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Polynomial Inequalities and their Applications to Extension Problems for Classes of Differentiable Functions

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Polynomial Inequalities and their Applications to Extension Problems for Classes of Differentiable Functions" submitted by Gordon Kwok in partial fulfillment of the requirements for the degree of MASTER OF SCIENCE.

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Abstract

Polynomials play a significant role in many fields of mathematics from Algebraic Number Theory and Algebraic Geometry to Applied Analysis, Fourier Analysis, Convex Geometry and Computer Science. In many problems related to different areas of mathematics one often uses the, so-called, polynomial inequalities. Recently there has been considerable interest on extending the classical (Bernstein/Markov) polynomial inequalities to higher dimensional cases. Originally, such inequalities have appeared in Approximation Theory and for a long time have been considered as technical tools for proofs of Bernstein type direct and inverse theorems. At the present time polynomial inequalities have found a lot of important applications in areas which are well apart from Approximation Theory.

In the present work, we will survey different types of polynomial inequalities, both univariate and multivariate cases. Some proofs of the basic theorems will be presented, and all results are presentations of published results. Also, we present applications of the polynomial inequalities to some Whitney type problems on characterization of trace spaces for certain classes of differentiable functions.

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Chapter 1

INTRODUCTION

Polynomials play a significant role in many fields of mathematics from Algebraic Number Theory and Algebraic Geometry to Applied Analysis, Fourier Analysis, Convex Geometry and Computer Science. The Fundamental Theorem of Algebra and finding solutions of polynomial equations are basic examples of theorems and problems arising from the study of polynomials. It is worth noting that in many problems related to different areas of Analysis one often uses the so-called polynomial inequalities. Roughly speaking such inequalities estimate the growth of a polynomial in $\mathbb{R}^n$. For instance, in the one-dimensional case the natural question we may ask, is *how large can* $\|p\|_{[-1,1]} := \max_{x\in[-1,1]} |p(x)|$ *be if* p *is a real polynomial on* $\mathbb{R}$ *of degree* n *and*

$$|\{x \in [-1,1] : |p(x)| \leq 1\}| \geq 2 - s? \tag{1.0.1}$$

Here $|U|$ denotes the Lebesgue measure of $U \subset \mathbb{R}$, and $s \geq 0$ is a real number.

The classical Remez inequality [1] proved in 1936 gives an answer to this question. This result is formulated in the next section.

In general, an extension of a one-dimensional polynomial inequality to the multi-dimensional case or to a more sophisticated class of functions is highly non-trivial and requires some additional analytic and geometric arguments. Such extensions play a central role in the proof of other important inequalities, such as Bernstein, Markov, Nikolskii, and Schur type inequalities.

In this work we will present the highlights of the theory of Remez type inequalities.

1.1 History

One of the first polynomial inequalities was proved by Chebychev at the end of 19th century. It states that for every real polynomial p on $\mathbb{R}$ of degree n and a pair of intervals $[c,d] \subset [a,b]$ the inequality

$$\sup_{x\in[a,b]} |p(x)| \le T_n\left(\frac{2+\lambda}{\lambda}\right) \sup_{x\in[c,d]} |p(x)| \tag{1.1.1}$$

holds with $\lambda := \frac{d-c}{b-a}$ and $T_n(x) := \cos(n \arccos x)$, the Chebychev polynomial of degree n.

Bernstein, A. Markov, Remez who came after Chebychev, each proved a different type of polynomial inequality. For instance, the classical Bernstein inequality concerns polynomials of a complex variable $z \in \mathbb{C}$ of degree n. It states that for such a polynomial p and every $R > 1$

$$\sup_{|z|\le R} |p(z)| \le R^n \sup_{|z|\le 1} |p(z)|. \tag{1.1.2}$$

Let us note that inequalities (1.1.1) and (1.1.2) are sharp.

In turn, the classical A. Markov inequality compares the supremum norm of a real polynomial of degree n on an interval $[a, b]$ with the supremum norm of its derivative there:

$$\sup_{x\in[a,b]} |p'(x)| \le \left(\frac{2n^2}{b-a}\right) \sup_{x\in[a,b]} |p(x)|. \tag{1.1.3}$$

This inequality was proved by A. Markov as an answer to a question raised by the Russian chemist D. Mendeleev who applied its particular case (for polynomials of degree two) to investigate the perfect percentage of alcohol content for vodka. He discovered that it is 38%. However, since spirits in his time were taxed on their strength, the percentage was rounded up to 40 to simplify the tax computation.

In the 1920s-1930s Polya and Remez initiated the study of polynomial inequalities on measureable subsets in $\mathbb{R}$. As a result, Remez, in his 1936 paper, generalized the Chebychev inequality by replacing the subinterval $[c, d]$ by an arbitrary measurable subset $E \subseteq [a, b]$.

Theorem 1. *(Remez) For a measurable set $E \subseteq [a, b]$, $|E| > 0$, and a real polynomial p of degree n,*

$$\sup_{x\in[a,b]} |p(x)| \le T_n\left(\frac{2(b-a)}{|E|} - 1\right) \sup_{x\in E} |p(x)|. \tag{1.1.4}$$

Equality in (1.1.4) *holds if and only if* $E = [a, a+\delta]$ *and* $p(x) = AT_n(\frac{2(x-a)}{\delta} - 1)$ *or* $E = [b-\delta, b]$ *and* $p(x) = AT_n(\frac{2(b-x)}{\delta} - 1)$*, where* $A \in \mathbb{R}$ *and* $0 < \delta < b - a$.

A multivariate generalization of the Remez inequality was proved by Yu. Brudnyi and Ganzburg [2] in the 1970s.

Theorem 2. *(Brudnyi-Ganzburg) Let* $V \subset \mathbb{R}^d$ *be a convex body and* $\omega \subset V$ *be a measurable subset of Lebesgue measure* $|\omega| > 0$*. Then for a real polynomial* p *on* $\mathbb{R}^d$ *of degree* n*,*

$$\sup_V |p| \le T_n\left(\frac{1+\sqrt[d]{1-\lambda}}{1-\sqrt[d]{1-\lambda}}\right) \sup_\omega |p| \tag{1.1.5}$$

where $\lambda := \frac{|\omega|}{|V|}$.

This inequality coincides with the classical Remez inequality if $d = 1$ and is sharp in any dimension.

The further development of the theory of Remez type inequalities is due to Yu. Brudnyi [3, 4, 5], Ganzburg [8], Erdelyi and Borwein [56], A. Brudnyi [5, 6, 7, 20, 21, 27, 28], C. Fefferman and Narasimhan [17], [18], Roytwarf and Yomdin [9] and many other mathematicians.

1.2 Application

Polynomial inequalities work as a main tool in different areas of mathematics. Originally, univariate Bernstein and Markov type inequalities for polynomials have appeared in Approximation Theory and for a long time have been considered as technical tools for proofs of Bernsteins type direct and inverse theorems. At the present time polynomial inequalities have found a lot of important applications in areas which are well apart from Approximation Theory. We will only briefly mention several of these areas.

The papers of V. Milman, Gromov [10], Bourgain [11], Kannan, Lovász, and Simonovits [12] apply polynomial inequalities with different integral norms to study some problems of Convex Geometry (in particular, the famous Slice Problem).

In the papers of Yu. Brudnyi, Pawlucki, Pleśniak [13, 14] and the books of DeVore and Sharpley [15] and Jonsson and Wallin [16] Chebychev-Bernstein and related Markov type inequalities are used to explore a wide range of properties of the classical spaces of smooth functions including Sobolev type embeddings and trace theorems, extensions and differentiablility.

The papers of C. Fefferman and Narasimhan [17, 18] on Bernstein's type inequalities for traces of polynomials to algebraic varieties were inspired by and would have important applications to some basic problems of the theory of subelliptic differential equations.

The paper of Bos, Levenberg, P. Milman, and Taylor [19] discovers a profound relation between the exponents in the tangential Markov inequalities for restrictions of polynomials to a smooth manifold $M \subset \mathbb{R}^d$ and the property of M to be an algebraic manifold.

Applications of polynomial inequalities to Cartwright type theorems for entire functions are presented in the papers of A. Brudnyi [20, 21], B. Levin [22], Logvinenko [23] and Katznel'son [24].

In the papers of Nazarov, Sodin and Volberg [25, 26] these inequalities are used to estimate the distribution of zeros of certain families of random analytic functions.

An application of polynomial inequalities to the second part of Hilbert's sixteenth problem concerning the number of limit cycles of planar polynomial vector fields was obtained by A. Brudnyi [27, 28].

Finally, in the papers of I. Vinogradov [29] a specific case of the upper estimates of trigonometric integrals based on Polya-type polynomial inequalities is obtained. Such estimates play an important role in some areas of Number Theory, Analysis (some problems of uniqueness and convergence of trigonmetric series, theory of orthogonal polynomials, differential properties of functions), Probability, and Mathematical Statistics.

1.3 Overview

In the present work, we will survey different types of polynomial inequalities, both univariate and multivariate cases. Some proofs of the basic theorems will be presented. Also, we will describe polynomial type inequalities for holomorphic functions. Finally we present applications of the polynomial inequalities to some problems of characterization of trace spaces for certain classes of differentiable functions.

Chapter 2

REMEZ-TYPE INEQUALITIES

2.1 Univariate Remez's Inequality

First, consider the Chebychev polynomials $T_n(x)$ which are defined as follows:

$$\begin{aligned} T_n(x) &:= \cos(n \arccos(x)), \quad x \in [-1,1], \quad \text{or, equivalently,} \\ T_n(x) &:= \frac{1}{2}\left(\left(x+\sqrt{x^2-1}\right)^n + \left(x-\sqrt{x^2-1}\right)^n\right), \quad x \in \mathbb{C}. \end{aligned} \tag{2.1.1}$$

The Chebychev polynomials were introduced by the famous Russian mathematician Pafnutii L'vovich Chebychev (1821 - 1894).

It has been shown already by Chebychev that $T_n(x)$ is the fastest growing polynomial outside $[-1,1]$. In other words,

$$\max\left\{|p(\xi)| \;:\; p \in P_n,\; \|p\|_{[-1,1]} \le 1\right\} = T_n(\xi), \quad \forall\, |\xi| \ge 1, \;\; \xi \in \mathbb{R} \tag{2.1.2}$$

where P_n is the set of all polynomials of degree n.

Let s be an arbitrary fixed positive number. For every $p \in P_n$ define the set

$$M(p) := \{x \in [-1, 1+s] : |p(x)| \le 1\}. \tag{2.1.3}$$

Clearly $M(p)$ consists of mutually disjoint closed subintervals of $[-1, 1+s]$ (some of these subintervals can be single points). Let $|M(p)|$ be the measure of $M(p)$. Next we consider

$$P_n(s) := \{p \in P_n : |M(p)| \ge 2\}. \tag{2.1.4}$$

In 1936, Remez [1] established the following

$$\sup_{p \in P_n(s)} \|p\| = \|T_n\|. \tag{2.1.5}$$

where $\|\cdot\|$ is the supremum norm over $[-1, 1+s]$, and, by the definition of Chebychev polynomials, $\|T_n\| = T_n(1+s)$.

This result implies the inequality of Theorem 1 of section 1. Indeed, let p be a polynomial of degree n on $[a,b] \subset \mathbb{R}$ and $E \subseteq [a,b]$ be a mesurable subset. We set

$$s := \frac{2(b-a)}{|E|} - 2$$

and consider a polynomial $\tilde{p}$ on $[-1, 1+s]$ defined by the formula

$$\frac{p(x)}{\sup_{t\in E}|p(t)|} := \tilde{p}\left(\frac{2(x-a)}{|E|} - 1\right), \quad x \in [a,b].$$

Then $M(\tilde{p})$ contains the set $\widetilde{E} := \{x \in [-1, 1+s] \ : \ x = \frac{2(t-a)}{|E|} - 1, \ t \in E\}$ of measure 2. Thus (2.1.5) implies in this case

$$\sup_{x\in[a,b]} |p(x)| := \left(\sup_{x\in[1,1+s]} |\tilde{p}(x)|\right)\left(\sup_{x\in E}|p(x)|\right) \leq$$

$$T_n(1+s) := T_n\left(\frac{2(b-a)}{|E|} - 1\right)\sup_{x\in E}|p(x)|$$

as is required.

So let us prove now (2.1.5).

Proof. We follow the proof given in [30]. Note that for any fixed $x \in [-1, 1+s]$ the quantity

$$\mu(x) := \sup\{|p(x)| : p \in P_n(s)\} \tag{2.1.6}$$

is attained for some polynomial from $P_n(s)$. We shall show first that $\mu(x) \leq \mu(1+s)$ for each $x \in [-1, 1+s]$. Indeed, let x be an interior point of $[-1, 1+s]$ and let p be the extremal polynomial for this point, i.e., $p \in P_n(s)$ and $|p(x)| = \mu(x)$. Introduce the polynomials

$$p_1(x) := p(\alpha(x)), \quad p_2(x) := p(\beta(x)), \tag{2.1.7}$$

where $\alpha : [-1, 1+s] \to [-1, x]$ and $\beta : [-1, 1+s] \to [x, 1+s]$ are the linear transformations. Let M_1 and M_2 be the parts of $M(p)$ situated in $I_1 := [-1, x]$ and $I_2 := [x, 1+s]$, respectively. Assuming that $|M_i| < \lambda |I_i|$ for $i = 1, 2$ and $\lambda = 2/(2+s)$, we would get $|M| = |M_1 + M_2| < \lambda |I_1 + I_2| = \lambda(2+s) = 2$, a contradiction. Therefore $|M_i|/|I_i| \geq \lambda$ at least for one i, say for $i = 1$. Then $|M(p_1)| \geq 2$ and hence $p_1 \in P_n(s)$. This yields

$$\mu(x) = |p(x)| = |p_1(1+s)| \leq \mu(1+s). \tag{2.1.8}$$

Therefore the Remez inequality will be proved if we show that

$$|p(1+s)| \leq T_n(1+s) \quad \forall p \in P_n(s). \tag{2.1.9}$$

In order to show this, denote by $-1 = \eta_0 < \eta_1 < \ldots < \eta_n = 1$ the extremal points of T_n. We have

$$T_n(\eta_k) = (-1)^{n-k} \quad k = 0, \ldots, n. \tag{2.1.10}$$

Let $x_0 < x_1 < \ldots < x_n$ be the points of $M(p)$ which coincide with $\eta_0, \ldots, \eta_n$ after we press $M(p)$ to the left, i.e., to the interval $[-1, M(p) - 1]$. By the Lagrange interpolation formula

$$|p(1+s)| \leq \sum_{k=0}^{n} \prod_{i=0, i \neq k}^{n} \frac{|1+s-x_i|}{|x_k - x_i|} \tag{2.1.11}$$

since $|p(x_i)| \leq 1$. Now taking into account the obvious inequalities $|1+s-x_i| \leq |1+s-\eta_i|$, $|x_k - x_i| \geq |\eta_k - \eta_i|$ and (2.1.10), we get

$$|p(1+s)| \leq \sum_{k=0}^{n} \prod_{i=0, i \neq k}^{n} \frac{|1+s-\eta_i|}{|\eta_k - \eta_i|} = T_n(1+s). \tag{2.1.12}$$

□

2.2 Multivariate Remez's Inequality

To prove the Brudnyi-Ganzburg inequality [2] (i.e., a multivariate version of the Remez inequality) formulated in Theorem 2 we will use the following geometric fact.

Fix an inner point x_0 of the body $V \subset \mathbb{R}^d$ and let $0 < \lambda \leq 1$. Let l stand for a ray emanating from x_0. By mes_1 we denote the linear Lebesgue measure on l. Consider the extreme problem

$$\gamma_d(\lambda) = \sup_{|\omega| \geq \lambda} \operatorname*{ess\,inf}_{l} \frac{mes_1(V \cap l)}{mes_1(\omega \cap l)} \tag{2.2.1}$$

where the sup is taken over all measurable $\omega \subset V$ satisfying

$$\frac{|\omega|}{|V|} \geq \lambda.$$

Lemma 1. *The following identity holds:*

$$\gamma_d(\lambda) = \frac{1}{1 - \sqrt[d]{1-\lambda}}. \tag{2.2.2}$$

Proof. Let us introduce in $\mathbb{R}^d$ a spherical system of coordinates with center x_0: $(r, \phi) = (r, \phi_1, \dots, \phi_{d-1})$. Let $r = H(\phi) = H(\phi_1, \dots, \phi_{d-1})$ be the equation of the surface of the boundary ∂V of V. Let us examine the set $\widetilde{\omega}$ which in the coordinates (r, ϕ) is defined by

$$\beta_d(\lambda) H(\phi) \leq r \leq H(\phi), \quad \beta_d(\lambda) := 1 - \sqrt[d]{1-\lambda}. \tag{2.2.3}$$

It is easy to calculate that $|\widetilde{\omega}| = \lambda$ and, for almost every ray l emanating from x_0,

$$\frac{mes_1(V \cap l)}{mes_1(\widetilde{\omega} \cap l)} = \frac{1}{1 - \beta_d(\lambda)}. \tag{2.2.4}$$

Therefore it remains to show that there is no set $\omega \subset V$ with $|\omega| \geq \lambda$ satisfying the inequality

$$\operatorname*{ess\,inf}_{l} \frac{mes_1(V \cap l)}{mes_1(\omega \cap l)} > \frac{1}{1 - \beta_d(\lambda)} \tag{2.2.5}$$

Suppose (2.2.5) holds for some ω, $|\omega| \geq \lambda$. Comparing (2.2.4) with (2.2.5) we obtain that for almost every l

$$mes_1(\widetilde{\omega} \cap l) = mes_1(\omega \cap l) + \epsilon(l) \tag{2.2.6}$$

where $\epsilon(l) > 0$. Using ω, we construct a "symmetrized" set $\omega^{(s)}$: On each such ray l we put on one side a segment of length $mes_1(\omega \cap l)$ in such a way that it lies in the set V and one of its ends coincides with the point of intersection of l with the boundary of V. By virtue of (2.2.6) the set $\omega^{(s)}$ lies strictly inside $\widetilde{\omega}$, and therefore $|\omega^{(s)}| < |\widetilde{\omega}| = \lambda$. On the other hand, by the monotonicity of r^{d-1}

$$\int_{\omega \cap l} r^{d-1} dr \leq \int_{\omega^{(s)} \cap l} r^{d-1} dr \tag{2.2.7}$$

and, integrating both sides with respect to ϕ, we conclude that $|\omega| \leq |\omega^{(s)}|$. Thus $|\omega| < \lambda$, which contradicts our assumption. Therefore the lemma is proved. □

Let us prove now Theorem 2.

Proof. Let $\omega \subset V$, $\frac{|\omega|}{|V|} \geq \lambda$ and $p \in P_{n,d}$ belong to the set of real polynomials on $\mathbb{R}^d$ of degree n. Assume that $||p||_{C(V)} := \sup_V |p| = |p(x_0)|$, $x_0 \in V$. Consider the restriction of p to a ray l emanating from x_0. Applying to the restriction Theorem 1, we have

$$||p||_{C(V)} = |p(x_0)| \leq T_n \left(\frac{2mes_1(V \cap l)}{mes_1(\omega \cap l)} - 1 \right) ||p||_{C(\omega \cap l)}. \tag{2.2.8}$$

Taking ess inf with respect to l on the right-hand side of the above inequality, and then sup over all measurable $\omega \subset V$, $\frac{|\omega|}{|V|} \geq \lambda$, we obtain (using monotonicity of $T_n(x)$ for $|x| \geq 1$) the inequality

$$||p||_{C(V)} \leq T_n \left(2\gamma_d(\lambda) - 1\right) ||p||_{C(\omega)}, \text{ or equivalently,}$$

$$||p||_{C(V)} \leq T_n \left(2 \left(\frac{1}{1 - \beta_d(\lambda)} \right) - 1 \right) = T_n \left(\frac{1 + \beta_d(\lambda)}{1 - \beta_d(\lambda)} \right).$$

This completes the proof of Theorem 2.

