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Solutions of the Equation of Motion with Absorption for some Common Sources

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Solutions of the Equation of Motion with Absorption for some Common Sources

by

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Abstract

Different kinds of solutions of the equation of motion (EOM) in a perfectly elastic medium have been derived and also have been widely recognized. However, in fact, we can hardly find an ideal medium without absorption. And, anelasticity of the earth causes physical dispersion of seismic waves. Dispersion resulting from absorption in the propagation medium has been included in the approximations of solutions of the equation of motion for some common sources (a directed point force, double-couple-without-moment forces and a shear-dislocation force) by replacing the velocity or slowness with the complex version. A velocity-frequency relation in the form of $\frac{1}{v(\omega)} = \frac{1}{v(\omega_{0r})}\left[1 - \frac{1}{\pi Q} ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{i}{2Q}\right]$ has been used, where ω is the angular frequency, $v(\omega_{0r})$ is the phase velocity at the reference frequency ω_{0r} . These approximations match very well with the exact numerical results, and the anelastic waveforms have significant differences in amplitude and shape than the elastic ones. Therefore, developing new solutions of the EOM with absorption is a very meaningful thing.

Keywords: equation of motion, absorption, dispersion.

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Dedication

I would like to dedicate this work to my beloved Mom and Dad for supporting me throughout my thesis.

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List of Symbols, Abbreviations and Nomenclature

Symbol	Definition
EOM	Equation of motion
K-K	Kramers-Krönig
FT	Fourier transform
IFT	Inverse Fourier transform
DFT	Discrete Fourier transform
FFT	Fast Fourier transform
IFFT	Inverse fast Fourier transform
Q	Quality factor
ω	Angular frequency
ω_{0r}	Reference angular frequency
u	Displacement
ρ	Density
σ_{ij}	Stress tensor
α	Velocity of P-wave
β	Velocity of S-wave
λ, μ	Lame constant
δ_{ij}	Kronecker delta function
M_{jk}	Moment tensor
f	frequency
r	Propagation distance
x	The vector from the origin to the receiver
γ	A unit vector in the direction of x
k	Wavenumber
$\bar{u}$	The averaged displacement discontinuity

CHAPTER 1: INTRODUCTION AND LITERATURE REVIEW

1.1 Thesis Overview

In the past few decades, geophysicists have developed different solutions of the elastic equation of motion for different sources, e.g., the solutions of the equation of motion showed in Aki and Richards (2002), and these solutions have also been widely recognized. However, our earth is not an ideal elastic body. Influence of absorption and dispersion caused by anelasticity should be considered.

The objective of this project is to develop new solutions of the equation of motion with absorption and dispersion effect for some common sources by converting the solutions of the elastic equation of motion into frequency domain and replacing velocities with the complex versions including absorption and dispersion terms in frequency domain, and then transform the new solutions back into time domain.

Relevant background knowledge will be introduced in the following contents in chapter 1. Chapter 2 will give the basic methodology and algorithm about how to deal with this problem. And, chapter 3, 4, and 5 will display the new solutions I have got for three different models (a directed point force model, a double-couple-without-moment source model, and a shear-dislocation source model), respectively. After that, this project will be summarized in chapter 6.

1.2 Equation of Motion

The equation of motion is an equation that describes the propagation characteristics of seismic waves in a medium as a set of mathematical functions expressed in terms of dynamic variables. The nature of these waves can be determined by solving the EOM (equation of motion). By analyzing the changes in waveforms during the propagation process, we can further deduce the structure and lithology of the medium and the properties of the seismic source. The equation of motion, in component form is given by

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2}\ , \quad i = 1, 2, 3. \ \text{ or } \ \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2}\ , \quad i = 1, 2, 3. \tag{1}$$

The equation (1) is the general equation of motion. If the summation convention and the simplified notation for partial derivatives is used, $\frac{\partial \sigma_{ij}}{\partial x_j} = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial \sigma_{i1}}{\partial x_1} + \frac{\partial \sigma_{i2}}{\partial x_2} + \frac{\partial \sigma_{i3}}{\partial x_3} = \sigma_{i1,1} + \sigma_{i2,2} + \sigma_{i3,3} = \sigma_{ij,j}$ and $\frac{\partial^2 u_i}{\partial t^2} = \ddot{u}_i$, then the equation (1) is written as

$$\sigma_{ij,j} + f_i = \rho \ddot{u}_i\ , \quad i = 1, 2, 3. \ \text{ or } \ \sum_{j=1}^{3} \sigma_{ij,j} + f_i = \rho \ddot{u}_i\ , \quad i = 1, 2, 3. \tag{2}$$

where $\mathbf{f} = \mathbf{f}(\mathbf{x}, \mathrm{t})$ is the body force density, $\rho = \rho(\mathbf{x})$ is the density, $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}, \mathrm{t})$ is the physical stress tensor, $\mathbf{x}$ is position, and t is time. $\mathbf{u} = \mathbf{u}(\mathbf{x}, \mathrm{t})$ is the displacement and $\ddot{\boldsymbol{u}} = \partial^2 \boldsymbol{u}/\partial t^2$ is the acceleration.

Over the past few decades, many kinds of the solutions of the equation of motion in a perfectly elastic medium have been derived and also have been widely recognized. For example, the solution of the EOM for a directed point force (see, e.g., Aki and Richards I, 2002, p. 72, eq. 4.23, and Achenbach, 1973, p. 100) is

$$u_i(\mathbf{x}, t) = \frac{(3\gamma_i\gamma_j - \delta_{ij})}{4\pi\rho r^3} \int_{r/\alpha}^{r/\beta} \tau s(t - \tau)\, d\tau + \frac{\gamma_i\gamma_j}{4\pi\rho\alpha^2 r} s\left(t - \frac{r}{\alpha}\right) + \frac{(\delta_{ij} - \gamma_i\gamma_j)}{4\pi\rho\beta^2 r} s(t - \frac{r}{\beta})$$

$$\equiv\ u_i{}^N + u_i{}^P + u_i{}^S, \qquad i = 1, 2, 3 \tag{3}$$

where u_i is the i-th component of displacement, $\mathbf{x}$ is the vector from the origin to the observation point (i.e., the location of the seismometer), and r is the distance between origin and observation point, and $\boldsymbol{\gamma} \equiv \mathbf{x}/r$ is a unit vector in the direction of $\mathbf{x}$ (i.e., $\gamma_n = x_n/r$ is the direction cosine between $\mathbf{x}$ and the x_n axis). Since the wave travels from the origin to $\mathbf{x}$, the vectors $\mathbf{x}$ and $\boldsymbol{\gamma}$ are in the direction of wave propagation. j indicates the direction of the point force. α is the velocity of P-wave and β is the velocity of S-wave,

$$\alpha = v_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}\,, \qquad \beta = v_s = \sqrt{\frac{\mu}{\rho}} \tag{4}$$

and ρ is density. δ_{ij} is the Kronecker delta (named after Leopold Kronecker) and it is defined as follows:

$$\delta_{ij} = \begin{cases} 0 & if\ i \neq j \\ 1 & if\ i = j \end{cases} \tag{5}$$

The first term, u_i^N, is the near-field term. It dominates over the other two terms at small values of r. u_i^N term is a convolution integral. This means that if the source pulse s(t) has finite duration T (i.e., $\mathrm{s(t)} \neq 0$ only for $0 \leq t \leq T$), then u_i^N is non-zero only in the range $(r/\alpha) \leq t \leq (r/\beta) + T$. The second term, u_i^P, is the far-field P wave term. And the third term, u_i^S, is the far-field S wave term.

Note that the far-field terms decay as $1/r$ due to the geometrical spreading of the wavefronts. The near-field term appears to decay as $1/r^3$ but it actually does not decay that fast because r appears in the limits of the integral, making the integral a function of *r*. For example, for an impulse (or, in general, a pulse s(t) of short duration), the near-field term decays roughly as $1/r^2$. Thus, it is the dominant term in u_i for small *r* (i.e., in the near-field). However, note also that the terms “near-field” and “far-field” have meaning only if the source pulse is “short” enough, i.e., if the pulse duration is significantly smaller than the typical travel-times, or equivalently, if the receiver distance *r* is large enough. Otherwise, for “long” pulses (or small *r*), both the near-field and far-field terms are important at all *r*, i.e., neither dominates (Krebes, 2004).

Geophysicists recognized that most earthquakes are caused by shear faulting (e.g., Gilbert, 1884; Lawson, 1908; Reid, 1910), and a shear fault in anisotropic elastic medium is equivalent to a distribution of double-couple mechanisms over the fault surface (Maruyama, 1963; Burridge and Knopoff, 1964).

Besides that, a wide variety of processes, including unsteady fluid flow, shear faulting on ring structures, and tensile faulting are particularly likely in geothermal and volcanic environments, where observed non-DC observations are commonest (Julian et al., 1998).

Therefore, in addition to the solution of the equation of motion for a directed point force I mentioned above, I will also discuss about the solutions of the EOM for a double-couple-without-moment source mechanism, and a shear dislocation source.

For the general cases we mentioned above, both of them can be derived from the solution of the EOM (6), which gives the radiation from any moment tensor **M** (the formula can be found in Aki and Richards, 2002, eq. 4.29, p. 77):

$$\begin{aligned} u_i(\mathbf{x},t) = {} & \frac{\left(15\gamma_i\gamma_j\gamma_k - 3\gamma_i\delta_{jk} - 3\gamma_j\delta_{ik} - 3\gamma_k\delta_{ij}\right)}{4\pi\rho}\frac{1}{r^4}\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau M_{jk}(t-\tau)\,d\tau \\ & + \frac{\left(6\gamma_i\gamma_j\gamma_k - \gamma_i\delta_{jk} - \gamma_j\delta_{ik} - \gamma_k\delta_{ij}\right)}{4\pi\rho\alpha^2}\frac{1}{r^2}M_{jk}\left(t-\frac{r}{\alpha}\right) \\ & - \frac{\left(6\gamma_i\gamma_j\gamma_k - \gamma_i\delta_{jk} - \gamma_j\delta_{ik} - 2\gamma_k\delta_{ij}\right)}{4\pi\rho\beta^2}\frac{1}{r^2}M_{jk}\left(t-\frac{r}{\beta}\right) \\ & + \frac{\gamma_i\gamma_j\gamma_k}{4\pi\rho\alpha^3}\frac{1}{r}\dot{M}_{jk}\left(t-\frac{r}{\alpha}\right) - \frac{\gamma_k\left(\gamma_i\gamma_j - \delta_{ij}\right)}{4\pi\rho\beta^3}\frac{1}{r}\dot{M}_{jk}\left(t-\frac{r}{\beta}\right) \end{aligned} \tag{6}$$

The near-field terms in this displacement field are proportional to $r^{-4}\int_{r/\alpha}^{r/\beta}\tau M_{jk}(t-\tau)\,d\tau$, and the far-field terms are proportional to $r^{-1}\dot{M}_{jk}(t-r/\alpha)$ (P-waves) or to $r^{-1}\dot{M}_{jk}(t-r/\beta)$ (S-waves). Present in (6) are some terms proportional to $r^{-2}M_{jk}(t-r/\alpha)$ and $r^{-2}M_{jk}(t-$

r/β). Since their asymptotic properties, at small and large values of r, are intermediate to the asymptotic properties of the near-field and far-field displacements, we can naturally call these the intermediate-field terms. This is, however, a slightly misleading name, since there is no intermediate range of distances in which these terms dominate, so it is common to include them with the near-field terms. Vidale *et al.* (1995) pointed out an unusual example where an effect of these intermediate terms is observable at great distance from a very large deep earthquake.

M_{jk} is called the moment tensor. The moment tensor is a second-rank tensor, which describes a superposition of nine elementary force systems, with each component of the tensor giving the moment of one force system. The diagonal components M_{11}, M_{22}, and M_{33} correspond to linear dipoles the exert no torque, and the off-diagonal elements M_{12}, M_{13}, M_{21}, M_{23}, M_{31}, and M_{32} correspond to force couples. It it usually assumed that the moment tensor is symmetric ($M_{12} = M_{21}$, $M_{13} = M_{31}$, $M_{23} = M_{32}$), so that the force couples exert no net torque, in which case only six moment tensor components are independent (Julian *et al*., 1998). M_{jk} represents a force couple in the sense that as $\xi_k \rightarrow 0$, $F_j \rightarrow \infty$ in such a way that M_{jk} remains finite. $M_{jk}(t) \equiv F_j(t)\xi_k$ is the moment, or torque, produced by the *j*-th component of the force $\boldsymbol{F}$, applied at $\boldsymbol{\xi}$, about the origin – see Fig.1(a) below. Also, $\dot{M}_{jk}(t) \equiv \dot{F}_j(t)\xi_k$. Diagrams representing the components M_{jk} are shown in Fig.1(b).

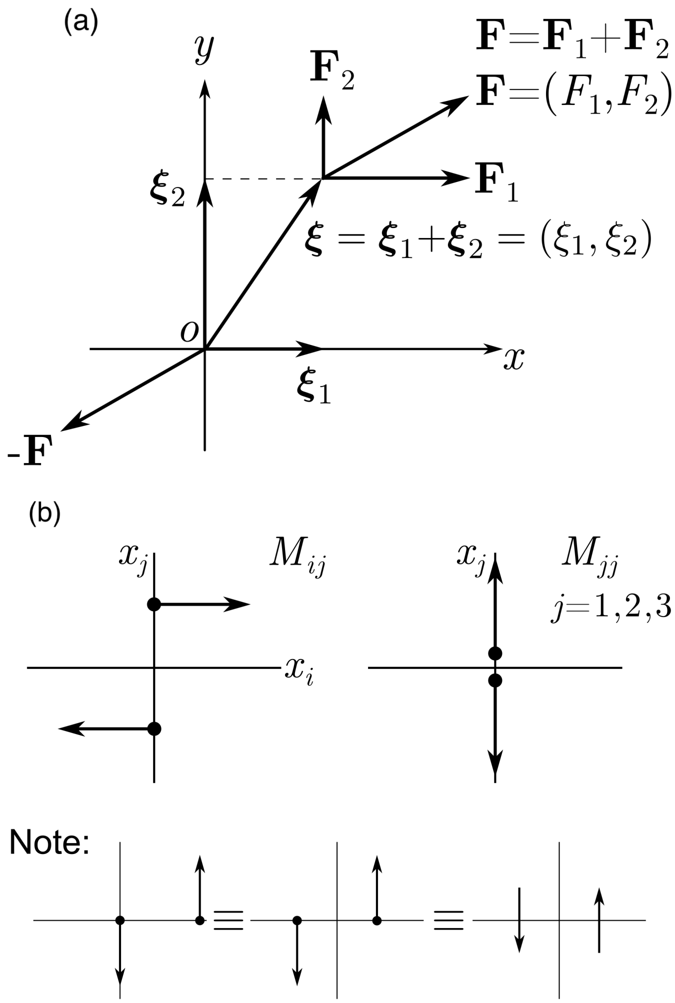


Fig.1. (a) Physical meaning of the moment tensor ($F_1\xi_2 = M_{12}$ =torque about origin O due to $\vec{F}_1$, and $F_2\xi_1 = M_{21}$ =torque about origin O due to $\vec{F}_2$). (b) Force couples corresponding to the moment tensor components. (the figure can be found in Krebes, 2004, Figure DPF-6)

Aki and Richards (2002) also turn the expression (6), for the displacement field radiated by a shear dislocation, from its Cartesian form into a form that naturally brings out the radial and transverse components of motion (see, eq. 4.30, p. 78).

$$u_i(\mathbf{x}, t) = \frac{\left(15\gamma_i\gamma_j\gamma_k v_k - 3v_i\gamma_j - 3\delta_{ij}\gamma_k v_k\right)}{2\pi\rho r^4}\mu A \int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau\bar{u}_j(t-\tau)\, d\tau$$

$$+\frac{\left(6\gamma_i\gamma_j\gamma_k v_k - v_i\gamma_j - \delta_{ij}\gamma_k v_k\right)}{2\pi\rho\alpha^2 r^2}\mu A\bar{u}_j\left(t-\frac{r}{\alpha}\right)$$

$$-\frac{\left(12\gamma_i\gamma_j\gamma_k v_k - 3v_i\gamma_j - 3\delta_{ij}\gamma_k v_k\right)}{4\pi\rho\beta^2 r^2}\mu A\bar{u}_j\left(t-\frac{r}{\beta}\right)$$

$$+\frac{\gamma_i\gamma_j\gamma_k v_k}{2\pi\rho\alpha^3 r}\mu A\bar{u}_{j'}\left(t-\frac{r}{\alpha}\right)$$

$$-\frac{2\gamma_i\gamma_j\gamma_k v_k - v_i\gamma_j - \delta_{ij}\gamma_k v_k}{4\pi\rho\beta^3 r}\mu A\bar{u}_{j'}\left(t-\frac{r}{\beta}\right) \quad (7)$$

The averaged displacement discontinuity, $\bar{\boldsymbol{u}}$, is then parallel to the fault surface: $\bar{\boldsymbol{u}}\cdot\boldsymbol{v} = 0$, where $\boldsymbol{v}$ is normal to the fault surface. And $M_{jk} = \mu(\bar{u}_j v_k + \bar{u}_k v_j)A$ for a fault with area A.

In recent research, when we refer to the solutions of the EOM for seismic waves for some common source (e.g., the solutions of the EOM for a directed point force, a double-couple-without-moment source, and a shear-dislocation source (Equation (3) and Equation (6)) we mentioned above) we always think of the ideal situation, i.e., a perfectly elastic medium, with no absorption.

However, in fact, we can hardly find an ideal medium in practical applications. In real materials, wave energy is absorbed due to internal friction or anelasticity (a short table about quality factor Q and velocity for different materials has been showed in Table 1). In addition, typical values of Q in the Earth are: Q = 5-20 for near surface layers, Q = 20-100 for the upper crust, Q = 50-150 for the lower crust, and Q = 100-500 for the upper mantle. For most materials, Q ≫1. Absorption is frequency dependent, i.e., different frequencies are absorbed by different amounts. One consequence of this is that the waveform changes with distance travelled. Therefore, developing new solutions of the equation of motion with absorption is a very meaningful thing.

Table 1. *Q* and velocity for reference materials*

Material	Q	V(km/sec)[1]	Mode	Frequency range
Aluminum	200,000	5.00	Longitudinal resonance	1 to 200 kHz
	5,900	6.32	P-wave pulse	3.1 to 7.5 MHz
	7,630	6.32	P-wave pulse	5 to 15 MHz
	19,400	3.10	S-wave pulse	3.5 to 4.5 MHz
	17,200	3.10	S-wave pulse	3 to 6.8 MHz
Brass	655	3.48	Flexural resonance	—
Copper				
Unannealed	2,180	3.81	Longitudinal resonance	2.5 to 30 kHz
	4,380	2.32	Torsional resonance	3 to 30 kHz
	1,770	4.76	P-wave pulse	15 to 65 MHz
Annealed	5,830	5.01	P-wave pulse	25 to 75 MHz
Lead	36	1.21	Longitudinal resonance	1.6 to 15 kHz
	34	0.69	Torsional resonance	1 to 9 kHz
Magnesium	965	5.77	P-wave pulse	7 to 76 MHz
Nickel	980	4.90	Flexural resonance	12 to 33 Hz
Steel	1,850	5.20	Flexural resonance	2 to 8 Hz
Celluloid	7	2.81	Flexural resonance	0.5 to 18 Hz
Fused Quartz	44,500	3.76	S-wave pulse	5 to 19 Hz
Glass	490	5.36	Flexural resonance	12 to 27 Hz
Glass (Pyrex)	1,860	5.17	Longitudinal resonance	10 kHz
Glass (Soda lime)	1,450	4.54	Longitudinal resonance	5.6 to 6.1 kHz
	1,340	2.84	Torsional resonance	3.6 to 64 kHz
Lucite[2]	23	2.11	Longitudinal resonance	1 kHz
Plexiglas[3]	20	2.59	Longitudinal resonance	10 kHz
Polystyrene	240	2.24	Longitudinal resonance	20 to 60 kHz
Air				
Dry	562	0.343	Resonance	100 Hz
	3,485	0.343		10 kHz
100 percent humidity	4,139	0.345	Resonance	100 Hz
	1,434	0.345		10 kHz
Water				
Fresh (17°C)	210,000	1.48	Resonance	100 kHz
Salt (36 ppm)	63,000	1.52		150 kHz

* These data are taken primarily from the compilations of Bradley and Fort (1966), Knopoff (1964), and the Chemical Rubber Company Handbook of Chemistry and Physics.

[1] Velocities represent typical values for the mode of excitation listed.

[2] Winkler, K., 1979, Ph.D. thesis, Stanford Univ.

[3] Johnston, D.H., 1978, Ph.D. thesis, Massachusetts Institute of Technology.

Table.1 Q and velocity for reference materials (Johnston, 1981, Table 1)

1.3 Absorption

All the solutions of the EOM we talked above are all defined in a perfectly elastic medium. However, from many researches, the fact that seismic waves attenuate with distance and that free oscillations decay with time indicate that the Earth is not an ideal elastic body. A lot of works have shown the differences between wave propagation in anelastic media and in elastic media, such as Lockett (1962), Cooper and Reiss (1966), Cooper (1967), Shaw and Bugl (1969), Buchen (1971), Schoenberg (1971), Borcherdt (1973, 1977), Krebes (1983), and Krebes and Daley (2007), and Carcione (2007).

Other processes which attenuate waves include geometrical spreading, reflection and transmission, diffraction, and scattering. But in this project, we will only consider absorption.

In real materials, wave energy is absorbed due to internal friction or anelasticity. Anelastic media are sometimes called dissipative media. Absorption is frequency dependent, i.e., different frequencies are absorbed by different amounts. One consequence of this is that the waveform changes with distance travelled. And we are accustomed to using the quality factor Q to express the effect of absorption on the waveform.

A dimensionless parameter useful for describing energy loss is Q^{-1}. In some treatments, $2\pi Q^{-1}$ is defined as the ratio of the loss in energy density per cycle of forced oscillation to the peak energy density stored during the cycle (e.g. Zener, 1948; Kolsky, 1963; Knopoff, 1964b; Anderson et al., 1965). In some other literatures, “mean” is used instead of “peak” (e.g. Buchen, 1971; O’Connell and Budiansky, 1978).

We first review the basic theory associated with this problem. Most of this theory is taken from Krebes (2004).

A dimensionless frequency-dependent parameter Q, which called the quality factor, is used as the standard measure of inverse-attenuation in nowadays, see (Pilant, 1979).

Consider a sinusoidal wave passing through a volume of material. Let E be the peak strain energy stored in the volume per cycle. Let A be the peak amplitude of the oscillation. In some treatments, “mean” is used instead of “peak”. Let $-\Delta E$ and $-\Delta A$ be the energy and amplitude, respectively, lost in each cycle due to anelasticity (ΔE and ΔA are $<$ 0). Then the quality factor Q is defined as follows:

$$\frac{1}{\mathrm{Q}(\omega)} = -\frac{\Delta E}{2\pi E} \quad or \quad \frac{1}{Q(\omega)} = -\frac{\Delta A}{\pi A} \tag{8}$$

The second equation comes from the fact that $E \sim A^2$, meaning $\delta E/E \sim 2\delta A/A$. For a perfectly elastic medium, $\mathrm{Q} = \infty$ and the loss factor 1/Q = 0.

Normally, it is assumed that absorption is a linear phenomenon, so that Fourier analysis can be used. Also, there are, technically, two types of Q: temporal and spatial.

Q_{temp} is obtained from the decay in time of the peak amplitude of a signal measured at a fixed spatial location (e.g., a seismic trace, a standing wave, the Earth's free oscillation data).

Q_{spat} is obtained from the decay in space of the peak amplitude of a signal measured at a fixed time (e.g., a photo of a medium in wave motion). Q_{spat} is used in seismic wave propagation studies because we are interested in the spatial decay of a wave due to anelasticity.

The "decay" mentioned above refers to decay due to absorption, not geometrical spreading or any other effect. Generally, $Q_{spat} \neq Q_{temp}$. For example, consider an absorbing medium containing a point source producing a continuous sine wave, $\sin[\omega_0 t]$. A receiver somewhere in the medium would also record a continuous sine wave. Its amplitude would be < 1 (due to absorption occurring between the source and receiver), but it would be a constant amplitude, i.e., the sine wave on the receiver trace would not be damped, meaning $Q_{temp} = \infty$. However, the medium is anelastic, i.e., a photograph of the medium at a fixed time would show the peak amplitude of the sine wave decaying with distance, meaning Q_{spat} is finite. Since the medium is anelastic, we want Q to be finite in value, so Q_{spat} would be used here. For another example, consider the opposite case, i.e., consider the source pulse to be a decaying sine wave, but the medium to be perfectly elastic ($= \infty$). The receiver trace would show a decaying sine wave, meaning Q_{temp} is finite. But we want Q to be infinite since the medium is perfectly elastic. Hence, in both examples, and generally in wave absorption studies, Q_{spat} is used. Consequently, from here on, we will drop the subscript "spat" on Q, and assume that "Q" is the spatial Q.

Consider a photograph or snapshot of an absorbing medium taken while a wave is travelling in the x direction. Consider a small distance δx. Let λ be the wavelength and V the wave speed. Assume Q ≫ 1 (weak absorption), which is true for most materials. Assuming weak absorption also means that A and λ do not change much over a cycle, meaning that the concepts of "amplitude" and "wavelength" are still physically meaningful. We then have

$$\Delta \mathrm{A} = \frac{change}{cycle} = \frac{change}{distance} \cdot \frac{distance}{cycle} = \frac{\delta A}{\delta x}\lambda \rightarrow \frac{dA}{dx}\lambda = \frac{dA}{dx}\frac{2\pi V}{\omega} \tag{9}$$

Substituting this into (8) then gives

$$\frac{dA}{dx} = -\left[\frac{w}{2VQ}\right]A \;\; \Rightarrow \;\; A(x) = A_0 \exp\left[-\frac{\omega x}{2VQ}\right] \tag{10}$$

A similar argument for temporal Q gives $A(\mathrm{t}) = A_0 \exp[-\omega t/2Q]$ (with $\mathrm{Q} = Q_{temp}$). Consider a 1D sinusoidal wave

$$u = A\exp[i(Kx - \omega t)], \quad K = \omega/V \tag{11}$$

in an absorbing medium. Substituting (10) into this gives

$$u = A_0 \exp[-ax]\exp[i(Kx - \omega t)] = u = A_0 \exp[i(kx - \omega t)],$$

$$a = \frac{\omega}{2VQ}, \quad k = K + ia = \left(\frac{\omega}{V}\right)\left(1 + \frac{i}{2Q}\right). \tag{12}$$

This shows that absorption can be included in wave motion by making the wavenumber complex (with the frequency ω being real). Equivalently, one can make the slowness complex, i.e., if $k = \omega/v$, where v is the complex wave speed, then

$$k = \omega/v \quad \Rightarrow \quad \frac{1}{v} = \frac{1}{V}\left(1 + \frac{i}{2Q}\right)$$

$$\text{or, for } Q \gg 1, \qquad v = V\left(1 - \frac{i}{2Q}\right) \tag{13}$$

Where the rule $\{(1+z)^{\epsilon} \approx 1 + \epsilon z, \;\; |\epsilon z| \ll 1\}$ has been used to get the last equation in (13). And, where V and Q are arbitrary positive real-valued constants. The imaginary parts of the complex velocities are the absorption terms.

The equation (13) are also well-known formulas for the complex velocity corresponding to a non-causal model of dissipative medium, which includes the effects of anelasticity by making the P-wave and S-wave velocities complex (e.g., Kennett, 1975; O'Neill and Hill, 1979).

Generally, an absorbing medium is dispersive, i.e., V = V(ω) and Q = Q(ω). Absorption can be included for a temporal Q by making ω complex (and keeping the wavenumber real).

But for seismic body waves, Q is nearly independent of frequency. The essentially constant-Q condition has been discussed by Kolsky (1960) and Knopoff (1964b). It has been found that the parameter Q depends only very slightly on frequency in a broad range of frequencies (Lomnitz, 1957; Liu et al., 1976; Kjartansson, 1979).

Lots of researches about the quality factor Q have been introduced in last few decades.

Anderson and Archambeau (1964) and Anderson et al. (1965) investigated a range of simple Q models and concluded that there was a low-Q zone at the top of the mantle, that Q increased with depth in the mantle, and that losses in pure compression were negligible compared to those in shear. The Q for shear waves in the upper 400 km of the mantle seems to vary from about 50 to about 150. The Q for mantle Rayleigh waves is greater than the Q for mantle Love waves (Anderson et al., 1965).

Anderson and Hart (1978) presented a family of models to determine a self-consistent earth model (Jeffreys, 1965; Davles, 1967; Akopyan et al., 1975; Liu et al., 1976; Anderson et al., 1976; Hart et al., 1976, 1977) and to help identify the normal modes spectral peaks (Gilbert and Dziewonski, 1975). They also indicated that there is a low-Q upper mantle layer underlying a high-Q lithosphere. And Q smoothly increase with depth over most of the lower mantle and a low-Q zone at the base of the mantle.

The Q of the outer core is apparently extremely high. Most of studies infer a Q of greater than 4000 (e.g., Buchbinder, 1971; Sacks, 1972; Muller, 1973; Qamar and Eisenberg, 1974.), and an average inner core Q around 500 (see, e.g., Buchbinder, 1971; Qamar and Eisenberg, 1974; Buland and Gilbert, 1978).

Further information about Q can be found in Knopoff, 1964b.

1.4 Dispersion

We have talked about how to include absorption effect into solutions of the equation of motion above, but to be physically realistic, one must also include dispersion in the calculations, to ensure causality.

Anelasticity of the earth causes physical dispersion of seismic waves. The significant effect of physical dispersion on surface wave phase and group velocities and free oscillation periods has been discussed many times, e.g. Jeffreys (1965), Davies (1967), Randall (1976), Kanamori and Anderson (1977).

It has been found that the effects of velocity dispersion due to attenuation are rather significant and well-measurable, see, e.g., Liu et al. (1976), Richards (1979).

Although Kennett (1975) suggests that above approximation should be adequate for band-limited source functions unless Q is very low, later researches (e.g. Kanamori and Anderson, 1977; O'Neill and Hill, 1979) indicate that the physical dispersion of seismic waves is required by causality and linearity when Q is nearly frequency-independent. Otherwise, significant errors may be introduced into computed travel times and wave forms by a noncausal approximation.

The consequences of dispersions are that one cannot directly compare body wave, surface wave, and free oscillation data or compare laboratory ultrasonic and shock wave data with seismic data, unless corrections are made for phase velocity dispersion arising from anelasticity (Liu et al., 1976).

And Strick also indicated in 1970 that the effect of absorption is to round off the apparent (visual) onset as well as the peak of the impulse response.

Nonlinear friction is commonly assumed to be the dominant attenuation mechanism, especially in crustal rocks (see, e.g., McDonal et al., 1958; White, 1966; Gordan and Davis, 1968; Johnston and Toksoz, 1977; Lockner et al., 1977; Tutuncu et al., 1998; Zhao and Cai, 2001; Mashinskii, 2006; Sleep et al. 2017). However, the fact is that a satisfactory nonlinear friction model for attenuation has never been developed to the point where meaningful predictions could be made about the propagation of waves (Kjartansson, 1979).

In addition to the linear models, some other models have been created to express dispersion effect. For example, based on the matrix formulation devised by Thomson (1950) and Haskell (1953), various surface-wave dispersion computations have been described by Dorman, Ewing, and Oliver (1960); Press, Harkrider, and Seafeldt (1961); Harkrider (1964); Ben-Menahem and Harkrider (1964); and Schwab and Knopoff (1970). While Randall (1967) also discussed this problem based on Knopoff's (1964a) method.

Several linear models of causal absorption conform well with Q-constant theory in the seismic frequency range. The most commonly used are the models of Futterman (1962), Lomnitz (1957) and Kjartansson (1979).

Dispersion relations of the Kramers-Krönig (K-K) type (Kramers, 1927; Krönig, 1926) is well known in electric circuit theory (Bode, 1945; Guilleman, 1949). Futterman (1962) gave an almost constant Q model based on the K-K type:

$$v(\omega) = v(\omega_r)\left[1 + \frac{1}{\pi Q} ln\left(\frac{\omega}{\omega_r}\right)\right] \quad or \quad \frac{1}{v(\omega)} = \frac{1}{v(\omega_r)}\left[1 - \frac{1}{\pi Q} ln\left(\frac{\omega}{\omega_r}\right)\right] \qquad (14)$$

where $v(\omega)$ is the phase velocity of body waves, surface waves, or free oscillations, ω is the angular frequency, $\omega = 2\pi f$, $v(\omega_r)$ is the phase velocity at the reference frequency ω_r, and Q is the quality factor appropriate for the wave considered.

In the formulas (14), Futterman assumed that Q is large enough so that we can drop terms that are second-order and higher in $1/Q$ and keep only first-order terms. Then, the rule (18) can be used to verify these two formulas are consistent.

Kanamori and Anderson (1977) also did some examinations of various absorption, and their models leaded us to the conclusion that expression (14) must be used for correcting the effect of physical dispersion arising from anelasticity.

Except for the Futterman's dispersion model, Lomnitz's model and Kjartansson's model (see, Lomnitz, 1957, and Kjartansson, 1979) are also commonly used.

Lomnitz (1957, 1962) derived a model for wave attenuation with Q approximately independent of frequency for large Q based on a logarithmic creep function and Boltzmann's after-effect equation (e.g., Gross, 1968).

Lomnitz's model is given by:

$$\frac{1}{v(\omega)} = \frac{1}{v(\omega_a)}\left[1 - \frac{2}{\pi Q} ln\left(\frac{\omega}{\omega_a}\right)\right]^{1/2} \tag{15}$$

where ω_a is some very high frequency (say, $\omega_a = 10^{10}$Hz).

O'Neill and Hill (1979) incorporates Lomnitz's causal formulation of anelasticity into the Fuchs-Muller-Kennett reflectivity program (Fuchs and Muller, 1971; Kennett, 1975), and finds for waves passing through a highly attenuating upper-mantle low-velocity zone (Q=35), the first pulse is larger and more abrupt, and the peaks and troughs are somewhat later for causal absorption than for noncausal absorption.

Kolsky (1956) and Lomnitz (1957) also gave linear descriptions of the absorption that could account for the observed frequency independence and were also consistent with other independent observations of the transient creep in rocks and the change in shape of pulses propagation through thin rods.

Kjartansson (1979) discussed some nearly constant Q theories (e.g., Lomnitz, 1957; Futterman, 1962; Strick, 1970; Liu et al., 1976) and found none of them could provide a better description of the attenuation in actual rocks than the constant Q theory (e.g., Bland, 1960; Strick, 1967; Kjartansson, 1979) does.

Kjartansson's model is given by:

$$\frac{v(\omega)}{v(\omega_r)} = \left(\frac{\omega}{\omega_r}\right)^{\gamma} \tag{16}$$

where $\gamma = \frac{1}{\pi}\tan^{-1}\left(\frac{1}{Q}\right)$.

The properties of these models are well described in the literature, e.g., Dziewonski (1979), Mavko et al. (1979), Pilant (1979) and Savage and O'Neill (1975). Let us only mention that the model by Kjartansson may be used for all frequencies, from zero to infinity. However, Futterman's model and Lomnitz's model are band-limited, so they fail for very low and very high frequencies.

Laboratory experiments on many solids have shown that, up to moderately high frequencies, the dimensionless quantity Q is indeed independent of frequency to a very good approximation. For example, Cerveny et al. (1982) indicated that the frequency range used in our body wave computations is usually rather narrow. In this narrow frequency range all the three models yielded practically the same results in all cases.

Therefore, for simplicity, we can present here only the results obtained by Futterman's model.

Applying relation (14) to the elastic velocity v_0 in the complex velocity formula (13), gives

$$v(\omega) = v_0(\omega)\left(1 - \frac{i}{2Q}\right) = v(\omega_{0r})\left[1 + \frac{1}{\pi Q} ln\left(\frac{\omega}{\omega_{0r}}\right)\right]\left(1 - \frac{i}{2Q}\right)$$

$$\approx v(\omega_{0r})\left[1 + \frac{1}{\pi Q} ln\left(\frac{\omega}{\omega_{0r}}\right) - \frac{i}{2Q}\right] \tag{17}$$

In the formulas above, we assumed that Q is large enough so that we can drop terms that are second-order and higher in $1/Q$ (i.e., terms in $1/Q^2$, etc.) and keep only first-order terms (i.e., terms of order $1/Q$). That means that we are using the rule

$$(1 + x)^n \approx 1 + nx, \ \ |x| \ll 1 \qquad \left[\text{Example: } (1 + x)^{-1/2} \approx 1 - \frac{1}{2}x\right] \tag{18}$$

So, in formula (17), only first-order terms in $1/Q$ were kept. Or, equivalently, again keeping only first-order terms in $1/Q$,

$$\frac{1}{v(\omega)} \approx \frac{1}{v(\omega_{0r})}\left[1 - \frac{1}{\pi Q} ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{i}{2Q}\right] \tag{19}$$

which we could also obtain from equation (13).

The absorption related to the dispersion by the dispersion relation is usually called the causal absorption.

In some treatments of dispersion, Q follows a similar rule (instead of being constant), i.e.,

$$\frac{1}{Q(\omega)} = \frac{1}{Q(\omega_r)}\left[1 + \frac{1}{\pi Q(\omega_r)} ln\left(\frac{\omega}{\omega_r}\right)\right] \quad \text{or} \quad Q(\omega) = Q(\omega_r)\left[1 - \frac{1}{\pi Q(\omega_r)} ln\left(\frac{\omega}{\omega_r}\right)\right] \tag{20}$$

In this rule, Q is nearly constant, because the logarithm ln (x) increases very slowly with x. Note also that $v(\omega)Q(\omega)$ is constant (to first order in $1/Q$) in this model, which is similar to Futterman's model (1962).

1.5 The Fourier Transform

To add absorption and dispersion effect into the solutions of the EOM, the first step is converting them into frequency domain from time domain.

The Fourier transform decomposes a function of time (a signal) into the frequencies that make it up.

The Fourier transform, and inverse Fourier transform of $g(t)$ are defined as

$$\bar{g}(\omega) = \int_{-\infty}^{\infty} g(t)e^{i\omega t}\, dt \tag{21a}$$

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{g}(\omega)e^{-i\omega t}\, d\omega \tag{21b}$$

The integral over t (time) is the forward Fourier transform, and the integral over ω (frequency, $\omega = 2\pi/f$) is the inverse Fourier transform.

$\bar{g}(\omega)$ is the frequency spectrum of $g(t)$. Since it is in general a complex number, it can be written as

$$\bar{g}(\omega) = \mathrm{Re}(\bar{g}) + i\mathrm{Im}(\bar{g}) = |\bar{g}(\omega)|e^{i\varphi(\omega)} \tag{22a}$$

, where $|\bar{g}(\omega)|$ is the amplitude spectrum of $g(t)$ and $\varphi(\omega)$ is the phase spectrum of $g(t)$, and where

$$|\bar{g}(\omega)| = \sqrt{[\mathrm{Re}(\bar{g})]^2 + [\mathrm{Im}(\bar{g})]^2} \tag{22b}$$

$$\varphi(\omega) = \tan^{-1}[\mathrm{Im}(\bar{g})/\mathrm{Re}(\bar{g})] \tag{22c}$$

If $c(t)$ is the convolution of two continuous signals $x(t)$ and $g(t)$, so $c(t)$ is defined as:

$$c(t) = x(t) * g(t) \equiv \int_{-\infty}^{\infty} x(\tau)g(t-\tau)d\tau$$

Convolution is used to model seismic traces. The Fourier transform (FT) of $c(t)$ is calculated as follows:

$$\bar{c}(\omega) = \int_{-\infty}^{\infty} x(t) * g(t)\, e^{i\omega t} dt = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau)g(t-\tau)\, e^{i\omega t} d\tau\ dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} g(t-\tau)\, e^{i\omega t} dt\right] d\tau = \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} g(u)\, e^{i\omega u} du\right] e^{i\omega\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{i\omega\tau}\, d\tau \times \bar{g}(\omega) = \bar{x}(\omega)\bar{g}(\omega)$$

Therefore, we can obtain the convolution theorem (Bracewell, 1965, p. 108),

$$\mathrm{FT}\,\{x(t) * g(t)\} = \bar{x}(\omega)\, \bar{g}(\omega) \tag{23}$$

And, we can also proof that if FT $\bar{g}(\omega)$ of a function $g(t)$ satisfies $\bar{g}(-\omega) = \bar{g}(\omega)^*$, then $g(t)$ is real, i.e., $g(t) = g(t)^*$. Proof:

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{g}(\omega)e^{-i\omega t}\,d\omega \;\Longrightarrow\; g(t)^* = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{g}(\omega)^* e^{i\omega t}\,d\omega \;\Longrightarrow$$

$$g(t)^* = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{g}(-\omega)e^{i\omega t}\,d\omega = \frac{1}{2\pi}\int_{\infty}^{-\infty} \bar{g}(v)e^{-ivt}\,(-dv) \;\Longrightarrow$$

$$g(t)^* = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{g}(v)e^{-ivt}\,dv = g(t) \tag{24a}$$

Similarly, if $g(t)$ is real (i.e., $g(t) = g(t)^*$) then $\bar{g}(-\omega) = \bar{g}(\omega)^*$. Proof:

$$\bar{g}(\omega) = \int_{-\infty}^{\infty} g(t)e^{i\omega t}\,dt \;\Longrightarrow\; \bar{g}(\omega)^* = \int_{-\infty}^{\infty} g(t)^* e^{-i\omega t}\,dt \;\Longrightarrow$$

$$\bar{g}(\omega)^* = \int_{-\infty}^{\infty} g(t)e^{-i\omega t}\,dt = \bar{g}(-\omega) \tag{24b}$$

Besides that, in MATLAB, we will use the “fft” function for a forward Fourier transform and the “ifft” function for an inverse Fourier transform. Both of the two functions are based on the theory of discrete Fourier transform (DFT).

We first review the basic theory for the DFT. Most of the theory is taken from Krebes (2011). When we apply the DFT theory, x will refer to the time t and wavenumber k will refer to the frequency f.

Consider a real function $h(x)$ which is “causal”, i.e., $h(x) = 0$ for $x < 0$. Suppose $h(x)$ is sampled at N points with a sample interval of Δx, giving

$$\begin{gathered} (h_0, h_1, \cdots, h_{N-1}), \qquad h_m \equiv h(x_m) \\ with\ x_m = m\Delta x, \quad m = 0,1,2,\cdots,N-1 \end{gathered} \tag{25}$$

The function $h(x)$ is sampled between $x = 0$ and $x = (N-1)\Delta x$. We assume that either $h(x)$ is zero beyond this range, or that N sampled points are more or less typical of what $h(x)$ looks like everywhere.

Similarly, $H(k)$ (k is the wavenumber), the spectrum of $h(x)$, can be sampled with N points as well. Assume N is an even number. Because $H(k)$ is non-zero for both positive and negative wavenumbers k, the N samples would be spread more or less evenly over both negative and positive wavenumbers as follows:

$$\left(H_{-\frac{N}{2}+1}, H_{-\frac{N}{2}+2}, \cdots, H_{\frac{N}{2}}\right), \qquad H_n \equiv H(k_n), \qquad k_n = n\Delta k,$$
$$n = -\frac{N}{2}+1, -\frac{N}{2}+2, \cdots, \frac{N}{2}-1, \frac{N}{2}. \tag{26}$$

For example, for $N = 8$, we would have

$$(H_{-3}, H_{-2}, H_{-1}, H_0, H_1, H_2, H_3, H_4), \qquad H_n \equiv H(k_n) = H(n\Delta k), \qquad k_n = n\Delta k,$$
$$n = -3, -2, -1, 0, 1, 2, 3, 4. \tag{27}$$

H_n can be obtained from h_m via a discrete Fourier transform (DFT), as explained below.

We know from sampling theory that is the spectrum $H(k)$ of a continuous function $h(x)$ is band-limited to wavenumbers that lie in the range $(-k_{Nyq},\ +K_{Nyq})$, where

$$k_{Nyq} = \frac{1}{2\Delta x} = \text{Nyquist wavenumber} \tag{28}$$

i.e., $H(k)$ is zero outside this range, then the continuous function $h(x)$ can be completely recovered from its samples h_m. If $H(k)$ is not band-limited to this range, then the values of $H(k)$ outside of the range will be added to the values of $H(k)$ inside of the range, giving an inaccurate distorted spectrum. The wavenumbers outside the range are disguised or aliased as wavenumbers inside the range, and hence are erroneously added to the range, giving a false spectrum. Clearly, the smaller Δx is, the larger is k_{Nyq}, meaning that Δx, the sampling interval, must be chosen small enough so that the Nyquist range $(-k_{Nyq},\ +K_{Nyq})$ covers the whole spectrum, with the spectrum being zero (or at least approximately zero) outside of the Nyquist range.

When given the samples h_m, we assume that $h(x)$ has been properly sampled, i.e., with a small enough Δx so that the spectrum is band-limited to the Nyquist range. We then perform a DFT on h_m, and inspect the spectrum values H_n. If these values do not go to zero as $k \to k_{Nyq}$, then aliasing has likely taken place, rendering the spectrum inaccurate and distorted, and $h(x)$ needs to be resampled with a smaller interval first before a DFT is performed.

Evidently then, the wavenumber range of the discrete spectrum H_n is confined to the Nyquist range. This means that the highest wavenumber in the range, i.e., the one for $n = N/2$ in (26), would be the Nyquist wavenumber k_{Nyq}:

$$k_{\frac{N}{2}} = \frac{N}{2}\Delta k = k_{Nyq} = \frac{1}{2\Delta x} \quad \Rightarrow \quad \Delta k = \frac{1}{N\Delta x} \tag{29}$$

Which is the relationship between the two sampling intervals.

We now compute H_n from h_m using a DFT.

$$H_n = \int_{-\infty}^{\infty} h(x)e^{-2\pi i k_n x}\, dx \approx \sum_{m=0}^{N-1} h(x_m)e^{-2\pi i k_n x_m}\, \Delta x = \Delta x \sum_{m=0}^{N-1} h_m e^{-2\pi i m n \Delta k \Delta x}$$

$$\Rightarrow \quad H_n = \Delta x \sum_{m=0}^{N-1} h_m e^{-2\pi i m n/N} \quad \Rightarrow \quad H_n{}^* = \Delta x \sum_{m=0}^{N-1} h_m e^{2\pi i m n/N} \tag{30a}$$

Where (29) was used in the last line. This is the DFT of $h(x)$.

