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Partitioning Trees

by

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Abstract

This paper examines the variations of the Halpern-Lauchli theorem that have come into being since it was originally devised in 1966, with emphasis on the perfect tree formulation (*LP*) by Laver and Pincus. We begin by looking at improvements on the one-tree version of *LP* together with analogues to 'profusely-branching' and 'perfectly-profuse' trees. A proof for the two-tree version of *LP* is presented, and combinatorial and analytical applications of *LP* are given. Finally, we look at the infinite formulations of *LP* and some remaining open problems in the field.

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Chapter 1

Introduction

Intrinsic to the study of mathematics, and Ramsey theory in particular, is a search for order in structures. Ramsey theory could be described as the search for substructures which are more 'orderly' in some sense than the relational structures from which they are drawn, while retaining as much of the relational complexity of the original structure as possible. A simple Ramsey-esque exercise is to show that in a group of six people there is a clique, or an 'anticlique', of three people. The Hales-Jewett theorem, another mainstay of Ramsey theory, says that in a language that contains all words made up of an alphabet of t letters and in which there are r 'parts of speech', for any integer m a number n can be found so that there exists a word of length n with m variable letters, such that whatever letters in the alphabet are substituted, the word thus developed will be the same part of speech.

In this thesis the search for order will be in trees, or finite sequences of trees. Upon coloring the nodes, or sequences of nodes, of these trees, we will seek subtrees having a similar structure to the original, but with greater chromatic organization. Before proceeding further, we will describe in greater detail the notions of 'tree' that we will be using in this thesis.

1.1 Trees

A *relational structure* is a set of elements, called *vertices*, together with a relation on this set. In a binary relational structure the relation is composed of ordered pairs of these nodes, referred to as *edges*. If G is a binary relational structure, then G can be represented by an ordered pair $(V, \leq)$, where V is the set of vertices and $\leq$ is the relation.

A tree, $T = (T, \leq)$, is a transitive binary relational structure with the following two distinguishing features:

1. T has a $\leq$-minimal element, known as the *root* of T, and

2. if $x \in V$, then the set of predecessors of x, or $\{y \in T : y \leq x\}$, is well-ordered.

In discussing trees certain set-theoretic notations will be useful:

- $n = \{0, 1, 2, \ldots, n-1\}$
- ω is the set of all finite numbers
- If A is a set and B is a cardinal, then $[A]^B$ is the set of all subsets of A of size B.

Certain other terminology will be used throughout the thesis. If T is a tree, and $x \in T$, then $P(x) = \{y \in T : y < x\}$ and $S(x) = \{y \in T : y > x\}$. We refer to $P(x)$ as the *predecessors* of x and $S(x)$ as its *successors*. For a set of nodes A, $P(A)$ is the set of all predecessors of members of A, and $S(A)$ is the set of their successors. The subtree T_x of T contains all nodes of T which are comparable to x. In other words, $T_x = P(x) \cup \{x\} \cup S(x)$.

The level of a node x in T, or lev(x), is the cardinality of $P(x)$. The nth level of T, or $T(n)$, is the set $\{x \in T : \text{lev}(x) = n\}$. We call a set of nodes a *level set* if each of its nodes is found in the same level. The set of immediate successors of x, or $IS(x)$, is the set of all nodes y in T with $y > x$ and lev(y) =lev$(x) + 1$. A node $x \in T$ is called *n-branching* if $|IS(x)| = n$. A *fork* is a two-branching node.

If A and B are sets of nodes in T, then $A \geq B$, or A *dominates* B, if for all x in B, there exists y in A such that $y \geq x$. If $A \subseteq T(k)$ and $B \subseteq T(m)$ for some $k \geq m$, then $A \geq^n B$, or A *n-dominates* B if for all $x \in B$, there exist n successors of x in A.

A *finitistic tree* is a tree T such that for all nodes $x \in T$:

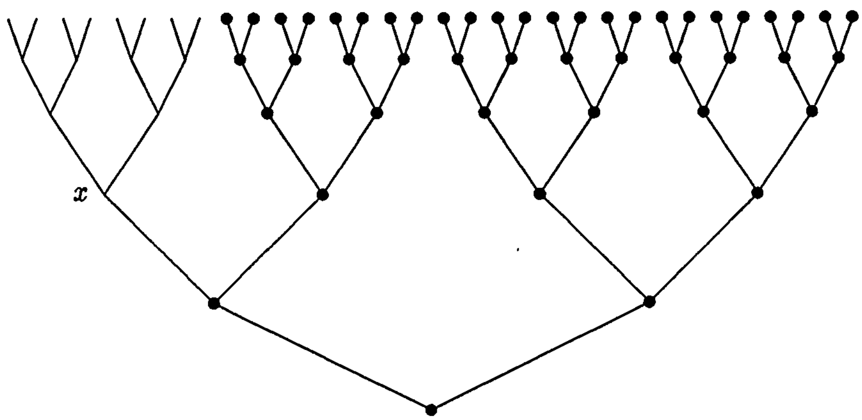
- $1 \leq |IS(x)| < \omega$, and
- lev$(x) < \omega$.

This thesis focuses on three kinds of trees, all of which are finitistic.

Our discussion of a subtree of a finitistic tree T will be restricted to 'closed downward' subsets of the vertices under the inherited ordering. When 'pruning' a tree T to find a monochromatic subtree, we will say that we *remove* a node x from T if we replace T with T', where T' is the largest finitistic subtree of T which does not include the node x. In Figure 1.1, the black dots represent the nodes remaining in $2^{<\omega}$ upon removing x.

If $\langle A_i : i \in d\rangle$ is a sequence of sets, then $\prod_{i\in d} A_i$ is the product of this collection. In other words it is the set of all d-tuples $\langle a_i : i \in d\rangle$ such that for all i in d, $a_i \in A_i$. For trees, we will be interested primarily in sequences of same-level nodes, and so we adopt a different notation. Let $L \in [\omega]^\omega$ and $\langle T_i : i \in d\rangle$. Then the *level product* of

Figure 1.1: The removal of a node from the complete binary tree.



$\langle T_i : i \in d \rangle$ over L, $\bigotimes_{i\in d}^{L} T_i$, is defined as follows:

$$\bigotimes_{i\in d}^{L} T_i = \bigcup_{j\in L} \left(\prod_{i\in d} T_i(j) \right).$$

If no set L is specified, it will be assumed that the product in question is the level product over ω.

1.1.1 Perfect Trees

The *complete binary tree* is the finitistic tree in which every node is a fork. It is also referred to as $2^{<\omega}$, as its nodes can be represented as finite sequences of 0s and 1s. A *perfect tree* is a subtree T of $2^{<\omega}$ in which $S(x)$ contains a fork for all $x \in T$. A *perfect subtree* of T is a perfect tree T' which is a subtree of T. We say, for short, $S \overset{p}{\subseteq} T$.

1.1.2 Profusely-Branching Trees

A *profusely-branching tree* is a finitistic tree T for which there is a sequence $\langle a_n : n \in \omega \rangle$ of natural numbers such that

- for some $b > 0$ and all $n \in \omega$, $a_n \leq b \cdot n$,
- $\lim_{n \to \infty} a_n = \infty$, and
- all nodes in $T(n)$ are a_n-branching.

Let T be a profusely-branching tree, $A \subseteq T(m)$ and $B \subseteq T(n)$ be level sets of nodes with $A \leq B$, and $\langle b_k : k \in \omega \rangle$ be a sequence of natural numbers. Let $C = P(B)/P(A)$. Then B is a $\langle b_n \rangle$-*extension* of A if for all $x \in C \cap T(k)$, $m \leq k \leq n-1$, x has b_k many immediate successors in C.

1.1.3 Perfectly-Profuse Trees

A *perfectly-profuse tree*, intended as a compromise between perfect and profusely-branching trees, is a finitistic tree in which $S(x)$ contains a node whose branching is n or larger, for all $n \in \omega$ and all $x \in T$.

1.2 The Halpern-Lauchli Theorem

1.2.1 The Boolean Prime Ideal Theorem

If B is a Boolean algebra, then an *ideal* on B is a subset I of B such that:

- $\mathbf{0} \in I$ and $\mathbf{1} \notin I$,
- $\boldsymbol{u},\boldsymbol{v} \in I \Rightarrow \boldsymbol{u}+\boldsymbol{v} \in I$, and

- $\boldsymbol{u}$,$\boldsymbol{v}\in B$, $\boldsymbol{u}\in I$, and $\boldsymbol{v}\leq \boldsymbol{u}\Rightarrow \boldsymbol{v}\in I$.

If I is an ideal on B such that either $\boldsymbol{u}$ or $-\boldsymbol{u}$ is in I for all $\boldsymbol{u} \in \boldsymbol{B}$, then I is called a *prime ideal*. The Boolean Prime Ideal Theorem (PIT) says that every ideal on B can be extended to a prime ideal.

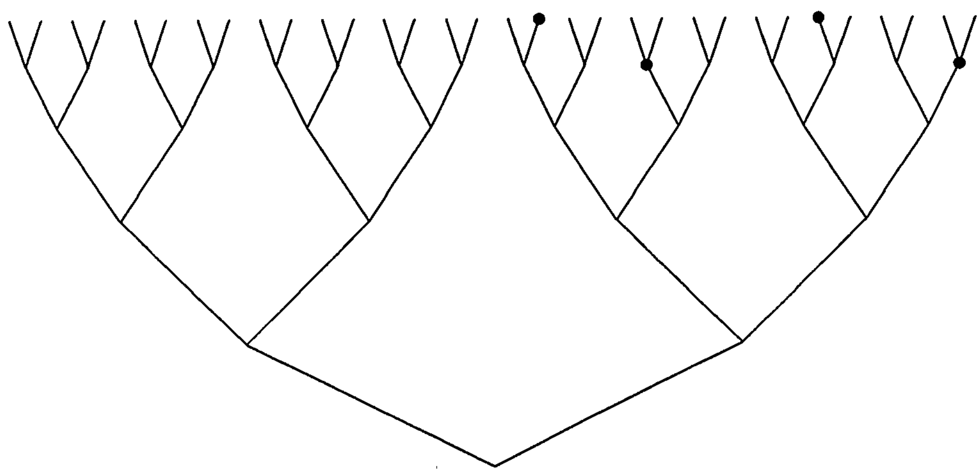
The proof of PIT depends on the Axiom of Choice (AC), and PIT can often be used in proofs in place of the AC, some examples being the proofs of the Hahn-Banach Theorem, compactification theorems, and the Completeness Theorem and Compactness Theorem of first order logic. For these reasons a natural question to ask is whether PIT is equivalent to AC or weaker.