□

Let us show that the inequality of Theorem 2 is sharp on the class of compact convex bodies. To this end, we define V to be a circular cone of height one, say

$$V := \left\{ x \in \mathbb{R}^d \ : \ x_1^2 \le \sum_{j=2}^{d} x_i^2, \quad 0 \le x_1 \le 1 \right\}.$$

Fix $\lambda \in (0,1)$ and let V_h be a subcone of V of height h where $h \in (0,1)$ is determined by the condition $|V \setminus V_h| = \lambda |V|$; then $h = \sqrt[d]{1-\lambda}$. Set now $S := V \setminus V_h$ and let

$$p(x) := T_n \left(\frac{2x_1 - 1 - h}{1-h} \right)$$

be the Chebychev polynomial associated to interval $[h, 1]$. Then $|S| = \lambda |V|$ and

$$\max_V |p| = T_n \left(\frac{1+h}{1-h} \right) = T_n \left(\frac{1+\sqrt[d]{1-\lambda}}{1-\sqrt[d]{1-\lambda}} \right) \max_S |p|,$$

that is, the inequality of Theorem 2 becomes equality.

Corollary 1. *Under the assumption of Theorem 2,*

$$\max_V |p| \le \frac{1}{2} \left(\frac{4d}{\lambda} \right)^k \max_S |p|.$$

Proof. The function $\lambda \mapsto 1 - \sqrt[d]{1-\lambda}$ is convex on $(0,1]$ and therefore

$$\frac{1+\sqrt[d]{1-\lambda}}{1-\sqrt[d]{1-\lambda}} \le \frac{2d}{\lambda} - 1.$$

This, the definition of T_n and its monotonicity on $[1, \infty)$ imply the result.

□

2.3 Remez type inequality for integral norms

The inequality of Theorem 2 may be generalized to integral norms as follows.

Corollary 2. *Let $0 < r \le q \le \infty$ and let S be a subset of a convex body $V \subset \mathbb{R}^d$ of relative measure $\lambda := \frac{|S|}{|V|} \in (0,1]$. Then for every polynomial p of degree n the inequality*

$$\left\{ \frac{1}{|V|} \int_V |p|^q \, dx \right\}^{\frac{1}{q}} \le (rn+1)^{\frac{1}{r}} \gamma(n,d) \lambda^{-k} \left\{ \frac{1}{|S|} \int_S |p|^r \, dx \right\}^{\frac{1}{r}} \tag{2.3.1}$$

holds with $\gamma(n,d) := \frac{1}{2}(4d)^n$.

Proof. It suffices to consider the case $q = \infty$. Due to the homogeneity of (2.3.1) we may assume that

$$\max_V |p| = 1.$$

Further, for $t \in (0, 1]$, define the level set for p by

$$L_t := \{x \in V \ : \ |p(x)| \le t\}.$$

Applying to this subset inequality of Corollary 1 we get

$$1 = \max_V |p| \le \gamma(n, d) \left(\frac{|V|}{|L_t|}\right)^n \cdot t$$

and then derive from here the inequality

$$|L_t| \le |V| (\gamma(n, d) t)^{\frac{1}{n}}. \tag{2.3.2}$$

To proceed we need the notion of *rearrangement*, see, e.g., [54], section 1.8.

Let (Σ, μ) be a measure space and $f : \Sigma \to \mathbb{R}$ be a μ-measurable function. A nonincreasing function $m(f) : (0, \infty) \to \mathbb{R}_+ \cup \{\infty\}$ is then given by

$$m(f; t) := \mu\{\sigma \in \Sigma \ : \ |f(\sigma)| > t\},$$

while the rearrangement $f^* : (0, \mu(\Sigma)] \to \mathbb{R}_+ \cup \{\infty\}$ is defined by

$$f^* := \inf\{t \ : \ m(f; t) \le s\}.$$

Functions f and f^* are *equimeasurable*; therefore, for $0 < r < \infty$,

$$\int_0^{\mu(\Sigma)} (f^*(s))^r \, ds = \int_\Sigma |f|^r \, d\mu.$$

Using these definitions we relate $|L_t|$ to the rearrangement of the restriction $p|_V$. Actually,

$$|L_t| = |V| - m(p|_V; t)$$

and therefore the inverse to the function $t \mapsto |L_t|$ is equal to $t \mapsto (p|_V)^*(|V| - t)$. This inverse is estimated by (2.3.2) to give

$$(p|_V)^*(|V| - t) \ge \frac{1}{\gamma(n, d)} \left(\frac{t}{|V|}\right)^n.$$

It remains to note that for $S \subset V$ and $0 \le t \le |S|$,

$$(p|_S)^*(t) \ge (p|_V)^*(t)$$

and therefore

$$\int_0^{|S|} \left[\frac{1}{\gamma(n,d)} \left(\frac{t}{|V|} \right)^k \right]^r dt \le \int_0^{|S|} [(p|_S)^*(|S|-t)]^r \, dt = \int_0^{|S|} [(p|_S)^*(t)]^r dt = \int_S |p|^r \, dx.$$

Integrating and raising to the power $\frac{1}{r}$ we get the inequality

$$\frac{1}{(rn+1)^{\frac{1}{r}}} \cdot \frac{1}{\gamma(n,d)} \left(\frac{|S|}{|V|} \right)^n \le \left(\frac{1}{|S|} \int_S |p|^r \, dx \right)^{\frac{1}{r}}$$

which is equivalent to (2.3.1) with $q = \infty$. □

Remark 1. Let $S = V$; then (2.3.1) yields the *inverse Hölder inequality* for polynomials. The constant obtained is, up to a numerical factor, optimal for $r \le 1$ and $q = \infty$, but may be essentially improved for other values of r and q, see the paper [55] by Carberry and J. Wright, and references therein.

Chapter 3

REMEZ TYPE INEQUALITIES FOR HOLOMORPHIC FUNCTIONS

The purpose of this chapter is to describe Remez type inequalities for holomorphic functions. We define the local degree of a holomorphic function which expresses its geometric properties and generalizes the degree of a polynomial. This notion is central in our consideration. It allows us to obtain better constants in Remez type inequalities even in the standard polynomial case.

We proceed with the formulation of the main results of this chapter.

3.1 A generalized Remez inequality

Let $B_c(0,1) \subset B_c(0,r) \subset \mathbb{C}^n$ be a pair of open complex Euclidean balls of radii 1 and r respectively centered at 0. Denote by $\mathcal{O}_r$ the set of holomorphic functions defined on $B_c(0,r)$. Let $l_x \subset \mathbb{C}^n(=\mathbb{R}^{2n})$ be a real straight line passing through $x \in B_c(0,1)$. Further, let $I \subset l_x \cap B_c(0,1)$ be an interval and $\omega \subset I$ be a measurable subset.

Theorem 3. *For any $f \in \mathcal{O}_r$, there is a constant $d = d(f,r) > 0$ such that for any $\omega \subset I \subset l_x \cap B_c(0,1)$*

$$\sup_I |f| \le \left(\frac{4|I|}{|\omega|}\right)^d \sup_\omega |f|. \tag{3.1.1}$$

Example 1. *As an application of the above theorem we obtain local inequalities for quasipolynomials.*

Definition 1. *Let $f_1, \dots, f_k \in (\mathbb{C}^n)^*$ be complex linear functionals. A quasipolynomial with spectrum $f_1, \dots, f_k$ is a finite sum*

$$f(z) = \sum_{i=1}^{k} p_i(z) e^{f_i(z)} \tag{3.1.2}$$

where $p_i \in \mathbb{C}[z_1, \dots, z_n]$. The expression $\sum_{i=1}^{k}(1 + deg(p_i))$ is said to be the degree of f.

Proposition 1. *Let f be a quasipolynomial of degree m and l_x be a real straight line passing through $x \in B_c(0,1)$. Then there is an absolute constant $c > 0$ such that the inequality*

$$\sup_I |f| \le \left(\frac{4|I|}{|\omega|}\right)^{c(\sqrt{k}M+m)} \tag{3.1.3}$$

holds for any interval $I \subset l_x \cap B_c(0,1)$ and any measurable subset $\omega \subset I$. Here $M :=$ $\max_i \left\{ \|f_i\|_{L^2(\mathbb{C}^n)} \right\}$.

Definition 2. *The best constant d in inequality* (3.1.1) *will be called the Chebyshev degree of the function $f \in \mathcal{O}_r$ in $B_c(0,1)$ and will be denoted by $d_f(r)$.*

All constants in the inequalities formulated below depend upon the possibility to obtain an effective bound of the Chebyshev degree in (3.1.1). The following result gives such a bound in terms of the local geometry of f.

We say that a univariate holomorphic function f defined in a disk is p - valent if it assumes no value more than p-times there. We also say that f is 0-valent if it is a constant. For any $t \in [1, r)$ let L_t denote the set of one-dimensional complex affine spaces $l \subset \mathbb{C}^n$ such that $l \cap B_c(0,t) \ne \emptyset$.

Definition 3. *Let $f \in \mathcal{O}_r$. The number*

$$v_f(t) := \sup_{l \in L_t} \left\{ \text{valency of } f|_{l \cap B_c(0,t)} \right\} \tag{3.1.4}$$

is said to be the valency of f in $B_c(0,t)$.

Proposition 2. *For any $f \in \mathcal{O}_r$ and any t, $1 \le t < r$, the valency $v_f(t)$ is finite. There is a constant $c = c(r) > 0$ such that $d_f(r) \le c v_f\left(\frac{1+r}{2}\right)$.*

Remark 2. *For any holomorphic polynomial $p \in \mathbb{C}[z_1, \dots, z_n]$ of degree at most k the classical Remez inequality implies $d_p(r) \le k$ while in many cases Proposition 2 yields a sharper estimate.*

From Theorem 2, one can obtain the following inequality

$$\sup_V |p| \le \left(\frac{4n|V|}{|\omega|}\right)^k \sup_\omega |p|. \tag{3.1.5}$$

In this section we formulate a generalization of inequality (3.1.5). Let $B(0,1) \subset B_c(0,1)$ be the real Euclidean unit ball.

Theorem 4. *For any convex body $V \subset B(0,1)$, any measurable subset $\omega \subset V$, $|\omega| > 0$, and any $f \in \mathcal{O}_r$ the inequality*

$$\sup_V |f| \leq \left(\frac{4n|V|}{|\omega|}\right)^{d_f(r)} \sup_\omega |f| \tag{3.1.6}$$

holds.

The following corollary is a version of the log-BMO-property for analytic functions [6, 57]

Corollary 3. *Under the hypothesis of Theorem 4 the inequality*

$$\frac{1}{|V|}\int_V \left|\ln \frac{|f|}{\|f\|_V}\right| dx \leq Cd_f(r)\ \ln(n) \tag{3.1.7}$$

holds with an absolute constant C, where $\|f\|_V := \sup_V |f|$.

Our next application of inequality (3.1.1) is a generalization of Bourgain's polynomial inequality [11].

Theorem 5. *Let $V \subset B(0,1)$ be a convex body and $\widetilde{d_f}(r)$ be the smallest integer $\geq d_f(r)$. There are positive absolute constants c_1, c_2 such that the following inequality*

$$\left|\left\{x \in V : |f(x)| > \frac{\lambda}{|V|}\int_V |f(x)|dx\right\}\right| \leq c_1\ exp(-\lambda^{c_2/\widetilde{d_f}(r)})|V| \tag{3.1.8}$$

holds for any $f \in \mathcal{O}_r$. In particular,

$$\|f\|_{L^\Phi(V,dx)} \leq (c_1+1)\,\|f\|_{L^1(V,dx)} \tag{3.1.9}$$

where L^Φ refers to the Orlicz space with the Orlicz function $\Phi(t) = exp(t^{c_2/\widetilde{d_f}(r)}) - 1$.

Let us recall that an Orlicz space is a type of a function space which generalizes L^p spaces. The spaces are named for W. Orlicz who discovered them in 1931. Here is the definition of them.

Suppose that μ is a σ-finite measure on a set X, and $\Phi : [0,\infty) \to [0,\infty)$ is a convex function such that

$$\frac{\Phi(x)}{x} \to \infty, \quad \text{as} \quad x \to \infty,$$
$$\frac{\Phi(x)}{x} \to 0, \quad \text{as} \quad x \to 0+.$$

Let L_*^Φ be the space of measurable functions $f : X \to \mathbb{R}$ such that the integral

$$\int_X \Phi(|f|)d\mu < \infty,$$

where as usual functions which agree almost everywhere are identified.

This may not be a vector space (it may fail to be closed under scalar multiplication). The vector space of functions spanned by L_*^Φ is the Orlicz space, denoted L^Φ. To define a norm on L^Φ, let Ψ be the Young complement of Φ; that is,

$$\Psi(x) = \int_0^x (\Phi')^{-1}(t)dt.$$

The norm is then given by

$$||f||_\Phi = \sup\left\{||fg||_{L^1} \ : \ \int_X \Psi \circ |g| d\mu \le 1\right\}.$$

Furthermore, the space L^Φ is precisely the space of measurable functions for which this norm is finite.

An equivalent norm is defined on L^Φ by

$$||f||'_\Phi := \inf\left\{k \in (0,\infty) \ : \ \int_X \Phi(|f|/k)d\mu \le 1\right\}.$$

Orlicz spaces generalize L^p spaces in the sense that if $\Phi(t) = t^p$, then $||u||_\Phi = ||u||_{L^p}$, so $L^\Phi(X) = L^p(X)$.

Remark 3. *The original Bourgain's inequality for polynomials contains the degree of the polynomial instead of* $\widetilde{d}_f(r)$.

As a corollary of inequality (3.1.1) we also obtain the reverse Hölder inequality with constant which does not depend on the dimension (this result does not follow from Theorem 4).

Corollary 4.

$$\left(\frac{1}{|V|}\int_V |f(x)|^s dx\right)^{1/s} \le c(\widetilde{d}_f(r), s)\frac{1}{|V|}\int_V |f(x)|dx \quad (f \in \mathcal{O}_r, s \in \mathbb{Z}_+) \qquad (3.1.10)$$

The following example shows that in the polynomial case our inequalities might be sharper than those of [2, 11].

Example 2. *Let $f \in \mathcal{O}_r$ be such that $\sup_{B_c(0,r)} |f| < 1$. Let ϕ be a holomorphic non-polynomial function univalent in an open neighbourhood U of $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \le 1\}$. Then using Proposition 2 and Proposition 3 below yields $d_{\phi\circ f}(r) \le c(r)v_f\left(\frac{1+r}{2}\right)$. Consider a polynomial approximation h_k of ϕ such that deg $h_k = k$ and h_k is also univalent on $\mathbb{D}$. Assume now that $f \in \mathcal{O}_r$ is a polynomial. Then deg$(h_k \circ f) = k \cdot$ deg f. Further, apply Brudnyi-Ganzburg and Bourgain's polynomial inequalities to the polynomial $h_k \circ f$. Then the exponents in these inequalities will be equivalent to $k \cdot$ deg f and $1/(k \cdot$ deg $f)$, respectively. However, in our generalizations of the above inequalities these exponents contain numbers $d_{h_k\circ f}(r)$ and $1/\widetilde{d}_{h_k\circ f}(r)$ with $d_{h_k\circ f}(r) \le c(r)$deg f and this is essentially better for all sufficiently large k.*

3.2 Proofs of Theorem 4 and Proposition 2

We begin with some auxiliary results used in the proof.

3.2.1 Parametrization of straight lines in the ball

Let $B_c(0,s)$, $1 < s < r$, be an open complex Euclidean ball. For any $x \in B_c(0,s)$ consider the complex straight line $l_{x,v} = \left\{x + vz\sqrt{s^2 - |x|^2}; \langle x, v\rangle = 0, |v| = 1, z \in \mathbb{C}\right\}$ passing through x. Here $|\cdot|$ denotes the Euclidean norm and $\langle\cdot,\cdot\rangle$ the inner product on $\mathbb{C}^n$. In this way we parametrize the set L_s of all complex straight lines passing through points of $B_c(0,s)$. Let f be a holomorphic function from $\mathcal{O}_r$. Consider the function

$$F(z, x, v, s) = f(x + vz\sqrt{s^2 - |x|^2}) \quad (z \in \mathbb{D}). \qquad (3.2.1)$$

Then $F(\cdot, x, v, s)$ is the restriction of f to $l_{x,v} \cap B_c(0,s)$. Note also that for any $t < s$ the inequality

$$\frac{s^2 - |x|^2}{t^2 - |x|^2} \geq \left(\frac{s}{t}\right)^2 \tag{3.2.2}$$

holds. This implies that the set $\left\{x + vz\sqrt{s^2 - |x|^2}; \langle x, v\rangle = 0, |v| = 1, z \in \frac{t}{s}\mathbb{D}\right\}$ contains disk $l_{x,v} \cap B_c(0,t)$. Set

$$M(x, v, s, t) = \sup_{\frac{t}{s}\mathbb{D}} \ln |F(\cdot, x, v, s)|\,. \tag{3.2.3}$$

Definition 4. *The number*

$$b_f(s, t, r) := \sup_{x,v} \{M(x, v, s, t) - M(x, v, s, 1)\} \tag{3.2.4}$$

is said to be the Bernstein index of $f \in \mathcal{O}_r$.

3.2.2 Bernstein index and Remez inequality

Assume that $F(\cdot, x, v, s)(= f|_{l_{x,v} \cap B_c(0,s)})$ has valency m on $\frac{t}{s}\mathbb{D}$. Assume also that $1 < t < s$. By Theorem (2.1.3) and Corollary (2.3.1) of [9] (see also [6] Lemma (3.1)), there is a constant $A = A(t) > 0$ such that

$$M(x, v, s, \frac{1+t}{2}) - M(x, v, s, 1) \leq Am. \tag{3.2.5}$$

Then we apply the main inequality of Theorem (1.1) of [6] to the function $|F|$ obtaining that there is a constant $c = c(t, A) > 0$ such that the inequality

$$\sup_{I'} |F| \leq \left(\frac{4|I'|}{|\omega'|}\right)^{cm} \sup_{\omega'} |F| \tag{3.2.6}$$

is valid for any interval $I' \subset [-1/s, 1/s]$ and any measurable set $\omega' \subset I'$.

Since $l_{x,v} \cap B_c(0,1) \subset \left\{x + vz\sqrt{s^2 - |x|^2}; \langle x, v\rangle = 0, |v| = 1, z \in \frac{1}{s}\mathbb{D}\right\}$, (3.2.6) implies inequality (3.1.1) with exponent cm for f restricted to the real straight line $l_x \subset l_{x,v}$.

3.2.3 Proofs of Theorem 4 and Proposition 2

Proof. Let $1 < t < r$ and $f \in \mathcal{O}_r$. First we prove inequality $v_f(t) < \infty$.