And,

$$h_m = \Delta k \sum_{n=0}^{N-1} H_n e^{2\pi i m n/N} \tag{30b}$$

This is the inverse discrete Fourier transform.

The sums for H_n and h_m are easily computed. In the sum for h_m for a specific value of m, there are N complex multiplications. Computing this sum N times (for all the N different values of m) would mean N^2 operations in total to obtain all the h_m. However, a fast-efficient scheme for computing the sums, known as the Fast Fourier Transform (FFT) has been developed. FFT is the approach which used in Matlab functions. The FFT exploits the mathematical properties of the DFT to greatly reduce the number of computations required. For the FFT, only about $N\log_2 N$ operations are required in total to obtain all the h_m. For example, if $N = 2^6 = 64$, then $N^2 = 4096$, but $N\log_2 N = 2^6\log_2 2^6 = 6 \times 2^6\log_2 2 = 6 \times 2^6 = 384$, which is a great reduction in the number of operations required.

$$H_n = \widehat{H}_n \Delta x, \qquad \widehat{H}_n = \sum_{m=0}^{N-1} h_m e^{-2\pi imn/N} \tag{31a}$$

$$h_m = \frac{1}{N}\sum_{n=0}^{N-1} \widehat{H}_n e^{2\pi imn/N} \tag{31b}$$

Most FFT programs compute $\widehat{H}_n$ in (31a) as the FFT, and h_m in (31b) as the invers FFT. This means that the FFT result must be multiplied by Δx to get the true value H_n of the spectrum. But the true value for h_m is given directly by the inverse FFT in (31b). Of course, $\widehat{H}_n$ in (31a) is the true value of spectrum divided by Δx, which must be remembered when sampling a spectrum and applying an inverse FFT.

CHAPTER 2: METHODOLOGY AND ALGORITHM

2.1 Principle

Because absorption and dispersion are frequency dependent, if we want to add their influence into the solutions of the EOM, firstly, we should translate the solutions of the EOM for some common sources without absorption in time domain into frequency domain, and then transform back. For no dispersion, the anelastic waveforms usually have the same form and shape as the elastic ones. When we include dispersion in general, it is not possible to do the integrals analytically. But they can be done numerically, and one of the steps in my thesis is to write a program to compute the same results with velocity dispersion. One finds that the anelastic waveforms have a different shape than the elastic ones.

In general, the correct way to approach this problem is to transform the viscoelastic equation of motion into the frequency domain (one obtains the same thing as when one transforms the elastic equation of motion into the frequency domain, except that the elastic modulus μ and λ are complex, meaning the velocities are complex), then solve it in the frequency domain (with the complex modulus), and then transform back into the time domain. But instead, what we did is to transform the solution of the elastic equation of motion into the frequency domain, then replace the velocity or slowness with the complex version, then transform back.

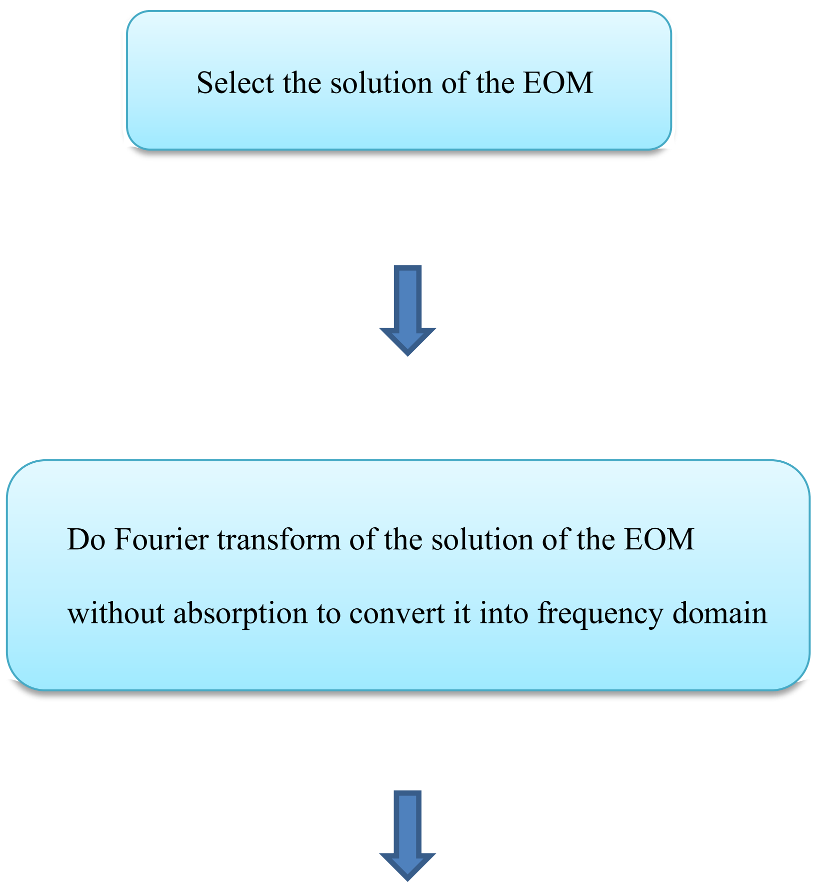
However, if we go through the calculation on pages 80-84 of the book "Seismic Ray Theory" by V. Cerveny (2001), we can see that the solution of the elastic equation of motion in the frequency domain applies to the viscoelastic case as well if we make the velocity complex. And we can also go through the derivation of the Weyl integral for viscoelasticity on page 76 of that book, which is,

$$\frac{exp[-i\omega(t-r/c)]}{r}=\frac{i\omega}{2\pi}\iint_{-\infty}^{\infty}\frac{dp_1dp_2}{p_3}exp[-i\omega(t-p_ix_i)] \tag{32}$$

and we will find that Weyl integral applies to the viscoelastic case as well (where the velocity in the Weyl integral is complex).

Therefore, we can justify that our approach gives the correct result.

2.2 Work Flow



Replace v with $v_0\left(1-\frac{i}{2Q}\right)$ to involve absorption effect, or replace v with $v(\omega_{0r})\left[1+\frac{1}{\pi Q}ln\left(\frac{\omega}{\omega_{0r}}\right)-\frac{i}{2Q}\right]$ to involve absorption and dispersion effect

$\Downarrow$

Do inverse Fourier transform (IFT) of the new solutions of the EOM (where the velocity is complex) to transform them back to time domain.

$\Downarrow$

Create programs by Matlab to compute the exact results with velocity dispersion numerically to verify the accuracy of our approximations

2.3 Sources and Models

2.3.1 Source pulse $\bar{s}(\omega)$ and s(t)

For all the situations we are going to talk about, they all need a suitable source which has been defined in frequency domain.

In order to make the calculation easier, we will assume that all the situations discussed in this thesis use the same expression of source $\bar{s}(\omega)$,

$$\bar{s}(\omega) = Ai\omega e^{-a|\omega|} \tag{33}$$

where a is a positive constant has the units of seconds, and A is also a constant has the units of mass *length.

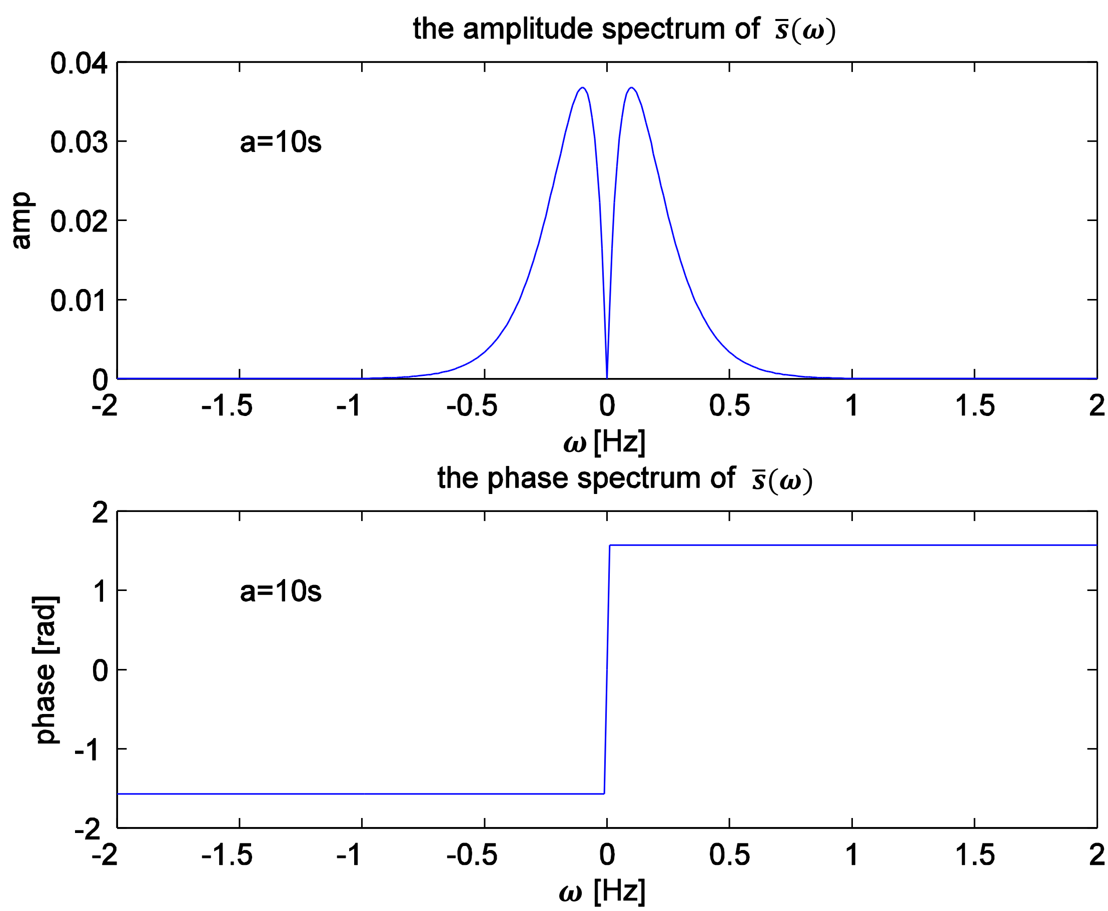


Fig.2. The plots of the amplitude and phase spectra of $\bar{s}(\omega)$

The above Fig.2 shows the plots of the amplitude spectrum, $|A\omega|e^{-a|\omega|}$, and phase spectrum of $\bar{s}(\omega)$, where we assume $A = 1$ and $a = 10s$.

Also using inverse Fourier transform to convert $\bar{s}(\omega)$, formula (33), into time domain, then we could get the source pulse $\mathrm{s(t)}$ in time domain,

$$\mathrm{s(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} Ai\omega e^{-a|\omega|} e^{-i\omega t}\, d\omega = \frac{2atA}{\pi(a^2 + t^2)^2} \tag{34}$$

and the source pulse $\mathrm{s(t)}$ is plotted in Fig.3 for $a = 10\mathrm{s}$,

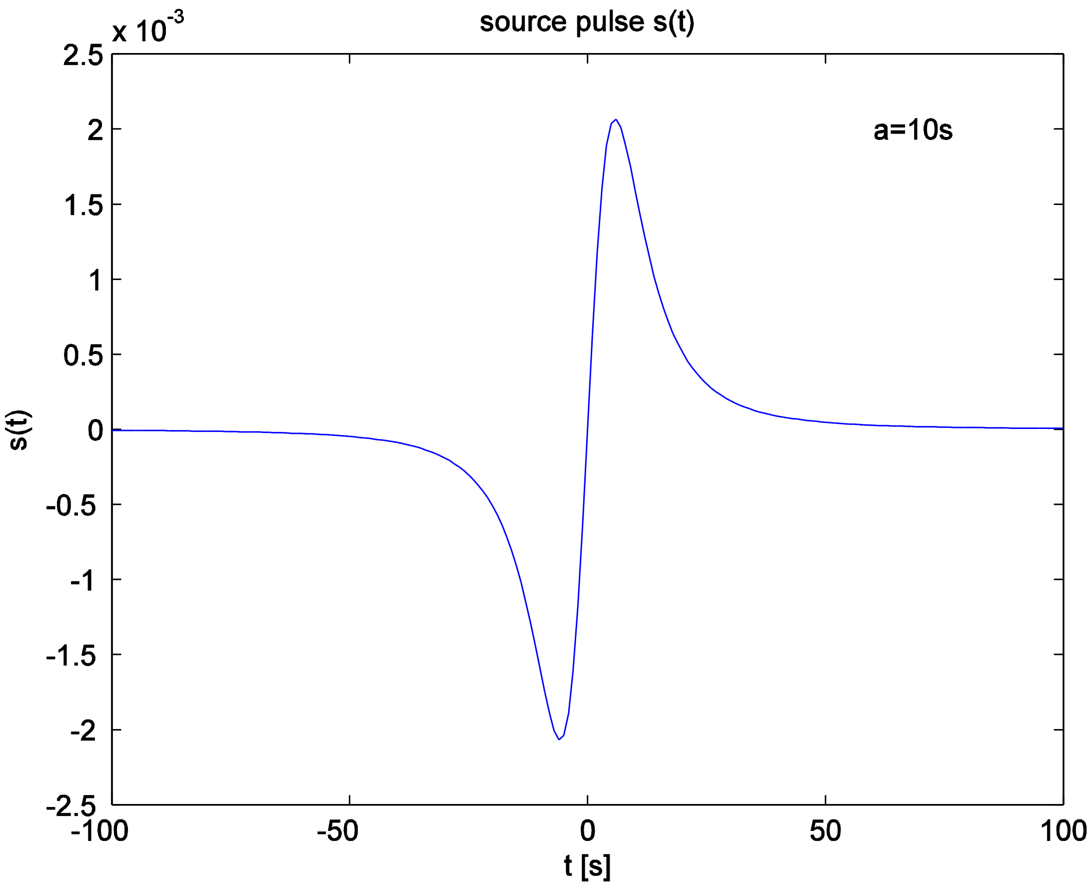


Fig.3. The source pulse in time domain s(t)

$\bar{s}(\omega)$ and $s(t)$ are the source pulse in frequency domain and time domain which we will use for a directed point force. For the double-couple-without-moment forces and a dislocation force, we will assume $\overline{M_0}(\omega) = \bar{s}(\omega)$ and $M_0(t) = s(t)$ as well. And we should also notice that this source is non-causal, but for our purposes, this is acceptable because it allows us to derive mathematical formulas for the absorption response, and because we can think of the source pulse as being effectively causal if we take the arrival point of the pulse at a point on the negative side where it is effectively zero, e.g., at t = -50 s in Figure 3.

2.3.2 A directed point force

To simplify the question, for a directed point force, we will only talk about the x component of displacement of a point force in x direction in this time.

Figure 4 shows what this situation looks like in a Cartesian coordinate system.

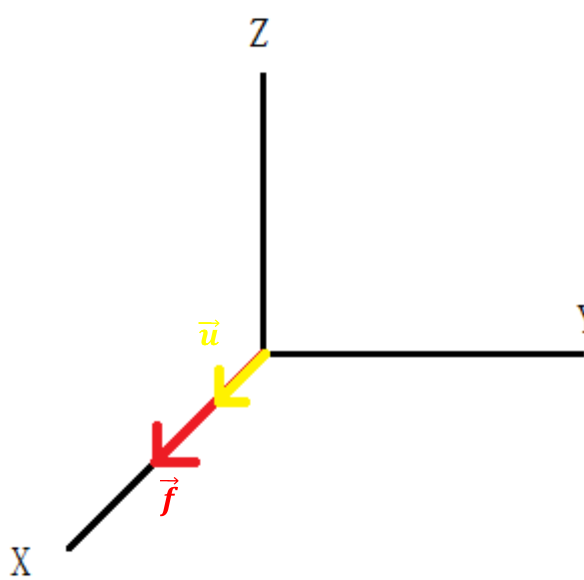


Fig.4. A directed point force in x direction

Substitute this situation into equation (3), where u_i is the i-th component of displacement, so in there $u_i = u_1$. Let us assume $\mathbf{f} = s(t)\delta(\mathbf{x})\boldsymbol{e_1}$ (the point force is in the x direction, i.e.,$e_j = e_1$). Then, we can obtain

$$u_1(\mathbf{x}, t) = \frac{(3\gamma_1\gamma_1 - \delta_{11})}{4\pi\rho r^3}\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau s(t-\tau)\, d\tau + \frac{\gamma_1\gamma_1}{4\pi\rho\alpha^2 r} s\left(t - \frac{r}{\alpha}\right) + \frac{(\delta_{11} - \gamma_1\gamma_1)}{4\pi\rho\beta^2 r} s(t - \frac{r}{\beta})$$

$$= \frac{1}{2\pi\rho r^3}\int_{r/\alpha}^{r/\beta} \tau s(t-\tau) d\tau + \frac{1}{4\pi\rho\alpha^2 r}\ s(t - \frac{r}{\alpha}) \tag{35}$$

where the Kronecker delta δ_{ij} is a piecewise function of variables i and j which we have indicated in equation (5). For example, $\delta_{12} = 0$, whereas $\delta_{11} = 1$.

And we have known $\gamma_n = x_n/r$ from above, so $\gamma_1 = x_1/r$. In our assumption, $\mathbf{x} = (x_1, 0,0)$, and $r = \sqrt{x^2 + y^2 + z^2} = \sqrt{{x_1}^2 + 0^2 + 0^2} = x_1$. Hence, we can obtain $\gamma_1 = \frac{x_1}{r} = 1$.

The S-wave moves as a shear or transverse wave, so motion is perpendicular to the direction of wave propagation. While, in isotropic and homogeneous solids, the mode of propagation of a P-wave is always longitudinal; thus, the particles in the solid vibrate along the axis of propagation (the direction of motion) of the wave energy. Since we are talking about the component of displacement along the direction of wave propagation, we can only see the near-field term and the far-field P wave term in the above equation (35). The particle motion is only perpendicular to the direction of wave propagation for the far-field S wave term.

2.3.3 A double-couple-without-moment source

The double-couple-without-moment is now accepted as the best body-force model of an earthquake source. It is composed of two single couples with opposite moments.

For example, in this thesis we are going to consider a vertical fault coinciding with the *xz* plane. From Fig.1(b), the only non-zero components of the moment tensor are then $M_{12} = M_{21} = M_0$. And To use a more general formula, for a receiver lying on a circle surrounding the origin, then we have ${\gamma_1}^2 + {\gamma_2}^2 = 1$ and $\gamma_3 = 0$. Applying (6) to obtain the displacement due to the double-couple gives:

$$
\begin{aligned}
u_i(\mathbf{x}, t) = {} & \frac{(15\gamma_i\gamma_1\gamma_2 - 3\gamma_i\delta_{12} - 3\gamma_1\delta_{i2} - 3\gamma_2\delta_{i1})}{4\pi\rho}\frac{1}{r^4}\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau M_{12}(t-\tau)\,d\tau \\
& + \frac{(6\gamma_i\gamma_1\gamma_2 - \gamma_i\delta_{12} - \gamma_1\delta_{i2} - \gamma_2\delta_{i1})}{4\pi\rho\alpha^2}\frac{1}{r^2}M_{12}\left(t-\frac{r}{\alpha}\right) \\
& - \frac{(6\gamma_i\gamma_1\gamma_2 - \gamma_i\delta_{12} - \gamma_1\delta_{i2} - 2\gamma_2\delta_{i1})}{4\pi\rho\beta^2}\frac{1}{r^2}M_{12}\left(t-\frac{r}{\beta}\right) \\
& + \frac{\gamma_i\gamma_1\gamma_2}{4\pi\rho\alpha^3}\frac{1}{r}\dot{M}_{12}\left(t-\frac{r}{\alpha}\right) - \frac{\gamma_2(\gamma_i\gamma_1 - \delta_{i1})}{4\pi\rho\beta^3}\frac{1}{r}\dot{M}_{12}\left(t-\frac{r}{\beta}\right) \\
& + \frac{(15\gamma_i\gamma_2\gamma_1 - 3\gamma_i\delta_{21} - 3\gamma_2\delta_{i1} - 3\gamma_1\delta_{i2})}{4\pi\rho}\frac{1}{r^4}\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau M_{21}(t-\tau)\,d\tau \\
& + \frac{(6\gamma_i\gamma_2\gamma_1 - \gamma_i\delta_{21} - \gamma_2\delta_{i1} - \gamma_1\delta_{i2})}{4\pi\rho\alpha^2}\frac{1}{r^2}M_{21}\left(t-\frac{r}{\alpha}\right) \\
& - \frac{(6\gamma_i\gamma_2\gamma_1 - \gamma_i\delta_{21} - \gamma_2\delta_{i1} - 2\gamma_1\delta_{i2})}{4\pi\rho\beta^2}\frac{1}{r^2}M_{21}\left(t-\frac{r}{\beta}\right) \\
& + \frac{\gamma_i\gamma_2\gamma_1}{4\pi\rho\alpha^3}\frac{1}{r}\dot{M}_{21}\left(t-\frac{r}{\alpha}\right) - \frac{\gamma_1(\gamma_i\gamma_2 - \delta_{i2})}{4\pi\rho\beta^3}\frac{1}{r}\dot{M}_{21}\left(t-\frac{r}{\beta}\right)
\end{aligned}
$$

$$
\begin{aligned}
u_i(\mathbf{x}, t) = {} & \frac{(15\gamma_i\gamma_1\gamma_2 - 3\gamma_1\delta_{i2} - 3\gamma_2\delta_{i1})}{2\pi\rho}\frac{1}{r^4}\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau M_0(t-\tau)\,d\tau \\
& + \frac{(6\gamma_i\gamma_1\gamma_2 - \gamma_1\delta_{i2} - \gamma_2\delta_{i1})}{2\pi\rho\alpha^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\alpha}\right) \\
& - \frac{(12\gamma_i\gamma_1\gamma_2 - 3\gamma_1\delta_{i2} - 3\gamma_2\delta_{i1})}{4\pi\rho\beta^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\beta}\right)
\end{aligned}
$$

$$+\frac{\gamma_i\gamma_2\gamma_1}{2\pi\rho\alpha^3}\frac{1}{r}\dot{M}_0\left(t-\frac{r}{\alpha}\right)-\frac{(2\gamma_i\gamma_1\gamma_2-\gamma_1\delta_{i2}-\gamma_2\delta_{i1})}{4\pi\rho\beta^3}\frac{1}{r}\dot{M}_{21}\left(t-\frac{r}{\beta}\right) \qquad (36)$$

Then the three components (x, y and z components) of the displacement on the receiver point are:

$$u_1(\mathbf{x},t)=\frac{(15{\gamma_1}^2\gamma_2-3\gamma_2)}{2\pi\rho}\frac{1}{r^4}\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}}\tau M_0(t-\tau)\,d\tau$$

$$+\frac{(6{\gamma_1}^2\gamma_2-\gamma_2)}{2\pi\rho\alpha^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\alpha}\right)-\frac{(12{\gamma_1}^2\gamma_2-3\gamma_2)}{4\pi\rho\beta^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\beta}\right)$$

$$+\frac{{\gamma_1}^2\gamma_2}{2\pi\rho\alpha^3}\frac{1}{r}\dot{M}_0\left(t-\frac{r}{\alpha}\right)-\frac{(2{\gamma_1}^2\gamma_2-\gamma_2)}{4\pi\rho\beta^3}\frac{1}{r}\dot{M}_0\left(t-\frac{r}{\beta}\right) \qquad (37a)$$

$$u_2(\mathbf{x},t)=\frac{(15\gamma_1{\gamma_2}^2-3\gamma_1)}{2\pi\rho}\frac{1}{r^4}\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}}\tau M_0(t-\tau)\,d\tau$$

$$+\frac{(6\gamma_1{\gamma_2}^2-\gamma_1)}{2\pi\rho\alpha^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\alpha}\right)-\frac{(12\gamma_1{\gamma_2}^2-3\gamma_1)}{4\pi\rho\beta^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\beta}\right)$$

$$+\frac{\gamma_1{\gamma_2}^2}{2\pi\rho\alpha^3}\frac{1}{r}\dot{M}_0\left(t-\frac{r}{\alpha}\right)-\frac{(2\gamma_1{\gamma_2}^2-\gamma_1)}{4\pi\rho\beta^3}\frac{1}{r}\dot{M}_0\left(t-\frac{r}{\beta}\right) \qquad (37b)$$

$$u_3(\mathbf{x},t)=0 \qquad (37c)$$

Since we assume the source is on the origin and the receiver point is on the xy plane somewhere, so ${\gamma_1}^2+{\gamma_2}^2=1$ and $\gamma_3=0$. Because $cos^2\theta+sin^2\theta=1$ as well, we can set $\gamma_1=\cos\theta$ and $\gamma_2=\sin\theta$, where θ is the angle from the x axis.

$$u_1(\mathbf{x},t)=\frac{(15cos^2\theta\sin\theta-3\sin\theta)}{2\pi\rho}\frac{1}{r^4}\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}}\tau M_0(t-\tau)\,d\tau$$

$$+\frac{(6cos^2\theta\sin\theta-\sin\theta)}{2\pi\rho\alpha^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\alpha}\right)$$

$$-\frac{(12cos^2\theta\sin\theta-3\sin\theta)}{4\pi\rho\beta^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\beta}\right)$$

$$+\frac{cos^2\theta\sin\theta}{2\pi\rho\alpha^3}\frac{1}{r}\dot{M}_0\left(t-\frac{r}{\alpha}\right)-\frac{(2cos^2\theta\sin\theta-\sin\theta)}{4\pi\rho\beta^3}\frac{1}{r}\dot{M}_0\left(t-\frac{r}{\beta}\right)$$

$$u_2(\mathbf{x},t)=\frac{(15\cos\theta\, sin^2\theta-3\cos\theta)}{2\pi\rho}\frac{1}{r^4}\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}}\tau M_0(t-\tau)\,d\tau$$

$$+\frac{(6\cos\theta\, sin^2\theta-\cos\theta)}{2\pi\rho\alpha^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\alpha}\right)$$

$$-\frac{(12\cos\theta\, sin^2\theta-3\cos\theta)}{4\pi\rho\beta^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\beta}\right)$$

$$+\frac{\cos\theta\, sin^2\theta}{2\pi\rho\alpha^3}\frac{1}{r}\dot{M}_0\left(t-\frac{r}{\alpha}\right)-\frac{(2\cos\theta\, sin^2\theta-\cos\theta)}{4\pi\rho\beta^3}\frac{1}{r}\dot{M}_0\left(t-\frac{r}{\beta}\right)$$

$$u_3(\mathbf{x},t)=0$$

Simplify the above equations, then we will get:

$$u_1(\mathbf{x},t)=\frac{(9+15\cos2\theta)\sin\theta}{4\pi\rho}\frac{1}{r^4}\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}}\tau M_0(t-\tau)\,d\tau$$

$$+\frac{(2+3\cos2\theta)\sin\theta}{2\pi\rho\alpha^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\alpha}\right)-\frac{(3+6\cos2\theta)\sin\theta}{4\pi\rho\beta^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\beta}\right)$$

$$+\frac{(1+\cos2\theta)\sin\theta}{4\pi\rho\alpha^3}\frac{1}{r}\dot{M}_0\left(t-\frac{r}{\alpha}\right)-\frac{cos\,2\theta\, sin\,\theta}{4\pi\rho\beta^3}\frac{1}{r}\dot{M}_0\left(t-\frac{r}{\beta}\right) \tag{38a}$$

$$u_2(\mathbf{x},t)=\frac{(9-15\cos2\theta)\cos\theta}{4\pi\rho}\frac{1}{r^4}\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}}\tau M_0(t-\tau)\,d\tau$$

$$+\frac{(2-3\cos2\theta)\cos\theta}{2\pi\rho\alpha^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\alpha}\right)-\frac{(3-6\cos2\theta)\cos\theta}{4\pi\rho\beta^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\beta}\right)$$

$$+\frac{(1-2\cos2\theta)\cos\theta}{4\pi\rho\alpha^3}\dot{M}_0\left(t-\frac{r}{\alpha}\right)-\frac{\cos2\theta\cos\theta}{4\pi\rho\beta^3}\frac{1}{r}\dot{M}_0\left(t-\frac{r}{\beta}\right) \tag{38b}$$

$u_3(\mathbf{x}, t) = 0$ (38c)

Since we assume the source is on the origin and the receiver point is on the xy plane somewhere, and the moment tensor $M_{12} = M_{21} = M_0$ are perpendicular to the z-axis, so in this situation, the displacement on the z direction is zero.

2.3.4 A shear-dislocation source

From the generality of formula (6), which gives the radiation from any moment tensor M, we shall often specialize to cases where M arises from a shear dislocation (7).

In this case, if we assume the xy plane is the ground surface, and if the fault plane is the xz plane, then using $M_{jk} = \mu(\bar{u}_j v_k + \bar{u}_k v_j)A$ from Aki and Richards (2002), where the $\boldsymbol{v}$ vector is perpendicular to the fault surface, and $\bar{\boldsymbol{u}} \cdot \boldsymbol{v} = 0$, we get $v_1 = v_3 = 0$, and $\bar{u}_2 = 0$, and so we have:

$$M_{12} = M_{21} = \mu\bar{u}_1 v_2 A \quad \text{and} \quad M_{23} = M_{32} = \mu\bar{u}_3 v_2 A$$

All other M_{jk} are zero.

So M_{12}, M_{21}, M_{23}, M_{32} are not zero for the shear dislocation case. But with four non-zero components of the moment tensor, this could be a lot of work.

In Aki and Richards (2002, p.78), the paragraph under equation (4.30), they "choose the x-axis to be the direction of slip, so that $\bar{\boldsymbol{u}} = (\bar{u}_1, 0, 0)$". Therefore, to simplify this question, we could assume the fault lies in the (x_1, x_3) plane, i.e. $\boldsymbol{v} = (0,1,0)$, and choose the x_1 axis to be the direction of slip as well, i.e. $\bar{\boldsymbol{u}}=(\bar{u}_1,0,0)$. Then, we will have only two non-zero components of the moment tensor, $M_{12} = M_{21} = \mu\bar{u}_1 A$.

If so, substitute this situation into equation (7), then we have

$$u_i(\mathbf{x},t) = \frac{(15\gamma_i\gamma_1\gamma_2 v_2 - 3v_i\gamma_1 - 3\delta_{i1}\gamma_2 v_2)}{2\pi\rho r^4}\mu A \int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau\bar{u}_1(t-\tau)\,d\tau$$

$$+\frac{(6\gamma_i\gamma_1\gamma_2 v_2 - v_i\gamma_1 - \delta_{i1}\gamma_2 v_2)}{2\pi\rho\alpha^2 r^2}\mu A\bar{u}_1\left(t-\frac{r}{\alpha}\right)$$

$$-\frac{(12\gamma_i\gamma_1\gamma_2 v_2 - 3v_i\gamma_1 - 3\delta_{i1}\gamma_2 v_2)}{4\pi\rho\beta^2 r^2}\mu A\bar{u}_1\left(t-\frac{r}{\beta}\right)$$

$$+\frac{\gamma_i\gamma_1\gamma_2 v_2}{2\pi\rho\alpha^3 r}\mu A\bar{u}_1{}'\left(t-\frac{r}{\alpha}\right)-\frac{2\gamma_i\gamma_1\gamma_2 v_2 - v_i\gamma_1 - \delta_{i1}\gamma_2 v_2}{4\pi\rho\beta^3 r}\mu A\bar{u}_1{}'\left(t-\frac{r}{\beta}\right)$$

$$+\frac{(15\gamma_i\gamma_2\gamma_1 v_1 - 3v_i\gamma_2 - 3\delta_{i1}\gamma_1 v_1)}{2\pi\rho r^4}\mu A \int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau\bar{u}_2(t-\tau)\,d\tau$$

$$+\frac{(6\gamma_i\gamma_2\gamma_1 v_1 - v_i\gamma_2 - \delta_{i2}\gamma_1 v_1)}{2\pi\rho\alpha^2 r^2}\mu A\bar{u}_2\left(t-\frac{r}{\alpha}\right)$$

$$-\frac{(12\gamma_i\gamma_2\gamma_1 v_1 - 3v_i\gamma_2 - 3\delta_{i2}\gamma_1 v_1)}{4\pi\rho\beta^2 r^2}\mu A\bar{u}_2\left(t-\frac{r}{\beta}\right)$$

$$+\frac{\gamma_i\gamma_2\gamma_1 v_1}{2\pi\rho\alpha^3 r}\mu A\bar{u}_2{}'\left(t-\frac{r}{\alpha}\right)-\frac{2\gamma_i\gamma_2\gamma_1 v_1 - v_i\gamma_1 - \delta_{i2}\gamma_1 v_1}{4\pi\rho\beta^3 r}\mu A\bar{u}_2{}'\left(t-\frac{r}{\beta}\right)$$

$$u_i(\mathbf{x},t) = \frac{(15\gamma_i\gamma_1\gamma_2 - 3v_i\gamma_1 - 3\delta_{i1}\gamma_2)}{2\pi\rho r^4}\mu A \int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau\bar{u}_1(t-\tau)\,d\tau$$

$$+\frac{(6\gamma_i\gamma_1\gamma_2 - v_i\gamma_1 - \delta_{i1}\gamma_2)}{2\pi\rho\alpha^2 r^2}\mu A\bar{u}_1\left(t-\frac{r}{\alpha}\right)$$

$$-\frac{(12\gamma_i\gamma_1\gamma_2 - 3v_i\gamma_1 - 3\delta_{i1}\gamma_2)}{4\pi\rho\beta^2 r^2}\mu A\bar{u}_1\left(t-\frac{r}{\beta}\right)$$

$$+\frac{\gamma_i\gamma_1\gamma_2}{2\pi\rho\alpha^3 r}\mu A\bar{u}_1{}'\left(t-\frac{r}{\alpha}\right)-\frac{2\gamma_i\gamma_1\gamma_2 - v_i\gamma_1 - \delta_{i1}\gamma_2}{4\pi\rho\beta^3 r}\mu A\bar{u}_1{}'\left(t-\frac{r}{\beta}\right) \qquad (39)$$

Assume the source is on origin and the receiver point is on the xy plane somewhere, so ${\gamma_1}^2 + {\gamma_2}^2 = 1$ and $\gamma_3 = 0$.

Then the three components of the displacement on the receiver point are,

$$u_1(\mathbf{x},t) \ = \frac{(15{\gamma_1}^2\gamma_2 - 3\gamma_2)}{2\pi\rho r^4}\mu A \int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau \bar{u}_1(t-\tau)\, d\tau$$

$$+\frac{(6{\gamma_1}^2\gamma_2 - \gamma_2)}{2\pi\rho\alpha^2 r^2}\mu A \bar{u}_1\left(t-\frac{r}{\alpha}\right) - \frac{(12{\gamma_1}^2\gamma_2 - 3\gamma_2)}{4\pi\rho\beta^2 r^2}\mu A \bar{u}_1\left(t-\frac{r}{\beta}\right)$$

$$+\frac{{\gamma_1}^2\gamma_2}{2\pi\rho\alpha^3 r}\mu A {\bar{u}_1}'\left(t-\frac{r}{\alpha}\right) - \frac{2{\gamma_1}^2\gamma_2 - \gamma_2}{4\pi\rho\beta^3 r}\mu A {\bar{u}_1}'\left(t-\frac{r}{\beta}\right) \tag{40a}$$

$$u_2(\mathbf{x},t) \ = \frac{(15\gamma_1{\gamma_2}^2 - 3\gamma_1)}{2\pi\rho r^4}\mu A \int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau \bar{u}_1(t-\tau)\, d\tau$$

$$+\frac{(6\gamma_1{\gamma_2}^2 - \gamma_1)}{2\pi\rho\alpha^2 r^2}\mu A \bar{u}_1\left(t-\frac{r}{\alpha}\right) - \frac{(12\gamma_1{\gamma_2}^2 - 3\gamma_1)}{4\pi\rho\beta^2 r^2}\mu A \bar{u}_1\left(t-\frac{r}{\beta}\right)$$

$$+\frac{\gamma_1{\gamma_2}^2}{2\pi\rho\alpha^3 r}\mu A {\bar{u}_1}'\left(t-\frac{r}{\alpha}\right) - \frac{2\gamma_1{\gamma_2}^2 - \gamma_1}{4\pi\rho\beta^3 r}\mu A {\bar{u}_1}'\left(t-\frac{r}{\beta}\right) \tag{40b}$$

$$u_3(\mathbf{x},t) \ = 0 \tag{40c}$$

Again, since ${\gamma_1}^2 + {\gamma_2}^2 = 1$ and $\gamma_3 = 0$, and $cos^2\theta + sin^2\theta = 1$, we can assume $\gamma_1 = \cos\theta \ \ and \ \gamma_2 = \sin\theta$, where θ is the angle from the x axis.

$$u_1(\mathbf{x},t) = \frac{(15cos^2\theta\sin\theta - 3\sin\theta)}{2\pi\rho}\frac{1}{r^4}\mu A \int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau \bar{u}_1(t-\tau)\, d\tau$$

$$+\frac{(6cos^2\theta\sin\theta - \sin\theta)}{2\pi\rho\alpha^2}\frac{1}{r^2}\mu A \bar{u}_1\left(t-\frac{r}{\alpha}\right)$$

$$-\frac{(12cos^2\theta\sin\theta - 3\sin\theta)}{4\pi\rho\beta^2}\frac{1}{r^2}\mu A \bar{u}_1\left(t-\frac{r}{\beta}\right)$$

$$+\frac{cos^2\theta\sin\theta}{2\pi\rho\alpha^3}\frac{1}{r}\mu A {\bar{u}_1}'\left(t-\frac{r}{\alpha}\right) - \frac{(2cos^2\theta\sin\theta - \sin\theta)}{4\pi\rho\beta^3}\frac{1}{r}\mu A {\bar{u}_1}'\left(t-\frac{r}{\beta}\right)$$

$$u_2(\mathbf{x},t) = \frac{(15\cos\theta\, sin^2\theta - 3\cos\theta)}{2\pi\rho}\frac{1}{r^4}\mu A\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau\bar{u}_1(t-\tau)\,d\tau$$

$$+\frac{(6\cos\theta\, sin^2\theta - \cos\theta)}{2\pi\rho\alpha^2}\frac{1}{r^2}\mu A\bar{u}_1\left(t-\frac{r}{\alpha}\right)$$

$$-\frac{(12\cos\theta\, sin^2\theta - 3\cos\theta)}{4\pi\rho\beta^2}\frac{1}{r^2}\mu A\bar{u}_1\left(t-\frac{r}{\beta}\right)$$

$$+\frac{\cos\theta\, sin^2\theta}{2\pi\rho\alpha^3}\frac{1}{r}\mu A\bar{u}_1{}'\left(t-\frac{r}{\alpha}\right)-\frac{(2\cos\theta\, sin^2\theta - \cos\theta)}{4\pi\rho\beta^3}\frac{1}{r}\mu A\bar{u}_1{}'\left(t-\frac{r}{\beta}\right)$$

$$u_3(\mathbf{x},t) = 0$$

Simplify the above equations, then we will get:

$$u_1(\mathbf{x},t) = \frac{(9+15\cos 2\theta)\sin\theta}{4\pi\rho}\frac{1}{r^4}\mu A\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau\bar{u}_1(t-\tau)\,d\tau$$

$$+\frac{(2+3\cos 2\theta)\sin\theta}{2\pi\rho\alpha^2}\frac{1}{r^2}\mu A\bar{u}_1\left(t-\frac{r}{\alpha}\right)$$

$$-\frac{(3+6\cos 2\theta)\sin\theta}{4\pi\rho\beta^2}\frac{1}{r^2}\mu A\bar{u}_1\left(t-\frac{r}{\beta}\right)$$

$$+\frac{(1+\cos 2\theta)\sin\theta}{4\pi\rho\alpha^3}\frac{1}{r}\mu A\bar{u}_1{}'\left(t-\frac{r}{\alpha}\right)-\frac{cos\,2\theta\, sin\,\theta}{4\pi\rho\beta^3}\frac{1}{r}\mu A\bar{u}_1{}'\left(t-\frac{r}{\beta}\right) \quad (41a)$$

$$u_2(\mathbf{x},t) = \frac{(9-15\cos 2\theta)\cos\theta}{4\pi\rho}\frac{1}{r^4}\mu A\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau\bar{u}_1(t-\tau)\,d\tau$$

$$+\frac{(2-3\cos 2\theta)\cos\theta}{2\pi\rho\alpha^2}\frac{1}{r^2}\mu A\bar{u}_1\left(t-\frac{r}{\alpha}\right)$$

$$-\frac{(3-6\cos 2\theta)\cos\theta}{4\pi\rho\beta^2}\frac{1}{r^2}\mu A\bar{u}_1\left(t-\frac{r}{\beta}\right)$$

$$+\frac{(1-2\cos 2\theta)\cos\theta}{4\pi\rho\alpha^3}\mu A\bar{u}_1{}'\left(t-\frac{r}{\alpha}\right)-\frac{\cos 2\theta\cos\theta}{4\pi\rho\beta^3}\frac{1}{r}\mu A\bar{u}_1{}'\left(t-\frac{r}{\beta}\right) \quad (41b)$$

$$u_3(\mathbf{x}, t) = 0 \tag{41c}$$

CHAPTER 3: A DIRECTED POINT FORCE

3.1 Converting the solution to frequency domain

We have discussed above that the first thing we should do is to do Fourier transform of the solution of the equation of motion for a directed point force, i.e. equation (35),

$$u_1(\mathbf{x},t) = \frac{1}{2\pi\rho r^3}\int_{r/\alpha}^{r/\beta} \tau s(t-\tau)d\tau + \frac{1}{4\pi\rho\alpha^2 r}\, s(t-\frac{r}{\alpha}) \tag{35}$$

to convert it into frequency domain.

$$\bar{u}_1(\mathbf{x},\omega) = \int_{-\infty}^{\infty}\left[\frac{1}{2\pi\rho r^3}\int_{r/\alpha}^{r/\beta} \tau s(t-\tau)d\tau + \frac{1}{4\pi\rho\alpha^2 r}\, s(t-\frac{r}{\alpha})\right] e^{i\omega t}\mathrm{dt}$$

$$= \frac{1}{2\pi\rho r^3}\int_{-\infty}^{\infty}\left[\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau s(t-\tau)d\tau\right] e^{i\omega t}\mathrm{dt} + \frac{1}{4\pi\rho r}\int_{-\infty}^{\infty}\frac{1}{\alpha^2} s(t-\frac{r}{\alpha})e^{i\omega t}dt \tag{42}$$

Firstly, we can use the convolution theorem (equation 23) to solve the first part of the integration, i.e., $\mathrm{I} = \frac{1}{2\pi\rho r^3}\int_{-\infty}^{\infty}\left[\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau s(t-\tau)d\tau\right] e^{i\omega t}\mathrm{dt}$.

$$\mathrm{I} = \frac{1}{2\pi\rho r^3}\int_{r/\alpha}^{r/\beta}\left[\int_{-\infty}^{\infty} s(t-\tau)e^{i\omega t}dt\right]\tau\mathrm{d}\tau$$

Assume $v = t - \tau \quad \Rightarrow$

$$= \frac{1}{2\pi\rho r^3}\int_{r/\alpha}^{r/\beta}\left[\int_{-\infty}^{\infty} s(v)e^{i\omega(v+\tau)}d(v+\tau)\right]\tau\mathrm{d}\tau$$

$$= \frac{1}{2\pi\rho r^3}\int_{r/\alpha}^{r/\beta}\left[\int_{-\infty}^{\infty} s(v)e^{i\omega v}dv\right]\tau e^{i\omega\tau}\mathrm{d}\tau$$

and, $\displaystyle\int_{-\infty}^{\infty} s(v)e^{i\omega v}dv = \int_{-\infty}^{\infty} s(t)e^{i\omega t}dt = FT\{s(t)\} = \bar{s}(\omega)$

$$\mathrm{I} = \frac{1}{2\pi\rho r^3}\int_{r/\alpha}^{r/\beta} \tau e^{i\omega\tau}\,\mathrm{d}\tau \times \bar{s}(\omega) \tag{43a}$$

Then, let us assume that

$$f(\tau) = \begin{cases} \tau\,, & \dfrac{r}{\alpha} \le \tau \le \dfrac{r}{\beta} \\ 0\,, & others \end{cases} \tag{43b}$$

Substituting this into the (43a), we will get

$$\mathrm{I} = \frac{1}{2\pi\rho r^3}\int_{-\infty}^{\infty} f(\tau)\, e^{i\omega\tau}\, \mathrm{d}\tau \times \bar{s}(\omega) = \frac{1}{2\pi\rho r^3}\ \bar{f}(\omega)\, \bar{s}(\omega) \tag{43c}$$

So, to obtain the solution of the first part, we only need to compute $\bar{f}(\omega)$ and $\bar{s}(\omega)$, and then multiply them together.

$$\bar{f}(\omega) = \int_{-\infty}^{\infty} f(\tau)\, e^{i\omega\tau}\, \mathrm{d}\tau = \frac{1}{i\omega}\int_{r/\alpha}^{r/\beta} \tau\ \mathrm{d}e^{i\omega\tau} = \left(\frac{1}{\omega} - \frac{r}{\beta} i\right)\frac{e^{i\omega\frac{r}{\beta}}}{\omega} - \left(\frac{1}{\omega} - \frac{r}{\alpha} i\right)\frac{e^{i\omega\frac{r}{\alpha}}}{\omega} \tag{44}$$

And we have assumed $\bar{s}(\omega)$ and $s(t)$ in Chapter 2, equation (33) and (34), i.e.,

$$\bar{s}(\omega) = Ai\omega e^{-a|\omega|} \tag{33}$$

$$s(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Ai\omega e^{-a|\omega|} e^{-i\omega t}\, d\omega = \frac{2atA}{\pi(a^2+t^2)^2} \tag{34}$$

Therefore,

$$\mathrm{I} = \frac{1}{2\pi\rho r^3}\ \bar{f}(\omega)\, \bar{s}(\omega) = \frac{A}{2\pi\rho r^3}\left[\left(\frac{1}{\omega} - \frac{r}{\beta} i\right)\frac{e^{i\omega\frac{r}{\beta}}}{\omega} - \left(\frac{1}{\omega} - \frac{r}{\alpha} i\right)\frac{e^{i\omega\frac{r}{\alpha}}}{\omega}\right] i\omega e^{-a|\omega|}$$

$$\mathrm{I} = \frac{A}{2\pi\rho r^3}\left[\left(\frac{i}{\omega} + \frac{r}{\beta}\right) e^{i\omega\frac{r}{\beta}} - \left(\frac{i}{\omega} + \frac{r}{\alpha}\right) e^{i\omega\frac{r}{\alpha}}\right]\ e^{-a|\omega|} \tag{45}$$

For the second term of equ. (42), $\mathrm{k} = \frac{1}{4\pi\rho r}\int_{-\infty}^{\infty}\frac{1}{\alpha^2} s(t - \frac{r}{\alpha}) e^{i\omega t} dt$,

$$\mathrm{k} = \frac{1}{4\pi\rho\alpha^2 r}\int_{-\infty}^{\infty} s(t - \frac{r}{\alpha}) e^{i\omega\left(t - \frac{r}{\alpha}\right)} d(t - \frac{r}{\alpha})\, e^{i\omega\frac{r}{\alpha}}$$

$$= \frac{1}{4\pi\rho\alpha^2 r}\bar{s}(\omega) e^{i\omega\frac{r}{\alpha}} = \frac{Ai\omega}{4\pi\rho\alpha^2 r} e^{-a|\omega|} e^{i\omega\frac{r}{\alpha}} \tag{46}$$

Substituting (45) and (46) into (42), then we will get the solution of the EOM for a directed point force in an elastic medium in frequency domain.

$$\bar{u}_1(\mathbf{x},\omega) = \frac{A}{2\pi\rho r^3}\left[\left(\frac{i}{\omega}+\frac{r}{\beta}\right)e^{i\omega\frac{r}{\beta}} - \left(\frac{i}{\omega}+\frac{r}{\alpha}\right)e^{i\omega\frac{r}{\alpha}}\right] e^{-a|\omega|} + \frac{Ai\omega}{4\pi\rho\alpha^2 r}e^{-a|\omega|}e^{i\omega\frac{r}{\alpha}} \quad (47)$$

Also, substitute $s(t)$ into equation (35) to get the solution in an elastic medium in time domain to compare with results including absorption later.

$$u_1(\mathbf{x},t) = \frac{A}{2\pi\rho r^3}\int_{r/\alpha}^{r/\beta} \tau \frac{2a(t-\tau)}{\pi(a^2+(t-\tau)^2)^2}\, d\tau + \frac{A}{4\pi\rho\alpha^2 r}\frac{2a\left(t-\frac{r}{\alpha}\right)}{\pi\left(a^2+\left(t-\frac{r}{\alpha}\right)^2\right)^2}$$

$$= \frac{A}{2\pi\rho r^3}\left[\frac{a\tau}{\pi(a^2+(t-\tau)^2)} - \frac{\tan^{-1}\left(\frac{\tau-t}{a}\right)}{\pi}\right]\Bigg|_{\frac{r}{\alpha}}^{\frac{r}{\beta}} + \frac{A}{4\pi\rho\alpha^2 r}\frac{2a\left(t-\frac{r}{\alpha}\right)}{\pi\left(a^2+\left(t-\frac{r}{\alpha}\right)^2\right)^2}$$

$$= \frac{A}{2\pi^2\rho r^3}\left[ar\left(\frac{1}{\beta\left(a^2+\left(t-\frac{r}{\beta}\right)^2\right)} - \frac{1}{\alpha\left(a^2+\left(t-\frac{r}{\alpha}\right)^2\right)}\right)\right.$$
$$\left. + \tan^{-1}\left(\frac{\frac{r}{\alpha}-t}{a}\right) - \tan^{-1}\left(\frac{\frac{r}{\beta}-t}{a}\right)\right] + \frac{A}{2\pi^2\rho\alpha^2 r}\frac{a\left(t-\frac{r}{\alpha}\right)}{\left(a^2+\left(t-\frac{r}{\alpha}\right)^2\right)^2} \quad (48)$$

3.2 The solutions of the EOM including absorption but no dispersion

The expression $v = v_0\left(1-\frac{i}{2Q}\right)$, is a low-loss approximation. It is based on the assumption that $Q \gg 1$. The formula for v for any Q (including small Q) is a much more complicated function of Q. For example, see equations 14-19 in Krebes and Daley (2007), or equations 5, 6, 9, 10 in Krebes (1983).

