THE UNIVERSITY OF CALGARY

VECTOR FIELDS ON FLAG MANIFOLDS

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE

DEGREE OF DOCTOR OF PHILOSOPHY

Department of Mathematics and Statistics

Calgary, Alberta

June 1985

THE UNIVERSITY OF CALGARY

FACULTY OF GRADUATE STUDIES

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ABSTRACT

The parallelizability question for spheres was solved in 1958 independently by M. Kervaire and J. Milnor. The vector field problem for spheres and projective spaces was settled by J.F. Adams in 1962. W.A. Sutherland proved in 1964 that the (real) Stiefel manifolds $V_{n,k}$ are parallelizable for $k \geq 2$.

The Grassmann manifolds are a natural generalization of the projective spaces, and are themselves particular cases of the much wider class of F-flag manifolds. These manifolds attracted closer attention of mathematicians in the mid seventies.

In this thesis, stable parallelizability and parallelizability of F-flag manifolds, $F = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and most of the closely related flag^{+} manifolds have been determined. Some of these are known results, others are new. The proofs given here are new, and in the case of existing results seem to be conceptually simpler. They also unify a variety of cases.

With varying degrees of success estimations for the span of Grassmann manifolds have been obtained. Some very general estimates on the span of flag manifolds have also been given.

Formulae for Stiefel-Whitney classes of tensor products, exterior and symmetric powers of Euclidean vector bundles have been obtained in full generality.

ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to Dr. Peter Zvengrowski for his invaluable guidance, for editing and correcting the manuscript and, above all, for his kindness and understanding, particularly at moments of darkness and despair, throughout the course of my research.

I would like to thank Drs. K. Varadarajan, H.K. Farahat, and M.J. Gotay whose courses were very helpful during my research. In particular, I wish to thank Professor Varadarajan who initiated me to the exciting study of Characteristic Classes.

I wish to thank the Department of Mathematics for their financial support, without which this work would not have been possible.

I wish to thank Mrs. Tae Nosal for her meticulous and speedy typing.

To My Parents,

Mr. and *Mrs. R. Sankaranarayanan*

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CHAPTER ONE

INTRODUCTION

§1. Basic Concepts and Definitions.

Let M be a smooth (i.e. C^∞ differentiable) manifold of dimension d, and let $\tau(M)$ denote its tangent bundle. Span of M, written span M, is defined to be the largest integer k for which $\tau(M)$ is bundle equivalent to $\eta \oplus k\varepsilon$ for some real vector bundle η of rank (d-k) over M, where $k\varepsilon$ denotes a trivial k-plane bundle over M. Span M is a diffeomorphism invariant of the manifold M, (but not in general a homeomorphism invariant.) Equivalently, span M is the maximum number of everywhere linearly independent vector fields defined on M. If span M = dim M, then M is said to be parallelizable. M is defined to be stably parallelizable, or a π-manifold, if $\tau(M)$ is stably trivial as a real vector bundle, i.e. $\tau(M) \oplus r\varepsilon$ is a trivial vector bundle for some $r \geq 0$. In this thesis we attempt to determine the span, parallelizability and stable parallelizability of Grassmann manifolds, and the closely related flag and flag$^+$ manifolds.

The concept of a fibre bundle $\xi = (E,B,F,p,G)$, where E is the total space, B the base space, F the fibre space, $p : E \longrightarrow B$ the projection of ξ, and G the structure group, is used here in the same sense as in Steenrod's book [39]. When G is the full group of homeomorphisms of F, ξ is referred to as a locally trivial bundle. Mostly we deal with $\mathbb{F}$-vector bundles ($\mathbb{F} = \mathbb{R}$ the reals, $\mathbb{C}$ the complex numbers, or $\mathbb{H}$ the quaternions) in which case F is the (left) vector space $\mathbb{F}^n$. The structure group G is the group $GL(\mathbb{F}^n)$, or, in case ξ admits a "Hermitian" metric, G is the group $U(\mathbb{F}^n)$, where $U(\mathbb{R}^n) = O(n)$. (the orthogonal group), $U(\mathbb{C}^n) = U(n)$ (the unitary group) and $U(\mathbb{H}^n) = Sp(n)$ (the symplectic group).

By the standard orientation on $\mathbb{R}^n$ we mean the orientation of $\mathbb{R}^n$ given by the canonical ordered basis $e_1,\ldots,e_n$ of $\mathbb{R}^n$, where $e_i = (x_{i1},\ldots,x_{in})$ and $x_{ij} = \delta_{ij}$. The standard Hermitian product on $\mathbb{F}^n$ for $\mathbb{F} = \mathbb{R},\mathbb{C}$ or $\mathbb{H}$ is defined as $\langle x,y\rangle = \sum_{1\le i\le n} x_i \cdot \bar{y}_i$ for $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n) \in \mathbb{F}^n$, where $\bar{x}_i$ denotes the conjugate of x_i in $\mathbb{F}$.

Let H be a closed subgroup of a Lie group G. Then G/H, with the quotient topology, is a manifold of dimension dim G − dim H. It has a unique smoothness structure such that the canonical projection π: $G \longrightarrow G/H$ is the projection of a differentiable bundle with fibre H and group G/H_o, where H_o is the largest normal subgroup of G contained in H. The manifolds of type G/H will be assumed to have this smoothness structure. We write "$G/H \cong G'/H'$" to mean that G/H is diffeomorphic to G'/H'.

For vector bundles ξ and η over a topological space X, $\xi \approx \eta$ if ξ and η are bundle equivalent. The notation $\approx$ is also used to denote equivalence in the appropriate K-group of the space X. We use the same symbol ξ to denote both the vector bundle and the equivalence class it represents in the K-group or the $\tilde{K}$-group of X, and to avoid confusion we make explicit in which sense the expression "$\xi \approx \eta$" is to be interpreted. Equivalence in $\tilde{K}$-group is denoted as "$\sim$". Thus, $\tau(S^3) \approx 3\varepsilon$ as vector bundles, $\tau(S^n) \approx n\varepsilon$ in $KO(S^n)$, and $\tau(S^n) \sim 0$ in $\widetilde{KO}(S^n)$ for any n.

By a vector field on a smooth manifold M we mean a continuous cross section $s : M \longrightarrow TM$, where TM denotes the total space of the tangent bundle $\tau(M)$ of M. Following E. Thomas [44], by a k-field on M we mean a collection of k vector fields $s_1,\ldots,s_k$ on M which is linearly independent at each point of M. If a collection $s_1,\ldots,s_k$ of vector field on M is linearly dependent at some (finitely many) points on M, then it will be called a k-field with (finite) singularities.

§2. Functors of Vector Bundles.

Let $\mathcal{V}$ denote the category of all finite dimensional vector spaces over $\mathbb{R}$ and vector space isomorphisms. Note that for any two vector spaces V and W in $\mathcal{V}$, the set (possibly empty) of all isomorphisms of V onto W has a natural topology. Define a functor $T : \mathcal{V} \longrightarrow \mathcal{V}$ to be continuous if T(f) depends continuously on f. Continuity of a functor $T : \mathcal{V} \times \cdots \times \mathcal{V} \longrightarrow \mathcal{V}$ covariant in the first r variables and contravariant in the remaining variables is defined similarly. Given a continuous functor T in k variables and real vector bundles $\xi_1,\ldots,\xi_k$ over a topological space B, one obtains a new vector bundle $T(\xi_1,\ldots,\xi_k)$ over B whose fibre F_b over $b \in B$ is the vector space

$T(F_1(b),\dots,F_k(b))$. Here $F_i(b)$ denotes the fibre of ξ_i over $b \in B$. For details see [32].

THEOREM 2.1

Let $\theta : T \longrightarrow T'$ be a natural equivalence of two continuous functors T, T' of $\mathcal{V} \times \cdots \times \mathcal{V}$ into $\mathcal{V}$. Then, for real vector bundles $\xi_1,\dots,\xi_k$ over B, $T(\xi_1,\dots,\xi_k)$ and $T'(\xi_1,\dots,\xi_k)$ are isomorphic vector bundles.

Proof:

Let $\xi = T(\xi_1,\dots,\xi_k)$. Let E, F, π denote the total space, fibre space and projection of ξ respectively. ξ', E', F', π' are defined similarly. Define $g_\theta : E \longrightarrow E'$ as follows:

$g_\theta(X) = \theta(F_1(b),\dots,F_k(b))(x)$ where $F_i(b)$ denotes the fibre of ξ_i over b, $1 \le i \le k$, $b = \pi(x)$. It follows readily that $g_\theta \mid \pi^{-1}(b)$ is an isomorphism of the vector space F_b onto F'_b for each $b \in B$, and that $\pi' \circ g_\theta = \pi$. The continuity of g_θ is easily verified. Thus g_θ defines a bundle isomorphism from ξ onto ξ'.

COROLLARY 2.2.

Let ξ, η and ζ be real vector bundles with the same base space B, with rank $\xi = m$, rank $\eta = n$, and rank $\zeta = 1$. Then we have the following vector bundle isomorphisms:

(i) $$\Lambda^k(\xi \oplus \eta) \approx \bigoplus_{i+j=k} \Lambda^i(\xi) \otimes \Lambda^j(\eta).$$

(ii) $$S^k(\xi \oplus \eta) \approx \bigoplus_{i+j=k} S^i(\xi) \otimes S^j(\eta).$$

(iii) $\Lambda^{\circ}(\zeta) \approx \varepsilon$, $\Lambda^{1}(\zeta) \approx \zeta$, $\Lambda^{k}(\zeta) \approx 0$ for $k \geq 2$,

$S^{\circ}(\zeta) \approx \varepsilon$, $S^{k}(\zeta) \approx \zeta \otimes \cdots \otimes \zeta$ (k-times).

Here Λ^k denotes the k^{th}-exterior power functor and S^k, the k^{th} symmetric power functor.

Proof:

Isomorphisms (i) and (ii) are functorially valid when ξ and η are replaced by any V and W in $\mathcal{V}$. Isomorphism (iii) is functorially valid if ζ is replaced by any one dimensional vector space L, ε by $\mathbb{R}$ and 0 by the vector space 0. Thus in each case these isomorphisms define natural transformations of suitable functors of $\mathcal{V} \times \mathcal{V}$ or $\mathcal{V}$ into $\mathcal{V}$. The rest of the proof now follows from Theorem 2.1.

In like manner, one could consider the category $\tilde{\mathcal{V}}$ of all oriented, finite dimensional (real) inner product spaces and vector space isomorphisms which preserve inner products and orientations. Continuity of a functor $T : \tilde{\mathcal{V}} \times \cdots \times \tilde{\mathcal{V}} \longrightarrow \tilde{\mathcal{V}}$ is defined as before. Let $\xi_1,\ldots,\xi_k$ be oriented vector bundles over B in the sense of Milnor [32]. (We will in fact generalize this in §6. Compare 6.5. See Remark 6.2.) Assume that $\xi_1,\ldots,\xi_k$ possess Euclidean metrics. As before one obtains a new vector bundle $\xi = T\,(\xi_1,\ldots,\xi_k)$ which is oriented and has a Euclidean metric. Further, if two functors T and T' are equivalent, then ξ and $\xi' = T'(\xi_1,\ldots,\xi_k)$ are isomorphic bundles.

PROPOSITION 2.3.

Let ξ be an orientable vector bundle over B in the sense of Milnor [32]. If ξ possesses a Euclidean metric then $\Lambda^k(\xi) \approx \Lambda^{n-k}(\xi)$ as vector bundles, where $n = \text{rank } \xi$.

Proof:

Choose a specific orientation and a specific Euclidean metric on ξ. Then each fibre of ξ has an orientation and an inner product.

The functors Λ^k, $\Lambda^{n-k} : \tilde{\mathcal{V}} \longrightarrow \tilde{\mathcal{V}}$ are naturally equivalent, the equivalence being established by the Hodge star operation $*$. Therefore it follows that Λ^k and Λ^{n-k} induce isomorphic vector bundles when applied to an oriented vector bundle of rank n with an Euclidean metric. Thus $\Lambda^k(\xi) \approx \Lambda^{n-k}(\xi)$ as vector bundles, as was to be shown.

COROLLARY 2.4.

Let ξ be as in Proposition 2.3 above. If rank $\xi = 2$, then $\Lambda^2(\xi) \approx \varepsilon$. If rank $\xi = 3$, then $\Lambda^2(\xi) \approx \xi$, as vector bundles.

Proof:

Immediate from the above Proposition, and the facts that $\Lambda^o(\xi) \approx \varepsilon$, $\Lambda^1(\xi) \approx \xi$ as vector bundles for any vector bundle ξ.

§3. Known Results on Span.

In the present section some well-known theorems on the span of a smooth compact manifold are stated. The reader is referred to standard sources in the literature for their proofs. These theorems link geometric properties of a manifold with its algebraic invariants.

Throughout this section by an orientable manifold is meant a manifold whose tangent bundle is orientable in the sense of Milnor [32]. (Compare 6.1, and Theorem 6.5.)

The following theorem was first proved in full generality by H. Hopf [20] in 1926. As consequences a smooth compact manifold of odd dimension admits a 1-field - i.e., a nowhere vanishing vector field, and span $S^{2n} = 0$.

THEOREM 3.1

Let M be a smooth compact manifold, and let $\chi(M)$ denote its Euler characteristic. Then span $M \geq 1$ if and only if $\chi(M) = 0$.

If there exists a nowhere vanishing vector field on M its flow, which is homotopic to the identity map of M, has no fixed points. Hence its Lefschetz number (cf. [10]) must be zero. But the Lefschetz number of any map homotopic to the identity map of M is $\chi(M)$. Thus the necessity part of the theorem follows. The converse part is more difficult. See [2].

Let $\chi^*(M)$ denote $\frac{1}{2}\chi(M) \in \mathbf{Q}$ when M is even dimensional and $\sum_{0\leq i\leq r} \dim H_i(M;\mathbf{Z}_2) \bmod 2 \in \mathbf{Z}_2$ if $\dim M = 2r + 1$. $\chi^*(M)$ is called the semi-characteristic of M. Closely related is the invariant real Kervaire semicharacteristic k(M) defined as follows: $k(M) = \Sigma \operatorname{rank} H_{2j}(M;\mathbf{Z}_2) \pmod 2$. The signature $\sigma(M)$ of a compact oriented smooth manifold M is defined to be zero if $\dim M \equiv 1,2$ or $3 \bmod 4$. If dim $M = 4k$, $\sigma(M)$ is defined as follows: Let $\mu \in H_{4k}(M;Q)$ be the (rational) fundamental class of M (see Appendix A of [32]). Let $a_1,\ldots,a_m$ be a basis for $H^{2k}(M;Q)$ such that the Kronecker index $\langle a_i \cup a_j,\mu\rangle = 0$ if $i \neq j$. $\sigma(M)$ = number of positive entries minus the number of negative entries of the diagonal matrix $(\langle a_i \cup a_j,\mu\rangle)$.

THEOREM 3.2 (Bredon-Kosinski).

Let M^d be stably parallelizable. Then (i) either M is parallelizable or span M = span S^d. (ii) for $d \neq 1,3$ or 7, M is parallelizable if and only if $\chi^*(M) = 0$.

The reader is referred to [9] for a proof. Classical results in linear algebra due to Radon [34], and Hurwitz [21] show that span $S^{n-1} \geq \rho(n) - 1$. Here ρ denotes the Radon-Hurwitz function. See §23. Using K-theory, J.F. Adams [1] proved in 1961 that span $S^{n-1} = \rho(n) - 1$.

Our next theorem gives conditions for the existence of a 2-field in the case of a compact smooth orientable M.

THEOREM 3.3

Let M be a compact smooth orientable manifold of dimension d. Then span $M \geq 2$ if and only if (i) when $d \equiv 1 \bmod 4$, one has $w_{d-1}(\mu) = 0$, $k(M) = 0$; (ii) when $d \equiv 2 \bmod 4$, $\chi(M) = 0$; (iii) $d \equiv 3 \bmod 4$ (iv) when $d \geq 4$ and $d \equiv 0 \bmod 4$, $\chi(M) = 0$, and $\sigma(M) \equiv 0 \bmod 4$.