Fix a number s satisfying $t < s < r$. For any $x \in B_c(0,s)$ consider the complex straight line $l_{x,v} = \left\{x + vz\sqrt{s^2 - |x|^2}; \langle x, v\rangle = 0, |v| = 1, z \in \mathbb{C}\right\}$ passing through x. Let $K := \{(x,v) \in B_c(0,s) \times S^{2n-1}; \langle x,v\rangle = 0\}$. Further, for $f \in \mathcal{O}_r$ consider the function F defined by Definition 4. Then F is ananlytic on $\mathbb{D} \times K$ and $F(\cdot, x, v, s)$ is holomorphic on $\mathbb{D}$ for any $(x,v) \in K$. Let $K_1 \subset K$ be a compact subset that consists of points with the first coordinate from $\overline{B_c(0,t)}$. In particular, the set of lines $l_{x,v}$ with $x \in B_c(0,t)$ coincides with L_t (defined just before Definition 3). Assume without loss of generality that $\sup_{B_c(0,s)} |f| = 1$ and consider the analytic function $\widetilde{F}(z,x,v,s,w) = F(z,x,v,s) - w$ defined on $\mathbb{D} \times K \times 2\mathbb{D}$. Set

$$
\begin{aligned}
f_1(x,v,r,w) &= \sup_{z \in \frac{2t}{t+s}\mathbb{D}} \ln |\widetilde{F}(z,x,v,s,w)| \\
f_2(x,v,r,w) &= \sup_{z \in \frac{t}{s}\mathbb{D}} \ln |\widetilde{F}(z,x,v,s,w)|.
\end{aligned}
\tag{3.2.7}
$$

Fix $(x,v,w) \in K_1 \times \overline{\mathbb{D}}$. If $\widetilde{F}(\cdot, x, v, s, w)$ is not a constant then the number of its zeros in $\frac{t}{s}\overline{\mathbb{D}}$ is estimated by the Jensen inequality

$$
\#\left\{z \in \frac{t}{s}\overline{\mathbb{D}} : \widetilde{F}(z,x,v,s,w) = 0\right\} \le c'(f_1(x,v,r,w) - f_2(x,v,r,w)) \tag{3.2.8}
$$

with $c' = c'(s,t) > 0$. Note also that by (3.2.2), the above number of zeros gives an upper bound for the number of points $y \in l_{x,v} \cap B_c(0,t)$ such that $f(y) = w$. Since $K_1 \times \overline{\mathbb{D}}$ is compact, the Bernstein theorem of C. Fefferman and R. Narasimhan [58] and the Hadamard three-circle theorem imply that there is a constant $C = C(\widetilde{F}, K_1 \times \overline{\mathbb{D}}) > 0$ such that

$$
f_1(x,v,r,w) - f_2(x,v,r,w) \le C \tag{3.2.9}
$$

for any $(x,v,w) \in K_1 \times \overline{\mathbb{D}}$. This inequality yields $v_f(t) \le c'C$ (see Definition 3).

It remains to prove the inequality $d_f(r) \le c(r) v_f\left(\frac{1+r}{2}\right)$. We will do it in a parallel way with the proof of Theorem 4.

Let $x \in B_c(0,1)$ and $l_x \subset \mathbb{C}^n$ be a real straight line passing through x. Let $I \subset l_x \cap B_c(0,1)$ be an interval and $\omega \subset I$ be a measurable subset. Set $s = \frac{1+r}{2}$, $t = \frac{1+s}{2}$ and denote by $l_x^c = \left\{y + vz\sqrt{s^2 - |y|^2}; \langle y, v\rangle = 0, |v| = 1, z \in \mathbb{C}\right\}$ the complex straight line containing l_x, where $y \in l_x$ is such that $dist(0, l_x) = |y|$. By definition function $F(\cdot, y, v, s) = f|_{l_x^c \cap B_c(0,s)}$ determined by (3.2.1) has valency $\le v_f(s)$ on $\frac{t}{s}\mathbb{D}$. Therefore Bernstein index $b_f(s, \frac{1+t}{2}, r) \le Av_f(s)$ for $A = A(r) > 0$. Finally, inequality (3.2.6) and arguments of section (3.2.2) show that the inequality of Theorem 4 is valid with $d \le cv_f(s)$, $c = c(r) > 0$. This implies that

$$d_f(r) \le cv_f\left(\frac{1+r}{2}\right). \tag{3.2.10}$$

□

Remark 4. *In order to estimate the Chebyshev degree we can also use instead of* $v_f(\frac{1+r}{2})$ *an appropiriate Bernstein index* $b_f(r) = b_f(s(r), t(r), r)$. *Then* $d_f(r) \le \widetilde{c}b_f(r) \le cv_f(\frac{1+r}{2})$ *with some* $\widetilde{c} = \widetilde{c}(r) > 0$.

3.3 Properties of Chebyshev Degree

We formulate further inequalities between the Chebyshev degree and valency. In the following proposition the constant $c = c(r)$ is the same as in Proposition 2.

Proposition 3. *1. Let* $f \in \mathcal{O}_r$ *and* $f(B_c(0,r)) \subset \mathbb{D} \subset \mathbb{C}$. *Let* ϕ *be a holomorphic function defined in an open neighbourhood* $U \supset \overline{\mathbb{D}}$. *Assume the* ϕ *has valency* k *in* U. *Then*

$$d_{\phi\circ f}(r) \le ckv_f\left(\frac{1+r}{2}\right). \tag{3.3.1}$$

2. Let $h := e^g \in \mathcal{O}_r$. *Then*

$$d_{1/h}(r) \le cv_h\left(\frac{1+r}{2}\right). \tag{3.3.2}$$

3. There is a constant $c_1 = c_1(r) > 0$ *such that*

$$d_{fg}(r) \le c_1\left(v_f\left(\frac{1+r}{2}\right) + v_g\left(\frac{1+r}{2}\right)\right) \tag{3.3.3}$$

for any $f, g \in \mathcal{O}_r$.

Consider the differential operator $(a, D) = \sum_{i=1}^{n} a_i D_i$, where $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$, $D_i := \frac{d}{d_{z_i}}$, $i = 1, \ldots, n$ and $z_1, \ldots, z_n$ are coordinates on $\mathbb{C}^n$. Set $f_{m,a} := (a, D)^m(f)$

Proposition 4 (The Rolle Theorem). *Let $f \in \mathcal{O}_r$. Assume that for any $a \in \mathbb{C}^n$ the valency of $f_{m,a}$ satisfies $v_{f_{m,a}}\left(\frac{1+3r}{4}\right) \leq M$. Then there is a constant $c_2 = c_2(r) > 0$ such that*

$$d_f(r) \leq c_2\,(m + M)\,. \tag{3.3.4}$$

Proof of Proposition 3. 1. According to the definition of the valency we have $v_{\phi \circ f}\left(\frac{1+r}{2}\right) \leq k v_f\left(\frac{1+r}{2}\right)$, where k is valency of ϕ. Then $d_{\phi \circ f}(r) \leq c k v_f\left(\frac{1+r}{2}\right)$ by Proposition 2.

2. The statement follows from Proposition 2 and the identity $v_{1/h}\left(\frac{1+r}{2}\right) = v_h\left(\frac{1+r}{2}\right)$ for $h = e^g$.

3. According to the results of Section 3.2.1 it suffices to prove the statement for univariate holomorphic functions $F(\cdot, x, v, s) = f|_{l_{x,v}}$ and $G(\cdot, x, v, s) = g|_{l_{x,v}}$. We consider a more general situation.

Assume that $\mathbb{D}_{r_1} \subset \mathbb{D}_{r_2} \subset \mathbb{C}$, $r_1 < r_2$, are disks centered at 0 of radii r_1, r_2, respectively. Further, assume that f, g are holomorphic in $\mathbb{D}_{r_2}$ of valency a and b, respectively. We prove that there is a constant $c = c(r_1, r_2) > 0$ such that Chebyshev degree $d_{fg}(r_1)$ of fg in $\mathbb{D}_{r_1} \leq c(a + b)$. Let $K = \left\{z \in \mathbb{C} : \frac{r_1+r_2}{2} \leq |z| \leq \frac{r_1+3r_2}{4}\right\}$ be an annulus in $\mathbb{D}_{r_2}$ and

$$g' = \frac{\ln|g| - \sup_{\mathbb{D}_{r_2}} \ln|g|}{\sup_{\mathbb{D}_{r_2}} \ln|g| - \sup_{\mathbb{D}_{r_1}} \ln|g|}. \tag{3.3.5}$$

Repeating word-for-word the arguments of Lemma 2.3 of [6] we can find a number $C = C(r_1, r_2) > 0$ and a circle $S \subset K$ centered at 0 such that

$$\inf_S g' \geq -C. \tag{3.3.6}$$

Going back to $|g|$ we obtain

$$\inf_S |g| \geq \sup_{\mathbb{D}_{r_2}} |g| \left(\frac{\sup_{\mathbb{D}_{r_1}} |g|}{\sup_{\mathbb{D}_{r_2}} |g|}\right)^C. \tag{3.3.7}$$

This implies

$$\frac{\sup_{\mathbb{D}_{r_2}}|fg|}{\sup_S|fg|} \le \frac{\sup_{\mathbb{D}_{r_2}}|f|\sup_{\mathbb{D}_{r_2}}|g|}{\sup_S|f|\inf_S|g|} \le \frac{\sup_{\mathbb{D}_{r_2}}|f|}{\sup_S|f|}\cdot\left(\frac{\sup_{\mathbb{D}_{r_2}}|g|}{\sup_{\mathbb{D}_{r_1}}|g|}\right)^C. \tag{3.3.8}$$

Finally, according to Section 3.2.2, there is a constant $B = B(r_1, r_2) > 0$ such that

$$\frac{\sup_{\mathbb{D}_{r_2}}|f|}{\sup_S|f|} \le \frac{\sup_{\mathbb{D}_{r_2}}|f|}{\sup_{\mathbb{D}_{\frac{r_1+r_2}{2}}}|f|} \le B^a \quad and \quad \frac{\sup_{\mathbb{D}_{r_2}}|g|}{\sup_{\mathbb{D}_{r_1}|g|}} \le B^b. \tag{3.3.9}$$

Thus we get

$$\frac{\sup_{\mathbb{D}_{r_2}}|fg|}{\sup_{\mathbb{D}_{\frac{r_1+r_2}{2}}}|fg|} \le \frac{\sup_{\mathbb{D}_{r_2}}|fg|}{\sup_S|fg|} \le \widetilde{B}^{a+b} \tag{3.3.10}$$

with $\widetilde{B} = \widetilde{B}(r_1, r_2, B) > 0$. Then inequality (3.2.6) applied to $|fg|$ implies the inequality of Theorem 3 with exponent $c(a+b)$, $c = c(r_1, r_2, B) > 0$. Therefore $d_{fg}(r_1) \le c(a+b)$.

In the multivariate case the above arguments estimate an appropriate Bernstein index of fg by sum of Bernstein indeces of f and g. These indeces can be estimated by $c_1 v_f(\frac{1+r}{2})$ and $c_1 v_g(\frac{1+r}{2})$ with some $c_1 = c_1(r) > 0$. Thus according to Remark 4, $d_{fg}(r) \le c'(r)(v_f(\frac{1+r}{2}) + v_g(\frac{1+r}{2}))$. This completes the proof of (3).

Proposition 3 is proved. □

Proof of Proposition 4. First, we recall the relation between Bernstein index and Bernstein classes.

Definition 5. *Let $f(z) = \sum_{i=0}^{\infty} a_i z^i$ be holomorphic in the disk $\mathbb{D}_R$, $R > 1$. We say that f belongs to the Bernstein class $B^2_{N,R,c}$, if for any $j > N$,*

$$|a_j|R^j \le c \max_{0\le i\le N} |a_i|R^i. \tag{3.3.11}$$

According to Corollary 2.3.1 of [9], if the m^{th} derivative $f^{(m)}$ of f is M-valent then $f^{(m+1)} \in B^2_{M-1,\frac{1+3R}{4},c^M}$ with $c := c(R) > 0$. Moreover, from Definition 5 it follows that $f \in B^2_{m+M,\frac{1+3R}{4},c^M}$. Then Theorem 2.1.3 of [9] based on the last implication yields

$$\sup_{\mathbb{D}_{\frac{1+R}{2}}} \leq a^{m+M} \sup_{\mathbb{D}_1} |f| \tag{3.3.12}$$

for some constant $a = a(R) > 1$.

We proceed with the proof of the proposition. As in the proof of Proposition 3 it suffices to prove the result for restriction F_l of f to a complex line l passing through a point of $B_c(0,1)$. Then the condition of the proposition implies that m^{th} derivative $F_l^{(m)}$ of F_l has valency at most M in the larger disk $l \cap B_c(0, \frac{1+3R}{4})$. Therefore the required result follows immediately from inequality (3.3.12) (an estimate for Bernstein index) and arguments of section 3.2.2.

The proof of the proposition is complete. $\square$

3.4 Proofs

Proof of Proposition 1. Let $l_y^c = \left\{y + vz\sqrt{4-|y|^2}; \langle y, v\rangle = 0, |v| = 1, z \in \mathbb{C}\right\}$ be a complex straight line passing through a point $y \in B_c(0,1)$. Consider the restriction F of the quasipolynomial $f(z) = \sum_{i=1}^k p_i(z)e^{f_i(z)}$ to l_y^c. Then F is a univariate quasipolynomial of the form

$$F(z) = q_i(z)e^{f_i(y)}e^{z\sqrt{4-|y|^2}f_i(v)} \quad (q_i \in \mathbb{C}[z]) \tag{3.4.1}$$

of degree $\leq m$. We estimate valency of F in disk $\mathbb{D}_2 := 2\mathbb{D}$ (i.e. we estimate the number of zeros of $F + c$ for any $c \in \mathbb{C}$). Note that $F + c$ is also a quasipolynomial of degree $\leq m+1$. Further, by definition $\max_i \{|f_i(v)|\} \leq M$ implying $\sqrt{4-|y|^2}f_i(v) \in \mathbb{D}_{2M}$ for any i. Then by Theorem 2 in [59] the number of zeros of $F + c$ in $\mathbb{D}_2$ less than or equal to $m + \frac{2}{\pi}(\sqrt{k+1}+1)\cdot 16M < 32(\sqrt{k+1}M + m)$. This and Proposition 2 yields

$$d_f(2) \leq cv_f(3/2) \leq c'(\sqrt{k+1}M + m) \tag{3.4.2}$$

with an absolute constant $c' > 0$. The required inequality follows from the definition of Chebyshev degree. $\square$

Proof of Theorem 4. Let $V \subset B(0,1)$ be a convex body, $\lambda \subset V$ be a measurable subset and $f \in \mathcal{O}_r$. Take a point $x \in V$ such that

$$|f(x)| = \sup_V |f|. \tag{3.4.3}$$

(Without loss of generality we may assume that x is an interior point of V; for otherwise, apply the arguments below to an interior point $x_\epsilon \in V$, $\epsilon > 0$, such that $|f(x_\epsilon)| > \sup_V |f| - \epsilon$ and then take the limit when $\epsilon \to 0$). According to Lemma 3 of [2] there is a ray l with origin at x such that

$$\frac{mes_l(l \cap V)}{mes_l(l \cap \lambda)} \leq \frac{n|V|}{|\lambda|}. \tag{3.4.4}$$

Let l' be the real straight line containing l. Applying inequality (3.1.1) to $f|_{l'}$ with $I := l \cap V$ and $w := l \cap \lambda$ and then inequality (3.4.4) lead to the required result. □

Remark 5. *Assume that $\omega \subset V$ is a pair of Euclidean balls of radii R_1 and R_2, respectively. Then the ray l in (3.4.4) can be chosen such that the constant in the inequality of Theorem 4 will be $\left(\frac{4R_1}{R_2}\right)^{d_f}$.*

Proof of Corollary 3. Before we begin the proof, we will define rearrangements of functions. We consider k-measurable functions f defined on the set $\Omega \subseteq \mathbb{R}^n$, equipped with the k-dimensional measure $|E|_k$, $1 \leq k \leq n$, for $E \subseteq \Omega$. For instance, $k = n-1$ if $\Omega = S^n$ and $k = n$ if $\Omega = \mathbb{R}^n$ or Ω is the bounded domain in $\mathbb{R}^n$.

Definition 6. *For each f on the bounded set $\Omega \subset \mathbb{R}^n$ we define its increasing rearrangement $f^* := [0, |\Omega|_k] \to [0, \infty]$ by $f^*(t) := f^*(t, \Omega) := \sup\{\tau \geq 0 : E_\tau \leq t\}$, where $E_\tau := |\{x \in \Omega : |f(x)| \leq \tau\}|$. Similarly, for each f on $\Omega \subseteq \mathbb{R}^n$ we define its decreasing rearrangement f_* by $f_*(t) := \inf\{\tau \geq 0 : I_\tau \leq t\}$, where $I_\tau := |\{x \in \Omega : |f(x)| > \tau\}|$.*

Let $V \subset B(0,1)$ be a convex body and $f \in \mathcal{O}_r$. For the distribution function $D_f(t) := mes\{x \in V : |f(x)| \leq t\}$ the inequality of Theorem 4 acquires the form

$$D_f(t) \leq 4n|V| \left(\frac{t}{\|f\|_V}\right)^{\frac{1}{d_f(r)}}. \tag{3.4.5}$$

The required result follows from the above inequality and the identity

$$\int_V \left| \ln \frac{|f|}{\|f\|_V} \right| dx = \int_0^{|V|} \left| \ln \frac{f_*}{\|f\|_V} \right| dx. \tag{3.4.6}$$

where $f_* = \inf\{s : D_f(s) \geq t\}$. □

Proof of Theorem 5. Let $V \subset B(0,1)$ be a convex body. For a real straight line l, $l \cap V \neq \emptyset$, and an interval $I \subset l \cap V$ inequality (3.1.1) implies

$$mes\left\{t \in I : |f(t)| \geq 10^{-\tilde{d}_f(r)} \|f\|_I\right\} > \frac{|I|}{2} \tag{3.4.7}$$

holds for any $f \in \mathcal{O}_r$ with $\|f\|_I = \sup_I |f|$. Applying the same arguments as in the original proof of Bourgain's inequality for polynomials [11] but based on the above inequality instead of Lemma 3.1 of [11] one obtains the required result. The second part of Theorem 5 follows from the distributional inequality of the theorem and the definition

$$\|f\|_{L^\Phi(V,dx)} := \inf\left\{A \geq 0 : \int_V \Phi\left(\frac{|f|}{A}\right) dx \leq 1\right\}. \tag{3.4.8}$$

□

Proof of Corollary 4. The reverse Hölder inequality (3.1.10) follows straightforwardly from the distributional inequality of Theorem 5. □

3.5 Concluding Remark

If $f_1, \ldots, f_k$ are functions from $\mathcal{O}_r$ and p is a holomorphic polynomial of degree d then for $h = p(f_1, \ldots, f_k)$ its degree $d_h(r)$ is bounded by a constant depending on d, r and $f_1, \ldots, f_k$. This follows from results of [58] and arguments used in the proof of Proposition 2. However, it is difficult to obtain an explicit estimate for $d_h(r)$ even in the case of naturally defined functions f_i (e.g., taken as solutions of some systems of ODEs). Assume, e.g., that $f_1 = z_1, \ldots, f_n = z_n$ are coordinate functions on $\mathbb{C}^n$ and $k \geq n$. Then the inequality $d_h(r) \leq cd$ holds for any polynomial p of degree d with c which does not depend on d if and only if $f_{n+1}, \ldots, f_k$ are algebraic functions [6, 60].

Chapter 4

REMEZ TYPE INEQUALITIES ON AHLFORS REGULAR SETS

4.1 Hausdorff measure and Hausdorff dimension

To formulate and prove the results of this chapter we require the definitions of the Hausdorff measure and of the Hausdorff dimension. *Hausdorff dimension* (also known as the *Hausdorff-Besicovitch dimension*) is an extended non-negative real number associated to any metric space. It was introduced in 1918 by the mathematician Felix Hausdorff. Many of the technical developments used to compute the Hausdorff dimension for highly irregular sets were obtained by Abram Samoilovitch Besicovitch. Less frequently it is also called the *capacity dimension* or *fractal dimension* (the latter is somewhat misleading as there are many other choices of definition).

Intuitively, the dimension of a set (for example, a subset of Euclidean space) is the number of independent parameters needed to describe a point in the set. One mathematical concept which closely models this naive idea is that of topological dimension of a set. For example a point in the plane is described by two independent parameters (the Cartesian coordinates of the point), so in this sense, the plane is two-dimensional. As one would expect, topological dimension is always a natural number.

However, topological dimension behaves in quite unexpected ways on certain highly irregular sets such as fractals. For example, the Cantor set has topological dimension zero, but in some sense it behaves as a higher dimensional space. Hausdorff dimension gives another way to define dimension, which takes the metric into account.

To define the Hausdorff dimension for X as a non-negative real number, we first consider the number $N(r)$ of balls of radius at most r required to cover X completely. Clearly, as r gets smaller $N(r)$ gets larger. Very roughly, if $N(r)$ grows in the same way as $1/r^d$ as r is squeezed down towards zero, then we say X has dimension d. In fact the rigorous definition of Hausdorff dimension is somewhat roundabout, since it first defines an entire family of covering measures for X. It turns out that Hausdorff dimension refines the concept of topological dimension and also relates it to other properties of the space such as area or volume.

