Size-Depth Tradeoffs for Algebraic Formulae*

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Abstract

We prove some tradeoffs between the size and depth of algebraic formulae. In particular, we show that, for any fixed $\epsilon > 0$, any algebraic formula of size S can be converted into an equivalent formula of depth $O(\log S)$ and size $O(S^{1+\epsilon})$. This result is an improvement over previously-known results where, to obtain the same depth bound, the formula-size is $\Omega(S^{\alpha})$, with $\alpha \geq 2$.

1 Introduction

A classical result, due to Brent (1974), implies that for any algebraic formula there is an algebraic circuit of "small" depth and "similar" size that computes the same function. More precisely, if the formula has size S then the circuit has depth $O(\log S)$ and size $O(S)$. This result holds for formulae over any field. We believe that a natural question to consider is whether for any algebraic formula there is an equivalent *formula* of small depth and similar size.

Since any circuit of depth $O(\log S)$ can be transformed into a formula of the same depth and with size polynomial in S, it follows immediately from Brent's result that there is also a formula of depth $O(\log S)$ and size $S^{O(1)}$ that computes the same function as the original formula of size S. Applying this to the specific circuits that result from Brent's construction yields formulae with size as large as $\Omega(S^{\alpha})$, with $\alpha \geq 2$. Simple changes in Brent's construction may improve the exponent, but straightforward modifications do not appear to result in exponents arbitrarily close to one.

A widely-investigated problem that is related to Brent's result, as well as our work, is the "formula evaluation problem," where the goal is to construct a "universal formula evaluator" algorithm. Such an algorithm takes as input a description of a formula, with all of its inputs specified, and produces as output the value of the formula. Parallel algorithms for this problem have been proposed by Cook and Gupta (1985); Miller and Reif (1985); Buss (1987); Buss, Cook, Gupta, and Ramachandran (1989); and Kosaraju and Delcher (1990). These yield *NC* algorithms for the problem that also produce, for any given formula of size S, a *circuit* of depth $O(\log S)$. When these circuits are expressed as formulae, the sizes are $\Omega(S^{\alpha})$, for various $\alpha \geq 2$. In the case of division-free formulae, the exponents are smaller, but nevertheless bounded above one. As an example, in Section 5, we exhibit division-free formulae for which Miller and Reif's method produces formulae with such a polynomial size blowup.

In this paper, we show that, over any field $\mathcal{F}$, for any fixed $\epsilon > 0$, for any formula of size S with operations from $\{+, -, \times, \div\} \cup \mathcal{F}$, there are equivalent formulae with:

- Depth $O(\log S)$ and size $O(S^{1+\epsilon})$.
- Depth $O(\log^{1+\epsilon} S)$ and size $S^{1+O\left(\frac{1}{\log\log S}\right)}$.

*A preliminary version of this paper was presented at the 32nd Annual ACM Symposium on Foundations of Computer Science, San Juan, Puerto Rico (1991). This research was supported in part by NSERC of Canada.

- Depth $O(S^\epsilon)$ and size $O(S)$

In the latter result, the method we use will add new variables to the formula when the field size is less than S. Also, for Boolean formulae with operations from $\{\wedge, \vee, \neg\}$, we obtain similar conditions as above.

The techniques that we use include a multi-level extension of Brent's tree-decomposition method, as well as other restructuring methods.

The organization of the remainder of this paper is as follows. Section 2 contains basic definitions and notation. Section 3 contains the main result (expressed in Theorem 6, and interpreted in Corollary 7). Section 4 concerns additional results that apply for special classes of formulae, such as division-free, Boolean, and "simple" formulae. Section 5 describes some specific formulae that appear to exhibit increases in size when their depth is reduced, and some known lower bounds on the size-depth tradeoff due to Commentz-Walter (1979) and Commentz-Walter and Sattler (1980).

2 Definitions and Notation

Algebraic Formulae: For a field $\mathcal{F}$, a *formula* over $(\mathcal{F}, +, -, \times, \div)$ of depth d is defined as follows. A depth 0 formula is either c, for some $c \in \mathcal{F}$ (a *constant*) or x_u, for some $u \in \{1, 2, \ldots\}$ (an *input*). For $d > 0$, a depth d formula is $(F * G)$, where $* \in \{+, -, \times, \div\}$, F and G are formulae of depth d_F and d_G respectively and $d = \max(d_F, d_G) + 1$. The *size* of a formula F, denoted as $|F|$, is, informally, the number of occurrences of inputs and constants in the formula. More formally, a depth 0 formula has size one, and $|(F * G)| = |F| + |G|$ ($* \in \{+, -, \times, \div\}$).

A formula over $(\mathcal{F}, +, -, \times, \div)$ corresponds to a rational function in $\mathcal{F}(x_1, ..., x_n)$ (for some n) in a natural way, provided that it does not involve a division by a formula equivalent to zero. For formulae F and G, $F \equiv G$ denotes that they correspond to the same rational function. Hence, $\equiv$ denotes equivalence in the function semantics sense.

Division-Free Formulae: A *division-free formula* is one that has no divisions. Clearly, division-free formulae correspond to polynomials.

Simple Formulae: A *simple formula* is one that is division-free, and for which at least one argument of each multiplication operation is either an input or a constant. Thus, a depth 0 simple formula is either a constant or an input, and for depth $d > 0$ a depth d simple formula is $(F * G)$ where F and G are simple formulae, $* \in \{+, -, \times\}$, and if $* = \times$ then either F or G has depth 0.

Extended Formulae: In order to denote decompositions of a formula, we define an *extended formula*, which is allowed to take *auxiliary inputs*, which are input symbols that are not from $\{x_1, x_2, \ldots\}$. For clarity, in extended formulae, we write all auxiliary inputs as "arguments" to the formula. For example, the extended formula $F(y)$ has auxiliary input symbol y. If G is a formula, then $F(G)$ denotes the formula $F(y)$ modified by substituting G for the symbol y.

The *size* of an extended formula is defined recursively as above, except that auxiliary inputs are not counted (that is, an auxiliary input has size 0). Also, we use special terminology to denote the number of occurrences of auxiliary variables in extended formulae. For $\mathcal{A} \subseteq \{y_1, ..., y_m\}$, $|G(y_1, ..., y_m)|_{\mathcal{A}}$ denotes precisely the total number of occurrences of inputs from $\mathcal{A}$ in $G(y_1, ..., y_m)$. In particular, an extended formula $G(y_1, ..., y_m)$ is *read-once* with respect to an auxiliary input y_i if and only if $|G(y_1, ..., y_m)|_{\{y_i\}} = 1$. Note that, for formulae $F_1, ..., F_m$,

$$|G(F_1, ..., F_m)| = |G(y_1, ..., y_m)| + \sum_{i=1}^{m} |F_i| \cdot |G(y_1, ..., y_m)|_{\{y_i\}}.$$

Binary Strings: As usual, for $k \geq 0$, $\{0, 1\}^k$ denotes all binary strings of length k. Furthermore, ε denotes the empty string, and $\{0, 1\}^{\leq k}$ denotes all binary strings of length less than or equal to k.

3 Main Result

Our main result is Theorem 6 (Corollary 7 presents some consequences of this result).

Brent's result is partially based on the following lemma, which concerns ways of partitioning trees into pieces of various sizes.