That also means that $\frac{1}{v} = \frac{1}{v_0}\left(1 + \frac{i}{2Q}\right)$ for $Q \gg 1$. Note that the $\frac{1}{Q^2}$ terms have been dropped, because $Q \gg 1$.

Therefore, we will have

$$\frac{1}{v^2} = \frac{1}{{v_0}^2}\left(1 + \frac{i}{2Q}\right)^2 \approx \frac{1}{{v_0}^2}\left(1 + \frac{i}{Q}\right) \tag{49}$$

That means that in the equation (47), we should replace $\frac{1}{v}$ with $\frac{1}{v} = \frac{1}{v_0}\left(1 + \frac{i}{2Q}\right)$, and $\frac{1}{v^2}$ with $\frac{1}{v^2} = \frac{1}{{v_0}^2}\left(1 + \frac{i}{Q}\right)$, to be consistent with the low-loss approximation.

$$\bar{u}_1(\mathbf{x}, \omega) = \frac{A}{2\pi\rho r^3}\left[\left(\frac{i}{\omega} + \frac{r}{\beta}\right)e^{i\omega\frac{r}{\beta}} - \left(\frac{i}{\omega} + \frac{r}{\alpha}\right)e^{i\omega\frac{r}{\alpha}}\right] e^{-a|\omega|} + \frac{Ai\omega}{4\pi\rho\alpha^2 r} e^{-a|\omega|} e^{i\omega\frac{r}{\alpha}} \tag{47}$$

In our solution above, we can find two kinds of velocities, i.e., speed α of compressional wave and speed β of shear wave. And the quality factor Q_α and Q_β for them respectively are different from each other. There, we assume Q_α and Q_β to be nearly independent of frequency for seismic body waves, i.e., Q_α and Q_β are nearly constant.

So, let us replace $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ as:

$$\frac{1}{\alpha} = \frac{1}{\alpha_0}\left(1 + \frac{i}{2Q_\alpha}\right), \qquad \frac{1}{\alpha^2} = \frac{1}{{\alpha_0}^2}\left(1 + \frac{i}{Q_\alpha}\right) \tag{50a}$$

$$\frac{1}{\beta} = \frac{1}{\beta_0}\left(1 + \frac{i}{2Q_\beta}\right), \qquad \frac{1}{\beta^2} = \frac{1}{{\beta_0}^2}\left(1 + \frac{i}{Q_\beta}\right) \tag{50b}$$

To avoid causality, and to keep linearity (so that Fourier analysis can be used), we must include velocity dispersion, but here we make the approximation that velocity is independent of frequency,

After replacing, then we will get,

$$\bar{u}_1(\mathbf{x},\omega) = \frac{A}{2\pi\rho r^3}\left[\left(\frac{i}{\omega}+\frac{r}{\beta_0}\left(1+\frac{i}{2Q_\beta}\right)\right)e^{i\omega\frac{r}{\beta_0}\left(1+\frac{i}{2Q_\beta}\right)}\right.$$

$$\left.-\left(\frac{i}{\omega}+\frac{r}{\alpha_0}\left(1+\frac{i}{2Q_\alpha}\right)\right)e^{i\omega\frac{r}{\alpha_0}\left(1+\frac{i}{2Q_\alpha}\right)}\right]e^{-a|\omega|}$$

$$+\frac{Ai\omega}{4\pi\rho r}\frac{1}{{\alpha_0}^2}\left(1+\frac{i}{Q_\alpha}\right)e^{-a|\omega|}e^{i\omega\frac{r}{\alpha_0}\left(1+\frac{i}{2Q_\alpha}\right)} \tag{51}$$

The next step we should do is to do inverse Fourier transform (IFT) of the new solution in frequency domain (51) to back to time domain.

Do inverse Fourier transform (IFT) of (51),

$$u_1(\mathbf{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{A}{2\pi\rho r^3}\left[\left(\frac{i}{\omega}+\frac{r}{\beta_0}\left(1+\frac{i}{2Q_\beta}\right)\right)e^{i\omega\frac{r}{\beta_0}(1+\frac{i}{2Q_\beta})}\right.\right.$$

$$\left.-\left(\frac{i}{\omega}+\frac{r}{\alpha_0}\left(1+\frac{i}{2Q_\alpha}\right)\right)e^{i\omega\frac{r}{\alpha_0}\left(1+\frac{i}{2Q_\alpha}\right)}\right]$$

$$\left.+\frac{Ai\omega}{4\pi\rho r}\frac{1}{{\alpha_0}^2}\left(1+\frac{i}{Q_\alpha}\right)e^{i\omega\frac{r}{\alpha_0}\left(1+\frac{i}{2Q_\alpha}\right)}\right\}e^{-a|\omega|}\mathrm{e}^{-\mathrm{i}\omega\mathrm{t}}d\omega \tag{52a}$$

As we said before, if a function $g(t)$ is real, then $\bar{g}(-\omega) = \bar{g}(\omega)^*$, and also the fact that for a complex number z, we have $z + z^* = 2Re(z)$.

Then one may write:

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{g}(\omega)e^{-i\omega t}\,d\omega = \frac{1}{\pi}Re\left\{\int_0^{\infty}\bar{g}(\omega)e^{-i\omega t}\,d\omega\right\}$$

In words, if $g(t)$ is real, then in the inverse Fourier transform that gives $g(t)$, one may replace the integral from $-\infty$ to ∞ with 2 times the real part of the integral from 0 to ∞.

And, even though the quality factor Q and velocity terms are strictly functions of frequency ω, we assume they are constant here, so we can take them outside the integral.

Therefore, we can rewrite the equation (52a) and solve it as:

$$u_1(\mathbf{x},t)=\frac{1}{2\pi^2\rho r^2\beta_0}Re\left\{\int_0^\infty\left(1+\frac{i}{2Q_\beta}\right)e^{-\left(\frac{r}{2Q_\beta\beta_0}+a\right)\omega}e^{i\left(\frac{r}{\beta_0}-t\right)\omega}d\omega\right\}$$

$$-\frac{1}{2\pi^2\rho r^2\alpha_0}Re\left\{\int_0^\infty\left(1+\frac{i}{2Q_\alpha}\right)e^{-\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)\omega}e^{i\left(\frac{r}{\alpha_0}-t\right)\omega}d\omega\right\}$$

$$+\frac{1}{2\pi^2\rho r^3}Re\left\{\int_0^\infty\frac{i}{\omega}e^{-a\omega}\left(e^{-\frac{r}{2Q_\beta\beta_0}\omega}e^{i\left(\frac{r}{\beta_0}-t\right)\omega}-e^{-\frac{r}{2Q_\alpha\alpha_0}\omega}e^{i\left(\frac{r}{\alpha_0}-t\right)\omega}\right)d\omega\right\}$$

$$+\frac{1}{4\pi^2\rho r\alpha_0{}^2}Re\left\{\int_0^\infty i\omega e^{-\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)\omega}e^{i\left(\frac{r}{\alpha_0}-t\right)\omega}d\omega\right\}$$

$$-\frac{1}{4\pi^2\rho r\alpha_0{}^2Q_\alpha}Re\left\{\int_0^\infty \omega e^{-\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)\omega}e^{i\left(\frac{r}{\alpha_0}-t\right)\omega}d\omega\right\}$$

Applying Euler's formula, $e^{ix}=cosx+isinx$ to the above equation:

$u_1(\mathbf{x},t)$

$$=\frac{1}{2\pi^2\rho r^2\beta_0}Re\left\{\int_0^\infty\left(1+\frac{i}{2Q_\beta}\right)e^{-\left(\frac{r}{2Q_\beta\beta_0}+a\right)\omega}\left[\cos\left(\frac{r}{\beta_0}\omega-t\omega\right)+\mathrm{i}\sin\left(\frac{r}{\alpha_0}\omega-t\omega\right)\right]d\omega\right\}$$

$$-\frac{1}{2\pi^2\rho r^2\alpha_0}Re\left\{\int_0^\infty\left(1+\frac{i}{2Q_\alpha}\right)e^{-\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)\omega}\left[\cos\left(\frac{r}{\alpha_0}\omega-t\omega\right)+\mathrm{i}\sin\left(\frac{r}{\alpha_0}\omega-t\omega\right)\right]d\omega\right\}$$

$$+\frac{1}{2\pi^2\rho r^3}Re\left\{\int_0^\infty\frac{i}{\omega}e^{-a\omega}\left(e^{-\frac{r}{2Q_\beta\beta_0}\omega}\left[\cos\left(\frac{r}{\beta_0}\omega-t\omega\right)+\mathrm{i}\sin\left(\frac{r}{\beta_0}\omega-t\omega\right)\right]\right.\right.$$

$$\left.\left.-e^{-\frac{r}{2Q_\alpha\alpha_0}\omega}\left[\cos\left(\frac{r}{\alpha_0}\omega-t\omega\right)+\mathrm{i}\sin\left(\frac{r}{\alpha_0}\omega-t\omega\right)\right]\right)d\omega\right\}$$

$$+\frac{1}{4\pi^2\rho r\alpha_0{}^2}Re\left\{\int_0^\infty i\omega e^{-\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)\omega}\left[\cos\left(\frac{r}{\alpha_0}\omega-t\omega\right)+\mathrm{i}\sin\left(\frac{r}{\alpha_0}\omega-t\omega\right)\right]d\omega\right\}$$

$$-\frac{1}{4\pi^2\rho r\alpha_0{}^2Q_\alpha}Re\left\{\int_0^\infty \omega e^{-\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)\omega}\left[\cos\left(\frac{r}{\alpha_0}\omega-t\omega\right)+\mathrm{i}\sin\left(\frac{r}{\alpha_0}\omega-t\omega\right)\right]d\omega\right\}$$

$$= \frac{1}{2\pi^2\rho r^2\beta_0}\int_0^\infty e^{-\left(\frac{r}{2Q_\beta\beta_0}+a\right)\omega}\cos\left(\left(\frac{r}{\beta_0}-t\right)\omega\right)-\frac{1}{2Q_\beta}e^{-\left(\frac{r}{2Q_\beta\beta_0}+a\right)\omega}\sin\left(\left(\frac{r}{\beta_0}-t\right)\omega\right)d\omega$$

$$-\frac{1}{2\pi^2\rho r^2\alpha_0}\int_0^\infty e^{-\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)\omega}\cos\left(\left(\frac{r}{\alpha_0}-t\right)\omega\right)-\frac{1}{2Q_\alpha}e^{-\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)\omega}\sin\left(\left(\frac{r}{\alpha_0}-t\right)\omega\right)d\omega$$

$$+\frac{1}{2\pi^2\rho r^3}\int_0^\infty \frac{1}{\omega}e^{-\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)\omega}\sin\left(\left(\frac{r}{\alpha_0}-t\right)\omega\right)-\frac{1}{\omega}e^{-\left(a+\frac{r}{2Q_\beta\beta_0}\right)\omega}\sin\left(\left(\frac{r}{\beta_0}-t\right)\omega\right)d\omega$$

$$-\frac{1}{4\pi^2\rho r\alpha_0{}^2}\int_0^\infty \omega e^{-\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)\omega}\sin\left(\left(\frac{r}{\alpha_0}-t\right)\omega\right)d\omega$$

$$-\frac{1}{4\pi^2\rho r\alpha_0{}^2Q_\alpha}\int_0^\infty \omega e^{-\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)\omega}\cos\left(\left(\frac{r}{\alpha_0}-t\right)\omega\right)d\omega$$

For Spiegel (1968), we know

$$\int_0^\infty e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$$

$$\int_0^\infty e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$$

$$\int_0^\infty \frac{e^{-ax}\sin bx}{x}dx = \tan^{-1}\frac{b}{a}$$

$$\int_0^\infty xe^{-ax}\cos bx\,dx = \frac{a^2-b^2}{(a^2+b^2)^2}$$

$$\int_0^\infty xe^{-ax}\sin bx\,dx = \frac{2ab}{(a^2+b^2)^2}$$

Therefore,

$$u_1(\mathbf{x},t) = \frac{A}{2\pi^2\rho r^2}\left(\frac{\frac{1}{\beta_0}\left(a+\frac{t}{2Q_\beta}\right)}{\left(a+\frac{r}{2Q_\beta\beta_0}\right)^2+\left(\frac{r}{\beta_0}-t\right)^2}-\frac{\frac{1}{\alpha_0}\left(a+\frac{t}{2Q_\alpha}\right)}{\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)^2+\left(\frac{r}{\alpha_0}-t\right)^2}\right)$$

$$+\frac{A}{2\pi^2\rho r^3}\left(\tan^{-1}\frac{\frac{r}{\alpha_0}-t}{a+\frac{r}{2Q_\alpha\alpha_0}}-\tan^{-1}\frac{\frac{r}{\beta_0}-t}{a+\frac{r}{2Q_\beta\beta_0}}\right)$$

$$-\frac{A}{2\pi^2\rho r{\alpha_0}^2}\frac{\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)\left(\frac{r}{\alpha_0}-t\right)}{\left(\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)^2+\left(\frac{r}{\alpha_0}-t\right)^2\right)^2}$$

$$-\frac{A}{4\pi^2\rho r{\alpha_0}^2Q_\alpha}\frac{\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)^2-\left(\frac{r}{\alpha_0}-t\right)^2}{\left(\left(a+\frac{r}{2Q_\alpha\alpha_0}\right)^2+\left(\frac{r}{\alpha_0}-t\right)^2\right)^2} \tag{52b}$$

Now, we got the new solution of the EOM for a directed point force which includes absorption but no dispersion, i.e., equation (52b). (52b) is a new and exact result, and that even though it is absorption without dispersion, it could be applied in cases where absorption (and therefore dispersion) is small, to estimate the effect of absorption.

If we let Q_α and Q_β in (52b) go to infinity, so $1/Q_\alpha \to 0,\ 1/Q_\beta \to 0$, then we will have,

$$u_1(\mathbf{x},t)=\frac{A}{2\pi^2\rho r^2}\left(\frac{\frac{a}{\beta_0}}{a^2+\left(\frac{r}{\beta_0}-t\right)^2}-\frac{\frac{a}{\alpha_0}}{a^2+\left(\frac{r}{\alpha_0}-t\right)^2}\right)$$

$$+\frac{A}{2\pi^2\rho r^3}\left(\tan^{-1}\frac{\frac{r}{\alpha_0}-t}{a}-\tan^{-1}\frac{\frac{r}{\beta_0}-t}{a}\right)$$

$$-\frac{A}{2\pi^2\rho r{\alpha_0}^2}\frac{a\left(\frac{r}{\alpha_0}-t\right)}{\left(a^2+\left(\frac{r}{\alpha_0}-t\right)^2\right)^2}$$

which leads to the correct result for the elastic case (48).

In order to show the influence of the absorption on the wave propagation process, we will develop Matlab code to generate plots of the solution of the equation of motion for a directed point force $\mathrm{s(t)}$ for different receiver distance r ($r = 0.3, 1, 2, 5, 10\ \ km$) without absorption (48) and with absorption (52b), respectively.

There, the parameters I used in the figures are $a = 0.02s, \rho = 1kg/m^3, A = 1kg * km, Q_\alpha = 40, Q_\beta = 20, \alpha_0 = 5\ km/s$ and $\beta_0 = 3\ km/s$.

The measurements of Q around near-surface have been discussed by Ewing and Press (1954 a, b), Sato (1958), Pandit and Savage (1973) and so on. $Q_\alpha = 40, Q_\beta = 20$ are values for Q in the near-surface and are more like the values of Q in exploration geophysics. That is also why we make the distances more like the distances one sees in exploration geophysics.

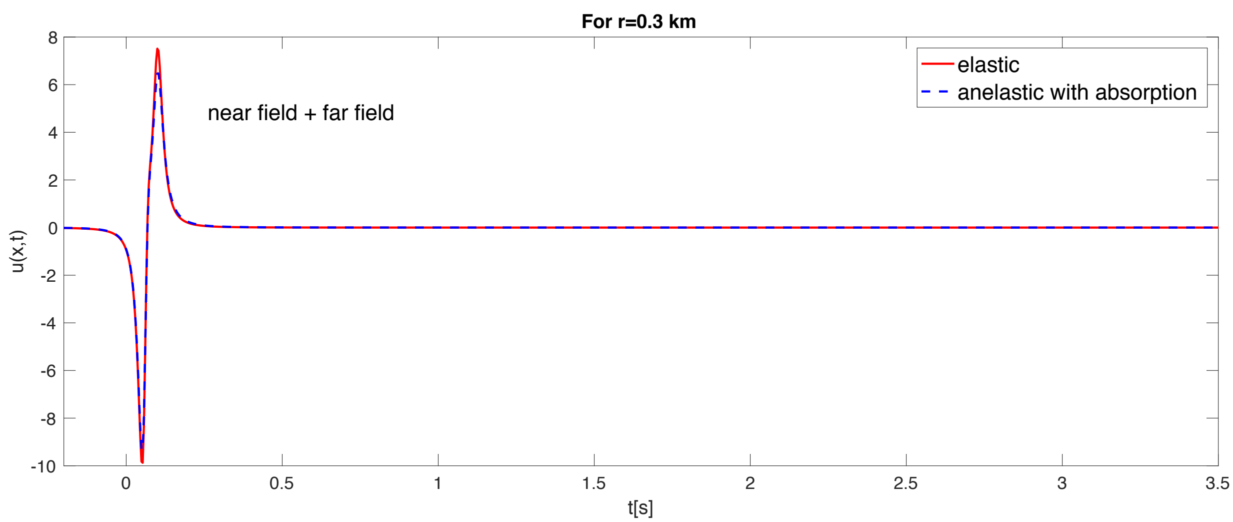


Fig.5a. The solution of the EOM for a directed point force with absorption (r=0.3km)

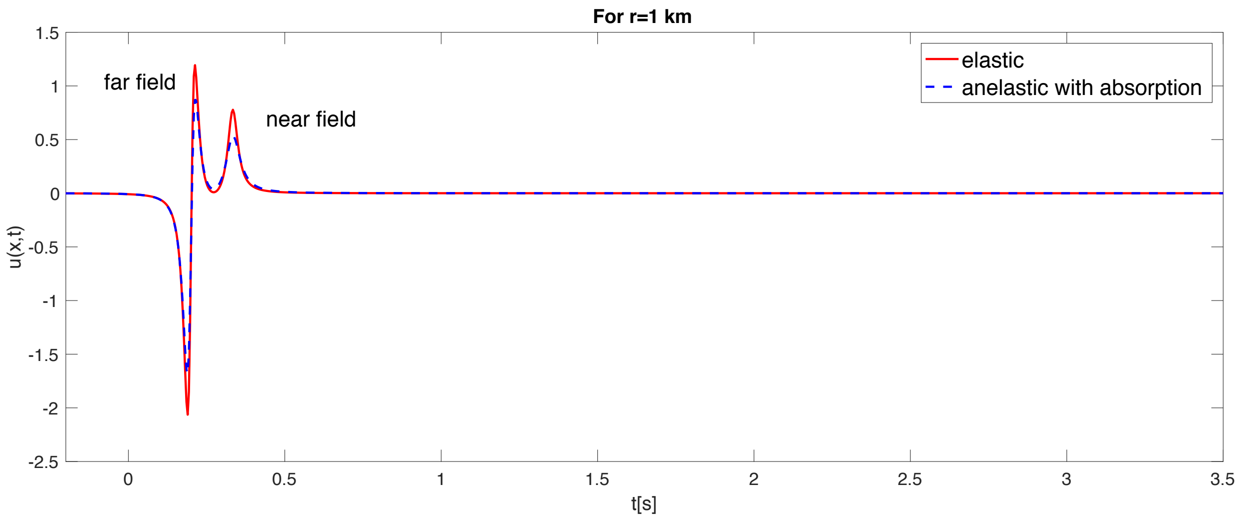


Fig.5b. The solution of the EOM for a directed point force with absorption (r=1km)

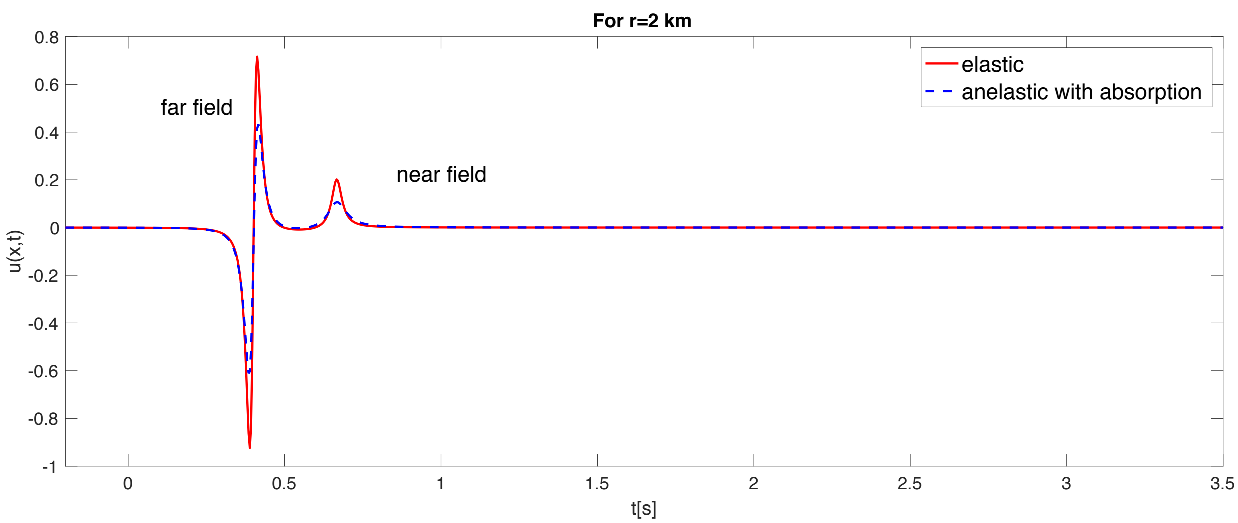


Fig.5c. The solution of the EOM for a directed point force with absorption (r=2km)

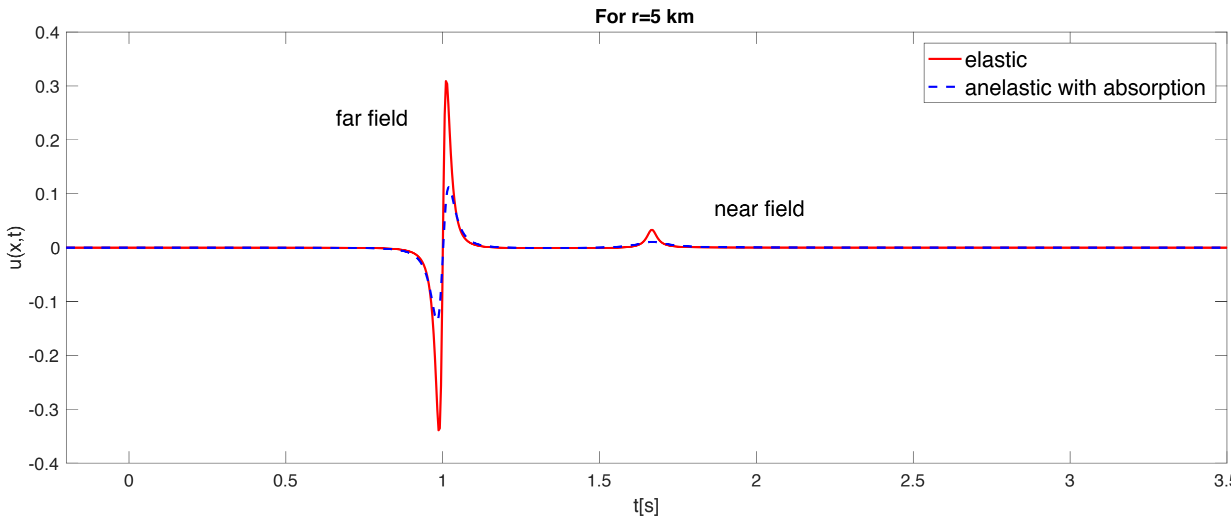


Fig.5d. The solution of the EOM for a directed point force with absorption (r=5km)

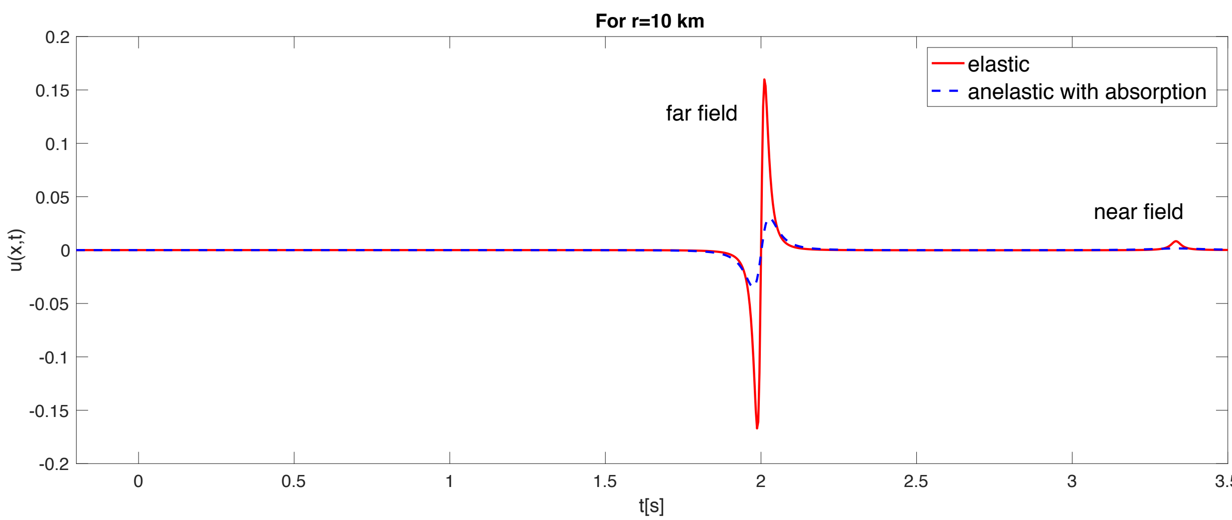


Fig.5e. The solution of the EOM for a directed point force with absorption (r=10km)

The red curves are the results of elastic waveforms, and the blue curves are anelastic waveforms. The effect of absorption becomes more and more obvious as the propagation distance increases.

At small values of distance, e.g., r=0.3km, the near-field term and far-field term cannot be distinguished. While, for a relative long distance, e.g., r=2km, they can be distinguished easily. And for a longer distance, e.g., r=10km, the far-field term dominates over the near-field term (especially in an anelastic medium).

3.3 The solutions of the EOM including absorption and dispersion

3.3.1 Find the analytical approximation

As I mentioned before, the expression (50) we used before are not quite physically realistic, because they do not include velocity dispersion.

We have been assuming that both velocity v and the quality factor Q are independent of frequency f. But to be physically realistic, one must include dispersion in the calculations, to ensure causality.

A velocity-frequency relation in the form of $\frac{1}{v(\omega)} = \frac{1}{v(\omega_{0r})}\left[1 - \frac{1}{\pi Q} ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{i}{2Q}\right]$ has been discussed in Chapter 1(equation 14). For seismic body waves, both v and Q vary with frequency f. But often Q is nearly constant (i.e., independent of frequency f). Q is not necessarily nearly constant for all types of seismic wave problems, but let us start with assuming Q is effectively constant.

Still, note that the $\frac{1}{Q^2}$ terms have been dropped, because $Q \gg 1$.

Therefore, we will have

$$\frac{1}{v^2(\omega)} = \frac{1}{{v_0}^2(\omega)}\left(1+\frac{i}{Q}\right) = \frac{1}{v^2(\omega_{0r})}\left[1-\frac{1}{\pi Q}ln\left(\frac{\omega}{\omega_{0r}}\right)\right]^2\left(1+\frac{i}{Q}\right)$$
$$\approx \frac{1}{v^2(\omega_{0r})}\left[1-\frac{2}{\pi Q}ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i}{Q}\right] \tag{53}$$

So, let us replace $\frac{1}{\alpha} \to \frac{1}{\alpha(\omega)}$ and $\frac{1}{\beta} \to \frac{1}{\beta(\omega)}$ as:

$$\frac{1}{\alpha(\omega)} = \frac{1}{\alpha(\omega_{0r})}\left[1-\frac{1}{\pi Q_\alpha}ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i}{2Q_\alpha}\right],$$
$$\frac{1}{\alpha^2(\omega)} = \frac{1}{\alpha^2(\omega_{0r})}\left[1-\frac{2}{\pi Q_\alpha}ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i}{Q_\alpha}\right] \tag{54a}$$

$$\frac{1}{\beta(\omega)} = \frac{1}{\beta(\omega_{0r})}\left[1-\frac{1}{\pi Q_\beta}ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i}{2Q_\beta}\right],$$
$$\frac{1}{\beta^2(\omega)} = \frac{1}{\beta^2(\omega_{0r})}\left[1-\frac{2}{\pi Q_\beta}ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i}{Q_\beta}\right] \tag{54b}$$

Substitute (54) into the solution of the EOM for a directed point force in an elastic medium, i.e., equation (47):

$$\bar{u}_1(\mathbf{x},\omega) = \frac{A}{2\pi\rho r^3}\left[\left(\frac{i}{\omega}+\frac{r}{\beta(\omega_{0r})}\left[1-\frac{1}{\pi Q_\beta}ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i}{2Q_\beta}\right]\right)e^{i\omega\frac{r}{\beta(\omega_{0r})}\left[1-\frac{1}{\pi Q_\beta}ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i}{2Q_\beta}\right]}\right.$$
$$\left.-\left(\frac{i}{\omega}+\frac{r}{\alpha(\omega_{0r})}\left[1-\frac{1}{\pi Q_\alpha}ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i}{2Q_\alpha}\right]\right)e^{i\omega\frac{r}{\alpha(\omega_{0r})}\left[1-\frac{1}{\pi Q_\alpha}ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i}{2Q_\alpha}\right]}\right]e^{-a|\omega|}$$
$$+\frac{Ai\omega}{4\pi\rho r}\frac{1}{\alpha^2(\omega_{0r})}\left[1-\frac{2}{\pi Q_\alpha}ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i}{Q_\alpha}\right]e^{i\omega\frac{r}{\alpha(\omega_{0r})}\left[1-\frac{1}{\pi Q_\alpha}ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i}{2Q_\alpha}\right]}e^{-a|\omega|} \tag{55a}$$

Note that in the above formula, the argument of the exponential is non-linear in ω, making it impossible to integrate. However, if one considers a signal spectrum $\bar{s}(\omega)$ that is narrow-band and centered on the reference frequency ω_{0r} (i.e., a spectrum that is zero except for frequencies near ω_{0r}), then one can use the following rule to make the argument linear in ω, meaning that it can be integrated:

$$lnx \approx \frac{x-1}{x}, \qquad x\ near\ 1, \qquad \Rightarrow \qquad \omega ln\left(\frac{\omega}{\omega_{0r}}\right) \approx \omega\left[\frac{(\omega/\omega_{0r})-1}{(\omega/\omega_{0r})}\right] = \omega - \omega_{0r} \qquad (56)$$

And we suppose that this is what was done in the papers by Cerveny and Frangie (1980).

Applying this rule to (55a),

$$\bar{u}_1(\mathbf{x},\omega) = \frac{A}{2\pi\rho r^3}\left[\left(\frac{i}{\omega}+\frac{r}{\beta(\omega_{0r})}\left[1-\frac{\omega ln\left(\frac{\omega}{\omega_{0r}}\right)}{\pi Q_\beta \omega}+\frac{i}{2Q_\beta}\right]\right)e^{i\frac{r}{\beta(\omega_{0r})}\left[\omega-\frac{1}{\pi Q_\beta}\omega ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i\omega}{2Q_\beta}\right]}\right.$$

$$\left.-\left(\frac{i}{\omega}+\frac{r}{\alpha(\omega_{0r})}\left[1-\frac{\omega ln\left(\frac{\omega}{\omega_{0r}}\right)}{\pi Q_\alpha \omega}+\frac{i}{2Q_\alpha}\right]\right)e^{i\frac{r}{\alpha(\omega_{0r})}\left[\omega-\frac{1}{\pi Q_\alpha}\omega ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i\omega}{2Q_\alpha}\right]}\right]e^{-a|\omega|}$$

$$+\frac{Ai}{4\pi\rho r}\frac{1}{\alpha^2(\omega_{0r})}\left[\omega-\frac{2}{\pi Q_\alpha}\omega ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i\omega}{Q_\alpha}\right]e^{i\frac{r}{\alpha(\omega_{0r})}\left[\omega-\frac{1}{\pi Q_\alpha}\omega ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i\omega}{2Q_\alpha}\right]}e^{-a|\omega|}$$

$$=\frac{A}{2\pi\rho r^3}\left[\left(\frac{i}{\omega}+\frac{r}{\beta(\omega_{0r})}\left[1-\frac{\omega-\omega_{0r}}{\pi Q_\beta \omega}+\frac{i}{2Q_\beta}\right]\right)e^{i\frac{r}{\beta(\omega_{0r})}\left[\omega-\frac{1}{\pi Q_\beta}(\omega-\omega_{0r})+\frac{i\omega}{2Q_\beta}\right]}\right.$$

$$\left.-\left(\frac{i}{\omega}+\frac{r}{\alpha(\omega_{0r})}\left[1-\frac{\omega-\omega_{0r}}{\pi Q_\alpha \omega}+\frac{i}{2Q_\alpha}\right]\right)e^{i\frac{r}{\alpha(\omega_{0r})}\left[\omega-\frac{1}{\pi Q_\alpha}(\omega-\omega_{0r})+\frac{i\omega}{2Q_\alpha}\right]}\right]e^{-a|\omega|}$$

$$+\frac{Ai}{4\pi\rho r}\frac{1}{\alpha^2(\omega_{0r})}\left[\omega-\frac{2}{\pi Q_\alpha}(\omega-\omega_{0r})+\frac{i\omega}{Q_\alpha}\right]e^{i\frac{r}{\alpha(\omega_{0r})}\left[\omega-\frac{1}{\pi Q_\alpha}(\omega-\omega_{0r})+\frac{i\omega}{2Q_\alpha}\right]}e^{-a|\omega|}$$

$$=\frac{A}{2\pi\rho r^3}\left[\left(\frac{i}{\omega}+\frac{r}{\beta(\omega_{0r})}\left[1-\frac{1}{\pi Q_\beta}+\frac{\omega_{0r}}{\pi Q_\beta \omega}+\frac{i}{2Q_\beta}\right]\right)e^{i\frac{r}{\beta(\omega_{0r})}\left[\omega-\frac{\omega}{\pi Q_\beta}+\frac{\omega_{0r}}{\pi Q_\beta}+\frac{i\omega}{2Q_\beta}\right]-a|\omega|}\right.$$

$$\left.-\left(\frac{i}{\omega}+\frac{r}{\alpha(\omega_{0r})}\left[1-\frac{1}{\pi Q_\alpha}+\frac{\omega_{0r}}{\pi Q_\alpha \omega}+\frac{i}{2Q_\alpha}\right]\right)e^{i\frac{r}{\alpha(\omega_{0r})}\left[\omega-\frac{\omega}{\pi Q_\alpha}+\frac{\omega_{0r}}{\pi Q_\alpha}+\frac{i\omega}{2Q_\alpha}\right]-a|\omega|}\right]$$

$$+\frac{A}{4\pi\rho r}\frac{i}{\alpha^2(\omega_{0r})}\left[\omega-\frac{2\omega}{\pi Q_\alpha}+\frac{2\omega_{0r}}{\pi Q_\alpha}+\frac{i\omega}{Q_\alpha}\right]e^{i\frac{r}{\alpha(\omega_{0r})}\left[\omega-\frac{\omega}{\pi Q_\alpha}+\frac{\omega_{0r}}{\pi Q_\alpha}+\frac{i\omega}{2Q_\alpha}\right]-a|\omega|}$$

$$=\frac{A}{2\pi\rho r^3}\left[\left(\frac{1}{\omega}\left(i+\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)+\frac{r}{\beta(\omega_{0r})}\left(1-\frac{1}{\pi Q_\beta}+\frac{i}{2Q_\beta}\right)\right)\right.$$

$$\times\ e^{i\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}}e^{i\frac{r}{\beta(\omega_{0r})}\left[\omega-\frac{\omega}{\pi Q_\beta}+\frac{i\omega}{2Q_\beta}\right]-a|\omega|}$$

$$-\left(\frac{1}{\omega}\left(i+\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)+\frac{r}{\alpha(\omega_{0r})}\left(1-\frac{1}{\pi Q_\alpha}+\frac{i}{2Q_\alpha}\right)\right)$$

$$\left.\times\ e^{i\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}} e^{i\frac{r}{\alpha(\omega_{0r})}\left[\omega-\frac{\omega}{\pi Q_\alpha}+\frac{i\omega}{2Q_\alpha}\right]-a|\omega|}\right]$$

$$+\frac{A}{4\pi\rho r}\frac{i}{\alpha^2(\omega_{0r})}\left[\omega\left(1-\frac{2}{\pi Q_\alpha}+\frac{i}{Q_\alpha}\right)+\frac{2\omega_{0r}}{\pi Q_\alpha}\right]$$

$$\times\ e^{i\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}} e^{i\frac{r}{\alpha(\omega_{0r})}\left[\omega-\frac{\omega}{\pi Q_\alpha}+\frac{i\omega}{2Q_\alpha}\right]-a|\omega|} \tag{55b}$$

Then, do inverse Fourier transform (IFT) of (55b). And again, if $g(t)$ is real, then in the inverse Fourier transform that gives $g(t)$, one may replace the integral from $-\infty$ to ∞ with 2 times the real part of the integral from 0 to ∞.

$$u_1(\mathbf{x},t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{A}{2\pi\rho r^3}\left[\left(\frac{1}{\omega}\left(i+\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)+\frac{r}{\beta(\omega_{0r})}\left(1-\frac{1}{\pi Q_\beta}+\frac{i}{2Q_\beta}\right)\right)\right.\right.$$

$$\times\ e^{i\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}} e^{i\frac{r}{\beta(\omega_{0r})}\left[\omega-\frac{\omega}{\pi Q_\beta}+\frac{i\omega}{2Q_\beta}\right]-a|\omega|}$$

$$-\left(\frac{1}{\omega}\left(i+\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)+\frac{r}{\alpha(\omega_{0r})}\left(1-\frac{1}{\pi Q_\alpha}+\frac{i}{2Q_\alpha}\right)\right)$$

$$\left.\times\ e^{i\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}} e^{i\frac{r}{\alpha(\omega_{0r})}\left[\omega-\frac{\omega}{\pi Q_\alpha}+\frac{i\omega}{2Q_\alpha}\right]-a|\omega|}\right]$$

$$+\frac{A}{4\pi\rho r}\frac{i}{\alpha^2(\omega_{0r})}\left[\omega\left(1-\frac{2}{\pi Q_\alpha}+\frac{i}{Q_\alpha}\right)+\frac{2\omega_{0r}}{\pi Q_\alpha}\right]$$

$$\left.\times\ e^{i\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}} e^{i\frac{r}{\alpha(\omega_{0r})}\left[\omega-\frac{\omega}{\pi Q_\alpha}+\frac{i\omega}{2Q_\alpha}\right]-a|\omega|}\right\} e^{-i\omega t} d\omega$$

$$=\frac{A}{2\pi^2\rho r^3} Re\left\{\int_0^{\infty}\left[\frac{1}{\omega}\left(i+\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)+\frac{r}{\beta(\omega_{0r})}\left(1-\frac{1}{\pi Q_\beta}+\frac{i}{2Q_\beta}\right)\right]\right.$$

$$\times\ e^{i\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}} e^{-\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)\omega} e^{i\omega\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)}$$

$$-\left[\frac{1}{\omega}\left(i+\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)+\frac{r}{\alpha(\omega_{0r})}\left(1-\frac{1}{\pi Q_\alpha}+\frac{i}{2Q_\alpha}\right)\right]$$

$$\times\ e^{i\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}} e^{-\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)\omega} e^{i\omega\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)} d\omega\Bigg\}$$

$$+\frac{A}{4\pi^2 \rho r} Re \int_0^\infty \frac{i}{\alpha^2(\omega_{0r})}\left[\omega\left(1-\frac{2}{\pi Q_\alpha}+\frac{i}{Q_\alpha}\right)+\frac{2\omega_{0r}}{\pi Q_\alpha}\right]$$

$$\times\ e^{i\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}} e^{-\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)\omega} e^{i\omega\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)} d\omega\Bigg\}$$

$$=\frac{A}{2\pi^2 \rho r^3} Re\left\{\int_0^\infty \frac{1}{\omega}\left(i+\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\left[\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)+i\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\right]\right.$$

$$\times\left[\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)+i\sin\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)\right]$$

$$\times\ e^{-\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)\omega}$$

$$-\frac{1}{\omega}\left(i+\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\left[\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)+i\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\right]$$

$$\times\left[\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)+i\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)\right]$$

$$\left.\times\ e^{-\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)\omega}\right\}$$

$$+\frac{A}{2\pi^2 \rho r^3} Re \int_0^\infty \frac{r}{\beta(\omega_{0r})}\left(1-\frac{1}{\pi Q_\beta}+\frac{i}{2Q_\beta}\right)$$

$$\times\left[\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)+i\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\right]$$

$$\times\left[\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)+i\sin\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)\right]$$

$$\times\ e^{-\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)\omega}$$

$$-\frac{r}{\alpha(\omega_{0r})}\left(1-\frac{1}{\pi Q_\alpha}+\frac{i}{2Q_\alpha}\right)\left[\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)+i\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\right]$$

$$\times\ \left[\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)\omega\right)+i\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)\omega\right)\right]$$

$$\times\ e^{-\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)\omega}d\omega\Bigg\}$$

$$+\frac{A}{4\pi^2\rho r}Re\left\{\int_0^\infty\frac{\omega}{\alpha^2(\omega_{0r})}\left(i-\frac{2i}{\pi Q_\alpha}-\frac{1}{Q_\alpha}\right)\right.$$

$$\times\left[\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)+i\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\right]$$

$$\times\ \left[\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)\omega\right)+i\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)\omega\right)\right]$$

$$\times\ e^{-\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)\omega}d\omega\Bigg\}$$

$$+\frac{A}{2\pi^3\rho r}Re\int_0^\infty\frac{i\omega_{0r}}{\alpha^2(\omega_{0r})Q_\alpha}\ \left[\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)+i\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\right]$$

$$\times\ \left[\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)\omega\right)+i\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)\omega\right)\right]$$

$$\times\ e^{-\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)\omega}d\omega$$

$$=\frac{A}{2\pi^2\rho r^3}\int_0^\infty\left\{\left[-\frac{1}{\omega}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)\omega\right)\right.\right.$$

$$-\frac{1}{\omega}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)\omega\right)$$

$$+\frac{1}{\omega}\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{1}{\omega}\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)\Bigg]$$

$$\times\ e^{-\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)\omega}$$

$$-\Bigg[-\frac{1}{\omega}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{1}{\omega}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$+\frac{1}{\omega}\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{1}{\omega}\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)\Bigg]$$

$$\times\ e^{-\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)\omega}\Bigg\}d\omega$$

$$+\frac{A}{2\pi^2\rho r^3}\int_0^\infty\Bigg\{\Bigg[\frac{r}{\beta(\omega_{0r})}\left(1-\frac{1}{\pi Q_\beta}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{r}{\beta(\omega_{0r})}\left(1-\frac{1}{\pi Q_\beta}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{r}{2Q_\beta \beta(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{r}{2Q_\beta \beta(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)\Bigg]$$

$$\times\ e^{-\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)\omega}$$

$$-\left[\frac{r}{\alpha(\omega_{0r})}\left(1-\frac{1}{\pi Q_\alpha}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)\right.$$

$$-\frac{r}{\alpha(\omega_{0r})}\left(1-\frac{1}{\pi Q_\alpha}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{r}{2Q_\alpha \alpha(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$\left.-\frac{r}{2Q_\alpha \alpha(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)\right]$$

$$\left.\times\ e^{-\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)\omega}\right\}d\omega$$

$$+\frac{A}{4\pi^2\rho r}\int_0^\infty\left\{\left[-\frac{\omega}{\alpha^2(\omega_{0r})}\left(1-\frac{2}{\pi Q_\alpha}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)\right.\right.$$

$$-\frac{\omega}{\alpha^2(\omega_{0r})}\left(1-\frac{2}{\pi Q_\alpha}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{\omega}{\alpha^2(\omega_{0r})Q_\alpha}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$\left.+\frac{\omega}{\alpha^2(\omega_{0r})Q_\alpha}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)\right]$$

$$\left.\times\ e^{-\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)\omega}\right\}d\omega$$

$$+\frac{A}{2\pi^3\rho r}\int_0^\infty\left\{\left[-\frac{\omega_{0r}}{\alpha^2(\omega_{0r})Q_\alpha}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)\right.\right.$$

$$- \frac{\omega_{0r}}{\alpha^2(\omega_{0r})Q_\alpha} \sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right) \cos\left(\left(\frac{r}{\alpha(\omega_{0r})} - \frac{r}{\pi Q_\alpha \alpha(\omega_{0r})} - t\right)\omega\right)\Bigg]$$

$$\times\ e^{-\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)\omega}\Bigg\} d\omega$$