The Hausdorff dimension is one measure of the dimension of an arbitrary metric space; this includes complicated spaces such as fractals.

Suppose (X, d) is a metric space with metric d. As mentioned above, we are interested in counting the number of balls of some radius necessary to cover a given set. It is possible to try to do this directly for many sets (leading to so-called box counting dimension), but Hausdorff's insight was to approach the problem indirectly using the theory of measure developed earlier in the century by Henri Lebesgue and Constantin Carathéodory. In order to deal with the technical details of this approach, Hausdorff defined an entire family of measures on subsets of X, one for each possible dimension $s \in [0, \infty)$.

Let C be the class of all subsets of X; for each positive real number s, let p_s be the function $A \longrightarrow diam(A)^s$ on C. The Hausdorff outer measure of dimension s, denoted $\mathcal{H}_s$, is the outer measure corresponding to the function p_s on C.

Thus, for any subset E of X

$$\mathcal{H}_{s\delta} := \inf \left\{ \sum_{i=1}^{\infty} diam(A_i)^s \right\}$$

where the infimum is taken over sequences $\{A_i\}_i$ which cover E by sets each with diameter $\leq \delta$. This quantity is non-decreasing as $\delta \longrightarrow 0$. The s-dimensional Hausdorff outer measure is defined as

$$\mathcal{H}_s := \limsup_{\delta \longrightarrow 0} \mathcal{H}_{s\delta}.$$

We can succinctly (though not in a very useful way) describe the value $\mathcal{H}_s(E)$ as the infimum of all $h > 0$ such that for all $\delta > 0$, E can be covered by countably many closed sets of diameter $\leq \delta$; and the sum of the s-th powers of these diameters is less than or equal to h.

The function $s \longrightarrow \mathcal{H}_s(E)$ is non-increasing. In fact, it turns out that for all values of s, except possibly one, $\mathcal{H}_s(E)$ is either 0 or ∞. We say E has positive finite Hausdorff dimension if, and only if, there is a real number $0 < d < \infty$ such that if $s < d$ then $\mathcal{H}_s(E) = \infty$ and if $s > d$, then $\mathcal{H}_s(E) = 0$. If $\mathcal{H}_s(E) = 0$ for all positive s, then E has Hausdorff dimension 0. Finally, if $\mathcal{H}_s(E) = \infty$ for all positive s, then E has Hausdorff dimension ∞. In other words,

$$dim_H(E) := \inf\{s \ : \ \mathcal{H}_s(E) = 0\} = \sup\{s \ : \ \mathcal{H}_s(E) = \infty\}.$$

The Hausdorff outer measure $\mathcal{H}_s$ is defined for all subsets of X. However, we can in general assert additivity properties, that is

$$\mathcal{H}_s(A \cup B) = \mathcal{H}_s(A) + \mathcal{H}_s(B).$$

for disjoint A, B only when A and B satisfy some additional condition, such as both being Borel sets (or more generally, that they are both measurable sets). From the perspective of assigning measure and dimension to sets with unusual metric properties such as fractals, however, this is not a restriction.

One can prove that $\mathcal{H}_s$ is a metric outer measure. Thus all Borel subsets of X are measurable and $\mathcal{H}_s$ is a countably additive measure on the σ-algebra of Borel sets.

Clearly, if (X, d) and (Y, e) are isomorphic metric spaces, then the corresponding Hausdorff measure spaces are also isomorphic. It is more useful to note however that Hausdorff measure even behaves well under certain bounded modifications of the underlying metric. Hausdorff measure is a Lipschitz invariant in the following sense: If d and d_1 are metrics on X such that for some $0 < C < \infty$ and all x, y in X,

$$C^{-1}d_1(x, y) \leq d(x, y) \leq Cd_1(x, y)$$

then the corresponding Hausdorff measures $\mathcal{H}_s$, $\mathcal{H}_{1s}$ satisfy

$$C^{-s}\mathcal{H}_{1s}(E) \leq \mathcal{H}_s(E) \leq C^s\mathcal{H}_{1s}(E).$$

for any Borel set E.

Note that if m is a positive integer, the m dimensional Hausdorff measure of $\mathbb{R}^n$ is a rescaling of the usual m-dimensional Lebesgue measure $\mathcal{L}_m$ which is normalized so that the Lebesgue measure of the m-dimensional unit cube $[0,1]^m$ is 1. In fact, for any Borel set E,

$$\mathcal{L}_m(E) = 2^{-m}\frac{\pi^{m/2}}{\Gamma(m/2+1)}\mathcal{H}_m(E).$$

4.2 Remez type inequalities on fractal sets

In an actively developing field of modern mathematics, analysis on fractal sets, see, e.g., [31] and references therein, one requires a generalization of the Remez inequality for fractal sets. In such a generalization ω is a subset of Lebesgue measure 0 in a Euclidean ball $B \subset \mathbb{R}^n$. Since zero sets of real polynomials on $\mathbb{R}^n$ have Hausdorff dimension $\le n-1$, to obtain a finite bound for $\sup_B |p|/\sup_\omega |p|$ one assumes also that the Hausdorff dimension of ω is more than $n-1$. Further, it is natural to estimate the above ratio by a function depending on the Hausdorff measures of B and ω. Specifically, let $\mathcal{H}_s$ denote the s-Hausdorff measure on $\mathbb{R}^n$, $0 < s \le n$; in particular, $\mathcal{H}_n$ coincides with the Lebesgue measure $\mathcal{L}_n$ on $\mathbb{R}^n$ up to a factor depending only on n. In this chapter we study Remez type inequalities of the following form

$$\sup_B |p| \le \phi(\lambda) \sup_\omega |p|, \tag{4.2.1}$$

where p is a real polynomial on $\mathbb{R}^n$ or a holomorphic polynomial on $\mathbb{C}^n$, B is a Euclidean ball in $\mathbb{R}^n$ or $\mathbb{C}^n$, respectively, and $\omega \subset B$ is a subset of finite Hausdorff s-measure with $n-1 < s \le n$ in the real case and $2n-2 < s \le 2n$ in the complex one. Also,

$$\lambda := \frac{\{\mathcal{H}_s(\omega)\}^{m/s}}{\mathcal{H}_m(B)},$$

where $m = n$ in the real case and $m = 2n$ in the complex case.

For many applications (related, e.g., to reverse Hölder inequalities or BMO-properties of functions) it is crucial that ϕ in (4.2.1) is a power function in λ. Inequalities of the form (4.2.1) with such a function will be referred to as *strong Remez type inequalities*. However, in applications related to trace and extension theorems for classical spaces of differentiable functions, see, in particular, [32], [33], [34], it suffices to use inequalities of the form (4.2.1) with a function ϕ whose dependence of λ is not specified. In this case the only required information is the monotonicity of ϕ in λ. Such inequalities will be referred to as *weak Remez type inequalities.*

The existence of inequalities (4.2.1) for $n = 1$ was first demonstrated in [7] where strong Remez type inequalities were proved for *(Ahlfors)* s-regular sets ω in $\mathbb{R}$ or $\mathbb{C}$ with $0 < s \leq 1$ for real p and with $0 < s \leq 2$ for holomorphic ones. Moreover, it was proved in [7] Proposition 3 that s-regularity is necessary for the validity of such an inequality.

Let us recall the definition of Ahlfors regular sets.

For a subset $K \subset \mathbb{R}^n$ and a point $x \in K$ by $B_r(x; K)$ we denote the intersection with K of an open Euclidean ball in $\mathbb{R}^n$ centered at x of radius r.

Definition 7. *A subset $K \subset \mathbb{R}^n$ is said to be (Ahlfors) s-regular if there is a positive number a such that for every $x \in K$ and $0 < r \leq diam(K)$*

$$\mathcal{H}_s(B_r(x; K)) \leq ar^s. \tag{4.2.2}$$

The class of these sets will be denoted by $\mathcal{A}_n(s, a)$.

Definition 8. *A subset $K \subset \mathcal{A}_n(s, a)$ is said to be an s-set if there is a positive number b such that for every $x \in K$ and $0 < r \leq diam(K)$*

$$br^s \leq \mathcal{H}_s(B_r(x; K)). \tag{4.2.3}$$

We denote this class by $\mathcal{A}_n(s, a, b)$.

The class of s-sets, in particular, contains compact Lipschitz s-manifolds (with integer s), Cantor type sets and self-similar sets (with arbitrary s), see, e.g., [16], page 29 and [35], Section 4.13.

In this chapter we establish inequalities of form (4.2.1) for s-regular sets $\omega \in \mathcal{A}_n(s, a)$ with ϕ depending also on s, n, $k := deg p$ and a. We prove strong Remez type inequalities for holomorphic polynomials using a technique of Algebraic Geometry. For the real case, strong Remez type inequalities are true for dimensions $n = 1, 2$ but the problem is open for $n > 2$. On the other hand, weak Remez type inequalities are valid in this case, see [5].

We start with strong Remez type inequalities for holomorphic polynomials on $\mathbb{C}^n$.

Let $X \subset \mathbb{C}^n$ belong to $\mathcal{A}_{2n}(s,a)$, $s = 2n-2+\alpha$, $\alpha > 0$. Let p be a holomorphic polynomial on $\mathbb{C}^n$ of degree k.

Theorem 6. *For any Euclidean ball $B \subset \mathbb{C}^n$ and an $\mathcal{H}_s$-measurable subset $\omega \subset X \cap B$ one has*

$$\sup_B |p| \le \left(\frac{c_1 \mathcal{H}_{2n}(B)}{\{\mathcal{H}_s(\omega)\}^{2n/s}}\right)^{c_2 k} \sup_\omega |p|$$

where c_1 depends on a, n, k, α and $c_2 > 0$ depends on α.

Corollary 5. *Let $X \in \mathcal{A}_{2n}(s,a,b)$. Let $B = B_r(x;X)$, $x \in X$, $r > 0$, and $\omega \subset B$ be $\mathcal{H}_s$-measurable. Then for a holomorphic polynomial p of degree k the following is true:*

$$\sup_B |p| \le \left(\frac{c_1 \mathcal{H}_s(B)}{\mathcal{H}_s(\omega)}\right)^{c_2 k} \sup_\omega |p|$$

where c_1 depends on a, b, n, k, α and c_2 depends on α.

Corollary 6. *Let $X \subset \mathbb{C}^n$ be an s-set with s as above. Then for any holomorphic polynomial p the function $\ln|p| \in BMO(X, \mathcal{H}_s)$. In other words,*

$$\sup_{x\in X, r>0} \left\{\frac{1}{\mathcal{H}_s(B_r(x;X))}\int_{B_r(x;X)} \left|\ln|p| - \frac{1}{\mathcal{H}_s(B_r(x;X))}\int_{B_r(x;X)} \ln|p|\, d\mathcal{H}_s\right| d\mathcal{H}_s\right\} < \infty$$

Another corollary is the following reverse Hölder inequality.

Corollary 7. *Under assumptions of Theorem 6 for $1 \le l \le \infty$ one has*

$$\left(\frac{1}{\mathcal{H}_s(B_r(x;X))}\int_{B_r(x;X)} |p|^l\, d\mathcal{H}_s\right)^{1/l} \le C\left(\frac{1}{\mathcal{H}_s(B_r(x;X))}\int_{B_r(x;X)} |p|\, d\mathcal{H}_s\right)$$

where C depends on k, n, α, a and b.

Let us present now a general form of weak Remez type inequalities for real polynomials on $\mathbb{R}^n$.

Theorem 7. *Assume that $U \subset \mathbb{R}^n$ is a bounded open set and $\omega \subset U$ belongs to $\mathcal{A}_n(s,a)$ with $n-1 < s \le n$. Assume also that*

$$\lambda := \frac{\{\mathcal{H}_s(\omega)\}^{n/s}}{\mathcal{H}_n(U)} > 0.$$

Then there exists a constant $C > 1$ such that for every polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ of degree k

$$\left(\frac{1}{\mathcal{H}_n(U)} \int_U |p|^r \, d\mathcal{H}_n\right)^{1/r} \leq C \left(\frac{1}{\mathcal{H}_s(\omega)} \int_\omega |p|^q \, d\mathcal{H}_s\right)^{1/q}. \tag{4.2.4}$$

Here $0 < q, r \leq \infty$ and C depends on U, n, q, r, s, k, a and λ and is increasing in $1/\lambda$. In particular, for $q = r = \infty$ we obtain the weak Remez type inequality of the form (4.2.1).

4.3 Complex Algebraic Varieties and s-sets

In this section we use some standard facts of Complex Algebraic Geometry. For the background and the proofs see, e.g., books [36] and [37].

By $\mathbb{CP}^n$ we denote the n-dimensional complex projective space with homogeneous coordinates $(z_0 : \cdots : z_n)$. The complex vector space $\mathbb{C}^n$ is a dense open subset of $\mathbb{CP}^n$ defined by $z_0 \neq 0$. The hyperplane at ∞, $H := \{(z_0 : \cdots : z_n) \in \mathbb{CP}^n \ : \ z_0 = 0\}$, can be naturally identified with $\mathbb{CP}^{n-1}$ and $\mathbb{CP}^n = \mathbb{C}^n \cup H$.

A closed subset $X \subset \mathbb{C}^n$ defined as the set of zeros of a family of holomorphic polynomials on $\mathbb{C}^n$ is called an *affine algebraic variety.* By $dim_{\mathbb{C}}X$ we denote the (complex) *dimension* of X, i.e., the maximum of complex dimensions of complex tangent spaces at smooth points of X.

Assume that an affine algebraic variety $X \subset \mathbb{C}^n$ has pure dimension $k \geq 1$, i.e., dimensions of complex tangent spaces at smooth points of X are the same. Then its closure $\overline{X}$ in $\mathbb{CP}^n$ is a projective variety of pure dimension k, and $dim_{\mathbb{C}}(H \cap \overline{X}) = k - 1$.

Any linear subspace of dimension $n-k$ in $\mathbb{CP}^n$ meets $\overline{X}$, but there is a linear subspace $L \subset H$ of dimension $n - k - 1$ such that $L \cap \overline{X} = \emptyset$. Moreover, for a generic $(n-k)$-dimensional subspace of $\mathbb{CP}^n$ its intersection with $\overline{X}$ consists of a finite number of points. The number of these points is called the *degree* of $\overline{X}$ and is denoted $deg\,\overline{X}$. For instance, if X as above is defined as the set of zeros of holomorphic polynomials $p_1, \ldots, p_{n-k}$ on $\mathbb{C}^n$ of degrees $d_1, \ldots, d_{n-k}$, respectively, then by the famous Bezout theorem $deg\,\overline{X} \leq d_1 \cdots d_{n-k}$.

Let $L \subset H$ be a linear subspace of dimension $n-k-1$ which does not intersect $\overline{X}$. This subspace defines a projection $\phi_L : \mathbb{CP}^n \to \mathbb{CP}^k$ as follows.

Fix a linear subspace of dimension k in $\mathbb{CP}^n$ disjoint from L. We will simply call it $\mathbb{CP}^k$. If $w \in \mathbb{CP}^n \setminus L$, then w and L span an $(n-k)$-dimensional linear subspace which meets $\mathbb{CP}^k$ in a unique point $\phi_L(w)$. The map ϕ_L sends w to $\phi_L(w)$. Further, $\mathbb{C}^n \subset \mathbb{CP}^n \setminus L$, and, with a suitable choice of linear coordinates, $\phi_L|_{\mathbb{C}^n} : \mathbb{C}^n \to \mathbb{C}^k$ is the standard projection: $(z_1, \dots, z_n) \mapsto (z_1, \dots, z_k)$.

The map $\phi_L|_X : X \to \mathbb{C}^k$ is a surjection and is a branched covering over $\mathbb{C}^k$ whose order μ, i.e., the number of points $\phi_L^{-1}(y) \cap X$ for a generic $y \in \mathbb{C}^k$, is $deg\,\overline{X}$. Then X is a complex subvariety of a pure k-dimensional algebraic variety $\widetilde{X}$ defined as the set of zeros of holomorphic polynomials p_i, $1 \le i \le n-k$, of the form

$$p_i(z_1, \dots, z_n) = z_{k+i}^{\mu} + \sum_{1 \le l \le \mu} b_{il}(z_1, \dots, z_k) z_{k+i}^{\mu-l} \tag{4.3.1}$$

where b_{il} is a holomorphic polynomial of degree $\le l$ on $\mathbb{C}^k$. Moreover, let $S \subset \mathbb{C}^k$ be the branch locus of $\phi_L|_X$. If $w \in \mathbb{C}^k \setminus S$, then $b_{il}(w)$ is the l-th elementary symmetric function in $z_{k+i}(w^{(1)}), \dots, z_{k+i}(w^{(\mu)})$, where $\phi_L^{-1}(w) \cap X = (w^{(1)}, \dots, w^{(\mu)})$. (Recall that the elementary symmetric functions s_i in $\xi_1, \dots, \xi_\mu$ are defined from the identity $\prod_{1 \le l \le \mu}(t - \xi_l) = t^\mu + s_1 t^{\mu-1} + \cdots + s_\mu$ of polynomials in variable t.) Since $dim_{\mathbb{C}} X = dim_{\mathbb{C}} \widetilde{X} = k$, X is the union of some irreducible components of $\widetilde{X}$.

Next, the Fubini-Studi metric on $\mathbb{CP}^n$ is a Riemannian metric defined by the associated $(1,1)$-form $\omega := \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\ln(|z_0|^2 + \cdots |z_n|^2)$, $(z_0 : \cdots : z_n) \in \mathbb{CP}^n$. For a k-dimensional projective variety $\overline{X}$ as above the (k,k)-form $\wedge^k\omega$ determines a Borel measure $\mu_{\overline{X}}$ on $\overline{X}$,

$$\mu_{\overline{X}}(U) := \int_U \wedge^k \omega \tag{4.3.2}$$

where $U \subset \overline{X}$ is a Borel subset. Moreover,

$$\mu_{\overline{X}}(\overline{X}) = deg\,\overline{X}, \tag{4.3.3}$$

see, e.g., [37], Chapter 1.5.

Let $\omega_e := \frac{\sqrt{-1}}{2}\sum_{1 \le i \le n} dz_i \wedge d\overline{z}_i$ be the Euclidean Kähler form determining the Euclidean metric on $\mathbb{C}^n$. Then ω and ω_e are equivalent on every compact subset $K \subset \mathbb{C}^n$

where the constants of equivalence depend on K and n only. In particular, the Fubini-Studi and the Euclidean metrics, and the (k,k)-forms $\wedge^k\omega_e$ and $\wedge^k\omega$ are equivalent on every such K. Let $\mu_{e,X}$ be a Borel measure on a pure k-dimensional affine algebraic variety X defined by the formula

$$\mu_{e,X}(U) := \int_U \wedge^k \omega_e \tag{4.3.4}$$

where $U \subset X$ is a Borel subset. Then for every compact subset $K \subset \mathbb{C}^n$ the measures $\mu_{\overline{X}}|_{K\cap X}$ and $\mu_{e,X}|_{K\cap X}$ are equivalent with the constants of equivalence depending on K, k and n only.

Let us establish a relation between complex algebraic varieties and s-sets.

Theorem 8. *Let $X \subset \mathbb{C}^n$ be an affine algebraic variety of pure dimension $k \geq 1$ such that $deg\,\overline{X} \leq \mu$. Then $X \in \mathcal{A}_{2n}(2k,a,b)$ where a and b depend on k, μ and n only.*

Proof. We will prove that

$$br^{2k} \leq \mu_{e,X}(B_r(x;X)) \leq ar^{2k} \tag{4.3.5}$$

with a and b depending on k, μ and n only, where X satisfies the assumptions of the theorem, $x \in X$ and $\mu_{e,X}$ is the measure on X determined in (4.3.4). From here applying [16], Section II.1.2, Theorem 1, we get the desired statement.

