Optimal Data Structures for Spherical Multiresolution Analysis and Synthesis

J. A. Rod Blais
Dept. of Geomatics Engineering
Pacific Institute for the Mathematical Sciences
University of Calgary, Calgary, AB
www.ucalgary.ca/~blais

Introduction

- **Geopotential and related fields are global with continuous spectra**
- **Multiresolution analysis and synthesis require regular array data**
- **Spherical Quad Tree (SQT) data structures are most appropriate**
- **Spherical Harmonic Wavelets (SHWs) using Transforms (SHTs)**
- **Equiangular discretizations are most common in applications**
- **Equitriangular (near equiareal) are often desirable in practice**
- **Other equidistribution strategies are much less appropriate**
- **Simulations with Geopotential Models (GMs) are discussed**
- **Concluding Remarks**

Multiresolution Analysis on Sphere (S^2)

A multiresolution analysis of $L^2(S^2)$ is a nested sequence of subspaces $\{V_n : n = 0, \pm 1, \pm 2, \ldots\}$ such that

- $L^2(S^2) \supset \ldots \supset V_1 \supset V_0 \supset V_{-1} \supset \ldots \supset \varnothing$
- $f(t) \in V_n \Leftrightarrow f(s \cdot t) \in V_{n-1}, \; s > 1$ **(dilations)**
- $f(t) \in V_n \Leftrightarrow f(t-k) \in V_n$ **(translations)**
- $\exists$ **a scaling or smoothing function** $\varphi(t) = \sqrt{s} \sum_k g_k \varphi(s \cdot t - k)$ **such that $\{\varphi(t-k)\}$ is an orthonormal basis of V_0**
- $\exists$ **a detail or wavelet function** $\psi(t) = \sqrt{s} \sum_k h_k \psi(s \cdot t - k)$ **that is a quadratic conjugate of the scaling function $\varphi(t)$**

Multiresolution Analysis of GMs

$$\begin{array}{ccccccccccc} GM_\infty & \rightarrow & \ldots & \rightarrow & GM_N & \rightarrow & GM_{N/2} & \rightarrow & \ldots & \rightarrow & GM_0 \\ & \searrow & & \searrow & & \searrow & & \searrow & & \searrow & \\ & & \ldots & & \Delta M_N & & \Delta M_{N/2} & & \ldots & & \Delta M_0 \end{array}$$

using the lowpass filter kernel with decimation (binary scaling)

$$K_L : GM_N \rightarrow GM_{N/2} \quad \text{and} \quad k_L : u_N(\theta,\lambda) \mapsto u_{N/2}(\theta,\lambda)$$

and highpass filter kernel with decimation (binary scaling)

$$K_H : GM_N \rightarrow \Delta M_{N/2} \quad \text{and} \quad k_H : u_N(\theta,\lambda) \mapsto v_{N/2}(\theta,\lambda)$$

satisfying Bezout's theorem: $K_L^* K_L + K_H^* K_H = I$

for reconstruction purposes: $K_L^* GM_{N/2} + K_H^* \Delta M_{N/2} = GM_N$

Example: SHT of Order N ≡ 2B

$$
\mathbf{CS} = \left(\begin{array}{cccc|cccc}
c_{00} & s_{11} & \cdots & s_{B-1,1} & s_{B+1,1} & s_{B+1,1} & \cdots & s_{N-1,1} \\
c_{10} & c_{11} & \cdots & s_{B-1,2} & s_{B+1,2} & s_{B+1,2} & \cdots & s_{N-1,2} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
c_{B-1,0} & c_{B-1,1} & \cdots & c_{B-1,B-1} & s_{B,B-1} & s_{B+1,B-1} & \cdots & s_{N-1,B-1} \\
\cline{1-4}
c_{B,0} & c_{B,1} & \cdots & c_{B,B-1} & c_{B,B} & s_{B+1,B} & \cdots & s_{N-1,B} \\
c_{B+1,0} & c_{B+1,1} & \cdots & c_{B+1,B-1} & c_{B+1,B} & c_{B+1,B+1} & \cdots & s_{N-1,B+1} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
c_{N-1,0} & c_{N-1,1} & \cdots & c_{N-1,B-1} & c_{N-1,B} & c_{N-1,B+1} & \cdots & c_{N-1,N-1}
\end{array} \right)
$$

