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Discrete Spherical Harmonic Transforms for Equiangular Grids of Spatial and Spectral Data

Research Article

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Abstract:

Spherical Harmonic Transforms (SHTs) which are non-commutative Fourier transforms on the sphere are critical in global geopotential and related applications. Among the best known global strategies for discrete SHTs of band-limited spherical functions are Chebychev quadratures and least squares for equiangular grids. With proper numerical preconditioning, independent of latitude, reliable analysis and synthesis results for degrees and orders over 3800 in double precision arithmetic have been achieved and explicitly demonstrated using white noise simulations. The SHT synthesis and analysis can easily be modified for the ordinary Fourier transform of the data matrix and the mathematical situation is illustrated in a new functional diagram. Numerical analysis has shown very little differences in the numerical conditioning and computational efforts required when working with the two-dimensional (2D) Fourier transform of the data matrix. This can be interpreted as the spectral form of the discrete SHT which can be useful in multiresolution and other applications. Numerical results corresponding to the latest Earth Geopotential Model EGM 2008 of maximum degree and order 2190 are included with some discussion of the implications when working with such spectral sequences of fast decreasing magnitude.

Keywords:
Fourier transforms • geocomputations • geopotential modeling • spherical harmonics

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1. Introduction

On the spherical Earth as on the celestial sphere, array computations can be done for regional and global domains using planar and spherical formulations. Quadratures and least-squares estimation are used to convert continuous integral formulations into summations over data lattices. Spherical topologies are quite different from planar ones and these have important implications in the computational aspects of array data processing.

Spherical geocomputations for regional domains of even continental extents can be reduced to planar computations and under assumptions of stationarity or shift invariance, discrete array computations can be optimized using Fast Fourier Transforms (FFTs). Specifically, convolution operations for filtering and other data processing applications thereby require only $O(NlogN)$ instead of $O(N^2)$ operations for N data in one dimension, $O(N^2 \log N)$ instead of $O(N^4)$ operations for $N \times N$ data in two dimensions, and so on. Furthermore, open-source FFT software packages such as FFTW (Frigo and Johnson, 2005) have been fully optimized to take advantage of multithreading facilities on High Performance Computing (HPC) platforms.

For global applications, Gaussian, equiangular and other similar regular grids can be used for spherical quadratures and discrete convolutions. Various quadrature strategies are available in the literature going back to Gauss and Neumann, in addition to least-squares estimation techniques

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(e.g. Swartztrauber (1979), Colombo (1981), Dilts (1985), Sneeuw (1994), Molenkamp (1999), Holmes and Featherstones (2002), Healy et al. (2003), Jekeli et al. (2007)). Other approaches have also been used for discretization and analysis of functions on the sphere using triangular and curvilinear tessellations based on inscribed regular polytopes (see e.g. Gorski et al. (2005), Blais (2007, 2010)). Depending on the applications, these strategies may be preferable to the equiangular ones which will be discussed in the following.

The associated Legendre functions for high degrees and orders are computationally very challenging. Without any normalization, one can hardly compute SHTs of degrees and orders over 50 or so in double precision arithmetic. With proper normalization such as the geodetic one used in the following computations, one can achieve degrees and orders to around 1800 in double precision arithmetic (Blais and Provins 2003) and over 3600 in quadruple precision arithmetic (i.e. REAL*16) (Blais et al. 2005 and 2006). With proper numerical preconditioning independent of the latitude, the Legendre functions can be evaluated reliably for degrees and orders over 3800 in double precision arithmetic (Blais 2008). This has been demonstrated explicitly in synthesis and analysis computations using unit spectral coefficients (i.e. white noise simulations) with equiangular grids that do not include the poles. In other words, the previously published results using Chebychev quadrature and least squares have been extended to degrees and orders over 3800 working in double precision arithmetic. This is very important for numerous applications in geocomputations for ground resolutions of about 5 km. It should be emphasized that much higher limits in terms of degrees and orders are achievable when only synthesis is required as often seen in the literature (e.g. Jekeli et al. 2007). One key objective in this research is to have spherical harmonic synthesis and analysis fully compatible in the mathematical sense for band-limited spherical harmonic functions.

The SHT synthesis has been modified for the ordinary Fourier transform of the global equiangular data matrix. For band-limited spherical harmonic functions, these results are also shown to be applicable to the SHT analysis and the mathematical situation is illustrated in a functional diagram. This new transform can be interpreted as the spectral form of the discrete SHT which can be useful in multiresolution and other applications. Numerical simulations and analysis have confirmed that there is essentially no difference in computational efforts and numerical conditioning between the original and modified SHTs. These results are also demonstrated with the latest Earth Geopotential Model EGM 2008 of maximum degree and order 2190. Some numerical analysis has been included with comments of the implications for other practical applications.