Since $deg\,\overline{X} = deg\,\overline{x+X}$ and $\mu_{e,X}(U) = \mu_{e,x+X}(x+U)$ for all $x \in \mathbb{C}^n$ and all Borel subsets $U \subset X$, without loss of generality we may assume that $0 \in X$ and prove (4.3.5) for $B_r(0;X)$ only. Since $\mu_{e,\lambda X}(\lambda U) = \lambda^{2k}\mu_{e,X}(U)$ and $deg\,\overline{\lambda X} = deg\,\overline{X}$ for $\lambda > 0$, $x \in \mathbb{C}^n$, and Borel subsets $U \subset \mathbb{C}^n$, it suffices to prove that

$$b \leq \mu_{e,X}(B_1(0;X)) \leq a \tag{4.3.6}$$

where a and b depend on k, μ and n only.

First we will prove the left-side inequality in (4.3.6). Let $\{X_l\}_{l\in\mathbb{N}}$ be a sequence of affine algebraic varieties containing 0 and satisfying the hypotheses of the theorem such that

$$\inf_X \mu_{e,X}(B_1(0;X)) = \lim_{l\to\infty} \mu_{e,X_l}(B_1(0;X_l)). \tag{4.3.7}$$

Here the infimum is taken over all X containing 0 and satisfying the conditions of the theorem. Consider the sequence $\{\overline{X}_l\}_{l\in\mathbb{N}}$ of pure k-dimensional projective subvarieties of $\mathbb{CP}^n$. Since $\mathbb{CP}^n$ is a compact manifold, one can choose a subsequence of $\{\overline{X}_l\}_{l\in\mathbb{N}}$ converging in the Hausdorff metric defined on compact subsets of $\mathbb{CP}^n$ to a compact set, say, Y. Without loss of generality we may assume that $\{\overline{X}_l\}_{l\in\mathbb{N}}$ itself converges to Y.

Lemma 2. *There are a linear subspace $L \subset \mathbb{CP}^n$ of dimension $n-k-1$ and a number $N \in \mathbb{N}$ such that $L \cap (\{\overline{X}_l\}_{l\geq N} \cup Y) = \emptyset$.*

Proof. We prove the result by induction on $n-k$, the codimension of $\overline{X}_l$ in $\mathbb{CP}^n$.

For $n-k=1$ every $\overline{X}_l$ being a projective hypersurface of degree $\leq \mu$ is defined as the set of zeros of a holomorphic homogeneous polynomial p_l of degree $\leq \mu$:

$$\overline{X}_l := \{(z_0 : \cdots : z_n) \in \mathbb{CP}^n \ : \ p_l(z_0,\dots,z_n) = 0\}.$$

Without loss of generality we may assume that l_2-norms of vectors of coefficients of all p_l are 1. Then we can choose a subsequence $\{p_{l_s}\}_{s\in\mathbb{N}}$ that converges uniformly on compact subsets of $\mathbb{C}^{n+1}$ to a nontrivial (holomorphic) homogeneous polynomial p of $deg\, p \leq \mu$. Next, if $y \in Y$, then by the definition of the Hausdorff convergence there is a sequence of points $\{x_l\}_{l\in\mathbb{N}}$, $x_l \in X_l$, such that $\lim_{l\to\infty} x_l = y$. In particular, if $y = (y_0 : \cdots : y_n)$ and $x_l = (x_{0l} : \cdots : x_{nl})$ with $\max_{0\leq i\leq n} |y_i| \leq 1$, $\max_{0\leq i\leq n} |x_{il}| \leq 1$, $l \in \mathbb{N}$, then

$$p(y_0,\dots,y_n) = \lim_{s\to\infty} p_{l_s}(x_{0l_s},\dots,x_{nl_s}) = 0.$$

Since $p \neq 0$, the latter implies that Y belongs to a projective hypersurface in $\mathbb{CP}^n$. In particular, Y is nowhere dense in $\mathbb{CP}^n$. Thus there is $z \in \mathbb{CP}^n \setminus Y$. And so there is a neighbourhood U of Y in $\mathbb{CP}^n$ which does not contain z. By the definition of the Hausdorff convergence this implies that there is a number $N \in \mathbb{N}$ such that $\{\overline{X}_l\}_{l\geq N} \subset U$ completing the proof of the lemma for $n-k=1$.

Suppose now that the result is proved for $n-k>1$ and prove it for $n-k+1$.

Since every $\overline{X}_l$ is contained in a projective hypersurface in $\mathbb{CP}^n$ of degree $\leq \mu$, by the induction hypothesis there are a number $N' \in \mathbb{N}$ and a point $y \in \mathbb{CP}^n$ such that $y \notin \{\overline{X}_l\}_{l \geq N'} \cup Y$. The point y determines a projection $\phi_y : \mathbb{CP}^n \setminus \{y\} \to \mathbb{CP}^{n-1}$ as described above (with $L := \{y\}$). Set $X_l' = \phi_y(\overline{X}_l)$, $l \geq N'$, and $Y' = \phi_y(Y)$. By the proper map theorem (see, e.g., [37], Chapter 0.2) and the Chow theorem (see, e.g., [37], Chapter 1.3) X_l' are projective subvarieties of $\mathbb{CP}^{n-1}$. Also, by the above construction, $dim_{\mathbb{C}} X_l' = dim_{\mathbb{C}} X_l$ and $deg\, X_l' \leq \mu$ for all $l \geq N'$. Moreover, $\{X_l'\}_{l \geq N'}$ converges in the Hausdorff metric defined on compact subsets of $\mathbb{CP}^{n-1}$ to Y', because ϕ_y is continuous in a neighbourhood of $\{\overline{X}_l\}_{l \geq N'} \cup Y$. Since the codimension of X_l' in $\mathbb{CP}^{n-1}$ is $n-k$, by the induction hypothesis there are an integer number $N \geq N'$ and a linear subspace $L' \subset \mathbb{CP}^{n-1}$ of dimension $n-k-1$ which does not intersect $\{X_l'\}_{l \geq N} \cup Y'$. Then $L = \phi_y^{-1}(L') \cup \{y\}$ is a linear subspace of $\mathbb{CP}^n$ of dimension $n-k$ which does not intersect $\{\overline{X}_l\}_{l \geq N} \cup Y$.

This completes the proof of the lemma. □

Further, since $0 \in \{X_l\}_{l \in \mathbb{N}} \cup Y$, there is a closed Euclidean ball $\overline{B}_{r_0}(0) \subset \mathbb{C}^n$ centered at 0 of radius $0 < r_0 \leq 1$ which does not intersect the L of the above lemma. Clearly,

$$\mu_{e,X_l}(B_1(0;X_l)) \geq \mu_{e,X_l}(B_{r_0}(0;X_l)), \quad l \in \mathbb{N}.$$

(As before, $B_{r_0}(0;X_l) := B_{r_0}(0) \cap X_l$.) Therefore to prove the left-side inequality in (4.3.6) it suffices to check that

$$\liminf_{l \to \infty} \mu_{e,X_l}(B_{r_0}(0;X_l)) > 0. \tag{4.3.8}$$

Recall that the Fubini-Studi metric is equivalent to the Euclidean metric on every compact subset $K \subset \mathbb{C}^n$ with the constants of equivalence depending on K and n only. Therefore there is a closed ball B in the Fubini-Studi metric centered at 0 and of radius $s_0 > 0$ depending on r_0 and n only such that $B \subset \overline{B}_{r_0}(0)$. Since μ_{e,X_l} is equivalent to $\mu_{\overline{X}_l}$ on $\overline{B}_{r_0}(0;X_l)$ with the constants of equivalence depending on r_0, k and n only (see the above discussion) inequality (4.3.8) follows from the inequality

$$\liminf_{l \to \infty} \mu_{\overline{X}_l}(B \cap X_l) > 0. \tag{4.3.9}$$

Let us check the last inequality. Diminishing, if necessary, r_0 we can find a hyperplane $L' \subset \mathbb{CP}^n$ which contains L from Lemma 2 and does not intersect $\overline{B}_{r_0}(0)$. Let $T : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be a unitary transformation which induces an isometry $\overline{T} : \mathbb{CP}^n \to \mathbb{CP}^n$ sending L' to the hyperplane at ∞, H. Then $\overline{T}(B)$ is a closed ball (in the Fubini-Studi metric) in $\mathbb{C}^n = \mathbb{CP}^n \setminus H$. By the definition of T, $deg\, T(\overline{X}_l) = deg\, \overline{X}_l$ and $\mu_{\overline{X}_l}(U) = \mu_{T(\overline{X}_l)}(T(U))$ for a Borel subset $U \in \overline{X}_l$. These facts and the above equivalence of μ_{e,X_l} and $\mu_{\overline{X}_l}$ on compact subsets of $\mathbb{C}^n$ show that in the proof of (4.3.9) without loss of generality we may assume that $L' = H$.

Now, consider the projection $\Phi_L : \mathbb{C}^n \to \mathbb{C}^k$ determined as above. Choosing suitable coordinates on $\mathbb{C}^n$ we may and will assume that Φ_L coincides with the projection $(z_1, \dots, z_n) \mapsto (z_1, \dots, z_k)$. Then $X_l := \overline{X}_l \setminus H$ are algebraic subvarieties of algebraic varieties $\widetilde{X}_l$ defined as sets of zeros of families of polynomials p_{il}, $1 \le i \le n-k$, $l \ge N$, of the form (4.3.1). Moreover, since $L \cap (\{\overline{X}_l\}_{l \ge N} \cup Y) = \emptyset$, the definition of p_{il}, see the above discussion, shows that for every i polynomials p_{il}, $l \ge N$, are uniformly bounded on compact subsets of $\mathbb{C}^n$. Since $deg\, p_{il} \le \mu$, we can find a subsequence $\{l_s\}_{s\in\mathbb{N}} \subset \mathbb{N}$ such that $\{p_{il_s}\}_{s\in\mathbb{N}}$ converge uniformly on compact subsets of $\mathbb{C}^n$ to polynomials p_i, $deg\, p_i \le \mu$, of the form (4.3.1), $1 \le i \le n-k$, and

$$\lim_{s\to\infty} \mu_{\overline{X}_{l_s}}(B \cap X_{l_s}) = \liminf_{l\to\infty} \mu_{\overline{X}_l}(B \cap X_l).$$

This implies easily that $Y \cap \mathbb{C}^n$ with Y from Lemma 2 is contained in the pure k-dimensional algebraic variety $\widetilde{Y}$ defined as the set of zeros of polynomials p_i, $1 \le i \le n-k$.

In what follows by $\Delta_r^l := \{(z_1, \dots, z_l) \in \mathbb{C}^l \ :\ \max_{1\le i\le l} |z_i| < r\}$ we denote the open polydisk in $\mathbb{C}^l$ centered at 0 of radius r.

Since, by the definition, $\widetilde{Y}$ is a finite branched covering over $\mathbb{C}^k$ and $0 \in \widetilde{Y}$, there is a polydisk $\Delta_\epsilon^n = \Delta_\epsilon^{n-k} \times \Delta_\epsilon^k$ such that $\Delta_\epsilon^n \cap \widetilde{Y} \subset B \cap \widetilde{Y}$ and $\Phi_L : \Delta_\epsilon^n \cap \widetilde{Y} \to \Delta_\epsilon^k$ is a finite branched covering over Δ_ϵ^k (for similar arguments see, e.g., the proof of the preparatory Weierstrass theorem in [37], Chapter 0.1). From here using the fact that $\{p_{il_s}\}$ converges uniformly on compact subsets of $\mathbb{C}^n$ to p_i for all i and diminishing, if necessary, ϵ we obtain analogously that there is a number $N_0 \in \mathbb{N}$ such that $\Delta_\epsilon^n \cap \widetilde{X}_{l_s} \subset B \cap \widetilde{X}_{l_s}$ and $\Phi_L : \Delta_\epsilon^n \cap \widetilde{X}_{l_s} \to \Delta_\epsilon^k$ are finite branched coverings over Δ_ϵ^k for all $s \ge N_0$. But $\Delta_\epsilon^n \cap X_{l_s}$

is a (closed) complex subvariety of $\Delta_\epsilon^n \cap \widetilde{X}_{l_s}$, and $\Phi_L(\Delta_\epsilon^n \cap X_{l_s})$ is an open subset of Δ_ϵ^k (because $0 \in X_{l_s}$ and $\Phi_L : X_{l_s} \to \mathbb{C}^k$ is a finite branched covering). Thus by the proper map theorem, $\Phi_L(\Delta_\epsilon^n \cap X_{l_s}) = \Delta_\epsilon^k$. (Here we used the fact that the map $\Phi_L : \Delta_\epsilon^n \cap X_{l_s} \to \Delta_\epsilon^k$ is proper, because $\Phi_L : \Delta_\epsilon^n \cap \widetilde{X}_{l_s} \to \Delta_\epsilon^k$ is proper and $X_{l_s} \cap \Delta_\epsilon^n$ is a complex subvariety of $\widetilde{X}_{l_s} \cap \Delta_\epsilon^n$.)

Let $\mathcal{L}_{2k}$ be the Lebesgue measure on $\mathbb{C}^k$. Then by the definition of $\mu_{\overline{X}_{l_s}}$ there is a constant $c > 0$ depending on μ, k and n only such that

$$\mu_{\overline{X}_{l_s}}(B \cap X_{l_s}) \geq c\mathcal{L}_{2k}(\Phi_L(B \cap X_{l_s})).$$

But for $s \geq N_0$ we have

$$\mathcal{L}_{2k}(\Phi_L(B \cap X_{l_s})) \geq \mathcal{L}_{2k}(\Delta_\epsilon^k) = \pi^k \epsilon^{2k} > 0.$$

The combination of the last two inequalities completes the proof of (4.3.9) and thus the proof of the left-side inequality in (4.3.6).

The right-side inequality in (4.3.6) is obtained as follows, see (4.3.3),

$$\mu_{e,X}(B_1(0;X)) \leq c(n,k)\mu_{\overline{X}}(B_1(0;X)) \leq c(n,k)\mu_{\overline{X}}(\overline{X}) = c(n,k)deg\,\overline{X} \leq c(n,k)\mu.$$

The proof of Theorem 8 is complete. $\square$

4.4 Covering lemmas

The proof of the Strong Remez type inequality for holomorphic polynomials presented in [5] is based on a deep generalization of the classical Cartan Lemma [38] discovered by Gorin [39]. Let us present a more general version of this result.

Let X be a pseudometric space with pseudometric d. By $\mathcal{F} := \{\overline{B}_r(x) \subset X \ : \ d(x,y) \leq r,\ x,y \in X, r \geq 0\}$ we denote the set of closed balls in X. Let $\xi : \mathcal{F} \to \mathbb{R}_+$ be a function satisfying the following two properties:

1.
$$\xi(\overline{B}_{r'}(x)) \leq \xi(\overline{B}_{r''}(x)) \quad \text{for all} \quad x \in X,\ r' \leq r''.$$

2. There is a numerical constant A such that for any collection of mutually disjoint balls $\{B_i\} \subset \mathcal{F}$,

$$\sum_{i\geq 1} \xi(B_i) \leq A$$

Consider a continuous strictly increasing nonnegative function ϕ on $[0,\infty)$, $\phi(0)=0$, $\lim_{t\to\infty}\phi(t) > A$ which will be called a *majorant.*

For each point $x \in X$ we set $\tau(x) = \sup\{t \ : \ \xi(\overline{B}_t(x)) \geq \phi(t)\}$. It is easy to see that $\xi(\overline{B}_{\tau(x)}(x)) = \phi(\tau(x))$ and $\sup_x \tau(x) \leq \phi^{-1}(A) < \infty$.

A point $x \in X$ is said to be *regular* (with respect to ξ and ϕ) if $\tau(x) = 0$, i.e., $\xi(\overline{B}_t(x)) < \phi(t)$ for all $t > 0$. The next result shows that the set of regular points is sufficiently large for an arbitrary majorant ϕ.

Lemma 3. *Fix $\gamma \in (0, 1/2)$. There is a sequence of balls $B_k = \overline{B}_{t_k}(x_k)$, $k = 1, 2, \ldots$, which collectively cover all irregular points such that $\sum_{k\geq 1}\phi(\gamma t_k) < A$ (i.e., $t_k \to 0$).*

The proof of this lemma for ξ being a finite Borel measure on a metric space X is given by Gorin [39]. His argument works also in the general case.

Proof. Let $0 < \alpha < 1$, $\beta > 2$ be such that $\gamma < \alpha/\beta$. We set $B_0 = \emptyset$ and assume that the balls $B_0, \ldots, B_{k-1}$ have been constructed. If $\tau_k = \sup\{\tau(x) \ : \ x \notin B_0 \cup \cdots \cup B_{k-1}\}$, then there exists a point $x_k \notin B_0 \cup \ldots B_{k-1}$ such that $\tau(x_k) \geq \alpha\tau_k$. We set $t_k = \beta\tau_k$ and $B_k = \overline{B}_{t_k}(x_k)$. Clearly, the sequence τ_k (and thus also t_k) does not increase. The balls $\overline{B}_{\tau_k}(x_k)$ are pairwise disjoint. Indeed, if $l > k$ then $x_l \notin B_k$, i.e., the pseudodistance between x_l and x_k is greater than $\beta\tau_k > 2\tau_k \geq \tau_k + \tau_l$. Thus $\overline{B}_{\tau_k}(x_k) \cap \overline{B}_{\tau_l}(x_l) = \emptyset$ by the triangle inequality for d. Now,

$$\sum_{k\geq 1}\phi(\gamma t_k) < \sum_{k\geq 1}\phi(\alpha\tau_k) \leq \sum_{k\geq 1}\phi(\tau(x_k)) = \sum_{k\geq 1}\xi(\overline{B}_{\tau_k}(x_k)) \leq A;$$

consequently, $\tau_k \to 0$, i.e., for each point x, not belonging to the union of the balls B_k, $\tau(x) = 0$, i.e., x is a regular point. In addition, $t_k = \beta\tau_k \to 0$. $\square$

Remark 6. (1) According to the Caratheodory construction, see, e.g., [40], Chapter 2.10, there is a finite Borel measure on X whose restriction to $\mathcal{F}$ is ξ.

(2) Assume that ξ is the restriction to $\mathcal{F}$ of a Borel measure μ on X with support $\{x_1, \dots, x_n\}$. Then, as it follows from the proof, the number of the balls B_k in this case is $\leq n$ and the balls $\overline{B}_{\tau_k}(x_k)$, $k \geq 1$, cover the support of μ. For otherwise, there is $\overline{B}_{\tau_k}(x_k)$ which does not meet $\{x_1, \dots, x_n\}$. Then $\overline{B}_{\tau(x_k)}(x_k)$ does not meet $\{x_1, \dots, x_n\}$, as well. Consequently, $\mu(\overline{B}_{\tau(x_k)}(x_k)) = 0$, a contradiction with the choice of x_k.

Let X be a pseudometric space with pseudometric d. For every $x \in X$ we set $S_x := \{y \in X \ : \ d(x,y) = 0\}$. Let μ be a Borel measure on X with $\mu(X) = k < \infty$ such that

$$\int_X \ln^+ d(x,\xi)\, d\mu(\xi) < \infty \quad \text{for all} \quad x \in X,$$

where $\ln^+ t := \max(0, \ln t)$. Then we define

$$u(x) = \begin{cases} \displaystyle\int_X \ln d(x,\xi)\, d\mu(\xi), & \text{if} \quad \mu(S_x) = 0 \\ \\ -\infty, & \text{if} \quad \mu(S_x) > 0. \end{cases}$$

By definition, every Lebesgue integral $\int_X \ln d(x,\xi)\, d\mu(\xi)$ exists but may be equal to $-\infty$. In this case we define $u(x) = -\infty$.