A first step in solving this problem occurred when Halpern showed in [4] that PIT did hold in a model of set theory without foundation, constructed by Mostowski, in which AC did not. It was seen from this that PIT is not equivalent to the Axiom of Choice in ZF without the axiom of foundation. The proof that they are not equivalent in full ZF uses the original Halpern-Lauchli theorem; indeed it was concocted for use in this proof. Several different formulations of the Halpern-Lauchli theorem were derived after this, leading to many more applications. The formulations fall into two basic categories: the matrix theorems, from which the proofs are drawn, and the tree theorems, which are more conducive to combinatorial applications.

1.2.2 The Matrix Theorems

If T is a finitistic tree, then $A \subseteq T$ is (h,k)-*dense* if there exists a node $x \in T(h)$ such that A dominates $T_x(h+k)$. An example of a $(1,2)$-dense set in the complete binary tree is represented by the black dots in Figure 2.1. If $\langle T_i : i \in d\rangle$ is a d-tuple

Figure 1.2: A (1,2)-matrix in the complete binary tree.



of finitistic trees, and $A_i \subseteq T_i$, then $\prod_{i\in d} A_i$ is an (h,k)-*matrix* if for each i, A_i is (h,k)-dense in T_i.

The original formulation of the Halpern-Lauchli Theorem, which will be referred to as *HL* or *HL*(d) (where d is the length of the sequence of trees in the theorem), is as follows:

> Let $\langle T_i : i \in d\rangle$ be a d-tuple of finitistic trees, and let $\prod_{i\in d} T_i = G_0 \cup G_1$. Then one of the following must be true:
>
> 1. for all k there exists a $(0,k)$-matrix in $\prod_{i\in d} T_i \cap G_0$, or
> 2. there is an h such that for all $k > h$ there exists an (h,k)-matrix in $\prod_{i\in d} T_i \cap G_1$.

This was later improved upon by Laver (1969, unpublished) and Pincus [5] when they showed that $\prod_{i\in d} T_i$ can be replaced in the theorem with $\bigotimes^{\omega}_{i\in d} T_i$. In other words the 'monochromatic' (h,k)-matrices can be found with the added property that all their nodes are in the same level. The Laver-Pincus subtree formulations of the Halpern-Lauchli Theorem follow from this level-cognizant version of the theorem.

1.2.3 The Subtree Theorems

The theorem usually referred to as the Laver-Pincus Theorem deals with strong subtrees. If T is a finitistic tree and $S \subseteq T$ is a finitistic tree under the ordering inherited from T then S is a strong subtree of T if there is an increasing function $f : \omega \to \omega$ (the level function) such that $S(n) \subseteq T(f(n))$ and for each n, for all $x \in S \cap T(n)$ and $y \in IS(x,T)$, there exists z in S such that $y \leq z$. A level family is a sequence of strong subtrees of a sequence of trees all related to the trees from which they are drawn by the same level function.

The Laver-Pincus Theorem says the following:

> Let d be finite and let $\langle T_i : i \in d\rangle$ be a d-tuple of trees. Let $\bigotimes^{\omega}_{i\in d} T_i = G_0 \cup G_1$. Then there exist a level family $\langle S_i : i \in d\rangle$ of strong subtrees of T_i, $i \in d$, and $k \in 2$ such that $\bigotimes^{\omega}_{i\in d} S_i \subseteq G_k$.

In this thesis we are more interested in a weaker version that is again due to the Laver-Pincus matrix formulation, and which will be referred to as *LP*, or *LP*(d):

> Let $\langle T_i : i \in d\rangle$ be a d-tuple of perfect trees, and let $\bigotimes^{\omega}_{i\in d} T_i = G_0 \cup G_1$. Then there exist $k \in 2$ and $L \in [\omega]^{\omega}$ and for all $i \in d$ there exists $T_i' \overset{p}{\subseteq} T_i$ such that $\bigotimes^{L}_{i\in d} T_i' \subseteq G_k$.

If 'perfectly-profuse' is substituted for 'perfect' in *LP*(d), the resulting theorem still follows from *HL*(d). We do not know if this is true for profusely-branching trees though. We include a proof of $LP(1)$ for profusely-branching trees; whether it is true for any $d > 1$ remains an open problem.

1.3 The Structure of the Thesis

The primary focus of this thesis is the perfect tree formulation of $HL(d)$ by Laver and Pincus ($LP(d)$), which is concerned with finding perfect subtrees of finitely many perfect trees such that the level product of the subtrees over some $A \in [\omega]^\omega$ is monochromatic. In Chapter 2 we will look at the simplest version of this theorem: the fact that a finitely-colored perfect tree has a monochromatic perfect subtree. We will also examine the connection between the density of nodes of a certain color and the existence of a perfect subtree of that color. In Chapter 3 we look at analagous situations for profusely-branching trees.

A constructive proof for the more difficult two-tree version of LP is presented in Chapter 4. Chapter 5 comprises the proofs of two applications of $LP(d)$ for d finite. Finally, the sixth chapter consists of an overview of the infinite versions of the Halpern-Lauchli Theorem. The original contributions contained in the thesis include Theorem 3.3 and the proofs of Theorems 2.4, 3.5, and 4.3.

Chapter 2

Single-Tree Theorems: Perfect Trees

In this chapter we present the proof of the most humble version of the Laver-Pincus theorem: that if the nodes of one perfect tree T are finitely-colored, then there exists a perfect subtree T' of T and a set $[L]^\omega$ such that $T'(L)$ is monochromatic. Upon knowing this, it is interesting to look for ways in which the search can be restricted. First we explore the relation between a high density of nodes of a certain color with the existence of a pefect subtree of that color. To conclude our investigation of single trees, we look at an ultrafilter in which the set L, as described above, can necessarily be found.

2.1 The Laver-Pincus Theorem: One Tree

The proof that a finitely-colored perfect tree contains a monochromatic perfect subtree, or $LP(1)$, is facilitated by the following lemma, the one-tree version of the Laver-Pincus matrix theorem:

Lemma 2.1 *Let T be a perfect tree with nodes colored red and green, and let $L \in [\omega]^\omega$ be a set of levels. Then either:*

1. *for all h there exists $j \in L$ and $A \subseteq T(j)$ such that all nodes in A are colored green and $A \geq T(h)$, or*
2. *there exists $x \in T$ such that for all h there exists $j \in L$ and $A \subseteq T_x(j)$ such that A is red and $A \geq T_x(h)$.*

Proof Assume that '1' does not hold. Then there exists h such that for all $j \in L$ there is no green subset of $T(j)$ dominating $T(h)$. In other words for all $j \in L$, there exists $x \in T(h)$ with all nodes of $T_x(j)$ colored red. As there are finitely many nodes in $T(h)$, for one such node, say x_0, there is an infinite subset of L, say L_0, such that $T_{x_0}(j)$ is red for all $j \in L_0$. It is clear then that '2' holds. □

Theorem 2.2 (LP(1)) *Let T be a perfect tree colored red and green and L be an infinite set of levels. Then there is a perfect subtree T' of T and an infinite set of levels $L' \in [L]^\omega$ such that $T'(L')$ is monochromatic.*

Proof We will assume that '1' holds in the statement of the Laver-Pincus matrix formulation, as the proof is analagous if '2' is true. We will define a nested sequence $\langle T_k \rangle_{k=0}^{\infty}$ of perfect subtrees of T and a sequence of levels $\langle \ell_k \rangle_{k=0}^{\infty}$ such that $T'(\ell_k)$ is colored green for all $k \in \omega$, where $T' = \bigcap_{k=0}^{\infty} T_k$.

It follows from '1' that there exists $\ell_0 \in L$ and $A \subseteq T(\ell_0)$ colored green such that $A \geq T(0)$. Remove all nodes in $T(\ell_0)$ that are not in A. The resulting tree is T_0.

Upon having defined levels ℓ_k and perfect trees T_k for $k \in n$, we define T_n as follows. Since T_{n-1} is a perfect tree, there is a level, say ℓ'_{n-1}, such that $T_{n-1}(\ell'_{n-1}) \geq^2 T_{n-1}(\ell_{n-1})$. By '1', again, there exists a level $\ell_n \in L$, and a set of nodes $A_n \subseteq T_{n-1}(\ell_n)$ colored green such that $A_n \geq T_{n-1}(\ell'_{n-1})$. Then T_n is defined by $\bigcup_{x \in A_n} (T_{n-1})_x$.

Notice that no nodes in or below $T_{n-1}(\ell'_{n-1})$ are removed to create T_n. This guarantees T_n retains all nodes in $T_{n-1}(\ell_{n-1})$ as well as two successors of each of these nodes, and thereby guarantees that $\bigcap_{k \in \infty} T_k$ will be a perfect tree. Notice also that for subtrees created in this manner, if $x \in T_n$, $\text{lev}(x) \geq \ell_n$ and $y > x$ in T_{n-1}, then $y \in T_n$, and so if '1' is true for T_{n-1}, it is also true for T_n.

The intersection of these trees is a perfect tree with the desired properties, and so the proof of the theorem is complete in the case when '1' holds in Theorem 2.1. The proof is similar in the case when '2' is true. □

2.2 Density Theorems

The Laver-Pincus Theorem tells us that any two-colored perfect tree contains a monochromatic perfect subtree. The next theorem, proved first in the doctoral dissertation of Olga Yiparaki (see [12]), shows that it probably contains perfect subtrees of both colors. More specifically, in order for a tree to have only green subtrees, the density of green nodes per level must approach 1 as you move up the tree. We present a new proof for this theorem here.