The literature on spherical harmonic transforms is quite extensive and a number of software packages in different computer languages for different applications are readily available for research purposes. Among the best known packages are the following five with web references:

1. Spherepack from UCAR (http://www.cisl.ucar.edu/css/software/spherepack);
2. SpharmonicKit from Dartmouth College (http://www.cs.dartmouth.edu/~geelong/sphere/);
3. SHTOOLS from the Institut de Physique du Globe de Paris (http://www.ipgp.fr/~wieczor/SHTOOLS/www/accuracy.html);
4. HEALPix from JPL (http://healpix.jpl.nasa.gov/healpixSoftwareDocumentation.shtml);
5. ccSHT from Lawrence Berkeley National Lab (http://crd.lbl.gov/~cmc/ccSHTlib/doc).

Obviously these software packages have been developed for different applications and the maximum degrees and orders mentioned are generally about 2800 for synthesis and analysis. Our research objectives are in terms of numerical efficiency and maximum degrees and orders in white noise simulations of synthesis and analysis as demonstrated by error RMS values in the spectral and spatial domains. General geodetic applications are obviously also important.

2. Continuous and Discrete Spherical Harmonic Transforms

The Fourier expansion of a function $f(\theta, \lambda)$ on the sphere S^2 is given by

$$f(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{|m| \leqslant n} f_{n,m} \; Y_n^m(\theta, \lambda) \tag{1}$$

using colatitude θ and longitude λ, where the orthogonal basis functions $Y_n^m(\theta, \lambda)$ are called the spherical harmonics of degree n and order m. In particular, the Fourier or spherical harmonic coefficients appearing in the preceding expansion are obtained as inner products

$$f_{n,m} = \int_{S^2} f(\theta, \lambda) \; \bar{Y}_n^m(\theta, \lambda) \; d\sigma \tag{2}$$

with the overbar denoting the complex conjugate with $\mathrm{d}\sigma$ denoting the standard rotation invariant measure $\mathrm{d}\sigma = \sin\ \theta\ \mathrm{d}\theta\ \mathrm{d}\lambda$ on S^2. In most practical applications, the functions $\mathrm{f}(\theta,\lambda)$ are (spherically) band-limited in the sense that only a finite number of those coefficients are nonzero, i.e. $f_{n,m} \equiv 0$ for all degrees $n \geq N$ and orders $|m| \leq n$. Hence, using the regular equiangular grid $\theta_j = \mathrm{j}\pi/J$ and $\lambda_k = k2\pi/K, j = 0, \ldots, J-1, k = 0, \ldots, K-1$, with J and K to be specified later on, spherical harmonic synthesis can be formulated as

$$f(\theta_j, \lambda_k) = \sum_{n=0}^{N-1} \sum_{|m| \leqslant n} f_{n,m} \; Y_n^m(\theta_j, \lambda_k) \tag{3}$$

and using some appropriate spherical quadrature, the corresponding spherical harmonic analysis can be formulated as

$$f_{n,m} = \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} q_j \; f(\theta_j, \lambda_k) \; \bar{Y}_n^m(\theta_j, \lambda_k) \qquad (4)$$

for quadrature weights q_j as discussed by various authors e.g. Driscoll and Healy (1994); Sneeuw (1994); Blais and Provins (2002). The usual geodetic spherical harmonic formulation is given as

$$f(\theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} [\tilde{c}_{nm} \cos m\lambda + \tilde{s}_{nm} \sin m\lambda] \, \tilde{P}_{nm}(\cos\theta) \qquad (5)$$

where

$$\left\{ \begin{matrix} \tilde{c}_{nm} \\ \tilde{s}_{nm} \end{matrix} \right\} = \frac{1}{4\pi} \int_{S^2} f(\theta, \lambda) \left\{ \begin{matrix} \cos m\lambda \\ \sin m\lambda \end{matrix} \right\} \tilde{P}_{nm}(\cos\theta) \, d\sigma \qquad (6)$$

with the geodetically normalized Legendre functions $\tilde{P}_{nm}(\cos\theta)$expressed in terms of the usual spherical harmonics $Y_n^m(\theta, \lambda)$ (see e.g. (Heiskanen and Moritz, 1967) and (Blais and Provins 2002) for details). The tilde "$\sim$" will be used to indicate geodetic normalization in the following.

Explicitly, using the geodetic formulation and convention, one has for synthesis,

$$f(\theta, \lambda) = \sum_{n=0}^{N-1} \sum_{m=0}^{n} [\tilde{c}_{nm} \cos m\lambda + \tilde{s}_{nm} \sin m\lambda] \, \tilde{P}_{nm}(\cos\theta) \qquad (7)$$

and for analysis, using complex analysis,

$$\begin{aligned} &\tilde{c}_{nm} + i\tilde{s}_{nm} = \\ &= \tfrac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} f(\theta, \lambda) \, (\cos m\lambda + i \sin m\lambda) \, \tilde{P}_{nm}(\cos\theta) \sin\theta d\theta d\lambda \\ &= \int_0^{\pi} [u_m(\theta) + iv_m(\theta)] \, \tilde{P}_{nm}(\cos\theta) \sin\theta d\theta \end{aligned} \qquad (8)$$

where

$$u_m(\theta) + iv_m(\theta) = \frac{1}{4\pi} \int_0^{2\pi} f(\theta, \lambda) \, (\cos m\lambda + i \sin m\lambda) \, d\lambda \qquad (9)$$

which is simply the parallel-wise Fourier transform of the spatial array data.