Multiresolution Synthesis of GMs

For a potential function $u(\theta,\lambda)$ in GM_∞

$$u(\theta,\lambda) = u_0(\theta,\lambda) + k_H^* v_0(\theta,\lambda) + \ldots + k_H^* v_N(\theta,\lambda) + \ldots$$
$$= u_N(\theta,\lambda) + \ldots$$

and with more observational information,

$$u(\theta,\lambda) = u_N(\theta,\lambda) + k_H^* v_N(\theta,\lambda) + k_H^* v_{2N}(\theta,\lambda) + \ldots$$

Recall that with the usual (geodetic) conventions,

$$u_N(\theta,\lambda) = \sum_{n=0}^{N-1} \sum_{|m|\le n} u_{n,m} Y_n^m(\theta,\lambda)$$
$$= \sum_{n=0}^{N-1} \sum_{m=0}^{n} \left(\tilde{c}_{nm} \cos m\lambda + \tilde{s}_{nm} \sin m\lambda \right) \tilde{P}_{nm}(\cos\theta)$$

Estimation of Spectral Coefficients

Chebychev Quadrature (CQ) Methods:

- Equiangular grids of e.g. 2Nx4N points for $\Delta\theta = \Delta\lambda$
- With 2N equispaced parallels and at least N equispaced meridians
- Usually excluding the poles for numerical stability

» Providing least degree N per data (N^2 coefts, $O(N^3)$ oper'ns)

Least-Squares (LS) Methods:

- Equiangular grids of e.g. Nx2N points for $\Delta\theta = \Delta\lambda$
- With at least N parallels and at least N equispaced meridians
- Usually excluding the poles for numerical stability
- Multiple least-squares estimation per order

» Requiring least data per degree N (N^2 coefts , $O(N^4)$ oper'ns)

SHT Using Chebychev Quadrature

SHT:

$$\begin{pmatrix} z_{11} & z_{12} & \cdots & z_{1K} \\ z_{21} & z_{22} & \cdots & z_{2K} \\ \cdots & \cdots & \cdots & \cdots \\ z_{J1} & z_{J2} & \cdots & z_{JK} \end{pmatrix} \rightarrow \begin{pmatrix} \tilde{c}_{00} & \tilde{s}_{11} & \cdots & \tilde{s}_{\nu 1} \\ \tilde{c}_{10} & \tilde{c}_{11} & \cdots & \tilde{s}_{\nu 2} \\ \cdots & \cdots & \cdots & \cdots \\ \tilde{c}_{\nu 0} & \tilde{c}_{\nu 1} & \cdots & \tilde{c}_{\nu\nu} \end{pmatrix}$$

with

$$\begin{Bmatrix} \tilde{c}_{nm} \\ \tilde{s}_{nm} \end{Bmatrix} = \sum_{j=1}^{J} \sum_{k=1}^{K} \tilde{q}_j \, z_{jk} \begin{Bmatrix} \cos m\lambda_k \\ \sin m\lambda_k \end{Bmatrix} \tilde{P}_{nm}(\cos\theta_j)$$

and

$$z_{jk} = \sum_{n=0}^{\nu} \sum_{m=0}^{n} (\tilde{c}_{nm} \cos m\lambda_k + \tilde{s}_{nm} \sin m\lambda_k) \, \tilde{P}_{nm}(\cos\theta_j)$$

in which q_j denote the Chebychev quadrature weights [Blais, 2011].

SHT Using Least Squares

From the previous synthesis formulation

$$z_{jk} = \text{"const."} \cdot \underset{k=1,K}{\text{IDFT}} \left\{ \sum_{n=m}^{\nu} (\tilde{c}_{nm} + i\,\tilde{s}_{nm}) \tilde{P}_{nm}(\cos\theta_j) \right\}$$

one has for unknown coefficients $\tilde{c}_{nm}$ and $\tilde{s}_{nm}$

$$\underset{k=1,K}{\text{DFT}}[z_{jk}] = \text{"const."} \cdot \left\{ \sum_{n=m}^{\nu} (\tilde{c}_{nm} + i\,\tilde{s}_{nm}) \tilde{P}_{nm}(\cos\theta_j) \right\}$$

implying a least-squares problem per order m, with

$\theta_j = j\,\pi/N, \quad j = 0, 1, \ldots, N-1$

$\lambda_k = k\,\pi/N, \quad k = 0, 1, \ldots, 2N-1$

for $\tilde{c}_{nm}$ and $\tilde{s}_{nm}$ with $m \le n$, $n = 0, 1, \ldots, N-1$ [Blais, 2011].