The *density* of green nodes in a level is, as one might expect, the number of green nodes divided by the total number of nodes in that level. In the language of the following proofs, $d_g(T(n)) = g(T(n))/t(T(n))$, where $T(n)$ is the level at which we are looking. The limit of the density over a set of levels will be of interest, and shorthand for $\lim_{n\to\infty} d_g(T_x(a_n))$ will be $L_x(a_n)$ for a sequence $\langle a_n : n \in \omega\rangle$.

A node x in T is a called a *start* if $L_x(n) \neq 0$. More specifically, an $\langle a_n\rangle$*-start* is a start x for which $L_x(a_n)$ exists and does not equal zero. The number of starts in $T(k)$ is $s(T(k))$ and the number of $\langle a_n\rangle$-starts in level k is specified by $s(a_n, T(k))$. The set of $\langle a_n\rangle$-starts in $T(k)$ will be referred to as $S(a_n, T(k))$. The density of $\langle a_n\rangle$-starts in level k, $d_s(a_n, T(k))$, like the density of greens, is the number of starts over the total number of nodes in $T(k)$. Finally, a node which is both a start and green will be referred to as a *green start*. The notation for these fruitful hybrids will be analagous to the start notation, replacing "s" or "S" with "gs" or "GS", respectively.

Lemma 2.3 *Let $T = 2^{<\omega}$ be colored red and green. If $\lim_{n\to\infty} d_g(T(a_n)) = L$ and for all nodes $x \in T$ there is no subsequence $\langle b_n \rangle$ of $\langle a_n \rangle$ with $L_x(b_n) > B$, then for each level k, there is a subsequence $\langle c_n \rangle$ of $\langle a_n \rangle$ such that $d_s(c_n, T(k)) \geq L/B$.*

Proof Let $T(k) = \{x_i\}_{i \in t(T(k))}$. A subsequence $\langle c_n \rangle$ of $\langle a_n \rangle$ exists such that $d_g(T_{x_i}(c_n))$ converges for all $i \in t(T(k))$. Let $S(c_n, T(k)) = \{x_i\}_{i \in s(c_n, T(k))}$. Then

$$\begin{aligned} L &= \lim_{n\to\infty} d_g(T(a_n)) \\ &= \lim_{n\to\infty} d_g(T(c_n)) \\ &= \lim_{n\to\infty} \frac{\sum_{i \in t(T(k))} d_g(T_{x_i}(c_n))}{t(T(k))} \\ &= \frac{\sum_{i \in t(T(k))} \lim_{n\to\infty} d_g(T_{x_i}(c_n))}{t(T(k))} \\ &= \frac{\sum_{i \in s(c_n, T(k))} \lim_{n\to\infty} d_g(T_{x_i}(c_n))}{t(T(k))} \\ &\leq \frac{s(c_n, T(k)) \cdot B}{t(T(k))}. \end{aligned}$$

We are left with the desired result — that is, that $L/B \leq d_s(c_n, T(k))$. □

Theorem 2.4 *If T is the complete binary tree and $d_g(T(n)) \not\to 0$ then there is a perfect subtree T' of T and an infinite sequence $\langle a_n \rangle$ for which $d_g(T'(a_n)) = 1$ for $n \in \omega$.*

Proof Let $\{x_0, x_1, \ldots, x_{h-1}\}$ be $\langle b_n \rangle$-starts in $T(n_0)$ for some sequence $\langle b_n \rangle$. Let $j_0 \in \omega$ be minimal such that there exist $y_0 \in T_{x_0}$ and a subsequence $\langle c_n^0 \rangle$ of $\langle b_n \rangle$

such that $L_{y_0}(c_n^0) > 1/(j_0+1)$. For $m \in \{1,2,\ldots,h-1\}$, let $j_m \in \omega$ be minimal such that there exist $y_m \in T_{x_m}$ and a subsequence $\langle c_n^m \rangle$ of $\langle c_n^{m-1} \rangle$ satisfying $L_{y_m}(c_n^m) > 1/(j_m+1)$.

Let $\langle c_n^{h-1} \rangle = \langle c_n \rangle$. For all $m \in h$, for any $z \in T_{y_m}$, there is no subsequence $\langle z_n \rangle$ of $\langle c_n \rangle$ such that $L_z(z_n) > 1/j_m$. We have, by Lemma 2.4, that for all $k > n_0$ and $m \in h$ there is a subsequence $\langle c_n^{k,m} \rangle$ of $\langle c_n \rangle$ such that $d_s(c_n^{k,m}, T_{y_m}(k)) > j_m \cdot L_{y_m}(c_n)$. As h is finite it is not difficult to see that we can find $\langle c_n^k \rangle$ for which for all $k > n_0$ and $m \in h$,

$$d_s(c_m^k, T_{y_m}(k)) > j_m \cdot L_{y_m}(c_n) > j_m/(j_m+1).$$

Now since for all m in h, $L_{y_m}(c_n) > 1/(j_m+1)$, and h is finite, there exists $a > 0$ and $N \in \omega$ such that for $k > N$, $d_g(T_{y_m}(c_k)) > 1/(j_m+1) + a$ for all $m \in h$. So for $k > N$, $d_{gs}(c_n^k, T_{y_m}(c_k)) \geq a$. For k large enough, $t(T_{y_m}(c_k)) \geq 2/a$ for all $m \in h$, and $gs(c_n^k, T_{y_m}(c_k)) \geq 2$.

We have shown that there is a set of nodes which 2-dominates the set of starts we began with, and which contains only starts which converge to a nonzero limit on the same set of levels. Now it is easy to construct a perfect tree with infinitely many levels having only green nodes. As $d_g(n,T) \not\to 0$, the base node of the tree is a start, and so there is a level above it, say n_1, and a set of levels $\langle c_n^1 \rangle$, such that $T(n_1)$ has at least two green $\langle c_n^1 \rangle$-starts . Remove all other nodes in $T(n_0)$. Let the resulting tree be T_1.

In the jth step, we start with $T_{j-1}(n_{j-1})$ containing only green $\langle c_n^{j-1} \rangle$-starts. There is a level above n_{j-1} which contains two green starts dominating each node in $T(n_{j-1})$. Call this level n_j. Remove all nodes in $T(n_j)$ that are not green starts, and call the remaining tree T_j. The intersection T' of the subtrees T_j created in each step of the pruning, is such that $\forall j \in \omega$, $d_g(n_j, T') = 1$. □

Note The above theorem does not hold for perfect trees in general. If the density of greens in a perfect tree does not converge to zero, it is not necessarily true that it contains a perfect green subtree, as the following example will demonstrate. One can prove, however, in a similar fashion to the above that any profusely-branching tree T for which $\lim_{n\to\infty} d_g(T) \neq 0$ has a perfectly-profuse green subtree. A more "symmetric" perfect tree, i.e. one in which the branching of a node depends only on its level, or a perfectly-profuse tree can also be shown to have a perfect or perfectly-profuse (respectively) green subtree.

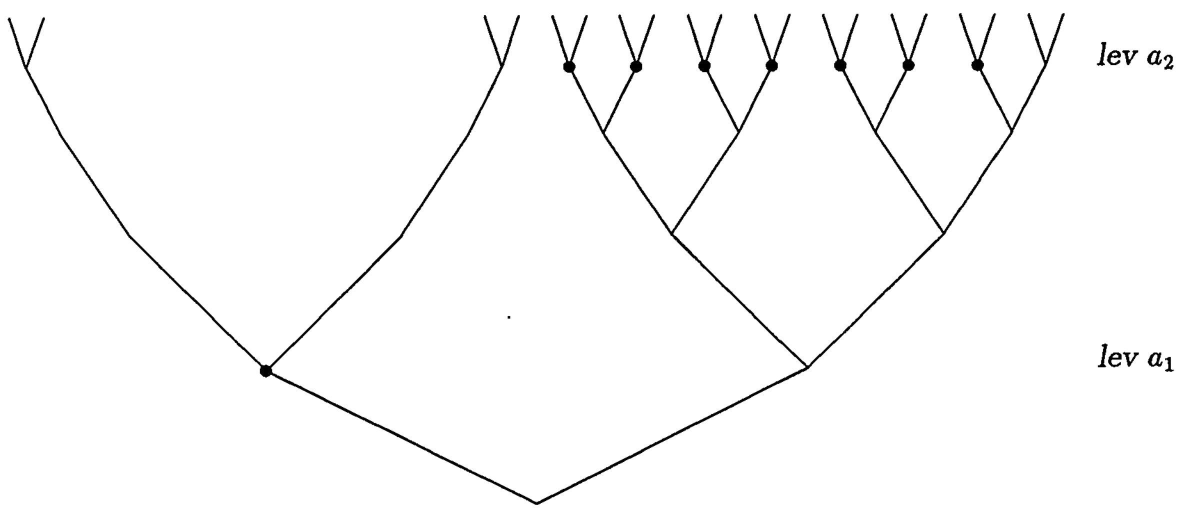
Example 2.5 *There exists a perfect tree T with $\lim_{n\to\infty} d_g(T(n)) \neq 0$ with no perfect green subtree.*

The example will be of a perfect tree T, a set of levels $\langle a_n \rangle$, and a coloring of T such that $\lim_{n\to\infty} d_g(T(a_n)) = 1$ but T contains no green perfect subtree. We define the three objects of interest such that $d_g(T(a_n)) \geq n/(n+1)$.

Let the base of T be a fork, and let $a_1 = 1$. Then $t(T(a_1)) = 2$, and we color the left node green. Upon having chosen a_n and defined T up to that level and colored $\lceil t(T(a_n)) \cdot n/(n+1) \rceil$ nodes green, we continue building the tree, level by level, so that all nodes at level a_n are forks, and above level a_n only nodes which are not successors of green nodes are forks; the rest are one-branching.