From a mathematical handbook (Spiegel, 1968), we know that:

$$\int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2 + b^2}$$

$$\int_0^\infty e^{-ax} \cos bx\, dx = \frac{a}{a^2 + b^2}$$

$$\int_0^\infty \frac{e^{-ax} \sin bx}{x} dx = \tan^{-1} \frac{b}{a}$$

$$\int_0^\infty x e^{-ax} \cos bx\, dx = \frac{a^2 - b^2}{(a^2 + b^2)^2}$$

$$\int_0^\infty x e^{-ax} \sin bx\, dx = \frac{2ab}{(a^2 + b^2)^2}$$

Applying these integrals, then we will have:

$$u_1(\mathbf{x}, t) = \frac{A}{2\pi^2 \rho r^3} \times$$

$$\left[-\left(\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right) + \frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})} \sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\right) \tan^{-1}\left(\frac{\frac{r}{\beta(\omega_{0r})} - \frac{r}{\pi Q_\beta \beta(\omega_{0r})} - t}{\frac{r}{2Q_\beta \beta(\omega_{0r})} + a}\right)\right.$$

$$\left. + \left(\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right) + \frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})} \sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\right) \tan^{-1}\left(\frac{\frac{r}{\alpha(\omega_{0r})} - \frac{r}{\pi Q_\alpha \alpha(\omega_{0r})} - t}{\frac{r}{2Q_\alpha \alpha(\omega_{0r})} + a}\right)\right]$$

$$+ \frac{A}{2\pi^2 \rho r^3} \times$$

$$\left\{\frac{\left[\frac{r}{\beta(\omega_{0r})}\left(1-\frac{1}{\pi Q_\beta}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)-\frac{r}{2Q_\beta \beta(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\right]\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)}{\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2}\right.$$

$$-\frac{\left[\frac{r}{\beta(\omega_{0r})}\left(1-\frac{1}{\pi Q_\beta}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\right]\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2}$$

$$-\frac{\left[\frac{r}{2Q_\beta \beta(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\right]\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2}$$

$$-\frac{\left[\frac{r}{\alpha(\omega_{0r})}\left(1-\frac{1}{\pi Q_\alpha}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)-\frac{r}{2Q_\alpha \alpha(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\right]\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)}{\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^2}$$

$$+\frac{\left[\frac{r}{\alpha(\omega_{0r})}\left(1-\frac{1}{\pi Q_\alpha}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\right]\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^2}$$

$$\left.+\frac{\left[\frac{r}{2Q_\alpha \alpha(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\right]\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^2}\right\}$$

$$+\frac{A}{4\pi^2\rho r}\times$$

$$\left\{-\frac{\left(1-\frac{2}{\pi Q_\alpha}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)}{\alpha^2(\omega_{0r})}\frac{2\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^2\right]^2}\right.$$

$$-\frac{\left(1-\frac{2}{\pi Q_\alpha}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)}{\alpha^2(\omega_{0r})}\frac{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2-\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^2}$$

$$-\frac{1}{\alpha^2(\omega_{0r})Q_\alpha}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\frac{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2-\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^2}$$

$$\left.+\frac{1}{\alpha^2(\omega_{0r})Q_\alpha}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\frac{2\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^2}\right\}$$

$$-\frac{A}{2\pi^3\rho r}\times$$

$$\left\{\frac{\frac{\omega_{0r}}{\alpha^2(\omega_{0r})Q_\alpha}\left[\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)\right]}{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}\right.$$

$$\left.+\frac{\frac{\omega_{0r}}{\alpha^2(\omega_{0r})Q_\alpha}\left[\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)\right]}{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}\right\}$$

$$+\frac{A}{2\pi^2\rho r^3}\int_0^\infty\left\{\left[-\frac{1}{\omega}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)\omega\right)\right.\right.$$

$$\left.+\frac{1}{\omega}\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)\omega\right)\right]$$

$$\times\ e^{-\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)\omega}$$

$$-\left[-\frac{1}{\omega}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)\omega\right)\right.$$

$$+\frac{1}{\omega}\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)\Bigg]$$

$$\left.\times\ e^{-\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)\omega}\right\}d\omega \tag{57a}$$

Now, what we need to consider is about how to solve the integrals M,

$$M=\frac{A}{2\pi^2\rho r^3}\int_0^\infty\left\{\left[-\frac{1}{\omega}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)\right.\right.$$

$$\left.+\frac{1}{\omega}\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)\right]$$

$$\times\ e^{-\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)\omega}$$

$$-\left[-\frac{1}{\omega}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)\right.$$

$$\left.+\frac{1}{\omega}\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)\right]$$

$$\left.\times\ e^{-\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)\omega}\right\}d\omega \tag{58}$$

In (58), we have four integrals, each of which has the form $\int_0^\infty \frac{1}{\omega}\cos(a\omega)\,e^{b\omega}d\omega$. This cannot be done, because the integrand goes to infinity as ω goes to zero.

Therefore, we can try to calculate them approximately by approximating the integrand with a simpler function that can be integrated. Plot the integrand for different parameter values to see if it can be replaced approximately with something simpler, like a straight-line function or a parabola.

To see if the integrand can be replaced approximately with something simpler, we first plot the integrand versus ω. Consider the range which Q is constant, I assumed the lowest angular frequency is 0.3Hz, and the sampling interval is 1.

If we assume parameters are $\mathrm{r} = 150\mathrm{km}$, $\mathrm{a} = 1\mathrm{s}$, $Q_\alpha = 180$, $Q_\beta = 150$, $\alpha = 5km/s$, $\beta = 3km/s$, $\omega_{0r} = 1Hz$, $t = 1s$, $\rho = 1kg/m^3$, $\omega = 0.3 - 50Hz$, then the integrand of M looks like below:

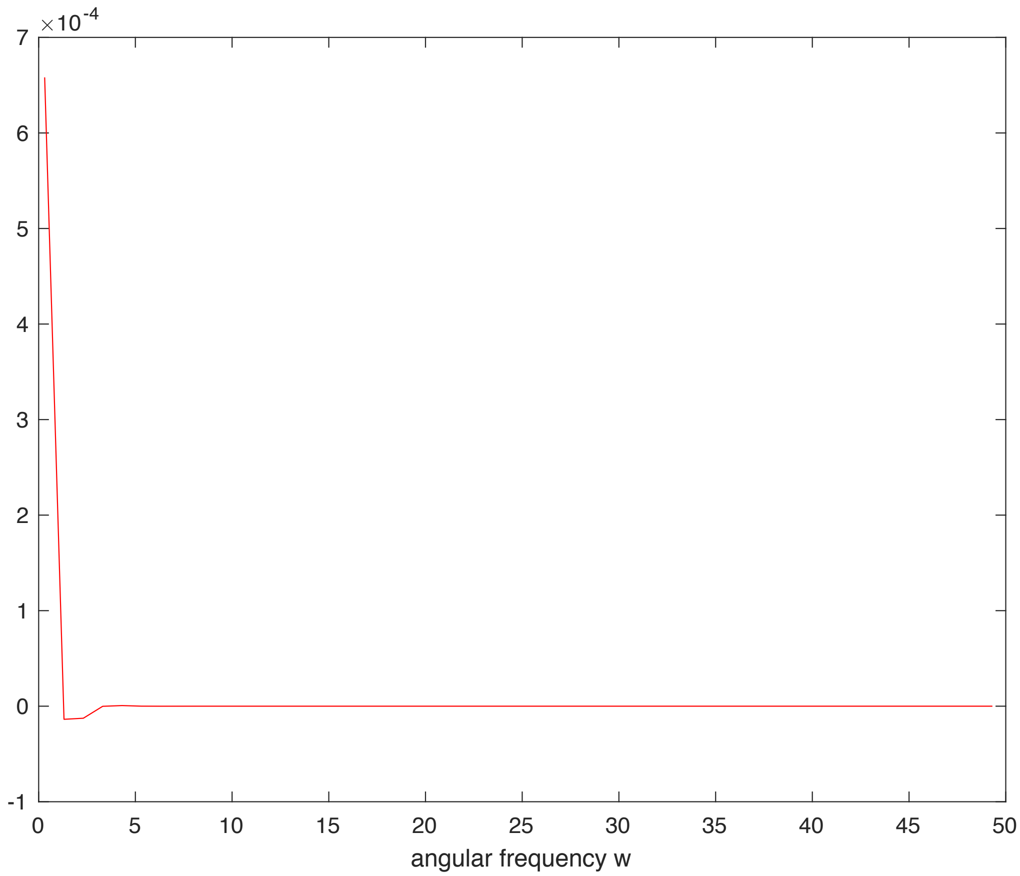


Fig.6a. The integrand of M ($\omega = 0.3 - 50Hz$)

To amplify the part of the above figure to see more detail from 0.3 to 5 Hz, so we are going to reduce the interval to 0.01Hz, and the integrand looks like,

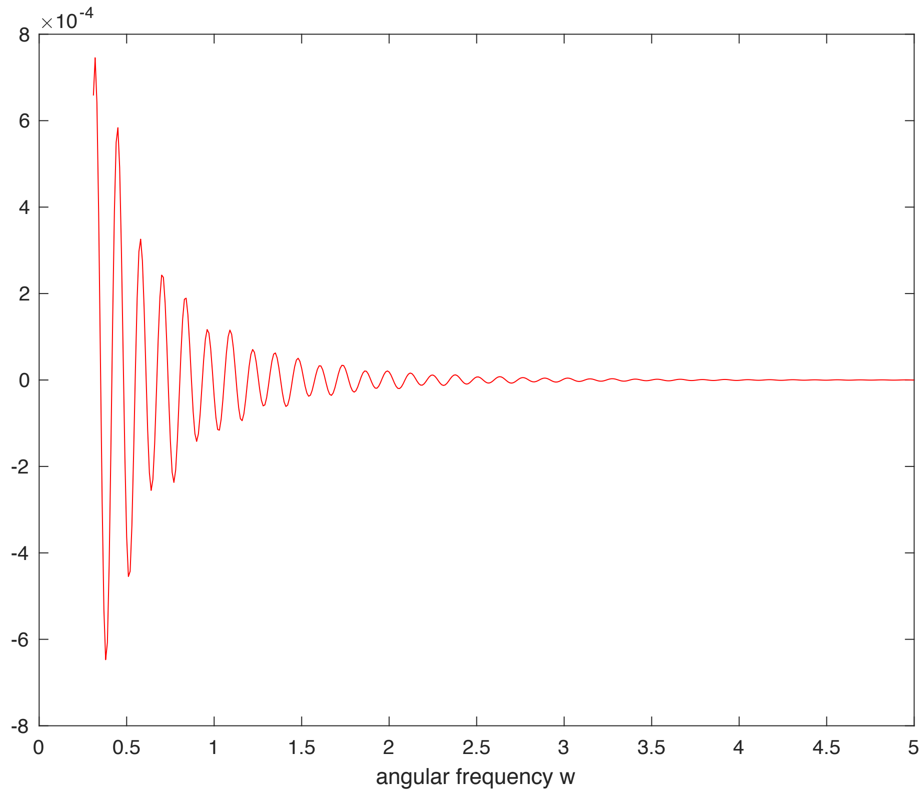


Fig.6b. The integrand of M ($\omega = 0.3 - 5Hz$)

We can see from Fig.6b that it is basically a damped sinusoid. After trying some different formulas and parameters, I found a function which is very similar to the integrand:

$$\mathrm{S1} = 0.0017e^{-2.5\omega}\sin(47.5\omega) \tag{59}$$

Plot the approximation (59) versus ω as well,

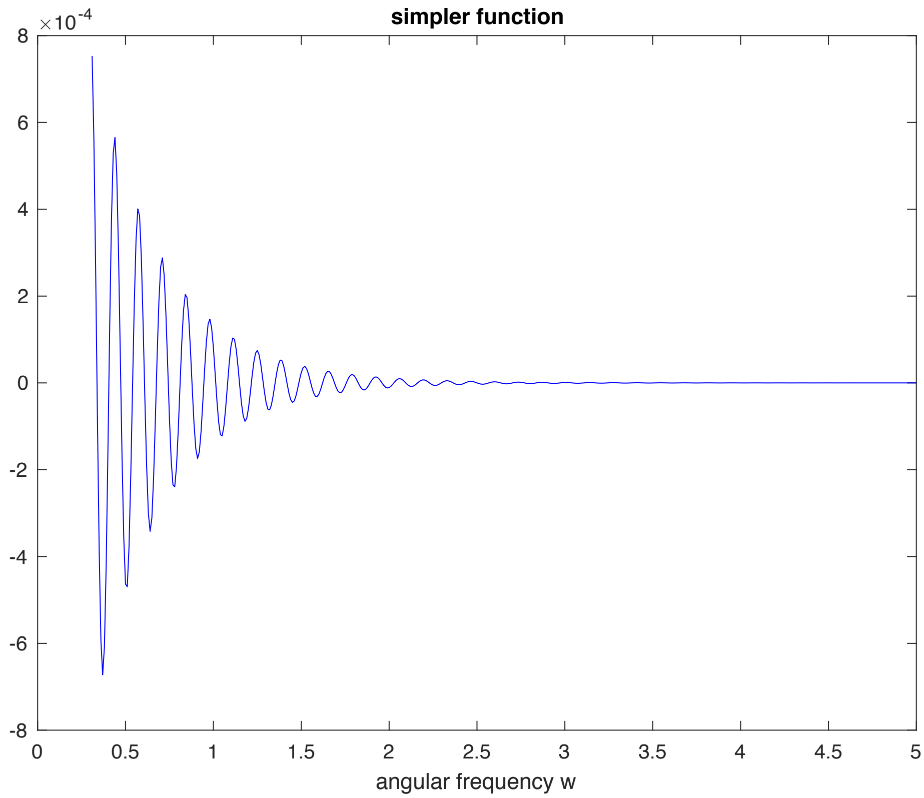


Fig.6c. The simpler function S1 ($\omega = 0.3 - 5Hz$)

Put the integrand and the approximation together for comparison in Fig.6d ($\omega = 0.3 - 50Hz$) and Fig.6e ($\omega = 0.3 - 50Hz$). Red curves indicate the integrand and blue curves indicate the simpler function. We can see that they coincide well in both pictures.

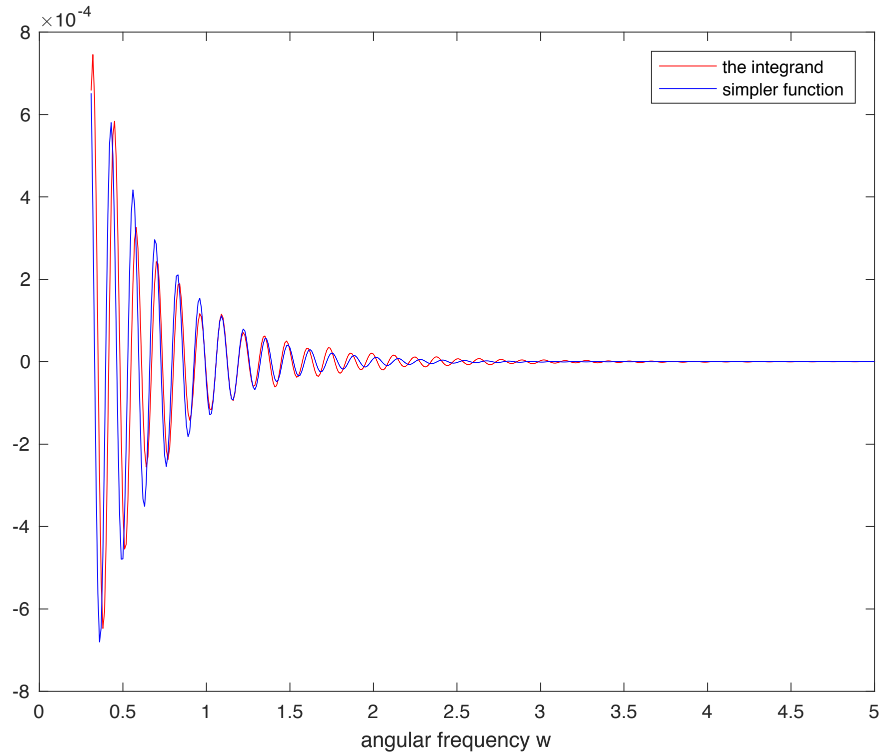


Fig.6d. The integrand of M and the simpler function S1 ($\omega = 0.3 - 5Hz$)

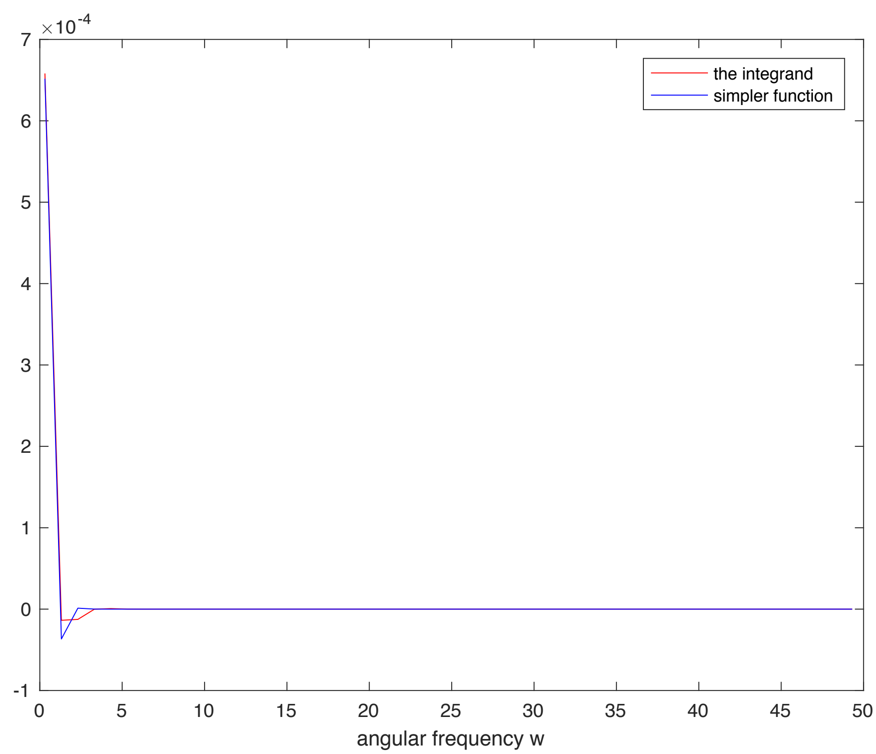


Fig.6e. The integrand of M and the simpler function S1 ($\omega = 0.3 - 50Hz$)

I also tried some different parameters in the integrand to test if a damped sinusoid function can work for all of them, and I can say it works well.

I will show another case here, which the parameters in the integrand are $\mathrm{r} = 10\mathrm{km},\ \mathrm{a} = 1\mathrm{s},\ Q_\alpha = 40,\ Q_\beta = 20,\ \alpha = 5km/s,\ \beta = 3km/s,\ \omega_{0r} = 1Hz,\ t = 1s,\ \rho = 1kg/m^3$.

In this case, the simpler approximation is:

$$\mathrm{S2} = 0.00035e^{-3\omega}\sin(-3.7\omega) \tag{60}$$

Show the result together like Fig.6d and Fig.6e in Fig.7a and Fig.7b:

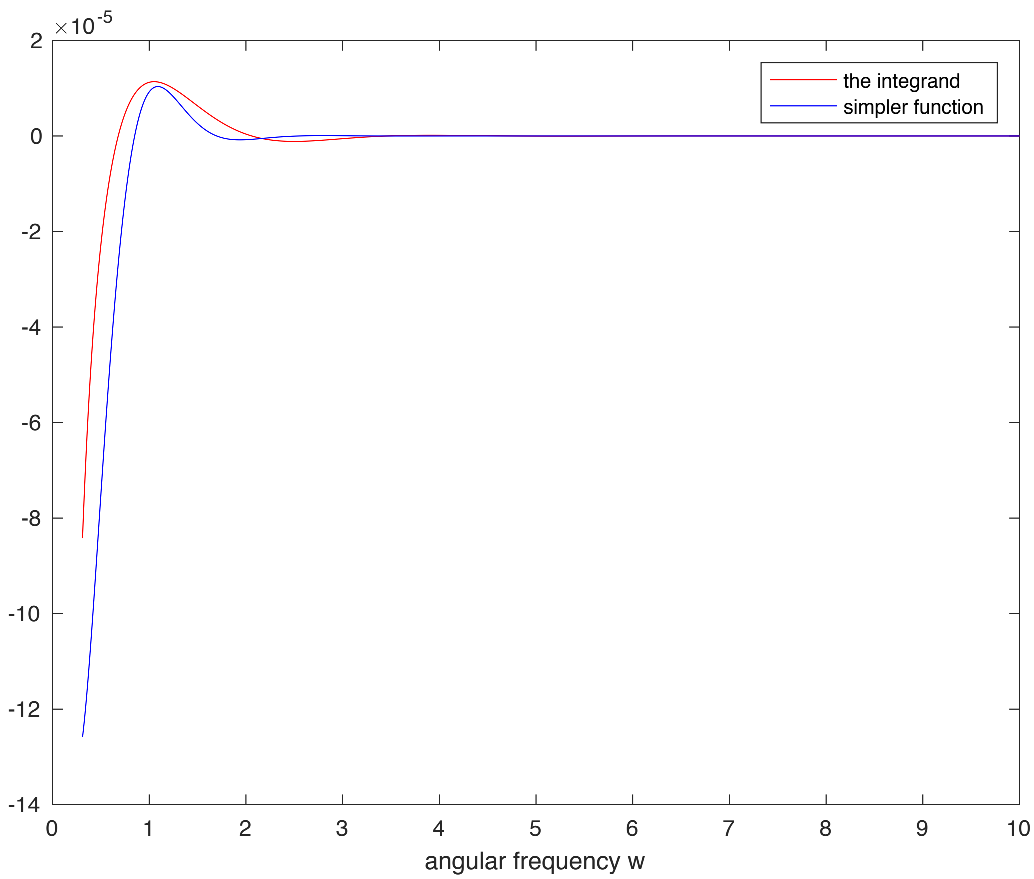


Fig.7a. The integrand of M and the simpler function S2 ($\omega = 0.3 - 10Hz$)

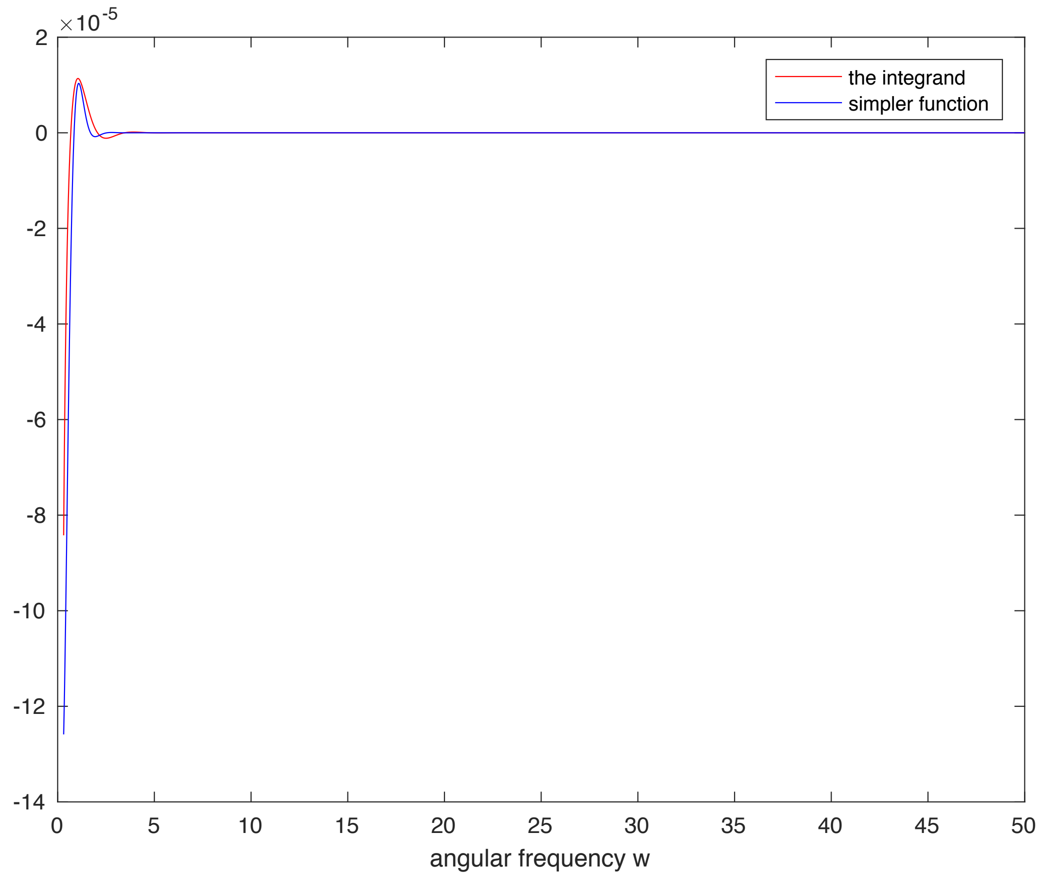


Fig.7b. The integrand of M and the simpler function S2 ($\omega = 0.3 - 50Hz$)

If we use this simpler function $S1$(or $S2$) to replace the integrand in the integral M, and then solve the integral, we will have:

For $S1$:

$$\frac{A}{2\pi^2\rho r^3} Re\left\{\int_0^\infty 0.0017e^{-2.5\omega}\sin(47.5\omega)\,d\omega\right\}$$

$$= \frac{1}{2\pi^2 150^3} 0.0017 \frac{47.5}{2.5^2 + 47.5^2} \approx 5.36 \times 10^{-13}$$

For $S2$:

$$\frac{A}{2\pi^2\rho r^3} Re\left\{\int_0^\infty 0.00035e^{-3\omega}\sin(-3.5\omega)\,d\omega\right\}$$

$$= \frac{1}{2\pi^2 10^3} 0.00035 \frac{-3.5}{3^2 + -3.5^2} \approx -2.92 \times 10^{-9}$$

This answer is much smaller than the other parts in (57a) (around 10^-7 for $S1$, 10^-5 for $S2$), so it is alright to directly ignore the integral M. I tried some other medium parameters as well, even though values of the results were very different, all of the values of the integral M in different medium were much smaller than other parts of the approximation.

Then, the new solution of the EOM for a directed point force with dispersion is:

$$u_1(\mathbf{x}, t) = \frac{A}{2\pi^2 \rho r^3} \times$$

$$\left[-\left(\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})} \right) + \frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})} \sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})} \right) \right) \tan^{-1}\left(\frac{\frac{r}{\beta(\omega_{0r})} - \frac{r}{\pi Q_\beta \beta(\omega_{0r})} - t}{\frac{r}{2Q_\beta \beta(\omega_{0r})} + a} \right) \right.$$

$$\left. + \left(\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})} \right) + \frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})} \sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})} \right) \right) \tan^{-1}\left(\frac{\frac{r}{\alpha(\omega_{0r})} - \frac{r}{\pi Q_\alpha \alpha(\omega_{0r})} - t}{\frac{r}{2Q_\alpha \alpha(\omega_{0r})} + a} \right) \right]$$

$$+ \frac{A}{2\pi^2 \rho r^3} \times$$

$$\left\{ \frac{\left[\frac{r}{\beta(\omega_{0r})} \left(1 - \frac{1}{\pi Q_\beta} \right) \cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})} \right) - \frac{r}{2Q_\beta \beta(\omega_{0r})} \sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})} \right) \right] \left(\frac{r}{2Q_\beta \beta(\omega_{0r})} + a \right)}{\left(\frac{r}{2Q_\beta \beta(\omega_{0r})} + a \right)^2 + \left(\frac{r}{\beta(\omega_{0r})} - \frac{r}{\pi Q_\beta \beta(\omega_{0r})} - t \right)^2} \right.$$

$$- \frac{\left[\frac{r}{\beta(\omega_{0r})} \left(1 - \frac{1}{\pi Q_\beta} \right) \sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})} \right) \right] \left(\frac{r}{\beta(\omega_{0r})} - \frac{r}{\pi Q_\beta \beta(\omega_{0r})} - t \right)}{\left(\frac{r}{2Q_\beta \beta(\omega_{0r})} + a \right)^2 + \left(\frac{r}{\beta(\omega_{0r})} - \frac{r}{\pi Q_\beta \beta(\omega_{0r})} - t \right)^2}$$

$$-\frac{\left[\frac{r}{2Q_\beta\beta(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)\right]\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2}$$

$$-\frac{\left[\frac{r}{\alpha(\omega_{0r})}\left(1-\frac{1}{\pi Q_\alpha}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)-\frac{r}{2Q_\alpha\alpha(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\right]\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)}{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}$$

$$+\frac{\left[\frac{r}{\alpha(\omega_{0r})}\left(1-\frac{1}{\pi Q_\alpha}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\right]\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}$$

$$\left.+\frac{\left[\frac{r}{2Q_\alpha\alpha(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\right]\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}\right\}$$

$$+\frac{A}{4\pi^2\rho r}\times$$

$$\left\{-\frac{\left(1-\frac{2}{\pi Q_\alpha}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)}{\alpha^2(\omega_{0r})}\frac{2\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^2}\right.$$

$$-\frac{\left(1-\frac{2}{\pi Q_\alpha}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)}{\alpha^2(\omega_{0r})}\frac{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2-\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^2}$$

$$-\frac{1}{\alpha^2(\omega_{0r})Q_\alpha}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\frac{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2-\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^2}$$

$$+\frac{1}{\alpha^2(\omega_{0r})Q_\alpha}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\frac{2\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^2}\Bigg\}$$

$$-\frac{A}{2\pi^3\rho r}\times$$

$$\left\{\frac{\frac{\omega_{0r}}{\alpha^2(\omega_{0r})Q_\alpha}\left[\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)\right]}{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}\right.$$

$$\left.+\frac{\frac{\omega_{0r}}{\alpha^2(\omega_{0r})Q_\alpha}\left[\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)\right]}{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}\right\} \tag{57b}$$

To simply this complicated expression, let us assume $F_v=\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}, H_v=\frac{r}{2Q_v v(\omega_{0r})}+a, and\ G_v=\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t$, substitute these relations into (57b), then we could have a simpler expression:

$$u_1(\mathbf{x},t)=$$

$$\frac{A}{2\pi^2\rho r^3}\left[-\left(\cos(F_\beta)+F_\beta\sin(F_\beta)\right)\tan^{-1}\left(\frac{G_\beta}{H_\beta}\right)+\left(\cos(F_\alpha)+F_\alpha\sin(F_\alpha)\right)\tan^{-1}\left(\frac{G_\alpha}{H_\alpha}\right)\right]$$

$$+\frac{A}{2\pi^2\rho r^3}\frac{\left(\frac{r}{\beta(\omega_{0r})}-\frac{F_\beta}{\omega_{0r}}\right)\left(H_\beta\cos(F_\beta)-G_\beta\sin(F_\beta)\right)-\left(H_\beta-a\right)\left(H_\beta\sin(F_\beta)+G_\beta\cos(F_\beta)\right)}{{H_\beta}^2+{G_\beta}^2}$$

$$-\frac{A}{2\pi^2\rho r^3}\frac{\left(\frac{r}{\alpha(\omega_{0r})}-\frac{F_\alpha}{\omega_{0r}}\right)\left(H_\alpha\cos(F_\alpha)-G_\alpha\sin(F_\alpha)\right)-\left(H_\alpha-a\right)\left(H_\alpha\sin(F_\alpha)+G_\alpha\cos(F_\alpha)\right)}{{H_\alpha}^2+{G_\alpha}^2}\Bigg]$$

$$+\frac{A}{4\pi^2\rho r}\frac{\left[\frac{\sin(F_\alpha)}{Q_\alpha}-\left(1-\frac{2}{\pi Q_\alpha}\right)\cos(F_\alpha)\right]2H_\alpha G_\alpha-\left[\frac{\cos(F_\alpha)}{Q_\alpha}+\left(1-\frac{2}{\pi Q_\alpha}\right)\sin(F_\alpha)\right]\left({H_\alpha}^2-{G_\alpha}^2\right)}{\alpha^2(\omega_{0r})\left({H_\alpha}^2+{G_\alpha}^2\right)^2}$$

$$-\frac{A}{2\pi^2\rho r^2}\frac{F_\alpha(\cos(F_\alpha)\,G_\alpha+\sin(F_\alpha)\,H_\alpha)}{\alpha(\omega_{0r})\left({H_\alpha}^2+{G_\alpha}^2\right)} \tag{61}$$

(61) is a new approximate result with absorption and dispersion. Even though it is an approximation, approximations are still useful in that they can be used to study which factors are important and also to sometimes reduce computation time.

3.3.2 Compare with exact numerical results

To verify the accuracy of the approximation (61), we created a program by Matlab to compute the exact results with velocity dispersion numerically. The basic idea is using "ifft" function in Matlab to apply inverse fast Fourier transform to (55a) to get some exact numerical results in time domain.

All the codes I used can be found in appendix.

To use the codes, one of the problem is how to choose parameters in them.

In this case, we will still simulate a near-surface area, like in exploration geophysics. Therefore, we assume $r = 0.3, 1, 2, 5, 10\ km$, $Q_\alpha = 40, Q_\beta = 20, \alpha_0 = 5\ km/s$ and $\beta_0 = 3\ km/s$. Also, the pulse width should be relative small, e.g., 0.05 s, so the value of "a" could not be too large, then we assume $a = 0.02s$ here.

As for ω_{0r}, this is the reference angular frequency, and $v(\omega_{0r})$ is the known wave speed at ω_{0r}. If the angular frequency vector goes from $\omega = 1Hz$ to 100*Hz*, and if ω_{0r}=1*Hz*, then the lowest v is at the lowest angular frequency $\omega = 1Hz$. But if you change ω_{0r} to 10*Hz*, then the term log (ω/ω_{0r}) will be negative for $\omega = 1Hz$ to 10*Hz*, and the lowest v will change to something smaller. This also affects the travel times.

Since the approximation we used for dispersion is only good for a narrow spectrum $\bar{u}_1(\mathbf{x},\omega)$ with ω near ω_{0r}, it will be better to plot the spectrum $\bar{u}_1(\mathbf{x},\omega)$ vs. ω to see if it is narrow and the frequencies ω are near ω_{0r}.

And we should also check to see if the approximation $\omega * \ln(\omega/\omega_{0r}) \approx \omega - \omega_{0r}$ is satisfied for frequencies within the range where $\bar{u}_1(\mathbf{x},\omega)$ has appreciable non-zero values. For example, find the maximum value, $\bar{u}_1(\mathbf{x},\omega)_{max}$, which occurs at some angular frequency ω_{max}, then check to see if the frequencies ω for which $\bar{u}_1(\mathbf{x},\omega)_{max}/\bar{u}_1(\mathbf{x},\omega) < 100$, say, satisfy $\omega * \ln(\omega/\omega_{0r}) \approx \omega - \omega_{0r}$. That means, we can look at the value of the ratio, $\omega\ln(\omega/\omega_{0r})/(\omega - \omega_{0r})$ to see if this ratio around 1 for the frequencies where $\bar{u}_1(\mathbf{x},\omega)$ is not zero. Also, it is better to test if $\omega/\omega_{0r} \approx 1$ as well.

After trying some different combinations of parameters, I found that ω_{0r} should always be chosen as the value of ω at the "peak", i.e., the maximum value of $\bar{u}_1(\mathbf{x},\omega)$, of the amplitude spectrum of $\bar{u}_1(\mathbf{x},\omega)$. That way, all the ω values on the curve on either side of the peak will, on average, be closer to ω_{0r}, and so the approximation will be better.

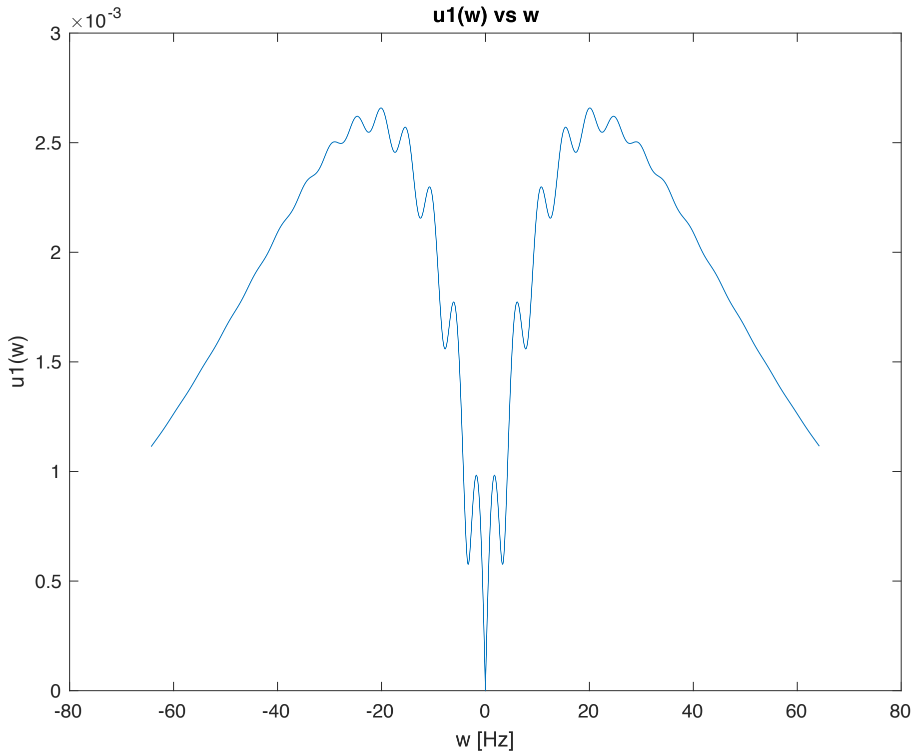


Fig.8a. $\bar{u}_1(\mathrm{x}, \omega)$ vs. ω

($a = 0.02s, Q_\alpha = 40, Q_\beta = 20, \alpha_0 = 5\ km/s$ and $\beta_0 = 3\ km/s$)

An example has been showed in Fig.8 above, where $a = 0.02s, Q_\alpha = 40, Q_\beta = 20, \alpha_0 = 5\ km/s$ and $\beta_0 = 3\ km/s$.

In the graph above, the peak is around $\omega = 20Hz$, therefore, for this graph, we should choose $\omega_{0r} = 20Hz$ (or something close to that). The significant frequency range for this graph, is 10 to 50 Hz. So if $\omega_{0r} = 20Hz$, and if we pick a frequency inside this range, e.g., $\omega = 40Hz$, then $\omega/\omega_{0r} = 2$, which is not close to 1, but the value of $|\bar{u}_1(\mathbf{x}, \omega)|$ at $\omega = 40Hz$ will also be much smaller than $\bar{u}_1(\mathbf{x}, \omega)_{max}$, meaning it will probably have a small negative effect on the approximation, and so the approximation could still be fairly good. Similarly, high value of

$\omega\ln(\omega/\omega_{0r})/(\omega-\omega_{0r})$ is 1.53, which is not close to 1, but the value of $|\bar{u}_1(\mathbf{x},\omega)|$ is also much smaller than $\bar{u}_1(\mathbf{x},\omega)_{max}$, so the approximation could still be fairly good.

Just to clarify, if we always choose ω_{0r} at the peak (which means different ω_{0r} for different "a"), then the approximation works for all "a". But if we choose only one ω_{0r} (e.g., 1 Hz), then the approximation only works for some "a" but is bad for other "a", because for different "a", the peak of spectrum also changes, while the other parameters do not affect the position of peak much. For example, if we let $a = 1s, r = 10km,\ Q_\alpha = 40, Q_\beta = 20, \alpha_0 = 5\ km/s$ and $\beta_0 = 3\ km/s$, then the spectrum is,

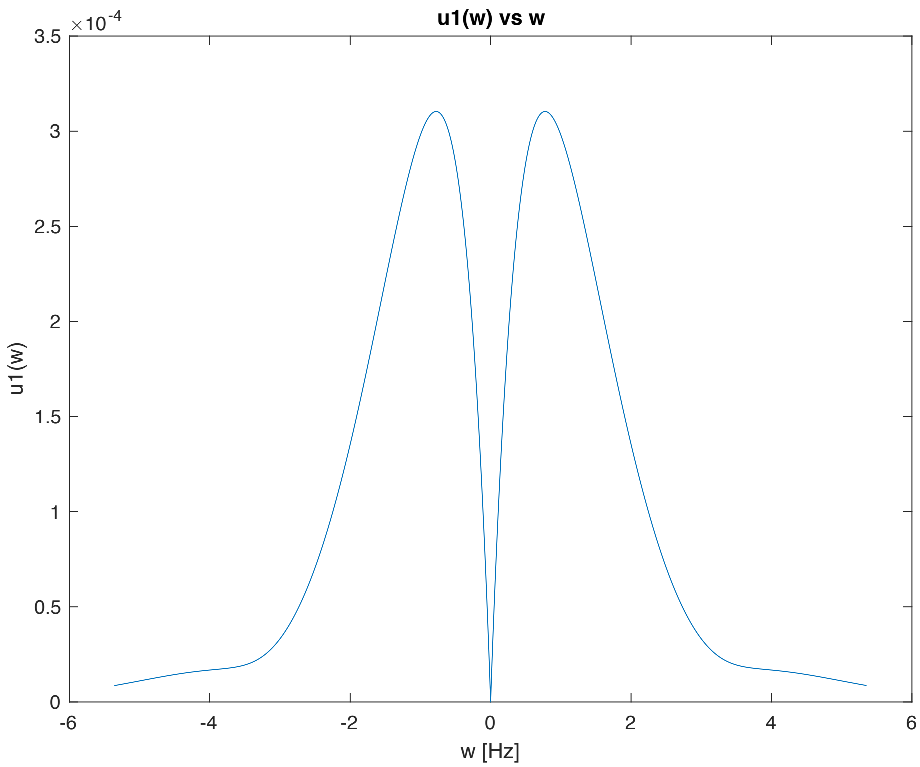


Fig.8b. $\bar{u}_1(\mathbf{x},\omega)$ vs. ω

($a = 1s, Q_\alpha = 40, Q_\beta = 20, \alpha_0 = 5\ km/s$ and $\beta_0 = 3\ km/s$)

In Fig.8b, we can see the peak has changed around $\omega = 1Hz$ when we only change a from 0.02s to 1s. If we keep a constant and change other parameters, like $a = 1s, r = 20km$, $Q_\alpha = 100, Q_\beta = 50, \alpha_0 = 15\ km/s$ and $\beta_0 = 10\ km/s$. The spectrum is showed in Fig.8c. The peak is still around $\omega = 1Hz$.

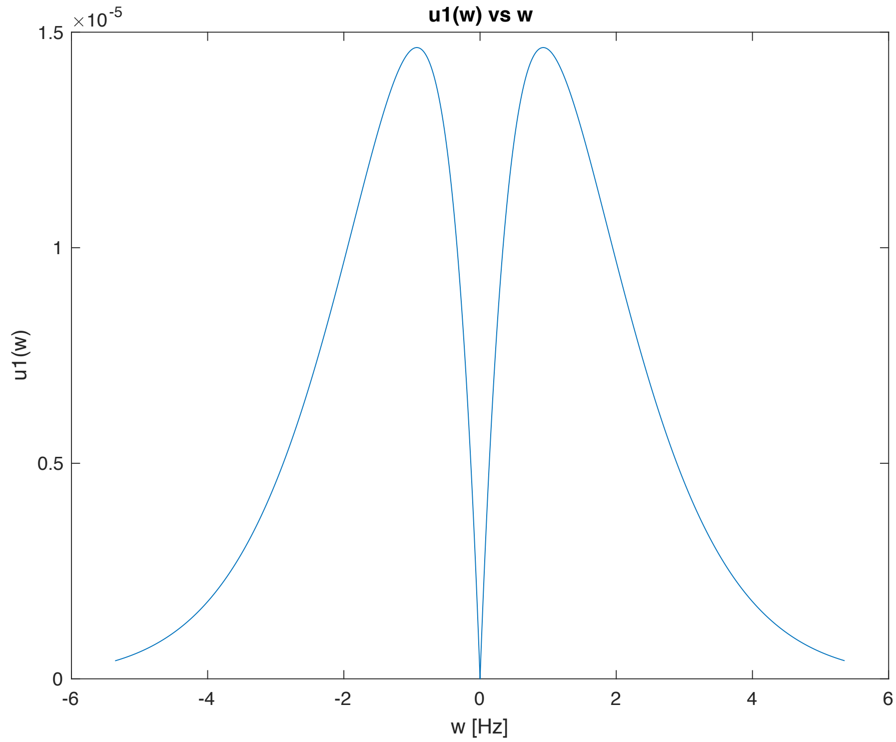


Fig.8c. $\boldsymbol{\bar{u}_1(\mathrm{x}, \omega)}$ **vs.** $\boldsymbol{\omega}$
$\boldsymbol{(a = 1s, Q_\alpha = 100, Q_\beta = 50, \alpha_0 = 15\ km/s}$ **and** $\boldsymbol{\beta_0 = 10\ km/s)}$

Also, there is another problem we should think of. The spectrum $\bar{s}(\omega)$ has frequencies that go all the way to zero, and at $\omega = 0$, we have $\ln(\omega/\omega_{0r})$ going to –infinity. But in reality, Q is constant only over a finite range of frequencies (see Fig.9, after Figure 3 in Liu *et al.,* 1976), and the velocity v obeys the ln formula inside this range, and is constant outside of the range.

