often than place opportunities. This is likely due to the lottery mentality of the crowd. If they were to bet a favorite to show, it is unlikely they would be satisfied with the payoff (even if it was a bet which offered a positive expected gain). Perhaps there is slightly less of this feeling towards place wagers, since they pay more. Yet some of this opinion must exist, as there were still 64 place bets which offered value.

Of concern, however, is that during the last half of the study there was basically a levelling off of the bankroll. The actual reason for this is unclear, although it is important to remember that his study occurred in the middle of winter when poorer quality horses run and poor weather conditions exist and that we are charting a stochastic phenomenon.

Perhaps there should be a point where profit should be taken and the gambler start over with a new \$200 bankroll. Very few people's utility could stand losing (or even making the wager) a \$336 bet as occurred on race 10, January 11. Again this is for each bettor to determine according to his or her utility function.

Overall, the study did what was hoped for. The actual number of successful wagers versus the expected number were extremely close showing good accuracy in the estimates, while at no time was the gambler's bankroll in jeopardy.

It must be noted, however, that these wagers are calculated based on final pools giving the gambler better information that he would have in reality. However, by making wagers and calculations as close to post time as possible, this problem is greatly reduced. It is likely that the change in the relative attractiveness of the bet will be small as the vast majority of wagers have already been made. Also, this change will often be to the bettor's advantage, an occurerence observed frequently by myself in testing the material.

Finally, and most importantly, the study shows the viability in playing the races for profit, something few other gambling opportunities offer. As Mark Twain said:

> "It is a difference of opinion
> that makes horse races".

It is this difference of opinion which makes possible the occurrence of profitable wagers.

3.4 CONCLUSION

The main emphasis of this thesis was to determine optimal betting strategies for the different types of wagering decisions a punter faces.

Although the suggested system showed healthy profits, the system was made completely mechanical to preserved the objectivity of the analysis. In actual practice, however, it is far more advisable for the bettor to become as familiar as possible with the sport. There is a multitude of information available and by using it along with the theories preented here, a horseplayer has a high probability of success.

Once a person has the experience and knowledge to be able to set his own probabilities, then far more prospective wagers will be available. Almost every race will have win wagers which have betting value. For these the multiple-win bet strategy is advised. There will also be place, show, exactor, quinella, and daily double wagers advised. It would be impossible for the bettor to handle all of the decisions involved, in the short time he has between races, so perhaps by enlisting the aid of other gamblers to form a betting team, the opportunities could be exploited to a maximum.

An important prerequisite is to have a professional attitude and be able to do exactly what the strategies require. A bettor must have confidence in the methods to know that it will succeed eventually. It is hard for gamblers to have these qualities, especially after losing a large wager by a horse's nose, or a driver's stupidity.

Other considerations in using this style of wagering is to reflect on the log-normality of the random variable X (wealth) shown by Thorpe [9]. As an alternative to fractional Kelly wagering (when the gambler is not comfortable with the size of the bet), one could examine the variance of Kelly-type wagers. Then by reducing the variance of his wealth, our gambler can reduce his risk. However, clearly a fractional Kelly strategy, such as 50% of the suggested wager, obtains the desired result as the gambler is in fact pretending he has only half of his actual wealth. The other half is kept as a reserve fund. This type of strategy is discussed by Ziemba and Hausch [10] for those so inclined.

Another consideration to be kept in mind is our bettor's utility function. If this function can be determined in concert with Kelly style betting, the results would be most rewarding. Utility functions can be difficult, if not impossible, to determine, so my advice would be for a profit-taking point determined by the gambler. When his bets become too large for his liking (or personal utility), an adjustment must be made. This can be by profit-taking or by decreasing the fraction of the Kelly wagers being made.