Hence using data equispaced in longitude and the corresponding Discrete Fourier Transform (DFT) and Inverse DFT, one can write for each parallel,

$$\{f(\theta, \lambda_k)\} \xrightarrow[per\ parallel]{DFT_\lambda} \{u_m(\theta) + iv_m(\theta)\} \xrightarrow[per\ parallel]{DFT_\lambda^{-1}} \{\widehat{f(\theta, \lambda_k)}\} \qquad (10)$$

and more generally, for complex data,

$$\{f(\theta, \lambda_k) + ig(\theta, \lambda_k)\} \xrightarrow[per\ parallel]{DFT_\lambda} \{u_m(\theta) + iv_m(\theta)\} \xrightarrow[per\ parallel]{DFT_\lambda^{-1}} \{\widehat{f(\theta, \lambda_k) + ig(\theta, \lambda_k)}\}$$

which will be seen to be important when experimenting with arbitrary spectral coefficients $\tilde{c}_{nm}$ and $\tilde{s}_{nm}$. Furthermore, the ordinary discrete Fourier transform implies that Eq. (10) is valid and exact in exact arithmetic for complex data $f(\theta, \lambda_k)$ which may not be common in geophysical applications but can arise in simulations with arbitrary spectral coefficients $\tilde{c}_{nm}$ and $\tilde{s}_{nm}$.

Correspondingly, for each meridian, with some appropriate Chebychev Quadrature (CQ) or Least Squares (LS) to be described explicitly below,

$$\{\tilde{c}_{nm} + i\tilde{s}_{nm}\} \xrightarrow{SYNTHESIS} \{u_m(\theta) + iv_m(\theta)\} \xrightarrow{CQ\ or\ LS} \{\widehat{\tilde{c}_{nm} + i\tilde{s}_{nm}}\} \qquad (11)$$

in which the synthesis is only partial, i.e. in the Fourier domain. Notice that in general for (spherical) band limit N, N rows of isolatitude data are required with LS while $2N$ rows of equispaced isolatitude data are required with the CQ, and at least N equispaced data are required (although $2N$ or even $4N$ data are common in practice) for DFT per parallel. Explicitly,

$$u_m(\theta) + iv_m(\theta) = \sum_{n=m}^{N-1} (\tilde{c}_{nm} + i\tilde{s}_{nm}) \tilde{P}_{nm}(\cos\theta) \qquad (12)$$

for $2N$ isolatitudes with $\Delta\theta = \pi/2N$ for CQ and N isolatitudes with $\Delta\theta = \pi/N$ for LS. In longitude, only N equispaced points with $\Delta\lambda = 2\pi/N$ for both CQ and LS in the following experimentation. In practice, it is often desirable to have $\Delta\theta = \Delta\lambda$ so that $4N$ equispaced isolongitudes are used with CQ and $2N$ are used with LS. This is achieved with appropriate zero padding of the $\mathrm{u}_m(\theta_j) + \mathrm{iv}_m(\theta_j)$ for each parallel. A shift in latitude of the grids by half $\Delta\theta$ is also often implemented to exclude the poles and allow the use of hemispherical symmetries in the associated Legendre functions $P_{nm}(\cos\,(\pi - \theta)) = (\text{-}1)^{n+m} P_{nm}(\cos\,\theta)$. These choices of equiangular grids and other options are discussed explicitly in e.g. Blais et al. (2005 and 2006).

Hence, for equiangular (complex) data $\{f(\theta_j, \lambda_k) + ig(\theta_j, \lambda_k)\}$, SHT is a two-step analysis transformation

$$SHT : \{f(\theta_j, \lambda_k) + ig(\theta_j, \lambda_k)\} \xrightarrow[per\ parallel]{DFT_\lambda} \{u_m(\theta_j) + iv_m(\theta_j)\} \xrightarrow{CQ\ or\ LS} \{\tilde{c}_{nm} + i\tilde{s}_{nm})\}$$

with the inverse being the synthesis transformation for $\mathrm{u}_m(\theta_j) + \mathrm{iv}_m(\theta_j)$ followed by the inverse DFT per parallel or row. More discussion of the data arrays will be included in the simulation examples.