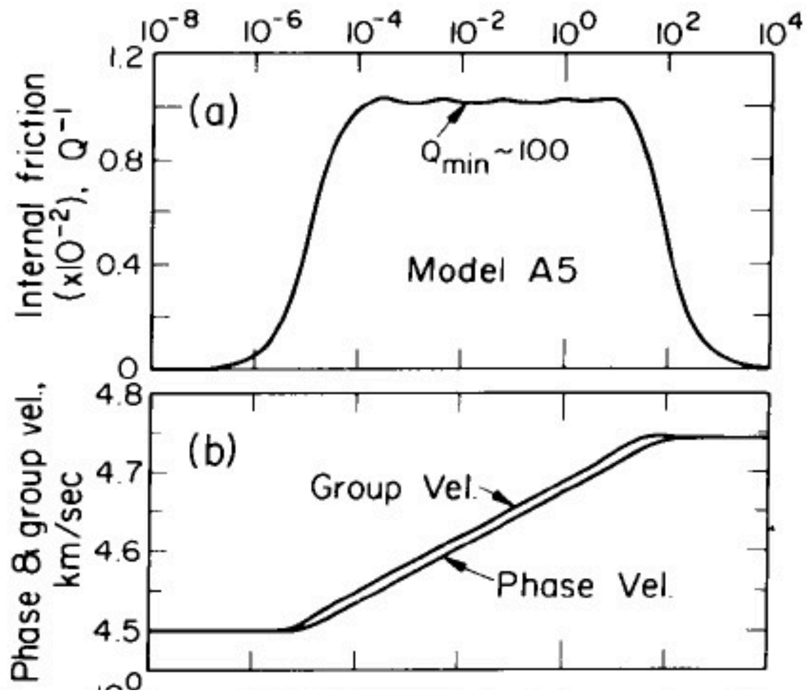


Fig.9. (a) Internal friction coefficient as a function of frequency,
(b) phase and group velocity dispersion. (Liu et al., 1976, figure 3)

So, for numerical modeling purposes, we could pick the lower and upper limits of this frequency range and use the ln formula for v in this range only. So v would not go to –infinity.

About the frequency range, for example, if we let the range from 10^{-2} to 10^{5}, then Q is constant only in this range and v is constant outside this range. Since I need to use the frequency from –Nyquist frequency to Nyquist frequency in my code, according to symmetry, I will have another range from -10^{-2} to -10^{5}. So, inside the ranges, we could let the velocity v obeys the ln formula. While outside the ranges, v is constant, and we could use the equation without dispersion which we have derived before.

And, for the same medium, we should use the same velocity vs. frequency function, but $v(\omega_{0r})$ would be different if ω_{0r} is different.

Then, I used the Matlab code to do IFFT to the solution of the anelastic EOM in frequency domain (55a) to compute the exact results with velocity dispersion in time domain numerically and compare the approximation and exact numerical result for different distances in Fig.10a-e. The red lines indicate exact numerical results, and the black dash lines indicate the approximation (61). We can see they coincide well.

Therefore, we could say that our approximation (61) can be a good expression for the solution of the EOM for a directed point force including absorption and dispersion.

However, we should notice that even though we involved dispersion effect to ensure causality, the waveforms are still non-causal (source pulse appears before zero second, e.g., Fig. 10a), because the function of source we used is a non-causal source. As mentioned previously, this is however acceptable because for our purposes, we can think of the source pulse we used as being causal if we merely shift the starting point of the pulse to a point on the negative side where the pulse is effectively zero, e.g., to the point t = -50 s in Figure 3. The effects of dispersion can still be seen with this pulse.

The parameters I used in the below figures are $a = 0.02s$, $\omega_{0r} = 20Hz$, $\rho = 1kg/m^3$, $A = 1kg * km$, $r = 0.3, 1, 2, 5, 10\ km$, $Q_\alpha = 40, Q_\beta = 20, \alpha_0 = 5\ km/s$ and $\beta_0 = 3\ km/s$.

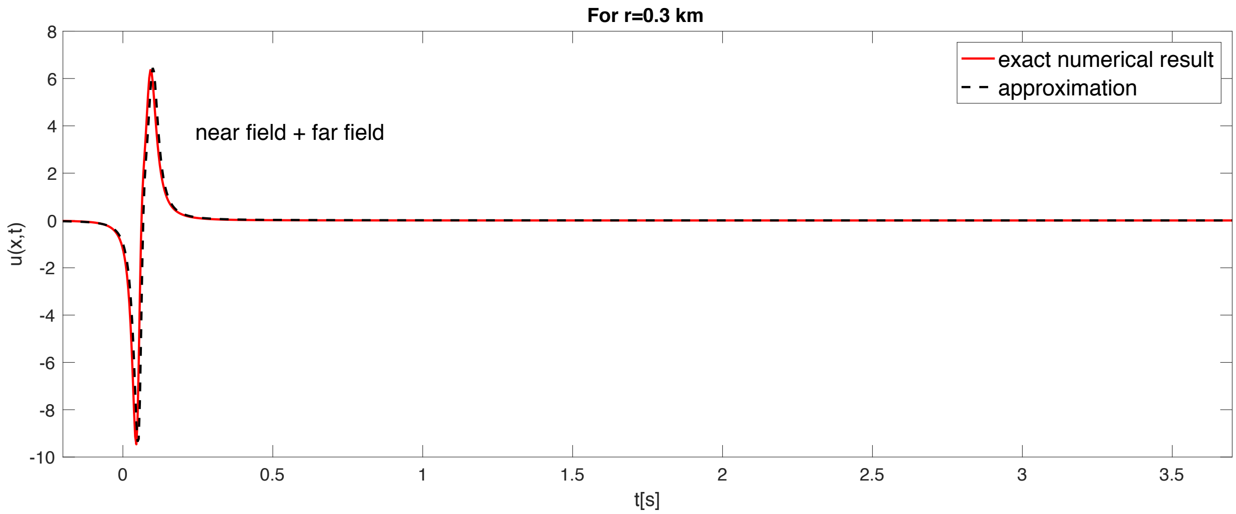


Fig.10a. Comparison of the approximation (61) and exact numerical result for the solution of the EOM for a directed point force with dispersion (r=0.3km)

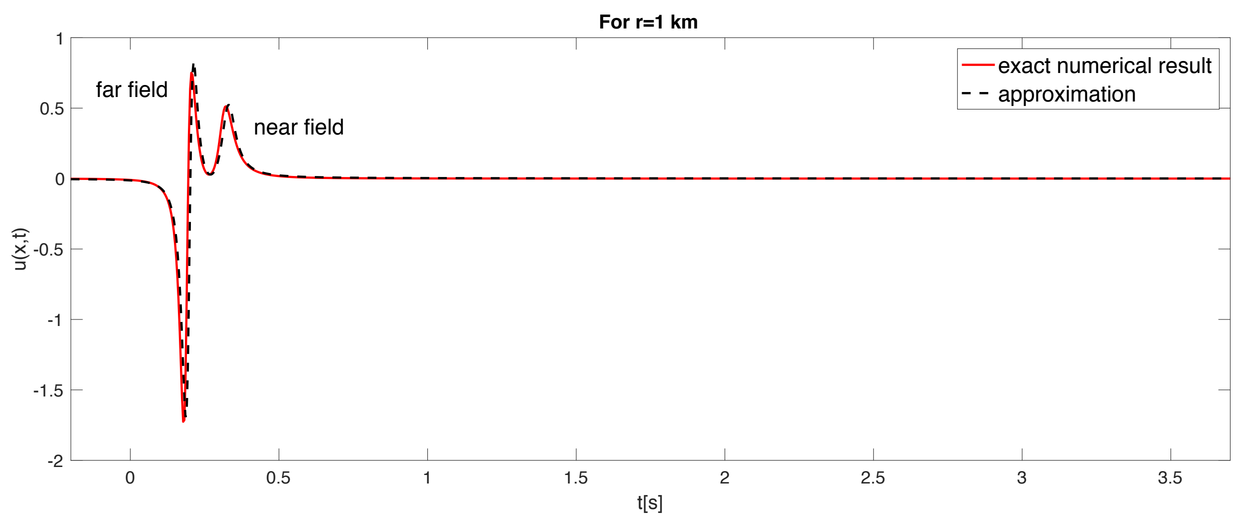


Fig.10b. Comparison of the approximation (61) and exact numerical result for the solution of the EOM for a directed point force with dispersion (r=1km)

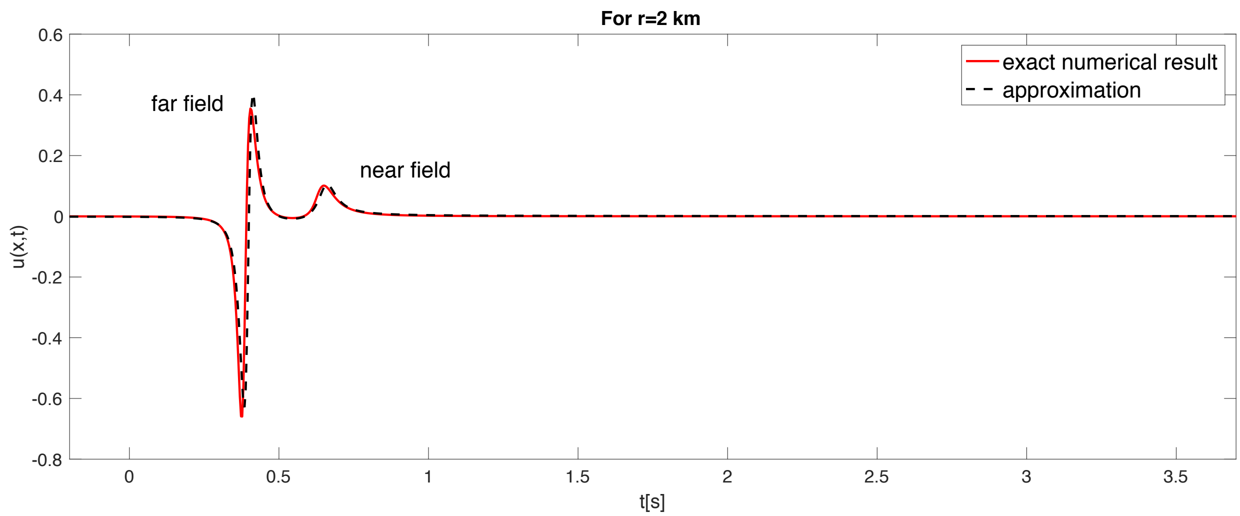


Fig.10c. Comparison of the approximation (61) and exact numerical result for the solution of the EOM for a directed point force with dispersion (r=2km)

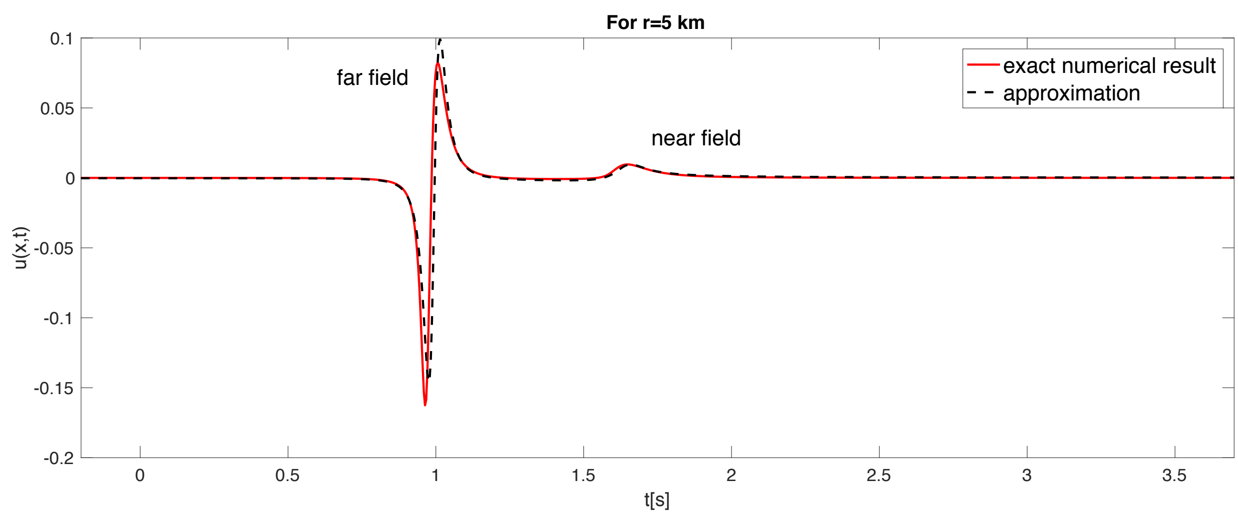


Fig.10d. Comparison of the approximation (61) and exact numerical result for the solution of the EOM for a directed point force with dispersion (r=5km)

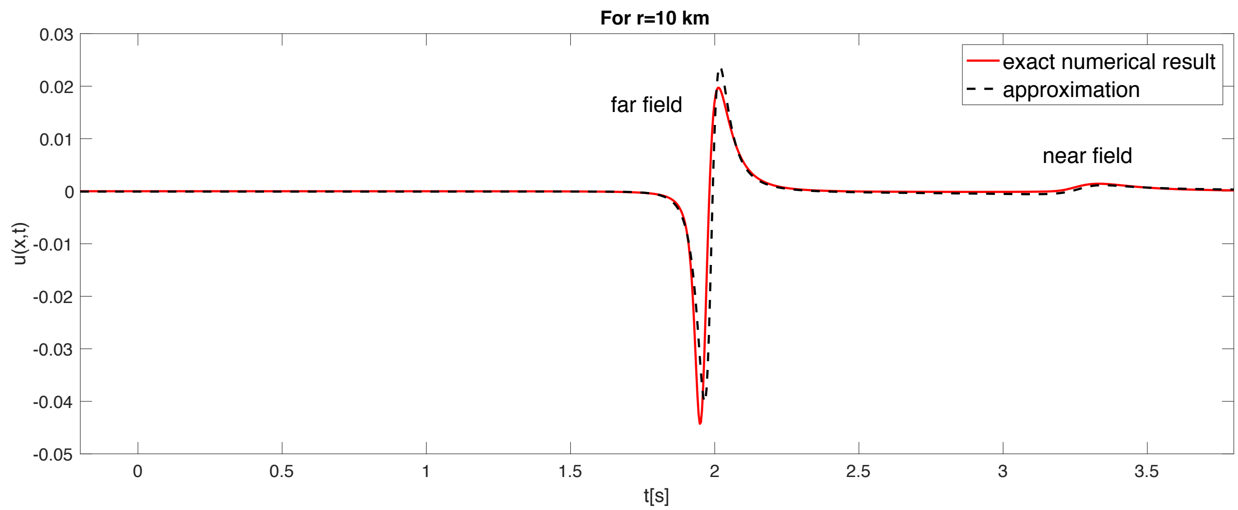


Fig.10e. Comparison of the approximation (61) and exact numerical result for the solution of the EOM for a directed point force with dispersion (r=10km)

Then, let us use the same parameters with above Fig.10a-e and see the differences between seismic waves in elastic medium (48) and in anelastic medium with absorption and dispersion effect (61).

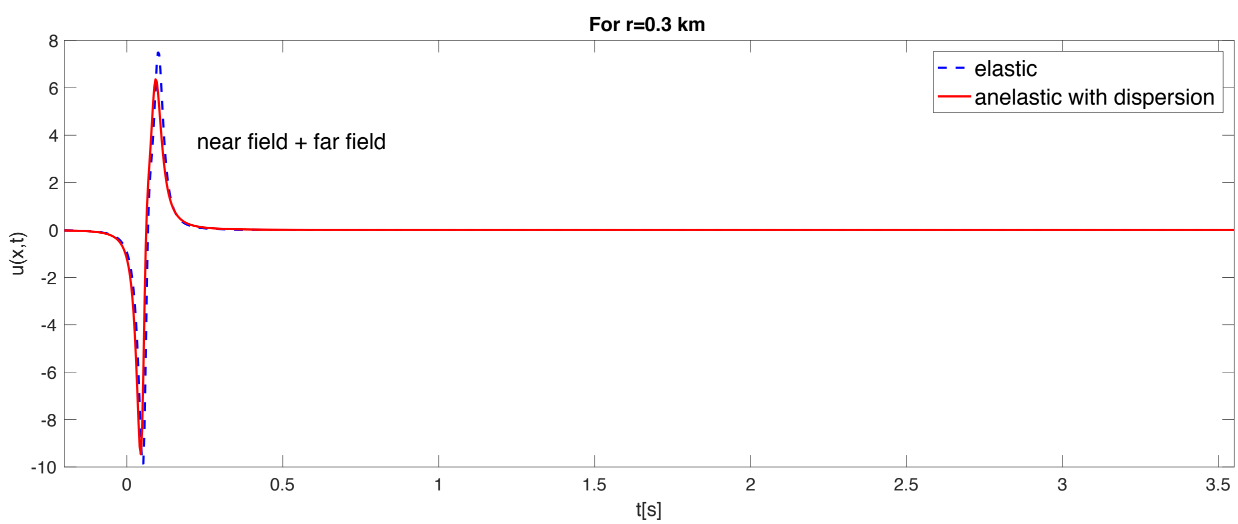


Fig.11a. Comparison of the approximate anelastic solution of the EOM for a directed point force with dispersion (61) and the elastic one (48) (r=0.3km)

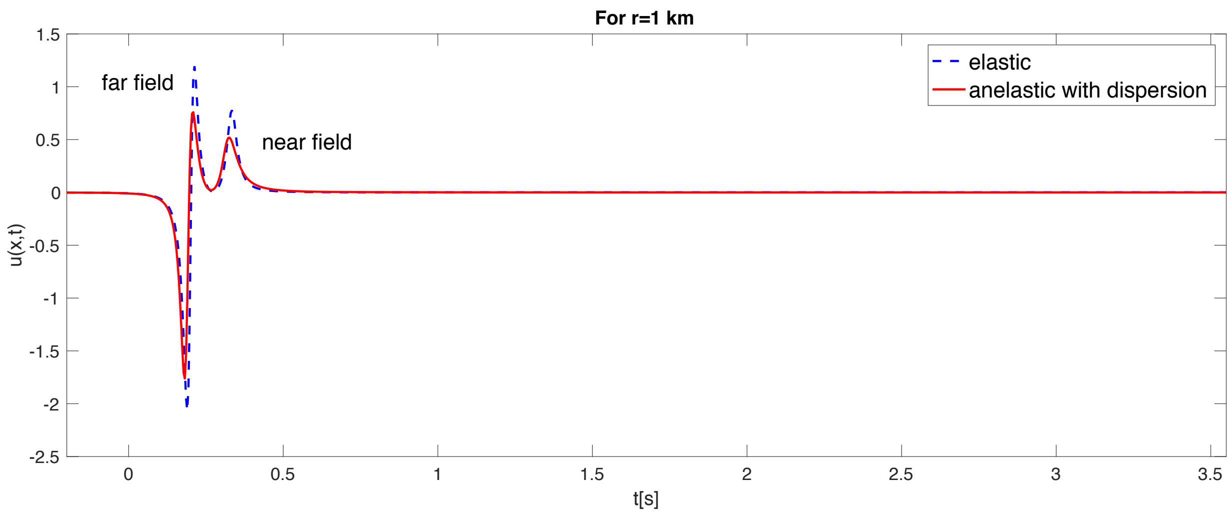


Fig.11b. Comparison of the approximate anelastic solution of the EOM for a directed point force with dispersion (61) and the elastic one (48) (r=1km)

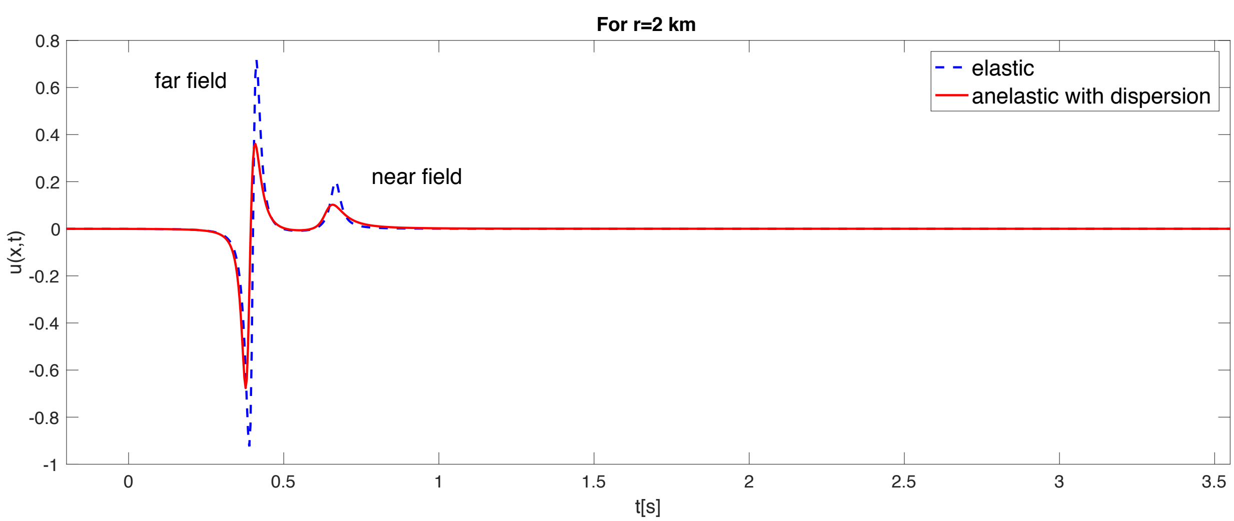


Fig.11c. Comparison of the approximate anelastic solution of the EOM for a directed point force with dispersion (61) and the elastic one (48) (r=2km)

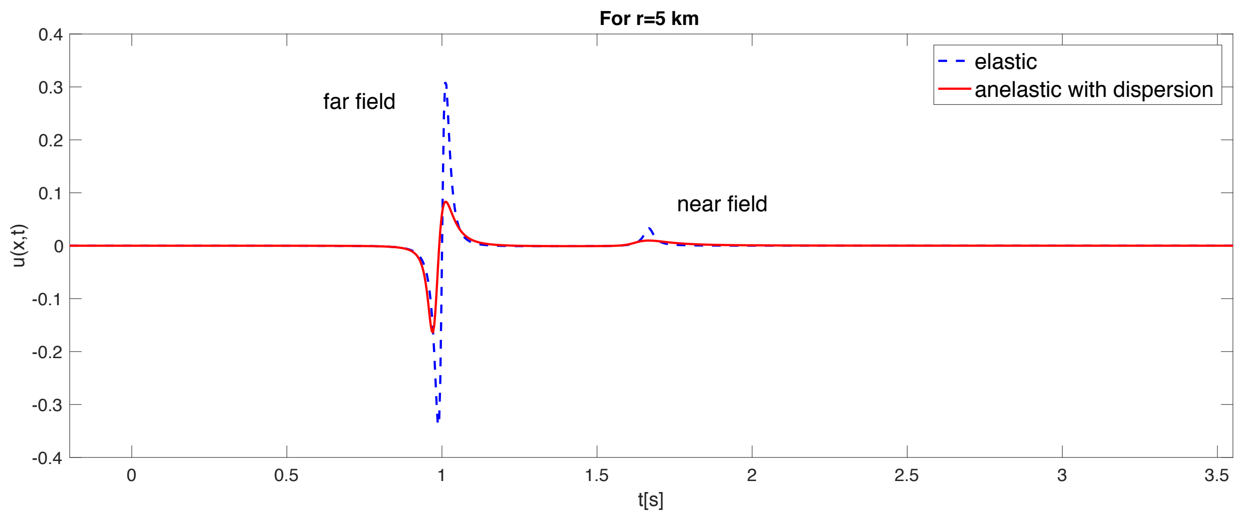


Fig.11d. Comparison of the approximate anelastic solution of the EOM for a directed point force with dispersion (61) and the elastic one (48) (r=5km)

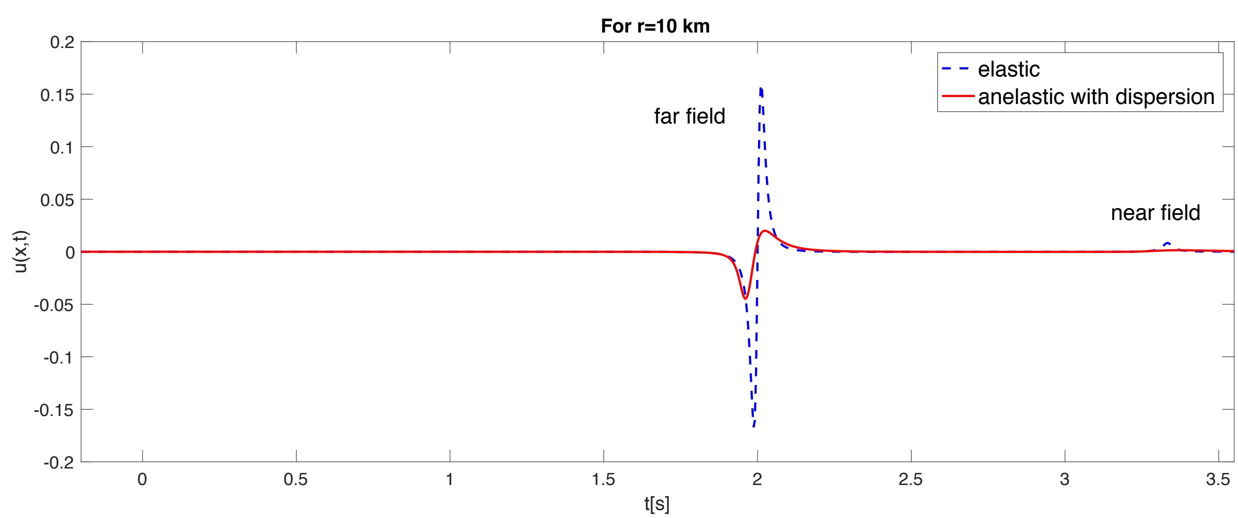


Fig.11e. Comparison of the approximate anelastic solution of the EOM for a directed point force with dispersion (61) and the elastic one (48) (r=10km)

In above pictures, Fig.11a-e, the dash blue curves are the results of elastic waveforms, and the red curves are anelastic waveforms including dispersion influence. The effect of absorption and dispersion becomes more and more obvious as the propagation distance increases. This trend is very similar to the solutions with only absorption effect but no dispersion.

However, for no dispersion, i.e., Fig.5a-e, the anelastic waveforms have the similar form and shape as the elastic ones, because the pulse does not oscillate much. While, when we include dispersion, the shape of anelastic waveforms is a little bit different from the elastic ones. It looks like the waveforms are wider in anelastic medium.

3.4 Effect of dispersion

To see the effect of dispersion, let us plot the exact solution only having absorption and no dispersion (52b) and the approximate solution have both absorption and dispersion (61) in same pictures.

For the parameters, we are still using $a = 0.02s, \omega_{0r} = 20Hz, \rho = 1kg/m^3, A = 1kg * km, r = 0.3, 1, 2, 5, 10\ \ km, Q_\alpha = 40, Q_\beta = 20, \alpha_0 = 5\ km/s$ and $\beta_0 = 3\ km/s$.

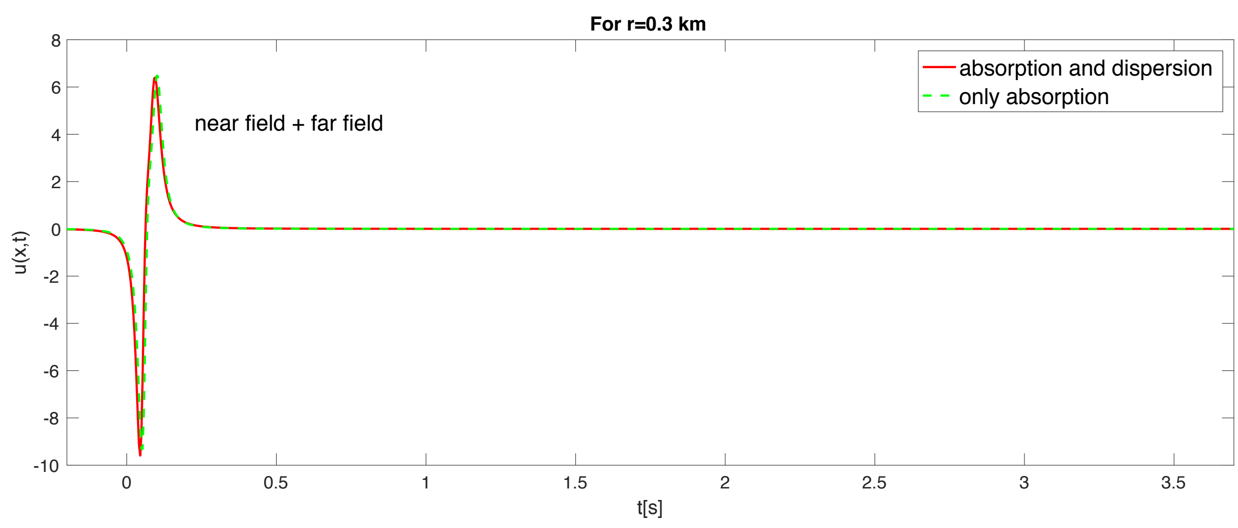


Fig.12a. Comparison of the approximate anelastic solution of the EOM for a directed point force with dispersion (61) and the solution only having absorption and no dispersion (52b) (r=0.3km)

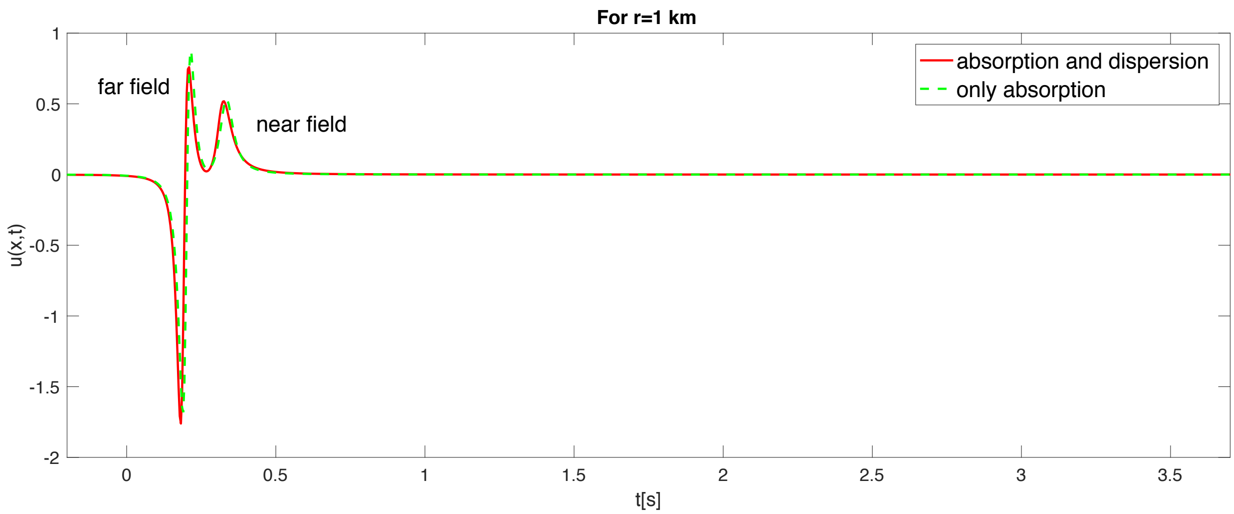


Fig.12b. Comparison of the approximate anelastic solution of the EOM for a directed point force with dispersion (61) and the solution only having absorption and no dispersion (52b) (r=1km)

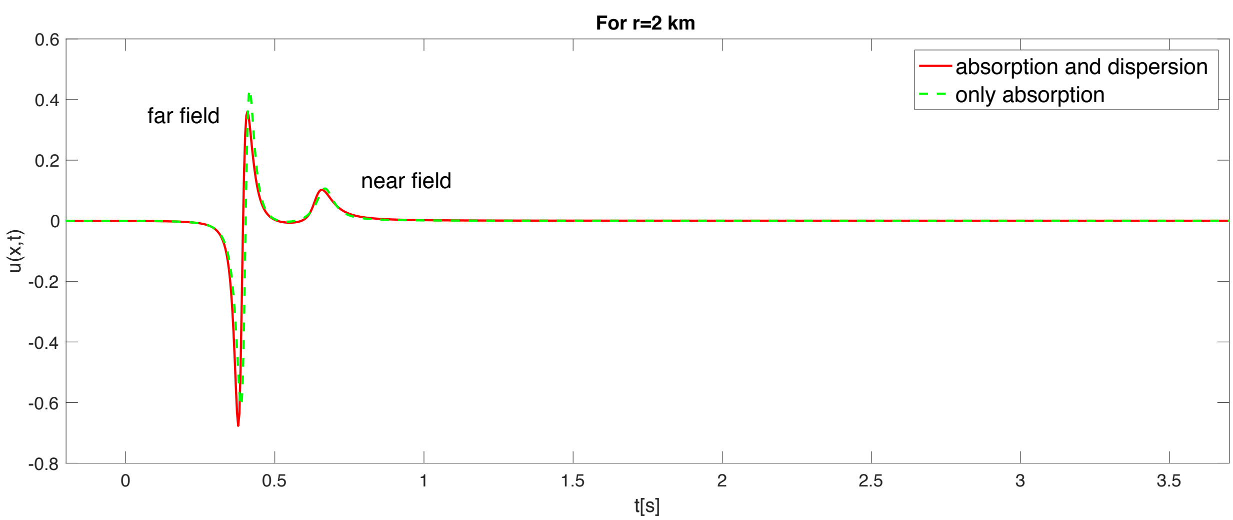


Fig.12c. Comparison of the approximate anelastic solution of the EOM for a directed point force with dispersion (61) and the solution only having absorption and no dispersion (52b) (r=2km)

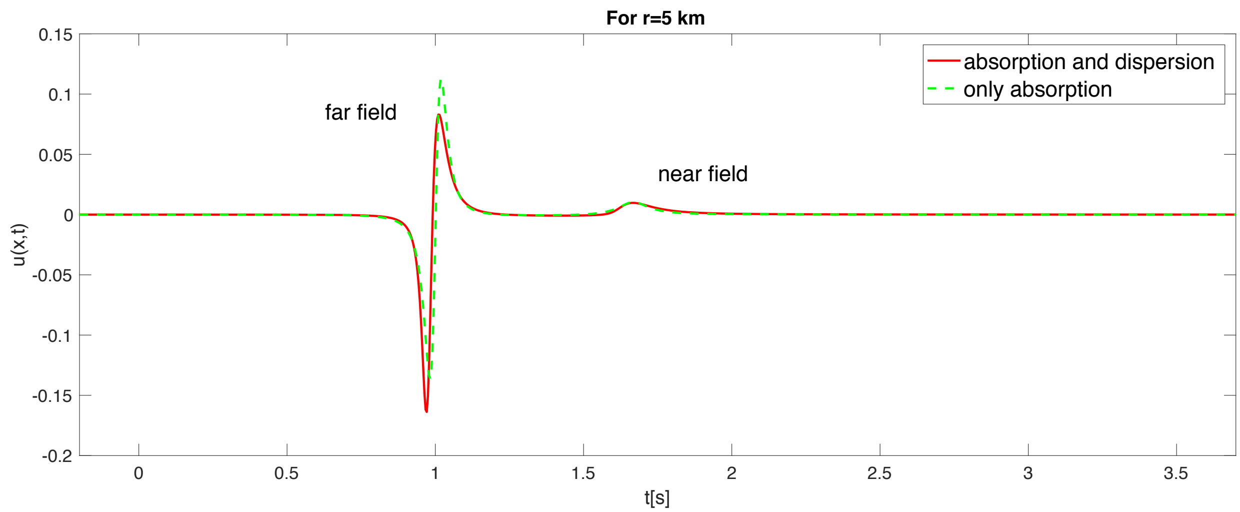


Fig.12d. Comparison of the approximate anelastic solution of the EOM for a directed point force with dispersion (61) and the solution only having absorption and no dispersion (52b) (r=5km)

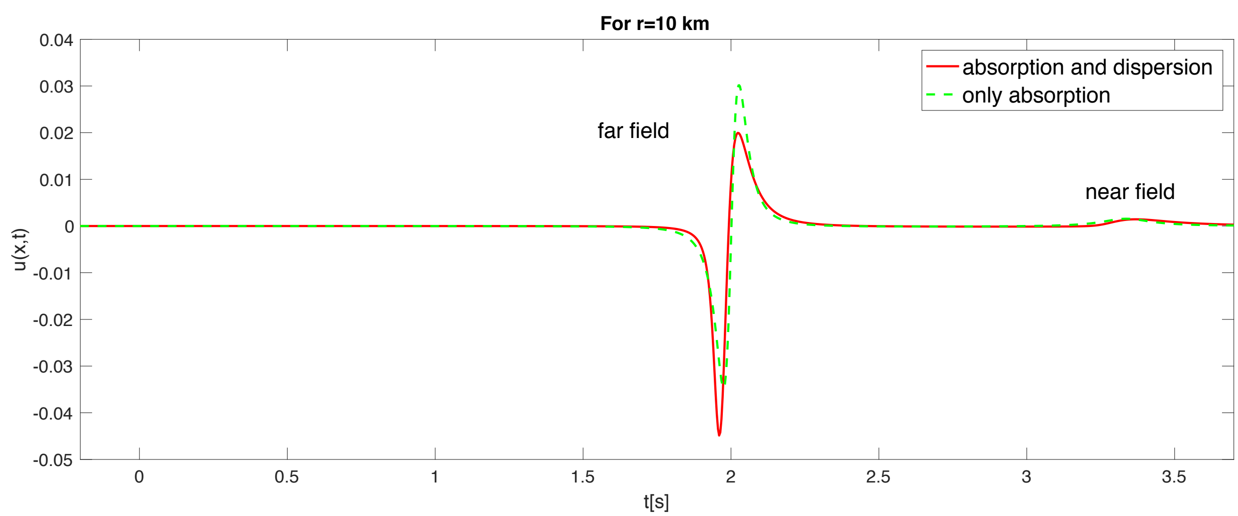


Fig.12e. Comparison of the approximate anelastic solution of the EOM for a directed point force with dispersion (61) and the solution only having absorption and no dispersion (52b) (r=10km)

In Fig.12a-e, green dash lines indicate the solution of the EOM for a directed point force including only absorption no dispersion (52b), and red lines show the solution which has both influence of absorption and dispersion. We can easily tell the differences between them, and it is obvious that dispersion has a greater impact on far-field term.

To see more specific, let us amplify the far-field term part in Fig.12e:

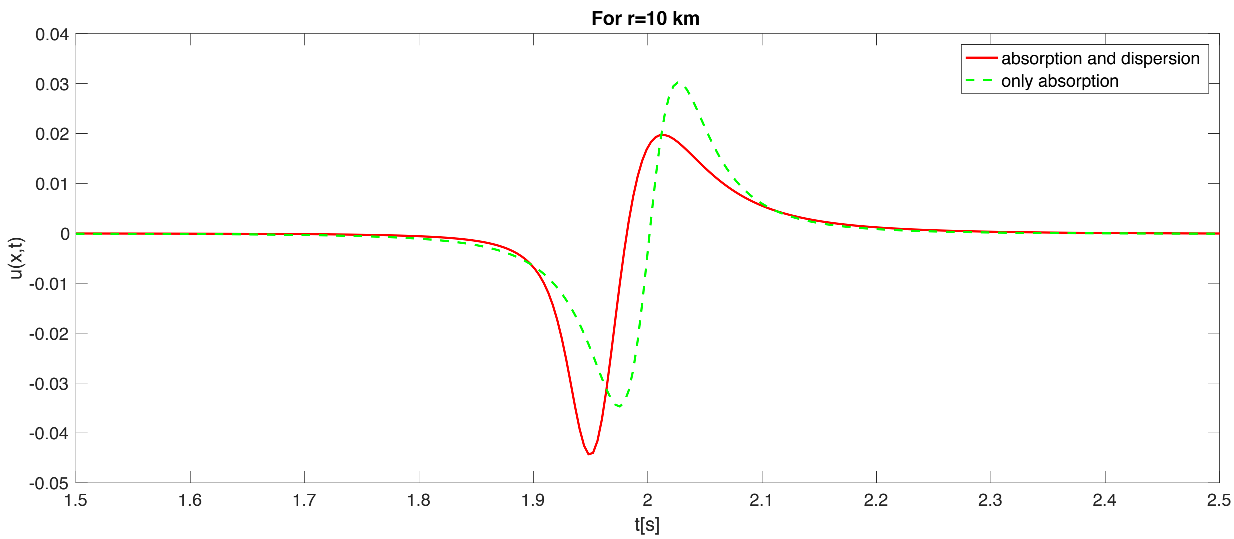


Fig.12f. Amplification of the far-field term in Fig.12e

We can see from Fig.12f that the red wave (with dispersion) arrived before the green one (without dispersion). And the shape of them are also different.

In general, compared to the elastic wave under dispersion effect, both the arrive time and the shape of the no dispersion one is more like the waveform in elastic medium.

CHAPTER 4: A DOUBLE-COUPLE-WITHOUT-MOMENT SOURCE

In chapter 4 and chapter 5, we will follow the same steps in the chapter 3 to derive new solutions of the EOM for a double-couple-without-moment source and a shear-dislocation source, respectively.

But in these two chapters, we will only talk about the dispersion situation, because to be physically realistic, the new solutions must include velocity dispersion.

4.1 Converting the solution to frequency domain

This time, we are going to do Fourier transform of the solution of the equation of motion for a double-couple-without-moment source in perfect elastic medium to convert it into frequency domain.

As we said in chapter 2, we are going to consider a vertical fault coinciding with the *xz* plane. The only non-zero components of the moment tensor are then $M_{12} = M_{21} = M_0$. And To use a more general formula, for a receiver lying on a circle surrounding the origin, then we have ${\gamma_1}^2 + {\gamma_2}^2 = 1$ and $\gamma_3 = 0$.