We stop when we reach a level where the ratio of all but one of the non-successors of green nodes to the total number of nodes at that level exceeds $(n+1)/(n+2)$. We then color red all nodes which succeed green nodes and the rightmost node that doesn't, and the remaining nodes are colored green. Repeating this process infinitely

Figure 2.1: The perfect tree described in Example 2.5



often gives us our tree and our levels, and upon coloring any extraneous node red, our coloring. As no green node is a successor of another green node, it is clear that there is no green perfect subtree. □

2.3 HLT Sets

Up to this point we have been satisfied with finding any infinite set of levels of a tree whose nodes are monochromatic. We now look at restricting the range of our search. A subset $\mathcal{L}$ of the power set of ω is called *HLT(K)*, where K can be one of P (perfect), PB (profusely-branching) or PP (perfectly-profuse), if given any tree of type K there exists a subtree of the same type which is monochromatic on a set of levels $L \in \mathcal{L}$. For example, in our proof of $LP(1)$ we have shown that for any infinite set A, the power set of A is *HLT*. We turn our attention now to a collection of sets called the *Ramsey ultrafilter*.

A *filter* on ω is a subset $\mathcal{U}$ of $\mathcal{P}(\omega)$ such that

1. the empty set is not in $\mathcal{U}$, but ω is,
2. if two sets A and B are in $\mathcal{U}$, then $A \cap B \in \mathcal{U}$, and
3. if $A \in \mathcal{U}$ and $A \subseteq B$, then $B \in \mathcal{U}$.

An *ultrafilter* is a maximal filter. Equivalently, a filter on ω is an ultrafilter if for all $A \subseteq \omega$ either A or ω/A is in the filter. The *principal ultrafilter generated by the element* $a \in \omega$ is the set of all subsets of ω containing a.

A Ramsey ultrafilter $\mathcal{R}$ on ω can be defined by either of the following characterizations:

1. If ω is partitioned into pieces P_k, $k \in \omega$, then either one of P_k is in $\mathcal{R}$ or there exists a set $A \in \mathcal{R}$, the 'selector', such that $|A \cap P_k| \leq 1$ for all $k \in \omega$.
2. If $f : [\omega]^2 \to 2$, then there exists $H \in \mathcal{R}$ such that f is constant on $[H]^2$.

The following is another result from the dissertation of Yiparaki:

Theorem 2.6 *Every Ramsey ultrafilter is* HLT(P).

Proof Let $\mathcal{R}$ be a non-principal Ramsey ultrafilter (the theorem is trivially true for principal ultrafilters), and T be a perfect tree colored red and green. For $s \in T$ let $G_s = \{n \in \omega : s \text{ has 2 green successors in } T(n)\}$.

Case 1: $G_s \in \mathcal{R}$ for all $s \in T$.

Let $G_k = \bigcap_{s \in T(k)} G_s$, $k \in \omega$. As a finite intersection of sets in $\mathcal{R}$, G_k is also in $\mathcal{R}$. Define a function $f : [\omega]^2 \to 2$ as follows:

$$f(m < n) = \begin{cases} 1 & \text{if } n \in G_m \\ 0 & \text{otherwise} \end{cases}$$

As $\mathcal{R}$ is Ramsey, there exists a set $H \in \mathcal{R}$ such that f is constant on H. Let $k \in H$ and $\ell \in G_k \cap H$. Then $f(k, \ell) = 1$, and as k and ℓ are both in H, $f(m < n) = 1$ for all m and n in H.

Let $H = \langle h_0, h_1, \ldots \rangle$ (in increasing order). Choose $s_0 \in T(h_0)$. Let $\{s_0\} = S_0$. Upon having chosen S_i, a set of nodes in $T(h_i)$, choose two green successors in $T(h_{i+1})$ of each node in S_i. Let the set of nodes thus chosen be S_{i+1}. Let $T = \bigcup_{i \in \omega} P(S_i)$. Then T is green on $H/\{h_0\}$, a set in $\mathcal{R}$.

Case 2: There exists $s \in T$ such that $G_s \notin \mathcal{R}$.

As $G_s \notin \mathcal{R}$, we have that $R_s = \omega / G_s \in \mathcal{R}$. For every level n in R_s, $T_s(n)$ has at most one green node. Let $\langle a_n \rangle$ be defined such that $a_0 = 0$ and $T_s(a_n) \geq^3 T_s(a_{n-1})$. As $\mathcal{R}$ is Ramsey, the partition $\{[a_n, a_{n+1}) : n \in \omega\}$ must have a selector S in $\mathcal{R}$. Let $S \cap R_s = \langle r_n : n \in \omega \rangle$. One of the sets $R_0 = \{r_{2n} : n \in \omega\}$ and $R_1 = \{r_{2n+1} : n \in \omega\}$ is found in ω; without loss of generality, R_0 does. Now each node in a level of R_0 has two red successors in each level of R_0 above it, and the proof is completed as in Case 1. $\square$

Chapter 3

Profusely-Branching Trees

The following is an analogue to the Laver-Pincus theorem, due independently to Laflamme and Shelah (see [11]), for profusely-branching trees. It is, in fact, analogous only to the one-tree case of the theorem, and it has not yet been determined whether the theorem holds true for any larger number of trees.

Theorem 3.1 (Laflamme, Shelah) *Let T be a profusely-branching tree with nodes colored red and green. Then there exists a profusely-branching subtree T' of T and an infinite $A \subseteq \omega$ such that $\bigcup_{n \in A} T'(n)$ is monochromatic.*

Proof The proof consists of attempting to find a green subtree, and upon failing this, showing that a red subtree can be found. Let G_1 be the set of nodes including only the root of the tree. In general, having defined in the previous step G_k, a same-level set of nodes each of which has 2k-1 immediate successors in T, we will look for G_{k+1}, a k-extension of G_k, satisfying:

- for each node $x \in \bigcup_{y \in G_{k+1}} T_y$, $IS(x) \geq 2(k+1) - 1$, and
- all nodes in G_{k+1} are colored green.

Let $G_k \subseteq \ell_k$ and let T_k be the profusely-branching subtree of T derived by removing all nodes in ℓ_k that are not in G_k.

If this search is successful infinitely many times, the result is a set of subtrees $\{T_k : k \in \omega\}$ for which $T' = \cap_{k \in \infty} T_k$ is a profusely-branching tree and $T'(\ell_k)$ is green for all $k \in \omega$.

Assume instead that we have failed at some stage. Then there is a $k \in \omega$ and a level set G_k of nodes which has no green k-extension for which each node and its successors are $(\geq 2(k+1)-1)$-branching. In other words, for all n large enough so that $a_n \geq 2(k+1)-1$, there exists $x \in G_k$ such that x has no k-extension into $T(n)$. It follows that there is a node $x_0 \in T(l_k)$ and an infinite set of levels A_0 such that x has no green k-extensions into A_0. We will find T', a red profusely branching subtree of T, in T_{x_0}. We will first require a lemma.

Lemma 3.2 *Let T be a profusely branching tree. If $x \in T$ and all of its successors have $\ell \geq 2k-1$ immediate successors and no green k-extensions into any level $T(n)$ in $A \in [\omega]^\omega$, then for all $n \in A$ there exists a red $\ell-(2k-2)$-extension R_n of x to level n. Moreover, for each $n \in A$ there exists $B \in [A]^\omega$ such that each node in R_n has no green k-extension to any level in B.*

Proof Let T be an $\langle a_n \rangle$-branching tree, and let $x \in T(j)$. Let $m \in A$; we will look for $R_m \subseteq T_x(m)$ and $B \in [A]^\omega$ as described in the lemma.

For each $n \in A$, x has no green k-extension into $T(n)$, and so there are at most $k-1$ immediate successors of x with a k-extension into $T(n)$, as any more would provide x with a k-extension into $T(n)$. It follows that for some set $A_1 \in [A]^\omega$ these will be the same $k-1$ immediate successors, which we now remove, resulting in $IS(x)$ containing no nodes with a green k-extension into $T(n)$, for all $n \in A_1$.

As we are focusing specifically on $T(m)$, we will also remove the at most $k-1$ immediate successors of x with a k-extension into $T(m)$. This leaves x with at least $a_j - (2k-2)$ immediate successors. As $a_j \geq 2k-1$, x is left with at least one immediate successor. None of these successors have a green k-extension into $T(m)$ or a level in A_1.

In the nth step, for $n \in m-j$, we find, in the same way, at least $a_{j+n} - (2k-2)$ immediate successors of each of the remaining nodes in $T(j+n)$ from the previous step, such that none have a green k-extension into $T(m)$ or a level in $A_n \in [A_{n-1}]^\omega$. If we let R_m be the nodes remaining in $T(m)$ after the $m-j-1$st step, and $B = A_{m-j-1}$, then they satisfy the requirements of the lemma. □

Upon applying the lemma to x, the resulting tree, without loss of generality, is $\langle b_n \rangle$-branching, where

$$b_n = \begin{cases} (a_n - (2k-2)) & n \in \{j, j+1, \ldots, m-1\} \\ a_n & \text{elsewhere} \end{cases}$$

We may now again use the lemma on the nodes in R, found in the first iteration of the lemma, as none have green k-extensions into $T(n)$, for $n \in B$. Upon infinitely many iterations of the lemma we are left with a tree that is 1-branching up to x and $a_n - (2k-2)$-branching for all $n \geq j$ – in other words, a red profusely-branching subtree of T. □

Theorem 3.3 *If T is a profusely-branching tree, then there exists a 2-coloring of T so that $\lim_{n\to\infty} d_g(n,T) = 1$, but no green profusely-branching subtree of T exists.*

Proof Let T be a profusely-branching tree, whose branching is specified by the sequence $\langle a_n \rangle$ (i.e. each node in the nth level of T is a_n-branching). We define a two-coloring Δ_m of T ($\geq m$) as follows:

- All nodes in $T(m)$ are colored red.
- If $n \geq m$ and $T(n)$ has been colored, then $T(n+1)$ is colored as follows. All immediate successors of green nodes in $T(n)$ are colored green, as is the leftmost successor of each red node in $T(n)$. All other nodes in $T(n+1)$ are colored red.

We also define the coloring $\Delta_{(m,n)}$, $n > m$, to be the restriction of Δ_m to T ($\geq n$). For future reference, note that under this coloring $d_g(k,T)$ is increasing, and $d_g(k,T) = d_g(k,T_x)$ for $k \geq n$ and $x \in T(m)$.