The proof uses obstruction theory and known results on the index of a 2-field with finitely many singularities. See the survey article by

E. Thomas [44] for details. In all our applications of the above theorem, $d \equiv 1 \bmod 4$. In order to simplify the computation of $k(M)$ in these cases, we use the following theorem.

THEOREM 3.4.

Let M be a compact orientable manifold of dimension $4n + 1$. Then $\chi^*(M) - k(M)$ equals the Stiefel-Whitney number $w_2w_{4n-1}[M]$.

Proof:

Refer to [28].

In our applications, M will be a boundary manifold, i.e. the boundary of a compact smooth manifold with boundary, so that all its Stiefel-Whitney numbers are zero. Consequently $\chi^*(M)$ will equal $k(M)$.

Let F denote one of the division rings $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and let $FG_{n,k} = G_k(F^n)$, $1 \leq k \leq n-1$, denote the Grassmann manifold of k-planes in F^n. Then $FG_{n,k}$ is a compact smooth manifold of dimension $d\cdot k(n-k)$ where $d = \dim_{\mathbb{R}} F$. We write $G_{n,k}$ to abbreviate $\mathbb{R}G_{n,k}$, and $\gamma^F_{n,k}$, (or simply $\gamma_{n,k}$ if $F = \mathbb{R}$) denotes the canonical F-vector bundle over $FG_{n,k}$ of rank k whose fibre over an arbitrary point $A \in FG_{n,k}$ is the F-vector space A. With respect to the usual "Hermitian" product on F^n, $\beta^R_{n,k} = (\gamma^F_{n,k})^{\perp}$ is the complementary bundle of rank $n - k$. Its fibre over $A \in FG_{n,k}$ is the vector space $A^{\perp}$.

Consider the Ehresmann cell structure on $FG_{n,k}$ as given in [32], which is applicable for $F = \mathbb{R},\mathbb{C},\mathbb{H}$. The proof of the following lemma can be found in [19], page 452.

LEMMA 3.5.

Let $n = 2t$ or $2t + 1$ and $k = 2p$ or $2p + 1$. Let N_e and N_o denote respectively the number of cells in $FG_{n,k}$ of even and odd F-dimension. Then

$$N_e = N_o = \frac{1}{2}\begin{bmatrix} n \\ k \end{bmatrix} \text{ for n even and k odd,}$$

$$\left.\begin{aligned} N_e &= \frac{1}{2}\left(\begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} t \\ p \end{bmatrix}\right) \\ N_o &= \frac{1}{2}\left(\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} t \\ p \end{bmatrix}\right) \end{aligned}\right\} \text{ otherwise}$$

THEOREM 3.6.

$\chi(FG_{n,k}) = 0$ if and only if $F = \mathbb{R}$, n even and k odd. Consequently span $FG_{n,k} \geq 1$ if and only if $F = \mathbb{R}$, n even and k odd.

Proof:

We use the well-known property that the Euler characteristic of a finite CW complex X is equal to $\sum_{i\geq 0} (-1)^i c_i$ where c_i stands for the number of i-cells in X. Now using Lemma 3.5, and the fact that a cell of F-dimension i is a cell of real dimension di, $d = \dim_{\mathbb{R}} F$, we see readily that $\chi(FG_{n,k}) = 0$ if and only if $F = \mathbb{R}$, n even and k odd. The rest of the proof now follows from Theorem 3.1.

Let $\tilde{G}_{n,k}$ $1 \leq k \leq n-1$, denote the smooth manifold of oriented k-planes $\tilde{A}$ in $\mathbb{R}^n$. $\tilde{G}_{n,k}$ is called the oriented Grassmann manifold. The map $p : \tilde{G}_{n,k} \longrightarrow G_{n,k}$ that forgets the orientation is a (universal) double covering of $G_{n,k}$ (for $n > 2$). Therefore we obtain

COROLLARY 3.8.

$\chi(\tilde{G}_{n,k}) = 2\chi(G_{n,k})$. Thus span $\tilde{G}_{n,k} \geq 1$ if and only if n is even and k odd.

We conclude this section with the following

Example 3.9.

span $\tilde{G}_{6,3} \geq$ span $G_{6,3} \geq 2$.

Proof:

Since $p : \tilde{G}_{6,3} \longrightarrow G_{6,3}$ is a local diffeomorphism, span $\tilde{G}_{6,3} \geq$ span $G_{6,3}$. We now use Theorem 3.3. Note that $\dim G_{6,3} = 9 \equiv 1 \bmod 4$. It will be shown in Theorem 24.6 that $w(G_{6,3}) = 1 + w_1^2(\gamma_{6,3})$. Hence $w_8(G_{6,3}) = 0$. By Theorem 3.4, it follows that $k(G_{6,3}) = \chi^*(G_{6,3})$. Now

$$\chi^*(G_{6,3}) \equiv \sum_{0\leq i\leq 4} \dim H_i(G_{6,3};\mathbf{Z}_2) \bmod 2.$$

$$\equiv \sum_{0\leq i\leq 4} \dim H^{2i}(G_{6,3};\mathbf{Z}_2) \bmod 2 \text{ by Poincaré}$$

duality, and the Künneth formula. Let $P(M;t) = \sum_{i\geq 0} a_i t^i$ where $a_i = \dim H^i(M;\mathbf{Z}_2)$. $P(M,t)$ is known as the $\mathbf{Z}_2$-Poincaré polynomial of M. In this notation,

$$\chi^*(G_{6,3}) \equiv (\tfrac{1}{2}(P(G_{6,3};1) + P(G_{6,3},-1)) \bmod 2.$$

$$\equiv (\tfrac{1}{2} P(G_{6,3};1)) \bmod 2, \text{ as}$$

$$P(G_{6,3};-1) = \chi(G_{6,3}) = 0.$$

Now $P(G_{6,3};1) = 20$ using Lemma 3.5 and the knowledge of $H^*(G_{6,3};\mathbf{Z}_2)$ from Exercise 7-B of [32]. Hence $\chi^*(G_{6,3}) = 0$, completing the proof.

§4. Methods, Notations, and Results.

In this thesis, we obtain solutions to stable parallelizability and parallelizability questions for the real, complex, and quaternionic flag manifolds as well as most of the flag$^+$ manifolds, of which the oriented Grassmann manifolds are special cases. The vector field problem for Grassmann manifolds has also been considered. Span of $G_{n,k}$ has been determined for infinitely many "nontrivial" cases - i.e., the cases where n is even and k odd.

Before we state the precise results, let us describe the methods employed to obtain them.

Let M^d be a smooth manifold, with tangent bundle τM. Let $w_i \in H^i(M;\mathbf{Z}_2)$ denote the i^{th} Stiefel-Whitney class $w_i(M) = w_i(\tau M)$ of M. If the stable geometric dimension of τM is $\leq k$ - i.e., $\tau M \approx \xi \oplus (d-k)\varepsilon$ in $KO(M)$ - then $w_j(M) = 0$ for $j > k$. It follows that span $M \leq m$, if $w_{d-m}(M) \neq 0$. This is one of the most useful ways to obtain upper bounds for span M, and has been found to be effective in many cases.

If M is a (stably) parallelizable manifold and N is a submanifold of M with trivial normal bundle, a simple argument show that N is stably trivial. Thus to show that a certain manifold M is not stably parallelizable, we search for a suitable submanifold N, known a priori to be not a π-manifold, with trivial normal bundle in M. This elementary concept has been exploited in Chapters 3 and 4 to obtain results for the non-stable parallelizability of many of the manifolds considered here. In certain cases the concept of n-universality (see, e.g. [39]) has been used to guarantee the existence of maps with certain properties. The existence of such maps are then shown to lead to a contradiction, should the manifold in question be stably parallelizable. This is probably a new method.

In the other direction, for positive results on stable parallelizability, the functor Λ^2 on vector bundles has been found to be quite useful. Once stable parallelizability has been established the Bredon-Kosinski Theorem 3.2 can be applied to decide the parallelizability of M. In many cases this is no easy task. To circumvent the difficulty, a homotopy theoretic approach has been made to establish directly the parallelizability of certain flag[+] manifolds in Chapter 4. This method has been used by Zvengrowski [48] to prove parallelizability of the Stiefel manifolds $V_{n,2}$.

The earlier proofs of known results that have been re-established here use calculations, often lengthy and involving Schubert calculus, of Stiefel-Whitney classes. The proofs of stable parallelizability of $\tilde{G}_{n,k}$ and related manifolds given by I.D. Miatello and R.J. Miatello in [30] make heavy use of Lie group theory that is conceptually quite involved. Their proof that $\tilde{G}_{2^r,2}$, $r > 2$, is not parallelizable seems to contain an error. It may be noted that as far as possible computations involving Stiefel-Whitney classes have been avoided in Chapter 3 and 4. Geometric arguments have been preferred to the use of theorems from the theory of Lie groups.

In Chapter 2 the concept of orientability of (real) vector bundle has been generalised. Using this and the splitting principle, formulae for Stiefel-Whitney classes of tensor products, exterior and symmetric powers of vector bundles have been obtained, under the modest assumption that the vector bundles in question admit Euclidean metrics. Our proof is similar to that of Theorem 4.4.3 of [18] in that it is axiomatic. No assumptions have been made about the base spaces. To the best of the author's knowledge, formulae for Stiefel-Whitney classes of symmetric powers of a vector bundle (that admits a Euclidean metric) are not found in the literature.

Some of our results overlap with those of Korbaš [24], Leite and I.D. Miatello [27], I.D. Miatello and R.J. Miatello [30] and Stong [42]. It will be mentioned clearly which of our results are not new in the appropriate context. In all cases our proofs are new and apparently much simpler.

F denotes one of the division rings $\mathbb{R},\mathbb{C}$ or $\mathbb{H}$. Let $\mu = (n_1,\ldots,n_s)$ be a sequence of positive integers with $s \geq 2$. Let $1 \leq r \leq s$, and let $n = n_1+\cdots+n_s$. The following notations will be used in the sequel. Details will appear in §10 and §15.

$$FG(\mu) = U(F^n)/(U(F^{n_1}) \times \cdots \times U(F^{n_s})),$$

$$G(\mu;r) = O(n)/(SO(n_1) \times \cdots \times SO(n_r) \times O(n_{r+1}) \times \cdots \times O(n_s))$$

$FG(\mu)$ are called F-flag manifolds. When $F = \mathbb{R}$ we write $G(\mu)$ to abbreviate $FG(\mu)$. $G(\mu;r)$ are called the flag$^+$ manifolds, in the sense that they are flag manifolds with the additional structure of orientations on the first r components of each flag.

The following are the main results of this thesis:

4.1. $\mathbb{Z}$-orientability in the sense of Definition 6.1 of a (real) vector bundle ξ is equivalent to $w_1(\xi) = 0$. Also proofs of formulae for the Stiefel-Whitney classes of tensor products, exterior and symmetric powers of vector bundles in a general setting.

4.2. (Stable) parallelizability of flag manifolds: Assume $s \geq 3$ ($s = 2$ case is known due to Trew and Zvengrowski [45]). $FG(\mu)$ is stably parallelizable if and only if $n_1 = \cdots = n_s = 1$, and is parallelizable if and only if $n_1 = \cdots = n_s = 1$, $F = \mathbb{R}$.

4.3. (Stable) parallelizability of flag$^+$ manifolds: The oriented Grassmann manifold $\tilde{G}_{n,k}$, $2 \leq k \leq n-2$, is stably parallelizable if and only if $n = 2k$, $k = 2,3$ and is parallelizable only in the case $n = 6$, $k = 3$.

Now assume that $s \geq 3$, $1 \geq r \geq s$. The list below shows the relatively "small" subcollection of the flag$^+$ manifolds for which the parallelizability is not solved. More precisely, up to diffeomorphisms arising from the permutations of $n_1,\ldots,n_r$ and $n_{r+1},\ldots,n_s$, the flag$^+$ manifolds $G(n_1,\ldots,n_s;r)$ not classified below known to be either parallelizable, or stably parallelizable but non-parallelizable, or not stably parallelizable. See Theorems 17.1, 17.3 and 18.1.

The following manifolds are not known to be stably parallelizable:

(i) $G(6,1,\ldots,1;s-2)$ $n \geq 10$

(ii) $G(6,1,\ldots,1;s-3)$ $n \geq 9$

(iii) $G(1,\ldots,1,k;s-2)$ $k = 3,7,$ $s \geq 3.$

<u>4.4</u>. Span of flag manifolds:

(i) Span $G(n_1,\ldots,n_s) \geq \rho(n) - 1.$

(ii) Span $G(n_1,\ldots,n_s) \geq 1$ if and only at least two of the numbers $n_1,\ldots,n_s$ are odd.

(iii) Span $G_{m(2^n+1),3} = \rho(m) - 1$ if $m = 4,8,$ $r \geq 1$

≥ 2 if $m = 2.$

(iv) Span $G_{n,k} \geq 2$ only if $\begin{bmatrix} n \\ k \end{bmatrix} \equiv 0 \bmod 4.$

(v) $3 \leq$ Span $G_{6,3} \leq 7.$

The Chapters 2-6 are arranged as follows. Chapter 2 deals with orientability of real vector bundles and formulae for tensor product, exterior and symmetric powers of vector bundles. (Stable) parallelizability of real, complex and quaternionic flag manifolds is considered in Chapter 3. In Chapter 4, we address the same question for oriented Grassmann manifolds and flag$^+$ manifolds. Chapter 5 deals with $\mathbf{Z}_2$-cohomology of the flag manifolds of the type $\mu = (1,\ldots,1,n-k)$. Computational techniques developed here are used in Chapter 6 to obtain certain Stiefel-Whitney classes of Grassmann manifolds, lower and upper bounds for span $G_{n,k}$.

The following diagram shows how various manifolds that will be encountered here are related. (For the definition of $X_{n,k}$ see §10.)

$$\begin{array}{ccccc}
 & & \{\text{F-Flag Manifolds}\} & & \\
 & & \cup & & \\
\{\text{F - Projective spaces}\} & \subset & \{\text{F-Grassmann Manifolds}\} & & \\
 & & & & \\
\{\text{Oriented Grassmann Manifolds}\} & \subset & \{\text{Flag}^{+}\text{ Manifolds}\} & \supset & X_{n,2} \\
\cup & & \cup & & \\
\{\text{Spheres}\} & \subset & \{\text{Stiefel Manifolds}\} & &
\end{array}$$

CHAPTER TWO

GENERALITIES ON STIEFEL-WHITNEY CLASSES

§5. Introduction

In this chapter we establish formulae for the Stiefel-Whitney classes of tensor products, symmetric and exterior powers of real vector bundles which admit Euclidean metrics.

In [36] it is shown that a real vector bundle ξ over an arbitrary base space B is orientable if and only if $w_1(\xi) = 0$. This is proved in §6. In §12 of [32] it is proved that if ξ is orientable over a paracompact base space B then $w_1(\xi) = 0$. That the converse is also true for CW complexes is left as an exercise (see Problem 12A). Exercise H on Page 281 of [38] deals with Stiefel-Whitney classes of sphere bundles, and 3(d) of this exercise states that a sphere bundle η is orientable if and only if $w_1(\eta) = 0$. In case ξ possesses a Euclidean metric one can apply this result to the associated sphere bundle of ξ. But when B is not paracompact, it is not true in general that a vector bundle ξ over B possesses a Euclidean metric. See Example 6.8.

In §7 the splitting principle is proved using the Leray-Hirsch theorem.