Lemma 1 (Brent, 1974): *For any formula F and any m such that $1 < m \le |F|$ there exists an extended formula $G(y)$ that is read-once with respect to y, formulae U and V, and an operation $*$ such that:*

- $F = G(U * V)$.
- $|G(y)| \le |F| - m$ *and* $|U|,\ \ |V| < m$.

For a formula F, Brent applies Lemma 1, with $m = \lceil \frac{1}{2}|F| \rceil$, thereby "decomposing" F into three pieces $G(y)$, U, and V, each of size at most $\lceil \frac{1}{2}|F| \rceil$. Then, using a recursive technique, he:

- translates $G(y)$ into a circuit of size $O(|G(y)|)$ and depth $O(\log |G(y)|)$ that computes A, B, C, and D such that
$$G(y) \equiv \frac{(A \times y) + B}{(C \times y) + D};$$
- translates U into an equivalent circuit $\hat{U}$ of size $O(|U|)$ and depth $O(\log |U|)$, and similarly translates V into $\hat{V}$.

Finally, Brent expresses F as the required circuit by the identity

$$F \equiv \frac{(A \times (\hat{U} * \hat{V})) + B}{(C \times (\hat{U} * \hat{V})) + D}.$$

Lemma 2 (below) is a multi-level version of the decomposition that Brent uses. Informally, it states that every formula F can be partitioned into $4k - 3$ or fewer pieces, each of size at most $\lceil \frac{1}{k}|F| \rceil$.

Lemma 2: *For any formula F, and any positive integer k such that $2 \le k < |F|$, there exist finite sets of indices Interior, Border $\subset \{0,1\}^*$, extended formulae $G_\alpha(y)$ and operations $*_\alpha$ for all $\alpha \in$ Interior, and formulae G_β for all $\beta \in$ Border, such that these form a well defined decomposition of F:*

(a) $\varepsilon \in$ *Interior;*

(b) *Interior* $\cap$ *Border* $= \emptyset$*;*

(c) *For all $\alpha \in$ Interior, both $\alpha 0$ and $\alpha 1$ are in Interior $\cup$ Border;*

(d) *For all $\beta \in$ Border, neither $\beta 0$ nor $\beta 1$ is in Interior $\cup$ Border;*

(e) *For all $\alpha \in$ Interior, $|G_\alpha(y)| \le \lfloor \frac{1}{k}|F| \rfloor$ and, for all $\beta \in$ Border, $|G_\beta| \le \lceil \frac{1}{k}|F| \rceil$;*

(f) *If formula U_γ is defined recursively for all $\gamma \in$ Interior $\cup$ Border by the rule*

$$U_\gamma = \begin{cases} G_\gamma(U_{\gamma 0} *_\gamma U_{\gamma 1}) & \text{if } \gamma \in \textit{Interior} \\ G_\gamma & \text{if } \gamma \in \textit{Border} \end{cases}$$

then $F = U_\varepsilon$.

In addition, these form a small decomposition of F:

(g) *Interior* $\subseteq \{0,1\}^{\le k-2}$ *and* $|$*Interior*$| \le 2k - 2$*;*

(h) *Border* $\subseteq \{0,1\}^{\le k-1}$ *and* $|$*Border*$| \le 2k - 1$.

Proof: Let F be an arbitrary formula and let k be an integer such that $2 \leq k < |F|$. We shall first give a construction for the sets *Interior* and *Border*, the extended formulae $G_\alpha(y)$ and operations $*_\alpha$ for $\alpha \in$ *Interior*, and formulae G_β for $\beta \in$ *Border*, and demonstrate that these give a well defined decomposition of F. We shall then argue that this decomposition is small.

To begin, initialize *Interior* and *Border* to be empty, and set $U_\varepsilon = F$, so that $|U_\varepsilon| = |F| = |F| - |\varepsilon| \cdot \lceil \frac{1}{k}|F| \rceil$. Since $k \geq 2$, $|U_\varepsilon| > \lceil \frac{1}{k}|F| \rceil$.

To continue, let $\gamma \in \{0,1\}^*$ such that U_γ has been defined with $|U_\gamma| \leq |F| - |\gamma| \cdot \lceil \frac{1}{k}|F| \rceil$, and such that γ has not been added (yet) to either *Interior* or *Border* — ending the construction if no such string γ exists.

If $|U_\gamma| \leq \lceil \frac{1}{k}|F| \rceil$ then add γ to *Border* and set G_γ to be U_γ. Otherwise, add γ to *Interior* and apply Lemma 1 (Brent) to U_γ with $m = |U_\gamma| - \lfloor \frac{1}{k}|F| \rfloor$, to define an extended formula $G_\gamma(y)$ that is read-once with respect to y, an operation $*_\gamma$, and formulae $U_{\gamma 0}$, $U_{\gamma 1}$ such that

- $U_\gamma = G_\gamma(U_{\gamma 0} *_\gamma U_{\gamma 1})$;
- $|G_\gamma(y)| \leq |U_\gamma| - m = \lfloor \frac{1}{k}|F| \rfloor$;
- $|U_{\gamma 0}|, |U_{\gamma 1}| < m$.

Note that it follows that $|U_{\gamma 0}| \leq |F| - |\gamma 0| \cdot \lceil \frac{1}{k}|F| \rceil$ and $|U_{\gamma 1}| \leq |F| - |\gamma 1| \cdot \lceil \frac{1}{k}|F| \rceil$.

It is clear that properties (a) – (f) are established if this construction terminates. To see that it does, consider $\gamma \in \{0,1\}^*$ such that U_γ is defined during the construction. Clearly, $0 < |U_\gamma| \leq |F| - |\gamma| \cdot \lceil \frac{1}{k}|F| \rceil$, so $|\gamma| < k$. Therefore the construction does eventually halt, defining sets *Border* $\subseteq \{0,1\}^{\leq k-1}$ and *Interior* $\subseteq \{0,1\}^{\leq k-2}$, and extended formulae, operations and formulae such that properties (a) – (f) hold.

For $\gamma \in$ *Interior* define *Interior*$_\gamma \subseteq$ *Interior* as

$$\mathit{Interior}_\gamma = \mathit{Interior} \cap \{\delta \in \{0,1\}^* : \gamma \text{ is a prefix of } \delta\}.$$