Claim: If $x \in T(m)$, and $T(\geq n)$ is colored by $\Delta_{(m,n)}$, then $d_g(k,T_x) \to 1$.
Proof of Claim: Above level n, $\Delta_{(m,n)}$ and Δ_m are equivalent, and so we show that $\lim_{k\to\infty} d_g(k,T_x) = 1$ under Δ_m. It is easiest to look at the density of red nodes:

$$\begin{aligned} d_r(k,T_x) &= \tfrac{(a_m-1)\cdot(a_{m+1}-1)\ldots(a_{k-1}-1)}{a_m\cdot a_{m+1}\cdot\ldots\cdot a_{k-1}} \\ &= (1-1/a_m)\cdot(1-1/a_{m+1})\cdot\ldots\cdot(1-1/a_{k-1}). \end{aligned}$$

This product converges to 0 as k approaches infinity if and only if $\Sigma_{i=m}^{\infty} 1/a_i = \infty$ (see [2], page 196). As $1/a_i \geq 1/b\cdot i$ for all $i \in \omega$ and some positive number b, this series does diverge, and so $\lim_{k\to\infty} d_r(k,T_x) = 0$. It follows that $\lim_{k\to\infty} d_g(k,T_x) = 1$, and the proof of the claim is complete.

We can now inductively define a coloring of T which will satisfy the conditions of the theorem. We start by coloring T with $\Delta_{(0,1)}$. Note that under this coloring there are no green 2-extensions from a node in lev(1). Let $0 = k_0$ and $1 = k_1$. If in the jth step we have colored the tree by the coloring $\Delta_{(k_{j-1},k_j)}$, we choose x_j, a red node in $T(k_j)$. Let $k_{j+1} > k_j$ be chosen so that $d_g(k_{j+1}, T_{x_j}) > 1 - (1/2)^j$. Such a k_{j+1} exists as $\lim_{k\to\infty} d_g(k, T_{x_j}) = 1$. Also, for k larger than k_{j+1}, as the density is increasing under this coloring, $d_g(k, T) > 1 - (1/2)^j$. We rescind the previous coloring of $T(\geq k_{j+1})$ and recolor this part of the tree by $\Delta_{(k_j,k_{j+1})}$.

Upon completion of the induction a coloring has been constructed under which T contains no green subtree which above some level is 2-branching. It follows that it contains no green profusely-branching subtree. Also, as $\lim_{j\to\infty} d_g(k_j, T_{x_j}) = 1$, we have that $\lim_{j\to\infty} d_g(k_j, T) = 1$ and finally that $\lim_{n\to\infty} d_g(n, T) = 1$. $\square$

The final theorem of this chapter will show that Ramsey ultrafilters, already shown to be *HLT(P)*, are also *HLT(PB)*.

Lemma 3.4 *Let T be a profusely-branching tree colored red and green whose branching is defined by $\langle a_k : k \in \omega\rangle$. Let $s \in T(m)$ and $\langle b_k : k \in \omega\rangle$ be such that $0 \leq b_k \leq a_k$. Then if $\{s\}$ has no green $\langle b_k\rangle$-extension to $T(n)$, $n > m$, it must have a red $\langle a_k - b_k + 1\rangle$-extension to $T(n)$.*

Proof As $\{s\}$ has no green $\langle b_k\rangle$-extension to $T(n)$, it has at most $b_m - 1$ immediate successors with green $\langle b_k\rangle$-extensions to $T(n)$. In other words, it has at least $a_m - b_m + 1$ immediate successors without such extensions. By the same logic, each of these nodes has at least $a_{m+1} - b_{m+1} + 1$ immediate successors which do not have green $\langle b_k\rangle$-extensions to $T(n)$.

By repeating this process, we get eventually to a set of nodes in $T(n-1)$ with no green $\langle b_k\rangle$-extensions. In other words, at least $a_{n-1} - b_{n-1} + 1$

immediate successors of each of these nodes are red. The set thus formed is a red $\langle a_k - b_k + 1\rangle$-extension of $\{s\}$. $\square$

Theorem 3.5 *Every Ramsey ultrafilter is* HLT(PB).

Proof Let T be a profusely-branching tree colored red and green with branching defined by $\langle a_k \rangle$ and let $\mathcal{R}$ be a Ramsey ultrafilter. For $s \in T$ let $G_s = \{n \in \omega : s$ has a green $\lfloor a_k/3 \rfloor$-extension to $T(n)\}$. If $G_s \in \mathcal{R}$ for all $s \in T$, then the proof is similar to Case 1 of Theorem 2.6, and so we assume that there exists $s' \in T$ such that $G_{s'} \notin \mathcal{R}$.

Let $R_s = \omega / G_s$ for all $s \in T$. Then by Lemma 3.4 for each level $n \in R_s$ there exists a $\lceil 2a_k/3 \rceil$-extension from $\{s\}$ into $T(n)$. Let $F = \{s \in T : R_s \in \mathcal{R}\}$.

Claim: If $s \in F$, then at least 2/3 of its immediate successors are in F.

Proof: We know that R_s is in $\mathcal{R}$, and so every level in R_s has an $\lceil 2a_n/3 \rceil$-extension from $\{s\}$. Each such extension must dominate a $\lceil 2a_n/3 \rceil$-tuple of immediate successors of s. As there are finitely many such $\lceil 2a_n/3 \rceil$-tuples, this creates a finite partition of R_s. One of the parts of this partition must be found in $\mathcal{R}$. (Otherwise the complements of the parts would be in $\mathcal{R}$, and so would be their intersection, ω/R_s.) The $\lceil 2a_n/3 \rceil$ immediate successors of s corresponding to this part are all in F.

Let $R_n = \cap_{s \in T(n) \cap F} R_s$. Define a mapping $f : [\omega]^2 \to 2$ as follows:

$$f(m < n) = \begin{cases} 1 & \text{if } n \in R_m \\ 0 & \text{otherwise} \end{cases}$$

As $\mathcal{R}$ is Ramsey, there exists $H \in \mathcal{R}$ such that f is constant on $H = \{h_0, h_1, \ldots\}$ (in increasing order). Let $m \in H$, and $n \in R_m \cap H$. Then $f(m,n) = 1$, and so $f(k,\ell) = 1$ for all $k < \ell$ in H. As such, we can assume $h_0 = \text{lev(s')}$.

If $\{s\}$ has a $\lceil 2a_k/3 \rceil$-extension of nodes in F to level n and a red $\lceil 2a_k/3 \rceil$-extension to level n, then it is easy to see that it has an $\lceil a_k/3 \rceil$-extension to level n of red nodes in F. With this knowledge we can proceed to find a subtree of T which is $\langle \lceil a_k/3 \rceil \rangle$-branching from level h_0 up and which is monochromatic on $H/\{h_0\}$.

Let $\{s'\} = S_0$. As $\{s'\}$ has a $\lceil 2a_k/3 \rceil$- extension of nodes in F to level h_1 and a red $\lceil 2a_k/3 \rceil$-extension to level h_1, it must have a $\lceil a_k/3 \rceil$-extension to level h_1 of red nodes in F. In the nth step, having chosen a set of nodes $S_{n-1} \subseteq T_{s'}(h_{n-1})$, each of these nodes, by the same reasoning, has a $\lceil a_k/3 \rceil$-extension to level h_n of red nodes in F. Let the nodes thus chosen compose the set S_n.

The tree $\cap_{n\in\omega} P(S_n)$ is $\langle \lceil a_k/3 \rceil \rangle$-branching from level h_0 up, and contains only red nodes in the levels in H/h_0, which is a set in $\mathcal{R}$. $\square$

Chapter 4

The Laver-Pincus Theorem for Two Trees

In this chapter we will give a proof of *LP*(2) which uses the *HLT*-ness of a certain set. We begin with a Ramsey-like lemma for perfect trees.

The proof also uses a simple kind of subtree called a path. A *path* p in a tree T is a totally-ordered subset of T such that $|p \cap T(n)| = 1$ for all $n \in \omega$. The highest level at which two paths p and q in a tree have a common node is called the *forking level* of p and q, or $f(p, q)$.

A set of paths P is said to *generate* a tree T if $\cup P = T$. For example, one countable generating set of the complete binary tree is the set of all paths representable by binary sequences which are eventually zero. We begin with a Ramsey-like lemma for perfect trees.

Lemma 4.1 *Let P be a set of paths generating a perfect tree T, and let $\Delta : P \to r$, for some $r \in \omega$. Then there exist $P' \subseteq P$ and $T' \overset{p}{\subseteq} T$ such that P' generates T' and Δ is constant on P'.*

Proof It is sufficient, by the usual colorblindness arguments, to show that the lemma is true when Δ is a 2-coloring. Let $P_x = \{p \in P : x \in p\}$. If there exists $x \in T$ such that $\Delta^{-1}(0) \cap P_x = \emptyset$, then $P_x \subseteq \Delta^{-1}(1)$. In this case P_x is colored '1' by Δ and generates T_{x_0}, which is clearly a perfect subtree of T.

If, on the other hand, for all $x \in T$ there exists $p \in \Delta^{-1}(0) \cap P_x$, then a path can be selected from each such set to create a denumerable subset P' of P which generates T. □

If $P = \{p_i : i \in \omega\}$ is a set of paths generating a perfect tree, then let $G(P) = \{A \in [\omega]^\omega : \{p_i \in P : i \in A\}$ generates a perfect tree$\}$. For ℓ and n in ω, let $P_\ell(n) = \{k : f(p_\ell, p_k) \geq n\}$.

Lemma 4.2 *If P generates a perfect tree, then $G(P)$ is* HLT(P).

Proof Let T be a perfect tree colored red and green, and let s_0, the root of T be colored red (without loss of generality). Let $P = \{p_0, p_1, \ldots\}$ be a set of paths generating a perfect tree. We will look for a subtree of T that is red on all levels in a set in $G(P)$.