Theorem 8.3, which contains the main results of this chapter is proved using Theorem 6.7 and the results of §6. These, and analogous results for Chern classes in the case of complex vector bundles, are obtained in [7] under suitable restrictions on the base spaces, using entirely different techniques. The axiomatic approach of the proof of Theorem 8.3 parallels the one found in §4 of [18] in the case of complex vector bundles.

§6. Orientability of Vector Bundles.

An orientation on a real vector space V of dimension $n > 0$ is an equivalence class of ordered bases of V. Here two ordered bases $u_1, \ldots u_n$ and $v_1, \ldots, v_n$ are equivalent if and only if the matrix (a_{ij}) has positive determinant, where $v_i = \Sigma\, a_{ij} u_j$. Thus there are precisely two orientations on any real vector space $V \neq 0$. The vector space $\mathbb{R}^n$ has a standard orientation given by the canonical ordered basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$.

It is shown easily that choosing a specific orientation on a vector space $V \neq 0$ is equivalent to choosing a specific generator for the cohomology group $H^n(V, V_o; \mathbf{Z}) \approx \mathbf{Z}$. Here $V_o = V - \{0\}$. See Page 95 of [32].

Let R be a commutative ring with identity, 1_R. Let ξ be a real vector bundle of rank $n > 0$ over an arbitrary base space B. Let $E = E(\xi)$ denote the total space and E_o the complement of the zero cross section in E. Let $\pi : E \to B$ denote the projection of ξ and π_o the restriction of π to E_o. For $b \in B$, F_b denotes the fibre of ξ over b and $F_{b,o}$ the non-zero elements of F_b. Let j_b denote the inclusion $(F_b, F_{b_o}) \to (E, E_o)$.

DEFINITION 6.1

ξ is said to be R-orientable if there exists an assignment, called an R-orientation on ξ, of a preferred generator u_b^R of $H^n(F_b, F_{b,o}; R) \cong R$ to each $b \in B$ satisfying the following local compatibility condition:

For each $b_o \in B$, there exists a neighbourhood N of b_o and an element u_N^R in $H^n(\pi^{-1}(N), \pi_o^{-1}(N); R)$ such that $u_N^R | (F_b, F_{b,o}) = u_b^R$ for all b in N. Here $u_b^N = \mu_b^*(u_N^R)$ where μ_b is the inclusion $(F_b, F_{b,o}) \to (\pi^{-1}(N), \pi_o^{-1}(N))$. ξ is said to be orientable if it is $\mathbf{Z}$-orientable.

If ξ is R-orientable, we denote an R-orientation on ξ by $\{u_b^R\}_{b \in B}$ or by $\{u_b\}$ if R is clear from the context.

Note that if ξ is orientable, then it is R-orientable for every commutative ring R with identity. In fact, letting r denote the unique ring homomorphism $\mathbf{Z} \to R$, if $\{u_b^{\mathbf{Z}}\}$ is a $\mathbf{Z}$-orientation of ξ then $\{r_*(u_b^{\mathbf{Z}})\}$ gives an R-orientation of ξ where r_* denotes the map in cohomology induced by the homomorphism r between the coefficient groups.

Let R denote either $\mathbf{Z}_2$ or, if ξ is $\mathbf{Z}$-orientable, an arbitrary commutative ring with identity. Let u_b^R denote the unique non-zero element of $H^n(F_b, F_{b,o}; \mathbf{Z}_2) \cong \mathbf{Z}_2$, and if ξ is $\mathbf{Z}$-oriented with orientation $\{u_b^{\mathbf{Z}}\}$, let $u_b^R = r_*(u_b^{\mathbf{Z}})$. We have the following Thom Isomorphism Theorem. For a proof see § 10 of [32].

THEOREM 6.2

There exists a unique cohomology class u^R in $H^n(E,E_o;R)$ such that $j_b^*(u^R) = u_b^R$ for each $b \in B$. Moreover, the correspondence $y \mapsto y \cup u^R$ maps $H^j(E;R)$ isomorphically onto $H^{j+n}(E,E_o;R)$ for every integer j.

u^R is called the Thom class of ξ.

Remark 6.3

Our definition of $\mathbb{Z}$-orientability is weaker in general than the one used in p.96 of [32]. However the proof of the above theorem in the oriented case as given in §10 of [32] requires only that ξ be $\mathbb{Z}$-oriented in the sense formulated above.

Example 6.4.

Consider the deleted comb space

$D = I \times 0 \cup \{\frac{1}{n} \mid n \geq 1\} \times I \cup \{(0,1)\} \subset \mathbb{R} \times \mathbb{R}$ where $I = [0,1] \subset \mathbb{R}$.

D has two path components $P_1 = \{(0,1)\}$, $P_2 = D - P_1$. Hence $H^o(D;\mathbb{R}) \cong \mathbb{Z} \times \mathbb{Z}$. Using the Künneth formula we obtain $H^1(D \times (\mathbb{R},\mathbb{R}-0)) \cong (\mathbb{Z} \times \mathbb{Z}) \otimes H^1(\mathbb{R},\mathbb{R}-0;\mathbb{Z})$. Let α be the generator of $H^1(\mathbb{R},\mathbb{R}-0;\mathbb{Z}) \cong \mathbb{Z}$ that corresponds to the standard orientation on $\mathbb{R}$. Let $U = (-1,1) \otimes \alpha$. Now consider the trivial line bundle $\mathcal{E}$ over D whose total space is $D \times \mathbb{R}$. It is readily verified that $U|(F_b,F_{b,o})$ is a generator of $H^1(F_b,F_{b,o};\mathbb{Z})$ for each $b \in D$. U defines a $\mathbb{Z}$-orientation Ω of $\mathcal{E}$. In fact the orientation of $F_b = b \times \mathbb{R}$ given by $U|(F_b,F_{b,o})$ is the standard orientation on $b \times \mathbb{R} \cong \mathbb{R}$ if and only if $b \neq (0,1)$. Since D is connected, it follows that there is no neighbourhood of (0,1) over which there exists an *orientation preserving* local trivialization of the $\mathbf{Z}$-oriented vector bundle (ε,Ω). Therefore Ω is not an orientation in the sense of Milnor [32].

The following theorem shows, for example, that in the context of manifolds the concept of orientability used here coincides with that used in Milnor's book [32] page 96.

THEOREM 6.5.

Let B be locally path connected. Let ξ^n be a $\mathbf{Z}$-oriented vector bundle. Then, given any $b_o \in B$ there exists a neighbourhood (V,h) with $b_o \in V$ such that $h : V \times \mathbb{R}^n \longrightarrow \pi^{-1}(V)$ is orientation preserving – i.e. for each $b \in V$, $h_b\colon b \times \mathbb{R}^n \longrightarrow F_b$ maps the standard basis of $\mathbb{R}^n$ onto an ordered basis in the orientation of F_b.

Proof:

Let $u_b^{\mathbf{Z}}$ be the preferred generator of $H^n(F_b,F_{b,o};\mathbf{Z}) \cong \mathbf{Z}$ for each $b \in B$. Let $b_o \in B$ be given. Choose a neighbourhood (N,h) as in Definition 6.1. Since B is locally path connected there exists a path connected open set $V \subset N$ with $b_o \in V$. Denote $h|V$ by the same letter h. Let $u_V^{\mathbf{Z}} = u_N^{\mathbf{Z}} \mid (\pi^{-1}(V),\pi_o^{-1}(V))$. It follows readily that $u_V^{\mathbf{Z}} \mid (F_b,F_{b,o})$ $= u_b^{\mathbf{Z}}$ for all $b \in V$. Without loss of generality assume that h preserves orientation at b_o. Equivalently, we assume that $h_{b_o}^*(u_{b_o}^{\mathbf{Z}})$ = the generator α of $H^n(b_o \times (\mathbb{R}^n,\mathbb{R}^n-0);\mathbf{Z}) \cong H^n(\mathbb{R}^n,\mathbb{R}^n-0;\mathbf{Z}) \cong \mathbf{Z}$ that corresponds to the standard orientation on $\mathbb{R}^n \cong b_o \times \mathbb{R}^n$.

Since V is path connected,

$H^n(V \times (\mathbb{R}^n,\mathbb{R}^n-0);\mathbf{Z}) \cong \mathbf{Z} \otimes H^n(\mathbb{R}^n,\mathbb{R}^n-0;\mathbf{Z})$ by the Künneth formula.

Writing $h^*(u_V^{\mathbf{Z}})$ as $1 \otimes k\alpha$ for some integer $k \in \mathbf{Z}$, we see from the commutativity of the diagram below with $b = b_o$,

$$\begin{array}{ccc} H^n(\pi^{-1}(V),\pi_o^{-1}(V);\mathbf{Z}) & \xrightarrow{h_b^*} & H^n(V\times(\mathbb{R}^n,\mathbb{R}^n-0);\mathbf{Z}) \\ \Big\downarrow \text{"restriction"} & & \Big\downarrow \text{"restriction"} \\ H^n(F_b,F_{b,o};\mathbf{Z}) & \xrightarrow{h_b^*} & H^n(b\times(\mathbb{R}^n,\mathbb{R}^n-0);\mathbf{Z}) \end{array}$$

that $(1 \otimes k\alpha)|(b_o \times (\mathbb{R}^n,\mathbb{R}^n-0)) = \alpha$. Hence $k = 1$. Therefore for any $b \in V$, $(h^*(u_V^{\mathbf{Z}}))|(b \times (\mathbb{R}^n,\mathbb{R}^n-0)) = (1 \otimes \alpha)|(b \times (\mathbb{R}^n,\mathbb{R}^n-0)) = \alpha$. By the commutativity of the above diagram again, $h_b^*(u_b^{\mathbf{Z}}) = \alpha$. Equivalently, $h_b\colon b \times \mathbb{R}^n \longrightarrow F_b$ preserves orientation, as was to be shown.

Theorem 6.2 shows, among other things, that every real vector bundle ξ is $\mathbf{Z}_2$-orientable. The following proposition gives a criterion for orientability in terms of R-orientability. It is my pleasant duty to thank Prof. K. Varadarajan who helped me with the following proposition.

Let U(R) denote the group of units in R.

PROPOSITION 6.6

Let ξ be a real vector bundle of rank n over B and R a commutative ring with the property that U(R) is a group with two elements $\pm 1_R$. Then ξ is orientable if and only if ξ is R-orientable.

Proof:

From the comments in an earlier paragraph we see that to prove the proposition we have only to show that if ξ is R-orientable, then it is $\mathbf{Z}$-orientable.

Assume that ξ is R-orientable. Let $\{u_b^R\}$ be an R-orientation on ξ. Thus u_b^R is a generator of $H^n(F_b,F_{b,o};R) \cong R$ for each $b \in B$. Further, given $b \in B$, there exists an open set N in B with $b \in N$ and an element $u_N^R \in H^n(\pi^{-1}(N),\pi_o^{-1}(N);R)$ satisfying $\mu_c^*(u_N^R) = u_c^R$ for all $c \in N$. We assume, as we may, that $\xi|N$ is trivial. Let $\theta : \pi^{-1}(N) \to N \times \mathbf{R}^n$ yield a trivialization of $\xi|N$.

Let $\{N_\gamma\}_{\gamma\in\Gamma}$ denote the path components of N, so that $H^o(N_\gamma;R) \cong R$ for each $\gamma \in \Gamma$. Choose a generator $e^1 \in H^1(\mathbf{R},\mathbf{R}-0;\mathbf{Z}) \cong \mathbf{Z}$. Then $e^n = e'\times\ldots\times e' \in H^n(\mathbf{R}^n,\mathbf{R}^n-0;\mathbf{Z})$ is a generator of $H^n(\mathbf{R}^n, \mathbf{R}^n-0;\mathbf{Z}) \cong \mathbf{Z}$. Using the isomorphism $H^o(N_\gamma;\mathbf{Z}) \xrightarrow{\times e^n} H^n(N_\gamma \times \mathbf{R}^n,N_\gamma \times (\mathbf{R}^n-0);\mathbf{Z})$ and $H^o(N_\gamma;R) \xrightarrow{\times r_*(e^n)} H^n(N_\gamma \times \mathbf{R}^n,N_\gamma \times (\mathbf{R}^n-0);R)$ (see p.106 of [32]) we see that $H^n(N_\gamma \times \mathbf{R}^n,N_\gamma \times (\mathbf{R}^n-0);R) \cong R$. Let $\theta_\gamma = \theta|\pi^{-1}(N_\gamma)$. Then $\theta_\gamma^*: H^n(N_\gamma \times \mathbf{R}^n,N_\gamma \times (\mathbf{R}^n-0);\mathbf{Z}) \longrightarrow H^n(\pi^{-1}(N_\gamma),\pi_o^{-1}(N_\gamma);\mathbf{Z})$ and $\theta_\gamma^*: H^n(N_\gamma \times \mathbf{R}^n,N_\gamma \times (\mathbf{R}^n-0);R) \longrightarrow H^n(\pi^{-1}(N_\gamma), \pi^{-1}(N_\gamma);R)$ are isomorphisms. Identifying $H^n(\pi^{-1}(N),\pi_o^{-1}(N);R)$ with $\prod_{\gamma\in\Gamma} H^n(\pi^{-1}(N_\gamma),\pi_o^{-1}(N_\gamma);R)$ we see that $u_N^R = \prod_{\gamma\in\Gamma} u_\gamma^R$ where $u_\gamma^R \in H^n(\pi^{-1}(N_\gamma), \pi_o^{-1}(N_\gamma);R) \cong R$ is an R-generator for R. Since $r : \mathbf{Z} \longrightarrow R$ maps $U(\mathbf{Z})$ isomorphically onto $U(R)$, we see that there exists a unique generator $u_\gamma^{\mathbf{Z}}$ in $H^n(\pi^{-1}(N_\gamma),\pi_o^{-1}(N_\gamma);\mathbf{Z}) \cong \mathbf{Z}$ with the property that $r_*(u_\gamma^{\mathbf{Z}}) = u_\gamma^R$.

For the same reason as above, there exists a unique generator $u_b^{\mathbf{Z}}$ in $H^n(F_b, F_{b,o}; \mathbf{Z}) \cong \mathbf{Z}$ with $r_*(u_b^{\mathbf{Z}}) = u_b^R$ for each $b \in B$. We claim that the assignment of $u_b^{\mathbf{Z}}$ to b for each b in B is a $\mathbf{Z}$-orientation of ξ. In fact consider the element $u_N^{\mathbf{Z}} = \prod_{\gamma\in\Gamma} u_\gamma^{\mathbf{Z}}$ in $\prod_{\gamma\in\Gamma} H^n(\pi^{-1}(N_\gamma), \pi_o^{-1}(N_\gamma); \mathbf{Z}) \cong H^n(\pi^{-1}(N), \pi_o^{-1}(N); \mathbf{Z})$. Clearly, we have $r_*(u_N^{\mathbf{Z}}) = u_N^R$. From the commutativity of the following diagram

$$\begin{array}{ccc}
H^n(\pi^{-1}(N), \pi_o^{-1}(N); \mathbf{Z}) & \xrightarrow{\mu_c^*} & H^n(F_c, F_{c,o}; \mathbf{Z}) \\
{\scriptstyle r_*}\downarrow & & \downarrow{\scriptstyle r_*} \\
H^n(\pi^{-1}(N), \pi_o^{-1}(N); R) & \xrightarrow{\mu_c^*} & H^n(F_c, F_{c,o}; R)
\end{array}$$

for any $c \in N$, we see that $r_*\mu_c^*(u_N^{\mathbf{Z}}) = u_c^R$. But $\mu_c^*(u_N^{\mathbf{Z}})$ is a generator of $H^n(F_c, F_{c,o}; \mathbf{Z}) \cong \mathbf{Z}$. Since $u_c^{\mathbf{Z}}$ is the unique generator of $H^n(F_c, F_{c,o}; \mathbf{Z}) \cong \mathbf{Z}$ satisfying $r_*(u_c^{\mathbf{Z}}) = u_c^R$ we obtain $\mu_c^*(u_N^{\mathbf{Z}}) = u_c^{\mathbf{Z}}$ for all $c \in N$. This completes the proof of Proposition 6.6.