Then it is easily established by structural induction that

$$|\mathit{Interior}_\gamma| \leq \left\lceil \frac{2 \cdot |U_\gamma| \cdot k}{|F|} \right\rceil - 2$$

for all $\gamma \in$ *Interior*: If $\gamma \in$ *Interior* and both of $\gamma 0, \gamma 1$ are in *Border*, then

$$|\mathit{Interior}_\gamma| = 1 \leq \left\lceil \frac{2 \cdot |U_\gamma| \cdot k}{|F|} \right\rceil - 2,$$

since $|U_\gamma| > \frac{|F|}{k}$. If $\gamma \in$ *Interior* and exactly one (say, $\gamma 0$) of $\gamma 0$ and $\gamma 1$ is in *Interior*, then

$$|\mathit{Interior}_\gamma| = |\mathit{Interior}_{\gamma 0}| + 1 \leq \left\lceil \frac{2 \cdot |U_{\gamma 0}| \cdot k}{|F|} \right\rceil - 1 \leq \left\lceil \frac{2 \cdot |U_\gamma| \cdot k}{|F|} \right\rceil - 2,$$

since $|U_\gamma| - |U_{\gamma 0}| \geq \frac{|F|}{k}$. Finally, if all of γ, $\gamma 0$, and $\gamma 1$ are in *Interior*, then, since $|U_\gamma| \geq |U_{\gamma 0}| + |U_{\gamma 1}|$,

$$\begin{aligned}
|\mathit{Interior}_\gamma| &= |\mathit{Interior}_{\gamma 0}| + |\mathit{Interior}_{\gamma 1}| + 1 \\
&\leq \left\lceil \frac{2 \cdot |U_{\gamma 0}| \cdot k}{|F|} \right\rceil + \left\lceil \frac{2 \cdot |U_{\gamma 1}| \cdot k}{|F|} \right\rceil - 3 \\
&\leq \left(\left\lceil \frac{2 \cdot |U_{\gamma 0}| \cdot k}{|F|} + \frac{2 \cdot |U_{\gamma 1}| \cdot k}{|F|} \right\rceil + 1 \right) - 3 \\
&\leq \left\lceil \frac{2 \cdot |U_\gamma| \cdot k}{|F|} \right\rceil - 2, \qquad \text{as desired.}
\end{aligned}$$

Therefore, since $F = U_\varepsilon$ and *Interior* = *Interior*$_\varepsilon$, $|\mathit{Interior}| \leq \left\lceil \frac{2 \cdot |F| \cdot k}{|F|} \right\rceil - 2 = 2k - 2$, which is sufficient to establish property (g). Property (h) also follows because elements of the sets *Interior* and *Border* correspond respectively to the internal nodes and leaves of a binary tree, so $|\mathit{Border}| = |\mathit{Interior}| + 1$. □

In order to control the formula size, we use a different approach than Brent for restructuring extended formulae of the form $G(y)$. Lemma 3 (below) achieves the restructuring; however, it introduces new auxiliary inputs. In Lemmas 4 and 5, we show how to eliminate the auxiliary inputs (roughly, by substituting constants for them—or small polynomials, if the field is too small—with special care to avoid introducing divisions by a zero formula).

Lemma 3: *For any extended formula $G(y)$ that is read-once with respect to y there exists an extended formula $H(y, g_1, g_2, g_3, z_1, z_2, z_3)$ such that:*

- $G(y) \equiv H(y, G(z_1), G(z_2), G(z_3), z_1, z_2, z_3)$.
- $H(y, g_1, g_2, g_3, z_1, z_2, z_3)$ *is read-once with respect to y.*
- $|H(y, g_1, g_2, g_3, z_1, z_2, z_3)| = 0$.
- $|H(y, g_1, g_2, g_3, z_1, z_2, z_3)|_{\{g_1, g_2, g_3\}} \leq 44$.
- $|H(y, g_1, g_2, g_3, z_1, z_2, z_3)|_{\{z_1, z_2, z_3\}} \leq 42$.
- $\text{depth}(H(y, g_1, g_2, g_3, z_1, z_2, z_3)) \leq 9$.

Proof: Since $G(y)$ is read-once with respect to y, there exist formulae P, Q, and R such that either

$$G(y) \equiv \frac{(P \times y) + Q}{y + R}$$

or

$$G(y) \equiv (P \times y) + Q.$$

The existence of P, Q, and R can be shown by considering the functions computed along the path in $G(y)$ from y to its root: each such function is the quotient of two affine linear functions of y. Although this establishes the existence of P, Q, and R, this does not lead to an efficient way to construct these formulae: in the general case (with divisions), the resulting formula size may be exponential in $|G(y)|$. Instead, we use the method below.

First, we consider the case where

$$G(y) \equiv \frac{(P \times y) + Q}{y + R}.$$

Substituting three distinct new auxiliary variables z_1, z_2, and z_3 for y in this equation for $G(y)$ results in the system of equations

$$\begin{bmatrix} z_1 & 1 & -G(z_1) \\ z_2 & 1 & -G(z_2) \\ z_3 & 1 & -G(z_3) \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \equiv \begin{bmatrix} G(z_1) \times z_1 \\ G(z_2) \times z_2 \\ G(z_3) \times z_3 \end{bmatrix}.$$

Since z_1, z_2, and z_3 are new auxiliary variables and $G(y)$ is not of the form $(P \times y) + Q$,

$$\begin{vmatrix} z_1 & 1 & -G(z_1) \\ z_2 & 1 & -G(z_2) \\ z_3 & 1 & -G(z_3) \end{vmatrix} \not\equiv 0,$$

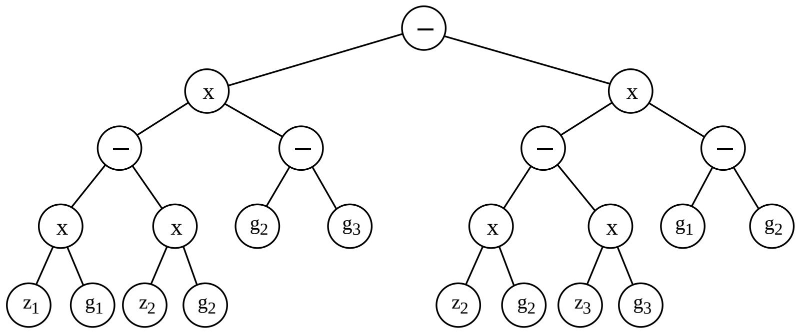
which implies that there are unique rational functions for P, Q, and R that satisfy the linear system. In particular,

$$P \equiv \frac{\begin{vmatrix} (z_3 \times G(z_3)) - (z_2 \times G(z_2)) & G(z_2) - G(z_3) \\ (z_2 \times G(z_2)) - (z_1 \times G(z_1)) & G(z_1) - G(z_2) \end{vmatrix}}{\begin{vmatrix} z_3 - z_2 & G(z_2) - G(z_3) \\ z_2 - z_1 & G(z_1) - G(z_2) \end{vmatrix}}.$$

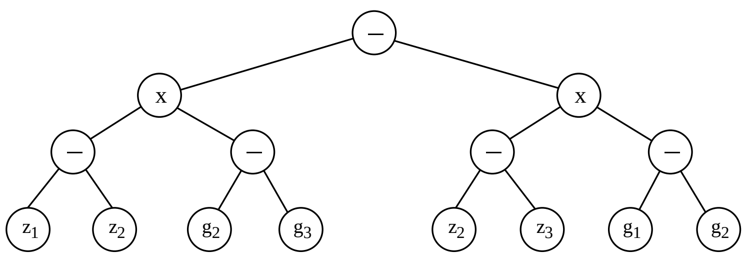
By examining the expressions for the above determinants, we deduce that[1]

$$P \equiv \frac{\hat{P}(\vec{G}(\vec{z}\,),\vec{z}\,)}{D(\vec{G}(\vec{z}\,),\vec{z}\,)},$$

where $\hat{P}(\vec{g},\vec{z})$ and $D(\vec{g},\vec{z})$ are as follows.