We begin by letting $S_0 = \{s_0\}$ and $\ell_0 = 0$. In the nth step, having previously defined a set of red nodes $S_{n-1} \subseteq T(l_{n-1})$ we look for S_n such that $S_n \geq^2 S_{n-1}$, $S_n \subseteq T(\ell_n)$ for some $\ell_n \in P_{\ell_{k_n}}(n)$ (where $k_n < n$ is such that $\max_{m<n} f(p_{k_n}, p_m)$ is minimal), and all nodes in S_n are colored red. If we are at no point impeded from continuing to the next step, then the tree $\cup_{n\in\omega} P(S_n)$ is a perfect tree which is red on the set of levels $\{\ell_n : n \in \omega\}$.

Say rather that there exists n and k_n such that for all $\ell \in P_{\ell_{k_n}}(n)$ there exists no set of red nodes $S_n \geq^2 S_{n-1}$ with $S_n \subseteq T(\ell)$. Then for each level $\ell \in P_{\ell_{k_n}}(n)$ there exists $x_\ell \in S_{n-1}$ such that $T_{x_\ell}(\ell)$ contains at most one red node. This induces a finite partition of $P_{l_{k_n}}$. As this set is in $G(P)$, by Lemma 4.1 there exists a member of this partition, say K, which is also in $G(P)$.

In other words, there exists $x \in S_{n-1}$ such that $T_x(k)$ contains at most one red node for all $k \in K$, and $\{p_k : k \in K\}$ generates a perfect subtree of the tree generated by P. Let k_0 be the smallest element of K. Remove the red node, if it exists, from $T_x(k_0)$. The remaining set is G_0. In the nth step choose $k_n \in K$ such that $T(k_n) \geq^3 G_{n-1}$ and $k_n \in P_{k_{j_n}}(n)$ (where, again, $j_n < n$ is such that $\max_{m<n} f(p_{j_n}, p_m)$ is minimal). Remove the red node in $T_x(k_n)$. Upon having completed this process we get $\cup_{n\in\omega} P(G_n)$, a perfect tree with only green nodes in any level in $\{k_n : n \in \omega\}$ – a set in $G(P)$. □

Theorem 4.3 (LP(2)) *Let T_0 and T_1 be perfect trees, and let $\bigotimes_{i\in 2}^{\omega} T_i$ be two-colored. Then there exist perfect subtrees T_0' of T_0 and T_1' of T_1 and a set of levels $L \in [\omega]^\omega$ such that $\bigotimes_{i\in 2}^{L} T_i'$ is monochromatic.*

Let $\bigotimes_{i\in 2}^{A} T_i$ be colored red and green. Let $P = \{p_n : n \in \omega\}$ be a generating set for T_1. Then by fixing a path $p_k \in P$ we can induce a coloring of T_0 (i.e., if $s \in T_0(n)$, then s is given the coloring of $(s, p_k(n))$ in the original coloring). By $LP(1)$ there must be a perfect subtree T_0^k of T_0 and a set $L_k \in [\omega]^\omega$ such that all nodes in $T_0^k(\ell)$ for all $\ell \in L_p$ have the same color under this coloring. If red, we write:

$$p \overset{r,L_k}{\rightarrow} T_0^k.$$

The r is replaced with a g if the nodes are green.

We will start by coloring the nodes in infinitely many levels of T_0, again red and green. Without loss of generality, let $p_0 \overset{r,L_0}{\rightarrow} T_0^0$ for some $T_0^0 \overset{p}{\subseteq} T_0$. Color the root of T_0^0 red. Let ℓ_1 be minimal in L_1 such that $T_0^0(\ell_1) \geq^2 T_0^0(0)$.

In the nth step, having in the previous step defined T_0^n and ℓ_n, using $LP(1)$, we will find T_0^{n+1}, a perfect subtree of T_0^n containing all nodes in $T_0^n(\ell_n)$ and $L_{n+1} \in [L_n]^\omega$ such that for all s in $T_0^n(\ell_n)$ either $p_n \overset{r,L_{n+1}}{\rightarrow} (T_0^{n+1})_s$ or $p_n \overset{g,L_{n+1}}{\rightarrow} (T_0^{n+1})_s$ and color each $s \in T_0^n(\ell_n)$ accordingly. Let ℓ_{n+1} be minimal in L_{n+1} such that linebreak $T_0^{n+1}(\ell_{n+1}) \geq^2 T_0^{n+1}(\ell_n)$.

Let $T_0^* = \cap_{n\in\omega} T_0^n$. We will apply Lemma 4.2 to T_0^* and get a tree T_0' and a set $\{\ell_{k_n} : n \in \omega\}$ on which this coloring is constant, say red. We can now find a subtree $T_1' \overset{p}{\subseteq} T_1$ and a set of levels $L \in [L']^\omega$ so that $\otimes_{i\in 2}^L T_i'$ is colored red. To facilitate this process we will look at a coloring of all nodes of T_1 in a level in $\{\ell_{k_n} : n \in \omega\}$. Color a node s '0' if its coloring with all nodes in T_0' at the same level is red. Color it '1' if not. We know then that for $n > m$, if $s = p_{k_m}(\ell_{k_n})$, then s is colored '0'.

We would like to find a perfect subtree of T_1' colored '0' at infinitely many levels. This is not difficult to do: we will generate it, path by path. Let $p_{k_{j_0}}$ be the first path. All nodes of this path with a level in $\{l_{k_n} : n \geq j_0\}$ are colored '0'. Having chosen the nth path, we will choose the next one, $p_{k_{j_{n+1}}}$, so that it branches with a previously chosen path with minimal highest fork.

We end up with a tree $T_1' = \cup_{n\in\omega} p_{k_{j_n}}$ and a set of levels $L = \{\ell_{k_{j_n}}\}$ such that $\otimes_{i\in 2}^L T_i'$ is colored red. $\square$

Chapter 5

Applications

In this chapter we will look at two results, one in Ramsey theory, the other in analysis, which follow from *LP*. The first, due to Laver in [8], was originally a conjecture by Galvin dealing with products of the rationals. Two ordered sets have the same order type if there is an order-preserving isomorphism from one to the other. As such, a countable linearly ordered set has the order type of the rationals if it has no greatest or least element, and between each pair of elements it contains another.

For example, the complete binary tree, under its 'left-to-right' ordering, has this order type. This ordering is defined as the lexicographic ordering of the ndoes as represented by their corresponding finite sequences followed by infinitely many 1s. The following lemma gives another example of how a set of this order type can be found. For the purpose of the proof, a left successor of a fork f is a successor of that fork which is itself a fork lexicographically smaller than f.

Lemma 5.1 *If the nodes of a perfect tree T are ordered lexicographically and $L \in [\omega]^\omega$, then $\bigcup_{n\in L} T(n)$ contains a subset with the order type of the rationals.*

Proof A set of nodes with the order type of the rationals is chosen by means of a nested sequence of sets: $A_0 \subseteq A_1 \subseteq \cdots$. We choose one node which composes A_0. Having chosen the nodes in A_{n-1}, we choose all of these nodes for A_n as well as other nodes so that in A_n none of the nodes in A_{n-1} is minimal or maximal and if $x < y$ are nodes in A_{n-1} then there exists a node z in A_n such that $x < z < y$. The set

$A = \bigcup_{n\in\omega} A_n$ has the order type of the rationals.

Let a_0 be a member of $\bigcup_{n\in L} T(n)$ with incomparable nodes to the left and right. Let $A_0 = \{a_0\}$. In the $n-1$st step of the construction let A_{n-1} have been chosen so that among the nodes incomparable to any of the nodes of A_{n-1} there exist lexicographic lower and upper bounds of A_{n-1} and nodes between every two nodes in A_{n-1}.

If x and y are nodes in A_{n-1}, z' is incomparable to each, and $x < z' < y$, then we choose a node z between x and y such that $z \in \bigcup_{n\in L} T(n)$ and z succeeds the left successor of a right successor of a fork above z'. This guarantees that there exist nodes uncomparable to x, y, and z both between x and z and between z and y. Upper and lower bounds are found similarly. The set $A = \bigcup_{n\in\omega} A_n$ has the order type of the rationals. □

Theorem 5.2 (Laver) *If η is the order type of the rationals, then:*

$$\begin{pmatrix} \eta \\ \eta \\ \cdot \\ \cdot \\ \cdot \\ \eta \end{pmatrix} \longrightarrow \begin{pmatrix} \eta \\ \eta \\ \cdot \\ \cdot \\ \cdot \\ \eta \end{pmatrix}^{\overbrace{(1,\ldots,1)}^{n}}_{<\aleph_0/n!}$$

In other words, for any finite coloring of $\mathbf{Q}^n$ there exist subsets X_i of $\mathbf{Q}$, $i < n$, with order type η, such that $\prod_{i\in n} X_i$ receives no more than $n!$ colors under the inherited coloring.

In the following lemma $\forall^{<\infty}$ is to be read as "for all but finitely many."

Lemma 5.3 *If Z_i has order type η for $i < n$ and $f : \prod_{i \in n} Z_i \to r < \omega$, then there exist subsets X_i of Z_i with order type η and $m < r$ such that*

$$\forall^{<\infty} x_0 \in X_0 \ldots \forall^{<\infty} x_{n-1} \in X_{n-1} \quad f(\langle x_0, \ldots x_{n-1} \rangle) = m.$$

Proof of Lemma The lemma is proved by induction. It is true for $n = 1$. (It is easy to show that the rationals, when finitely colored, have a monochromatic subset of the same order type.) Assume, inductively, that we know that the lemma holds for n. We will prove that it is true for $n + 1$.

Let $Z_0, \ldots, Z_n$ have the order type of the rationals, and let $f : \prod_{i \in n+1} Z_i \to r < \omega$. Let $2^{<\omega}$ be ordered lexicographically.

First we define an order-preserving function $z_n : 2^{<\omega} \to Z_n$. We will proceed to define order-preserving functions $z_i : 2^{<\omega} \to Z_i$, for $i < n$, satisfying a few other conditions, using the inductive assumption.