SHT Computations for Degree N

For gridded data with at least 2N equispaced isolatitude data

$$\left\{\mathbf{z}_{jk}\right\} \xrightarrow[\textbf{per row}]{\textbf{DFT}} \left\{\mathbf{u}_{jh} + i\mathbf{v}_{jh}\right\} \xrightarrow[\textbf{per row}]{\textbf{IDFT}} \left\{\hat{\mathbf{z}}_{jk}\right\}$$

For gridded data with 2N data per column with constant $\Delta\theta$

$$\left\{\mathbf{c}_{nm} + i\mathbf{s}_{nm}\right\} \xrightarrow{\Sigma} \left\{\mathbf{u}_{jh} + i\mathbf{v}_{jh}\right\} \xrightarrow[\textbf{Quadrature}]{\textbf{Chebychev}} \left\{\hat{\mathbf{c}}_{nm} + i\hat{\mathbf{s}}_{nm}\right\}$$

For gridded data with N data per column with variable $\Delta\theta$

$$\left\{\mathbf{c}_{nm} + i\mathbf{s}_{nm}\right\} \xrightarrow{\Sigma} \left\{\mathbf{u}_{jh} + i\mathbf{v}_{jh}\right\} \xrightarrow[\textbf{Squares}]{\textbf{Least}} \left\{\hat{\mathbf{c}}_{nm} + i\hat{\mathbf{s}}_{nm}\right\}$$

Note that only DFTs are generally invertible above.

Equidistribution on the Sphere

In general for an arbitrary partition of the sphere S^2,

$$\text{Discrepancy} = \sup_{\text{all } C\in S^2} \frac{\sum_{n=1}^{N} \chi_C(\xi_n)/N}{\lim_{N\to\infty} \sum_{n=1}^{N} \chi_C(\xi_n)/N}$$

in which $\chi_C(\zeta_k)$ denotes the characteristic function for the cell C.

- **With pseudo-random numbers, Disc. ~ $O((\log\log N)^{1/2}/N^{1/2})$**
- **With quasi-random numbers, Disc. ~ $O((\log N)^s/N)$ for dim. s**

Hence quasi-random (deterministic) sequences are advantageous in higher dimensional simulations [Morokoff & Caflisch, 1994]

Quad Tree Data Structures

Planar Quad Trees

- **Two-dimensional pyramidal binary data structures**
- **Planar domains can be square, rectangular or triangular**
- **Most often used for multiresolution analysis/synthesis**

Spherical Quad Trees

- **Spherical pyramidal binary data structures**
- **Spherical partitions can be equiangular or equitriangular**
- **Octahedrons and icosahedrons have equitriangular faces**
- **Densification can only be near equiareal in general**