Formula $\hat{P}(\vec{g},\vec{z})$



Formula $D(\vec{g},\vec{z})$

Clearly, $\text{depth}(\hat{P}(\vec{g},\vec{z}\,)) = 4$, $\text{depth}(D(\vec{g},\vec{z}\,)) = 3$, $|\hat{P}(\vec{g},\vec{z})| = 0$, $|\hat{P}(\vec{g},\vec{z})|_{\{\vec{g}\}} = 8$, $|\hat{P}(\vec{g},\vec{z})|_{\{\vec{z}\}} = 4$, $|D(\vec{g},\vec{z})| = 0$, $|D(\vec{g},\vec{z})|_{\{\vec{g}\}} = 4$, and $|D(\vec{g},\vec{z})|_{\{\vec{z}\}} = 4$. Similarly,

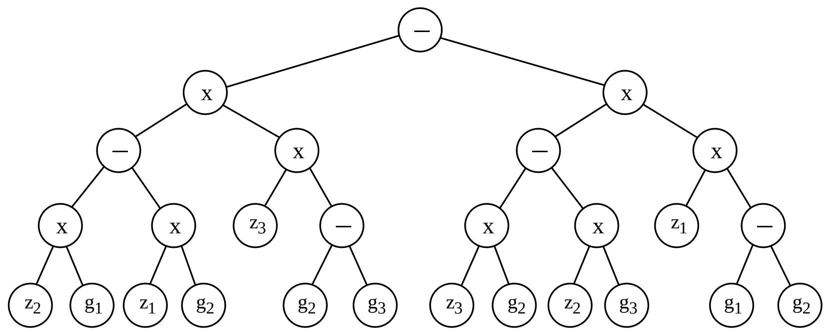
$$Q \equiv \frac{\hat{Q}(\vec{G}(\vec{z}\,),\vec{z}\,)}{D(\vec{G}(\vec{z}\,),\vec{z}\,)}$$

and

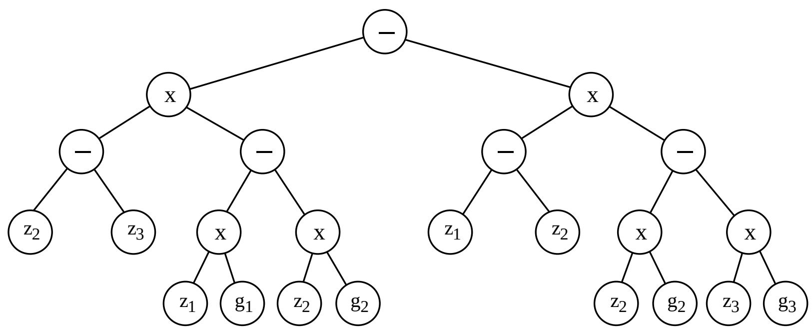
$$R \equiv \frac{\hat{R}(\vec{G}(\vec{z}\,),\vec{z}\,)}{D(\vec{G}(\vec{z}\,),\vec{z}\,)},$$

where $\hat{Q}(\vec{g},\vec{z})$ and $\hat{R}(\vec{g},\vec{z})$ are as shown below.

[1] Here $(\vec{G}(\vec{z}\,),\vec{z}\,)$ denotes $(G(z_1),G(z_2),G(z_3),z_1,z_2,z_3)$ and $(\vec{g},\vec{z}\,)$ denotes $(g_1,g_2,g_3,z_1,z_2,z_3)$.



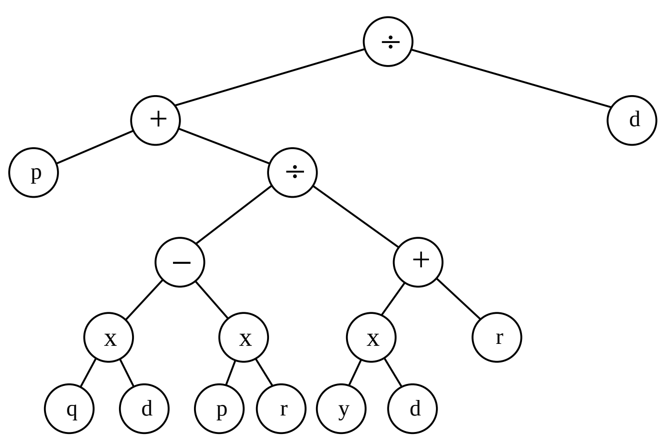
Formula $\hat{Q}(\vec{g},\vec{z})$



Formula $\hat{R}(\vec{g},\vec{z})$

It is easily checked that $\text{depth}(\hat{Q}(\vec{g},\vec{z})) = \text{depth}(\hat{R}(\vec{g},\vec{z})) = 4$, $|\hat{Q}(\vec{g},\vec{z})| = 0$, $|\hat{Q}(\vec{g},\vec{z})|_{\{\vec{g}\}} = 8$, $|\hat{Q}(\vec{g},\vec{z})|_{\{\vec{z}\}} = 6$, $|\hat{R}(\vec{g},\vec{z})| = 0$, $|\hat{R}(\vec{g},\vec{z})|_{\{\vec{g}\}} = 4$, and $|\hat{R}(\vec{g},\vec{z})|_{\{\vec{z}\}} = 8$.

Clearly, the desired $H(y,\vec{g},\vec{z})$ can be expressed in terms of $\hat{P}(\vec{g},\vec{z})$, $\hat{Q}(\vec{g},\vec{z})$, $\hat{R}(\vec{g},\vec{z})$, and $D(\vec{g},\vec{z})$; however, prior to completing the construction, we restructure the expression $((P \times y) + Q) \div (y + R)$ so that it is read-once with respect to y. This accomplished by performing a "polynomial division" of $y + R$ into $(P \times y) + Q$, resulting in the identity $((P \times y) + Q) \div (y + R) = \tilde{H}(y, P, Q, R, D)$, for the formula $\tilde{H}(y, p, q, r, d)$ given below.



Formula $\tilde{H}(y, p, q, r, d)$

Set $H(y,\vec{g},\vec{z}) = \tilde{H}(y, \hat{P}(\vec{g},\vec{z}), \hat{Q}(\vec{g},\vec{z}), \hat{R}(\vec{g},\vec{z}), D(\vec{g},\vec{z}))$; then it is easily checked that $\text{depth}(H(y,\vec{g},\vec{z})) = 9$,

$|H(y,\vec{g},\vec{z})| = 0$, $|H(y,\vec{g},\vec{z})|_{\{y\}} = 1$, $|H(y,\vec{g},\vec{z})|_{\{\vec{g}\}} = 44$, and $|H(y,\vec{g},\vec{z})|_{\{\vec{z}\}} = 42$, as claimed. Furthermore, it follows by the construction of these formulae that

$$G(y) \equiv H(y, \vec{G}(\vec{z}), \vec{z})$$

which completes the proof for the case where $G(y) \equiv ((P \times y) + Q) \div (y + R)$.

The simpler case where $G(y) \equiv (P \times y) + Q$ is handled similarly. □

The formulae introduced in Lemma 3 introduce new variables, z_1, z_2, and z_3. The next lemmas establish that these can be replaced by elements of the ground field $\mathcal{F}$, provided $\mathcal{F}$ is sufficiently large.