Enumerate $\bigotimes_{i \in n}^{\omega} 2^{<\omega}$ so that $\mathrm{lev}(\vec{s}_i) < \mathrm{lev}(\vec{s}_j) \Rightarrow i < j$. The ith coordinate of $\vec{s}_j$ will be called s_j^i. For each $\vec{s}_k \in \bigotimes_{i \in n+1}^{\omega} 2^{<\omega}$, we will define order-preserving functions $g_{s_k^i} : 2^{<\omega} \to Z_i$ for $i \in n$ and in doing so both create a coloring for $\prod_{i \in n} 2^{<\omega}$ and define functions z_i, $i < n$. In the following, let $Z_{j,i} = g''((2^{<\omega})_{s_j^i})$.

First let $g_{s_0^i} : 2^{<\omega} \to Z_i$ for all $i \in n$ so that there exists $m_0 < r$ such that

$$\forall^{<\infty} z_0 \in Z_{0,0} \ldots \forall^{<\infty} z_{n-1} \in Z_{0,n-1} \quad f(\langle z_0, \ldots, z_{n-1}, z_n(s_0^n) \rangle) = m_0.$$

Color $\vec{s}_0$ 'm_0', and let $z_0(s_0^i) = g_{s_0^i}(s_0^i)$.

For $j > 0$ and $i \in n$, if s_j^i is in level h, then let $g_{s_j^i} : 2^{<\omega} \to Z_{k,i}$, where k is maximal such that $s^i_k \leq s^i_j$, again so that there exists $m_j < r$ such that

$$\forall^{<\infty} z_0 \in Z_{j,0} \ldots \forall^{<\infty} z_{n-1} \in Z_{j,n-1} \quad f(\langle z_0, \ldots, z_{n-1}, z_n(s^n_j)\rangle) = m_j.$$

Color $\vec{s_j}$ 'm_j'. Upon having defined functions g_{s^i}, $i < n$ for all $\vec{s} \in \bigotimes^{\omega}_{i \in n+1} 2^{<\omega}$, define $z_i(s) = g_{s^i_j}(s)$ where j is maximal such that $s^i_j = s$.

By continuing this process, ad infinitum, we get a finite coloring of $\bigotimes^{\omega}_{i \in n} 2^{<\omega}$, and by $LP(n)$ we can find perfect subtrees T_i of $2^{<\omega}$ and $L \in [\omega]^{\omega}$ such that $\bigotimes^{L}_{i \in n} T_i$ is monochromatic . By Lemma 5.1 we can find subsets S_i of $T_i(L)$ which have the order type η for each $i \in n$. Their images under the functions z_i therefore also have the order type of the rationals . It is the product of these images which satisfies the theorem. $\square$

Proof of Theorem Apply the lemma $n!$ times to get $W_i \subseteq Z_i$ for $i < n$ each of order type η such that for each permutation π of n, there is $m_\pi < r$ such that:

$$\forall^{<\omega} w_{\pi(0)} \in W_{\pi(0)} \ldots \forall^{<\omega} w_{\pi(n-1)} \in W_{\pi(n-1)} \quad f(\langle w_0, \ldots w_{n-1}\rangle) = m_\pi.$$

Now we find $Y_i \subseteq W_i$ with order type η satisfying the theorem. We do this by choosing $\{y_j : j \in \omega\}$ successively. We choose y_j such that:

1. If $y_{j_i} \in Y_i$ for all $i \in n$, and $j_{\pi(0)} < j_{\pi(1)} < \ldots < j_{\pi(n-1)}$, then $f(\langle y_{j_0}\rangle, \ldots y_{j_{n-1}}\rangle) = m_\pi$, and

2. $\{y_j : j \in \omega\} \cap W_j$ has order type η.

Notice that with the choice of y_j, the number of options for y_{j+1} diminishes by a finite number (by 1). This is not sufficient to impede the choice of $\{y_j : j \in \omega\}$ satisfying 2.

If we let $Y_i = \{y_j : j \in \omega\} \cap W_i$, then the theorem is satisfied. □

In the previous application, perfect trees were useful because of their connection with the rational numbers. In the following theorem of Harrington's, the isomorphism from the paths in the complete binary tree to a perfect subset of $[0,1]$ is utilized in an analytical application of LP.

Lemma 5.4 *Let $\langle Q_i : i \in d\rangle$ be a d-tuple of perfect subsets of $I = [0,1]$ and for n in ω let f and g_n be continuous functions from $\prod_{i\in d} Q_i$ into I. Then for i in d there exist perfect $P_i \subseteq Q_i$, $R \in \{<,=,>\}$, and a subsequence $\langle g_{t_n}\rangle$ of $\langle g_n\rangle$ such that for each n and each $\vec{p} \in \prod_{i\in d} P_i$, $R(g_{t_n}(\vec{p}), f(\vec{p}))$.*

Proof We will inductively define $J_{i,s}$, for $i \in d$ and $s \in 2^{<\omega}$, where $J_{i,s}$ is a closed interval in Q_i, such that:

1. $s < t \Rightarrow J_{i,t} \subseteq J_{i,s}$,
2. s is incomparable to $t \Rightarrow J_{i,s}$ and $J_{i,t}$ are disjoint, and
3. $\vec{s} \in \bigotimes_{i\in d} 2^{<\omega}$ and $\text{lev}(\vec{s}) = n \Rightarrow$ there exists a relation $R_{\vec{s}} \in \{<,=,>\}$ such that for all $\vec{p} \in \prod_{i\in d} J_{i,s}$, $R_{\vec{s}}(g_n(\vec{p}), f(\vec{p}))$.

We begin by enumerating $\bigotimes_{i\in d}^{\omega} 2^{<\omega}$ such that $j < k$ if $\text{lev}(\vec{s_j}) < \text{lev}(\vec{s_k})$. The ith coordinate of $\vec{s_j}$ will be called s_j^i.

Define $J_{s_0^i}$, for $i < d$, so that it is a closed interval for which there exists $R_{\vec{s_0}} \in \{<,=,>\}$ such that for all $\vec{p} \in \prod_{i\in d} J_{s_0^i}$, $R_{\vec{s_0}}(g_0(\vec{p}), f(\vec{p}))$. (This is possible as g_0 and f are continuous.)

Having defined $J_{s_{j-1}^i}$ for i in d, we define $J_{s_j^i}$ where $\text{lev}(\vec{s_j}) = n$, such that

1. if $k < j$, then $s^i_k \leq s^i_j \Rightarrow J_{s^i_j} \subseteq J_{s^i_k}$,

2. if $k < j$ and s^i_k is incomparable to s^i_j, then $J_{s^i_j} \cap J_{s^i_k} = \emptyset$, and

3. if $\vec{s_j} \in \bigotimes_{i \in d} 2^{<\omega}$ and $\mathrm{lev}(\vec{s_j}) = n$, then there exists a relation $R_{\vec{s_j}} \in \{<, =, >\}$ such that for all $\vec{p} \in \prod_{i \in d} J_{s^i_j}$, $R_{\vec{s_j}}(g_n(\vec{p}), f(\vec{p}))$.

Upon finishing this induction, define $J_{i,s}$ to be $J_{s^i_j}$ where j is maximal such that $s^i_j = s$.

This process induces a 3-coloring (where the 'colors' are $<$, $=$, and $>$) of $\bigotimes^{\omega}_{i \in d} 2^{<\omega}$. By $LP(d)$, there exist perfect subtrees T_i of $2^{<\omega}$, $i \in d$, and $A \in [\omega]^{\omega}$ such that all n-tuples in $\bigotimes^{A}_{i \in d} T_i$ have the same coloring, say R. Let $P_i = \bigcup_{p \in [T_i]} \bigcap_{n \in A} J_{i,p(n)}$ (where $[T_i]$ represents the set of paths in T_i). Then P_i, $\langle g_n : n \in A \rangle$, and R satisfy the lemma. □

The following theorem was proved by Harrington (unpublished) and reformulated in [8].

Theorem 5.5 (Harrington) *Let $\langle f_n : n \in \omega \rangle$ be a sequence of continuous functions from I^d into $I = [0, 1]$. Then there are nonempty perfect subsets P_i of I and a subsequence $\langle g_n \rangle$ of $\langle f_n \rangle$ such that $\langle g_n \rangle$ is monotonic and uniformly convergent on $\prod_{i \in d} P_i$.*

Note In the original proof by Harrington, using a canonical version of LP, a single perfect subset P of I was found such that the sequence $\langle g_n : n \in \omega \rangle$ was uniformly convergent on P^d.

Proof To prove the theorem we will inductively choose an infinite subsequence $\langle g_n \rangle$ of $\langle f_n \rangle$ and perfect sets $J_{i,s}$ for $i \in d$ and $s \in 2^{<\omega}$ such that

1. if $s < t$, then $J_{i,t} \subseteq J_{i,s} \subseteq I$,

2. if s and t are incomparable then $J_{i,s} \cap J_{i,t} = \emptyset$, and

3. if $\vec{s} \in \bigotimes_{i \in d} 2^{<\omega}$ and $\text{lev}(\vec{s}) = n$ then there exists $R_{\vec{s}}$ such that for all $n' > n$, $R_{\vec{s}}(g_{n'}(\vec{p}), g_n(\vec{p}))$ for all $p \in \prod_{i \in d} J_{i,s^i}$.

We start by enumerating $\bigotimes_{i \in d}^{\omega} 2^{<\omega}$ as in the proof of the lemma. The first application of the lemma is to I^d, f_0 and $\langle f_n : n > 0 \rangle$ to get perfect $J_{s_0^i} \subseteq Q_i$, $A \in [\omega]^\omega$ and a subsequence $\langle f_{t_n^0} \rangle$ of $\langle f_n : n > 0 \rangle$ such that for all $n > 0$ and $\vec{p} \in \prod_{i \in d} J_{s_0^i}$, $R(f_{t_n^0}(\vec{p}), f_0(\vec{p}))$.

Then we choose two disjoint perfect subsets of $J_{s_0^i}$ for each i in d and call them $Q_{s_j^i}$ and $Q_{s_k^i}$, where j and k are minimal such that s_j^i and s_k^i are the two immediate successors of s_0^i, thus completing the first step of the induction.