We are now ready to prove the following main theorem of this section.

THEOREM 6.7.

Let ξ be a real vector bundle over an arbitrary base space B. Then ξ is orientable if and only if $w_1(\xi) = 0$.

Proof:

Assume that ξ has rank $n \geq 1$. We know that ξ is always $\mathbf{Z}_2$-orientable and that there exists a unique element $u^{\mathbf{Z}_2} \in H^n(E, E_o; \mathbf{Z}_2)$

with the property that $j_b^*(u^{\mathbf{Z}_2})$ is the non-zero element of $H^n(F_b, F_{b,o}; \mathbf{Z}_2) \cong \mathbf{Z}_2$ for all $b \in B$. Then, by definition, $w_1(\xi) = \phi^{-1}(Sq^1\, u^{\mathbf{Z}_2})$ where $\phi : H^1(B;\mathbf{Z}_2) \longrightarrow H^{n+1}(E,E_o;\mathbf{Z}_2)$ is the Thom isomorphism. Thus $w_1(\xi) = 0$ if and only if $Sq^1\, u^{\mathbf{Z}_2} = 0$. Consider the exact sequence

$$0 \longrightarrow \mathbf{Z}_2 \xrightarrow{s} \mathbf{Z}_4 \xrightarrow{\epsilon} \mathbf{Z}_2 \longrightarrow 0 \qquad \text{(i)}$$

where $s(1) = 2$ and ϵ is the canonical quotient map, that is $\epsilon(1) = 1$. Then $Sq^1 \colon H^n(E,E_o;\mathbf{Z}_2) \longrightarrow H^{n+1}(E,E_o;\mathbf{Z}_2)$ is same as the connecting homomorphism β in the exact cohomology sequence

$$\begin{aligned} \cdots \longrightarrow H^n(E,E_o;\mathbf{Z}_2) &\xrightarrow{s_*} H^n(E,E_o;\mathbf{Z}_4) \xrightarrow{\epsilon_*} H^n(E,E_o;\mathbf{Z}_2) \\ &\xrightarrow{\beta} H^{n+1}(E,E_o;\mathbf{Z}_2) \longrightarrow \cdots \qquad \text{(ii)} \end{aligned}$$

corresponding to the coefficient sequence (i). Hence $w_1(\xi) = 0 \Leftrightarrow \beta(u^{\mathbf{Z}_2}) = 0 \Leftrightarrow u^{\mathbf{Z}_2} = \epsilon_*(u^{\mathbf{Z}_4})$ for some element $u^{\mathbf{Z}_4} \in H^n(E,E_o;\mathbf{Z}_4)$.

Assume that $w_1(\xi) = 0$, and choose an element $u^{\mathbf{Z}_4} \in H^n(E,E_o;\mathbf{Z}_4)$ with $\epsilon_*(u^{\mathbf{Z}_4}) = u^{\mathbf{Z}_2}$. Under $\epsilon : \mathbf{Z}_4 \longrightarrow \mathbf{Z}_2$ the only elements of $\mathbf{Z}_4$ that get mapped onto $1 \in \mathbf{Z}_2$ are ± 1 of $\mathbf{Z}_4$ and they are the multiplicative units in the ring $\mathbf{Z}_4$. Define $u_b^{\mathbf{Z}_4} = j_b^*(u^{\mathbf{Z}_4})$ for all $b \in B$. From the commutativity of the diagram below,

$$\begin{tikzcd} H^n(E,E_o;\mathbf{Z}_4) \arrow[r, "j_b^*"] \arrow[d, "\epsilon_*"] & H^n(F_b,F_{b,o};\mathbf{Z}_4) \arrow[d, "\epsilon_*"] \\ H^n(E,E_o;\mathbf{Z}_2) \arrow[r, "j_b^*"] & H^n(F_b,F_{b,o};\mathbf{Z}_2) \end{tikzcd}$$

and the facts $H^n(F_b,F_{b,o};\mathbf{Z}_4) \cong \mathbf{Z}_4$; $H^n(F_b,F_{b,o};\mathbf{Z}_2) \cong \mathbf{Z}_2$ we see that $u_b^{\mathbf{Z}_4}$ is a generator of $H^n(F_b,F_{b,o};\mathbf{Z}_4) \cong \mathbf{Z}_4$ for all $b \in B$. This shows that the assignment of $u_b^{\mathbf{Z}_4}$ to each b in B is a $\mathbf{Z}_4$-orientation of ξ. Since $U(\mathbf{Z}_4) = \{\pm 1\}$, it follows by Proposition 6.6 that ξ is orientable.

Conversely, assume that ξ is orientable. Then by the Thom Isomorphism Theorem, there exists an element $u^{\mathbf{Z}_4} \in H^n(E,E_o;\mathbf{Z}_4)$ with $j_b^*(u^{\mathbf{Z}_4})$ a generator of $H^n(F_b,F_{b,o};\mathbf{Z}_4)$ for all $b \in B$. Then $\epsilon_* j_b^*(u^{\mathbf{Z}_4})$ is the unique non-zero element of $H^n(F_b,F_{b,o};\mathbf{Z}_2)$. The commutativity of the above diagram shows that the element $\epsilon_*(u^{\mathbf{Z}_4}) \in H^n(E,E_o;{}_2)$ has the property that $j_b^*\epsilon_*(u^{\mathbf{Z}_4})$ is the non-zero element of $H^n(F_b,F_{b,o};\mathbf{Z}_2)$. It follows therefore that $w_1(\xi) = 0$.

This completes the proof.

Example 6.8.

Let L^+ denote the Alexandroff half line and ξ the tangent bundle of L^+ with respect to some smoothness structure on L^+. Then ξ is an orientable line bundle as can be seen either directly or by using Theorem 6.7. But L^+ is a manifold that is not paracompact. Hence ξ does not admit a Euclidean metric. Since clearly every trivial vector bundle admits a Euclidean metric, ξ is an example of a line bundle that is orientable but not trivial.

The following theorem gives another characterisation of orientability of a real vector bundle ξ over an arbitrary base space B.

THEOREM 6.9.

Let ξ be a real vector bundle of rank $n \geq 1$ over an arbitrary base space B. Then ξ is orientable if and only if the line bundle $\Lambda^n(\xi)$ is orientable.

Proof:

By Theorem 6.7 we need only to show that $w_1(\xi) = 0$ if and only if $w_1(\Lambda^n(\xi)) = 0$. In fact we will prove that $w_1(\xi) = w_1(\Lambda^n(\xi))$. By Theorem 6.7 and the Whitney product formula, again, this is equivalent to showing that $\xi \oplus \Lambda^n(\xi)$ is orientable. Let $b \in B$. Let $v_1,\dots,v_n$ be any basis for F_b. Then the orientation of the vector space determined by the ordered basis $v_1,\dots,v_n$, $v_1\wedge \dots \wedge v_n$ is independent of the choice of the basis $v_1,\dots,v_n$ of F_b, and hence orients $F_b \oplus \Lambda^n(F_b)$ canonically.

Let $\xi' = \xi \oplus \Lambda^n(\xi)$, $E' = E(\xi')$, π' the projection of ξ' and F_b' the fibre, $F_b \oplus \Lambda^n(F_b)$, of ξ' over b for each $b \in B$. Let b_o be any element of B. Choose an open set N with $b_o \in N$ and $\xi|N$ trivial. Let $h : N \times \mathbb{R}^n \longrightarrow \pi^{-1}(N)$ be an explicit trivialization for $\xi|N$. Then $h' : N \times \mathbb{R}^{n+1} \longrightarrow \pi'^{-1}(N)$ is defined by

$h'(b,x,y) = (h(b,x), \Lambda^n(h_b)(y))$, for $b \in N$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, where $h_b = h|b\times\mathbb{R}^n$ and $\Lambda^n(h_b) : b \times \Lambda^n(\mathbb{R}) \longrightarrow \Lambda^n(F_b)$ is the map induced by

h_b. Note that h' is orientation preserving, that is, h_b' maps the standard oriented basis of $b \times \mathbb{R}^{n+1} = b \times (\mathbb{R}^n \oplus \Lambda^n(\mathbb{R}^n))$ onto the canonically oriented basis of F_b' for each $b \in N$. This implies that ξ' is orientable, completing the proof. (See Remark 6.3)

We record the following fact established in the above proof as a lemma.

LEMMA 6.10.

Let ξ be a real vector bundle of rank n over an arbitrary base space B. Then $w_1(\xi) = w_1(\Lambda^n(\xi))$.

Example 6.11.

Let ξ be a real vector bundle of rank 3 that admits a Euclidean metric. Then $\Lambda^2(\xi) \approx \xi \otimes \Lambda^3(\xi)$, as vector bundles.

Proof:

Let $\zeta = \Lambda^3(\xi)$. Then ζ admits a Euclidean metric. It is easily verified that the map $(x \otimes su) \wedge (y \otimes tu) \longrightarrow st(x \wedge y)$ for $x \otimes su$, $y \otimes tu$ of $\xi \otimes \zeta$ and u a unit vector (with respect to some Euclidean metric on ζ) is a bundle isomorphism. Thus $\Lambda^2(\xi) \approx \Lambda^2(\xi \otimes \zeta)$. Since $\xi \otimes \zeta$ is orientable by Lemma 6.10, we see that, according to Corollary 2.4

$$\Lambda^2(\xi \otimes \zeta) \approx \xi \otimes \zeta.$$

Hence
$$\Lambda^2(\xi) \approx \xi \otimes \zeta.$$

§7 The Splitting Principle.

In this section we continue to use the notations of §6.

Let ξ be a real vector bundle of rank $n \geq 1$ over an arbitrary base space B. We construct a fibre bundle $P(\xi)$ over B with fibre space $\mathbb{R}P^{n-1}$ as follows. Denoting both the fibre bundle and its total space by the same symbol $P(\xi)$, $P(\xi) = \bigcup_{b\in B} \{[x] \mid x \in F_{b,o}\}$. Here $[x]$ denotes the 1-dimensional subspace of F_b spanned by x, for $x \in F_{b,o}$. Thus $P(\xi)$ is the set of all equivalence classes of E_o under the equivalence relation: $x \sim x'$ for $x,x' \in E_o$ if and only if $\pi_o(x) = \pi_o(x')$ and $x = tx'$ for some $t \in \mathbb{R} - \{0\}$. $P(\xi)$ is then given the quotient topology. The map $p : P(\xi) \longrightarrow B$ induced by π_o is the projection of the bundle $P(\xi)$. Clearly $p^{-1}(b) \cong \mathbb{R}P^{n-1}$ for each $b \in B$. The $GL(n)$-action on ξ induces a $GL(n)$-action on $P(\xi)$. $P(\xi)$ is called the projective bundle associated to ξ.

We now assume that ξ possesses a Euclidean metric. Consequently $p^*(\xi)$, the induced bundle, also possesses a Euclidean metric. Consider the canonical line bundle ζ over $P(\xi)$ whose fibre over a point $[x] \in P(\xi)$ is the vector space $[x]$. Clearly ζ is a subbundle of $p^*(\xi)$. Since $p^*(\xi)$ possesses a Euclidean metric, ζ has a complementary subbundle η of rank $n - 1$. Thus $\zeta \oplus \eta \approx p^*(\xi)$.

Remark 7.1

More generally, given a Euclidean vector bundle ξ, and a flag manifold $G(\mu)$ with $|\mu| = n = \operatorname{rank} \xi$, one can associate with it a fibre bundle $G(\mu)(\xi)$ called the $G(\mu)$-bundle over B associated to ξ, with fibre space the real flag manifold $G(\mu)$. There exist canonical n_i-plane bundles ξ_i, $1 \le i \le s$, over $G(\mu)(\xi)$ where $\mu = (n_1,\ldots,n_s)$, sastisfying the following relation:

$\xi_1 \oplus \cdots \oplus \xi_s \approx p^*(\xi)$, where p is the projection $p : G(\mu)(\xi) \longrightarrow B$.

The main theorem of this section is the following:

THEOREM 7.2. (The Splitting Principle.)

Let ξ be a Euclidean vector bundle of rank n over an arbitrary base space B. Then there exists a space B' and a map $f : B' \longrightarrow B$ such that $f^*(\xi)$ splits as a Whitney sum of n line bundles $\xi_1,\ldots,\xi_n$ and $f^*: H^*(B;\mathbf{Z}_2) \longrightarrow H^*(B';\mathbf{Z}_2)$ is a monomorphism.

$f': B' \longrightarrow B$ or, by an abuse of terminology, B', will be called a splitting bundle of ξ in this case.

In order to prove the above theorem we need the following form of the Leray-Hirsch Theorem.

THEOREM 7.3.

Let $p : X \longrightarrow B$ be a locally trivial fibre bundle over an arbitrary base space B with fibre F. Let K be a field. Suppose that there exist finitely many elements $x_1,\ldots,x_r \in H^*(X ; K)$ such that $i_b^*(x_1),,,.,i_b^*(x_r)$ forms a K-basis for $H^*(p^{-1}(b);K)$ for each $b \in B$, where $i_b: p^{-1}(b) \longrightarrow X$ denotes the inclusion of the fibre over b.

Then

$$\Phi : H^*(B\ ;\ K) \otimes H^*(F\ ;\ K) \longrightarrow H^*(E\ ;\ K) \text{ defined by}$$

$$\Phi(y \otimes \alpha_i) = p^*(y) \cup x_i \text{ is a K-isomorphism.}$$

Here $\alpha_i = i_b^*(x_i)$, $1 \le i \le r$, under the identification of F with $p^{-1}(b)$ for some $b \in B$.

Method of Proof: First one proves the above theorem for product bundles where $p : X \longrightarrow B$ is the first projection, $X = B \times F$. Then, using an argument involving a Mayer-Vietoris sequence, one proves the theorem for bundles where $B = B_1 \cup B_2$ and the theorem is known to be valid for $p_i : p^{-1}(B_i) \longrightarrow B_i$, $p_i = p|p^{-1}(B_i)$, $i = 1,2$. An obvious induction shows that the theorem is valid when $B = B_1 \cup \cdots \cup B_k$ and the theorem is known to be valid for $p_i : p^{-1}(B_i) \longrightarrow B_i$, $1 \le i \le k$. Finally, in the general case one uses local triviality of the bundle and the fact that, since K is a field, $H^i(B;K)$ is naturally isomorphic to the inverse limit, $\varprojlim H^i(C;K)$, of $H^i(C;K)$ as C runs through all compact subsets of B.

Remarks 7.4.

The hypothesis of the above theorem can be weakened. It is easy to see that the theorem still holds if one assumes only that $i_b^*(x_1),\ldots,i_b^*(x_r)$ forms a K-basis for $H^*(p^{-1}(b);K)$ for <u>some</u> b in each path component of B.

PROPOSITION 7.5.

Let ξ be a real vector bundle of constant rank n over an arbitrary base space B. Let $a = w_1(\zeta)$ where ζ is the canonical line bundle over the projective bundle $P(\xi)$ of ξ. Then $H^*(P(\xi);\mathbf{Z}_2)$ is an $H^*(B;\mathbf{Z}_2)$-module freely generated by the basis elements $1,a,\ldots,a^{n-1}$, where $y\cdot\alpha = p^*(y) \cup \alpha$ for $y \in H^*(B;Z_2)$, $\alpha \in H^*(P(\xi);\mathbf{Z}_2)$.