Lemma 4:

(i) Suppose $G(z) = G_L(z) * G_R(z)$ for $* \in \{+, -, \times, \div\}$ and that no proper subformula of $G(z)$ is identically zero. Let d_L (respectively, d_R) be an upper bound on the degree in z of the numerator and denominator of the rational function $G_L(z)$ (respectively, $G_R(z)$). If $* \in \{+, -\}$ then there are at most $d_L + d_R$ elements $\zeta \in \mathcal{F}$ such that $G(\zeta)$ is identically zero but none of $G_L(\zeta)$, $G_R(\zeta)$, or any of their subformulae are identically zero. If $* \in \{\times, \div\}$ then there are no elements $\zeta \in \mathcal{F}$ such that $G(\zeta)$ is identically zero but none of $G_L(\zeta)$, $G_R(\zeta)$, or any of their proper subformulae are identically zero.

(ii) Suppose $G(z)$ is read-once with respect to z and that no subformula of $G(z)$ is identically zero. Then there are at most $1 + \text{depth}(G(z)) \le |G(z)|$ elements $\zeta \in \mathcal{F}$ such that a subformula of $G(\zeta)$ is identically zero.

Proof: The first claim is easily verified by expressing the numerator of $G(z)$ as a function of the numerators and denominators of $G_L(z)$ and $G_R(z)$. If $* \in \{+, -\}$ then this numerator is a nonzero polynomial with degree at most $d_L + d_R$ in z. If $* \in \{\times, \div\}$ then the numerator is a product of numerators of subformulae of $G_L(z)$ and $G_R(z)$.

The second claim can be proved using induction on the depth of $G(z)$ and the fact that, if $G(z)$ is read-once with respect to z, then every subformula of $G(z)$ that includes z is the quotient of two affine linear functions of z, while every other subformula of $G(z)$ has degree zero in z. □

Lemma 5: *Suppose the extended formula $G(y)$ is read-once with respect to y, $|\mathcal{F}| \ge |G(y)| + 9$, and let $\mathcal{S}$ be a subset of $\mathcal{F}$ with at least $|G(y)| + 9$ distinct elements. Then there exists an extended formula $\hat{H}(y, a_1, a_2, a_3)$ and formulae A_1, A_2, and A_3 such that:*

- $G(y) \equiv \hat{H}(y, A_1, A_2, A_3)$.
- $|\hat{H}(y, a_1, a_2, a_3)| \le 42$.
- $|\hat{H}(y, a_1, a_2, a_3)|_{\{a_1, a_2, a_3\}} \le 44$.
- *$\hat{H}(y, a_1, a_2, a_3)$ is read-once with respect to y.*
- $\text{depth}(\hat{H}(y, a_1, a_2, a_3)) \le 9$.
- $|A_1|, |A_2|, |A_3| \le |G(y)| + 1$.
- *The only constants occurring as a subformula of $\hat{H}(y, a_1, a_2, a_3)$, A_1, A_2, or A_3 either occur as a subformula of $G(y)$ or belong to $\mathcal{S}$.*

Proof: As argued in the proof of lemma 3, since $G(y)$ is read-once with respect to y, there exist formulae P, Q, and R such that either

$$G(y) \equiv \frac{(P \times y) + Q}{y + R}$$

or

$$G(y) \equiv (P \times y) + Q.$$

Suppose the second case holds, and let $H(y, g_1, g_2, g_3, z_1, z_2, z_3)$ be the formula that exists by applying Lemma 3 to $G(y)$. If, as in the proof of Lemma 3,

$$D(g_1, g_2, g_3, z_1, z_2, z_3) = \begin{vmatrix} z_3 - z_2 & g_2 - g_3 \\ z_2 - z_1 & g_1 - g_2 \end{vmatrix}$$

then, for any $\zeta_1, \zeta_2, \zeta_3 \in \mathcal{F}$, the formula $H(y, G(\zeta_1), G(\zeta_2), G(\zeta_3), \zeta_1, \zeta_2, \zeta_3)$ is well-defined and equivalent to $G(y)$ provided that $D(G(\zeta_1), G(\zeta_2), G(\zeta_3), \zeta_1, \zeta_2, \zeta_3)$ is well-defined and not equivalent to the zero function (again, see the proof of Lemma 3 for details).

We next show that there exist $\zeta_1,\ \zeta_2,\ \zeta_3 \in \mathcal{F}$ with the above properties. Lemma 4 (ii) implies that the set $\mathcal{F}$ includes at most $|G(y)|$ elements ζ such that either $G(\zeta)$ or one of its subformulae is identically zero. Applying this and the fact that $G(y)$ is read-once with respect to y (and, hence, the quotient of two affine linear functions) to Lemma 4 (i) implies that there are at most $|G(y)|+8$ elements $\zeta_1 \in \mathcal{F}$ such that $D(G(\zeta_1), G(z_2), G(z_3), \zeta_1, z_2, z_3)$ or one of its subformulae is identically zero. Since $|\mathcal{S}| > |G(y)| + 8$, such a ζ_1 can be found in $\mathcal{S}$. The same argument can be applied twice more to prove the existence of elements $\zeta_2,\ \zeta_3$ of $\mathcal{S}$ such that $D(G(\zeta_1), G(\zeta_2), G(z_3), \zeta_1, \zeta_2, z_3)$ and $D(G(\zeta_1), G(\zeta_2), G(\zeta_3), \zeta_1, \zeta_2, \zeta_3)$ are both well-defined and nonzero, as desired.

Now, set

$$\hat{H}(y, a_1, a_2, a_3) = H(y, a_1, a_2, a_3, \zeta_1, \zeta_2, \zeta_3),$$

$$A_1 = G(\zeta_1), \quad A_2 = G(\zeta_2), \quad A_3 = G(\zeta_3).$$

Clearly, $\hat{H}(y, A_1, A_2, A_3) \equiv H(y, G(\zeta_1), G(\zeta_2), G(\zeta_3), \zeta_1, \zeta_2, \zeta_3) \equiv G(y)$, the size bounds stated above for A_1, A_2, and A_3 follow immediately from the definitions of these formulae, and the size and depth bounds for $\hat{H}(a_1, a_2, a_3)$ and $\hat{H}(A_1, A_2, A_3)$ follow directly from those given for $H(y, g_1, g_2, g_3, y_1, y_2, y_3)$ in the statement of Lemma 3.

The simpler case where $G(y) \equiv (P \times y) + Q$ is handled similarly. $\square$

Theorem 6: *For any formula F of size $S \le |\mathcal{F}| - 9$, any subset $\mathcal{S}$ of $\mathcal{F}$ with size at least $S + 9$, and any integer $k \ge 2$, there exists a formula G that is equivalent to F and has depth bounded by*

$$(\tfrac{9\,k}{\log k}) \log S + 9\,k + 3$$

and size bounded by

$$64\, S^{1 + \frac{6}{\log k}},$$

such that the only constants appearing as a subformula of G either appear as a subformula of F or belong to $\mathcal{S}$.

For any formula F of size $S > |\mathcal{F}| - 9$, and any integer $k \ge 2$, there exists a formula G that is equivalent to F and has depth bounded by

$$(\tfrac{9\,k}{\log k} + 2) \log S + 9\,k + 11$$

and size bounded by

$$64\, S^{1 + \frac{6}{\log k}}\, (\tfrac{2 \log S}{\log |\mathcal{F}|} + 9).$$

Proof: Let F be an arbitrary formula of size $S \le |\mathcal{F}| - 9$, let $\mathcal{S}$ be a subset of $\mathcal{F}$ with at least $S + 9$ elements, and let k be an arbitrary integer such that $k \ge 2$. Since the formula F has the depth and size stated in the lemma if $k > \frac{S}{9}$, we will assume $k \le \frac{S}{9}$. Using Lemma 2 and Lemma 5, we shall show that F can be restructured in a particular way and then iterate this restructuring process on a series of subformulae.