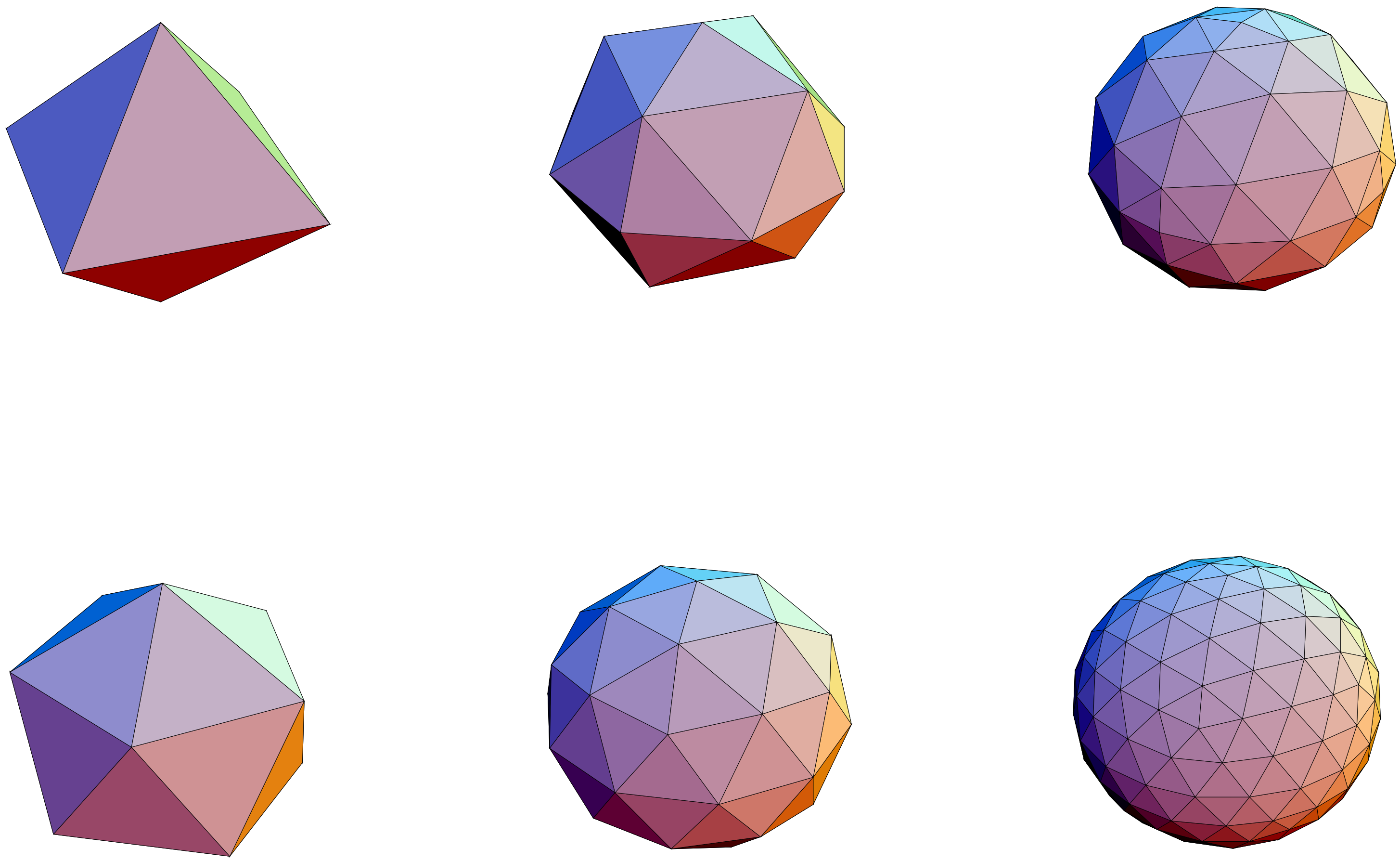
Near Equiareal Strategies

Platonic Solids (for exact regularity)