Now, in some applications with equiangular data $f(\theta_j, \lambda_k)$, it is important to consider the corresponding 2D Fourier transform, that is,

$$\{f(\theta_j,\lambda_k)\} \xrightarrow[per\ parallel]{DFT_\lambda} \{u_m(\theta_j)+iv_m(\theta_j)\} \xrightarrow[per\ meridian]{DFT_\theta} \{f(\theta_j,\lambda_k)\}^\wedge \tag{13}$$

in which $\{f(\theta_j,\lambda_k)\}^\wedge$ denotes the 2D Fourier transform of the array $\{f(\theta_j,\lambda_k)\}$. Notice that in general, for complex spatial data,

$$\{f(\theta_j,\lambda_k)+ig(\theta_j,\lambda_k)\} \xrightarrow[per\ parallel]{DFT_\lambda} \{u_m(\theta_j)+iv_m(\theta_j)\} \xrightarrow[per\ meridian]{DFT_\theta} \{f(\theta_j,\lambda_k)+ig(\theta_j,\lambda_k)\}^\wedge$$

which is also exact in exact arithmetic. Hence the functional diagram in Figure 1 which summarizes the mathematical situation with the separability of latitude and longitude formulations as well as the commutativity of the ordinary Fourier transforms.

It therefore follows that a modification of SHT for the Fourier transform of equiangular (complex) data $\{f(\theta_j,\lambda_k)+ig(\theta_j,\lambda_k)\}^\wedge$ can be defined as follows:

$$FHT:\{f(\theta_j,\lambda_k)+ig(\theta_j,\lambda_k)\}^\wedge \xrightarrow[per\ meridian]{DFT_\theta^{-1}} \{u_m(\theta_j)+iv_m(\theta_j)\} \xrightarrow{CQ\ or\ LS} \{\tilde{c}_{nm}+i\tilde{s}_{nm})\}$$

with the inverse being the synthesis transformation for $\mathrm{u}_m(\theta_j)$ $+\ \mathrm{iv}_m(\theta_j)$ followed by the DFT per meridian or column. More discussion of the data arrays will be included in the simulation examples.

$$\begin{array}{ccc} & \{\tilde{c}_{nm}+i\tilde{s}_{nm}\} & \\ & \updownarrow (11) & \\ & \{u_m(\theta_j)+iv_m(\theta_j)\} & \\ \nearrow DFT_\lambda & & \searrow DFT_\theta \\ \{f(\theta_j,\lambda_k)+ig(\theta_j,\lambda_k)\} & \xrightarrow[in\ 2D]{DFT} & \{f(\theta_j,\lambda_k)+ig(\theta_j,\lambda_k)\}^\wedge \end{array}$$

Figure 1. Functional diagram relating spherical harmonic coefficients and global array data.

Explicitly, the Chebychev Quadrature (CQ) is as follows

$$\tilde{c}'_{nm}+i\tilde{s}'_{nm} = \sum_{j=0}^{2N-1} q_j(u_{jm}+iv_{jm})\tilde{P}_{nm}(\cos\theta_j) \tag{14}$$

with $u_{jm} \equiv \mathrm{u}_m(\theta_j)$ and $v_{jm} \equiv \mathrm{v}_m(\theta_j)$, and CQ weights

$$q_j = \frac{1}{N}\sin\left((j+\tfrac{1}{2})\tfrac{\pi}{2N}\right)\sum_{h=0}^{N-1}\frac{1}{2h+1}\sin\left((2h+1)(j+\tfrac{1}{2})\tfrac{\pi}{2N}\right) \tag{15}$$

with $q_{2N-j} = q_j$ for $j = 0, 1, \ldots, N-1$ by hemispherical symmetry. These computations are roughly $O(N^3)$ for degree N. A brief overview of the derivation of these Chebychev weights can be found in Appendix A of Driscoll and Healy (1994).

The Least-Squares (LS) formulation per degree m is as follows

$$\sum_{n=m}^{N-1}\tilde{P}_{nm}(\cos\theta_j)\,(\tilde{c}''_{nm}+i\tilde{s}''_{nm}) = u_m(\theta_j)+iv_m(\theta_j) \tag{16}$$

with (shifted) isolatitudes $\theta_j = (j+\frac{1}{2})\frac{\pi}{N} = (2j+1)\pi/2N$ for $j = 0, 1, \ldots, N-1$. The least-squares computations for $\tilde{c}''_{nm}+i\tilde{s}''_{nm}$ per degree m are obviously very demanding and roughly $O(N^4)$. The elements in the corresponding normal matrices could be evaluated using the Christoffel-Darboux formula as shown in (Swarztrauber and Spotz 2000, Appendix B) based on (Hildebrand 1956) for more computational efficiency. More details of the LS formulation can be found in Blais et al. (2005, 2006).

3. Numerical Experimentation

The previous definitions of SHTs and FHTs are for band-limited spherical functions and these correspond to finite sequences of spherical harmonic coefficients. In practice, numerical experimentation and analysis are done using simulations for spectral coefficients of different degrees and orders. After spherical harmonic synthesis and analysis, an RMS of the spectral differences can be obtained and subsequently using a second synthesis of the computed coefficients, an RMS of the spatial differences results. The RMS quantities for both SHT and FHT have been analyzed for maximum degrees and orders. The analysis of their magnitudes and stability characteristics confirm the appropriateness of the mathematical formulations and computer code.