Let *Interior* $\subseteq \{0,1\}^{\le k-2}$, *Border* $\subseteq \{0,1\}^{\le k-1}$, $G_\alpha(y)$ and $*_\alpha$ (for all $\alpha \in$ *Interior*), and G_β (for all $\beta \in$ *Border*) be the result of applying Lemma 2 to F and k. Let $\hat{H}_\alpha(y, a_1^\alpha, a_2^\alpha, a_3^\alpha)$ and A_1^α, A_2^α, A_3^α ($\alpha \in$ *Interior*) be the result of applying Lemma 5 to $G_\alpha(y)$ ($\alpha \in$ *Interior*), respectively. Intuitively, the next step is to "reassemble" the formula F substituting $\hat{H}_\alpha(y, a_1^\alpha, a_2^\alpha, a_3^\alpha)$ in place of each $G_\alpha(y)$. Prior to doing this, we simplify our subscript notation as follows. Let

$$\Delta = (\{1,2,3\} \times \mathit{Interior}) \cup \mathit{Border},$$

and, for each $\delta \in \Delta$, let

$$w_\delta = \begin{cases} a_i^\alpha & \text{if } \delta = (i,\alpha) \in \{1,2,3\} \times \mathit{Interior} \\ g_\beta & \text{if } \delta = \beta \in \mathit{Border} \end{cases}$$

and

$$W_\delta = \begin{cases} A_i^\alpha & \text{if } \delta = (i,\alpha) \in \{1,2,3\} \times \mathit{Interior} \\ G_\beta & \text{if } \delta = \beta \in \mathit{Border}. \end{cases}$$

Now, define $E_\gamma(w_\delta : {\scriptstyle \delta\in\Delta})$ (for all $\gamma \in \mathit{Interior} \cup \mathit{Border}$) recursively as

$$E_\gamma(w_\delta : {\scriptstyle \delta\in\Delta}) = \begin{cases} \hat{H}_\gamma(E_{\gamma 0}(w_\delta : {\scriptstyle \delta\in\Delta}) *_\gamma E_{\gamma 1}(w_\delta : {\scriptstyle \delta\in\Delta}), a_1^\gamma, a_2^\gamma, a_3^\gamma) & \text{if } \gamma \in \mathit{Interior} \\ E_\gamma(w_\delta : {\scriptstyle \delta\in\Delta}) = g_\gamma & \text{if } \gamma \in \mathit{Border}. \end{cases}$$

It follows from the above and the properties of $\hat{H}_\alpha(y, a_1^\alpha, a_2^\alpha, a_3^\alpha)$ ($\alpha \in \mathit{Interior}$) that:

- $F \equiv E_\varepsilon(W_\delta : {\scriptstyle \delta\in\Delta})$.
- $\mathrm{depth}(E_\varepsilon(w_\delta : {\scriptstyle \delta\in\Delta})) \leq 9 \cdot k$.
- $|E_\varepsilon(w_\delta : {\scriptstyle \delta\in\Delta})| \leq 42\,|\mathit{Interior}|$.
- For all $\alpha \in \mathit{Interior}$,
$$|E_\varepsilon(w_\delta : {\scriptstyle \delta\in\Delta})|_{\{w_{(1,\alpha)}, w_{(2,\alpha)}, w_{(3,\alpha)}\}} \leq 44$$
and
$$|W_{(1,\alpha)}|, |W_{(2,\alpha)}|, |W_{(3,\alpha)}| \leq |G_\alpha(y)| + 1 \leq \lfloor \tfrac{1}{k}|F| \rfloor + 1.$$
- For all $\beta \in \mathit{Border}$, $|E_\varepsilon(w_\delta : {\scriptstyle \delta\in\Delta})|_{\{w_\beta\}} = 1$ and $|W_\beta| = |G_\beta| \leq \lceil \frac{1}{k}|F| \rceil$.
- $\displaystyle\sum_{\alpha\in\mathit{Interior}} |G_\alpha(y)| + \sum_{\beta\in\mathit{Border}} |G_\beta| = |F|$.

As well, $|\mathit{Interior}| \leq 2k$. Therefore,

- $\displaystyle |E_\varepsilon(W_\delta : {\scriptstyle \delta\in\Delta})| \leq 42\,|\mathit{Interior}| + 44 \sum_{\alpha\in\mathit{Interior}} (|G_\alpha(y)| + 1) + \sum_{\beta\in\mathit{Border}} |G_\beta| \leq 44|F| + 172k \leq 64|F|,$
since $k \leq \frac{|F|}{9}$.
- $\mathrm{depth}(E_\varepsilon(w_\delta : {\scriptstyle \delta\in\Delta})) \leq 9 \cdot k$.
- For all $\delta \in \Delta$, $|W_\delta| \leq \lfloor \frac{1}{k}|F| \rfloor + 1 \leq \frac{1}{k}S + 1$.

Now, by iterating this entire restructuring process on all the formulae W_δ ($\delta \in \Delta$) i times, we obtain extended formulae $E_\varepsilon^i(w_{\vec{\delta}} : {\scriptstyle \vec{\delta}\in\Delta^i})$ and formulae $W_{\vec{\delta}}$ (for all $\vec{\delta} \in \Delta^i$) such that:

- $F \equiv E_\varepsilon^i(W_{\vec{\delta}} : {\scriptstyle \vec{\delta}\in\Delta^i})$.
- $|E_\varepsilon^i(W_{\vec{\delta}} : {\scriptstyle \vec{\delta}\in\Delta^i})| \leq 64^i S$.
- $\mathrm{depth}(E_\varepsilon^i(w_{\vec{\delta}} : {\scriptstyle \vec{\delta}\in\Delta^i})) \leq 9 \cdot k \cdot i$.
- For all $\vec{\delta} \in \Delta^i$, $|W_{\vec{\delta}}| \leq (\frac{1}{k})^i S + 2$.

Therefore, after $i = \lceil \frac{\log S}{\log k} \rceil$ iterations, $|W_{\vec{\delta}}| \leq 3$ (for all $\vec{\delta} \in \Delta^i$) so

$$\begin{aligned}\text{depth}(E^i_\varepsilon(W_{\vec{\delta}} : \vec{\delta} \in \Delta^i)) &\leq 9\,k \left\lceil \frac{\log S}{\log k} \right\rceil + 3 \\ &\leq (\tfrac{9 \cdot k}{\log k}) \log S + 9\,k + 3\end{aligned}$$

and

$$|E^i_\varepsilon(W_{\vec{\delta}} : \vec{\delta} \in \Delta^i)| \leq 64^{\lceil \frac{\log S}{\log k} \rceil} S \leq 64\, S^{1+\frac{6}{\log k}}$$

as required.