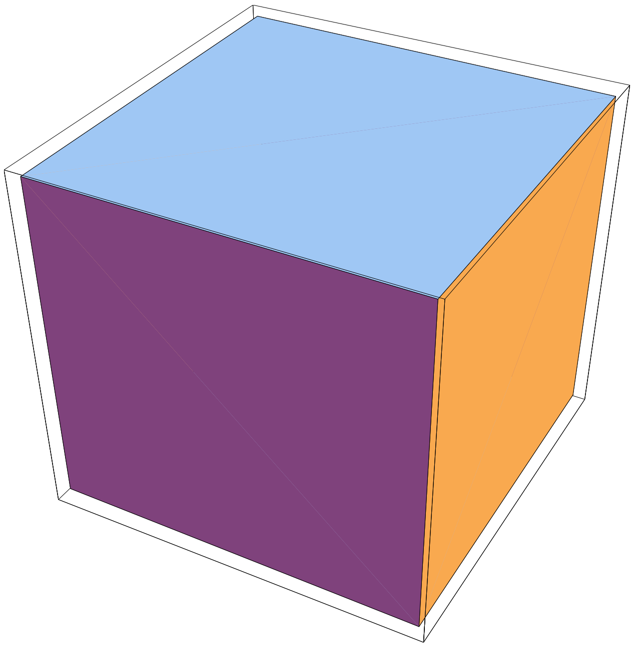
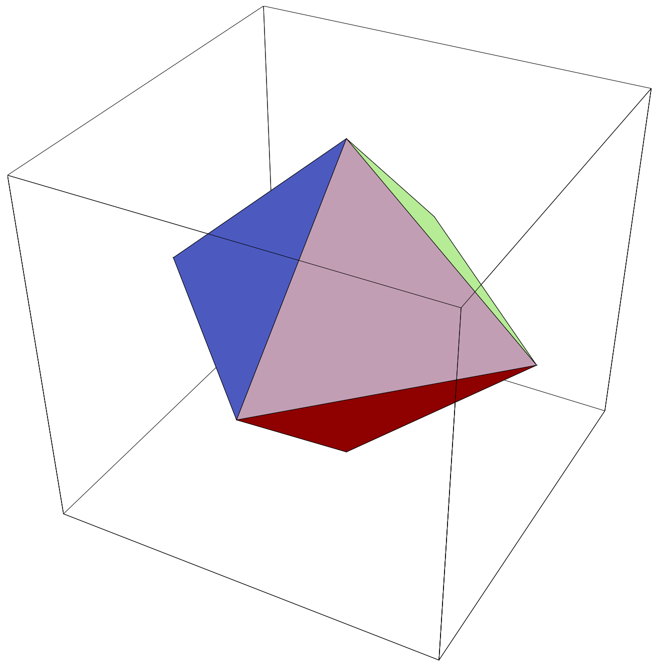
- **Tetrahedron (4 vertices, 4 edges, 4 faces)**
- **Cube (8 vertices, 12 edges, 6 faces)**
- **Octahedron (6 vertices, 12 edges, 8 faces)**
- **Dodecahedron (20 vertices, 30 edges, 12 faces)**
- **Icosahedron (12 vertices, 30 edges, 20 faces)**

Dualities (by interchanging faces and vertices)

- **Tetrahedron ↔ Tetrahedron**
- **Cube ↔ Octahedron**
- **Dodecahedron ↔ Icosahedron**



Cube and Octahedron

For the octahedron, radially projected face centres are equispaced on parallels enabling the use of FFTs in SHTs.

Octahedron Based Examples

Level One:

$$\begin{pmatrix} x & x & x & x \\ x & x & x & x \end{pmatrix}$$

Level Two:

$$\begin{pmatrix} x & & & x & & & x & & & x & & \\ x & x & x & x & x & x & x & x & x & x & x & x \\ x & x & x & x & x & x & x & x & x & x & x & x \\ x & & & x & & & x & & & x & & \end{pmatrix}$$

Level Three:

$$\begin{pmatrix} x & & & & & & & x & & & & & & & x & & & & & & & x & & & & & & \\ x & x & & & & & x & x & x & & & & & x & x & x & & & & & x & x & x & & & & & x \\ x & x & x & & & x & x & x & x & x & & & x & x & x & x & x & & & x & x & x & x & x & & & x & x \\ x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x \\ x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x \\ x & x & x & & & x & x & x & x & x & & & x & x & x & x & x & & & x & x & x & x & x & & & x & x \\ x & x & & & & & x & x & x & & & & & x & x & x & & & & & x & x & x & & & & & x \\ x & & & & & & & x & & & & & & & x & & & & & & & x & & & & & & \end{pmatrix}$$