Most of the experimentation has been done using unit spectral coefficients which correspond to white noise. Such white noise spectrum may not be physically meaningful but the objective here is the numerical reproducibility after synthesis/analysis in the spectral domain, and following a second synthesis, in the spatial domain. With more physically realizable spectra such as $1, \ldots, 1/n^2$ for degrees $n > 0, \ldots$, the RMS values are orders of magnitude smaller than for unit spectral coefficients. For the EGM 2008 and other similar models, the RMS values after synthesis/analysis and second synthesis are much smaller as those spectra generally decrease rapidly for large degrees n.

Starting with simulated unit spectral coefficients, the reconstructed coefficients using CQ and LS formulations are compared with the input coefficients and RMS values are computed for various degrees and orders. Similarly, after a second synthesis, RMS values are computed in the spatial domain. Schematically, using the Chebychev Quadrature (CQ),

$$\{\tilde{c}_{nm}+i\tilde{s}_{nm}\}_{N\times N}\xrightarrow[\text{using CQ}]{SHT^{-1}}\left\{\widehat{f_{jk}+ig_{jk}}\right\}_{2N\times N}\xrightarrow[\text{using CQ}]{SHT}\left\{\widehat{\tilde{c}_{nm}+i\tilde{s}_{nm}}\right\}_{N\times N}\xrightarrow[\text{using CQ}]{SHT^{-1}}\left\{\widehat{\widehat{f_{jk}+ig_{jk}}}\right\}_{2N\times N}$$

and correspondingly using Least Squares (LS),

$$\{\tilde{c}_{nm}+i\tilde{s}_{nm}\}_{N\times N}\xrightarrow[\text{using LS}]{SHT^{-1}}\left\{\widehat{f_{jk}+ig_{jk}}\right\}_{N\times N}\xrightarrow[\text{using LS}]{SHT}\left\{\widehat{\tilde{c}_{nm}+i\tilde{s}_{nm}}\right\}_{N\times N}\xrightarrow[\text{using LS}]{SHT^{-1}}\left\{\widehat{\widehat{f_{jk}+ig_{jk}}}\right\}_{N\times N}$$

The key results, all obtained in double precision arithmetic, are shown in Table 1 for degrees and orders to 3900. The results for degrees and orders to 1800 or so agree exactly with those double precision results published in (Blais et al, 2005 and 2006). For higher degrees and orders to 3900, these results also generally agree with the synthesis/analysis results in (Blais, 2008) corresponding to Eq. (11). The numerical stability in the full spherical harmonic synthesis and analysis is very good for simulations using unit spectral coefficients. Again, with physically realizable spectral coefficients decreasing in magnitude with increasing order, the corresponding RMS would be smaller.

The same synthesis/analysis and second synthesis simulations have also been done using the modified SHT, herein called FHT, for the ordinary Fourier transform of the data matrix, i.e. using the Chebychev Quadrature (CQ),

$$\{\tilde{c}_{nm}+i\tilde{s}_{nm}\}_{N\times N}\xrightarrow[\text{using CQ}]{FHT^{-1}}\left\{\widehat{f_{jk}+ig_{jk}}\right\}^{\wedge}_{2N\times N}\xrightarrow[\text{using CQ}]{FHT}\left\{\widehat{\tilde{c}_{nm}+i\tilde{s}_{nm}}\right\}_{N\times N}\xrightarrow[\text{using CQ}]{FHT^{-1}}\left\{\widehat{\widehat{f_{jk}+ig_{jk}}}\right\}^{\wedge}_{2N\times N}$$

and correspondingly using Least Squares (LS),

$$\{\tilde{c}_{nm}+i\tilde{s}_{nm}\}_{N\times N}\xrightarrow[\text{using } LS]{FHT^{-1}}\left\{\widehat{f_{jk}+ig_{jk}}\right\}^{\wedge}_{N\times N}\xrightarrow[\text{using } LS]{FHT}\left\{\widehat{\tilde{c}_{nm}+i\tilde{s}_{nm}}\right\}_{N\times N}\xrightarrow[\text{using } LS]{FHT^{-1}}\left\{\widehat{\widehat{f_{jk}+ig_{jk}}}\right\}^{\wedge}_{N\times N}$$

The differences in the corresponding RMS values in Tables 1 and 2 show a small improvement for the Fourier domain over the spatial domain which would be somewhat insignificant for practical applications.

In comparison with the corresponding SHT results for Table 1 in (Blais, 2008), the small differences are attributable to the fact that the latter are for the reproducibility of the spectral coefficients following Eq. (11) above using CQ and LS using unit spectral coefficients. The following Tables 1 and 2 are the RMS results for full synthesis/analysis and second synthesis with unit spectral coefficients. The resulting accuracies and numerical stability characteristics are essentially the same as the numerical properties of the DFTs are well known and documented in the literature. The DFT software used herein is the FFTW (the "Fastest Fourier Transform in the West") from M. Frigo and S.G. Johnson (2005), (www.fftw.org), on Linux Operating Systems and IMSL FFT (www.vni.com) on Microsoft Operating Systems.