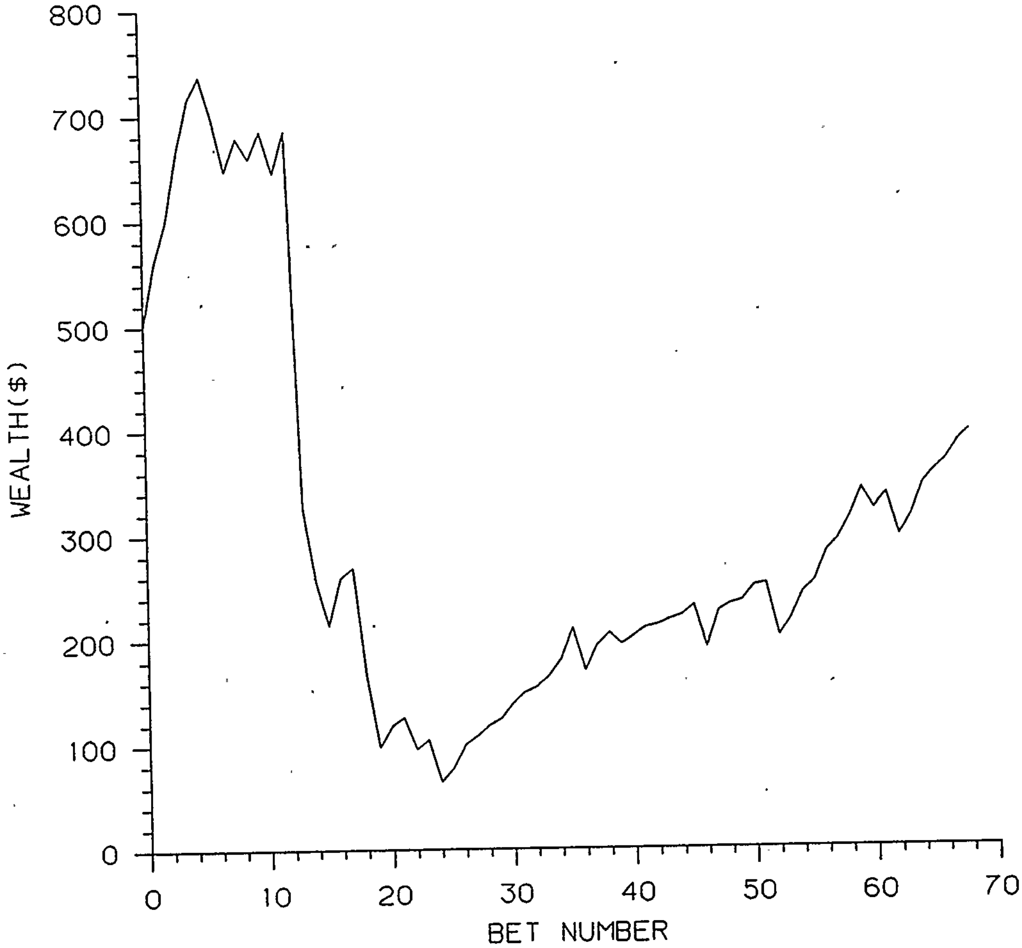
This suggestion is not made based solely on theory, but on personal experience. During the course of my research for this thesis, a series of field trips to the racetrack were necessary. Based on the theories suggested in section 3.3, approximately 65 races have been bet which offered value. Using a starting bankroll of $500, faithfully supplied by my trusting supervisor, the results were not overwhelmingly satisfying.

The first few wagers were satisfying followed by a losing streak highlighted by a particularly drastic loss of $360 on one extremely uncooperative animal. As our bankroll dropped below the $100 mark, an adjustment was deemed necessary (with support from my somewhat poorer supervisor). At that point, however, the study had only encompassed approximately 24 wagers, too small a sample to panic over, especially when dealing with a stochastic phenomenon.

A careful observance of our previous wagers, as well as those in Table 3.2, found that the majority of the problem was in wagering on young horses which have had little experience at the races. By eliminating these type of races (maiden) from consideration, it seemed plausible to hope for an improvement in performance. The bankroll surged from \$64 ot its' present value of \$393 in 65 wagers. We can now state that the original few wagers were an aberration, with the more extensive workout in Section 3.3 much more representative of the theory's performance.

That workout had good and bad periods, but rarely was there any concern over the original bankroll. It is stressed, however, that to best utilize the formulae presented, probabilities should be determined by the gambler. With a set of good estimates, a gambler becomes an investor whose likelihood of success is high.

FIGURE 3.2 - AT-TRACK WORKOUT OF STRATEGIES
68 plays (JAN-MAY 1987, Stampede Park, Calgary)



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