Proof:

First we note that, denoting by i_b the inclusion of the fibre over $b \in B$ into $P(\xi)$, $i_b^*(\zeta)$ is the line bundle over $p^{-1}(b) = P(F_b) \cong P(\mathbb{R}^n) = \mathbb{R}P^{n-1}$ whose fibre over $[x] \in p^{-1}(b)$ is the one-dimensional subspace $[x] \subset F_b$. Therefore, up to the identification $p^{-1}(b) \cong \mathbb{R}P^{n-1}$, $i_b^*(\zeta)$ is the canonical line bundle over $p^{-1}(b)$. Hence $1,i_b^*(a),\ldots,i_b^*(a^{n-1})$ forms a $\mathbf{Z}_2$-basis for $H^*(p^{-1}(b);\mathbf{Z}_2)$, since $w_1(i_b^*(\zeta)) = i_b^*(w_1(\zeta)) = i_b^*(a)$. The theorem now follows by applying Theorem 7.3.

We are now ready to prove Theorem 7.2.

Proof of Theorem 7.2:

If rank $\xi = 1$, take $B' = B$ and $f =$ the identity map. If rank $\xi = n \geq 2$, then take $X = P(\xi)$. As we have noted before, $p^*(\xi) \approx \zeta \oplus \eta$ for some bundle η of rank $n - 1$ and moreover η admits a Euclidean metric. Further, $p^*(H^*(B;\mathbf{Z}_2)) \subset H^*(X;\mathbf{Z}_2)$ is generated as an $H^*(B;\mathbf{Z}_2)$-module by the basis element 1, by Proposition 7.5. Hence p^* is a monomorphism.

By induction assume that there exists a topological space X' and

$n-1$ line bundles $\xi_2,\ldots,\xi_n$ over it, a map $f'\colon X' \longrightarrow X$ such that

$$f'^*(\eta) \approx \xi_2 \oplus \cdots \oplus \xi_n,$$

and that $f'^*\colon H^*(X;\mathbf{Z}_2) \longrightarrow H^*(X';\mathbf{Z}_2)$ is a monomorphism.

Now taking $B' = X'$, $f = p \circ f'$, and $\xi_1 = f'^*(\zeta)$ one easily verifies that the conclusions of Theorem 7.2 follow.

COROLLARY 7.6.

Let $\xi_1,\ldots,\xi_k$ be k real vector bundles over B, each of which possesses a Euclidean metric. Then there exists a space B' and a map $f : B' \longrightarrow B$ which induces a monomorphism f^* in $\mathbf{Z}_2$-cohomology such that $f^*(\xi_i)$ splits as a sum of n_i line bundles, $1 \le i \le k$, where $n_i = \operatorname{rank} \xi_i$.

Proof:

Follows from Theorem 7.2 by an induction on k.

Remarks 7.7

(i) Theorem 7.2 is usually proved under some restrictions on the base space. (cf. §5, Chapter 4 [33].) The theorem is used to define Stiefel-Whitney classes in §5, Chapter 16 of [22] and p.73, [18].

(ii) The assumption that ξ^n admit a Euclidean metric means that its structure group $GL(n,\mathbb{R})$ can be reduced to $O(n) \subset GL(n,\mathbb{R})$. If ξ^n is an (arbitrary) vector bundle over an arbitrary topological space B, we can only assert that there exists a space B' and a map $f : B' \longrightarrow B$ which induces a monomorphism $f^*\colon H^*(B;\mathbf{Z}_2) \longrightarrow H^*(B';\mathbf{Z}_2)$ and there are vector bundles $\xi_1,\ldots,\xi_n$, $\eta_1,\ldots,\eta_{n-1}$ over B' such that the following are exact sequences of vector bundles:

$$0 \longrightarrow \xi_i \longrightarrow \eta_{i-1} \longrightarrow \eta_i \longrightarrow 0, \quad 1 \le i \le n,$$

with $\eta_o = f^*(\xi)$, $\eta_n = 0$ and rank $\xi_i = 1$.

§8 Applications.

In this section we establish formulae for Stiefel-Whitney classes of $\Lambda^k(\xi)$, $S^k(\xi)$, and $\eta_1 \otimes \cdots \otimes \eta_r$ in terms of the Stiefel-Whitney classes of ξ, $\eta_1,\ldots,\eta_r$. It is assumed that $\xi,\eta_1,\ldots,\eta_r$ are all vector bundles over the same base space B and that each of them admits a Euclidean metric.

Choose B' and a map $f : B' \longrightarrow B$ such that $f^*\colon H^*(B;\mathbf{Z}_2) \longrightarrow H^*(B';\mathbf{Z}_2)$ is a monomorphism and there exist line bundles $\xi_1,\ldots,\xi_n$, η_{ij}, $1 \le j \le n_i$, $1 \le i \le r$, where $n =$ rank ξ, $n_i =$ rank η_i, such that

$$f^*(\xi) = \xi_1 \oplus \cdots \oplus \xi_n,$$

$$f^*(\eta_i) = \bigoplus_{1 \le j \le n_i} \eta_{ij} \, .$$

Such a choice is possible by Corollary 7.6. Let ζ^m denote the m-fold tensor product $\zeta \otimes \cdots \otimes \zeta$.

LEMMA 8.1

With the above notations, one has the following vector bundle isomorphisms.

(i) $f^*(\eta_1 \otimes \cdots \otimes \eta_r) \approx \bigoplus_{\substack{1 \le j_i \le n_i \\ 1 \le i \le r}} (\eta_{1j_1} \otimes \cdots \otimes \eta_{rj_r})$

(ii) For $1 \le k \le n$, $f^*(\Lambda^k(\xi)) \approx \bigoplus_{1 \le i_1 < \cdots < i_k \le n} (\xi_{i_1} \otimes \cdots \otimes \xi_{i_k})$.

(iii) For any $k \ge 1$,

$$f^*(S^k(\xi)) \approx \bigoplus_{\substack{0 \le k_1, \ldots, k_r \le k \\ k_1 + \cdots + k_r = k}} \xi_1^{k_1} \otimes \cdots \otimes \xi_r^{k_r}.$$

Proof:

(i) follows immediately from the fact that the tensor product distributes over Whitney sums.

(ii) follows from the repeated application of the formula

$$\Lambda^k(\alpha \oplus \beta) = \bigoplus_{i+j=k} \Lambda^i(\alpha) \otimes \Lambda^j(\beta)$$

and the fact that $\Lambda^o(\zeta) \approx \varepsilon$, $\Lambda^1(\zeta) \approx \zeta$, and all other exterior powers are zero for a line bundle ζ.

Note that the symmetric power $S^m(\zeta)$ of a line bundle ζ is bundle isomorphic to the m-fold tensor product ζ^m, for $m \ge 1$. Hence, using the formula (cf. Corollary 2.2(ii))

$$S^k(\alpha \oplus \beta) = \bigoplus_{i+j=k} S^i(\alpha) \otimes S^j(\beta), \qquad \text{(See §2)}$$

we obtain

$$f^*S^k(\xi) \approx S^k(\xi_1 \oplus \cdots \oplus \xi_r)$$

$$\approx \bigoplus_{\substack{0 \le k_1,\ldots,k_r \le k \\ k_1+\cdots+k_r=k}} \xi_1^{k_1} \otimes \cdots \otimes \xi_r^{k_r}.$$

This proves (iii).

Let $q_{n,k}$, $1 \le k \le n$, denote the unique polynomial in the variables $\sigma_1,\ldots,\sigma_n$ for which the following defining relation

$$q_{n,k}(\sigma_1,\ldots,\sigma_n) = \prod_{1 \le i_1 < \cdots < i_k \le n} (1 + x_{i_1} + \cdots + x_{i_k})$$

holds in $\mathbf{Z}_2[x_1,\ldots,x_n]$. Here σ_i denotes the i^{th} elementary symmetric polynomial in the indeterminates $x_1,\ldots,x_n$. Similarly we let $s_{n,k}$, $n,k \ge 1$, denote the unique polynomial defined by the relation

$$s_{n,k}(\sigma_1,\ldots,\sigma_n) = \prod_{\substack{o \le k_1,\ldots,k_r \le k \\ k_1+\cdots+k_r=k}} (1 + k_1x_1 + \cdots + k_rx_r)$$

in $\mathbf{Z}_2[x_1,\ldots,x_n]$.

Let $\sigma_j(i)$ denote the j^{th} elementary symmetric polynomial in indeterminates $y_{i1},\ldots,y_{in_i}$. Then it is easily seen that the polynomial

$$\prod_{1 \le i \le r} \prod_{1 \le j_i \le n_i} (1 + y_{1j_1} + \cdots + y_{rj_r})$$

in $\mathbf{Z}_2[y_{ij}\colon 1 \le j \le n_i,\ 1 \le i \le r]$ is symmetric in $y_{i1},\ldots,y_{in_i}$ and thus uniquely expressible as a polynomial in the variables $\sigma_j(i)$, $1 \le j \le n_i$, $1 \le i \le r$. Let $p_{n_1,\ldots,n_r}$ denote the unique polynomial defined by the relation

$$p_{n_1,\ldots,n_r}(\sigma_1(1),\ldots,\sigma_{n_1}(1),\ldots,\sigma_1(r),\ldots,\sigma_{n_r}(r))$$
$$= \prod_{1\le i\le r} \prod_{1\le j_i\le n_i} (1 + y_{1j_1} + \cdots + y_{rj_r}).$$

Examples 8.2

(i) $q_{n,1} = \prod_{1\le i\le n} (1+x_i) = 1 + \sigma_1 + \cdots + \sigma_n.$

(ii) $q_{3,2} = 1 + \sigma_1^2 + \sigma_2 + \sigma_1\sigma_2 + \sigma_3 = s_{3,2}.$

(iii) $p_{1,n}(y,\sigma_1',\ldots,\sigma_n') = \prod_{1\le i\le n} (1 + y + y_i)$

$$= (1 + y)^n + (1 + y)^{n-1}\sigma_1' + \cdots + (1 + y)\sigma_{n-1}' + \sigma_n'$$

where σ_i' is the i^{th}-elementary symmetric polynomial in $y_1,\ldots,y_n$.

(iv) Define ϕ_n to be $p_{n,n}(\sigma_1,\ldots,\sigma_n,\sigma_1,\ldots,\sigma_n)$.

Then $\phi_n = \left[\prod_{1\le i<j\le n} (1 + x_i + x_j)\right]^2$

$$= \prod_{1\le i<j\le n} (1 + x_i + x_j)^2$$

$$= (q_{n,2})^2 \qquad \text{since } 2x_i = 0.$$

We are now ready to prove the following main result of this section.

Let w_i denote $w_i(\xi)$ and $w_i(j) = w_i(\eta_j)$.

THEOREM 8.3

With the above notations,

(i) $w(\eta_1 \otimes \cdots \otimes \eta_r) =$

$p_{n_1,\ldots,n_r}(w_1(1),\ldots,w_{n_1}(1),\ldots,w_1(r),\ldots,w_{n_r}(r))$

(ii) $w(\Lambda^k(\xi)) = q_{n,k}(w_1,\ldots,w_n)$

(iii) $w(S^k(\xi)) = s_{n,k}(w_1,\ldots,w_n)$.

Proof of (ii):

By Lemma 6.10 and the Whitney product formula we have

$$w_1(\xi_{i_1} \otimes \cdots \otimes \xi_{i_k}) = w_1(\xi_{i_1} \oplus \cdots \oplus \xi_{i_k})$$
$$= w_1(\xi_{i_1}) + \cdots + w_1(\xi_{i_k}).$$

Therefore, writing $x_i = w_1(\xi_i)$ for $1 \le i \le n$,

(*) $$w(\xi_{i_1} \otimes \cdots \otimes \xi_{i_k}) = 1 + x_{i_1} + \cdots + x_{i_k}.$$

By naturality, another use of the Whitney product formula, Lemma 8.1, and (*), we now find

$$f^*(w(\Lambda^k(\xi)) = w(f^*\Lambda^k(\xi)) = w(\oplus(\xi_{i_1} \otimes \cdots \otimes \xi_{i_k}))$$
$$= \prod_{1 \le i_1 < \cdots < i_k \le n} (1 + x_{i_1} + \cdots + x_{i_k})$$
$$= q_{n,k}(\sigma_1,\ldots,\sigma_n).$$

Since f^* is a monomorphism and since $f^*(w_i) = \sigma_i$, it follows that $w(\Lambda^k(\xi)) = q_{n,k}(w_1,\ldots,w_n)$.

(i) and (iii) are established similarly.

Remarks 8.4

Theorem 8.3 (i), (ii) and analogous formulae for Chern and Pontrjagin classes are proved in [7] using group representations and the description of characteristic classes in terms of root systems of suitable Lie groups. It may be noted that their proof applies for a suitable class of base spaces for which the classification theorem for vector bundles holds. Exercise 7C in [32] gives another proof for $p_{m,n}$ that holds for paracompact base spaces. The above proof, which holds for any bundles admitting a Euclidean metric, parallels the proof of an analogous result for complex vector bundles given in §4.4 page 63, of [18].

To the best of the author's knowledge, the formula for $w(S^k(\xi))$ is not found in standard references.

In view of Theorem 8.3, Examples 8.2 (i) - (iii) can be rewritten to give formulae for Steifel-Whitney classes of vector bundles obtained from given vector bundles by applying Λ^k, S^k or by taking tensor products. Thus, by Example 8.2 (ii), $w(\Lambda^2(\xi)) = 1 + w_1^2 + w_2 + w_1 w_2 + w_3$ for a three plane bundle ξ (writing $w(\xi) = 1 = w_1 + w_2 + w_3$). We conclude this chapter with the following formula for $w(\Lambda^2(\xi))$ for an oriented four plane bundle ξ.

Example 8.5.

Let ξ be an orientable vector bundle of rank 4. Then $w(\Lambda^2(\xi)) = 1 + w_2^2 + w_3^2$ (again $w_i = w_i(\xi)$).

Proof:

By Theorem 8.3, $w(\Lambda^2(\xi)) = q_{4,2}\,(0, w_2, w_3, w_4)$.Hence, it suffices to compute $q_{4,2}$ modulo the ideal $\langle\sigma_1\rangle$ generated by σ_1. Equivalently, we let $\sigma_1 = x_1 + x_2 + x_3 + x_4 = 0$ in the computation of $\tilde{q}_{4,2}(\sigma_2,\sigma_3,\sigma_4) = q_{4,2}(0,\sigma_2,\sigma_3,\sigma_4)$.

Now $\tilde{q}_{4,2} = \prod_{1\le i<j\le 4} (1 + x_i + x_j)$

$$= \left[\prod_{1\le i<j\le 3} (1+x_i+x_j)\right]\cdot(1+x_1+x_4)(1+x_2+x_4)(1+x_3+x_4)$$

$$= \left[\prod_{1\le i<j\le 3} (1+x_i+x_j)\right]\cdot(1+x_2+x_3)(1+x_1+x_3)(1+x_1+x_2)$$

by substituting for x_4 from the relation $x_1 + x_2 + x_3 + x_4 = 0$. Thus

$$\tilde{q}_{4,2} = \prod_{1\le i<j\le 3} (1 + x_i + x_j)^2$$

$$= q_{3,2}^2 = 1 + \sigma_1'^4 + \sigma_2'^2 + \sigma_1'^2\sigma_2'^2 + \sigma_3'^2,$$

where $\sigma_i' = \sigma_i(x_1,x_2,x_3)$, by Example 8.2 (ii) above. By a straightforward and easy computation, or due to symmetry in x_1,x_2,x_3 and x_4, the σ_i' 's can be replaced by $\sigma_i = \sigma_i(x_1,x_2,x_3,x_4)$. Since $\sigma_1 = 0$ this yields

$$\tilde{q}_{4,2} = 1 + \sigma_2^2 + \sigma_3^2 .$$

Remark 8.6

By Example 8.2 (iv) and the above example, for an oriented 4-plane bundle ξ, one has $w(\xi \otimes \xi) = (1 + w_2(\xi) + w_3(\xi))^4$. This is in agreement with the formula for ϕ_4 announced in [30]. The direct computation of ϕ_4 is very difficult. Since $\eta \otimes \eta \approx (\eta \otimes \Lambda^4(\eta)) \otimes (\eta \otimes \Lambda^4(\eta))$ for any four plane bundle η and since $\eta \otimes \Lambda^4(\eta)$ is always orientable (see Lemma 6.10), we can compute $w(\eta \otimes \eta)$ using Examples 8.2 (iii), (iv) and the above formula for $\tilde{q}_{4,2}$. Indeed, in their application, $\eta = \tilde{\gamma}_{8,4}$. Thus the above formula simplifies the computation of $w(\tilde{\gamma}_{8,4} \otimes \tilde{\gamma}_{8,4})$ greatly.