Suppose now that $\mathcal{F}$ is finite and that F is a formula with size $S > |\mathcal{F}| - 9$. Let $\mathcal{E} = \mathcal{F}(x_1)$, and consider F as a formula of size S over the infinite field $\mathcal{E}$; the only "constants" arising as subformulae of F are x_1 and elements of the small field $\mathcal{F}$. Let $\mathcal{S} \subset \mathcal{E}$ include all elements of $\mathcal{F}[x_1]$ whose degree in x_1 is at most $\log_{|\mathcal{F}|}(S+9)$; clearly, $|\mathcal{S}| \geq S + 9$, and (by the claim for formulae over large fields) there exists a formula $\hat{G}$ equivalent to F that has depth bounded by

$$(\tfrac{9\,k}{\log k}) \log S + 9\,k + 3$$

and size bounded by

$$64\, S^{1+\frac{6}{\log k}},$$

such that the only constants appearing as a subformula of $\hat{G}$ either appear as a subformula of F (hence are x_1 or belong to $\mathcal{F}$) or belong to $\mathcal{S}$. Now, each element of $\mathcal{S}$ is equivalent to a formula (with constants in $\mathcal{F}$ and variable x_1) with size at most $2 \log_{|\mathcal{F}|}(S+9) + 1 \leq 2 \log_{|\mathcal{F}|} S + 9$, and depth at most $2 \log_{|\mathcal{F}|}(S+9) \leq 2 \log_{|\mathcal{F}|} S + 8$. Therefore, the formula $\hat{G}$ can be used to obtain an equivalent formula G with constants in $\mathcal{F}$ and variables $x_1, x_2, \ldots$ such that $|G| \leq |\hat{G}|(2 \log_{|\mathcal{F}|} S + 9)$ and $\text{depth}(G) \leq \text{depth}(\hat{G}) + 2 \log_{|\mathcal{F}|} S + 8$, as is required to prove the claim for the case that $\mathcal{F}$ is small. □

Corollary 7: *Over any field $\mathcal{F}$, for any fixed $\epsilon > 0$, for any formula of size S with operations from $\{+, -, \times, \div\} \cup \mathcal{F}$, there are equivalent formulae with:*

- *Depth $O(\log S)$ and size $O(S^{1+\epsilon})$.*
- *Depth $O(\log^{1+\epsilon} S)$ and size $S^{1+O(\frac{1}{\log \log S})}$*
- *Depth $O(S^\epsilon)$ and size $\begin{cases} O(S) & \text{if } |\mathcal{F}| \geq S \\ O(S \frac{\log S}{\log |\mathcal{F}|}) & \text{if } |\mathcal{F}| < S. \end{cases}$*

Proof: Apply Theorem 4 setting $k = 2^{\frac{6}{\epsilon}}$ (in the first case), $k = \log^\epsilon S$ (in the second case), and $k = S^{\frac{\epsilon}{2}}$ (in the third case). □

4 Additional Results

Division-Free Formulae: It should be noted that several parts of the proofs in Section 3 are significantly simpler in the case of division-free formulae. In particular, the process of introducing new variables to a formula and then eliminating these variables so as to avoid division by a zero formula (in Lemmas 3, 4, and 5) is unnecessary. Lemmas 3, 4, and 5 may be replaced by the following.

Lemma 8: *For a division-free extended formula $G(y)$ that is read-once with respect to y, there exists an extended formula $H(y, a_1, a_2)$, and formulae A_1 and A_2 such that:*

- *$G(y) \equiv H(y, A_1, A_2)$*
- *$|H(y, a_1, a_2)| = 0$*

- $|H(y,a_1,a_2)|_{\{a_1,a_2\}} = 3$
- $H(y,a_1,a_2)$ *is read-once with respect to* y
- $\text{depth}(H(y,a_1,a_2)) = 3$
- $|A_1|, |A_2| \le |G(y)| + 1$.

Proof: Clearly, since $G(y)$ is read-once with respect to y, there exist formulae P and Q such that

$$G(y) \equiv (P \times y) + Q.$$

By substituting $y = 0$ and $y = 1$ in the above equivalence, we obtain $Q \equiv G(0)$ and $P \equiv G(1) - G(0)$. Thus, by setting

$$H(y,a_1,a_1) = ((a_2 - a_1) \times y) + a_1,$$

and $A_1 = G(0)$ and $A_2 = G(1)$, the required properties are satisfied. □

Following this, all references to Lemma 5 in Theorem 6 may be replaced by references to Lemma 8.

Although some constants are smaller for the division-free case, the ultimate tradeoffs (expressed in Corollary 7) are the same, except that, in the third case of Corollary 7, the formula size can be linear, regardless of the size of the field.

Boolean Formulae: It is straightforward to adapt our techniques to Boolean formulae over the basis $\{\wedge, \vee, \neg\}$, since any such formula of size S is equivalent to a Boolean formula of size $O(S)$ over the basis $\{\wedge, \oplus, 1\}$: Each $\vee$ in a formula can be replaced by a $\wedge$ and three $\neg$'s, by De Morgan's law,

$$F \vee G \equiv \neg(\neg F \wedge \neg G);$$

then, each $\neg$ can be replaced by a $\oplus$ and a 1:

$$\neg F \equiv 1 \oplus F.$$

Now, since Boolean formulae over the basis $\{\wedge, \oplus, 1\}$ can be regarded as division-free formula over the field $GF(2)$, the above comments for division-free formula apply.

Simple Formulae: Kosaraju (1986) showed that any simple formula F is equivalent to a division free formula $\hat{F}$ of depth at most $\log|F| + 2\sqrt{\log|F|} + d$, for some constant d; clearly such a formula $\hat{F}$ can have size at most $c|F|^{1+\frac{2}{\sqrt{\log|F|}}} \in O(|F|^{1+\varepsilon})$ for some constant c and for arbitrary $\varepsilon > 0$. We show how Brent's construction can be modified to improve the bound on formula size implied by Kosaraju (1986) for the restructuring of simple formulae.

Theorem 9: *For any simple formula F there exists an equivalent division-free formula G (not generally simple) with depth at most $3\log|F|$ such that $|G| \le |F| + \frac{1}{2}|F|\log|F|$.*

Proof: We prove the result by induction on the size of F. The result is trivial if $|F| \le 2$, since it is sufficient to set $G = F$.

Suppose $|F| > 2$ and set $m = \lceil \frac{1}{2}|F| \rceil$. By Lemma 1 there exists an extended formula $G(y)$ that is read-once with respect to y, formulae U and V, and an operation $*$ such that $F = G(U * V)$, $|G(y)| \le |F| - m = \lfloor \frac{1}{2}|F| \rfloor$, and $|U|, |V| \le m - 1 \le \lfloor \frac{1}{2}|F| \rfloor$. Since F is simple, $G(y)$, U and V are simple as well. Furthermore, there exist formulae A and B such that $|A|, |B| \le |G(y)|$, A and B are both simple, and such that $G(y) \equiv (A \times y) + B$; Brent's construction ensures that $|U| + |V| + |B| \le |F|$. Since $G(y)$ is simple the only operation used in A will be $\times$; consequently, since at least one argument of every gate in A has depth 0, A is equivalent to a balanced formula $\hat{A}$ such that $|A| = |\hat{A}|$ and such that the depth of $\hat{A}$ is at most $\lfloor \log|A| \rfloor \le (\log|F|) - 1$. Now