Using the latest Earth Geopotential Model EGM 2008 of maximum degree and order 2190, available from http://earth-info.nima.mil/GandG/, is the most complete geopotential model for terrestrial applications such as geoid undulations, deflections of the vertical, etc. (see Pavlis et al., 2008 for details). The EGM 2008 spectrum is well known to decrease rapidly with increasing degrees implying much smaller RMS in the synthesis/analysis and second synthesis corresponding to the preceding white noise simulations. The following experimentation with EGM 2008 confirms the computational efficiency and reliability of the above described software:

$$\left\{\begin{matrix}\tilde{c}_{nm}+i\tilde{s}_{nm}\\ of EGM2008\end{matrix}\right\}_{2190\times 2190}\xrightarrow[\text{using CQ}]{SHT^{-1}}\left\{\widehat{f_{jk}+ig_{jk}}\right\}_{4380\times 2190}\xrightarrow[\text{using CQ}]{SHT}\left\{\begin{matrix}\widehat{\tilde{c}_{nm}+i\tilde{s}_{nm}}\\ of EGM\,2008\end{matrix}\right\}_{2190\times 2190}\xrightarrow[\text{using CQ}]{SHT^{-1}}\left\{\widehat{\widehat{f_{jk}+ig_{jk}}}\right\}_{4380\times 2190}$$

with RMS in the spectral domain of

$$RMS\left[\{\tilde{c}_{nm}+i\tilde{s}_{nm}\}-\{\widehat{\tilde{c}_{nm}+i\tilde{s}_{nm}}\}\right]_{2190\times 2190} = 3.25706991E-022$$

and the corresponding RMS in the spatial domain (on the unit sphere) is

$$RMS\left[\{\widehat{f_{jk}+ig_{jk}}\}-\{\widehat{\widehat{f_{jk}+ig_{jk}}}\}\right]_{4380\times 2190} = 2.14335602E-021$$

Using the modified SHT (herein called FHT) for the 2D DFT of the spatial data i.e. $\{f(\theta_j,\lambda_k)\}^{\wedge}$, one obtains

$$\left\{\begin{matrix}\tilde{c}_{nm}+i\tilde{s}_{nm}\\ of EGM2008\end{matrix}\right\}_{2190\times 2190}\xrightarrow[\text{using CQ}]{FHT^{-1}}\left\{\widehat{f_{jk}+ig_{jk}}\right\}^{\wedge}_{4380\times 2190}\xrightarrow[\text{using CQ}]{FHT}\left\{\begin{matrix}\widehat{\tilde{c}_{nm}+i\tilde{s}_{nm}}\\ of EGM\,2008\end{matrix}\right\}_{2190\times 2190}\xrightarrow[\text{using CQ}]{FHT^{-1}}\left\{\widehat{\widehat{f_{jk}+ig_{jk}}}\right\}^{\wedge}_{4380\times 2190}$$

Table 1. Numerical CQ and LS **SHT** Results for Synthesis/Analysis and Second Synthesis with Unit Spectral Coefficients on a PC Desktop in Double Precision Arithmetic.

Degrees N	CQ SHT RMS of Synthesis/Analysis and Second Synthesis (grid: $2N\times N$)		LS SHT RMS of Synthesis/Analysis and Second Synthesis (grid: $N\times N$)	
1000	0.12463916E-012	7.73103474E-012	3.93281287E-014	2.52035513E-011
2000	3.16718363E-012	3.29685442E-011	8.08325307E-014	1.02379555E-010
3000	6.72948908E-012	7.47029579E-011	1.16240398E-013	2.23369947E-010
3200	2.60215965E-012	1.41240416E-011	1.19171640E-013	2.43384839E-010
3400	3.86495948E-012	4.04349665E-011	1.15282097E-013	2.50333725E-010
3600	3.54526184E-012	3.00488397E-011	1.16869701E-013	2.68849687E-010
3700	3.59012376E-011	6.09252567E-011	1.20727075E-013	2.84276107E-010
3800	3.86992724E-004	1.72661284E-004	1.46767720E-013	3.54453854E-010
3900	8.92525237E-002	1.13286540E-002	1.56210407E-013	3.86462663E-010

Table 2. Numerical CQ and LS **FHT** Results for Synthesis/Analysis and Second Synthesis with Unit Spectral Coefficients on a PC Desktop in Double Precision Arithmetic.