Corollary 8. *Fix $\gamma \in (0, 1/2)$. Given $H > 0$, $s > 0$ there is a family of closed balls B_j with radii r_j satisfying*

$$\sum r_j^s < \frac{(H/\gamma)^s}{s}$$

such that

$$u(x) \geq k \ln\left(\frac{H}{e}\right) \quad \text{for all} \quad x \in X \setminus \bigcup_j B_j.$$

Proof. Let $\phi(t) = (pt)^s$ be a majorant with $p = \frac{(ks)^{1/s}}{H}$. We cover all irregular points of X by closed balls according to Lemma 3 (with $\xi = \mu$) and prove that the required inequality is valid for any regular point x. This will complete the proof. First, observe that $u(x)$ is finite for every regular point x by the definition of the Lebesgue integral and the regularity condition for the ϕ. Let $n(t;x) = \mu(\overline{B}_t(x))$ for such x. Then, for any $N \geq \max(1, H)$ we have by integration by parts

$$u(x) \geq \int_{\overline{B}_N(x)} \ln d(x,\xi)\, d\mu(\xi) = \int_0^N \ln t\, dn(t;x) = n(t;x) \ln t|_0^N - \int_0^N \frac{n(t;x)}{t}\, dt.$$

Since $n(t;x) < (pt)^s$, we obtain $\lim_{t\to 0+} n(t;x)\ln t = 0$ and therefore

$$u(x) \geq n(N;x)\ln N - \int_0^N \frac{n(t;x)}{t}\,dt.$$

In addition, $n(t;x) \leq n(N;x)$ for $t \leq N$. Therefore,

$$u(x) \geq n(N;x)\ln N - \int_0^H \frac{(pt)^s}{t}\,dt - \int_H^N \frac{n(N;x)}{t}\,dt =$$

$$n(N;x)\ln N - \frac{(pH)^s}{s} - n(N;x)\ln N + n(N;x)\ln H = -k + n(N;x)\ln H.$$

Letting here $N \to \infty$ and taking into account that $\lim_{N\to\infty} n(N;x) = k$ we obtain the required result. □

In the proof of the strong Remez inequality we use also the following result proved in [41].

Corollary 9. *Let f be a holomorphic function in the disk $|z| \leq 2eR$ $(R > 0)$ in $\mathbb{C}$, $f(0) = 1$ and η is an arbitrary positive number $\leq \frac{3}{2}e$. Then inside the disk $|z| \leq R$ but outside a family of closed disks $\overline{D}_{r_i}(z_i)$ centered at z_i of radii r_i such that $\sum r_i \leq 4\eta R$,*

$$\ln|f(z)| \geq -H(\eta)\ln M(2eR)$$

where

$$H(\eta) = 2 + \ln\left(\frac{3e}{2\eta}\right)$$

and

$$M(2eR) := \sup_{|z|\leq 2eR} |f|.$$

Remark 7. The proof is based on a particular case of Corollary 8 for μ being a sum of delta-measures, and the Harnack inequality for positive harmonic functions. According to Remark 6 (2), from the proof presented in [41] it follows that the number of disks $\overline{D}_{r_i}(z_i)$ does not exceed the number of zeros of f in the disk $|z| < 2R$ (which, by the Jensen inequality, is bounded from above by $[\ln M_f(2eR)]$) and, moreover, the disks $\overline{D}_{r_i/2}(z_i)$ cover the set of these zeros.

4.5 Strong Remez type inequalities

In this part we present the proof of Theorem 6 using the arguments from [5].

Proof. Let $X \subset \mathbb{C}^n$ be a closed subset of the class $\mathcal{A}_{2n}(s,a)$ where $s = 2n-2+\alpha$, $\alpha > 0$. Let p be a holomorphic polynomial on $\mathbb{C}^n$ of degree k. Let $B \subset \mathbb{C}^n$ be a closed Euclidean ball and $\omega \subset X \cap B$ be an $\mathcal{H}_s$-measurable subset. We must prove the inequality

$$\sup_B |p| \leq \left(\frac{c_1 \mathcal{H}_{2n}(B)}{\{\mathcal{H}_s(\omega)\}^{2n/s}} \right)^{c_2 k} \sup_\omega |p| \tag{4.5.1}$$

where c_1 depends on a, n, k, α and $c_2 > 0$ depends on α.

Since the ratio on the right-hand side of (4.5.1) is invariant with respect to dilations and translations of $\mathbb{C}^n$ and the class $\mathcal{A}_{2n}(s,a)$ is also invariant with respect to these transformations, without loss of generality we may assume that B is the closed unit ball centered at $0 \in \mathbb{C}^n$. Then we must prove that

$$\sup_B |p| \leq \left(\frac{\overline{c}_1}{\lambda^{2n/s}} \right)^{c_2 k} \sup_\omega |p| \tag{4.5.2}$$

where $\lambda := \mathcal{H}_s(\omega)$, $\overline{c}_1$ depends on a, n, k, α and $c_2 > 0$ depends on α.

By $Z_p \subset \mathbb{C}^n$ we denote the set of zeros of p. According to Theorem 8 we have $Z_p \in \mathcal{A}_{2n}(2n-2, a, b)$ for some a and b depending on n and k only. By $\mathcal{H}_{2n-2,\,p}$ we denote the Hausdorff $(2n-2)$-measure supported on Z_p. Let $B_1 \subset B_2$ be closed Euclidean balls centered at $0 \in \mathbb{C}^n$ of radii 2 and 10, respectively. Set

$$\mu := \mathcal{H}_{2n-2,\,p}|_{B_2}.$$

Since $Z_p \in \mathcal{A}_{2n}(2n-2, a, b)$, we have

$$\mu(B_r(x)) \geq b r^{2n-2} \quad \text{for all} \quad x \in B_1, \quad 0 \leq r \leq 5. \tag{4.5.3}$$

Let $H > 0$. Consider $\phi(t) := \frac{t^s}{H}$ as the majorant in Lemma 3. Then a point $x \in \mathbb{C}^n$ is regular with respect to ϕ and μ if $\mu(B_r(x)) < \frac{r^s}{H}$ for all $r > 0$. (Here we consider $\mathbb{C}^n$ with the Euclidean norm $|\cdot|$.)

Lemma 4. *There is a sequence of open Euclidean balls $B_{r_k}(x_k)$, $k = 1, 2, \ldots,$ which collectively cover all the irregular points such that*

$$\sum_{k \geq 1} r_k^s < 3H\mu(B_2).$$

Moreover, the distance $d(x)$ from a regular point x to the compact set $K := B_1 \cap Z_p$ is $\geq \min\left\{5, \left(\frac{bH}{2^s}\right)^{1/\alpha}\right\}$.

Proof. The first statement follows directly from Lemma 3. Let $y \in K$ be such that $|x - y| = d(x)$. Observe that condition (4.5.3) implies that $x \notin K$. For otherwise, we must have

$$br^{2n-2} < \frac{r^s}{H} \quad \text{for all} \quad 0 < r < 5$$

which is impossible. Thus $d(x) > 0$. Next, the ball centered at x of radius $2d(x)$ contains the ball centered at y of radius $d(x)$. Now from the regularity condition for x by (4.5.3) we get

$$b \min\{5, d(x)\}^{2n-2} \leq \mu(B_{2d(x)}(x)) \leq \frac{\{2d(x)\}^{2n-2+\alpha}}{H}.$$

This implies that

$$d(x) \geq \min\left\{5, \left(\frac{bH}{2^s}\right)^{1/\alpha}\right\}.$$

□

Continuing the proof of the theorem observe that by the definition of X,

$$\lambda := \mathcal{H}_s(\omega) \leq a2^s \tag{4.5.4}$$

(because if $\omega \subset X \cap B \neq \emptyset$, then ω is contained in a closed Euclidean ball of radius 2 centered at a point of X). Without loss of generality we may assume that $\lambda > 0$.

Lemma 5. *The set ω cannot be covered by a family $\{B_j\}$ of open Euclidean balls whose radii r_j satisfy*

$$\sum r_j^s < \frac{\lambda}{2^s a}.$$

Proof. Assume to the contrary that there is a family of balls $B_j := B_{r_j}(x_j)$, $j = 1, 2, \dots$, whose radii satisfy the inequality of the lemma which covers ω. Without loss of generality we may assume that each B_j meets ω. Then for every x_j choose $y_j \in \omega$ so that $|x_j - y_j| \le r_j$. Clearly, the family of balls $\{B_{2r_j}(y_j)\}$ also covers ω. From here, since $\omega \subset X \in \mathcal{A}_{2n}(s, a)$, we obtain

$$\lambda := \mathcal{H}_s(\omega) \le \sum \mathcal{H}_s(X \cap B_{2r_j}(y_j)) \le 2^s a \sum r_j^s < \lambda,$$

a contradiction. □

Further, note that $\mu(B_2)$ in Lemma 4 is bounded from above by a constant c depending on n and k only (because $Z_p \in \mathcal{A}_{2n}(2n-2, a, b)$ with a, b depending on n, k only). Thus choosing in this lemma $H := \frac{\lambda}{3c2^s}$ we obtain from Lemma 5 for some constant $\overline{c}$ depending on n, k:

Corollary 10. *There is a point $x \in \omega$ such that*

$$dist(x, Z_p) \ge \min\left\{1, (\overline{c}\lambda)^{1/\alpha}\right\}.$$

Proof. From the above lemmas it follows that there is $x \in \omega$ such that

$$dist(x, Z_p \cap B_1) \ge \min\left\{5, (\overline{c}\lambda)^{1/\alpha}\right\}.$$

Moreover, $x \in B$ and so $dist(x, Z_p \setminus B_1) \ge 1$; this implies the required. □

Let $z \in B$ be a point such that

$$M := \max_B |p| = |p(z)|.$$

Let l be the complex line passing through z and the point x from Corollary 10. Without loss of generality we may identify l with $\mathbb{C}$ so that z coincides with $0 \in \mathbb{C}$. Then, in this identification, the point x belongs to $\overline{D}_2(0)$, the closed disk of radius 2 centered at 0. Observe also that (under the identification) the set $B_1 \cap l$ contains $\overline{D}_1(0)$. Thus, by the classical Bernstein inequality for holomorphic polynomials

$$\max_{|z|\le 4e} |p| \le (4e)^k \max_{|z|\le 1} |p| \le (4e)^k \max_{B_1} |p| \le (8e)^k \max_B |p| := (8e)^k M.$$

Set $f = p/M$ and apply Corollary 9 with $R = 2$. According to this corollary for every $\eta \le 3e/2$ there is a family of closed disks $\overline{D}_{r_i}(z_i)$ such that $\sum r_i \le 8\eta$ and $\ln|f(z)| > -H(\eta)k\ln(8e)$ for any $|z| \le 2$ outside the above disks where $H(\eta) = 2 + \ln(3e/2\eta)$. Recall also that the number of these disks is $\le$ the number of zeros of f in $|z| \le 4$ and the disks $\overline{D}_{r_i/2}(z_i)$ cover the set of zeros of f there. In particular, if a point $z \in \overline{D}_1(0)$ satisfies $dist(z, Z_f) \ge 14\eta$ where Z_f is the set of zeros of f in $\mathbb{C}$, then it cannot belong to the union of disks $\overline{D}_{r_i}(z_i)$, and therefore $|f(z)|$ satisfies the above inequality. Choose $\eta := \min(1, (\overline{c}\lambda)^{1/\alpha})/14$. Then by Corollary 10, $dist(x, Z_f) \ge 14\eta$. Thus we have

$$\sup_{\omega} \ln|f| \ge \ln|f(x)| \ge -H(\eta)k\ln(8e).$$

We will consider two cases:

(1)

$$(\overline{c}\lambda)^{1/\alpha} \ge 1.$$

Then $\eta = \frac{1}{14}$ and

$$\sup_{\omega} \ln|f| \ge -(3 + \ln 21)k\ln(8e) > -20d.$$

This and (4.5.4) imply that

$$\sup_{B} |p| \le e^{20k} \sup_{\omega} |p| = \left(\frac{e^{20}}{\lambda^{2n/s}}\lambda^{2n/s}\right)^{k} \sup_{\omega} |p| \le \left(\frac{2^{2n}a^{2n/s}e^{20}}{\lambda^{2n/s}}\right)^{k} \sup_{\omega} |p|.$$

Thus, inequality (4.5.2) is proved in this case.

(2)

$$(\overline{c}\lambda)^{1/\alpha} < 1.$$

Then

$$\sup_{\omega} \ln|f| \ge -(c' - \ln\lambda^{1/\alpha})k\ln(8e)$$

where c' depends on n and k only. This yields

$$\sup_{B} |p| \le \left(\frac{\overline{c}_1}{\lambda^{2n/s}}\right)^{c_2 k} \sup_{\omega} |p|$$

where $\overline{c}_1 > 0$ depends on k, n and α and $c_2 > 0$ depends on α only.

The proof of Theorem 6 is complete. $\square$

Now, proof of Corollary 5 follows directly from the estimates obtained in cases (1) and (2) above and from the fact that $X \in \mathcal{A}_{2n}(s,a,b)$. Also, proofs of Corollaries 6 and 7 repeat word-for-word proofs of similar statements of Theorems 1 and 3 of [7] and are based on the inequality of Corollary 5, see this paper for details.

4.6 Weak Remez type inequalities

In this section we present the proof of Theorem 7.

Proof. We set for brevity

$$||p;\omega||_q = \left(\frac{1}{\mathcal{H}_s(\omega)}\int_\omega |p|^q\, d\mathcal{H}_s\right)^{1/q} \quad \text{and}$$

$$||p;U||_r = \left(\frac{1}{\mathcal{H}_n(U)}\int_U |p|^r\, d\mathcal{H}_n\right)^{1/r}.$$

Since the above functions are invariant with respect to dilations of $\mathbb{R}^n$, without loss of generality we may and will assume that $\mathcal{H}_n(U) = 1$.

Let $\Sigma(a,\lambda)$, $a, \lambda > 0$, be the class of subsets $\omega \in \mathcal{A}_n(s,a)$ of U satisfying

$$\{\mathcal{H}_s(\omega)\}^{n/s} \geq \lambda. \tag{4.6.1}$$

We must show that there is a positive constant $C = C(U,n,q,r,s,k,a,\lambda)$ such that for every real polynomial p of degree k on $\mathbb{R}^n$

$$||p;U||_r \leq C||p;\omega||_q. \tag{4.6.2}$$

Remark 8. Let C_0 be the optimal constant in (4.6.2). Since the class $\Sigma(a,\lambda)$ increases as λ decreases, C_0 increases in $1/\lambda$, as is required in the theorem.

If, on the contrary, the constant in (4.6.2) does not exist, one can find sequences of real polynomials $\{p_j\}$ of degrees k and sets $\{\omega_j\} \subset \Sigma(a,\lambda)$ so that

$$||p_j;U||_r = 1 \quad \text{for all} \quad j \in \mathbb{N}, \tag{4.6.3}$$

$$\lim_{j\to\infty} ||p_j;\omega_j||_q = 0. \tag{4.6.4}$$

Since all (quasi-) norms on the space of real polynomials of degree k on $\mathbb{R}^n$ are equivalent, (4.6.3) implies the existence of a subsequence of $\{p_j\}$ that converges uniformly on U to a polynomial $p \in \mathbb{R}[x_1, \dots, x_n]$ with $deg\, p \leq k$. Assume without loss of generality that $\{p_j\}$ itself converges uniformly to p. Then (4.6.3), (4.6.4) imply for this p that

$$||p; U||_r = 1, \tag{4.6.5}$$

$$\lim_{j \to \infty} ||p; \omega_j||_q = 0. \tag{4.6.6}$$

From this we derive the next result.

Lemma 6. *There is a sequence of closed subsets $\{\sigma_j\} \subset \overline{U}$ such that for every j larger than a fixed j_0 the following is true*

$$\{\mathcal{H}_s(\sigma_j)\}^{n/s} \geq \frac{\lambda}{2^{n/s}}. \tag{4.6.7}$$

Moreover,

$$\max_{\sigma_j} |p| \to 0 \quad \text{as} \quad j \to \infty. \tag{4.6.8}$$

Proof. Let first $q < \infty$. By the (probabilistic) Chebyshev inequality

$$\mathcal{H}_s(\{x \in \omega_j \ : \ |p(x)| \leq t\}) \geq \mathcal{H}_s(\omega_j) - \frac{\mathcal{H}_s(\omega_j)}{t^q} ||p; \omega_j||_q^q.$$

Pick here $t = t_j := ||p; \omega_j||_q^{1/2}$. Then by (4.6.6) the left-hand side is at least $\frac{1}{2}\mathcal{H}_s(\omega_j)$, for j sufficiently large. Denoting the closure of the set in the braces by σ_j we also have

$$\max_{\sigma_j} |p| = t_j \to 0 \quad \text{as} \quad j \to \infty.$$

If $q = \infty$, simply set $\sigma_j := \omega_j$ to produce (4.6.7) and (4.6.8). $\square$

Apply now the Hausdorff compactness theorem to find a subsequence of $\{\sigma_j\}$ converging to a closed subset $\sigma \subset \overline{U}$ in the Hausdorff metric. We assume without loss of generality that $\{\sigma_j\} \to \sigma$. By (4.6.8) this limit set is a subset of the zero set of p. Since p is nontrivial by (4.6.5), the dimension of its zero set is at most $n-1$; hence $\mathcal{H}_s(\sigma) = 0$ because $s > n-1$. Then for every $\epsilon > 0$ one can find a finite open cover of σ by open Euclidean balls B_i of radii r_i at most $r(\epsilon)$ so that

$$\sum_i r_i^s < \epsilon. \tag{4.6.9}$$

Let σ_δ be a δ-neighbourhood of σ such that

$$\sigma_\delta \subset \bigcup_i B_i \quad \text{and} \quad \delta < r(\epsilon).$$

Pick j so large that $\sigma_j \subset \sigma_\delta$. For every B_i intersecting σ_j choose a point $x_i \in B_i \cap \sigma_j$. Consider an open Euclidean ball $\widetilde{B}_i$ centered at x_i of radius twice that of B_i. Then $B_i \subset \widetilde{B}_i$ and $\{\widetilde{B}_i\}$ is an open cover of σ_j. Hence

$$\mathcal{H}_s(\sigma_j) \le \sum_i \mathcal{H}_s(\sigma_j \cap \widetilde{B}_i) \le a2^s \sum_i r_i^s$$

because $\omega_j \in \mathcal{A}_n(s,a)$. Together with (4.6.7) and (4.6.9) this implies that

$$\frac{1}{2}\lambda^{s/n} \le \mathcal{H}_s(\sigma_j) \le a2^s \sum_i r_i^s \le 2^s a\epsilon.$$

Letting $\epsilon \to \infty$ one gets a contradiction. $\square$

Remark 9. Strong Remez type inequalities for real polynomials from $\mathbb{R}[x]$ and Ahlfors regular subsets of $\mathbb{R}$ are proved in [7]. Inequalities of the form described in Theorem 6 are also valid for real polynomials on $\mathbb{R}^2$. The method of the proof of such inequalities is very similar to that of Theorem 6 and is based on the fact that an analytic compact connected curve in $\mathbb{R}^n$ is a 1-set. However, it is still an open question whether similar strong Remez type inequalities are valid for real polynomials on $\mathbb{R}^n$ for $n > 2$.

Chapter 5

REMEZ TYPE INEQUALITIES IN EXTENSION AND TRACE PROBLEMS

5.1 Whitney problems

This chapter deals with the basic problems of modern analysis, restrictions (i.e., so-called traces) and extensions of functions with a prescribed structure. For example, suppose that a certain meteorological phenomenom depended jointly on air pressure and temperature. Then fixing the temperature at say 25C, and studying how the phenomenom depends only on pressure, for that fixed temperature, is the simplest example of a restriction. More generally, one might restrict to the case when temperature plus pressure is a constant, and even when more complicated relationships hold. Extensions are the reverse process: given a functional relationship under some restricted conditions, what are the possible relationships when these restrictions are removed?