Data matrix: 2^L x $4(2^L-1)$ for level L=1, 2, 3, …

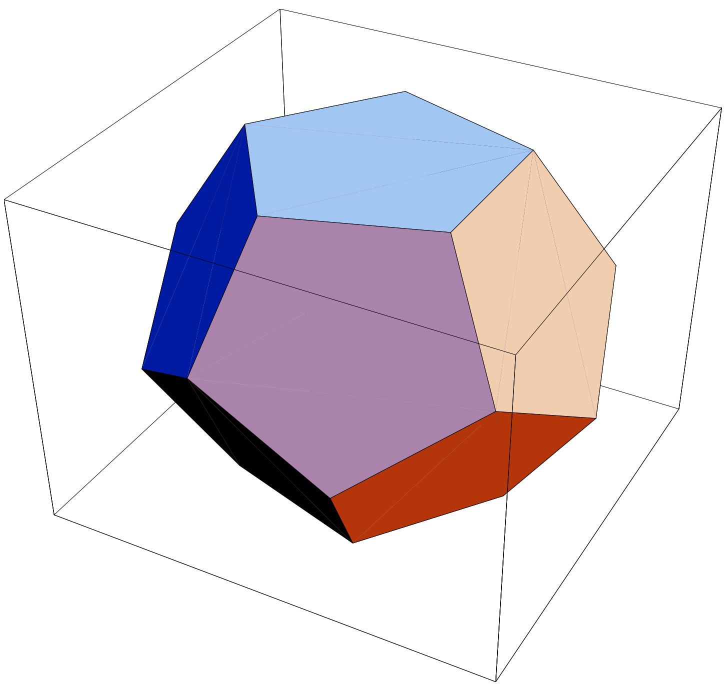
Octahedron Based EGM 2008

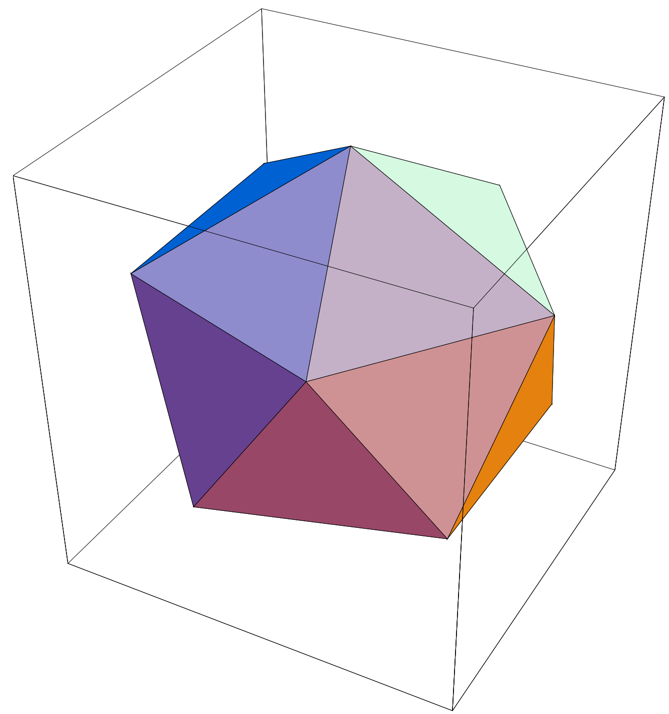
Level: Grid	CQ SHT without MASK		CQ SHT with MASK	
	Spectral RMS	Spatial RMS	Spectral RMS	Spatial RMS
1: 2 × 4	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
2: 4 × 12	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
3: 8 × 28	5.01518E-22	2.46658E-22	3.46314E-07	1.76088E-07
4: 16 × 60	4.60189E-22	2.71065E-22	2.95147E-07	1.35163E-07
5: 32 × 124	3.49076E-22	2.48811E-22	1.75164E-07	5.95464E-08
6: 64 × 252	3.27367E-22	1.97523E-22	9.36630E-08	2.82141E-08
7: 128 × 508	2.35428E-22	1.49245E-22	4.83148E-08	1.36872E-08
8: 256 × 1020	2.95569E-22	4.46778E-22	2.45261E-08	6.72713E-09
9: 512 × 2044	2.44512E-22	3.41138E-22	1.23503E-08	3.34493E-09
10: 1024 × 4092	1.86931E-22	1.98060E-22	6.19652E-09	1.65967E-09
11: 2048 × 8188	2.63594E-22	1.34133E-21	3.10356E-09	8.27754E-10
12: 4096 × 16380	1.79304E-22	4.37061E-22	1.93038E-09	2.61150E-11