In the mth step of the induction, we apply the lemma to perfect sets $Q_{s_m^i}$, $i \in d$, or, where this has not been defined, J'_{i,s_ℓ^i}, defined in the ℓth step, where $\ell < m$ is maximal such that $s_m^i = s_\ell^i$. Through the application of the lemma we get perfect sets J'_{i,s_m^i} and subsequence $\langle f_{t_n^{m+1}} : n \in \omega \rangle$ of $\langle f_{t_n^m} : n > 0 \rangle$ such that for all n and all $\vec{p} \in \prod_{i \in d} J'_{s_m^i}$, $R_{\vec{s}}(f_{t_n^{m+1}}(\vec{p}), f_{t_0^m}(\vec{p}))$. We finish, if there is no $k > m$ such that $s_k^i = s_m^i$, by choosing two disjoint perfect subsets of J'_{i,s_m^i}, which will be called $Q_{s_j^i}$ and Q_{i,s_k^i} where j and k are minimal such that s_j^i and s_k^i are the immediate successors of s_m^i.

When the induction is complete, we are left with a 3-coloring of $\bigotimes_{i \in d} 2^{<\omega}$, $\vec{s} \to R_{\vec{s}}$. By $LP(d)$ we get $A \in [\omega]^\omega$ and perfect subtrees T_i of $2^{<\omega}$, $i \in d$, such that $\bigotimes_{i \in d}^{A} T_i$ is monochromatic.

Let $h_m = g_{t_0^m}$, and $P_i = \bigcup_{p_i \in [T_i]} \bigcap_{n \in \omega} J_{i,p_i(n)}$, a perfect set. Then we have that the sequence of functions $\langle h_m : m \in A \rangle$ is monotonic on $\prod_{i \in d} P_i$ and therefore pointwise convergent, as I is bounded. Let $\lim_{m \in A} h_m(x) = h(x)$.

It remains to show that there are perfect subsets of P_i on which $\langle h_m : m \in A \rangle$ converges uniformly. To this end we introduce some topological notions. A set has the Baire property if its symmetric difference with some open set is meager (the union of countably many nowhere dense sets). A function f has the Baire property if $f^{-1}(O)$ has the Baire property for every open set O.

Claim: The function h has the Baire property.

Proof of Claim: We start by showing that for all open sets $O \subseteq I$, $h^{-1}(O)$ is the countable union of G_δ sets. Let $\langle B_n : n \in \omega \rangle$ be the set of all closed intervals in O with rational endpoints. Then $h^{-1}(O) = \bigcup_{N \in \omega} \bigcup_{k \in \omega} \bigcap_{n \geq N} \{x : h_n(x) \in B_k\}$. So $h^{-1}(O)$ is the countable union of G_δ sets. It is easily seen that all open sets are Baire, the union of countably many Baire sets is Baire, and the complement of a Baire set is Baire. It follows that a G_δ set is Baire, as is the countable union of G_δ sets, and so h has the Baire property.

Claim: There exists a comeager subset H of $\prod_{i \in d} P_i$ such that h is continuous on H.

Proof of Claim: Let $\langle B_n : n \in \omega \rangle$ be a countable open basis for I. Choose M_n meager and U_n open such that $h^{-1}(B_n) \Delta U_n = M_n$. Let $H = \prod_{i \in d} P_i / \cup M_n$. Then H is comeager, and for each n, $h^{-1}(B_n) \cap H = U_n \cap H$ which is relatively open. So h is continuous on H, as suggested.

As H is comeager, it contains a subset $\prod_{i\in d} P_i'$, where P_i' is perfect, $i \in d$. Finally, by Dini's theorem, a monotononic pointwise-convergent sequence of functions on a perfect set is uniformly convergent, and the proof is complete. $\square$

Chapter 6

Infinite Versions of the Halpern-Lauchli Theorem

The most powerful subtree version of the Halpern-Lauchli Theorem deals with strong subtrees. A tree T' is called a strong subtree of a tree T if there is a function $f : \omega \to \omega$ such that for all $x \in T'(n)$, x is in $T(f(n))$, and for all x in T and $y \in IS(x,T)$ there exists a node $z \in T' \cap S(y,T)$. (These subtrees are "strong" when compared to the normal notion of subtree, which is that of any subset of a tree with its inherited ordering, and not in comparison to the 'closed-downward' definition of the subtree we have used in this thesis.)

The Laver-Pincus strong subtree formulation of the Halpern-Lauchli Theorem says the following:

> If $\langle T_i : i \in d\rangle$ is a finite sequence of finitistic trees, and $\bigotimes_{i\in d}^{\omega} T_i$ is r-colored, $r < \omega$, then there exist strong subtrees $T_i' \subseteq T_i$, $i < d$, such that $\bigotimes_{i\in d}^{\omega} T_i'$ is monochromatic under the inherited coloring.

This theorem cannot be extended to infinitely many trees, however. It is possible to partition $\bigotimes_{i\in\omega}^{\omega} 2^{<\omega}$, topologized by the Tychonoff product of discrete topologies, into two pieces C_0 and C_1 such that neither piece contains a perfect set of sequences. This is done using the Axiom of Choice. For all $n \in \omega$, first give the perfect sets in $/prod_{i\in\omega}2^{<\omega}(n)$ the well-ordering $\langle P_i : i \in 2^{\omega}\rangle$. Then put one element of P_0 in $C_0(n)$ and one in $C_1(n)$. Repeat this process, successively, with each perfect set, taking elements which have not been chosen previously (this is possible as each perfect set contains 2^ω elements). Now neither $C_0(n)$ nor $C_1(n)$ contains a perfect set. As such, neither contains a set $\prod_{i\in\omega} A_i$ such that $|A_i| \geq 2$ for all $i \in \omega$. We can then partition $\bigotimes_{i\in\omega}^{\omega} 2^{<\omega}$ into two pieces, $C_0 = \bigcup_{n\in\omega} C_0(n)$ and $C_1 = \bigcup_{n\in\omega} C_1(n)$, and it is easy to see that there is no sequence of strong subtrees $\langle T_i' : i \in \omega\rangle$ for which $\bigotimes_{i\in\omega}^{\omega} T_i'$ is contained in one piece of the partition.

It is seen from this example that the strong subtree version of the Halpern-Lauchli theorem does not work for infinitely many trees, and that in order to find a infinite subtree version of *HL* it was necessary to sacrifice either the 'strength' of the subtrees or, if strong subtrees were to be retained, the existence of a monochromatic level product of the subtrees. In other words, the successful candidate would need either a weaker notion of the subtree or a less comprehensive product.

The subtree formulation contrived by Halpern and Pincus weakened the idea of the product:

> Let $\langle T_i : i \in \omega\rangle$ be a sequence of finitistic trees, and let $\bigotimes_{i\in\omega}^{\omega} T_i = \bigcup_{j\in p} P_j$, $p < \omega$. Then there is an f-level family of strong subtrees T_i' of T_i, $i \in \omega$ and $j < \omega$, such that
>
> $$\forall d \in \omega \quad \forall^\infty n \in \omega \quad \forall \vec{x} \in \prod_{i\in d} T_i'(n) \quad \exists \vec{y} \in \prod_{i\geq d} T_i(f(n)) \quad \vec{x}^\wedge\vec{y} \in P_j.$$

Halpern and Pincus used this version to extend Theorem 5.2 in [5]. We will write $\mathbb{Q}^{\underline{\omega}}$ for the weak infinite power of $\mathbb{Q}$, that is to say all $\vec{x} \in \mathbb{Q}^\omega$ which are eventually zero.

> Let $\mathbb{Q}^\omega = \bigcup_{j<p} P_j$. Then there exists a sequence of sets of the order type of the rationals $\langle X_i : i \in \omega\rangle$ and finite $U_d \subseteq \mathbb{Q}^{\underline{\omega}}$, for $d < \omega$, and $j < p$

such that

$$\forall d \in \omega \quad \forall^\infty x_{d-1} \in X_{d-1} \ldots \forall^\infty x_0 \in X_0 \ \exists \vec{u} \in U_d \text{ such that } \vec{x}^\wedge \vec{u} \in P_j.$$

Two open problems exist with respect to this theorem: whether a singleton U_d can be found, and whether it is necessary to reverse the order of the X_i, $i \in d$, in the last line.

The Halpern-Pincus infinite version of the Halpern-Lauchli theorem seems to lack a certain esthetic appeal though. With a conjecture by Milliken in [10], the search for an infinite subtree version with a less awkward monochromatic level product continued. It was for this conjecture that the perfect tree was defined. The conjecture was that an infinite sequence of perfect trees, $\langle T_i : i \in \omega \rangle$, with its sequences of same-level nodes finitely colored, must have an infinite sequence of perfect subtrees T_i' of T_i whose level product, for some infinite set of levels, is monochromatic.

The conjecture, in other words, was what we have been referring to as $LP(\omega)$, and it inspired the reformulation of the Laver-Pincus finite matrix version as $LP(d)$. The infinite version was finally proved by Laver in [8]. It is not difficult to extend Harrington's application for $LP(d)$ using this theorem. In fact, Laver extended Theorem 5.5 to say the following.

> If $\langle f_i : I^\omega \to I \rangle_{i \in \omega}$ is a sequence of measurable functions or a sequence of functions with the Baire property, then there exist perfect $P_i \subseteq I$ for $i < \omega$ and a subsequence $\langle f_{i_n} \rangle$ whose restriction to $\prod_{i \in \omega} P_i$ is monotonically and uniformly convergent.

What is still lacking, though, at this point is an infinite matrix version. The counterexample to the posited infinite strong subtree version of Halpern-Lauchli shows also that we cannot find a level product matrix version (as the subtree version is a corollary to the matrix version). It presents no imposition, though, to proving the infinite version of the theorem which was the first of this kind: the Halpern-Lauchli theorem. We conclude our overview of this area with its most obdurate open problem:

HL(ω) Let $\langle T_i : i \in \omega \rangle$ be a sequence of perfect trees, and let $\prod_{i \in \omega} T_i = P_0 \cup P_1$. Then must one of the following be true?

1. For all k there exists a $(0, k)$-matrix in P_0.
2. There exists h such that for all k there is a (h, k)-matrix. in P_1.

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Notation