$$F \equiv F_1 = (\hat{A} \times (U * V)) + B.$$

By the inductive hypothesis U is equivalent to a formula $\hat{U}$ such the depth of $\hat{U}$ is at most $3\log|U|$ and $\hat{U} \leq |U| + \frac{1}{2}|U|\log|U|$. Similarly, V is equivalent to a formula $\hat{V}$ whose depth is at most $3\log|V|$ and whose size is at most $|V| + \frac{1}{2}|V|\log|V|$, and B is equivalent to a formula $\hat{B}$ with depth at most $3\log|B|$ and size at most $|B| + \frac{1}{2}|B|\log|B|$. Set

$$G = (\hat{A} \times (\hat{U} * \hat{V})) + \hat{B}.$$

Then $G \equiv F$, and the depth of G is the maximum of $1+\text{depth}(\hat{B})$, $2+\text{depth}(\hat{A})$, $3+\text{depth}(\hat{U})$, and $3+\text{depth}(\hat{V})$. Since the depths of $\hat{U}$, $\hat{V}$, and $\hat{B}$ are at most $3\log\lfloor(|F|/2)\rfloor$ and the depth of $\hat{A}$ is at most $(\log|F|)-1 \leq 3\log|F|-2$, the depth of G is at most $3\log|F|$. Also,

$$\begin{aligned} |G| &\leq |\hat{A}| + |\hat{U}| + |\hat{V}| + |\hat{B}| \\ &\leq \lfloor \tfrac{1}{2}|F| \rfloor + (|U| + |V| + |B|) + \tfrac{1}{2}(|U| + |V| + |B|)\log\lfloor \tfrac{1}{2}|F| \rfloor \\ &\leq \lfloor \tfrac{1}{2}|F| \rfloor + |F| + \tfrac{1}{2}|F|((\log|F|) - 1) \\ &\leq |F| + \tfrac{1}{2}|F|\log|F|, \end{aligned}$$

as desired. □

5 Specific Formulae and Known Lower Bounds

As mentioned in Section 1, parallel algorithms for the formula evaluation problem can be modified to transform formulae into small-depth circuits, which can in turn be transformed into formulae of the same depth. We conclude with an example illustrating that polynomial size blow-up can arise from this approach, even if one is restricted to division-free formulae. In particular, we shall exhibit a formula of size n such that when the formula evaluation algorithm of Miller and Reif (1985) is applied to it, the resulting formula is of size $\Omega(n^{1+\delta})$, for a fixed $\delta > 0$.

For each n, define the formula $F_n(x_1, x_2, ..., x_{2n+1})$ as

$$F_n(x_1, x_2, ..., x_{2n+1}) = (\cdots(((x_1 \times x_2) + x_3) \times x_4) + \cdots \times x_{2n}) + x_{2n+1}.$$

Clearly, $\text{depth}(F_n(x_1, ..., x_{2n+1})) = 2n$ and $|F_n(x_1, ..., x_{2n+1})| = 2n + 1$.

Commentz-Walter (1979) shows that, over the Boolean semi-ring $(\{0,1\}, \wedge, \vee)$ (where negations are disallowed), there are formulae equivalent to $F_n(x_1, ..., x_{2n+1})$ with depth $O(\log n)$, but that all such formulae have size $\Omega(n\log n)$. Furthermore, Commentz-Walter and Sattler (1980) show that, even if negations can be introduced, any formula of depth $O(\log n)$ that computes $F_n(x_1, ..., x_{2n+1})$ must have size $\Omega(\frac{n\log\log n}{\log\log\log\log n})$. (This nonmonotonic lower bound does not apply if $\oplus$ operations can be introduced.)

Now consider the formula $G_n^{(k)}$, where

$$G_n^{(1)}(x_1, \ldots, x_{2n+1}) = F_n(x_1, \ldots, x_{2n+1}),$$

for $F_n(x_1, x_2, \ldots, x_{2n+1})$ as above, and

$$G_n^{(k)}(x_1, \ldots, x_{(2n+1)^k}) = F(G_n^{(k-1)}(x_1^{(k-1)}), \ldots, G_n^{(k-1)}(x_{2n+1}^{(k-1)}))$$

for $k > 1$, where $x_i^{(k-1)} = (x_{(i-1)(2n+1)^{k-1}+1}, \ldots, x_{i(2n+1)^{k-1}})$, $i = 1, \ldots, 2n+1$.

For $n = 3$ the method of Miller and Reif balances $G_3^{(1)}(x_1, \ldots, x_7)$ to $(x_1x_2 + x_3)x_6 + x_4 + x_6x_5 + x_7$ which induces one more occurrence of the variable x_6. When we try to balance $G_3^{(k)}$ using Miller and Reif's method we induce two copies of each $G_3^{(i)}(x_6^{(i)})$ in each level of the formula $G_3^{(k)}$. This implies that if the balanced formula is of size $S(N)$, for $N = 7^k$, then

$$S(N) = 8S(N/7)$$

which implies that Miller and Reif's method gives a formula of size $n^{\frac{\log 8}{\log 7}} = n^{1.0686}$.

Acknowledgements

We thank Allan Borodin for pointing out the work of Commentz-Walter (1979) and Commentz-Walter and Sattler (1980).

References

Brent, R.P. (1974), "The Parallel Evaluation of General Arithmetic Expressions," *J. ACM*, Vol. 21, No. 2, pp. 201–206.

Bshouty, N.H., R. Cleve, and W. Eberly (1991), "Size-Depth Tradeoffs for Algebraic Formulae," *Proc. 32th Ann. IEEE Symp. on Foundations of Computer Sci.*, pp. 334–341.

Buss, S.R. (1987), "The Boolean Formula Value Problem is in ALOGTIME," *Proc. 19th Ann. ACM Symp. on Theory of Comput.*, pp. 123–131.

Buss, S.R., S. Cook, A. Gupta, and V. Ramachandran (1989), "An Optimal Parallel Algorithm for Formula Evaluation," Manuscript (submitted for publication).

Commentz-Walter, B. (1979), "Size-Depth Tradeoff in Monotone Boolean Formulae," *Acta Informatica* Vol. 12, pp. 227–243.

Commentz-Walter, B., J. Sattler (1980), "Size-Depth Tradeoff in Non-monotone Boolean Formulae," *Acta Informatica* Vol. 14, pp. 257–269.

Gupta, A. (1985), "A Fast Parallel Algorithm for Recognition of Parenthesis Languages," Master's thesis, University of Toronto.

Kosaraju, S.A. (1986), "Parallel Evaluation of Division-Free Arithmetic Expressions," *Proc. 18th Ann. ACM Symp. on Theory of Comput.*, pp. 231–239.

Kosaraju, S.A., and A.L. Delcher (1990), "A Tree Partitioning Technique With Applications to Expression Evaluation and Term Matching," *Proc. 31th Ann. IEEE Symp. on Foundations of Computer Sci.*, pp. 163–172.

Miller, G.L., and J. Reif (1985), "Parallel Tree Contraction and its Application," *Proc. 26th Ann. IEEE Symp. on Foundations of Computer Sci.*, pp. 478–489.

Muller, D.E., and F.P. Preparata (1976), "Restructuring of Arithmetic Expressions for Parallel Evaluation," *J. ACM*, Vol. 23, No. 3, pp. 534–543.