We already have three components of the displacement for the solution of the EOM for a double-couple-without-moment source in an elastic medium, i.e., (38a, b, and c). Here, we will add dispersion effect on the x component of the displacement, which is (38a):

$$u_1(\mathbf{x}, t) = \frac{(9 + 15\cos 2\theta)\sin\theta}{4\pi\rho}\frac{1}{r^4}\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau M_0(t-\tau)\, d\tau$$

$$+\frac{(2 + 3\cos 2\theta)\sin\theta}{2\pi\rho\alpha^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\alpha}\right) - \frac{(3 + 6\cos 2\theta)\sin\theta}{4\pi\rho\beta^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\beta}\right)$$

$$+\frac{(1+\cos 2\theta)\sin\theta}{4\pi\rho\alpha^3}\frac{1}{r}\dot{M}_0\left(t-\frac{r}{\alpha}\right)-\frac{\cos 2\theta\,\sin\theta}{4\pi\rho\beta^3}\frac{1}{r}\dot{M}_0\left(t-\frac{r}{\beta}\right) \tag{38a}$$

And, to make the calculations easier, we will assume source $M_0(t) = s(t)$, and $\overline{M_0}(\omega) = \bar{s}(\omega)$, which means

$$M_0(t) = \frac{2atA}{\pi(a^2+t^2)^2} \tag{62a}$$

$$\overline{M_0}(\omega) = Ai\omega e^{-a|\omega|} \tag{62b}$$

Use Fourier transform to convert $u_1(\mathbf{x}, t)$ into frequency domain,

$$\bar{u}_1(\mathbf{x},\omega) = \int_{-\infty}^{\infty}\left[\frac{(9+15\cos 2\theta)\sin\theta}{4\pi\rho}\frac{1}{r^4}\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}}\tau M_0(t-\tau)\,d\tau\right.$$

$$+\frac{(2+3\cos 2\theta)\sin\theta}{2\pi\rho\alpha^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\alpha}\right)-\frac{(3+6\cos 2\theta)\sin\theta}{4\pi\rho\beta^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\beta}\right)$$

$$\left.+\frac{(1+\cos 2\theta)\sin\theta}{4\pi\rho\alpha^3}\frac{1}{r}\dot{M}_0\left(t-\frac{r}{\alpha}\right)-\frac{\cos 2\theta\,\sin\theta}{4\pi\rho\beta^3}\frac{1}{r}\dot{M}_0\left(t-\frac{r}{\beta}\right)\right]e^{i\omega t}\mathrm{dt}$$

$$=\frac{(9+15\cos 2\theta)\sin\theta}{4\pi\rho}\frac{1}{r^4}\int_{-\infty}^{\infty}\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}}\tau M_0(t-\tau)\,d\tau\, e^{i\omega t}\mathrm{dt}$$

$$+\frac{(2+3\cos 2\theta)\sin\theta}{2\pi\rho\alpha^2}\frac{1}{r^2}\int_{-\infty}^{\infty}M_0\left(t-\frac{r}{\alpha}\right)e^{i\omega t}\mathrm{dt}$$

$$-\frac{(3+6\cos 2\theta)\sin\theta}{4\pi\rho\beta^2}\frac{1}{r^2}\int_{-\infty}^{\infty}M_0\left(t-\frac{r}{\beta}\right)e^{i\omega t}\mathrm{dt}$$

$$+\frac{(1+\cos 2\theta)\sin\theta}{4\pi\rho\alpha^3}\frac{1}{r}\int_{-\infty}^{\infty}\dot{M}_0\left(t-\frac{r}{\alpha}\right)e^{i\omega t}\mathrm{dt}$$

$$-\frac{\cos 2\theta\,\sin\theta}{4\pi\rho\beta^3}\frac{1}{r}\int_{-\infty}^{\infty}\dot{M}_0\left(t-\frac{r}{\beta}\right)e^{i\omega t}\mathrm{dt} \tag{63a}$$

Like what we have done in chapter 3, (43b) and (44), we can assume that

$$f(\tau) = \begin{cases} \tau\,, & \frac{r}{\alpha} \le \tau \le \frac{r}{\beta} \\ 0\,, & others \end{cases} \tag{43b}$$

and,

$$\bar{f}(\omega) = \int_{-\infty}^{\infty} f(\tau)\, e^{i\omega\tau}\,\mathrm{d}\tau = \frac{1}{i\omega}\int_{r/\alpha}^{r/\beta} \tau\ \mathrm{d}e^{i\omega\tau} = \left(\frac{1}{\omega} - \frac{r}{\beta} i\right)\frac{e^{i\omega\frac{r}{\beta}}}{\omega} - \left(\frac{1}{\omega} - \frac{r}{\alpha} i\right)\frac{e^{i\omega\frac{r}{\alpha}}}{\omega} \qquad (44)$$

Therefore,

$$\int_{-\infty}^{\infty}\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau M_0(t-\tau)\, d\tau\, e^{i\omega t}\mathrm{dt} = \int_{r/\alpha}^{r/\beta}\left[\int_{-\infty}^{\infty} M_0(t-\tau) e^{i\omega t} dt\right]\tau \mathrm{d}\tau$$

Assume $v = t - \tau \quad \Rightarrow$

$$= \int_{r/\alpha}^{r/\beta}\left[\int_{-\infty}^{\infty} M_0(v) e^{i\omega(v+\tau)} d(v+\tau)\right]\tau\mathrm{d}\tau = \int_{r/\alpha}^{r/\beta}\left[\int_{-\infty}^{\infty} M_0(v) e^{i\omega v} dv\right]\tau \mathrm{e}^{\mathrm{i}\omega\tau}\mathrm{d}\tau$$

$$= \bar{M}_0(\omega)\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau \mathrm{e}^{\mathrm{i}\omega\tau}\mathrm{d}\tau = \bar{M}_0(\omega)\int_{-\infty}^{\infty} f(\tau)\mathrm{e}^{\mathrm{i}\omega\tau}\mathrm{d}\tau = \bar{M}_0(\omega)\bar{f}(\omega)$$

$$\Rightarrow \quad \int_{-\infty}^{\infty}\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau M_0(t-\tau)\, d\tau\, e^{i\omega t}\mathrm{dt} = \bar{M}_0(\omega)\bar{f}(\omega) \qquad (64)$$

To do the integral $\int_{-\infty}^{\infty} \dot{M}_0\left(t - \frac{r}{v}\right) e^{i\omega t}\mathrm{dt}$, we can use the rule $\int uv'\,dx = uv - \int u'v\,dx$,

$$\int_{-\infty}^{\infty} \dot{M}_0\left(t - \frac{r}{v}\right) e^{i\omega t}\mathrm{dt} = M_0\left(t - \frac{r}{v}\right) e^{i\omega t}\Big|_{-\infty}^{\infty} - i\omega\int_{-\infty}^{\infty} M_0\left(t - \frac{r}{v}\right) e^{i\omega t}\mathrm{dt}$$

$$= M_0\left(t - \frac{r}{v}\right) e^{i\omega t}\Big|_{-\infty}^{\infty} - i\omega\bar{M}_0(\omega) e^{i\omega\frac{r}{v}}$$

Since we still assume $M_0(t) = \frac{2atA}{\pi(a^2+t^2)^2}$, and $\overline{M_0}(\omega) = Ai\omega e^{-a|\omega|}$, then we can get $\lim_{t\to\pm\infty} M_0(t) = 0$. So,

$$\int_{-\infty}^{\infty} \dot{M}_0\left(t - \frac{r}{v}\right) e^{i\omega t}\mathrm{dt} = -i\omega\bar{M}_0(\omega) e^{i\omega\frac{r}{v}} \qquad (65)$$

Substitute (64) and (65) into (63a),

$$\bar{u}_1(\mathbf{x},\omega) = \frac{(9+15\cos 2\theta)\sin\theta}{4\pi\rho}\frac{1}{r^4}\bar{M}_0(\omega)\bar{f}(\omega)$$

$$+\frac{(2+3\cos 2\theta)\sin\theta}{2\pi\rho\alpha^2}\frac{1}{r^2}\bar{M}_0(\omega)e^{i\omega\frac{r}{\alpha}} - \frac{(3+6\cos 2\theta)\sin\theta}{4\pi\rho\beta^2}\frac{1}{r^2}\bar{M}_0(\omega)e^{i\omega\frac{r}{\beta}}$$

$$-\frac{(1+\cos 2\theta)\sin\theta}{4\pi\rho\alpha^3}\frac{1}{r}i\omega\bar{M}_0(\omega)e^{i\omega\frac{r}{\alpha}} + \frac{cos\,2\theta\,sin\,\theta}{4\pi\rho\beta^3}\frac{1}{r}i\omega\bar{M}_0(\omega)e^{i\omega\frac{r}{\beta}}$$

Then, substitute (62) and (44) into it,

$$\bar{u}_1(\mathbf{x},\omega) = \frac{(9+15\cos 2\theta)\sin\theta}{4\pi\rho}\frac{A}{r^4}i\omega e^{-a|\omega|}\left[\left(\frac{1}{\omega}-\frac{r}{\beta}i\right)\frac{e^{i\omega\frac{r}{\beta}}}{\omega} - \left(\frac{1}{\omega}-\frac{r}{\alpha}i\right)\frac{e^{i\omega\frac{r}{\alpha}}}{\omega}\right]$$

$$+\frac{(2+3\cos 2\theta)\sin\theta}{2\pi\rho\alpha^2}\frac{A}{r^2}i\omega e^{-a|\omega|}e^{i\omega\frac{r}{\alpha}} - \frac{(3+6\cos 2\theta)\sin\theta}{4\pi\rho\beta^2}\frac{\mathrm{A}}{r^2}i\omega e^{-a|\omega|}e^{i\omega\frac{r}{\beta}}$$

$$-\frac{(1+\cos 2\theta)\sin\theta}{4\pi\rho\alpha^3}\frac{i\omega}{r}Ai\omega e^{-a|\omega|}e^{i\omega\frac{r}{\alpha}} + \frac{cos\,2\theta\,sin\,\theta}{4\pi\rho\beta^3}\frac{i\omega}{r}Ai\omega e^{-a|\omega|}e^{i\omega\frac{r}{\beta}}$$

$$= \frac{(9+15\cos 2\theta)\sin\theta}{4\pi\rho}\frac{A}{r^4}e^{-a|\omega|}\left[\left(\frac{i}{\omega}+\frac{r}{\beta}\right)e^{i\omega\frac{r}{\beta}} - \left(\frac{i}{\omega}+\frac{r}{\alpha}\right)e^{i\omega\frac{r}{\alpha}}\right]$$

$$+\frac{(2+3\cos 2\theta)\sin\theta}{2\pi\rho\alpha^2}\frac{A}{r^2}i\omega e^{-a|\omega|}e^{i\omega\frac{r}{\alpha}} - \frac{(3+6\cos 2\theta)\sin\theta}{4\pi\rho\beta^2}\frac{\mathrm{A}}{r^2}i\omega e^{-a|\omega|}e^{i\omega\frac{r}{\beta}}$$

$$+\frac{(1+\cos 2\theta)\sin\theta}{4\pi\rho\alpha^3}\frac{A}{r}\omega^2 e^{-a|\omega|}e^{i\omega\frac{r}{\alpha}} - \frac{cos\,2\theta\,sin\,\theta}{4\pi\rho\beta^3}\frac{A}{r}\omega^2 e^{-a|\omega|}e^{i\omega\frac{r}{\beta}} \tag{63b}$$

Also, substitute $M_0(t)$ into equation (38a) to get the solution in an elastic medium in time domain to compare with results including dispersion later:

$$u_1(\mathbf{x},t) = \frac{(9+15\cos 2\theta)\sin\theta}{4\pi\rho}\frac{1}{r^4}\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}}\tau\frac{2a(t-\tau)A}{\pi(a^2+(t-\tau)^2)^2}\,d\tau$$

$$+\frac{(2+3\cos 2\theta)\sin\theta}{2\pi\rho\alpha^2}\frac{1}{r^2}\frac{2a\left(t-\frac{r}{\alpha}\right)A}{\pi\left(a^2+\left(t-\frac{r}{\alpha}\right)^2\right)^2}$$

$$-\frac{(3+6\cos 2\theta)\sin\theta}{4\pi\rho\beta^2}\frac{1}{r^2}\frac{2a\left(t-\frac{r}{\beta}\right)A}{\pi\left(a^2+\left(t-\frac{r}{\beta}\right)^2\right)^2}$$

$$+\frac{(1+\cos 2\theta)\sin\theta}{4\pi\rho\alpha^3}\frac{1}{r}\left[\frac{2a\left(t-\frac{r}{\alpha}\right)A}{\pi\left(a^2+\left(t-\frac{r}{\alpha}\right)^2\right)^2}\right]'$$

$$-\frac{cos\,2\theta\;sin\,\theta}{4\pi\rho\beta^3}\frac{1}{r}\left[\frac{2a\left(t-\frac{r}{\beta}\right)A}{\pi\left(a^2+\left(t-\frac{r}{\beta}\right)^2\right)^2}\right]'$$

$$=\frac{(9+15\cos 2\theta)\sin\theta}{4\pi^2\rho}\frac{A}{r^4}\left[\frac{ar}{\beta\left(a^2+\left(t-\frac{r}{\beta}\right)^2\right)}-\tan^{-1}\left(\frac{\frac{r}{\beta}-t}{a}\right)-\frac{ar}{\alpha\left(a^2+\left(t-\frac{r}{\alpha}\right)^2\right)}\right.$$

$$\left.+\tan^{-1}\left(\frac{\frac{r}{\alpha}-t}{a}\right)\right]$$

$$+\frac{(2+3\cos 2\theta)\sin\theta}{\pi^2\rho\alpha^2}\frac{A}{r^2}\frac{a\left(t-\frac{r}{\alpha}\right)}{\left(a^2+\left(t-\frac{r}{\alpha}\right)^2\right)^2}$$

$$-\frac{(3+6\cos 2\theta)\sin\theta}{2\pi^2\rho\beta^2}\frac{\mathrm{A}}{r^2}\frac{a\left(t-\frac{r}{\beta}\right)}{\left(a^2+\left(t-\frac{r}{\beta}\right)^2\right)^2}$$

$$+\frac{(1+\cos 2\theta)\sin\theta}{2\pi^2\rho\alpha^3}\frac{aA}{r}\frac{\left(a^2+\left(t-\frac{r}{\alpha}\right)^2\right)-4\left(t-\frac{r}{\alpha}\right)^2}{\left(a^2+\left(t-\frac{r}{\alpha}\right)^2\right)^3}$$

$$-\frac{\cos 2\theta \sin\theta}{2\pi^2\rho\beta^3}\frac{aA}{r}\frac{\left(a^2+\left(t-\frac{r}{\beta}\right)^2\right)-4\left(t-\frac{r}{\beta}\right)^2}{\left(a^2+\left(t-\frac{r}{\beta}\right)^2\right)^3} \tag{66}$$

4.2 The solutions of the EOM including absorption and dispersion

A velocity-frequency relation in the form of $\frac{1}{v(\omega)}=\frac{1}{v(\omega_{0r})}\left[1-\frac{1}{\pi Q}ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i}{2Q}\right]$, equation (14), will be used again. We assume Q is nearly constant (independent of frequency) and keep only first-order terms.

We have derived velocity-frequency relations of $\frac{1}{v(\omega)}$ and $\frac{1}{v^2(\omega)}$, but in this chapter, we will also need a velocity-frequency relation of $\frac{1}{v^3(\omega)}$,

$$\begin{aligned}
\frac{1}{v(\omega)}&=\frac{1}{v(\omega_{0r})}\left[1-\frac{1}{\pi Q}ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i}{2Q}\right]\\
\frac{1}{v^2(\omega)}&=\frac{1}{v^2(\omega_{0r})}\left[1-\frac{1}{\pi Q}ln\left(\frac{\omega}{\omega_{0r}}\right)\right]^2\left(1+\frac{i}{2Q}\right)^2\\
&=\frac{1}{v^2(\omega_{0r})}\left[1-\frac{2}{\pi Q}ln\left(\frac{\omega}{\omega_{0r}}\right)\right]\left(1+\frac{i}{Q}\right)\approx\frac{1}{v^2(\omega_{0r})}\left[1-\frac{2}{\pi Q}ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i}{Q}\right]\\
\frac{1}{v^3(\omega)}&=\frac{1}{v^3(\omega_{0r})}\left[1-\frac{1}{\pi Q}ln\left(\frac{\omega}{\omega_{0r}}\right)\right]^3\left(1+\frac{i}{2Q}\right)^3\\
&=\frac{1}{v^3(\omega_{0r})}\left[1-\frac{3}{\pi Q}ln\left(\frac{\omega}{\omega_{0r}}\right)\right]\left(1+\frac{3i}{2Q}\right)\approx\frac{1}{v^3(\omega_{0r})}\left[1-\frac{3}{\pi Q}ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{3i}{2Q}\right]
\end{aligned} \tag{67}$$

So, let us replace $\frac{1}{\alpha}\to\frac{1}{\alpha(\omega)}$ and $\frac{1}{\beta}\to\frac{1}{\beta(\omega)}$ as:

$$\begin{aligned}
\frac{1}{\alpha(\omega)}&=\frac{1}{\alpha(\omega_{0r})}\left[1-\frac{1}{\pi Q_\alpha}ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i}{2Q_\alpha}\right],\\
\frac{1}{\alpha^2(\omega)}&=\frac{1}{\alpha^2(\omega_{0r})}\left[1-\frac{2}{\pi Q_\alpha}ln\left(\frac{\omega}{\omega_{0r}}\right)+\frac{i}{Q_\alpha}\right],
\end{aligned} \tag{68a}$$

$$\frac{1}{\alpha^3(\omega)} = \frac{1}{\alpha^3(\omega_{0r})}\left[1 - \frac{3}{\pi Q} ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{3i}{2Q}\right].$$

$$\frac{1}{\beta(\omega)} = \frac{1}{\beta(\omega_{0r})}\left[1 - \frac{1}{\pi Q_\beta} ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{i}{2Q_\beta}\right],$$

$$\frac{1}{\beta^2(\omega)} = \frac{1}{\beta^2(\omega_{0r})}\left[1 - \frac{2}{\pi Q_\beta} ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{i}{Q_\beta}\right], \tag{68b}$$

$$\frac{1}{\beta^2(\omega)} = \frac{1}{\beta^2(\omega_{0r})}\left[1 - \frac{3}{\pi Q} ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{3i}{2Q}\right].$$

Substitute the above equations into $\bar{u}_1(\mathbf{x}, \omega)$, i.e., (63b), then we will have:

$$\begin{aligned}
\bar{u}_1(\mathbf{x}, \omega) = {} & \frac{(9 + 15\cos 2\theta)\sin\theta}{4\pi\rho}\frac{A}{r^4}e^{-a|\omega|} \\
& \times \left[\left(\frac{i}{\omega} + \frac{r}{\beta(\omega_{0r})}\left[1 - \frac{1}{\pi Q_\beta} ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{i}{2Q_\beta}\right]\right) e^{\frac{i\omega r}{\beta(\omega_{0r})}\left[1 - \frac{1}{\pi Q_\beta} ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{i}{2Q_\beta}\right]} \right. \\
& \left. - \left(\frac{i}{\omega} + \frac{r}{\alpha(\omega_{0r})}\left[1 - \frac{1}{\pi Q_\alpha} ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{i}{2Q_\alpha}\right]\right) e^{\frac{i\omega r}{\alpha(\omega_{0r})}\left[1 - \frac{1}{\pi Q_\alpha} ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{i}{2Q_\alpha}\right]}\right] \\
& + \frac{(2 + 3\cos 2\theta)\sin\theta}{2\pi\rho}\frac{A}{r^2}\frac{i\omega e^{-a|\omega|}}{\alpha^2(\omega_{0r})}\left[1 - \frac{2}{\pi Q_\alpha} ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{i}{Q_\alpha}\right] e^{i\omega\frac{r}{\alpha(\omega_{0r})}\left[1 - \frac{1}{\pi Q_\alpha} ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{i}{2Q_\alpha}\right]} \\
& - \frac{(3 + 6\cos 2\theta)\sin\theta}{4\pi\rho}\frac{\mathrm{A}}{r^2}\frac{i\omega e^{-a|\omega|}}{\beta^2(\omega_{0r})}\left[1 - \frac{2}{\pi Q_\beta} ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{i}{Q_\beta}\right] e^{i\omega\frac{r}{\beta(\omega_{0r})}\left[1 - \frac{1}{\pi Q_\beta} ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{i}{2Q_\beta}\right]} \\
& + \frac{(1 + \cos 2\theta)\sin\theta}{4\pi\rho}\frac{A}{r}\frac{\omega^2 e^{-a|\omega|}}{\alpha^3(\omega_{0r})}\left[1 - \frac{3}{\pi Q_\alpha} ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{3i}{2Q_\alpha}\right] e^{i\omega\frac{r}{\alpha(\omega_{0r})}\left[1 - \frac{1}{\pi Q_\alpha} ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{i}{2Q_\alpha}\right]} \\
& - \frac{\cos 2\theta \sin\theta}{4\pi\rho}\frac{A}{r}\frac{\omega^2 e^{-a|\omega|}}{\beta^3(\omega_{0r})}\left[1 - \frac{3}{\pi Q_\beta} ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{3i}{2Q_\beta}\right] e^{i\omega\frac{r}{\beta(\omega_{0r})}\left[1 - \frac{1}{\pi Q_\beta} ln\left(\frac{\omega}{\omega_{0r}}\right) + \frac{i}{2Q_\beta}\right]}
\end{aligned} \tag{69a}$$

Apply the approximation (56) to it,

$$lnx \approx \frac{x-1}{x}, \qquad x\ near\ 1, \qquad \Rightarrow \quad \omega ln\left(\frac{\omega}{\omega_{0r}}\right) \approx \omega\left[\frac{(\omega/\omega_{0r}) - 1}{(\omega/\omega_{0r})}\right] = \omega - \omega_{0r} \tag{56}$$

$$\bar{u}_1(\mathbf{x},\omega) = \frac{(9+15\cos 2\theta)\sin\theta}{4\pi\rho}\frac{A}{r^4}e^{-a|\omega|}$$
$$\times\left[\left(\frac{i}{\omega}+\frac{r}{\beta(\omega_{0r})}\left[1-\frac{\omega-\omega_{0r}}{\omega\pi Q_\beta}+\frac{i}{2Q_\beta}\right]\right)e^{\frac{ir}{\beta(\omega_{0r})}\left[\omega-\frac{\omega-\omega_{0r}}{\pi Q_\beta}+\frac{i\omega}{2Q_\beta}\right]}\right.$$
$$\left.-\left(\frac{i}{\omega}+\frac{r}{\alpha(\omega_{0r})}\left[1-\frac{\omega-\omega_{0r}}{\omega\pi Q_\alpha}+\frac{i}{2Q_\alpha}\right]\right)e^{\frac{ir}{\alpha(\omega_{0r})}\left[\omega-\frac{\omega-\omega_{0r}}{\pi Q_\alpha}+\frac{i\omega}{2Q_\alpha}\right]}\right]$$
$$+\frac{(2+3\cos 2\theta)\sin\theta}{2\pi\rho}\frac{A}{r^2}\frac{1}{\alpha^2(\omega_{0r})}\left[i\omega-\frac{2i\omega-2i\omega_{0r}}{\pi Q_\alpha}-\frac{\omega}{Q_\alpha}\right]e^{-a|\omega|}e^{i\frac{r}{\alpha(\omega_{0r})}\left[\omega-\frac{\omega-\omega_{0r}}{\pi Q_\alpha}+\frac{i\omega}{2Q_\alpha}\right]}$$
$$-\frac{(3+6\cos 2\theta)\sin\theta}{4\pi\rho}\frac{\mathrm{A}}{r^2}\frac{1}{\beta^2(\omega_{0r})}\left[i\omega-\frac{2i\omega-2i\omega_{0r}}{\pi Q_\beta}-\frac{\omega}{Q_\beta}\right]e^{-a|\omega|}e^{i\frac{r}{\beta(\omega_{0r})}\left[\omega-\frac{\omega-\omega_{0r}}{\pi Q_\beta}+\frac{i\omega}{2Q_\beta}\right]}$$
$$+\frac{(1+\cos 2\theta)\sin\theta}{4\pi\rho}\frac{A}{r}\frac{1}{\alpha^3(\omega_{0r})}\left[\omega^2-\frac{3\omega^2-3\omega_{0r}\omega}{\pi Q_\alpha}+\frac{3i\omega^2}{2Q_\alpha}\right]e^{-a|\omega|}e^{i\frac{r}{\alpha(\omega_{0r})}\left[\omega-\frac{\omega-\omega_{0r}}{\pi Q_\alpha}+\frac{i\omega}{2Q_\alpha}\right]}$$
$$-\frac{\cos 2\theta\sin\theta}{4\pi\rho}\frac{A}{r}\frac{1}{\beta^3(\omega_{0r})}\left[\omega^2-\frac{3\omega^2-3\omega_{0r}\omega}{\pi Q_\alpha}+\frac{3i\omega^2}{2Q_\alpha}\right]e^{-a|\omega|}e^{i\frac{r}{\beta(\omega_{0r})}\left[\omega-\frac{\omega-\omega_{0r}}{\pi Q_\beta}+\frac{i\omega}{2Q_\beta}\right]} \quad (69b)$$

Then, do inverse Fourier transform of (69b),

$$u_1(\mathbf{x},t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\frac{(9+15\cos 2\theta)\sin\theta}{4\pi\rho}\frac{A}{r^4}e^{-a|\omega|}\right.$$
$$\times\left[\left(\frac{i}{\omega}+\frac{r}{\beta(\omega_{0r})}\left[1-\frac{\omega-\omega_{0r}}{\omega\pi Q_\beta}+\frac{i}{2Q_\beta}\right]\right)e^{\frac{ir}{\beta(\omega_{0r})}\left[\omega-\frac{\omega-\omega_{0r}}{\pi Q_\beta}+\frac{i\omega}{2Q_\beta}\right]}\right.$$
$$\left.-\left(\frac{i}{\omega}+\frac{r}{\alpha(\omega_{0r})}\left[1-\frac{\omega-\omega_{0r}}{\omega\pi Q_\alpha}+\frac{i}{2Q_\alpha}\right]\right)e^{\frac{ir}{\alpha(\omega_{0r})}\left[\omega-\frac{\omega-\omega_{0r}}{\pi Q_\alpha}+\frac{i\omega}{2Q_\alpha}\right]}\right]$$
$$+\frac{(2+3\cos 2\theta)\sin\theta}{2\pi\rho}\frac{A}{r^2}\frac{1}{\alpha^2(\omega_{0r})}\left[i\omega-\frac{2i\omega-2i\omega_{0r}}{\pi Q_\alpha}-\frac{\omega}{Q_\alpha}\right]e^{-a|\omega|}e^{i\frac{r}{\alpha(\omega_{0r})}\left[\omega-\frac{\omega-\omega_{0r}}{\pi Q_\alpha}+\frac{i\omega}{2Q_\alpha}\right]}$$
$$-\frac{(3+6\cos 2\theta)\sin\theta}{4\pi\rho}\frac{\mathrm{A}}{r^2}\frac{1}{\beta^2(\omega_{0r})}\left[i\omega-\frac{2i\omega-2i\omega_{0r}}{\pi Q_\beta}-\frac{\omega}{Q_\beta}\right]e^{-a|\omega|}e^{i\frac{r}{\beta(\omega_{0r})}\left[\omega-\frac{\omega-\omega_{0r}}{\pi Q_\beta}+\frac{i\omega}{2Q_\beta}\right]}$$
$$+\frac{(1+\cos 2\theta)\sin\theta}{4\pi\rho}\frac{A}{r}\frac{1}{\alpha^3(\omega_{0r})}\left[\omega^2-\frac{3\omega^2-3\omega_{0r}\omega}{\pi Q_\alpha}+\frac{3i\omega^2}{2Q_\alpha}\right]e^{-a|\omega|}e^{i\frac{r}{\alpha(\omega_{0r})}\left[\omega-\frac{\omega-\omega_{0r}}{\pi Q_\alpha}+\frac{i\omega}{2Q_\alpha}\right]}$$

$$-\frac{\cos 2\theta \sin\theta}{4\pi\rho}\frac{A}{r}$$

$$\times \frac{1}{\beta^3(\omega_{0r})}\left[\omega^2 - \frac{3\omega^2 - 3\omega_{0r}\omega}{\pi Q_\alpha} + \frac{3i\omega^2}{2Q_\alpha}\right] e^{-a|\omega|} e^{i\frac{r}{\beta(\omega_{0r})}\left[\omega - \frac{\omega-\omega_{0r}}{\pi Q_\beta} + \frac{i\omega}{2Q_\beta}\right]}\Bigg\} \times e^{-i\omega t} d\omega$$

And again, if $g(t)$ is real, then in the inverse Fourier transform that gives $g(t)$, one may replace the integral from $-\infty$ to ∞ with 2 times the real part of the integral from 0 to ∞.

$$u_1(\mathbf{x}, t) = \frac{A(9 + 15\cos 2\theta)\sin\theta}{4\pi^2\rho r^4}$$

$$\times Re\left\{\int_0^\infty \left[\left(\frac{i}{\omega} + \frac{r}{\beta(\omega_{0r})}\left[1 - \frac{\omega - \omega_{0r}}{\omega\pi Q_\beta} + \frac{i}{2Q_\beta}\right]\right) e^{\frac{ir}{\beta(\omega_{0r})}\left[\omega - \frac{\omega-\omega_{0r}}{\pi Q_\beta} + \frac{i\omega}{2Q_\beta}\right] - a\omega}\right.\right.$$

$$\left.\left. - \left(\frac{i}{\omega} + \frac{r}{\alpha(\omega_{0r})}\left[1 - \frac{\omega - \omega_{0r}}{\omega\pi Q_\alpha} + \frac{i}{2Q_\alpha}\right]\right) e^{\frac{ir}{\alpha(\omega_{0r})}\left[\omega - \frac{\omega-\omega_{0r}}{\pi Q_\alpha} + \frac{i\omega}{2Q_\alpha}\right] - a\omega}\right] e^{-i\omega t} d\omega\right\}$$

$$+\frac{A(2 + 3\cos 2\theta)\sin\theta}{2\pi^2\rho r^2}$$

$$\times Re\left\{\int_0^\infty \frac{1}{\alpha^2(\omega_{0r})}\left[i\omega - \frac{2i\omega - 2i\omega_{0r}}{\pi Q_\alpha} - \frac{\omega}{Q_\alpha}\right] e^{i\frac{r}{\alpha(\omega_{0r})}\left[\omega - \frac{\omega-\omega_{0r}}{\pi Q_\alpha} + \frac{i\omega}{2Q_\alpha}\right] - a\omega} e^{-i\omega t} d\omega\right\}$$

$$-\frac{A(3 + 6\cos 2\theta)\sin\theta}{4\pi^2\rho r^2}$$

$$\times Re\left\{\int_0^\infty \frac{1}{\beta^2(\omega_{0r})}\left[i\omega - \frac{2i\omega - 2i\omega_{0r}}{\pi Q_\beta} - \frac{\omega}{Q_\beta}\right] e^{i\frac{r}{\beta(\omega_{0r})}\left[\omega - \frac{\omega-\omega_{0r}}{\pi Q_\beta} + \frac{i\omega}{2Q_\beta}\right] - a\omega} e^{-i\omega t} d\omega\right\}$$

$$+\frac{A(1 + \cos 2\theta)\sin\theta}{4\pi^2\rho r}$$

$$\times Re\left\{\int_0^\infty \frac{1}{\alpha^3(\omega_{0r})}\left[\omega^2 - \frac{3\omega^2 - 3\omega_{0r}\omega}{\pi Q_\alpha} + \frac{3i\omega^2}{2Q_\alpha}\right] e^{i\frac{r}{\alpha(\omega_{0r})}\left[\omega - \frac{\omega-\omega_{0r}}{\pi Q_\alpha} + \frac{i\omega}{2Q_\alpha}\right] - a\omega} e^{-i\omega t} d\omega\right\}$$

$$-\frac{A\cos 2\theta \sin\theta}{4\pi^2\rho r}$$

$$\times Re\left\{\int_0^\infty \frac{1}{\beta^3(\omega_{0r})}\left[\omega^2 - \frac{3\omega^2 - 3\omega_{0r}\omega}{\pi Q_\alpha} + \frac{3i\omega^2}{2Q_\alpha}\right] e^{i\frac{r}{\beta(\omega_{0r})}\left[\omega - \frac{\omega-\omega_{0r}}{\pi Q_\beta} + \frac{i\omega}{2Q_\beta}\right] - a\omega}\ e^{-i\omega t} d\omega\right\}$$

$$= \frac{A(9+15\cos 2\theta)\sin\theta}{4\pi^2\rho r^4} Re\left\{\int_0^\infty \left[\frac{i}{\omega} + \frac{1}{\omega}\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})} + \frac{r}{\beta(\omega_{0r})}\left(1 - \frac{1}{\pi Q_\beta}\right) + \frac{ir}{2Q_\beta\beta(\omega_{0r})}\right]\right.$$

$$\times\ e^{i\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}} e^{-\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)\omega} e^{i\omega\left(\frac{r}{\beta(\omega_{0r})} - \frac{r}{\pi Q_\beta\beta(\omega_{0r})} - t\right)}$$

$$-\left[\frac{i}{\omega} + \frac{1}{\omega}\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})} + \frac{r}{\alpha(\omega_{0r})}\left(1 - \frac{1}{\pi Q_\alpha}\right) + \frac{ir}{2Q_\alpha\alpha(\omega_{0r})}\right]$$

$$\left.\times\ e^{i\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}} e^{-\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)\omega} e^{i\omega\left(\frac{r}{\alpha(\omega_{0r})} - \frac{r}{\pi Q_\alpha\alpha(\omega_{0r})} - t\right)} d\omega\right\}$$

$$+ \frac{A(2+3\cos 2\theta)\sin\theta}{2\pi^2\rho r^2} Re\left\{\int_0^\infty \left[\frac{i\omega}{\alpha^2(\omega_{0r})}\left(1 - \frac{2}{\pi Q_\alpha}\right) - \frac{\omega}{\alpha^2(\omega_{0r})Q_\alpha} + \frac{i2\omega_{0r}}{\pi Q_\alpha \alpha^2(\omega_{0r})}\right]\right.$$

$$\left.\times\ e^{i\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}} e^{-\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)\omega} e^{i\omega\left(\frac{r}{\alpha(\omega_{0r})} - \frac{r}{\pi Q_\alpha\alpha(\omega_{0r})} - t\right)} d\omega\right\}$$

$$- \frac{A(3+6\cos 2\theta)\sin\theta}{4\pi^2\rho r^2} Re\left\{\int_0^\infty \left[\frac{i\omega}{\beta^2(\omega_{0r})}\left(1 - \frac{2}{\pi Q_\beta}\right) - \frac{\omega}{\beta^2(\omega_{0r})Q_\beta} + \frac{i2\omega_{0r}}{\pi Q_\beta \beta^2(\omega_{0r})}\right]\right.$$

$$\left.\times\ e^{i\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}} e^{-\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)\omega} e^{i\omega\left(\frac{r}{\beta(\omega_{0r})} - \frac{r}{\pi Q_\beta\beta(\omega_{0r})} - t\right)} d\omega\right\}$$

$$+ \frac{A(1+\cos 2\theta)\sin\theta}{4\pi^2\rho r} Re\left\{\int_0^\infty \left[\frac{\omega^2}{\alpha^3(\omega_{0r})}\left(1 - \frac{3}{\pi Q_\alpha}\right) + \frac{3i\omega^2}{2\alpha^3(\omega_{0r})Q_\alpha} + \frac{3\omega_{0r}\omega}{\pi Q_\alpha \alpha^3(\omega_{0r})}\right]\right.$$

$$\left.\times\ e^{i\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}} e^{-\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)\omega} e^{i\omega\left(\frac{r}{\alpha(\omega_{0r})} - \frac{r}{\pi Q_\alpha\alpha(\omega_{0r})} - t\right)} d\omega\right\}$$

$$- \frac{A\cos 2\theta\sin\theta}{4\pi^2\rho r} Re\left\{\int_0^\infty \left[\frac{\omega^2}{\beta^3(\omega_{0r})}\left(1 - \frac{3}{\pi Q_\beta}\right) + \frac{3i\omega^2}{2\beta^3(\omega_{0r})Q_\beta} + \frac{3\omega_{0r}\omega}{\pi Q_\beta \beta^3(\omega_{0r})}\right]\right.$$

$$\times\ e^{i\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}} e^{-\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)\omega} e^{i\omega\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)} d\omega\Bigg\}$$

$$= \frac{A(9+15\cos 2\theta)\sin\theta}{4\pi^2\rho r^4} Re\left\{\int_0^\infty \left[\frac{i}{\omega}+\frac{1}{\omega}\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}+\frac{r}{\beta(\omega_{0r})}\left(1-\frac{1}{\pi Q_\beta}\right)+\frac{ir}{2Q_\beta \beta(\omega_{0r})}\right]\right.$$

$$\times\left[\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)+i\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\right]\times\ e^{-\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)\omega}$$

$$\times\left[\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)+i\sin\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)\right]$$

$$-\frac{A(9+15\cos 2\theta)\sin\theta}{4\pi^2\rho r^4} Re\left\{\int_0^\infty \left[\frac{i}{\omega}+\frac{1}{\omega}\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}+\frac{r}{\alpha(\omega_{0r})}\left(1-\frac{1}{\pi Q_\alpha}\right)+\frac{ir}{2Q_\alpha \alpha(\omega_{0r})}\right]\right.$$

$$\times\left[\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)+i\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\right]\times\ e^{-\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)\omega}$$

$$\times\ \left.\left[\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)+i\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)\right]\right\} d\omega$$

$$+\frac{A(2+3\cos 2\theta)\sin\theta}{2\pi^2\rho r^2} Re\left\{\int_0^\infty \left[\frac{i\omega}{\alpha^2(\omega_{0r})}\left(1-\frac{2}{\pi Q_\alpha}\right)-\frac{\omega}{\alpha^2(\omega_{0r})Q_\alpha}+\frac{i2\omega_{0r}}{\pi Q_\alpha \alpha^2(\omega_{0r})}\right]\right.$$

$$\times\left[\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)+i\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\right]\times\ e^{-\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)\omega}$$

$$\times\ \left.\left[\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)+i\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)\right]\right\} d\omega$$

$$-\frac{A(3+6\cos 2\theta)\sin\theta}{4\pi^2\rho r^2} Re\left\{\int_0^\infty \left[\frac{i\omega}{\beta^2(\omega_{0r})}\left(1-\frac{2}{\pi Q_\beta}\right)-\frac{\omega}{\beta^2(\omega_{0r})Q_\beta}+\frac{i2\omega_{0r}}{\pi Q_\beta \beta^2(\omega_{0r})}\right]\right.$$

$$\times\left[\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)+i\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\right]\times\ e^{-\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)\omega}$$

$$\times\left.\left[\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)+i\sin\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)\right]\right\} d\omega$$

$$+\frac{A(1+\cos 2\theta)\sin\theta}{4\pi^2\rho r} Re\left\{\int_0^\infty \left[\frac{\omega^2}{\alpha^3(\omega_{0r})}\left(1-\frac{3}{\pi Q_\alpha}\right)+\frac{3i\omega^2}{2\alpha^3(\omega_{0r})Q_\alpha}+\frac{3\omega_{0r}\omega}{\pi Q_\alpha \alpha^3(\omega_{0r})}\right]\right.$$

$$\times\left[\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)+i\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\right]\times\ e^{-\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)\omega}$$

$$\times\left[\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)+i\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)\right]\Bigg\}d\omega$$

$$-\frac{A\cos 2\theta\sin\theta}{4\pi^2\rho r}Re\left\{\int_0^\infty\left[\frac{\omega^2}{\beta^3(\omega_{0r})}\left(1-\frac{3}{\pi Q_\beta}\right)+\frac{3i\omega^2}{2\beta^3(\omega_{0r})Q_\beta}+\frac{3\omega_{0r}\omega}{\pi Q_\beta \beta^3(\omega_{0r})}\right]\right.$$

$$\times\left[\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)+i\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\right]\times\ e^{-\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)\omega}$$

$$\times\left[\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)+i\sin\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)\right]\Bigg\}d\omega$$

$$=\frac{A(9+15\cos 2\theta)\sin\theta}{4\pi^2\rho r^4}\int_0^\infty\left[-\frac{1}{\omega}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)\right.$$

$$-\frac{1}{\omega}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$+\frac{1}{\omega}\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{1}{\omega}\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$+\frac{r}{\beta(\omega_{0r})}\left(1-\frac{1}{\pi Q_\beta}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{r}{\beta(\omega_{0r})}\left(1-\frac{1}{\pi Q_\beta}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{r}{2Q_\beta \beta(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$\left.-\frac{r}{2Q_\beta \beta(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)\right]$$

$$\times\ e^{-\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)\omega}d\omega$$

$$-\frac{A(9+15\cos 2\theta)\sin\theta}{4\pi^2\rho r^4}\int_0^\infty\left[-\frac{1}{\omega}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)\right.$$

$$-\frac{1}{\omega}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$+\frac{1}{\omega}\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{1}{\omega}\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$+\frac{r}{\alpha(\omega_{0r})}\left(1-\frac{1}{\pi Q_\alpha}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{r}{\alpha(\omega_{0r})}\left(1-\frac{1}{\pi Q_\alpha}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{r}{2Q_\alpha \alpha(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$\left.-\frac{r}{2Q_\alpha \alpha(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)\right]$$

$$\times\ e^{-\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)\omega}d\omega$$

$$+\frac{A(2+3\cos 2\theta)\sin\theta}{2\pi^2\rho r^2}\times$$

$$\int_0^\infty\left[\frac{\omega}{\alpha^2(\omega_{0r})Q_\alpha}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)\right.$$

$$-\frac{\omega}{\alpha^2(\omega_{0r})Q_\alpha}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{\omega}{\alpha^2(\omega_{0r})}\left(1-\frac{2}{\pi Q_\alpha}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{\omega}{\alpha^2(\omega_{0r})}\left(1-\frac{2}{\pi Q_\alpha}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{2\omega_{0r}}{\pi Q_\alpha \alpha^2(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{2\omega_{0r}}{\pi Q_\alpha \alpha^2(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)\Bigg]$$

$$\times\ e^{-\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)\omega}d\omega$$

$$-\frac{A(3+6\cos 2\theta)\sin\theta}{4\pi^2\rho r^2}\times$$

$$\int_0^\infty\Bigg[\frac{\omega}{\beta^2(\omega_{0r})Q_\beta}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{\omega}{\beta^2(\omega_{0r})Q_\beta}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{\omega}{\beta^2(\omega_{0r})}\left(1-\frac{2}{\pi Q_\beta}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{\omega}{\beta^2(\omega_{0r})}\left(1-\frac{2}{\pi Q_\beta}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{2\omega_{0r}}{\pi Q_\beta \beta^2(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{2\omega_{0r}}{\pi Q_\beta \beta^2(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)\Bigg]$$

$$\times\ e^{-\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)\omega}d\omega$$

$$+\frac{A(1+\cos 2\theta)\sin\theta}{4\pi^2\rho r}\times$$

$$\int_0^\infty\Bigg[\frac{3\omega_{0r}\omega}{\pi Q_\alpha \alpha^3(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{3\omega_{0r}\omega}{\pi Q_\alpha \alpha^3(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$+\frac{\omega^2}{\alpha^3(\omega_{0r})}\left(1-\frac{3}{\pi Q_\alpha}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{\omega^2}{\alpha^3(\omega_{0r})}\left(1-\frac{3}{\pi Q_\alpha}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{3\omega^2}{2\alpha^3(\omega_{0r})Q_\alpha}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{3\omega^2}{2\alpha^3(\omega_{0r})Q_\alpha}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)\omega\right)\Bigg]$$

$$\times\ e^{-\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)\omega}d\omega$$

$$-\frac{A\cos 2\theta \sin\theta}{4\pi^2\rho r}\int_0^\infty\Bigg[\frac{3\omega_{0r}\omega}{\pi Q_\beta \beta^3(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{3\omega_{0r}\omega}{\pi Q_\beta \beta^3(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$+\frac{\omega^2}{\beta^3(\omega_{0r})}\left(1-\frac{3}{\pi Q_\beta}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{\omega^2}{\beta^3(\omega_{0r})}\left(1-\frac{3}{\pi Q_\beta}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{3\omega^2}{2\beta^3(\omega_{0r})Q_\beta}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\sin\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)$$

$$-\frac{3\omega^2}{2\beta^3(\omega_{0r})Q_\beta}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)\omega\right)\Bigg]$$

$$\times\ e^{-\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)\omega}d\omega$$

Because, from a mathematical handbook (Spiegel, 1968):

$$\int_0^\infty e^{-ax}\sin bx\,dx=\frac{b}{a^2+b^2}$$

$$\int_0^\infty e^{-ax}\cos bx\,dx=\frac{a}{a^2+b^2}$$

$$\int_0^\infty \frac{e^{-ax}\sin bx}{x}\,dx=\tan^{-1}\frac{b}{a}$$

$$\int_0^\infty xe^{-ax}\cos bx\,dx = \frac{a^2-b^2}{(a^2+b^2)^2}$$

$$\int_0^\infty xe^{-ax}\sin bx\,dx = \frac{2ab}{(a^2+b^2)^2}$$

$$\int_0^\infty x^2e^{-ax}\cos bx\,dx = \frac{2a^3-6ab^2}{(a^2+b^2)^3}$$

$$\int_0^\infty x^2e^{-ax}\sin bx\,dx = \frac{6a^2b-2b^3}{(a^2+b^2)^3}$$

Applying these integrals, then we will have:

$$u_1(\mathbf{x},t) = \frac{A(9+15\cos 2\theta)\sin\theta}{4\pi^2\rho r^4}\times$$

$$\Bigg\{-\left[\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)+\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\right]\tan^{-1}\frac{\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t}{\frac{r}{2Q_\beta \beta(\omega_{0r})}+a}$$

$$+\frac{\frac{r}{\beta(\omega_{0r})}\left(1-\frac{1}{\pi Q_\beta}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)}{\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2}$$

$$-\frac{\frac{r}{2Q_\beta \beta(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)}{\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2}$$

$$-\frac{\frac{r}{\beta(\omega_{0r})}\left(1-\frac{1}{\pi Q_\beta}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2}$$

$$-\frac{\frac{r}{2Q_\beta\beta(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2}$$

$$+\left[\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)+\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\right]\tan^{-1}\frac{\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t}{\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a}$$

$$-\frac{\frac{r}{\alpha(\omega_{0r})}\left(1-\frac{1}{\pi Q_\alpha}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)}{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}$$

$$+\frac{\frac{r}{2Q_\alpha\alpha(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)}{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}$$

$$+\frac{\frac{r}{\alpha(\omega_{0r})}\left(1-\frac{1}{\pi Q_\alpha}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}$$

$$\left.+\frac{\frac{r}{2Q_\alpha\alpha(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}\right\}$$

$$-\frac{A(9+15\cos 2\theta)\sin\theta}{4\pi^2\rho r^4}\int_0^\infty\left[-\frac{1}{\omega}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)\omega\right)\right.$$

$$\left.+\frac{1}{\omega}\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)\omega\right)\right]$$

$$\times\ e^{-\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)\omega}$$

$$-\left[-\frac{1}{\omega}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)\omega\right)\right.$$

$$\left.+\frac{r}{\alpha(\omega_{0r})}\left(1-\frac{1}{\pi Q_\alpha}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\cos\left(\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)\omega\right)\right]$$

$$\times\ e^{-\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)\omega}d\omega$$

$$+\frac{A(2+3\cos 2\theta)\sin\theta}{2\pi^2\rho r^2}\times$$

$$\left\{\frac{1}{\alpha^2(\omega_{0r})Q_\alpha}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\frac{2\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^2}\right.$$

$$-\frac{1}{\alpha^2(\omega_{0r})Q_\alpha}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\frac{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2-\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^2}$$

$$-\frac{\left(1-\frac{2}{\pi Q_\alpha}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)}{\alpha^2(\omega_{0r})}\frac{2\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^2}$$

$$-\frac{\left(1-\frac{2}{\pi Q_\alpha}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)}{\alpha^2(\omega_{0r})}\frac{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2-\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^2}$$

$$-\frac{2\omega_{0r}}{\pi Q_\alpha\alpha^2(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\frac{\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}$$

$$\left.-\frac{2\omega_{0r}}{\pi Q_\alpha\alpha^2(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\frac{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)}{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}\right\}$$

$$-\frac{A(3+6\cos 2\theta)\sin\theta}{4\pi^2\rho r^2}\times$$

$$\left\{\frac{1}{\beta^2(\omega_{0r})Q_\beta}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)\frac{2\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2\right]^2}\right.$$

$$-\frac{1}{\beta^2(\omega_{0r})Q_\beta}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)\frac{\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2-\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2\right]^2}$$

$$-\frac{\left(1-\frac{2}{\pi Q_\beta}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)}{\beta^2(\omega_{0r})}\frac{2\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2\right]^2}$$

$$-\frac{\left(1-\frac{2}{\pi Q_\beta}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)}{\beta^2(\omega_{0r})}\frac{\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2-\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2\right]^2}$$

$$-\frac{2\omega_{0r}}{\pi Q_\beta\beta^2(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)\frac{\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2}$$

$$\left.-\frac{2\omega_{0r}}{\pi Q_\beta\beta^2(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)\frac{\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)}{\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}\right\}$$

$$+\frac{A(1+\cos 2\theta)\sin\theta}{4\pi^2\rho r}\times$$

$$\left\{\frac{3\omega_{0r}}{\pi Q_\alpha\alpha^3(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\frac{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2-\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^2}\right.$$

$$-\frac{3\omega_{0r}}{\pi Q_\alpha\alpha^3(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\frac{2\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^2}$$

$$+\frac{\left(1-\frac{3}{\pi Q_\alpha}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)}{\alpha^3(\omega_{0r})}\frac{2\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^3}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^3}$$

$$-\frac{\left(1-\frac{3}{\pi Q_\alpha}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)}{\alpha^3(\omega_{0r})}\frac{6\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^3}$$

$$-\frac{\left(1-\frac{3}{\pi Q_\alpha}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)}{\alpha^3(\omega_{0r})}\frac{6\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^3}$$

$$+\frac{\left(1-\frac{3}{\pi Q_\alpha}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)}{\alpha^3(\omega_{0r})}\frac{2\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^3}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^3}$$

$$-\frac{3\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)}{2\alpha^3(\omega_{0r})Q_\alpha}\frac{6\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^3}$$

$$+\frac{3\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)}{2\alpha^3(\omega_{0r})Q_\alpha}\frac{2\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^3}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^3}$$

$$-\frac{3\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)}{2\alpha^3(\omega_{0r})Q_\alpha}\frac{2\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^3}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^3}$$

$$\left.+\frac{3\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)}{2\alpha^3(\omega_{0r})Q_\alpha}\frac{6\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^3}\right\}$$

$$-\frac{A\cos 2\theta\sin\theta}{4\pi^2\rho r}\times$$

$$\left\{\frac{3\omega_{0r}}{\pi Q_\beta\beta^3(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)\frac{\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2-\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2\right]^2}\right.$$

$$-\frac{3\omega_{0r}}{\pi Q_\beta\beta^3(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)\frac{2\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2\right]^2}$$

$$+\frac{\left(1-\frac{3}{\pi Q_\beta}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)}{\beta^3(\omega_{0r})}\frac{2\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^3}{\left[\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2\right]^3}$$

$$-\frac{\left(1-\frac{3}{\pi Q_\beta}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)}{\beta^3(\omega_{0r})}\frac{6\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2\right]^3}$$

$$-\frac{\left(1-\frac{3}{\pi Q_\beta}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)}{\beta^3(\omega_{0r})}\frac{6\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2\right]^3}$$

$$+\frac{\left(1-\frac{3}{\pi Q_\beta}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)}{\beta^3(\omega_{0r})}\frac{2\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^3}{\left[\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2\right]^3}$$

$$-\frac{3\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)}{2\beta^3(\omega_{0r})Q_\beta}\frac{6\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2\right]^3}$$

$$+\frac{3\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)}{2\beta^3(\omega_{0r})Q_\beta}\frac{2\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^3}{\left[\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2\right]^3}$$

$$-\frac{3\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)}{2\beta^3(\omega_{0r})Q_\beta}\frac{2\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^3}{\left[\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2\right]^3}$$

$$\left.+\frac{3\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)}{2\beta^3(\omega_{0r})Q_\beta}\frac{6\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2\right]^3}\right\}$$