CHAPTER THREE

FLAG MANIFOLDS

§9. Introduction.

That the only stably parallelizable real Grassmann manifolds are the obvious ones, namely, $G_1(\mathbb{R}^2)$, $G_1(\mathbb{R}^4) \cong G_3(\mathbb{R}^4)$ and $G_1(\mathbb{R}^8) \cong G_7(\mathbb{R}^8)$ which are all parallelizable, was first noted by T.Yoshida [46]. A proof using only elementary concepts and results from K-theory that also covers the complex and quaternionic Grassmann manifolds was found by Trew, Zvengrowski [45]. In this chapter we determine exactly which of the real, complex and quaternionic flag manifolds are stably parallelizable and parallelizable. See Theorem 11.1. Recently Korbaš [24] has obtained the same results for real flag manifolds, where the negative results are based on computations of Stiefel-Whitney classes.

In §10 we state, without proof, some known results which will be made use of in the proof of Theorem 11.1. In §11 we prove the main theorem of this chapter. In §12 an explicit trivialization for the "classical" real flag manifolds is given.

The results of this chapter appear in [37].

§10. Summary of Known Results.

First we introduce some notations. F will denote either the field $\mathbb{R}$ of real numbers, or $\mathbb{C}$ of complex numbers, or the division ring $\mathbb{H}$ of quaternions. Denote by μ a sequence $(n_1,\ldots,n_s)$ of positive integers, with $s \geq 2$. Denote by μ_r the subsequence $(n_1,\ldots,n_r)$, for $1 \leq r \leq s$. We write $|\mu_r| = n_1 + \cdots + n_r$ and $n = |\mu|$.

$FG(\mu)$ denotes the F-flag manifold $FG(n_1,\ldots,n_s)$ and will be called the F-flag manifold of type μ. The notation $G_F(n_1,\ldots,n_s)$ for the flag manifold $FG(\mu)$ is used by many authors. When $F = \mathbb{R}$, we write $G(\mu)$ instead of $\mathbb{R}G(\mu)$. When $\mu = (1,\ldots,1)$, $FG(\mu)$ are the so called "classical" flag manifolds.

Note that there are s canonically defined F-vector bundles $\xi_1^F(\mu),\ldots\xi_s^F(\mu)$ over $FG(\mu)$, with rank $\xi_i^F(\mu) = n_i$ for $1 \leq i \leq s$. The total space of $\xi_i^F(\mu)$ is $\{(\underline{A}, \nu) \mid \nu \in A_i,\ \underline{A} = (A_1,\ldots,A_s) \in FG(\mu)\}$ $\subset FG(\mu) \times F^n$ and the fibre of $\xi_i^F(\mu)$ over a flag $\underline{A} = (A_1,\ldots,A_s) \in FG(\mu)$ is the F-vector space A_i. In case $s = 2$, it is customary to denote $\xi_1^F(n_1,n_2)$ and $\xi_2^F(n_1,n_2)$ by γ_{n,n_1}^F and β_{n,n_1}^F respectively (see p.9). Denoting F-vector bundle isomorphism by $\approx_F$ it is clear that

$$\xi_1^F(\mu) \oplus \cdots \oplus \xi_s^F(\mu) \approx_F n\varepsilon^F$$

where $n\varepsilon^F$ denotes the trivial F-vector bundle of rank n.

The following theorem is due to Lam. See Corollary 1.2 of [26]. Let $\mathbb{Z}(F)$ denote the centre of F.

THEOREM 10.1

The tangent bundle $\tau^F(\mu)$ of $FG(\mu)$ is isomorphic to

$$\bigoplus_{1\leq i<j\leq s} \bar{\xi}_i^F(\mu) \otimes_F \xi_j^F(\mu)$$

as $\mathbb{Z}(F)$-vector bundles. Here $\bar{\xi}$ denotes the "conjugate" vector bundle of ξ.

Let us now focus on the Grassmann manifolds, the case s = 2. In this case the above formula gives the familiar description,

$$\tau^F_{n,k} \approx \bar{\gamma}^F_{n,k} \otimes_F \beta^F_{n,k}$$

where $\tau^F_{n,k}$ denotes the tangent bundle of $G_k(F^n) = FG(k,n-k)$. The following theorem, due to Trew and Zvengrowski [45], will be made use of in the proof of Theorm 11.1.

THEOREM 10.2

The only Grassmann manifolds that are stably parallilizable as real manifolds are $G_1(F^2)$, $G_1(\mathbb{R}^4) \cong G_3(\mathbb{R}^4)$, $G_1(\mathbb{R}^8) \cong G_7(\mathbb{R}^8)$, where $F = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

The above theorem is proved by imbedding FP^{n-k} in $G_k(F^n)$ in a natural way and showing that the pull back of the tangent bundle $\tau^F_{n,k}$ is not stably trivial, except in precisely the obvious cases noted in the theorem.

Let $V_{n,k}$ denote the (real) Stiefel manifold of orthonormal k-frames in the Euclidean space $\mathbb{R}^n$, with standard inner product. Let $X_{n,k}$ be the projective Stiefel manifold obtained from $V_{n,k}$ by identifying each $\alpha = (v_1,\ldots,v_k) \in V_{n,k}$ with $-\alpha = (-v_1,\ldots,-v_k)$ in $V_{n,k}$. $X_{n,k}$ is just the homogeneous space $O(n)/\{\pm I_k\}\} \times O(n-k)$.

Let $p : V_{n,k} \longrightarrow X_{n,k}$ denote the double covering map that maps $\alpha \in V_{n,k}$ onto $[\alpha] = \{\pm\alpha\}$ in $X_{n,k}$. Let ζ denote the associated line bundle over $X_{n,k}$. By Theorem 3.2 of [26] the tangent bundle of $X_{n,k}$ is stably equivalent to nk ζ. One can use the following proposition to decide the stable parallelizability of $X_{n,k}$ in many cases. For further results regarding (stable) parallelizability of $X_{n,k}$, see [4]. Also see Theorem 14.5.

PROPOSITION 10.3

Let $k < n$, and let $N = \min\{j \mid n-k < j \le n \text{ and } \binom{n}{j} \equiv 1 \bmod 2\}$. The cohomology algebra $H^*(X_{n,k}; \mathbf{Z}_2)$ is isomorphic, as a $\mathbf{Z}_2$-graded algebra, to

$$(\mathbf{Z}_2[y]/_{y^N}) \otimes V(y_{n-k},\ldots,y_{N-2},y_N,\ldots,y_{n-1})$$

where $y = w_1(\zeta)$, and $V(y_{n-k},\ldots,y_{N-2},y_N,\ldots,y_{n-1})$ is a suitable $\mathbf{Z}_2$-algebra which has $\{y_{n-k}^{\epsilon_1}\cdots y_{n-1}^{\epsilon_{k-1}} \mid \epsilon_j = 0 \text{ or } 1 \text{ for } 1 \le j \le k-1\}$ as an <u>additive</u> basis and $\deg y_j = j$.

Proof:

The proof of this proposition can be found in [13].

Let M^m, N^n be smooth paracompact manifolds. Let $f : M \longrightarrow N$ be a smooth map. f is said be an immersion (resp. submersion) if $(Tf)_x : T_xM \longrightarrow T_{f(x)}N$ is a monomorphism (resp. an epimorphism) for all $x \in M$. Here T_xM denotes the tangent space to M at x, and $(Tf)_x$ is the derivative of f at x.

Note that if $f : M \longrightarrow N$ is an immersion, then $f^*\tau(N)$ contains $\tau(M)$ as a subbundle and that if f is a submersion $f^*\tau(N)$ is a subbundle of $\tau(M)$. Let $\nu(M)$ denote the normal bundle in the former case and the "vertical" bundle in the latter case. Thus $f^*\tau(N) \approx \tau(M) \oplus \nu(M)$ when f is an immersion and, $\tau(M) \approx f^*(\tau(N)) \oplus \nu(M)$ when f is a submersion. We have the following lemma.

LEMMA 10.4.

Let $\nu(M)$ be trivial. Then M is stably parallelizable if N is stably parallelizable. In case f is a submersion and rank $\nu(M) \geq 1$, M is parallelizable if N is stably parallelizable.

Proof:

Let $k = \operatorname{rank} \nu(M)$. Let N be stably parallelizable.

In case f is an immersion, we have

$$n\varepsilon \sim f^*(\tau(N)) \approx \tau(M) \oplus \nu(M) \approx \tau(M) \oplus k\varepsilon.$$

Hence M is stably parallelizable.

In case f is a submersion, we have

$$\tau(M) \approx f^*(\tau(N)) \oplus \nu(M) \approx f^*(\tau(N)) \oplus k\varepsilon \sim f^*(\tau(N)).$$

Since $\tau(N) \sim 0$, it follows that $f^*(\tau(N)) \sim 0$. Thus M is stably parallelizable. If $k \geq 1$, then $\tau(N) \oplus k\varepsilon \approx (n+k)\varepsilon$. Therefore $f^*(\tau(N)) \oplus k\varepsilon$ is a trivial vector bundle. Since $\tau(M) \approx f^*(\tau(N)) \oplus k\varepsilon$, it follows that, in this case, M is parallelizable.

The above lemma is used to obtain results on stable parallelizability of flag manifolds.

§11. Stable Parallelizability of Flag Manifolds.

In this section our aim is to prove the following theorem. Notations are as in the previous section.

THEOREM 11.1.

Let $s \geq 3$, $\mu = (n_1,\ldots,n_s)$. Then,

(i) $FG(\mu)$ is stably parallelizable when $n_1 = \cdots = n_s = 1$, and, in this case, parallelizable only when $F = \mathbb{R}$.

(ii) If $n_i > 1$ for some i, then $FG(\mu)$ is not stably parallelizable.

Note: The case $s = 2$ is just that of Grassmann manifolds, for which the result is known by Theorem 10.2.

Proof of (i):

Let $\mu = (n_1,\ldots,n_s)$ with $n_i = 1$. The stable parallelizability of $FG(\mu)$ has been noted in [26]. The parallelizability of $\mathbb{R}G(\mu) \cong O(n)/O(1) \times \cdots \times O(1)$ follows from the fact that the quotient of a Lie group by a *finite* subgroup is parallelizable [p.502, 8]. However, an explicit trivialization for the tangent bundle of $\mathbb{R}G(\mu)$ is constructed in §12. To prove that $FG(\mu)$ is not parallelizable for $F = \mathbb{C}$ or $\mathbb{H}$, we show that the Euler characteristic in these cases is non-zero. Note that π_n: $FG(\mu) \longrightarrow FP^{n-1} \cong FG(n-1,1)$, the projection map that sends $(A_1,\ldots,A_n)$ to $A_n \in FP^{n-1}$, is the projection of a fibre bundle with fibre $FG(\mu_{s-1})$. This bundle is orientable for $F = \mathbb{C}$ or $\mathbb{H}$.

Further, $\chi(FP^m) > 0$ for $F = \mathbb{C},\mathbb{H}$ and $m \geq 1$. Using induction and the multiplicative property of Euler characteristic we see that $\chi(FG(\mu)) > 0$ for $F = \mathbb{C}$ or $\mathbb{H}$.

Proof of (ii):

Since $FG(n_1,\ldots,n_s) \cong FG(n_{i_1},\ldots,n_{i_s})$ where $\{i_1,\ldots,i_s\} = \{1,\ldots,s\}$ we assume, without loss of generality, that $n_1 \geq \cdots \geq n_s$. Now let $n_1 > 1$. By Theorem 10.1 one has

$$\tau^F(\mu) \approx_{\mathbf{Z}(F)} \bigoplus_{1\leq i<j\leq s} \bar{\xi}_i^F(\mu) \otimes \xi_j^F(\mu).$$

Now consider the inclusion $i : FG(\mu_{s-1}) \longrightarrow FG(\mu)$ which is induced by the identification $F^{|\mu|} \cong F^{|\mu_{s-1}|} \oplus F^{n_s}$. Clearly one has the following isomorphisms of F-vector bundles:

$$i^*(\xi_i^F(\mu)) \approx \xi_i^F(\mu_{s-1}) \quad \text{for } 1 \leq i \leq s-1$$

and

$$i^*(\xi_s^F(\mu)) \approx n_s\, \varepsilon^F.$$

Therefore, denoting stable equivalence of $\mathbf{Z}(F)$-vector bundles by $\sim$,

$$\begin{aligned} i^*(\tau^F(\mu)) &\approx i^*\Big[\bigoplus_{1\leq i<j\leq s} \bar{\xi}_i^F(\mu) \otimes \xi_j^F(\mu)\Big] \\ &\approx \bigoplus_{1\leq i<j\leq s-1} \bar{\xi}_i^F(\mu_{s-1}) \otimes \xi_j^F(\mu) \oplus \Big[\bigoplus_{1\leq i\leq s-1} \bar{\xi}_i^F(\mu_{s-1}) \otimes n_s \varepsilon^F\Big] \\ &\sim \tau^F(\mu_{s-1}), \text{ since } \bigoplus_{1\leq i\leq s-1} \bar{\xi}_i^F(\mu_{s-1}) \approx |\mu_{s-1}|\bar{\varepsilon}^F \approx |\mu_{s-1}|\varepsilon^F. \end{aligned}$$

Let j denote the composition of inclusions

$$FG(\mu_2) \xrightarrow{\ i\ } \cdots \xrightarrow{\ i\ } FG(\mu).$$

By applying i^* successively, we obtain

$$j^*(\tau^F(\mu)) \sim \tau^F(\mu_2).$$

Now (ii) follows from the negative results on the stable parallelizability of Grassmann manifolds except when $F = \mathbb{R}$, $n_2 = 1$ and $n_1 = 3$ or 7. See Theorem 10.2.

In the $(3,1,1)$ or $(7,1,1)$ cases, consider the double covering map

$$X_{n,2} \longrightarrow G(n-2,\ 1,1)$$

where $p([\alpha]) = (\{v_1,v_2\}^{\perp},\mathbb{R}v_1,\mathbb{R}v_2)$ for $\alpha = (v_1,v_2)$ in $V_{n,2}$, $[\alpha] = \{\pm\alpha\} \in X_{n,2}$.

As for any covering map,

$$p^*(\tau^{\mathbb{R}}(n-2,1,1)) \approx \tau(X_{n,2})$$

By Proposition 10.3, and using the fact $\tau(X_{n,2}) \sim 2n\zeta$, it follows that $w_2(X_{n,2}) \neq 0$ for $n = 5,9$, (since $\begin{bmatrix}10\\2\end{bmatrix}, \begin{bmatrix}18\\2\end{bmatrix} \equiv 1 \bmod 2$). Hence $X_{n,2}$ is not stably parallelizable for $n = 5,9$. Consequently, $G(3,1,1)$ and $G(7,1,1)$ are not stably parallelizable. This completes the proof of (ii).

Remark 11.2.

The top Chern class of $\mathbb{C}G(1,\ldots,1)$ is its Euler class. Since the Euler characteristic of $\mathbb{C}G(1,\ldots,1)$ is non-zero it follows that the top Chern class of $\mathbb{C}G(1,\ldots,1)$ is non-zero. Hence $\mathbb{C}G(1,\ldots,1)$ is not stably parallelizable as a *complex* manifold.

Remark 11.3.