Degrees N	CQ FHT RMS of Synthesis/Analysis and Second Synthesis (grid: $2N\times N$)		LS FHT RMS of Synthesis/Analysis and Second Synthesis (grid: $N\times N$)	
1000	1.24639700E-012	5.85214418E-015	3.92230629E-014	2.08092327E-014
2000	3.16718392E-012	1.17631266E-014	8.08980689E-013	4.12514658E-014
3000	6.72948833E-012	1.83162980E-014	1.16478400E-013	6.26858711E-014
3200	2.60215900E-012	4.13440121E-015	1.19336843E-013	6.48340211E-014
3400	3.86495292E-012	9.89864372E-015	1.15266963E-013	6.28269357E-014
3600	3.54526145E-012	6.99588177E-015	1.17272526E-013	6.49750224E-014
3700	3.59012377E-011	1.37156046E-014	1.20281259E-013	6.66999748E-014
3800	3.86992724E-004	4.54371799E-008	1.46264987E-013	7.52423113E-014
3900	8.92525237E-002	2.90478307E-006	1.55443019E-013	7.96237119E-014

with RMS in the spectral domain of

$$RMS\left[\{\tilde{c}_{nm}+i\tilde{s}_{nm}\}-\{\widehat{\tilde{c}_{nm}+i\tilde{s}_{nm}}\}\right]_{2190\times2190}=3.25722469E-022$$

and the corresponding RMS in the spatial domain is

$$RMS\left[\{f_{jk}\widehat{+ig}_{jk}\}^{\wedge}-\{f_{jk}\widehat{\widehat{+ig}}_{jk}\}^{\wedge}\right]_{4380\times2190}=6.92374626E-025$$

Using least squares, the same numerical experimentation can be done and the corresponding four RMS values are respectively

$$RMS\left[\{\tilde{c}_{nm}+i\tilde{s}_{nm}\}-\{\widehat{\tilde{c}_{nm}+i\tilde{s}_{nm}}\}\right]_{2190\times2190}=1.45227398E-023$$

$$RMS\left[\{f_{jk}\widehat{+ig}_{jk}\}-\{f_{jk}\widehat{\widehat{+ig}}_{jk}\}\right]_{2190\times2190}=2.00437035E-020$$

$$RMS\left[\{\tilde{c}_{nm}+i\tilde{s}_{nm}\}-\{\widehat{\tilde{c}_{nm}+i\tilde{s}_{nm}}\}\right]_{2190\times2190}=1.49249170E-023$$

$$RMS\left[\{f_{jk}\widehat{+ig}_{jk}\}^{\wedge}-\{f_{jk}\widehat{\widehat{+ig}}_{jk}\}^{\wedge}\right]_{2190\times2190}=8.39844042E-024$$

These results confirm the accuracy and numerical stability of SHT and FHT when using the current EGM 2008.

For the white noise simulations, the computer times are essentially as discussed in Blais (2008) when simply implementing Eq. (11) in the spectral domain only while the preceding results refer to the full spherical harmonic syntheses and analyses. For the experimentation using EGM 2008, the desktop PC computer times (using a COMPAQ FORTRAN 95 compiler under Microsoft XP with IMSL FFT) are given in Table 3. The computational efforts are respectively $O(N^3)$ with CQ and $O(N^4)$ with LS. Some optimization of the LS analysis code could be done as previously mentioned in Section 2.

Table 3. Computer Times Using CQ and LS **SHT** and **FHT** for Synthesis/Analysis and Second Synthesis with EGM 2008 Coefficients on a PC Desktop.

USING	Grid Size	Synthesis/Analysis	Second Synthesis
SHT with CQ	4380×2190	1028.297 sec.	523.188 sec.
FHT with CQ	4380×2190	1043.750 sec.	534.141 sec.
SHT with LS	2190×2190	91583.73 sec.	257.828 sec.
FHT with LS	2190×2190	91753.97 sec.	259.250 sec.

4. Concluding Remarks

For general applications, considerable work has been done on solving the computational complexities, and enhancing the speed of calculation of spherical harmonic transforms for different equiangular grids. The numerical problems of evaluating the associated Legendre functions for very high degrees and orders have been resolved using numerical preconditioning as detailed in Blais (2008). Explicitly, using simulated unit spectral coefficients for degrees and orders over 3800, full synthesis and analysis lead to numerically stable RMS errors. For more physically realizable spectra such as in geodetic applications, these simulation results can be expected to improve by at least a couple of orders of magnitude, as experienced in previous experimental work. Such results would perhaps be more indicative of the expected numerical accuracies in practice as exemplified by the results using the Earth Geopotential Model EGM 2008 of maximum degree and order 2190.

A new functional diagram shows the mathematical relationship between the discrete spherical harmonic transform and the corresponding 2D Fourier transform of equiangular grids of data on the sphere. Using the separability of the 2D Fourier transform, a modification of the conventional SHT, called FHT above, has been introduced to handle the 2D Fourier transform of equiangular data matrices. Experimentation has shown that the numerical accuracy and conditioning are not really different with FHT and SHT as the modification really consists in replacing the row wise (or parallel wise) Fourier transform by a column wise (or meridian wise) one. The implications can be very interesting for applications where the Fourier transform of the data matrix is more appropriate or convenient.

As enormous quantities of data are involved the gravity field and other (e.g. the Cosmic Microwave Background) applications, parallel and grid computations are imperative for these applications. Preliminary experimentation with parallel processing has already been done (Soofi and Blais, 2005) and these double precision results can readily be duplicated in parallel environments.