As we discussed in chapter 3, the integral in the form of $\int_0^\infty \frac{1}{\omega}\cos(a\omega)\, e^{b\omega} d\omega$ can be ignored. Therefore, we can directly move the integrals from the above formula, then a new solution including dispersion of the EOM for a double-couple-without-moment source will be:

$$u_1(\mathbf{x},t)=\frac{A(9+15\cos 2\theta)\sin\theta}{4\pi^2\rho r^4}\times$$

$$\left\{-\left[\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)+\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\right]\tan^{-1}\frac{\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t}{\frac{r}{2Q_\beta \beta(\omega_{0r})}+a}\right.$$

$$+\frac{\frac{r}{\beta(\omega_{0r})}\left(1-\frac{1}{\pi Q_\beta}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)}{\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2}$$

$$-\frac{\frac{r}{2Q_\beta \beta(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)}{\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2}$$

$$-\frac{\frac{r}{\beta(\omega_{0r})}\left(1-\frac{1}{\pi Q_\beta}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2}$$

$$-\frac{\frac{r}{2Q_\beta \beta(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2}$$

$$+\left[\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)+\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\right]\tan^{-1}\frac{\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t}{\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a}$$

$$-\frac{\frac{r}{\alpha(\omega_{0r})}\left(1-\frac{1}{\pi Q_\alpha}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)}{\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^2}$$

$$+\frac{\frac{r}{2Q_\alpha \alpha(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)}{\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^2}$$

$$+\frac{\frac{r}{\alpha(\omega_{0r})}\left(1-\frac{1}{\pi Q_\alpha}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^2}$$

$$\left.+\frac{\frac{r}{2Q_\alpha \alpha(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^2}\right\}$$

$$+\frac{A(2+3\cos 2\theta)\sin\theta}{2\pi^2\rho r^2}\times$$

$$\left\{\frac{1}{\alpha^2(\omega_{0r})Q_\alpha}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\frac{2\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^2\right]^2}\right.$$

$$
\begin{aligned}
&-\frac{1}{\alpha^2(\omega_{0r})Q_\alpha}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\frac{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2-\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^2}\\
&-\frac{\left(1-\frac{2}{\pi Q_\alpha}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)}{\alpha^2(\omega_{0r})}\frac{2\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^2}\\
&-\frac{\left(1-\frac{2}{\pi Q_\alpha}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)}{\alpha^2(\omega_{0r})}\frac{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2-\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^2}\\
&-\frac{2\omega_{0r}}{\pi Q_\alpha\alpha^2(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\frac{\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}\\
&\left.-\frac{2\omega_{0r}}{\pi Q_\alpha\alpha^2(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha\alpha(\omega_{0r})}\right)\frac{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)}{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}\right\}\\
-&\frac{A(3+6\cos 2\theta)\sin\theta}{4\pi^2\rho r^2}\times\\
&\left\{\frac{1}{\beta^2(\omega_{0r})Q_\beta}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)\frac{2\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2\right]^2}\right.\\
&-\frac{1}{\beta^2(\omega_{0r})Q_\beta}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta\beta(\omega_{0r})}\right)\frac{\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2-\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2\right]^2}
\end{aligned}
$$

$$-\frac{\left(1-\frac{2}{\pi Q_\beta}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)}{\beta^2(\omega_{0r})}\frac{2\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2\right]^2}$$

$$-\frac{\left(1-\frac{2}{\pi Q_\beta}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)}{\beta^2(\omega_{0r})}\frac{\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2-\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2\right]^2}$$

$$-\frac{2\omega_{0r}}{\pi Q_\beta\beta^2(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\frac{\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta\beta(\omega_{0r})}-t\right)^2}$$

$$\left.-\frac{2\omega_{0r}}{\pi Q_\beta\beta^2(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\frac{\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)}{\left(\frac{r}{2Q_\beta\beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}\right\}$$

$$+\frac{A(1+\cos 2\theta)\sin\theta}{4\pi^2\rho r}\times$$

$$\left\{\frac{3\omega_{0r}}{\pi Q_\alpha\alpha^3(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\frac{\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2-\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^2}\right.$$

$$-\frac{3\omega_{0r}}{\pi Q_\alpha\alpha^3(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)\frac{2\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^2}$$

$$+\frac{\left(1-\frac{3}{\pi Q_\alpha}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)}{\alpha^3(\omega_{0r})}\frac{2\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^3}{\left[\left(\frac{r}{2Q_\alpha\alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha\alpha(\omega_{0r})}-t\right)^2\right]^3}$$

$$-\frac{\left(1-\frac{3}{\pi Q_\alpha}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)}{\alpha^3(\omega_{0r})}\frac{6\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^2\right]^3}$$

$$-\frac{\left(1-\frac{3}{\pi Q_\alpha}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)}{\alpha^3(\omega_{0r})}\frac{6\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)^2\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^2\right]^3}$$

$$+\frac{\left(1-\frac{3}{\pi Q_\alpha}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)}{\alpha^3(\omega_{0r})}\frac{2\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^3}{\left[\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^2\right]^3}$$

$$-\frac{3\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)}{2\alpha^3(\omega_{0r})Q_\alpha}\frac{6\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)^2\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^2\right]^3}$$

$$+\frac{3\cos\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)}{2\alpha^3(\omega_{0r})Q_\alpha}\frac{2\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^3}{\left[\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^2\right]^3}$$

$$-\frac{3\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)}{2\alpha^3(\omega_{0r})Q_\alpha}\frac{2\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)^3}{\left[\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^2\right]^3}$$

$$\left.+\frac{3\sin\left(\frac{r\omega_{0r}}{\pi Q_\alpha \alpha(\omega_{0r})}\right)}{2\alpha^3(\omega_{0r})Q_\alpha}\frac{6\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)^2+\left(\frac{r}{\alpha(\omega_{0r})}-\frac{r}{\pi Q_\alpha \alpha(\omega_{0r})}-t\right)^2\right]^3}\right\}$$

$$-\frac{A\cos 2\theta \sin\theta}{4\pi^2\rho r}\times$$

$$\left\{\frac{3\omega_{0r}}{\pi Q_\beta \beta^3(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\frac{\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2-\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2\right]^2}\right.$$

$$-\frac{3\omega_{0r}}{\pi Q_\beta \beta^3(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)\frac{2\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2\right]^2}$$

$$+\frac{\left(1-\frac{3}{\pi Q_\beta}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)}{\beta^3(\omega_{0r})}\frac{2\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^3}{\left[\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2\right]^3}$$

$$-\frac{\left(1-\frac{3}{\pi Q_\beta}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)}{\beta^3(\omega_{0r})}\frac{6\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2\right]^3}$$

$$-\frac{\left(1-\frac{3}{\pi Q_\beta}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)}{\beta^3(\omega_{0r})}\frac{6\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2\right]^3}$$

$$+\frac{\left(1-\frac{3}{\pi Q_\beta}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)}{\beta^3(\omega_{0r})}\frac{2\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^3}{\left[\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2\right]^3}$$

$$-\frac{3\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)}{2\beta^3(\omega_{0r})Q_\beta}\frac{6\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2\right]^3}$$

$$+\frac{3\cos\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)}{2\beta^3(\omega_{0r})Q_\beta}\frac{2\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^3}{\left[\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2\right]^3}$$

$$-\frac{3\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)}{2\beta^3(\omega_{0r})Q_\beta}\frac{2\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^3}{\left[\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2\right]^3}$$

$$\left.+\frac{3\sin\left(\frac{r\omega_{0r}}{\pi Q_\beta \beta(\omega_{0r})}\right)}{2\beta^3(\omega_{0r})Q_\beta}\frac{6\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_\beta \beta(\omega_{0r})}+a\right)^2+\left(\frac{r}{\beta(\omega_{0r})}-\frac{r}{\pi Q_\beta \beta(\omega_{0r})}-t\right)^2\right]^3}\right\} \quad (70)$$

It is hard to see such a long formula, so, let us rewrite it to make it looks simpler.

If we assume

$$I(v(\omega_{0r}),Q_v)=$$

$$-\left[\cos\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)+\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)\right]\tan^{-1}\frac{\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t}{\frac{r}{2Q_v v(\omega_{0r})}+a}$$

$$+\frac{\left[\frac{r}{v(\omega_{0r})}\left(1-\frac{1}{\pi Q_v}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)-\frac{r}{2Q_v v(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)\right]\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)}{\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2+\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2}$$

$$-\frac{\frac{r}{v(\omega_{0r})}\left(1-\frac{1}{\pi Q_v}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2+\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2}$$

$$-\frac{\frac{r}{2Q_v v(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2+\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2} \quad (71a)$$

$J(v(\omega_{0r}), Q_v) =$

$$
\frac{1}{v^2(\omega_{0r})Q_v}\sin\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)\frac{2\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2+\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2\right]^2}
$$

$$
-\frac{1}{v^2(\omega_{0r})Q_v}\cos\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)\frac{\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2-\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2+\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2\right]^2}
$$

$$
-\frac{\left(1-\frac{2}{\pi Q_v}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)}{v^2(\omega_{0r})}\frac{2\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2+\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2\right]^2}
$$

$$
-\frac{\left(1-\frac{2}{\pi Q_v}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)}{v^2(\omega_{0r})}\frac{\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2-\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2+\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2\right]^2}
$$

$$
-\frac{2\omega_{0r}}{\pi Q_v v^2(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)\frac{\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)}{\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2+\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2}
$$

$$
-\frac{2\omega_{0r}}{\pi Q_v v^2(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)\frac{\left(\frac{r}{2Q_\alpha \alpha(\omega_{0r})}+a\right)}{\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2+\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2} \tag{71b}
$$

$K(v(\omega_{0r}), Q_v) =$

$$
\frac{3\omega_{0r}}{\pi Q_v v^3(\omega_{0r})}\cos\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)\frac{\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2-\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2+\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2\right]^2}
$$

$$
-\frac{3\omega_{0r}}{\pi Q_v v^3(\omega_{0r})}\sin\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)\frac{2\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2+\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2\right]^2}
$$

$$
\begin{aligned}
&+\frac{\left(1-\frac{3}{\pi Q_v}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)}{v^3(\omega_{0r})}\frac{2\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^3}{\left[\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2+\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2\right]^3}\\
&-\frac{\left(1-\frac{3}{\pi Q_v}\right)\cos\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)}{v^3(\omega_{0r})}\frac{6\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2+\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2\right]^3}\\
&-\frac{\left(1-\frac{3}{\pi Q_v}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)}{v^3(\omega_{0r})}\frac{6\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2+\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2\right]^3}\\
&+\frac{\left(1-\frac{3}{\pi Q_v}\right)\sin\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)}{v^3(\omega_{0r})}\frac{2\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^3}{\left[\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2+\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2\right]^3}\\
&-\frac{3\cos\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)}{2v^3(\omega_{0r})Q_v}\frac{6\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)}{\left[\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2+\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2\right]^3}\\
&+\frac{3\cos\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)}{2v^3(\omega_{0r})Q_v}\frac{2\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^3}{\left[\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2+\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2\right]^3}\\
&-\frac{3\sin\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)}{2v^3(\omega_{0r})Q_v}\frac{2\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^3}{\left[\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2+\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2\right]^3}\\
&+\frac{3\sin\left(\frac{r\omega_{0r}}{\pi Q_v v(\omega_{0r})}\right)}{2v^3(\omega_{0r})Q_v}\frac{6\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2}{\left[\left(\frac{r}{2Q_v v(\omega_{0r})}+a\right)^2+\left(\frac{r}{v(\omega_{0r})}-\frac{r}{\pi Q_v v(\omega_{0r})}-t\right)^2\right]^3} \qquad (71c)
\end{aligned}
$$

Substitute (71s) into (70), then (70) becomes:

$$
\begin{aligned}
u_1(\mathbf{x},t) = {} & \frac{A(9+15\cos 2\theta)\sin\theta}{4\pi^2\rho r^4}\left[I\big(\beta(\omega_{0r}),Q_\beta\big) - I(\alpha(\omega_{0r}),Q_\alpha)\right] \\
& + \frac{A(2+3\cos 2\theta)\sin\theta}{2\pi^2\rho r^2} J(\alpha(\omega_{0r}),Q_\alpha) - \frac{A(3+6\cos 2\theta)\sin\theta}{4\pi^2\rho r^2} J\big(\beta(\omega_{0r}),Q_\beta\big) \\
& + \frac{A(1+\cos 2\theta)\sin\theta}{4\pi^2\rho r} K(\alpha(\omega_{0r}),Q_\alpha) - \frac{A\cos 2\theta\sin\theta}{4\pi^2\rho r} K\big(\beta(\omega_{0r}),Q_\beta\big) \qquad (72)
\end{aligned}
$$

Again, to verify the accuracy of the approximation (72), we will use the Matlab code we mentioned before to compare the exact numerical results and our approximation.

In this case, we will still simulate a near-surface area, like in exploration geophysics. Therefore, we keep assuming $r = 0.3, 1, 2, 5, 10\ km$, $Q_\alpha = 40, Q_\beta = 20, \alpha_0 = 5\ km/s$ and $\beta_0 = 3\ km/s$, $a = 0.02s$. The new parameter "θ" in Matlab should be in radian form, so we choose $\theta = \pi/6$ in this time.

In this situation, if we plot the spectrum $\bar{u}_1(\mathbf{x},\omega)$, i.e., (69a), versus ω,

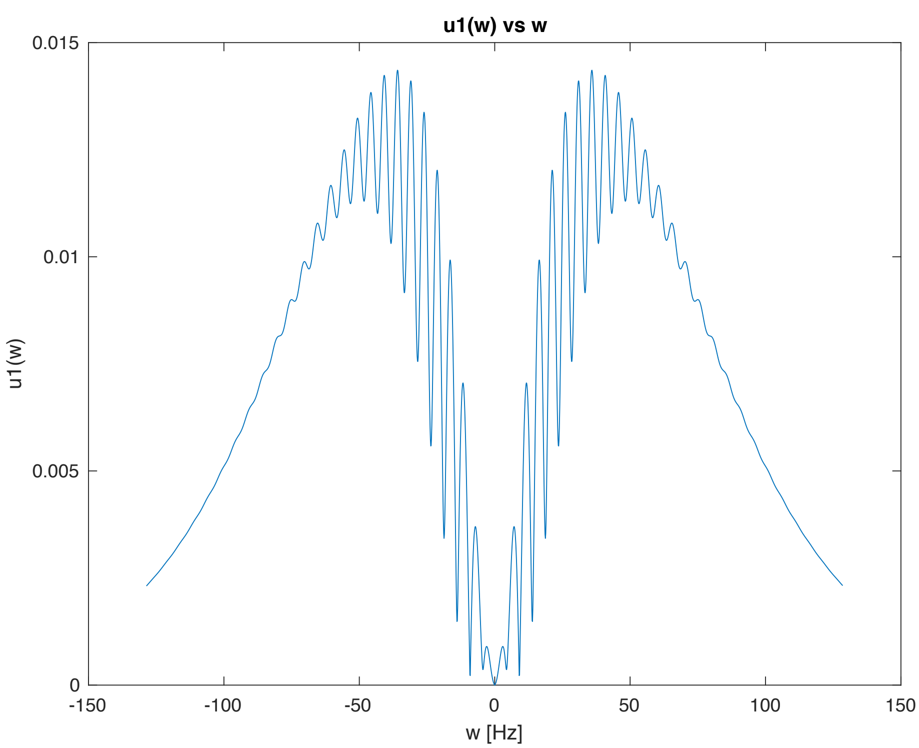


Fig.13. $\bar{u}_1(\mathbf{x},\omega)$ vs. ω
$(a = 0.02s, Q_\alpha = 40, Q_\beta = 20, \theta = \pi/6, \alpha_0 = 5\ km/s$ and $\beta_0 = 3\ km/s)$

We can find in Fig.13 that the peak is around $\omega = 40Hz$. Therefore, we should choose $\omega_{0r} = 40Hz$ (or something close to that). The significant frequency range for this graph, is around 20 to 80 *Hz*. As we talked before, if $\omega_{0r} = 40Hz$, and if we pick a frequency inside this range, e.g., $\omega = 80Hz$, then $\omega/\omega_{0r} = 2$, which is not close to 1, but the value of $|\bar{u}_1(\mathbf{x},\omega)|$ at $\omega = 40Hz$ is also much smaller than $\bar{u}_1(\mathbf{x},\omega)_{max}$, meaning it will probably have a small negative effect on the approximation, and so the approximation could still be fairly good.

Then, we used the Matlab code to compute the exact results with velocity dispersion numerically and compare the approximation and exact numerical result for different distances in Fig.14a-e. The parameters we used in the below figures are $a = 0.02s$, $\omega_{0r} = 40Hz$, $\rho = 1kg/m^3, A = 1kg * km$, $r = 0.3, 1, 2, 5, 10\ \ km$, $Q_\alpha = 40, Q_\beta = 20, \alpha_0 = 5\ km/s$ and $\beta_0 = 3\ km/s$, $\theta = \pi/6$.

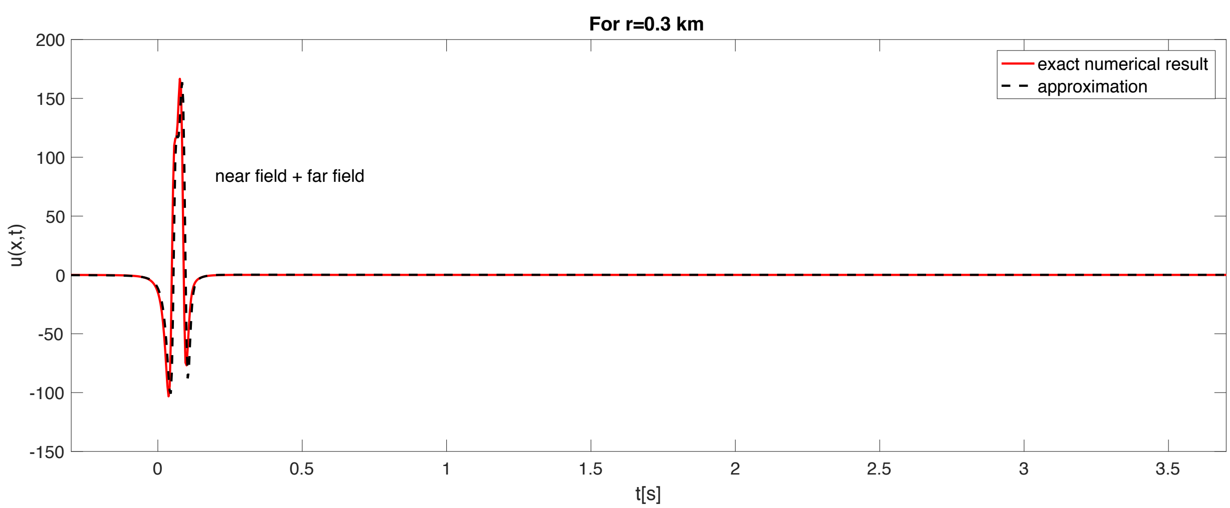


Fig.14a. Comparison of the approximation (72) and exact numerical result for the solution of the EOM for a double-couple-without-moment force with dispersion (r=0.3km)

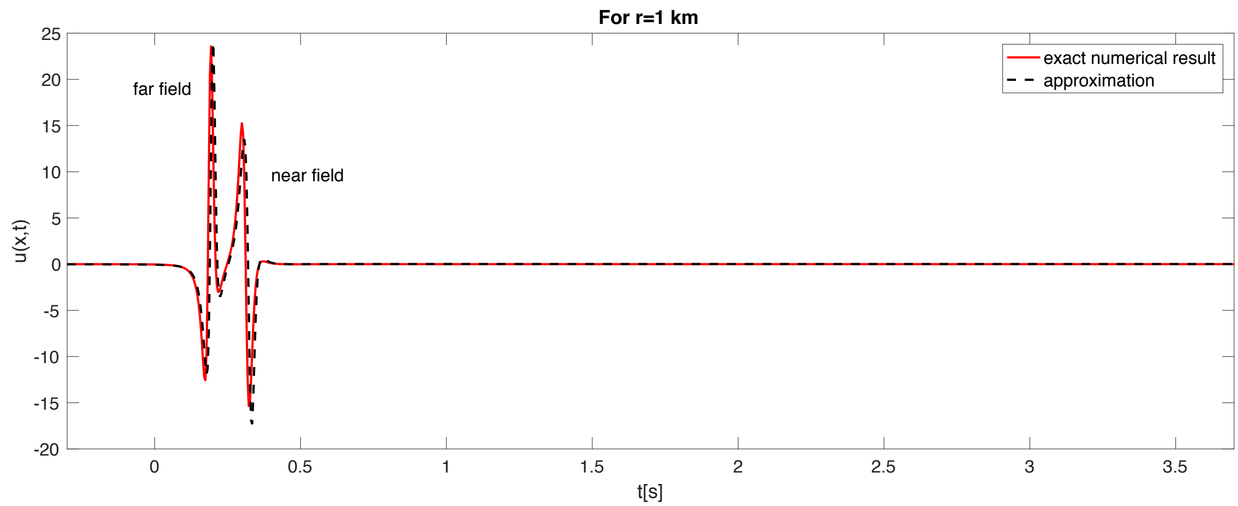


Fig.14b. Comparison of the approximation (72) and exact numerical result for the solution of the EOM for a double-couple-without-moment force with dispersion (r=1km)

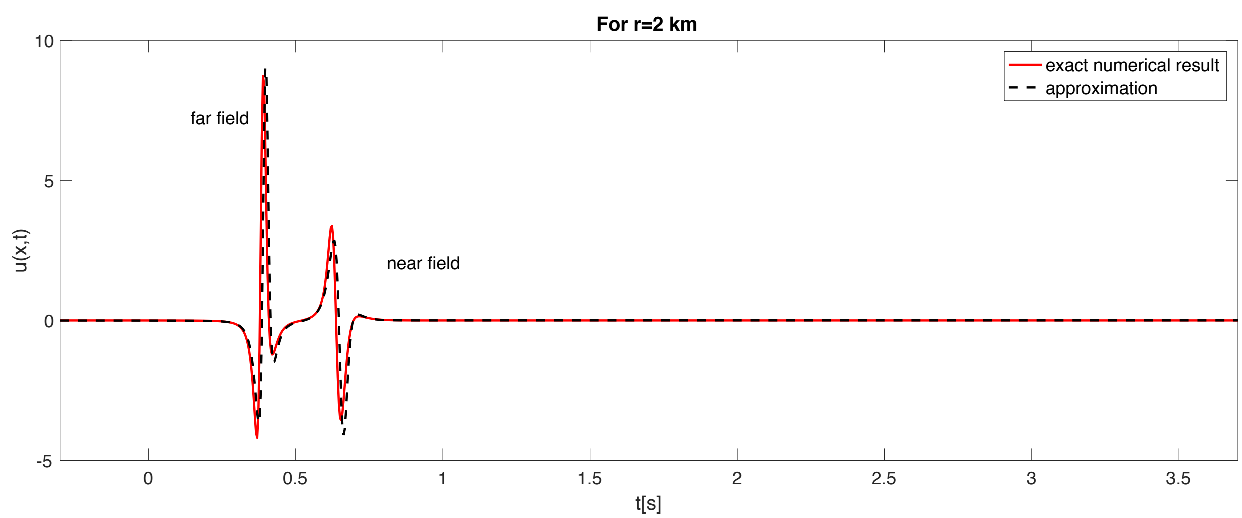


Fig.14c. Comparison of the approximation (72) and exact numerical result for the solution of the EOM for a double-couple-without-moment force with dispersion (r=2km)

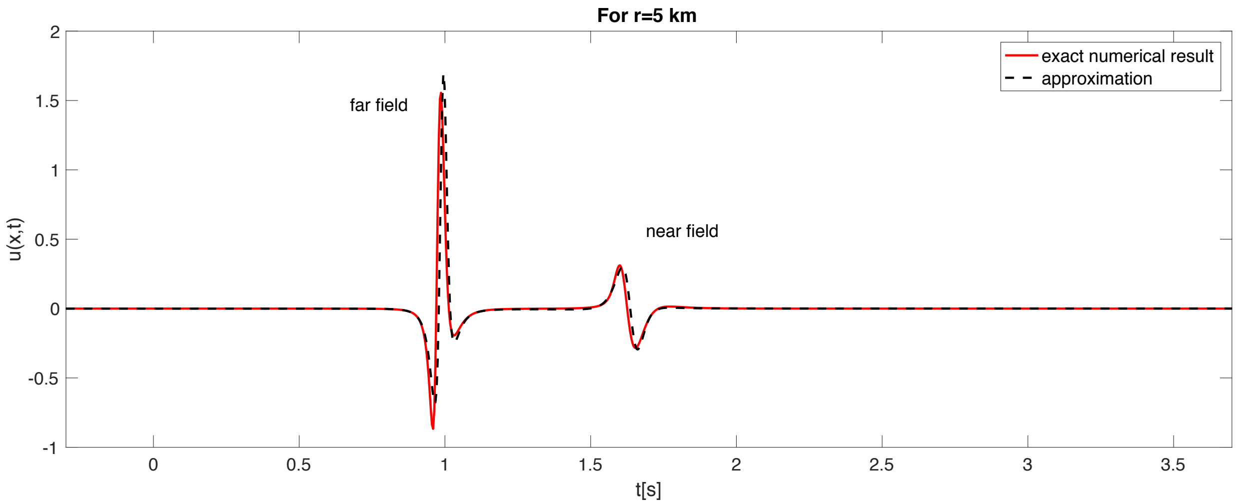


Fig.14d. Comparison of the approximation (72) and exact numerical result for the solution of the EOM for a double-couple-without-moment force with dispersion (r=5km)

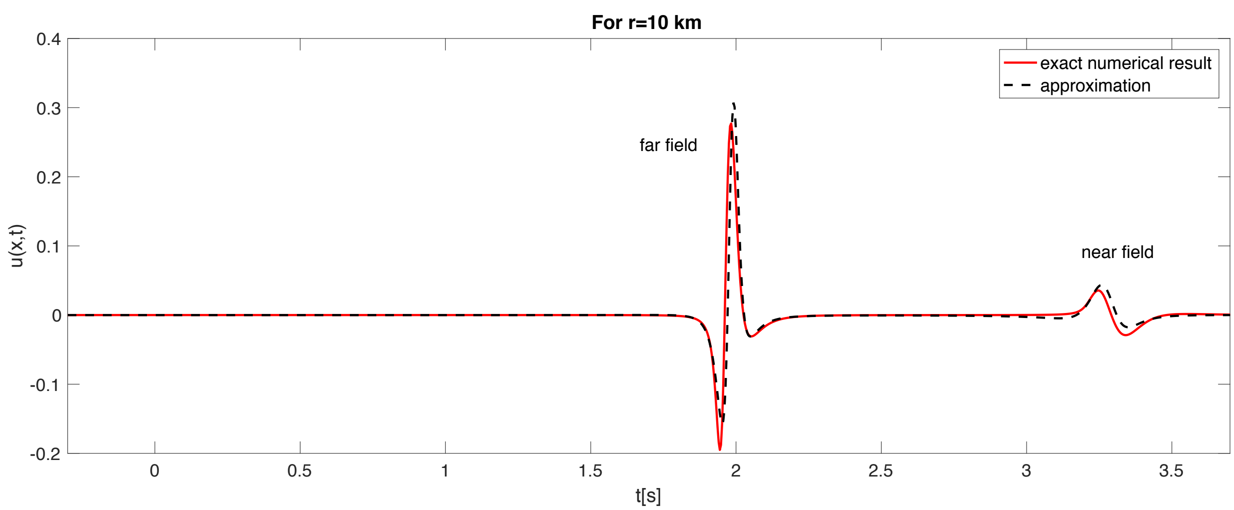


Fig.14e. Comparison of the approximation (72) and exact numerical result for the solution of the EOM for a double-couple-without-moment force with dispersion (r=10km)

The red lines indicate exact numerical results, and the black dash lines indicate the approximation (72). We can see they coincide well.

Therefore, we could say that our approximation (72) can be a good expression for the solution of the EOM for a double-couple-without-moment source including absorption and dispersion.

4.3 Compare the new solution of the EOM including absorption and dispersion with the elastic one

Then, again, let us use the same parameters with above Fig.14a-e, i.e., $a = 0.02s$, $\omega_{0r} = 40Hz$, $\rho = 1kg/m^3, A = 1kg * km$, $r = 0.3, 1, 2, 5, 10\ km$, $Q_\alpha = 40, Q_\beta = 20, \alpha_0 = 5\ km/s$ and $\beta_0 = 3\ km/s$, $\theta = \pi/6$, and see the differences between seismic waves in elastic medium (66) and in anelastic medium with absorption and dispersion effect (72).

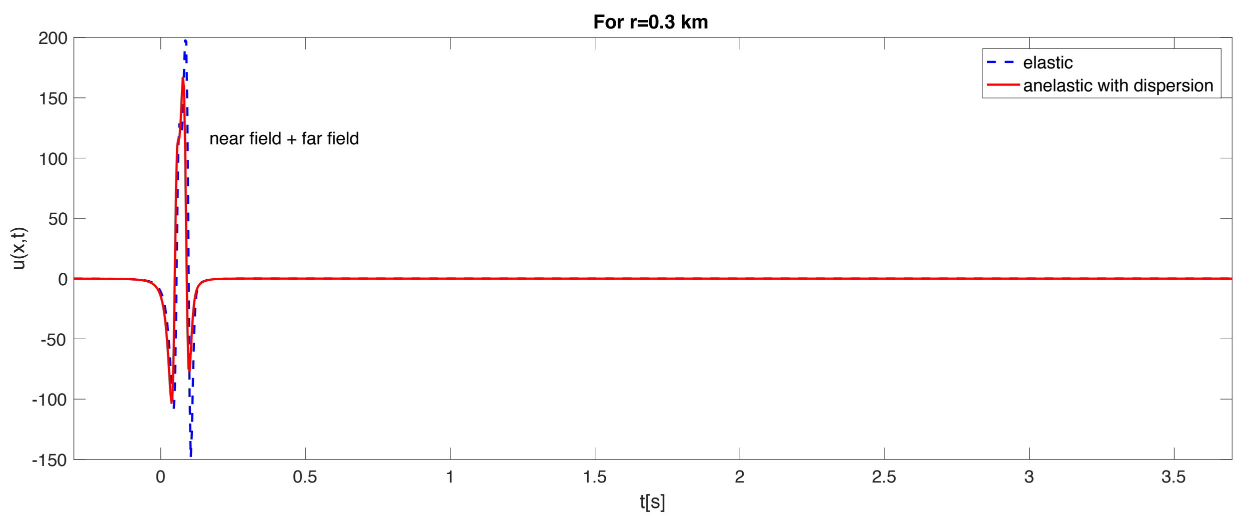


Fig.15a. Comparison of the approximate anelastic solution of the EOM for a double-couple-without-moment force with dispersion (72) and the elastic one (66) ($\boldsymbol{a = 0.02s}$, $\boldsymbol{\omega_{0r} = 40Hz}$, $\boldsymbol{\rho = 1kg/m^3, A = 1kg * km}$, $\boldsymbol{Q_\alpha = 40, Q_\beta = 20, \alpha_0 = 5\ km/s\ and\ \beta_0 = 3\ km/s, \theta = \pi/6, r = 0.3\ km}$)

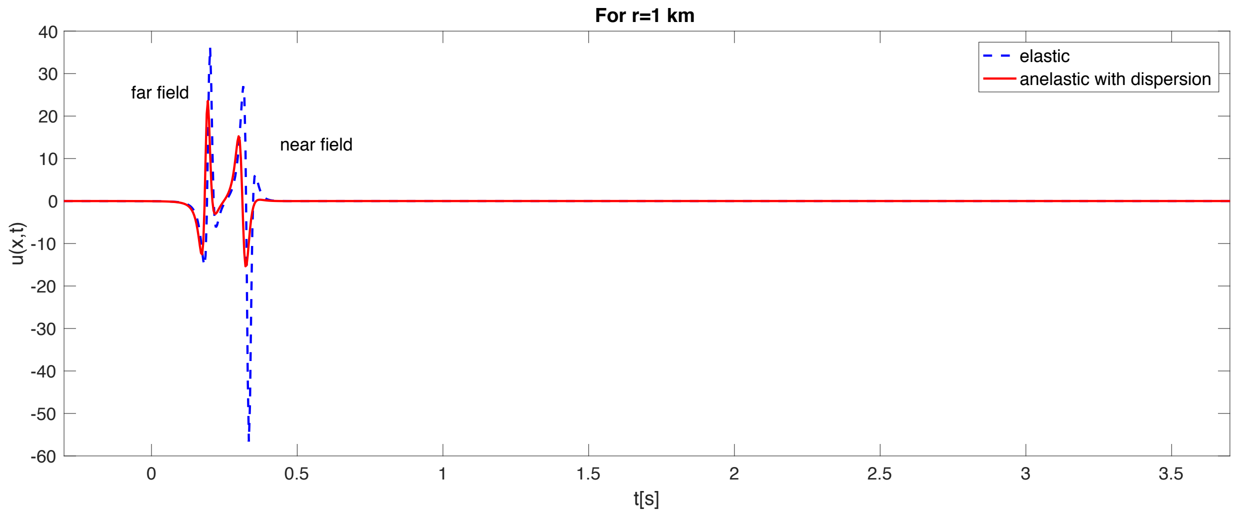


Fig.15b. Comparison of the approximate anelastic solution of the EOM for a double-couple-without-moment force with dispersion (72) and the elastic one (66) ($a = 0.02s$, $\omega_{0r} = 40Hz$, $\rho = 1kg/m^3$, $A = 1kg * km$, $Q_\alpha = 40$, $Q_\beta = 20$, $\alpha_0 = 5\ km/s$ *and* $\beta_0 = 3\ km/s$, $\theta = \pi/6$, $r = 1\ km$)

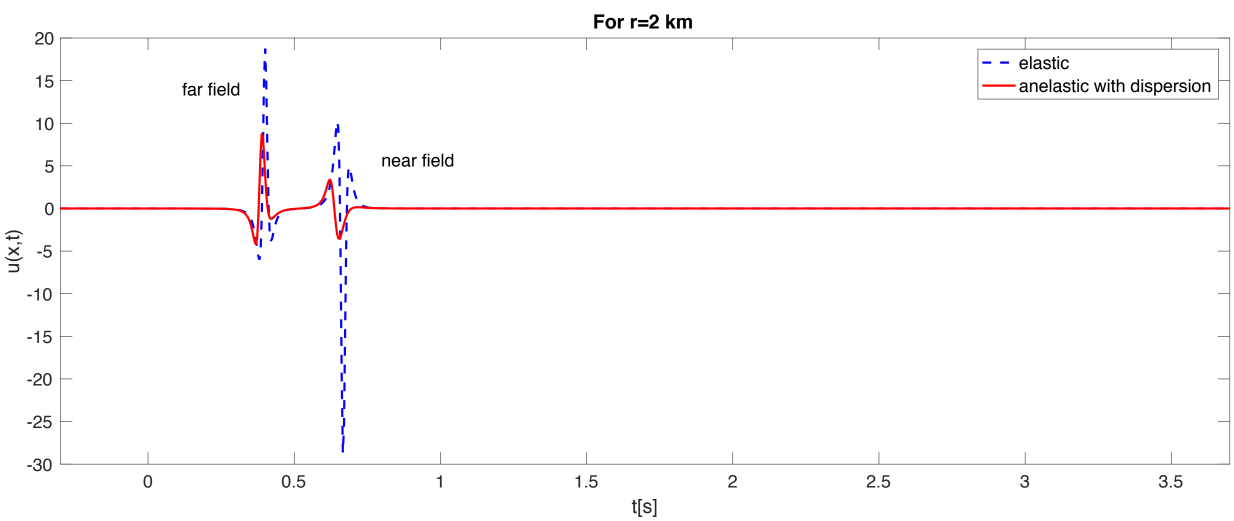


Fig.15c. Comparison of the approximate anelastic solution of the EOM for a double-couple-without-moment force with dispersion (72) and the elastic one (66) ($a = 0.02s$, $\omega_{0r} = 40Hz$, $\rho = 1kg/m^3$, $A = 1kg * km$, $Q_\alpha = 40$, $Q_\beta = 20$, $\alpha_0 = 5\ km/s$ *and* $\beta_0 = 3\ km/s$, $\theta = \pi/6$, $r = 2\ km$)

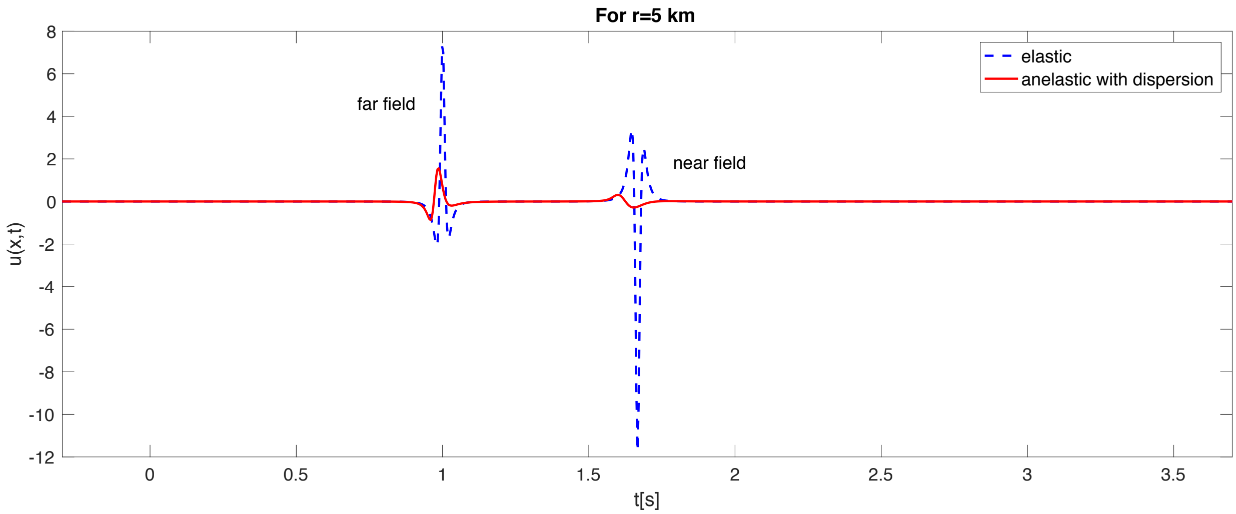


Fig.15d. Comparison of the approximate anelastic solution of the EOM for a double-couple-without-moment force with dispersion (72) and the elastic one (66) ($a = 0.02s$, $\omega_{0r} = 40Hz$, $\rho = 1kg/m^3, A = 1kg * km$, $Q_\alpha = 40, Q_\beta = 20, \alpha_0 = 5\ km/s\ and$ $\beta_0 = 3\ km/s$, $\theta = \pi/6$, $r = 5\ km$)

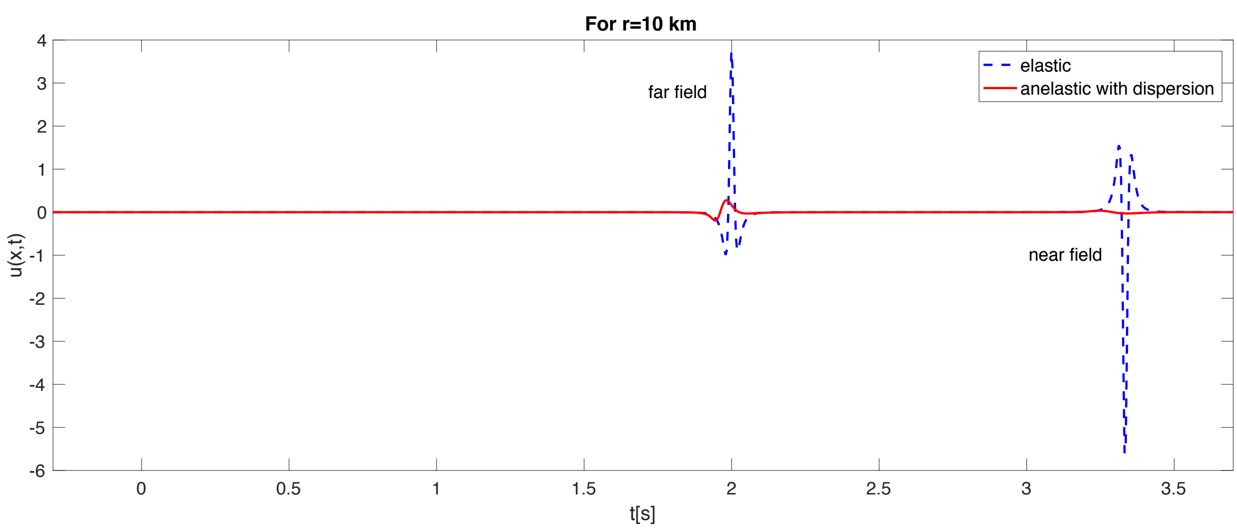


Fig.15e. Comparison of the approximate anelastic solution of the EOM for a double-couple-without-moment force with dispersion (72) and the elastic one (66) ($a = 0.02s$, $\omega_{0r} = 40Hz$, $\rho = 1kg/m^3, A = 1kg * km$, $Q_\alpha = 40, Q_\beta = 20, \alpha_0 = 5\ km/s\ and\ \beta_0 = 3\ km/s$, $\theta = \pi/6$, $r = 10\ km$)

The pulses in Fig.15d and Fig.15e are hard to see, so let us show the amplification of the pulses in the following graphs:

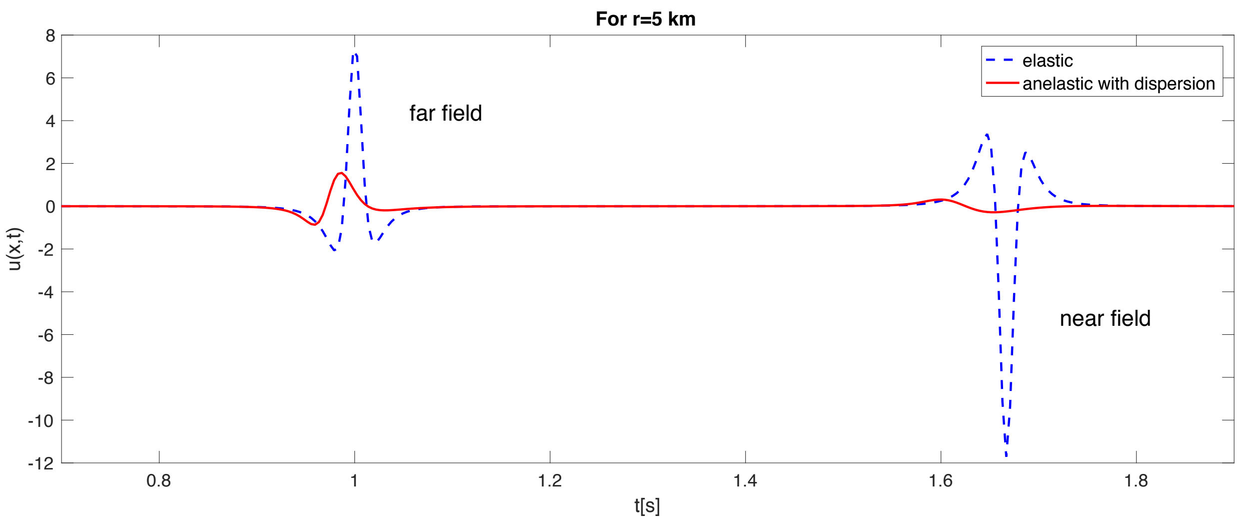


Fig.15f. Amplification of the pulses in Fig.15d

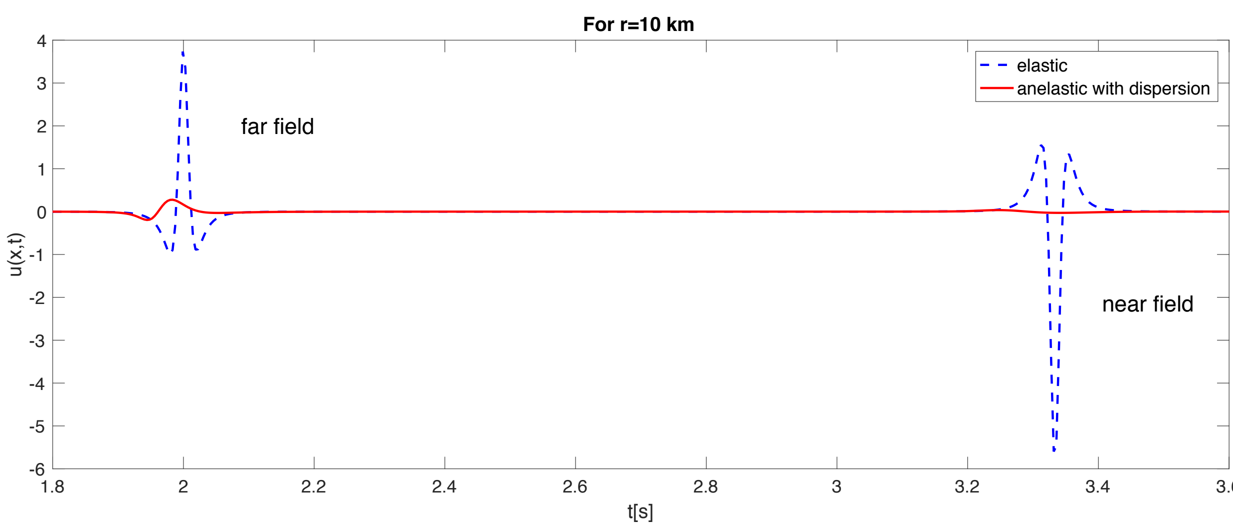


Fig.15g. Amplification of the pulses in Fig.15e

In above Fig.15a-e, the blue dash curves are the results of waveforms in elastic medium, and the red curves are anelastic waveforms including dispersion influence. The effect of absorption and dispersion becomes more and more obvious as the propagation distance increases. And, this time, not only the attenuation of amplitudes but also the changes of the waveforms can be observed easily, especially when distance is larger.