Octahedron Based EGM 2008

Level: Grid	LS SHT without MASK		LS SHT with MASK	
	Spectral RMS	Spatial RMS	Spectral RMS	Spatial RMS
1: 2 × 4	0.00000E+00	0.00000E+00	0.00000E+00	0.00000E+00
2: 4 × 12	1.69759E-22	5.94638E-22	4.37280E-07	1.47796E-06
3: 8 × 28	1.99773E-22	1.72819E-21	2.81856E-07	2.03906E-06
4: 16 × 60	1.31301E-22	2.13755E-21	1.69567E-07	1.82278E-06
5: 32 × 124	1.12135E-22	3.27313E-21	9.21266E-08	1.76746E-06
6: 64 × 252	7.95785E-23	4.51222E-21	4.79136E-08	1.73217E-06
7: 128 × 508	5.15702E-23	5.27782E-21	2.44238E-08	1.71213E-06
8: 256 × 1020	3.44812E-23	7.57625E-21	1.23245E-08	1.70243E-06
9: 512 × 2044	2.72378E-23	1.14566E-20	6.19003E-09	1.69697E-06
10: 1024 × 4092	1.82607E-23	1.49602E-20	3.10193E-09	1.69397E-06
11: 2048 × 8188	1.27162E-23	2.12524E-20	1.55270E-09	1.69232E-06

Dodecahedron and Icosahedron





For the icosahedron, radially projected face centres are equispaced on parallels enabling the use of FFTs in SHTs.

Icosahedron Based Examples

Level One:

$$\begin{pmatrix}
x & & x & & x & & x & & x & \\
x & x & x & x & x & x & x & x & x & x \\
x & & x & & x & & x & & x &
\end{pmatrix}$$

Level Two:

$$\begin{pmatrix}
x & & & & x & & & & x & & & & x & & & & x & & & \\
x & x & & x & x & x & & x & x & x & & x & x & x & & x & x & x & & x \\
x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x & x \\
x & x & & x & x & x & & x & x & x & & x & x & x & & x & x & x & & x \\
x & & & & x & & & & x & & & & x & & & & x & & &
\end{pmatrix}$$

Level Three:

$$\begin{pmatrix}
x & & & & & & & & x & & & & & & & & x & & & & & & & & x & & & & & & & & x & & & & & & & \\
x & x & & & & & & x & x & x & & & & & & x & x & x & & & & & & x & x & x & & & & & & x & x & x & & & & & & x \\
x & x & x & & & & x & x & x & x & x & & & & x & x & x & x & x & & & & x & x & x & x & x & & & & x & x & x & x & x & & & & x & x \\
x & x & x & x & & x & x & x & x & x & x & x & & x & x & x & x & x & x & x & & x & x & x & x & x & x & x & & x & x & x & x & x & x & x & & x & x & x \\
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x & x & & & & & & x & x & x & & & & & & x & x & x & & & & & & x & x & x & & & & & & x & x & x & & & & & & x \\
x & & & & & & & & x & & & & & & & & x & & & & & & & & x & & & & & & & & x & & & & & & &
\end{pmatrix}$$

Data matrix: $3\cdot 2^{L-1}$ x $10\cdot 2^{L-1}$ for level L=1, 2, 3, …

Icosahedron Based EGM 2008

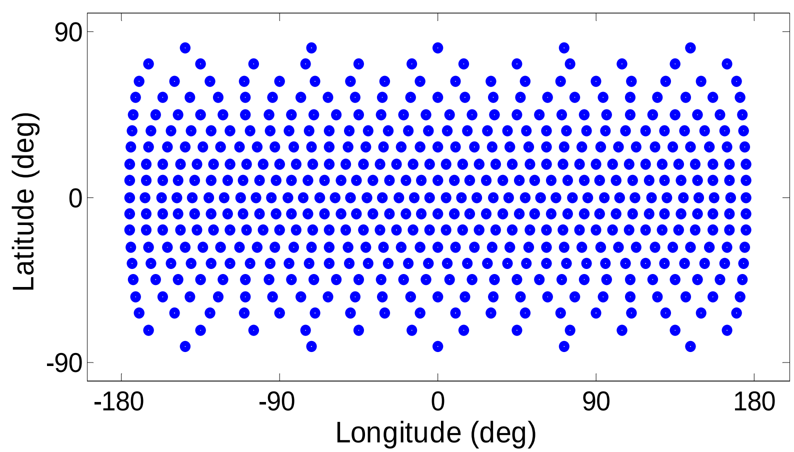
Level: Grid	CQ SHT without MASK		CQ SHT with MASK	
	Spectral RMS	Spatial RMS	Spectral RMS	Spatial RMS
2: 6 × 20	5.80442E-22	3.08926E-22	5.16965E-08	2.97412E-08
3: 12 × 40	5.36589E-22	3.10983E-22	2.34916E-07	1.17869E-07
4: 24 × 80	5.34168E-22	4.00308E-22	1.48875E-07	6.88029E-08
5: 48 × 160	3.90610E-22	2.79288E-22	8.43465E-08	2.84157E-08
6: 96 × 320	2.39380E-22	1.71708E-22	4.36315E-08	1.32942E-08
7: 192 × 640	2.34844E-22	2.53629E-22	2.21818E-08	6.42338E-09
8: 384 × 1280	3.76414E-22	7.51429E-22	1.11800E-08	3.14001E-09
9: 768 × 2560	2.44109E-22	5.55374E-22	5.61113E-09	1.55210E-09
10: 1536 × 5120	2.10235E-22	7.98760E-22	2.81076E-09	7.71222E-10
11: 3072 × 10240	2.67382E-22	1.17283E-21	1.40667E-09	3.84318E-10