The general, abstract forms of such problems are central in the purely mathematical fields of General and Algebraic Topology (continuous extensions), in Geometric Analysis (extension of Lipschitz functions), in Multivariate Differential Analysis and Harmonic Analysis (functions and maps of prescribed smoothness). We will present an approach to extension and trace problems for certain classes of smooth functions from the classical function spaces of modern analysis. Our approach is based on local polynomial approximation theory whose methods allow us to reduce the initial analytic problems to more simple problems. The methods and results of the area of extension and trace problems are the outcome of intensive work of many outstanding mathematicians of the twentieth century including Lebesgue, Brouwer, Whitney, Hestens, Calderon, to name but a few. Among recent striking applications of the subject considered we would like to mention those of Numerical Harmonic Analysis related to the reconstruction of signals and images given incomplete data.

Unlike extension problems of the nineteenth century dealing with uniquely determined solutions (Weierstrass' theory of analytic functions, Lagrange and Hermite interpolation, Dirichlet problem for Laplace equation etc.) the modern theory is working with incomplete data and infinitely many possible extensions. This makes the problem much more complicated; on the other hand, extension algorithms of modern theory dealing with incomplete data for recovering functions of prescribed inner structure have a much greater possibility for applications. The results and algorithms are of value in such diverse fields such as linear and nonlinear Partial Differential Equations (the equations that describe almost all physical phenomema in Science and Engineering), Numerical Analysis, Approximation Theory, Signal and Image Processing, and Computer Tomography.

We will present an application of Remez type inequalities to extension and trace problems for classes of differentiable functions following the paper of A. Brudnyi and Yu. Brudnyi [5].

For differentiable functions on $\mathbb{R}^n$ such problems were originally posed and studied by Whitney [42] in 1934. His methods have been then used in a variety of problems of Analysis. To discuss several results in this field we recall that $C_b^k(\mathbb{R}^n)$ and $C_u^k(\mathbb{R}^n)$ are the spaces of k-times continuously differentiable functions on $\mathbb{R}^n$ whose higher derivatives are, respectively, bounded or uniformly continuous. We also introduce the space $C^{k,\omega}(\mathbb{R}^n) \subset C^k(\mathbb{R}^n)$ defined by the seminorm

$$|f|_{C^{k,\omega}} := \max_{|\alpha|=k} \sup_{x,y\in\mathbb{R}^n} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{\omega(|x-y|)} . \tag{5.1.1}$$

Here $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing, equal to 0 at 0 and concave; we will write $C^{k,s}(\mathbb{R}^n)$ for $\omega(t) := t^s$, $0 < s \leq 1$.

Finally, $\Lambda^\omega(\mathbb{R}^n)$ stands for the Zygmund space defined by the seminorm

$$|f|_{\Lambda^\omega} := \sup_{x\neq y} \frac{|f(x) - 2f(\frac{x+y}{2}) + f(y)|}{\omega(|x-y|)}; \tag{5.1.2}$$

here $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is as in (5.1.1), but we assume now that $\omega(\sqrt{t})$ is concave.

Let now $S \subset \mathbb{R}^n$ be an arbitrary closed subset and X be one of the above introduced function spaces. Then $X|_S$ denotes the linear space of traces of functions from X to S

endowed with the seminorm

$$|f|_X^S := \inf\{|g|_X \ : \ g|_S = f\}. \tag{5.1.3}$$

The (Whitney) linear extension problem can be formulated as follows.

Does there exist a linear continuous extension operator from $X|_S$ into X?

One can also consider the *restricted linear extension problem* with S belonging to a fixed class of closed (metric) subspaces of $\mathbb{R}^n$.

Whitney's paper [43] is devoted to a criterion for a function $f \in C(S)$ with $S \subset \mathbb{R}$ to belong to the trace space $C_b^k(\mathbb{R})|_S$ and gives, in fact, a positive solution to the linear extension problem for $C_b^k(\mathbb{R})$. It was noted in [44] that Whitney's method gives the same result for the spaces $C^{k,\omega}(\mathbb{R})$ and $C_u^k(\mathbb{R})$.

The situation for the multidimensional case is much more complicated. The restricted problem, for the class of *compact* subsets of $\mathbb{R}^n$ was solved positively by G. Glaeser [45] for the space $C_b^1(\mathbb{R}^n)$ using a special construction of the geometry of subsets in $\mathbb{R}^n$. However, for the space $C_u^1(\mathbb{R}^n)$, $n \geq 2$, the linear extension problem fails to be true, see [44], Theorem 2.5. In [44] the linear extension problem was solved positively for the spaces $C^{1,\omega}(\mathbb{R}^n)$ and $\Lambda^\omega(\mathbb{R}^n)$. A recent breakthrough due to Ch. Fefferman [46] in the problem of a constructive characterization of the trace space $C^{k,1}(\mathbb{R}^n)|_S$, allowed him to solve the linear extension problem for the space $C^{k,\omega}(\mathbb{R}^n)$, see [47], [48] and [49].

We will present a solution of the Whitney extension and trace problems for a specific class of differentiable functions on $\mathbb{R}^n$ and their restrictions to Ahlfors regular subsets.

5.2 Morrey-Campanato spaces on Ahlfors regular sets

Let $X \subset \mathbb{R}^n$ be a measurable set of positive Hausdorff s-measure. By $\mathcal{K}_X$ we denote the family of closed cubes in $\mathbb{R}^n$ with centers at X and "radii" ($:= \frac{1}{2}$ lengthside) at most $4 diam\, X$. We write $Q_r(x)$ for the cube of radius r and center x and denote by $X_r(x)$ the set $Q_r(x) \cap X$ for $x \in X$.

In order to introduce the basic concept, Morrey-Campanato space on X, we denote by $L_q(X)$, $1 \leq q \leq \infty$, the linear space of $\mathcal{H}_s$-measurable functions on X equipped with norm

$$||f||_q := \left(\int_X |f|^q \, d\mathcal{H}_s\right)^{1/q}, \quad 0 \le q \le \infty, \tag{5.2.1}$$

and use the following

Definition 9. *The local best approximation of order $k \in \mathbb{Z}_+$ is a function $\mathcal{E}_k : L_q(X) \times \mathcal{K}_X \to \mathbb{R}_+$ given for $Q = Q_r(x)$ by*

$$\mathcal{E}_k(f;Q) := \inf_p \left\{ \frac{1}{\mathcal{H}_s(X_r(x))} \int_{X_r(x)} |f-p|^q \, d\mathcal{H}_s \right\}^{1/q} \tag{5.2.2}$$

where p runs over the space $\mathcal{P}_{k-1} \subset \mathbb{R}[x_1, \ldots, x_n]$ of polynomials of degree $k-1$.

For $k = 0$ we let $\mathcal{P}_{k-1} := \{0\}$; hence $\mathcal{E}_0(f;Q)$ is the normalized L_q-norm of f on $X_r(x)$.

Let now $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a monotone function on $\mathbb{R}_+ := (0, \infty)$ (it may be a constant).

Definition 10. *The (generalized) Morrey-Campanato space $\dot{C}_q^{k,\omega}(X)$ is defined by semi-norm*

$$|f|_{\dot{C}_q^{k,\omega}(X)} := \sup \left\{ \frac{\mathcal{E}_k(f;Q)}{\omega(r_Q)} \;:\; Q \in \mathcal{K}_X \right\}$$

where r_Q denotes the radius of Q.

For X being a domain in $\mathbb{R}^n$ and $s = n$ this space coincides with the Morrey space $\mathcal{M}_q^\lambda$ [50] (for $k = 0$, $\omega(t) = t^\lambda$, $-n < \lambda < 0$), the BMO-space [51] (for $k = 1$, $\omega(t) = const$) and the Campanato space [32] (for $k \ge 1$, $\omega(t) = t^\lambda$, $\lambda > 0$). These spaces play an important role in Harmonic Analysis and in the theory of PDEs.

To formulate the main result we also need

Definition 11. *Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be nondecreasing such that*

$$\omega(+0) = 0 \quad \text{and} \quad t \to \frac{\omega(t)}{t^k} \quad \text{be nonincreasing.}$$

This ω is said to be a quasipower k-majorant if

$$C_\omega := \sup_{t>0} \left\{ \frac{1}{\omega(t)} \int_0^t \frac{\omega(u)}{u} \, du \right\} < \infty.$$

The Lipschitz space $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ *of order* $k \geq 1$ *consists of locally bounded functions* $f : \mathbb{R}^n \to \mathbb{R}$ *such that the seminorm*

$$|f|_{\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)} := \sup\left\{\frac{|\Delta_h^k f(x)|}{\omega(|h|)} \ : \ x, h \in \mathbb{R}^n\right\}$$

is finite.

Here $|h|$ is the Euclidean norm of h and

$$\Delta_h^k f(x) := \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + jh).$$

Choosing in this definition $\omega(t) := t^\lambda$, $0 < \lambda \leq k$, we obtain the (homogeneous) Besov space $B_\infty^\lambda(\mathbb{R}^n)$ (a type of abstract space which occurs in spline and rational function approximations). It coincides with the Sobolev space $\dot{W}_\infty^k(\mathbb{R}^n)$ for $\lambda = k$, the Hölder space $\dot{C}^{l,\alpha}(\mathbb{R}^n)$ for $\lambda = l + \alpha$, l is an integer and $0 < \alpha < 1$, and with the Marchaud-Zygmund space for λ integer and $0 < \lambda < k$. In the last case, the corresponding seminorm is

$$|f|_{B_\infty^\lambda(\mathbb{R}^n)} := \max_{|\alpha|=\lambda-1} \sup_h \frac{||\Delta_h^2(D^\alpha f)||_{C(\mathbb{R}^n)}}{|h|}.$$

Recall that a Sobolev space is a vector space of functions equipped with a norm that is a combination of L_p norms of the function itself as well as its derivatives up to a given order. The derivatives are understood in a suitable weak sense to make the space complete, thus a Banach space. Intuitively, a Sobolev space is a Banach space or Hilbert space of functions with sufficiently many derivatives for some application domain, such as partial differential equations, and equipped with a norm that measures both the size and smoothness of a function.

Our main result is the following theorem which gives a solution of the corresponding Whitney extension problem.

Theorem 9. *Let* $X \subset \mathbb{R}^n$ *be an s-set with* $n - 1 < s \leq n$ *and* ω *be a quasipower k-majorant. Then there is a linear continuous extension operator* $T_k : \dot{C}_q^{k,\omega}(X) \to \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$.

In particular, $\dot{C}_q^{k,\omega}(X)$ *is isomorphic to the trace space* $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_X$.

5.3 Proof of Theorem 9

Proof. It is well known, see, e.g., [16], Proposition VIII.1, that the closure $\overline{X}$ of an s-set X is also an s-set and $\mathcal{H}_s(\overline{X} \setminus X) = 0$. Moreover, the spaces $\dot{C}_q^{k,\omega}(\overline{X})$ and $\dot{C}_q^{k,\omega}(X)$ are isometric. Thus without loss of generality we may and will assume in the proof that X is closed.

Given $f \in \dot{C}_q^{k,\omega}(X)$ we should find a function $\widetilde{f} : X \to \mathbb{R}$ which equals f modulo zero $\mathcal{H}_s$-measure and admits an extension to a function from $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$.

We begin with

Lemma 7. *Let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a quasipower k-majorant, see Definition 11. Let $t_j := 2^j$, $j \in \mathbb{Z}_+$. Then for every pair of integers $-\infty < i < i' < \infty$ we have*

$$\sum_{j=i}^{i'} \omega(t_j) \le c(k,\omega)\, \omega(t_{i'}). \tag{5.3.1}$$

Proof. By the monotonicity of ω

$$\omega(t_j) \le \frac{1}{\ln 2} \int_{t_j}^{t_{j+1}} \frac{\omega(u)}{u}\, du$$

and therefore the sum in (5.3.1) is at most

$$\frac{1}{\ln 2} \int_{t_i}^{t_{i'+1}} \frac{\omega(u)}{u}\, du \le \frac{1}{\ln 2} C_\omega\, \omega(t_{i'+1}) \le \frac{C_\omega}{\ln 2} 2^k \omega(t_{i'})$$

for some constant C_ω depending only on ω. □

Our next result reformulates Theorem 3.5 of the paper [44] concerning the trace of the space $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ to an arbitrary closed subset $X \subset \mathbb{R}^n$, to adopt it to our situation. The trace space denoted by $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_X$ consists of locally bounded functions $f : X \to \mathbb{R}$ and is equipped with seminorm

$$|f|_{\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_X} := \inf\{|g|_{\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)} \ : \ f = g|_X\}. \tag{5.3.2}$$

To formulate the result we need

Definition 12. *Let $X \subset \mathbb{R}^n$ and $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be as above, and $\mathcal{T}_\omega := \{t_i\}_{i\in\mathbb{Z}_+}$ be the sequence of Lemma 7.*

A family $\Pi := \{P_Q\}_{Q\in\mathcal{K}_X}$ of polynomials of degree $k-1$ is said to be a (k,ω,X)-chain if for every pair of cubes $Q \subset Q'$ from $\mathcal{K}_X$ which satisfy for some $i \in \mathbb{Z}$ the condition

$$t_i \leq r_Q < r_{Q'} \leq t_{i+2} \tag{5.3.3}$$

the inequality

$$\max_{x\in Q} |P_Q(x) - P_{Q'}(x)| \leq C\omega(r_{Q'}) \tag{5.3.4}$$

holds with a constant C independent of Q, Q' and i.

The linear space of such chains is denoted by $Ch(k,\omega,X)$. It is equipped with semi-norm

$$|\Pi|_{Ch} := \inf C$$

where the infimum is taken over all constants C in (5.3.4).

Recall that $\mathcal{K}_X$ is the family of closed cubes centered at X and of radii at most $4 diam\, X$. In the sequel c_Q and r_Q stand for the center and the radius of the cube Q.

Using the concept introduced and the related notations we now formulate the desired result.

Proposition 5. *(a) A locally bounded function $f : X \to \mathbb{R}$ belongs to $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_X$ if and only if there is a (k,ω,X)-chain $\Pi := \{P_Q\}_{Q\in\mathcal{K}_X}$ such that for every $Q \in \mathcal{K}_X$*

$$f(c_Q) = P_Q(c_Q). \tag{5.3.5}$$

Moreover, the following two-sided inequality

$$|\Pi|_{Ch} \approx |f|_{\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_X}$$

holds with constants independent of f.

(b) If, in addition, this chain depends on f linearly, then there is a linear extension operator $T_k : \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_X \to \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ such that

$$||T_k|| \leq O(1)|\Pi|_{Ch}.$$

Hereafter $O(1)$ denotes a constant depending only on inessential parameters. It may change from line to line and even in a single line.

Proof. In the above cited paper this result is proved under the assumption that inequality (5.3.4) holds for any pair of cubes $Q \subset Q'$ centered at X. The restrictions (5.3.3) and $r_Q,\ r_{Q'} \le 4 diam\, X$ may be not satisfied for this pair. In the forthcoming derivation we explain how these restrictions can be disregarded to apply the aforementioned Theorem 3.5 of [44] and in this way to complete the proof of the proposition.

Consider first the case of an unbounded X. Hence, the only restriction is now inequality (5.3.3) and we should show that if a (k, ω, X)-chain satisfies condition (5.3.4) under restriction (5.3.3), then (5.3.4) holds for *any* pair $Q \subset Q'$ from $\mathcal{K}_X$. Note that the necessity of conditions (5.3.4) and (5.3.5) trivially follows from that in the aforementioned Theorem 3.5 from [44]. So we should only prove their sufficiency.

Assume that $f \in l_\infty^{loc}(X)$ (the function space of locally bounded functions on X) and conditions (5.3.3)-(5.3.5) hold. Let $Q \subset Q'$ be a pair of cubes from $\mathcal{K}_X$ of radii r and r', respectively. Then for some indices $i \le i'$

$$t_i \le r \le t_{i+1} \quad \text{and} \quad t_{i'} \le r' \le t_{i'+1}.$$

If $i = i'$, then by (5.3.4)

$$\max_Q |P_Q - P_{Q'}| \le 2|\Pi|_{Ch}\omega(t_{i+1}) \le 2\left(\frac{t_{i+1}}{t_i}\right)^k \omega(r')|\Pi|_{Ch} = 2^{k+1}\omega(r')|\Pi|_{Ch}$$

as is required.

Let now $i < i'$ and r_j with $i \le j \le i'+1$ are given by

$$r_i := r, \quad r_{i'+1} = 2r' \quad \text{and} \quad r_j := t_j \quad \text{for} \quad i < j < i'+1.$$

Let Q_j be the cubes centered at c_Q of radii r_j, $i \le j < i'+1$, and $Q_{i'+1}$ be the cube centered at $c_{Q'}$ of radius $r_{i'+1}$. (In particular, $\{Q_j\}_{i \le j \le i'+1} \subset \mathcal{K}_X$ is an increasing sequence of cubes with $Q_i := Q$.) Then

$$\max_Q |P_Q - P_{Q'}| \le \sum_{j=i}^{i'} \max_{Q_{j+1}} |P_{Q_j} - P_{Q_{j+1}}|. \tag{5.3.6}$$

It is easily seen that (5.3.3) holds for every pair $Q_j \subset Q_{j+1}$, $i \le j \le i'$. Applying (5.3.4) to each of these pairs and then (5.3.1) and the definition of ω we estimate the right-hand side of (5.3.6) by

$$2|\Pi|_{Ch} \sum_{j=i}^{i'} \omega(r_{j+1}) \le O(1)|\Pi|_{Ch}\omega(t_{i'+2}) \le O(1)|\Pi|_{Ch}\omega(r').$$

Thus we conclude that inequality (5.3.4) holds for every pair $Q \subset Q'$ of cubes centered at X.

Let now $diam\, X < \infty$. The previous argument proves the required inequality

$$\max_Q |P_Q - P_{Q'}| \le C\omega(r_{Q'}) \tag{5.3.7}$$

for every pair $Q \subset Q'$ from $\mathcal{K}_X$ under the restriction $r_{Q'} \le 2diam\, X$. Fix a cube $\widetilde{Q} \in \mathcal{K}_X$ with $r_{\widetilde{Q}} = 2diam\, X$ and introduce a new family of polynomials $\{\overline{P}_Q\}$, where Q runs over the set of all cubes centered at X, by setting

$$\overline{P}_Q := \begin{cases} P_Q, & \text{if} \quad r_Q \le diam\, X \\ \\ P_{\widetilde{Q}} - P_{\widetilde{Q}}(c_Q) + f(c_Q), & \text{if} \quad r_Q > diam\, X. \end{cases} \tag{5.3.8}$$

We will prove that the new family satisfies the hypotheses of Theorem 3.5 from [44]. This will complete the proof of the proposition in this case.

Clearly, $\{\overline{P}_Q\}$ satisfies condition (5.3.5), and if the chain Π depends linearly on f, then $\{\overline{P}_Q\}_Q$ depends linearly on f, as well. So we must check only that (5.3.4) holds for $\{\overline{P}_Q\}$ for every pair $Q \subset Q'$ of cubes centered at X. According to (5.3.7) and (5.3.8) inequality (5.3.4) holds for this family for every pair of cubes $Q \subset Q'$ with $r_{Q'} \le diam\, X$. Assume now that $r_{Q'} \ge r_Q > diam\, X$. Then by (5.3.8) we have

$$\max_Q |\overline{P}_Q - \overline{P}_{Q'}| \le |P_{\widetilde{Q}}(c_Q) - f(c_Q)| + |P_{\widetilde{Q}}(c_{Q'}) - f(c_{Q'})| \le$$

$$\max_{Q_1} |P_{\widetilde{Q}} - P_{Q_1}| + \max_{Q_2} |P_{\widetilde{Q}} - P_{Q_2}| \le 2C\omega(r_{\widetilde{Q}}) \le O(1)C\omega(r_{Q'}).$$

Here Q_1 and Q_2 are some cubes from $\mathcal{K}_X$ centered at c_Q and $c_{Q'}$, respectively, and contained in Q. The last two inequalities follow from (5.3.7) and the definition of ω.

Finally, if $r_Q \leq diam\, X < r_{Q'}$, then $Q \subset \widetilde{Q}$ and so we have by (5.3.7) and by the definition of ω

$$\max_Q |\overline{P}_Q - \overline{P}_{Q'}| \leq \max_Q |P_Q - P_{\widetilde{Q}}| + |P_{\widetilde{Q}}(c_{Q'}) - f(c_{Q'})| \leq 2C\omega(r_{\widetilde{Q}}) \leq O(1)C\omega(r_{Q'})$$

as is required.