So far, we only discussed about the exploration geophysics situation, which is in the near-surface area. This should also be interesting to see what will happen in a relative deep area if we put dispersion influence into it.

For a deeper place, we will suppose the distance are $r = 20, 50, 100, 150, 200\ km$. The effect of absorption will weaken, so the quality factor Q should become bigger, then we could assume $Q_\alpha = 180, Q_\beta = 120$. To see waves more clearly, we will use a bigger a as well, e.g., $a = 1s$. And we can find from Fig.8b and Fig.8c that when $a = 1s$, ω_{0r} should be $\omega_{0r} = 1Hz$.

In general, the parameters we are going to use are $a = 1s$, $\omega_{0r} = 1Hz$, $\rho = 1kg/m^3, A = 1kg * km$, $r = 20, 50, 100, 150, 200\ km$, $Q_\alpha = 180, Q_\beta = 120, \alpha_0 = 5\ km/s$ and $\beta_0 = 3\ km/s, \theta = \pi/6$.

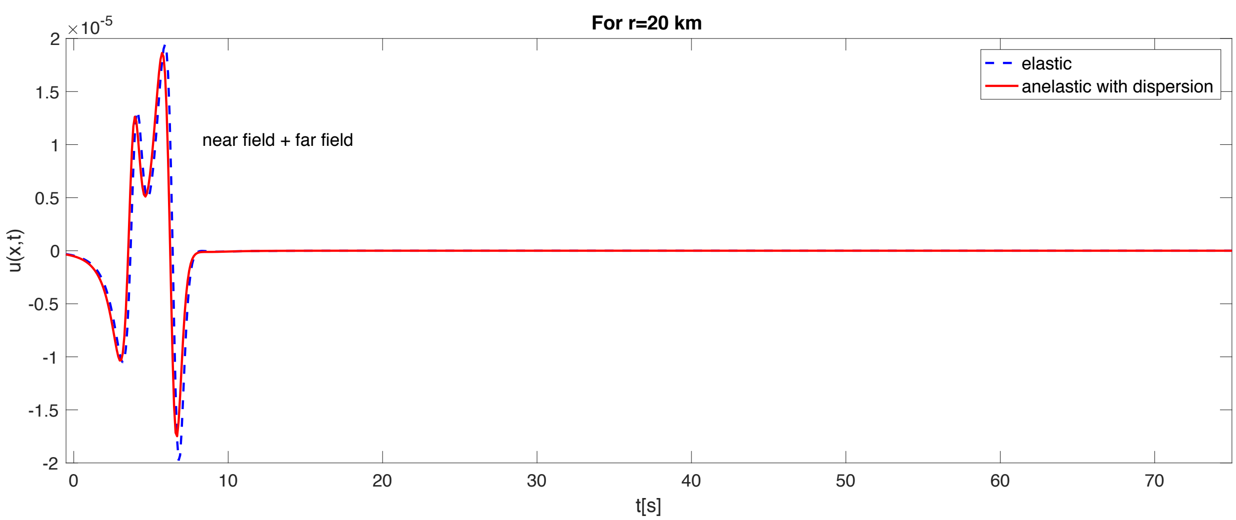


Fig.16a. Comparison of the approximate anelastic solution of the EOM for a double-couple-without-moment force with dispersion (72) and the elastic one (66) ($a = 1s$, $\omega_{0r} =$

$1Hz$, $\rho = 1kg/m^3, A = 1kg * km$, $Q_\alpha = 180, Q_\beta = 120, \alpha_0 = 5\ km/s$ *and* $\beta_0 = 3\ km/s$, $\theta = \pi/6$, $r = 20\ km$)**

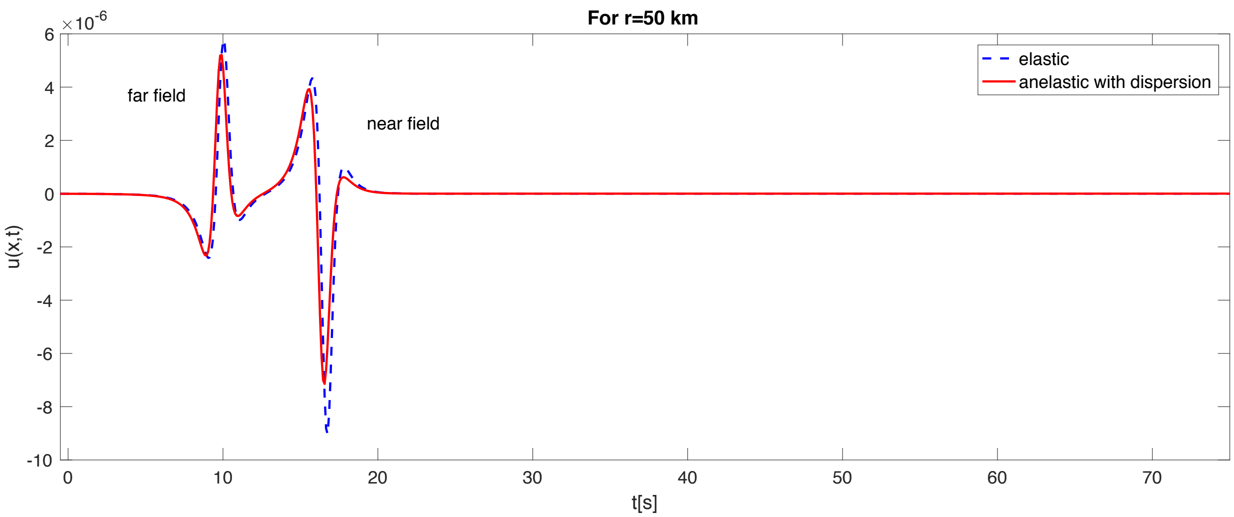


Fig.16b. Comparison of the approximate anelastic solution of the EOM for a double-couple-without-moment force with dispersion (72) and the elastic one (66) ($a = 1s$, $\omega_{0r} = 1Hz$, $\rho = 1kg/m^3, A = 1kg * km$, $Q_\alpha = 180, Q_\beta = 120, \alpha_0 = 5\ km/s$ *and* $\beta_0 = 3\ km/s$, $\theta = \pi/6$, $r = 50\ km$)

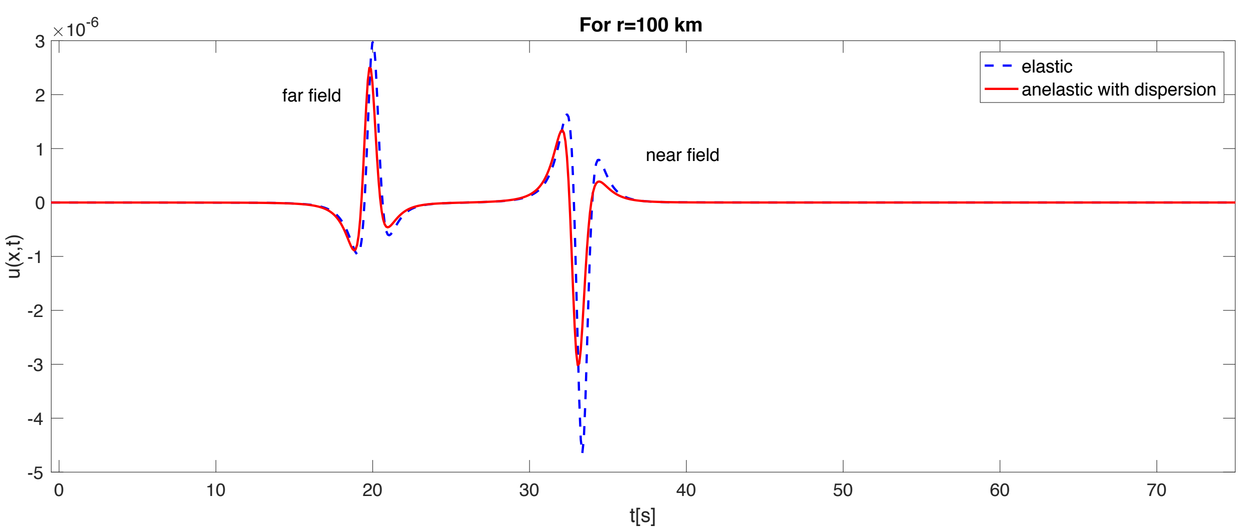


Fig.16c. Comparison of the approximate anelastic solution of the EOM for a double-couple-without-moment force with dispersion (72) and the elastic one (66) ($a = 1s$, $\omega_{0r} = 1Hz$, $\rho = 1kg/m^3$, $A = 1kg * km$, $Q_\alpha = 180$, $Q_\beta = 120$, $\alpha_0 = 5\ km/s$ and $\beta_0 = 3\ km/s$, $\theta = \pi/6$, $r = 100\ km$)

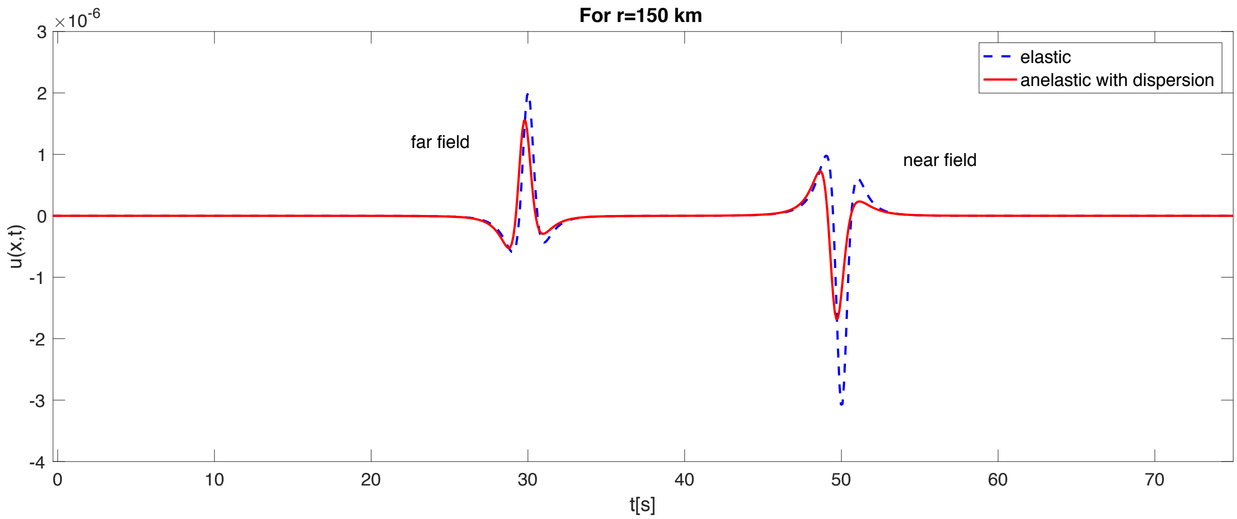


Fig.16d. Comparison of the approximate anelastic solution of the EOM for a double-couple-without-moment force with dispersion (72) and the elastic one (66) ($a = 1s$, $\omega_{0r} = 1Hz$, $\rho = 1kg/m^3$, $A = 1kg * km$, $Q_\alpha = 180$, $Q_\beta = 120$, $\alpha_0 = 5\ km/s$ and $\beta_0 = 3\ km/s$, $\theta = \pi/6$, $r = 150\ km$)

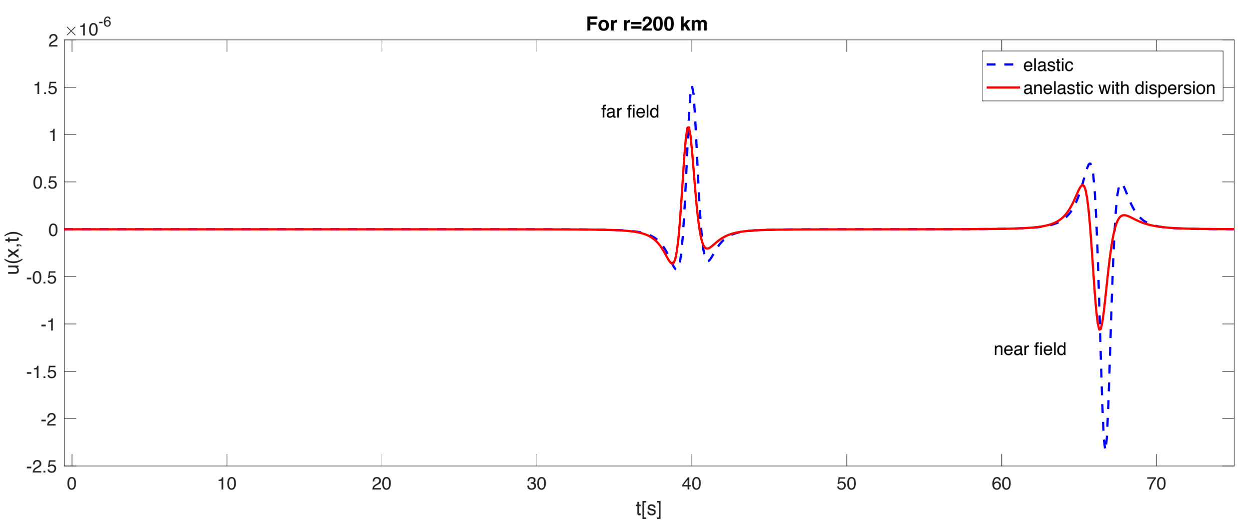


Fig.16e. Comparison of the approximate anelastic solution of the EOM for a double-couple-without-moment force with dispersion (72) and the elastic one (66) ($a = 1s$, $\omega_{0r} = 1Hz$, $\rho = 1kg/m^3$, $A = 1kg * km$, $Q_\alpha = 180$, $Q_\beta = 120$, $\alpha_0 = 5\ km/s$ and $\beta_0 = 3\ km/s$, $\theta = \pi/6$, $r = 200\ km$)

Still, the blue dash curves are the results of waveforms in elastic medium, and the red curves are anelastic waveforms including dispersion influence. The effect of absorption and dispersion becomes more and more obvious as the propagation distance increases. But in Fig.16a-e, we can see that when quality factor Q is large, even though distance is long, the attenuation and deformation of seismic waves is not that significant as when Q is small.

CHAPTER 5: A SHEAR-DISLOCATION SOURCE

In this chapter, we will only talk about the dispersion situation as well. Because to be physically realistic, the new solutions must include velocity dispersion.

Consider the situation we talked in chapter 2. If we assume the xy plane is the ground surface, and if the fault plane is the xz plane, then using $M_{jk} = \mu(\bar{u}_j v_k + \bar{u}_k v_j)A$ from Aki and Richards (2002), where the $\boldsymbol{v}$ vector is perpendicular to the fault surface, and $\bar{\boldsymbol{u}} \cdot \boldsymbol{v} = 0$, we get $v_1 = v_3 = 0$, and $\bar{u}_2 = 0$, and so we have:

$$M_{12} = M_{21} = \mu \bar{u}_1 v_2 A \quad \text{and} \quad M_{23} = M_{32} = \mu \bar{u}_3 v_2 A$$

All other M_{jk} are zero.

So $M_{12},\ M_{21},\ M_{23},\ M_{32}$ are not zero for the shear dislocation case. But with four non-zero components of the moment tensor, this could be a lot of work.

In Aki and Richards (2002, p.78), the paragraph under equation (4.30), they "choose the x-axis to be the direction of slip, so that $\bar{\boldsymbol{u}} = (\bar{u}_1, 0, 0)$". Therefore, to simplify this question, we could assume the fault lies in the (x_1, x_3) plane, i.e. $\boldsymbol{v} = (0,1,0)$, and choose the x_1 axis to be the direction of slip as well, i.e. $\bar{\boldsymbol{u}}=(\bar{u}_1,0,0)$. Then, we will have only two non-zero components of the moment tensor, $M_{12} = M_{21} = \mu \bar{u}_1 A$.

Assume the source is on origin and the receiver point is on the xy plane somewhere, so $\gamma_1{}^2 + \gamma_2{}^2 = 1$ and $\gamma_3 = 0$. Again, since $\gamma_1{}^2 + \gamma_2{}^2 = 1$ and $\gamma_3 = 0$, and $\cos\theta^2 + \sin\theta^2 = 1$, we can assume $\gamma_1 = \cos\theta \ \ and \ \gamma_2 = \sin\theta$, where θ is the angle from the x axis.

Based on this assumption, we derived the three components of the displacement on the receiver point (41a, b, and c). Here, we will add dispersion effect on the x component of the displacement, which is (41a):

$$u_1(\mathbf{x},t) = \frac{(9+15\cos 2\theta)\sin\theta}{4\pi\rho}\frac{1}{r^4}\mu A\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau\bar{u}_1(t-\tau)\,d\tau$$

$$+\frac{(2+3\cos 2\theta)\sin\theta}{2\pi\rho\alpha^2}\frac{1}{r^2}\mu A\bar{u}_1\left(t-\frac{r}{\alpha}\right)$$

$$-\frac{(3+6\cos 2\theta)\sin\theta}{4\pi\rho\beta^2}\frac{1}{r^2}\mu A\bar{u}_1\left(t-\frac{r}{\beta}\right)$$

$$+\frac{(1+\cos 2\theta)\sin\theta}{4\pi\rho\alpha^3}\frac{1}{r}\mu A\bar{u}_1{}'\left(t-\frac{r}{\alpha}\right)-\frac{cos\,2\theta\,sin\,\theta}{4\pi\rho\beta^3}\frac{1}{r}\mu A\bar{u}_1{}'\left(t-\frac{r}{\beta}\right) \quad (41a)$$

To make the calculations easier, we still assume source $M_0(t) = M_{12} = M_{21} = s(t) = \frac{2atB}{\pi(a^2+t^2)^2}$, and $\overline{M_0}(\omega) = \bar{s}(\omega) = Bi\omega e^{-a|\omega|}$ (we turned the constant A into B here to distinguish the constant A from the area A we used in (41a)). And according to the definition $M_0(t) = M_{12} = M_{21} = \mu\bar{u}_1 v_2 A$, we will have:

$$\bar{u}_1 = \frac{1}{\mu A}M_0(t) = \frac{2atB}{\mu A\pi(a^2+t^2)^2} \quad (73)$$

Substitute (73) into (41a),

$$u_1(\mathbf{x},t) = \frac{(9+15\cos 2\theta)\sin\theta}{4\pi\rho}\frac{1}{r^4}\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau M_0(t-\tau)\,d\tau$$

$$+\frac{(2+3\cos 2\theta)\sin\theta}{2\pi\rho\alpha^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\alpha}\right)$$

$$-\frac{(3+6\cos 2\theta)\sin\theta}{4\pi\rho\beta^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\beta}\right)$$

$$+\frac{(1+\cos 2\theta)\sin\theta}{4\pi\rho\alpha^3}\frac{1}{r}M_0{}'\left(t-\frac{r}{\alpha}\right)-\frac{cos\,2\theta\,sin\,\theta}{4\pi\rho\beta^3}\frac{1}{r}M_0{}'\left(t-\frac{r}{\beta}\right) \quad (74)$$

And the x-component displacement $u_1(\mathbf{x},t)$ (74) is exactly the same result with what I got from the case of a double-couple-without-moment forces (38a).

$$u_1(\mathbf{x},t) = \frac{(9+15\cos 2\theta)\sin\theta}{4\pi\rho}\frac{1}{r^4}\int_{\frac{r}{\alpha}}^{\frac{r}{\beta}} \tau M_0(t-\tau)\,d\tau$$

$$+\frac{(2+3\cos 2\theta)\sin\theta}{2\pi\rho\alpha^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\alpha}\right)-\frac{(3+6\cos 2\theta)\sin\theta}{4\pi\rho\beta^2}\frac{1}{r^2}M_0\left(t-\frac{r}{\beta}\right)$$

$$+\frac{(1+\cos 2\theta)\sin\theta}{4\pi\rho\alpha^3}\frac{1}{r}\dot{M}_0\left(t-\frac{r}{\alpha}\right)-\frac{cos\,2\theta\,sin\,\theta}{4\pi\rho\beta^3}\frac{1}{r}\dot{M}_0\left(t-\frac{r}{\beta}\right) \tag{38a}$$

If we review the assumptions we made in chapter 4 and chapter 5, we can see that, in both of them, we assumed a vertical fault coinciding with the *xz* plane, and the only non-zero components of the moment tensor are then $M_{12} = M_{21}$. Therefore, basically, under these assumptions, what we did in this chapter is identical to what we did for a double-couple-without-moment force in chapter 4. We do not need to do the whole works one more time. If we let M_{12}, M_{21}, M_{23}, M_{32} all be non-zero components of the moment tensor, then the answer should be quite different. But if so, the calculations could even be much more complicated than what we did in chapter 4.

CHAPTER 6: CONCLUSION

In real materials, wave energy is absorbed due to internal friction or anelasticity. Absorption is frequency dependent, i.e., different frequencies are absorbed by different amounts. One consequence of this is that the waveform changes with distance travelled. And, anelasticity of the earth causes physical dispersion of seismic waves. So, to be physically realistic, one must also include dispersion in the calculations, to ensure causality. Generally, an absorbing medium is dispersive, i.e., $V = V(\omega)$ and $Q = Q(\omega)$. And for seismic body waves, Q is nearly independent of frequency.

To obtain new solutions of the equation of motion for some common source (such as a directed point force, a double-couple-without-moment source, a shear-dislocation source and so on) with absorption or dispersion effect, we carried out the following derivation procedure:

1. Select the suitable solutions of the EOM in perfect elastic medium (all the solutions I used are all from Aki and Richard (2002)).

2. Do Fourier transform of the solutions of the EOM without absorption to convert it into frequency domain.

3. Replace v with $v_0\left(1-\frac{i}{2Q}\right)$ to involve absorption effect, or replace v with $v(\omega_{0r})\left[1+\frac{1}{\pi Q}ln\left(\frac{\omega}{\omega_{0r}}\right)-\frac{i}{2Q}\right]$ to involve absorption and dispersion effect.

4. Do inverse Fourier transform (IFT) of the new solutions of the EOM to convert it back into time domain. This step gives solutions with Q.

5. Create programs by Matlab to compute the exact results with velocity dispersion numerically to verify the accuracy of our approximations.

The new solutions I got have been showed in (52b) (only absorption, for a directed point force), (61) (absorption and dispersion, for a directed point force), and (72) (absorption and dispersion, for a double-couple-without-moment force).

Equation (52b) is a new and exact result, and that even though this new solution without dispersion, it could be applied in cases where absorption (and therefore dispersion) is small, to estimate the effect of absorption. On the other hand, (61) and (72) are new approximate results with absorption and dispersion. Although they are approximations, approximations are still useful in that they can be used to study which factors are important and also to sometimes reduce computation time.

The following table summarizes our solutions.

Solution	Equation	Comments	Elastic	Absorption	Dispersion	Exact	Approximate
Point force directed in x direction	(48)	Solution along x axis	✓			✓	
	(52b)			✓		✓	
	(61)			✓	✓		✓
double-couple-without-moment force causing a vertical fault in the xz plane; shear-dislocation source	(66)	$\boldsymbol{M_{12}} = \boldsymbol{M_{21}} = \boldsymbol{M_0}$, others zero	✓			✓	
	(72)			✓	✓		✓

Table.2 Summary of derived equations

After plotting and comparing waveforms of the new solutions of the EOM with absorption and dispersion effect and the elastic ones, we found that the effect of absorption and dispersion becomes more and more obvious as the propagation distance increases. And for no dispersion, the amplitude and shape of anelastic waveforms usually didn't change that significant as when include dispersion. We also verified that when quality factor Q is large, even though distance is long, the attenuation and deformation of seismic waves is not that significant as when Q is small.

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APPENDIX A: MATLAB CODE

This code has been used to plot Fig.5, Fig.10, Fig.11, Fig.12, Fig.14, Fig.15, and Fig.16. In the code, the parameter a1 represents a, w0 indicates the angular frequency ω_{0r}, rho is density ρ, theta is angle θ, rr means distance r, a is P-wave velocity α_0 and b is S-wave velocity β_0, and Qa is Q_α, Qb is Q_β.

Also, Hk in the following code could be the solution of the EOM including dispersion effect in frequency domain for a directed point force or a double-couple-without-moment source. Hifft is the exact numerical result which derived from Hk by doing IFFT. Besides that, point_appro and double_appro are the approximate anelasitc solution of the EOM with dispersion in time domain for a directed point force and a double-couple-without-moment source respectively. While, point_elastic and double_elastic show the elastic solution of the EOM for both of them. And, point_nodisper is the solution of the EOM for a directed point force which only includes influence of absorption.

The code is showed from here:

```
clear;

clc;


xhigh=4; xmin=-4; N=2048; dx = (xhigh-xmin)/(N-1); No2 = N/2;

kNyq = 1/(2*dx); dk = 1/(N*dx);

x = [xmin:dx:xhigh]';

t=x;
```

```
a1=0.02; A=1; rho=1; Qa=40; Qb=20; a=5; b=3; w0=20; theta=pi/6;
k = linspace((-kNyq)*2*pi,(kNyq-dk)*2*pi,N);
rr=[0.3,1,2,5,10];

for i=1:5
    r=rr(i);
    for n=1:N
    if (k(n)>0.01 && k(n)<1e05)|| (k(n)<-0.01 && k(n)>-1e05)
        %a directed point source
        Hk(n) = A./2./pi./rho./r.^3.*((1i./k(n)+r./b.*(1-
1./pi./Qb.*log(k(n)./w0)+1i./2./Qb)).*exp(1i.*k(n).*r./b.*(1-1./pi./Qb.*log(k(n)./w0)+1i./2./Qb))-
(1i./k(n)+r./a.*(1-1./pi./Qa.*log(k(n)./w0)+1i./2./Qa)).*exp(1i.*k(n).*r./a.*(1-
1./pi./Qa.*log(k(n)./w0)+1i./2./Qa))).*exp(-a1.*abs(k(n)))+A.*1i.*k(n)./4./pi./rho./a.^2./r.*(1-
2./pi./Qa.*log(k(n)./w0)+1i./Qa).*exp(1i.*k(n).*r./a.*(1-
1./pi./Qa.*log(k(n)./w0)+1i./2./Qa)).*exp(-a1.*abs(k(n)));

        %a double-couple without movement forces
%        Hk(n)=(9+15.*cos(2.*theta)).*sin(theta)./4./pi./rho.*A./r.^4.*exp(-
a1.*abs(k(n))).*...
%            ((1i./k(n)+r./b.*(1-1./pi./Qb.*log(k(n)./w0)+1i./2./Qb)).*exp(1i.*k(n).*r./b.*(1-
1./pi./Qb.*log(k(n)./w0)+1i./2./Qb))...
%            -(1i./k(n)+r./a.*(1-1./pi./Qa.*log(k(n)./w0)+1i./2./Qa)).*exp(1i.*k(n).*r./a.*(1-
1./pi./Qa.*log(k(n)./w0)+1i./2./Qa)))...
```

```
%           +(2+3.*cos(2.*theta)).*sin(theta)./2./pi./rho.*A./r.^2.*...
%           1./a.^2.*(1-2./pi./Qa.*log(k(n)./w0)+1i./Qa).*1i.*k(n).*exp(-
a1.*abs(k(n))).*exp(1i.*k(n).*r./a.*(1-1./pi./Qa.*log(k(n)./w0)+1i./2./Qa))...
%           -(3+6.*cos(2.*theta)).*sin(theta)./4./pi./rho.*A./r.^2.*...
%           1./b.^2.*(1-2./pi./Qb.*log(k(n)./w0)+1i./Qb).*1i.*k(n).*exp(-
a1.*abs(k(n))).*exp(1i.*k(n).*r./b.*(1-1./pi./Qb.*log(k(n)./w0)+1i./2./Qb))...
%           +(1+cos(2.*theta)).*sin(theta)./4./pi./rho.*A./r.*...
%           1./a.^3.*(1-3./pi./Qa.*log(k(n)./w0)+3.*1i./2./Qa).*k(n).^2.*exp(-
a1.*abs(k(n))).*exp(1i.*k(n).*r./a.*(1-1./pi./Qa.*log(k(n)./w0)+1i./2./Qa))...
%           -cos(2.*theta).*sin(theta)./4./pi./rho.*A./r.*...
%           1./b.^3.*(1-3./pi./Qb.*log(k(n)./w0)+3.*1i./2./Qb).*k(n).^2.*exp(-
a1.*abs(k(n))).*exp(1i.*k(n).*r./b.*(1-1./pi./Qb.*log(k(n)./w0)+1i./2./Qb));

    else
          %a direct point force
          Hk(n)=1./2./pi./rho./r.^3.*((j./k(n)+r./b).*exp(j.*k(n).*r./b)-
(j./k(n)+r./a).*exp(j.*k(n).*r./a)).*exp(-a1.*abs(k(n)))+j.*k(n)./4./pi./a.^2./r.*exp(-
a1.*abs(k(n))).*exp(j.*k(n).*r./a);

          %double-couple-without-moment
%           Hk(n)=(9+15.*cos(2.*theta)).*sin(theta)./4./pi./rho.*A./r.^4.*exp(-
a1.*abs(k(n))).*((1i./k(n)+r./b).*exp(1i.*k(n).*r./b)-(1i./k(n)+r./a).*exp(1i.*k(n).*r./a))...
```

```
%           +(2+3.*cos(2.*theta)).*sin(theta)./2./pi./rho./a.^2.*A./r.^2.*1i.*k(n).*exp(-a1.*abs(k(n))).*exp(1i.*k(n).*r./a)...
%           -(3+6.*cos(2.*theta)).*sin(theta)./4./pi./rho./b.^2.*A./r.^2.*1i.*k(n).*exp(-a1.*abs(k(n))).*exp(1i.*k(n).*r./b)...
%           +(1+cos(2.*theta)).*sin(theta)./4./pi./rho./a.^3.*A./r.*k(n).^2.*exp(-a1.*abs(k(n))).*exp(1i.*k(n).*r./a)...
%           -cos(2.*theta).*sin(theta)./4./pi./rho./b.^3.*A./r.*k(n).^2.*exp(-a1.*abs(k(n))).*exp(1i.*k(n).*r./b);
    end
end

Hk(k==0)=0;
Hifft = N*dk*ifft(ifftshift(Hk));
Hifft=Hifft(N:-1:1);
Hifftp = Hifft(1:No2+1); % the N/2 + 1 values for k >= 0 in the DFT.
Hifftn = Hifft(No2+2:N); % the N/2 - 1 values for k <  0 in the DFT.
Hifft = [Hifftn, Hifftp];

point_appro=A./2./pi.^2./rho./r.^3.*...
    (-(cos(r.*w0./pi./Qb./b)+r.*w0./pi./Qb./b.*sin(r.*w0./pi./Qb./b)).*atan((r./b-r./pi./Qb./b-x)./(r./2./Qb./b+a1))...
    +(cos(r.*w0./pi./Qa./a)+r.*w0./pi./Qa./a.*sin(r.*w0./pi./Qa./a)).*atan((r./a-r./pi./Qa./a-x)./(r./2./Qa./a+a1))+3.75*10^(-5))...
```

```
            +A./2./pi.^2./rho./r.^3.*...
            ((r./b.*(1-1./pi./Qb).*cos(r.*w0./pi./Qb./b)-
r./2./Qb./b.*sin(r.*w0./pi./Qb./b)).*(r./2./Qb./b+a1)./((r./2./Qb./b+a1).^2+(r./b-r./pi./Qb./b-
x).^2)...
            -(r./b.*(1-1./pi./Qb).*sin(r.*w0./pi./Qb./b)+r./2./Qb./b.*cos(r.*w0./pi./Qb./b)).*(r./b-
r./pi./Qb./b-x)./((r./2./Qb./b+a1).^2+(r./b-r./pi./Qb./b-x).^2)...
            -(r./a.*(1-1./pi./Qa).*cos(r.*w0./pi./Qa./a)-
r./2./Qa./a.*sin(r.*w0./pi./Qa./a)).*(r./2./Qa./a+a1)./((r./2./Qa./a+a1).^2+(r./a-r./pi./Qa./a-x).^2)...
            +(r./a.*(1-1./pi./Qa).*sin(r.*w0./pi./Qa./a)+r./2./Qa./a.*cos(r.*w0./pi./Qa./a)).*(r./a-
r./pi./Qa./a-x)./((r./2./Qa./a+a1).^2+(r./a-r./pi./Qa./a-x).^2))...
            +A./4./pi.^2./rho./r.*...
            (-(2-4./pi./Qa).*cos(r.*w0./pi./Qa./a)./a.^2./((r./2./Qa./a+a1).^2+(r./a-r./pi./Qa./a-
x).^2).^2.*(r./2./Qa./a+a1).*(r./a-r./pi./Qa./a-x)...
            -(1-2./pi./Qa).*sin(r.*w0./pi./Qa./a)./a.^2./((r./2./Qa./a+a1).^2+(r./a-r./pi./Qa./a-
x).^2).^2.*((r./2./Qa./a+a1).^2-(r./a-r./pi./Qa./a-x).^2)...
            -1./a.^2./Qa.*cos(r.*w0./pi./Qa./a)./((r./2./Qa./a+a1).^2+(r./a-r./pi./Qa./a-
x).^2).^2.*((r./2./Qa./a+a1).^2-(r./a-r./pi./Qa./a-x).^2)...
            +2./a.^2./Qa.*sin(r.*w0./pi./Qa./a)./((r./2./Qa./a+a1).^2+(r./a-r./pi./Qa./a-
x).^2).^2.*(r./2./Qa./a+a1).*(r./a-r./pi./Qa./a-x))...
            -A./2./pi.^3./rho./r.*...
            w0./a.^2./Qa.*(cos(r.*w0./pi./Qa./a).*(r./a-r./pi./Qa./a-
x)+sin(r.*w0./pi./Qa./a).*(r./2./Qa./a+a1))./((r./2./Qa./a+a1).^2+(r./a-r./pi./Qa./a-x).^2);
```

% double_appro=A.*(9+15.*cos(2.*theta)).*sin(theta)./4./pi.^2./rho./r.^4.*...

% (-(cos(r.*w0./pi./Qb./b)+r.*w0./pi./Qb./b.*sin(r.*w0./pi./Qb./b)).*atan((r./b-r./pi./Qb./b-t)./(r./2./Qb./b+a1))...

% +(r./b.*(1-1./pi./Qb).*cos(r.*w0./pi./Qb./b)-r./2./Qb./b.*sin(r.*w0./pi./Qb./b)).*(r./2./Qb./b+a1)./((r./2./Qb./b+a1).^2+(r./b-r./pi./Qb./b-t).^2)...

% -(r./b.*(1-1./pi./Qb).*sin(r.*w0./pi./Qb./b)+r./2./Qb./b.*cos(r.*w0./pi./Qb./b)).*(r./b-r./pi./Qb./b-t)./((r./2./Qb./b+a1).^2+(r./b-r./pi./Qb./b-t).^2)...

% +(cos(r.*w0./pi./Qa./a)+r.*w0./pi./Qa./a.*sin(r.*w0./pi./Qa./a)).*atan((r./a-r./pi./Qa./a-t)./(r./2./Qa./a+a1))...

% -(r./a.*(1-1./pi./Qa).*cos(r.*w0./pi./Qa./a)-r./2./Qa./a.*sin(r.*w0./pi./Qa./a)).*(r./2./Qa./a+a1)./((r./2./Qa./a+a1).^2+(r./a-r./pi./Qa./a-t).^2)...

% +(r./a.*(1-1./pi./Qa).*sin(r.*w0./pi./Qa./a)+r./2./Qa./a.*cos(r.*w0./pi./Qa./a)).*(r./a-r./pi./Qa./a-t)./((r./2./Qa./a+a1).^2+(r./a-r./pi./Qa./a-t).^2))...

% +A.*(2+3.*cos(2.*theta)).*sin(theta)./2./pi.^2./rho./r.^2.*...

% ((1./a.^2./Qa.*sin(r.*w0./pi./Qa./a)-(1-2./pi./Qa).*cos(r.*w0./pi./Qa./a)./a.^2).*2.*(r./2./Qa./a+a1).*(r./a-r./pi./Qa./a-t)./((r./2./Qa./a+a1).^2+(r./a-r./pi./Qa./a-t).^2).^2 ...

% -(1./a.^2./Qa.*cos(r.*w0./pi./Qa./a)+(1-2./pi./Qa).*sin(r.*w0./pi./Qa./a)./a.^2).*((r./2./Qa./a+a1).^2-(r./a-r./pi./Qa./a-t).^2)./((r./2./Qa./a+a1).^2+(r./a-r./pi./Qa./a-t).^2).^2 ...

% -2.*w0./pi./Qa./a.^2.*cos(r.*w0./pi./Qa./a).*(r./a-r./pi./Qa./a-t)./((r./2./Qa./a+a1).^2+(r./a-r./pi./Qa./a-t).^2)...

% -

2.*w0./pi./Qa./a.^2.*sin(r.*w0./pi./Qa./a).*(r./2./Qa./a+a1)./((r./2./Qa./a+a1).^2+(r./a-r./pi./Qa./a-t).^2))...

% -A.*(3+6.*cos(2.*theta)).*sin(theta)./4./pi.^2./rho./r.^2.*...

% ((1./b.^2./Qb.*sin(r.*w0./pi./Qb./b)-(1-2./pi./Qb).*cos(r.*w0./pi./Qb./b)./b.^2).*2.*(r./2./Qb./b+a1).*(r./b-r./pi./Qb./b-t)./((r./2./Qb./b+a1).^2+(r./b-r./pi./Qb./b-t).^2).^2 ...

% -(1./b.^2./Qb.*cos(r.*w0./pi./Qb./b)+(1-2./pi./Qb).*sin(r.*w0./pi./Qb./b)./b.^2).*((r./2./Qb./b+a1).^2-(r./b-r./pi./Qb./b-t).^2)./((r./2./Qb./b+a1).^2+(r./b-r./pi./Qb./b-t).^2).^2 ...

% -2.*w0./pi./Qb./b.^2.*cos(r.*w0./pi./Qb./b).*(r./b-r./pi./Qb./b-t)./((r./2./Qb./b+a1).^2+(r./b-r./pi./Qb./b-t).^2)...

% -

2.*w0./pi./Qb./b.^2.*sin(r.*w0./pi./Qb./b).*(r./2./Qb./b+a1)./((r./2./Qb./b+a1).^2+(r./b-r./pi./Qb./b-t).^2))...

% +A.*(1+cos(2.*theta)).*sin(theta)./4./pi.^2./rho./r.*...

% (3.*w0./pi./Qa./a.^3.*cos(r.*w0./pi./Qa./a).*((r./2./Qa./a+a1).^2-(r./a-r./pi./Qa./a-t).^2)./((r./2./Qa./a+a1).^2+(r./a-r./pi./Qa./a-t).^2).^2 ...

% -3.*w0./pi./Qa./a.^3.*sin(r.*w0./pi./Qa./a).*2.*(r./2./Qa./a+a1).*(r./a-r./pi./Qa./a-t)./((r./2./Qa./a+a1).^2+(r./a-r./pi./Qa./a-t).^2).^2 ...

% +((1-3./pi./Qa).*cos(r.*w0./pi./Qa./a)./a.^3-3.*sin(r.*w0./pi./Qa./a)./2./a.^3./Qa).*(2.*(r./2./Qa./a+a1).^3-6.*(r./2./Qa./a+a1).*(r./a-r./pi./Qa./a-t).^2)./((r./2./Qa./a+a1).^2+(r./a-r./pi./Qa./a-t).^2).^3 ...

```
% -((1-
3./pi./Qa).*sin(r.*w0./pi./Qa./a)./a.^3+3.*cos(r.*w0./pi./Qa./a)./2./a.^3./Qa).*(6.*(r./2./Qa./a+a1)
.^2.*(r./a-r./pi./Qa./a-t)-2.*(r./a-r./pi./Qa./a-t).^3)./((r./2./Qa./a+a1).^2+(r./a-r./pi./Qa./a-t).^2).^3)
...
% -A.*cos(2.*theta).*sin(theta)./4./pi.^2./rho./r.*...
% (3.*w0./pi./Qb./b.^3.*cos(r.*w0./pi./Qb./b).*((r./2./Qb./b+a1).^2-(r./b-r./pi./Qb./b-
t).^2)./((r./2./Qb./b+a1).^2+(r./b-r./pi./Qb./b-t).^2).^2 ...
% -3.*w0./pi./Qb./b.^3.*sin(r.*w0./pi./Qb./b).*2.*(r./2./Qb./b+a1).*(r./b-r./pi./Qb./b-
t)./((r./2./Qb./b+a1).^2+(r./b-r./pi./Qb./b-t).^2).^2 ...
% +((1-3./pi./Qb).*cos(r.*w0./pi./Qb./b)./b.^3-
3.*sin(r.*w0./pi./Qb./b)./2./b.^3./Qb).*(2.*(r./2./Qb./b+a1).^3-6.*(r./2./Qb./b+a1).*(r./b-
r./pi./Qb./b-t).^2)./((r./2./Qb./b+a1).^2+(r./b-r./pi./Qb./b-t).^2).^3 ...
% -((1-
3./pi./Qb).*sin(r.*w0./pi./Qb./b)./b.^3+3.*cos(r.*w0./pi./Qb./b)./2./b.^3./Qb).*(6.*(r./2./Qb./b+a1
).^2.*(r./b-r./pi./Qb./b-t)-2.*(r./b-r./pi./Qb./b-t).^3)./((r./2./Qb./b+a1).^2+(r./b-r./pi./Qb./b-
t).^2).^3);


point_elastic=A./2./pi.^2./rho./r.^3.*(a1.*r.*(1./(b.*(a1.^2+(t-r./b).^2))-1./(a.*(a1.^2+(t-
r./a).^2)))+atan((r./a-t)./a1)-atan((r./b-t)./a1))...
    +A./2./pi.^2./rho.*a1./(a.^2.*r).*(t-r./a)./(a1.^2+(t-r./a).^2).^2;


% double_elastic=A.*(9+15.*cos(2.*theta)).*sin(theta)./4./pi.^2./rho./r.^4.*...
```

```
%    (a1.*r./b./(a1.^2+(t-r./b).^2)-atan((r./b-t)./a1)-a1.*r./a./(a1.^2+(t-r./a).^2)+atan((r./a-t)./a1))...
%    +(2+3.*cos(2.*theta)).*sin(theta)./pi.^2./rho./a.^2./r.^2.*a1.*(t-r./a).*A./(a1.^2+(t-r./a).^2).^2 ...
%    -(3+6.*cos(2.*theta)).*sin(theta)./2./pi.^2./rho./b.^2./r.^2.*a1.*(t-r./b).*A./(a1.^2+(t-r./b).^2).^2 ...
%    +(1+cos(2.*theta)).*sin(theta)./2./pi.^2./rho./a.^3./r.*a1.*A.*((a1.^2+(t-r./a).^2)-4.*(t-r./a).^2)./(a1.^2+(t-r./a).^2).^3 ...
%    -cos(2.*theta).*sin(theta)./2./pi.^2./rho./b.^3./r.*a1.*A.*((a1.^2+(t-r./b).^2)-4.*(t-r./b).^2)./(a1.^2+(t-r./b).^2).^3;

point_nodisper=A./4./r.^2./pi.^2*(1./b.*(2.*a1+t./Qb)./((a1+r./2./Qb./b).^2+(r./b-t).^2)-1./a.*(2.*a1+t./Qa)./((a1+r./2./Qa./a).^2+(r./a-t).^2))...
   +A./2./pi.^2./r.^3.*(atan((r./a-t)./(a1+r./2./Qa./a))-atan((r./b-t)./(a1+r./2./Qb./b)))...
   -A./2./pi.^2./r./a.^2.*((a1+r./2./Qa./a).*(r./a-t)./((a1+r./2./Qa./a).^2+(r./a-t).^2).^2)...
   -A./4./pi.^2./r./a.^2./Qa.*((a1+r./2./Qa./a).^2-(r./a-t).^2)./((a1+r./2./Qa./a).^2+(r./a-t).^2).^2;

figure;
plot(x,point_elastic,'--b','Linewidth',2)
hold on;
plot(x,Hifft,'r','Linewidth',2)
hold on;
```

```
%plot(x,point_appro,'--k','Linewidth',2)

%hold on;

%plot(x,point_nodisper,'--g','Linewidth',2)

%legend('exact numerical result','approximation')

legend('elastic','anelastic with dispersion')

%legend('absorption and dispersion','only absorption')

title(['For r=',num2str(rr(i)),' km'])

xlabel('t[s]')

ylabel('u(x,t)')

end
```