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References

Blais J.A.R., 2010, Discrete Spherical Harmonic Transforms of Equidistributed Global Data. Manuscript under preparation.

Blais J.A.R., 2008, Discrete Spherical Harmonic Transforms: Numerical Preconditioning and Optimization. Computational Science -- ICCS 2008, Part II, edited by M. Bubak, G.D. van Albada, J.J. Dongarra and P.M.A. Sloot. Lecture Notes in Computer Science, Vol.5102:638-645, Springer-Verlag.

Blais, J.A.R., 2007, Optimal Spherical Triangulation for Global Multiresolution Analysis and Synthesis. Poster presentation at the 2007 Fall Meeting of the American Geophysical Union in San Francisco, CA.

Blais J.A.R. and Provins D.A., 2002, Spherical Harmonic Analysis and Synthesis for Global Multiresolution Applications. Journal of Geodesy, 76(1):29-35.

Blais J.A.R. and Provins D.A., 2003, Optimization of Computations in Global Geopotential Field Applications. Computational Science -- ICCS 2003, Part II, edited by P.M.A. Sloot, D. Abramson, A.V. Bogdanov, J.J. Dongarra, A.Y. Zomaya and Y.E. Gorbachev. Lecture Notes in Computer Science, Vol.2658:610-618. Springer-Verlag.

Blais J.A.R., Provins D.A. and Soofi M.A., 2006, Spherical Harmonic Transforms for Discrete Multiresolution Applications. Journal of Supercomputing, Vol.38:173-187.

Blais J.A.R., Provins D.A. and Soofi M.A., 2005, Optimization of Spherical Harmonic Transform Computations. Computational Science -- ICCS 2005, Part I, edited by V.S. Sunderam, G.D. van Albada, P.M.A. Sloot and J.J. Dongarra, Vol.3514:74-81. Springer-Verlag.

Colombo O., 1981, Numerical Methods for Harmonic Analysis on the Sphere. Report no. 310, Department of Geodetic Science and Surveying, The Ohio State University.

Dilts G.A., 1985, Computation of Spherical Harmonic Expansion Coefficients via FFTs. Journal of Computational Physics, 57:439-453.

Driscoll J.R. and Healy D.M. Jr., 1994, Computing Fourier Transforms and Convolutions on the 2-Sphere. Advances in Applied Mathematics, 15:202-250.

Frigo M. and S.G. Johnson, 2005, The Design and Implementation of FFTW, Proceedings of the IEEE 93(2):216-231. Invited paper, Special Issue on Program Generation, Optimization, and Platform Adaptation.

Gorski K.M., Hivon E., Banday A.J., Wandelt B.D., Hansen F.K., Reinecke M. and Bartelmann M. ,2005 , HEALPix: A Framework for High Resolution Discretization and Fast Analysis of Data Distributed on the Sphere. The Astronomical Journal, 622(2):759-771.

Healy D.M., Rockmore D., Kostelec P.J. and Moore S.S.B., 2003, FFTs for the 2-Sphere -- Improvements and Variations. J. Fourier Anal. Appl., 9(4):341-385.

Heiskanen W.A. and Moritz H., 1967, Physical Geodesy, W.H. Freeman and Company, San Francisco, 363.

Hildebrand F.B., 1956, Introduction to Numerical Analysis. McGraw-Hill, N.Y.

Holmes S.A. and Featherstone W.E., 2002, A unified approach to the Clenshaw Summation and the Recursive Computation of Very-High Degree and Order Normalised Associated Legendre Functions. J. Geodesy, 76(5):279-299.

Jekeli C., Lee J.K. and Kwon J.H., 2007, On the Computation and Approximation of Ultra-High-Degree Spherical Harmonic Series. J. Geodesy, 81(9):603-615.

Mohlenkamp M.J., 1999, A Fast Transform for Spherical Harmonics. J. Fourier Anal. Appl. 5(2/3):159-184.

Pavlis N.K., Holmes S.A., Kenyon S.C. and Factor J.K., 2008, An Earth Gravitational Model to Degree 2160: EGM2008. Presentation at the 2008 General Assembly of the European Geosciences Union, Vienna, Austria, April 13-18, 2008.

Sneeuw N., 1994, Global Spherical Harmonic Analysis by Least-Squares and Numerical Quadrature Methods in Historical Perspective. Geophys. J. Int. 118:707-716.

Soofi M.A. and Blais J.A.R., Parallel Computations of Spherical Harmonic Transforms, Oral presentation at the 2005 Annual Meeting of the Canadian Geophysical Union, Banff, Alberta, Canada.

Swarztrauber P.N., 1979, On the Spectral Approximation of Discrete Scalar and Vector Functions on the Sphere. SIAM J. Numer. Anal. 16(6):934-949.

Swarztrauber P.N. and Spotz W.F., 2000, Generalized Discrete Spherical Harmonic Transforms. J. Comp. Phys., 159:213-230.