In the case $F = \mathbb{R}$, many of our results follow from the work of Miatello-Miatello [30] by considering the covering map $f : \tilde{G}(n_1,\ldots,n_s) \longrightarrow G(n_1,\ldots,n_s)$ that forgets the orientations. However there are numerous cases where $\tilde{G}(n_1,\ldots,n_s)$ is stably parallelizable and hence gives no information about the stable parallelizability of the corresponding unoriented flag manifolds $G(n_1,\dots,n_s)$. (See §15 for the notation $\tilde{G}(n_1,\dots,n_s)$.)

§12. Parallelizability of $G(1,\dots,1)$.

We conclude this chapter by constructing an explicit trivialization for the tangent bundle $\tau^{\mathbb{R}}(1,\dots,1)$ of the "classical" real flag manifold $G(1,\dots,1)$.

For each pair of integers k,ℓ, $1 \le k < \ell \le n$ we will construct a tangent vector field $\varphi_{k,\ell}$ and show that these $\begin{bmatrix} n \\ 2 \end{bmatrix}$ vector fields are everywhere linearly independent. Since $\dim G(1,\dots,1) = \begin{bmatrix} n \\ 2 \end{bmatrix}$, the manifold is therefore parallelizable.

Let $\underline{a} = ([a_1],\dots,[a_n]) \in G(1,\dots,1)$ where $\{a_1,\dots,a_n\}$ is an orthonormal basis for $\mathbb{R}^n$, and $[a_i] = [-a_i] = \{a_i, -a_i\}$. Define $\varphi_{k,\ell}$ as follows: Writing $a_i = (a_{i1},\dots,a_{in}) \in \mathbb{R}^n$ for $1 \le i \le n$,

$$\varphi_{k\ell}(\underline{a}) = \sum_{1\le i<j\le n} (a_{ik}a_{j\ell} - a_{i\ell}a_{jk})a_i \otimes a_j$$

for $1 \le k < \ell \le n$.

It is clear that $\varphi_{k\ell}\colon G(1,\dots,1) \longrightarrow T^{\mathbb{R}}(1,\dots,1)$, the total space of $\tau^{\mathbb{R}}(1,\dots,1) \approx \bigoplus_{1\le i<j\le n} \xi_i \otimes \xi_j$ is well-defined and continuous.

Now consider the homomorphism $f : \bigoplus_{1\le i<j\le n} A_i \otimes A_j \longrightarrow \Lambda^2(\mathbb{R}^n)$ defined by

$$f(a_i \otimes a_j) = a_i \wedge a_j,$$

where $A_i = \mathbb{R}a_i$. Since $\{a_1,\ldots,a_n\}$ is an orthonormal basis for $\mathbb{R}^n$, $\{a_i \wedge a_j \mid 1 \le i < j \le n\}$ is an orthonormal basis for $\Lambda^2(\mathbb{R}^n)$. Therefore f preserves inner products and is an isomorphism. Now

$$\begin{aligned} f\Upsilon_{k\ell}(\underline{a}) &= \sum_{1\le i<j\le n} (a_{ik}a_{j\ell} - a_{i\ell}a_{jk})a_i \wedge a_j \\ &= u_k \wedge u_\ell \end{aligned}$$

where $u_k = \Sigma\, a_{ik}a_i = \Sigma\, a_{ik}a_{im}e_m = \Sigma\, \delta_{km}e_m = e_k$, $\{e_1,\ldots,e_n\}$ being the standard orthonormal basis for $\mathbb{R}^n$. Therefore

$$\{f\Upsilon_{k\ell}(\underline{a}) \mid 1 \le k < \ell \le n\} = \{e_k \wedge e_\ell \mid 1 \le k < \ell \le n\}$$

is an orthonormal basis for $\Lambda^2(\mathbb{R}^n)$. Consequently $\{\Upsilon_{k\ell}(\underline{a}) \mid 1 \le k < \ell \le n\}$ is an orthonormal basis for the tangent space at $\underline{a}$ to $G(1,\ldots,1)$. Since $\underline{a} \in G(1,\ldots,1)$ was arbitrary, it follows that $\{\Upsilon_{k\ell} \mid 1 \le k < \ell \le n\}$ is everywhere linearly independent.

CHAPTER FOUR

FLAG$^+$ MANIFOLDS AND THEIR PARALLELIZABILITY.

§13. Introduction.

The parallelizability of oriented Grassmann manifolds is known by the work of Miatello-Miatello [30]. Their results are based on computations of Stiefel-Whitney classes involving Schubert calculus, and techniques from the theory of Lie groups. The oriented flag manifolds were also considered in [30].

In the present chapter we consider the wider class of flag$^+$ manifolds. With a few exceptions, the question of stable parallelizability and parallelizability for these manifolds are solved. These results include those of [30], and the proofs are generally based on more geometric and conceptually simpler methods.

In §14 some well-known results that will be made use of in the subsequent sections of this chapter are stated without proof. In §15 flag$^+$ manifolds are defined. These manifolds first appeared in §3 of [26]. Here we establish some properties of these manifolds which will be made use of in §17. §16 deals with (stable) parallelizability of oriented Grassmann manifolds. The stable parallelizability results are obtained for most flag$^+$ manifold in §17, and the parallelizability results in §18.

§14 Some Well-known Results.

Recall that a principal G-bundle $\Upsilon = (E,q,B)$ is called n-universial if, given any n-complex K with a principal G-bundle ξ over K, and given a map $f : L \longrightarrow B$ such that $f^*(\Upsilon) \approx \xi|L$ with L a subcomplex of K, then f can be extended to a map $\bar{f} : K \longrightarrow B$ such that $\bar{f}^*(\Upsilon) \approx \xi$. Υ is called a universal G-bundle if it is n-universal for all n.

The proofs of Theorems 14.1 and 14.2 below can be found in §19 of [39].

THEOREM 14.1 (Classification Theorem):

Let K be an m-complex, $m < n$, and let $\Upsilon = (E,q,B)$ be an n-universal principal G-bundle with B path connected. Then the set of principal G-bundles over K is in bijective correspondence with the set $[K,B]$ of homotopy class of maps of K into B. Under this correspondence, $f : K \longrightarrow B$ corresponds to the bundle $f^*(\Upsilon)$.

THEOREM 14.2.

A principal G-bundle $\Upsilon = (E,q,B)$ is n-universal if and only if E is (n-1)-connected; that is, $\pi_i(E) = 0$ for $1 \leq i < n$, and E is path connected.

The following are well-known examples. (See [39], [32].)

Examples 14.3

(i) $\tilde{\Upsilon}_{n,k}$ over $\tilde{G}_{n,k}$ is (n-k)-universal for SO(k) and $\Upsilon_{n,k}$ over $G_{n,k}$ is (n-k)-universal for O(k).

(ii) $\tilde{\gamma}_{\infty,k}$ (resp. $\gamma_{\infty,k}$) over $\tilde{G}_{\infty,k}$ (resp. $G_{\infty,k}$) is a universal bundle for SO(k) (resp. O(k)).

The following result, due to B.J. Sanderson [35], is used to obtain non-parallelizability results for $\tilde{G}_{n,k}$. See also [12] and [19].

Let ω be the canonical complex line bundle over $\mathbb{C}P^2$. Let ξ denote its underlying real vector bundle. Denote by y the element $(\xi-2\varepsilon) \in KO(\mathbb{C}P^2)$. As in §11, "~" denotes "stably isomorphic".

PROPOSITION 14.4

$KO(\mathbb{C}P^2)$ is the truncated polynomial ring $\mathbb{Z}[y]$ with $y^2 = 0$. Thus ξ has infinite order and $\xi \otimes \xi \sim 4\xi$.

It will be seen that the class of flag$^+$ manifolds contain the projective Stiefel manifolds $X_{n,2}$. The parallelizability of projective Stiefel manifolds, except for $X_{12,8}$, is known from the works of Antoniano [3] and Zvengrowski [48]. See also [4]. We record their results here as a

THEOREM 14.5

(i) $X_{4,k}$, $X_{8,k}$, $X_{16,8}$, $X_{n,n}$, $X_{n,n-1}$ and $X_{2n,2n-2}$ are all paralelizable.

(ii) $X_{n,k}$ is not stably parallelizable if $(n,k) \neq (12,8)$ and is not listed in (i) above.

Remark 14.6

The parallelizability of $X_{n,n}$ and $X_{n,n-1}$ also follows from the fact

$G(1,\dots,1)$ are parallelizable and $X_{n,n-1}$ and $X_{n,n}$ are finite coverings of $G(1,\dots,1)$. Using the fact that $\tau(X_{n,k}) \sim nk\zeta$ where ζ is the canonical line bundle over $X_{n,k}$ (see §10) and Proposition 10.3, a Stiefel-Whitney class argument shows that many of $X_{n,k}$ are not π-manifolds.

§15 Flag$^+$ Manifolds.

Let $\mu = (n_1,\dots,n_s)$ be a sequence of positive integers with $s \geq 2$, and $1 \leq r \leq s$. Let $\mu_r = (n_1,\dots,n_r)$ $|\mu_r| = n_1 + \cdots + n_r$ for $1 \leq r \leq s$ and $n = |\mu|$.

DEFINITION 15.1

The flag$^+$ manifold $G(\mu;r)$ of type $(\mu;r)$ is the manifold $\{(\tilde{A}_1,\dots,\tilde{A}_r, A_{r+1},\dots,A_s) \mid (A_1,\dots,A_s) \in G(\mu)\}$, where $\tilde{A}_i$ is the vector space A_i with an orientation, for $1 \leq i \leq r$. When $r = s$, we require that the orientations in $\tilde{A}_i$ give the vector space $A_1 + \cdots + A_s = \mathbb{R}^n$ the standard orientation on $\mathbb{R}^n$, so that $G(\mu;s-1) = G(\mu;s)$. We write $\tilde{G}(\mu)$ to denote $G(\mu;s)$. A point of $G(\mu;s)$ will be referred to as a flag$^+$.

Examples 15.2

(i) When $s = 2$, $\tilde{G}(\mu)$ is just the oriented Grassmann manifold $\tilde{G}(n_1,n_2) = \tilde{G}_{n_1+n_2,n_1} \cong \tilde{G}_{n_1+n_2,n_2}$.

(ii) When $n_1 = \cdots = n_{s-1} = 1$, $\tilde{G}(\mu)$ is readily seen to be the (real) Stiefel manifold $V_{n,s-1}$.

(iii) $G(n-2,1,1;1)$ can be identified with the (real) projective Stiefel manifold $X_{n,2}$ as follows: identify $[\alpha] \in X_{n,2}$ for $\alpha = (a_1,a_2) \in V_{n,2}$ with $(\tilde{A},\mathbb{R}a_1,\mathbb{R}a_2)$ in $G(n-2,1,1;1)$ where $A = \{a_1,a_2\}^{\perp}$ and the orientation on it being given by an ordered basis $(b_1,\ldots,b_{n-2})$ of A such that $(b_1,\ldots,b_{n-2},a_1,a_2)$ (equivalently $(b_1,\ldots,b_{n-2},-a_1,-a_2)$) gives $\mathbb{R}^n$ its standard orientation.

The flag$^+$ manifold $G(\mu;r)$, $1 \le r \le s-1$, is just the homogeneous space $O(n)/(SO(n_1) \times \cdots \times SO(n_r) \times O(n_{r+1}) \times \cdots \times O(n_s))$. The map $f : G(\mu;r) \longrightarrow G(\mu)$ that maps each flag$^+$ $(\tilde{A}_1,\ldots,\tilde{A}_r,A_{r+1},\ldots,A_s)$ in $G(\mu;r)$ to the flag $(A_1,\ldots,A_s)$ is a 2^r-covering map for $1 \le r \le s-1$ and a 2^{s-1}-covering map when $r = s$. Denote by $\xi_i(\mu;r)$ the bundle $f^*(\xi_i(\mu))$ for $1 \le i \le s$. In case $r = s-1,s$ we write $\tilde{\xi}_i(\mu)$ instead of $\xi_i(\mu;r)$. Note that $\xi_i(\mu;r)$ is canonically oriented for $1 \le i \le r$, the fibre over $(\tilde{A}_1,\ldots,\tilde{A}_r,A_{r+1},\ldots,A_s)$ being the oriented vector space $\tilde{A}_i$. As for any covering map $f^*(\tau(\mu)) = \tau(\mu;r)$, the tangent bundle of $G(\mu;r)$. Thus we have the following description of the tangent bundle of $G(\mu;r)$ as noted by K.Y. Lam [26]:

$$\tau(\mu;r) \approx \bigoplus_{1\le i<j\le s} \xi_i(\mu;r) \otimes \xi_j(\mu;r).$$

Also we have $\oplus\, \xi_i(\mu;r) \approx n\varepsilon$.

Let $\chi(\mu;r)$ denote the Euler characteristic of $G(\mu;r)$.

THEOREM 15.3

(i) $\chi(\mu;r) = 0$ if and only if there are at least two odd numbers in the sequence $\mu = (n_1,\ldots,n_s)$.

(ii) $\chi(FG(\mu)) > 0$ for $F = \mathbb{C}, \mathbb{H}$.

Proof:

(i) Since $\tilde{G}(\mu) = G(\mu;s)$ is a finite covering space of $G(\mu;r)$, $\chi(\mu;r) = 0$ if and only if $\chi(\mu;s) = 0$. Therefore we need only prove the theorem for $r = s$.

When $s = 2$, $\tilde{G}(\mu)$ is the oriented Grassmann manifold $\tilde{G}_{n_1+n_2,n_1}$, for which we have seen in Corollary 3.7, Theorem 3.6 that $\chi(\tilde{G}_{n_1+n_2,n_1}) = 0$ if and only if n_1 is odd and $n_1 + n_2$ even. Thus the statement of the theorem is true in case $s = 2$.

Now let $s \geq 3$. Consider the bundle projection $p : \tilde{G}(\mu) \longrightarrow \tilde{G}(\mu')$ with $\mu' = (n_1 + n_2,n_3,\ldots,n_s)$ where $p(\tilde{A}_1,\ldots,\tilde{A}_s) = (\tilde{A}_1 + \tilde{A}_2,\tilde{A}_3,\ldots,\tilde{A}_s)$. The orientation on $A_1 + A_2$ is determined by the ordered basis $(a_{11},\ldots,a_{1n_1},a_{21},\ldots,a_{2n_2})$ where $(a_{i1},\ldots,a_{in_i})$ is in the orientation of $\tilde{A}_i$, $i = 1,2$. It is clear that the fibre of this bundle is

$$\frac{SO(n_1+n_2) \times SO(n_3) \times\cdots\times SO(n_s)}{SO(n_1) \times SO(n_2) \times\cdots\times SO(n_s)} \cong \frac{SO(n_1 + n_2)}{SO(n_1) \times SO(n_2)}$$
$$\cong \tilde{G}_{n_1+n_2,n_1}$$

Further, since $\tilde{G}(\mu')$ is simply connected, the bundle is orientable. Hence

$$\chi(\mu;s) = \chi(\mu') \cdot \chi(\tilde{G}_{n_1+n_2,n_1}) \qquad (*)$$

Now assume that there are at least two odd numbers in the set $\{n_1,\dots,n_s\}$. Without loss of generality, assume that n_1 and n_2 are odd. By (*) and the fact that $\chi(\tilde{G}_{n_1+n_2,n_1}) = 0$ it follows that $\chi(\mu;s) = 0$.

If at most one of the n_i's is odd, then $\chi(\tilde{G}_{n_1+n_2,n_1}) \neq 0$. Moreover, there is at most one odd number in the set $\{n_1+n_2,\dots,n_s\}$. Using (*) and an induction argument now completes the proof.

Proof of (ii):

Let $F = \mathbb{C}$ or $\mathbb{H}$. When $s = 2$, $FG(\mu) \cong FG_{n_1+n_2,n_1}$,

$$\chi(FG(\mu)) = \chi(FG(\mu')) \cdot \chi(FG_{n_1+n_2,n_1}).$$

The proof is now completed by an induction argument.