Hence, in both of these cases the assumptions of Theorem 3.5 from [44] hold. This completes the proof of the proposition. □

Now we outline the proof of Theorem 9. Given $f \in \dot{C}_q^{k,\omega}(X)$ where $X \subset \mathbb{R}^n$ is a closed s-set, $n-1 < s \leq n$, we will define a new function $\widetilde{f} : X \to \mathbb{R}$ such that

$$\widetilde{f}(x) = f(x) \quad \mathcal{H}_s - \textit{almost everywhere on } X. \tag{5.3.9}$$

We then apply Proposition 5 to this function to show that $\widetilde{f} \in \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_X$ to construct a linear extension operator from $\dot{C}_q^{k,\omega}(X)$ to $\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$. To this end we will find for the $\widetilde{f}$ a (k,ω,X)-chain linearly depending on f. In the definition of the desired chain we will use the following construction. Let $Q := Q_r(x) \in \mathcal{K}_X$. By the Kadets-Snobar theorem [52] there is a linear projection π_Q from the space $L_1(X_r(x); \mathcal{H}_s)$ onto the subspace of polynomials of degree $k-1$ restricted to $X_r(x) := Q_r(x) \cap X$ whose norm $||\pi_Q||_1 \leq \sqrt{d_{k,n}}$ where $d_{k,n}$ is the dimension of the space of polynomials of degree $k-1$ on $\mathbb{R}^n$. Set

$$P_Q(f) := \pi_Q(f). \tag{5.3.10}$$

Using the definitions of $\widetilde{f}$ and $\{P_Q(f)\}_{Q \in \mathcal{K}_X}$ we will show that the following is true.

Claim 1. *There exists a (k,ω,X)-chain $\widetilde{\Pi}(f) := \{\widetilde{P}_Q(f)\}_{Q \in \mathcal{K}_X}$ linearly depending on f and such that*

$$|\widetilde{\Pi}(f)|_{Ch} \leq O(1)|f|_{\dot{C}_q^{k,\omega}(\mathbb{R}^n)|_X}. \tag{5.3.11}$$

Claim 2. *For every $Q \in \mathcal{K}_X$*

$$\widetilde{f}(c_Q) = \widetilde{P}_Q(f)(c_Q). \tag{5.3.12}$$

Since the operator $f \mapsto \widetilde{P}_Q(f)$ is linear, these allow us to apply Proposition 5 and to conclude that $\widetilde{f} \in \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)|_X$, and there is a linear extension operator $T_k : \dot{C}_q^{k,\omega}(X) \to$

$\dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ satisfying

$$||T_k|| \leq O(1)$$

completing the first part of the proof of Theorem 9. The fact that the restriction to X of every $f \in \dot{\Lambda}^{k,\omega}(\mathbb{R}^n)$ belongs to $\dot{C}_q^{k,\omega}(X)$ follows easily from Proposition 5 and Definition 10. This proves also the second assertion of the theorem and completes its proof.

To realize this program we need several auxiliary results. The main tool in their proofs is the weak Remez type inequality for s-sets, see Theorem 7 and (4.2.4).

Lemma 8. *For every $Q = Q_r(x) \in \mathcal{K}_X$*

$$\left\{ \frac{1}{\mathcal{H}_s(X_r(x))} \int_{X_r(x)} |f - \pi_Q(f)|^q \, d\mathcal{H}_s \right\}^{1/q} \leq O(1)\mathcal{E}_k(f;Q). \tag{5.3.13}$$

Proof. Here and below for $Q = Q_r(x) \in \mathcal{K}_X$ by P_Q we denote a polynomial of degree $k-1$ satisfying

$$\left\{ \frac{1}{\mathcal{H}_s(X_r(x))} \int_{X_r(x)} |f - P_Q|^q \, d\mathcal{H}_s \right\}^{1/q} = \mathcal{E}_k(f;Q). \tag{5.3.14}$$

Then

$$f - \pi_Q(f) = (f - P_Q) + \pi_Q(f - P_Q),$$

and applying the triangle inequality we estimate the left-hand side in (5.3.13) as is required but with the factor $(1 + ||\pi_Q||_q)$ instead of $O(1)$. So it remains to show that $||\pi_Q||_q \leq O(1)$. However, for $q = 1$ this norm is bounded by $\sqrt{d_{k,n}}$ by the definition. On the other hand, the weak Remez type inequality, see (4.2.4), and the fact that X is an s-set, imply that

$$||\pi_Q(g)||_1 \approx ||\pi_Q(g)||_q$$

with the constants of equivalence independent of g and Q. Thus by the Hölder inequality we have

$$||\pi_Q(g)||_q \leq O(1)||\pi_Q(g)||_1 \leq O(1)||\pi_Q||_1||g||_1 \leq O(1)||g||_q.$$

$\square$

Lemma 9. *Let $Q = Q_r(x) \in \mathcal{K}_X$. Then there exists the limit*

$$\widetilde{f}(x) := \lim_{Q \to x} P_Q(x) \tag{5.3.15}$$

and, moreover,

$$|\widetilde{f}(x) - P_Q(x)| \le O(1)\omega(r)|f|_{\dot{C}_q^{k,\omega}(X)}. \tag{5.3.16}$$

Proof. Let i be defined by

$$t_i < r \le t_{i+1} \tag{5.3.17}$$

and for $j \le i$

$$Q_j := Q_{t_j}(x), \quad P_j := P_{Q_j}.$$

Recall that $\{t_j\}$ is the sequence of Lemma 7. We also set $Q_{i+1} := Q$ and $P_{i+1} := P_Q$. Since X is an s-set, the weak Remez type inequality (4.2.4) implies that

$$|P_{j+1}(x) - P_j(x)| \le O(1)|||P_{j+1} - P_j; X_j|||$$

where for simplicity we set

$$|||g; X_j||| := \left\{ \frac{1}{\mathcal{H}_s(X_j)} \int_{X_j} |g|^q \, d\mathcal{H}_s \right\}^{1/q} \quad \text{and} \quad X_j := Q_j \cap X.$$

Adding and subtracting f and remembering the definition of P_j, see (5.3.14), we estimate the right-hand side of the last inequality by

$$O(1)\{\mathcal{E}_k(f; Q_j) + |||f - P_{j+1}; X_j|||\}.$$

By definition, the first term is bounded by $\omega(t_j)|f|_{\dot{C}_q^{k,\omega}(X)}$ while the second one is at most

$$\left(\frac{\mathcal{H}_s(X_{j+1})}{\mathcal{H}_s(X_j)}\right)^{1/q} \mathcal{E}_k(f; Q_{j+1}) \le \left(\frac{a t_{j+1}^s}{b t_j^s}\right)^{1/q} \omega(t_{j+1})|f|_{\dot{C}_q^{k,\omega}(X)},$$

see the definition of s-sets in Capter 4.

Since, in turn, $t_{j+1}/t_j \le 2$, using the definition of ω we finally get

$$|P_{j+1}(x) - P_j(x)| \le O(1)\omega(t_j)|f|_{\dot{C}_q^{k,\omega}(X)}.$$

This, Lemma 7 and the choice of i, see (5.3.17), yield

$$\sum_{j\le i}|P_{j+1}(x)-P_j(x)|\le O(1)|f|_{\dot{C}_q^{k,\omega}(X)}\sum_{j\le i}\omega(t_j)\le O(1)|f|_{\dot{C}_q^{k,\omega}(X)}\omega(t_i)\le$$

$$O(1)\omega(r)|f|_{\dot{C}_q^{k,\omega}(X)}.$$

This implies easily that the limit

$$\widetilde{f}(x):=\lim_{Q\to x}P_Q(x)=P_{i+1}(x)+\sum_{j\le i}(P_j(x)-P_{j+1}(x))$$

exists and, moreover,

$$|\widetilde{f}(x)-P_Q(x)|\le O(1)\omega(r)|f|_{\dot{C}_q^{k,\omega}(X)}.$$

$\square$

Lemma 10. *The assertions of the previous lemma hold with the same $\widetilde{f}(x)$ for $P_Q(f)$ substituted for P_Q.*

Proof. By (5.3.10)

$$P_Q-P_Q(f)=\pi_Q(P_Q-f)$$

and then Lemma 8 and inequality (4.2.4) yield

$$|P_Q(x)-P_Q(f)(x)|\le O(1)\max_{Q\cap X}|P_Q-P_Q(f)|\le O(1)|||P_Q-P_Q(f);Q\cap X|||\le$$

$$O(1)\{\mathcal{E}_k(f;Q)+|||f-P_Q;Q\cap X|||\}\le O(1)\mathcal{E}_k(f;Q)\le O(1)\omega(r)|f|_{\dot{C}_q^{k,\omega}(X)}.$$

This immediately implies that

$$\lim_{Q\to x}P_Q(f)(x)=\lim_{Q\to x}P_Q(x)=\widetilde{f}(x)$$

and gives the required estimate of $|\widetilde{f}(x)-P_Q(f)(x)|$ by the right-hand side of (5.3.16). $\square$

Hereafter we assume for simplicity that

$$|f|_{\dot{C}_q^{k,\omega}(X)}=1. \tag{5.3.18}$$

In particular, in this case

$$\mathcal{E}_k(f;Q)\le\omega(r_Q),\quad Q\in\mathcal{K}_X. \tag{5.3.19}$$

Lemma 11. *Let $Q \subset K$ be cubes from $\mathcal{K}_X$ of radii r and R, respectively, $r < R \le 2diam\, X$. Let $\widetilde{K}$ be the cube centered at c_K of radius $2R$. Then it is true that*

$$\mathcal{E}_1(f;Q) \le O(1)\left\{r\int_r^{2R}\frac{\omega(t)}{t^2}\,dt + \frac{r}{R}|||f;Q\cap\widetilde{K}|||\right\}. \tag{5.3.20}$$

Proof. Choose $J \in \mathbb{N}$ from the condition

$$R \le 2^J r < 2R$$

and let Q_j be the cubes centered at c_Q and of radii $r_j := 2^j r$, $j = 0, 1, \dots J-1$, and $Q_J := \widetilde{K}$, $r_J := 2R$. Then $\{Q_j\}_{0\le j\le J} \subset \mathcal{K}_X$ is an increasing sequence of cubes. We also set $P_j := P_{Q_j}$, $0 \le j \le J$, see (5.3.14) for the definition of $P_Q \in \mathcal{P}_{k-1}$. Under these notations we get

$$\mathcal{E}_1(f;Q) \le \left\{\mathcal{E}_1(f-P_Q;Q) + \sum_{j=0}^{J-1}\mathcal{E}_1(P_{j+1}-P_j;Q) + \mathcal{E}_1(P_{\widetilde{K}};Q)\right\}. \tag{5.3.21}$$

The first summand clearly equals

$$\mathcal{E}_k(f;Q) \le \omega(r) \le O(1)\,r\int_r^{2R}\frac{\omega(t)}{t^2}\,dt$$

as is required.

To estimate the remaining terms we use two inequalities whose proofs are postponed to the end.

(A) Let p be a polynomial of degree $k-1$ and $Q \in \mathcal{K}_X$ be a cube of radius r. Then

$$\mathcal{E}_1(p;Q) \le O(1) r \max_{|\alpha|=1} |||D^\alpha p; Q\cap X|||. \tag{5.3.22}$$

(B) Let, in addition, $\widetilde{Q} \in \mathcal{K}_X$ be a cube of radius $\widetilde{r}$ containing Q. Then

$$\max_{|\alpha|=1}|||D^\alpha p; Q\cap X||| \le O(1)\frac{1}{\widetilde{r}}|||p : \widetilde{Q}\cap X|||. \tag{5.3.23}$$

Using these inequalities to estimate the j-th term in (5.3.21) we get

$$r^{-1}\mathcal{E}_1(P_{j+1}-P_j;Q) \le O(1)\frac{1}{r_j}|||P_{j+1}-P_j;Q_j\cap X|||.$$

By the definitions of s-sets, ω and (5.3.19), the norm on the right-hand side is at most

$$O(1)\frac{1}{r_j}\left(\mathcal{E}_k(f;Q_j)+\left(\frac{\mathcal{H}_s(Q_{j+1}\cap X)}{\mathcal{H}_s(Q_j\cap X)}\right)^{1/q}\mathcal{E}_k(f;Q_{j+1})\right)\le O(1)\frac{\omega(r_j)}{r_j}.$$

Moreover, by the definition of r_j we get

$$\frac{\omega(r_j)}{r_j}\le O(1)\int_{r_j}^{r_{j+1}}\frac{\omega(t)}{t^2}\,dt,\quad 0\le j\le J-1.$$

Summing the finally obtained estimates over j we then have

$$\sum_{j=0}^{J-1}\mathcal{E}_1(P_{j+1}-P_j;Q)\le O(1)\,r\int_r^{2R}\frac{\omega(t)}{t^2}\,dt.$$

Using now (5.3.22) and (5.3.23) we bound the last summand in (5.3.21) by

$$O(1)r\frac{|||P_{\widetilde{K}};\widetilde{K}\cap X|||}{R}\le O(1)r\frac{2\,|||f;\widetilde{K}\cap X|||}{R}$$

as is required.

To complete the proof of the lemma it remains to prove (5.3.22) and (5.3.23). By the hypothesis of (A) we get

$$\mathcal{E}_1(p;Q)\le\inf_{\widetilde{p}}||p-\widetilde{p}||_{C(Q)}\le O(1)r\max_{|\alpha|=1}||D^\alpha p||_{C(Q)}$$

where $\widetilde{p}$ runs over the space of polynomials of degree 0. The second of these inequalities is proved as follows. Using a homothety of $\mathbb{R}^n$ we replace Q by the unit cube $Q_0:=[0,1]^n$. The functions in p of the both parts of this inequality are norms on the finite-dimensional factor-space $\mathcal{P}_{k-1}/\mathcal{P}_0$ and therefore they are equivalent. This implies the desired inequality.

Continuing the derivation we now use the weak Remez type inequality, see (4.2.4), and the fact that X is an s-set to have

$$||D^\alpha p||_{C(Q)}\le O(1)|||D^\alpha p;Q\cap X|||$$

and this completes the proof of (5.3.22).

Inequality (5.3.23) is proved in a similar way by means of the Markov inequality. $\square$

Lemma 12. *$f = \widetilde{f}$ modulo $\mathcal{H}_s$-measure zero.*

Proof. Let $L(f)$ be the Lebesgue set of f, i.e., the set of points $x \in X$ such that

$$f(x) = \lim_{r \to 0} \frac{1}{\mathcal{H}_s(X_r(x))} \int_{X_r(x)} f \, d\mathcal{H}_s.$$

Since X is an s-set, the family of "balls" $\{X_r(x) \ : \ x \in X, \ 0 < r \leq 1\}$ satisfies axioms (i), (ii) in [53] page 8. Therefore the Corollary of Section I.3 from this book can be applied to our case with the measure $\mu := \mathcal{H}_s|_X$. By this Corollary

$$\mathcal{H}_s(X \setminus L(f)) = 0.$$

It remains to show that

$$f(x) = \widetilde{f}(x) \quad \text{for} \quad x \in L(f).$$

To this end choose a cube $Q = Q_r(x) \in \mathcal{K}_X$, $0 < r < 1$, and set

$$f_r(x) := \frac{1}{\mathcal{H}_s(X_r(x))} \int_{X_r(x)} f \, d\mathcal{H}_s.$$

By the triangle inequality, the weak Remez type inequality for $f_r(x) - P_Q$, see (4.2.4), and the fact that X is an s-set we obtain

$$|f_r(x) - P_Q(x)| \leq O(1)\{|||f - f_r(x); Q \cap X||| + \mathcal{E}_k(f, Q)\}. \tag{5.3.24}$$

But $f \mapsto f_r$ is a projection from $L_1(X_r(x))$ onto the space $\mathcal{P}_0$ of polynomials of degree 0 whose norm is 1. Applying an argument similar to that of Lemma 8 with this projection substituted for π_Q we obtain that

$$|||f - f_r(x); Q \cap X||| \leq O(1)\mathcal{E}_1(f; Q),$$

and therefore by Lemma 11 and (5.3.19) for a sufficiently small r the right-hand side of (5.3.24) is bounded by

$$O(1)\{\mathcal{E}_1(f; Q) + \mathcal{E}_k(f; Q)\} \leq O(1)\left\{r\left(\int_r^2 \frac{\omega(t)}{t^2}\, dt + |||f; K \cap X|||\right) + \omega(r)\right\}$$

for some fixed cube K of radius 1 containing Q. We conclude from here that for every $0 < \epsilon < 2$

$$\lim_{r\to 0} |f_r(x) - P_Q(x)| \le$$

$$O(1) \limsup_{r\to 0} \left(\omega(r) + r \left(\int_r^\epsilon \frac{\omega(t)}{t^2}\, dt + \int_\epsilon^2 \frac{\omega(t)}{t^2}\, dt + |||f; K \cap X||| \right) \right) =$$

$$O(1) \limsup_{r\to 0} \left(r \int_r^\epsilon \frac{\omega(t)}{t^2}\, dt \right) \le O(1)\omega(\epsilon).$$

Letting $\epsilon \to 0$ and noting that $\lim_{r\to 0} f_r(x) = f(x)$ for the Lebesgue point x and $\lim_{Q\to x} P_Q(x) = \widetilde{f}(x)$ we complete the proof of the lemma. □

Now we finalize the proof of Theorem 9. For $Q \in \mathcal{K}_X$ and the polynomial $P_Q(f)$ of degree $k-1$ defined in (5.3.10) we set

$$\widetilde{P}_Q(f) := P_Q(f) - P_Q(f)(c_Q) + \widetilde{f}(c_Q).$$

Then $\widetilde{P}_Q(f)(c_Q) = \widetilde{f}(c_Q)$ and Claim 2, see (5.3.12), is true for the family $\widetilde{\Pi}(f) := \{\widetilde{P}_Q\}_{Q\in\mathcal{K}_X}$. Show that Claim 1 is also true for $\widetilde{\Pi}(f)$.

Let $Q \subset Q'$ be cubes from $\mathcal{K}_X$ of radii $r < r'$ satisfying for some i the condition

$$t_i \le r < r' \le t_{i+2}.$$

By the weak Remez type inequality, see (4.2.4), and Lemma 8 we have

$$\max_Q |P_Q(f) - P_{Q'}(f)| \le O(1) \max_{X\cap Q} |P_Q(f) - P_{Q'}(f)| \le$$

$$O(1)|||P_Q(f) - P_{Q'}(f); X \cap Q||| \le O(1) \left\{ \mathcal{E}_k(f; Q) + \left(\frac{\mathcal{H}_s(Q' \cap X)}{\mathcal{H}_s(Q \cap X)} \right)^{1/q} \mathcal{E}_k(f; Q') \right\}.$$

Both of the best approximations are bounded by $\omega(r')|f|_{\dot{C}_q^{k,\omega}(X)}$ while, since X is an s-set, the ratio of $\mathcal{H}_s$-measures is at most

$$\left\{ \frac{a}{b} \left(\frac{r'}{r} \right)^s \right\}^{1/q} \le O(1) \left(\frac{t_{i+2}}{t_i} \right)^{s/q} \le O(1).$$

Hence, in this situation, see (5.3.18),

$$\max_{Q} |P_Q(f) - P_{Q'}(f)| \le O(1)\omega(r').$$

Moreover, by Lemma 10, see(5.3.16),

$$|\widetilde{f}(c_Q) - P_Q(f)(c_Q)| \le O(1)\omega(r).$$

Taking into account the definition of $\widetilde{P}_Q(f)$ we then obtain the inequality

$$\max_{Q} |\widetilde{P}_Q(f) - \widetilde{P}_{Q'}(f)| \le O(1)\omega(r')$$

as is required in the definition of a (k, ω, X)-chain.

This completes the proof of Claim 1 and therefore of Theorem 9. $\square$

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