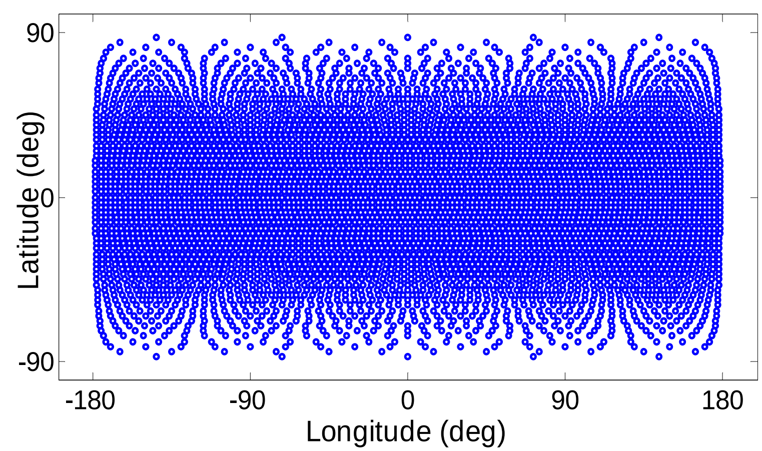
Icosahedron Based EGM 2008

Level: Grid	LS SHT without MASK		LS SHT with MASK	
	Spectral RMS	Spatial RMS	Spectral RMS	Spatial RMS
1: 3 × 10	1.25238E-22	2.11758E-22	1.25238E-22	2.59350E-22
2: 6 × 20	2.95140E-22	1.51008E-21	2.51053E-07	1.46517E-06
3: 12 × 40	1.52452E-22	2.20379E-21	1.52962E-07	1.68124E-06
4: 24 × 80	1.15572E-22	2.77977E-21	8.48097E-08	1.35730E-06
5: 48 × 160	1.05883E-22	4.66450E-21	4.37294E-08	1.27438E-06
6: 96 × 320	5.34825E-23	4.45502E-21	2.22020E-08	1.23198E-06
7: 192 × 640	4.06924E-23	6.62470E-21	1.11842E-08	1.20504E-06
8: 384 × 1280	3.88597E-23	1.15537E-20	5.61215E-09	1.19159E-06
9: 768 × 2560	2.02286E-23	1.24795E-20	2.81100E-09	1.18438E-06
10: 1536 × 5120	1.58626E-23	1.93084E-20	1.40673E-09	1.18052E-06

Reuter Grids

- Homogeneous point distributions include e.g. Reuter grids
- Fast computations can be performed using panel clustering
- New grids are required for each level of decomposition
- Not a pyramidal data structure for analysis and synthesis





Concluding Remarks

- Multiresolution analysis/synthesis can be done on $\mathbb{R}^2$ or S^2
- Equiangular grids are common but not always desirable
- Equitriangular (near equiareal) are generally convenient
- Other equidistributed point sets often have drawbacks
- SHTs (CQ & LS) analysis/synthesis give comparable RMS
- With EGM and like data, equitriangular grids work well
- With noise like data, some analysis problems may arise