COROLLARY 15.4

(i) Span $G(\mu;r)$ is positive if and only if there exist at least two odd numbers in the sequence $n_1,\dots,n_s$. (ii) Span $FG(\mu) = 0$ for $F = \mathbb{C},\mathbb{H}$.

Remarks 15.5

(i) Theorem 15.3 (ii) and the case $r = s$ of (i) also follow from the fact that the homogeneous space G/K where G is a compact connected Lie group and K a closed connected subgroup of G has vanishing Euler characteristic if and only if rank of K is less than that of G. See Vol. II [14].

(ii) Korbaš [24] has obtained Theorem 15.3 (i) and its corollary using the knowledge of the $\mathbf{Z}_2$-Poincarè polynomial of $G(\mu)$.

The following lemma will be used to show that certain Stiefel-Whitney classes are non-zero.

Let ς be a real line bundle with a Euclidean metric over a connected topological space B. Let $\tilde{B} \xrightarrow{p} B$ be the associated S°-bundle.

LEMMA 15.6

Let $w_1 = w_1(\varsigma)$. Then $p^*(\alpha) = 0$ for $\alpha \in H^*(B;\mathbf{Z}_2)$ if and only if α is in the ideal of $H^1(B;\mathbf{Z}_2)$ generated by w_1.

Proof:

Consider the Gysin exact sequence

$$\cdots \longrightarrow H^{i-1}(B;\mathbf{Z}_2) \xrightarrow{\cup w_1} H^i(B;\mathbf{Z}_2) \xrightarrow{p^*} H^i(\tilde{B};\mathbf{Z}_2) \longrightarrow H^i(B;\mathbf{Z}_2) \longrightarrow$$

$\cdots$

associated to the line bundle ς. For $\alpha \in H^*(B;\mathbf{Z}_2)$, $p^*(\alpha) = 0 \Leftrightarrow \alpha \in \ker p^* \Leftrightarrow \alpha \in \mathrm{Im}(\cup w_1) \Leftrightarrow \alpha$ is in the ideal of $H^*(B;\mathbf{Z}_2)$ generated by w_1.

Examples: 15.7

(i) The only relations among the Stiefel-Whitney classes $w_i = w_i(\tilde{\gamma}_{n,k})$ in $\tilde{G}_{n,k}$ are those that arise from the relation

$$(1 + w_2 + w_3 + \cdots + w_k) \cdot (1 + \bar{w}_2 + \cdots + \bar{w}_{n-k}) = 1$$

where $\bar{w}_i = w_i(\beta_{n,k})$.

Proof:

We know that the only relations among the Stiefel-Whitney classes $w_i(\gamma_{n,k})$ in $H^*(G_{n,k};\mathbf{Z}_2)$ are those that arise from the single inhomogeneous relation $w(\gamma_{n,k})\cdot w(\beta_{n,k}) = 1$. (See, for example, Problem 7B, [32].)

The example now follows from Lemma 15.6 by taking ζ to be the line bundle $\Lambda^k(\gamma_{n,k})$ so that $w_1(\zeta) = w_1(\gamma_{n,k})$ by Lemma 6.10.

(ii) The only relations among the Stiefel-Whitney classes $w_j(\xi_i(\mu;r))$, $1 \le i \le s$, in $H^*(G(\mu;r);\mathbf{Z}_2)$ are those that arise from the relation

$$\prod_{1\le i\le s} w(\xi_i(\mu;r)) = 1.$$

Proof of (ii) requires the knowledge of relations among $w_j(\xi_i(\mu))$, which is given in Proposition 20.2, in the following chapter. The proof is completed by induction on r, the case r = 1 being similar to Example 15.7 (i) above.

A more natural name for what has been called "flag$^+$ manifold" here might be "partially oriented flag manifold". But this may be misleading since many of the flag$^+$ manifolds are not orientable as manifolds. The concept of orientability of a manifold refers to its tangent bundle. Thus "partially oriented flag manifolds" could be interpreted as conveying that part of the flag manifold has been oriented. We think of flag$^+$ manifolds as flag manifolds with additional structures. The terminology we have chosen also achieves greater economy in space.

§16. Stable Parallelizability of $\tilde{G}_{n,k}$.

The main result of this section is the following:

THEOREM 16.1

Let $2 \le k \le n-2$. $\tilde{G}_{n,k}$ is stably parallelizable if and only if $(n,k) = (4,2)$ or $(6,3)$, and is parallelizable if and only if $(n,k) = (6,3)$.

Note that in case $k = 1$ or $n-1$, $\tilde{G}_{n,k} \cong S^{n-1}$ and the solution for their parallelizability was obtained independently by Kervaire [23], and Milnor, [31].

We first prove the following proposition:

PROPOSITION 16.2

Let $\mu = (n_1,\ldots,n_s)$, with $n_1 = \cdots = n_k = 2$ or 3 and $n_{k+1} = \cdots = n_s = 1$. Then $G(\mu;r)$ is stably parallelizable if $n_1 = 2$, $k \le r$; or $n_1 = 3$ and $r = s$.

Proof:

To simplify notations, let us write $\xi_i = \xi_i(\mu;r)$. As noted in §15, we have the following bundle isomorphism:

$$n\varepsilon \approx \xi_1 \oplus \cdots \oplus \xi_s.$$

Applying Λ^2-functor to both sides,

$$\begin{aligned}\begin{bmatrix} n \\ 2 \end{bmatrix}\varepsilon = \Lambda^2(n\varepsilon) &\approx \Lambda^2(\xi_1 \oplus \cdots \oplus \xi_s) \\ &\approx \bigoplus_{1\le i\le s} \Lambda^2(\xi_i) \oplus \Big(\bigoplus_{1\le i<j\le s} \xi_i \otimes \xi_j\Big)\end{aligned}$$

$$\approx \bigoplus_{1\le i\le k} \Lambda^2(\xi_i) \oplus \tau(\mu;r). \qquad (*)$$

The last bundle equivalence is due to the fact that $\Lambda^2(\zeta) = 0$ for a line bundle ζ, and the description of $\tau(\mu;r)$ noted in §15.

Since ξ_i are oriented for $1 \le i \le k$, $\Lambda^2(\xi_i) \approx \varepsilon$ if rank $\xi_i = 2$ and $\Lambda^2(\xi_i) \approx \Lambda^{3-2}(\xi_i) \approx \xi_i$ if rank $\xi_i = 3$. See §2. Therefore, if $n_1 = \cdots n_k = 2$, $k \le r$, from (*)

$$\begin{bmatrix} n \\ 2 \end{bmatrix}\varepsilon \approx k\varepsilon \oplus \tau(\mu;2)$$

proving that $G(\mu;r)$ is stably parallelizable in this case.

If $n_1 = \cdots = n_k = 3$, $r = s$ then $\xi_1 \oplus \cdots \oplus \xi_s \approx n\varepsilon$ implies that $\xi_1 \oplus \cdots \oplus \xi_k$ is stably trivial since for $k+1 \le j \le s$, ξ_j, being an oriented line bundle over a manifold, is trivial. From (*) we now obtain

$$\begin{aligned} \begin{bmatrix} n \\ 2 \end{bmatrix}\varepsilon &\approx \bigoplus_{1\le i\le k} \xi_i \oplus \tau(\mu;r) \\ &\sim \tau(\mu;r). \end{aligned}$$

This completes the proof.

We are now ready to prove the main theorem of this section.

Proof of Theorem 16.1:

The stable parallelizability of $\tilde{G}_{4,2} \cong \tilde{G}(2,2)$ and $\tilde{G}_{6,3} \cong \tilde{G}(3,3)$ follow from Proposition 16.2 above. The non-parallelizability of $\tilde{G}_{4,2}$ follows from the fact that span $\tilde{G}_{4,2} = 0$ as $\chi(\tilde{G}_{4,2}) \ne 0$, by Corollary 15.4. By Example 3.8 Span $\tilde{G}_{6,3} \ge 2$. Since dim $\tilde{G}_{6,3} = 9$, by Bredon-Kosinski's Theorem 3.2 it follows that $\tilde{G}_{6,3}$ is parallelizable.

As in the proof of Proposition 16.2, we see that

$$\begin{aligned} 10\varepsilon &\approx \Lambda^2(\tilde{\gamma}_{5,2}) \oplus \Lambda^2(\tilde{\beta}_{5,2}) \oplus \tilde{\tau}_{5,2} \\ &\approx \varepsilon \oplus \Lambda^1(\tilde{\beta}_{5,2}) \oplus \tilde{\tau}_{5,2} \\ &\sim \tilde{\beta}_{5,2} \oplus \tilde{\tau}_{5,2}, \end{aligned}$$

where $\tilde{\tau}_{n,k}$ denotes $\tau(\tilde{G}_{n,k})$. Therefore $\tilde{\tau}_{n,k} \sim \tilde{\gamma}_{5,2} \not\sim 0$. Thus $\tilde{G}_{5,2}$ and $\tilde{G}_{5,3} \cong \tilde{G}_{5,2}$ are not stably parallelizable.

Now only the cases $\tilde{G}_{n,k}$ with $n \geq 6$ and $(n,k) \neq (6,3)$ need to be considered. Assume as we may that $2k \leq n$. It follows that $n-k \geq 4 = \dim \mathbb{C}P^2$ in the cases presently under consideration. By Example 14.3 (i) and the Classification Theorem 14.1, there exists a map $g : \mathbb{C}P^2 \longrightarrow \tilde{G}_{n,k}$ such that $g^*(\tilde{\gamma}_{n,k}) \approx \xi \oplus (k-2)\varepsilon$, ξ being the underlying real 2-plane bundle of the canonical complex line bundle over $\mathbb{C}P^2$. Consequently, the following equalities hold in $KO(\mathbb{C}P^2)$:

$$\begin{aligned} g^*(\tilde{\beta}_{n,k}) &\approx g^*(n\varepsilon - \gamma_{n,k}) \\ &\approx n\varepsilon - g^*(\gamma_{n,k}) \\ &\approx (n - k + 2)\varepsilon - \xi. \end{aligned}$$

Thus

$$\begin{aligned} g^*(\tilde{\tau}_{n,k}) &\approx g^*(\tilde{\gamma}_{n,k} \otimes \tilde{\beta}_{n,k}) \\ &\approx g^*(\tilde{\gamma}_{n,k}) \otimes g^*(\tilde{\beta}_{n,k}) \\ &\approx (\xi \oplus (k-2)\varepsilon) \otimes ((n-k+2)\varepsilon - \xi) \\ &\approx (n-2k+4)\xi \oplus m\varepsilon - \xi\otimes\xi \end{aligned}$$

for a suitable m.

By Proposition 14.4, we obtain

$$\begin{aligned} g^*(\tilde{\tau}_{n,k}) &\approx (n-2k+4)\xi \oplus m\varepsilon - 4\xi \oplus 2\varepsilon \\ &\sim (n-2k)\xi, \end{aligned} \tag{*}$$

where '~' denotes the equality in $\widetilde{KO}$. From (*) and Proposition 14.4 again, it follows immediately that $\tilde{G}_{n,k}$ is not stably parallelizable if $2k \neq n$.

In case $n = 2k$, $k \geq 4$, consider the "inclusion"

$$\tilde{G}(4,4) = \tilde{G}_{8,4} \xrightarrow{j} \tilde{G}_{2k,k} = \tilde{G}(k,k)$$

where $j(\tilde{A},\tilde{B}) = (\tilde{X} + \tilde{A},\ \tilde{B} + \tilde{Y})$ where $\tilde{X} = \tilde{\mathbb{R}}^{k-4} \oplus 0 \oplus 0$ and $\tilde{Y} = 0 \oplus 0 \oplus \mathbb{R}^{k-4}$ in $\mathbb{R}^{2k} \cong \mathbb{R}^{k-4} \oplus \mathbb{R}^{8} \oplus \mathbb{R}^{k-4}$. It is clear that $j^*(\tilde{\gamma}_{2k,k}) \approx \tilde{\gamma}_{8,4} \oplus (k-4)\varepsilon$ and $j^*(\tilde{\beta}_{2k,k}) \approx \tilde{\beta}_{8,4} \oplus (k-4)\varepsilon$. Hence

$$\begin{aligned} j^*(\tilde{\tau}_{2k,k}) &\approx j^*(\tilde{\gamma}_{2k,k} \otimes \tilde{\beta}_{2k,k}) \approx j^*(\tilde{\gamma}_{2k,k}) \otimes j^*(\tilde{\beta}_{2k,k}) \\ &\approx (\tilde{\gamma}_{8,4} \oplus (k-4)\varepsilon) \otimes (\tilde{\beta}_{8,4} \oplus (k-4)\varepsilon) \\ &\approx (\tilde{\gamma}_{8,4} \otimes \tilde{\beta}_{8,4}) \oplus (k-4)\varepsilon \otimes (\tilde{\gamma}_{8,4} \oplus \tilde{\beta}_{8,4}) \oplus (k-4)^2\varepsilon \\ &\approx \tilde{\tau}_{8,4} \oplus (k-4)\varepsilon \otimes 8\varepsilon \oplus (k-4)^2\varepsilon \\ &\approx \tilde{\tau}_{8,4} \oplus ((k-4)^2 + (k-4)8)\varepsilon, \end{aligned}$$

using the relation $\tilde{\gamma}_{8,4} \oplus \tilde{\beta}_{8,4} \approx 8\varepsilon$. The above relation shows that $\tilde{G}_{2k,k}$ is stably trivial only if $\tilde{G}_{8,4}$ is. We will show that $w(\tilde{G}_{8,4}) \neq 1$. Indeed, from the relation

$$\tilde{\tau}_{8,4} \oplus (\tilde{\gamma}_{8,4} \otimes \tilde{\gamma}_{8,4}) \approx \tilde{\gamma}_{8,4} \otimes (\tilde{\beta}_{8,4} \oplus \tilde{\gamma}_{8,4}) \approx \tilde{\gamma}_{8,4} \otimes 8\varepsilon \approx 8\tilde{\gamma}_{8,4},$$

we obtain

$$w(\tilde{G}_{8,4}) = (w(\tilde{\gamma}_{8,4}))^8 \cdot (w(\tilde{\gamma}_{8,4} \otimes \tilde{\gamma}_{8,4}))^{-1}.$$

Using Examples 8.5 and 8.2 (iv)

$$w(\tilde{G}_{8,4}) = (1 + w_2^8 + w_3^8) \cdot (1 + w_2^4 + w_3^4)^{-1}.$$

where $w(\tilde{\gamma}_{8,4}) = 1 + w_2 + w_3 + w_4$. Thus

$$w_8(\tilde{G}_{8,4}) = w_2^4.$$

To show that $w_2^4 \neq 0$, we use Example 15.7(i). We have the following relation that generates all other relations among w_i's

$$(1 + w_2 + w_3 + w_4) \cdot (1 + \bar{w}_2 + \bar{w}_3 + \bar{w}_4) = 1,$$

where $\bar{w}_i = w_i(\tilde{\beta}_{8,4})$. Thus the only relations among the w_is in degree 8 are additively generated by the following relations:

$$w_3^2 w_2 + w_2^4 = 0$$
$$w_4^2 + w_4 w_2^2 = 0$$

It follows immediately that $w_2^4 \neq 0$. Consequently $\tilde{G}_{8,4}$, and hence $\tilde{G}_{2k,k}$, $k \geq 4$, are not stably parallelizable.

This completes the proof.

Remarks 16.3

(i) The above theorem has also been obtained by Miatello-Miatello [30]. Their proof of non-parallelizability is based on computations of Stiefel-Whitney classes involving Schubert calculus, and Pontrjagin classes. The proof that $\tilde{G}_{2^r,2}$ and $\tilde{G}_{2^r+2,3}$ are not stably parallelizable for $r \geq 3$ given in [30] seems to contain an error. The stable parallelizability of $\tilde{G}(3,\ldots,3)$, of which $\tilde{G}_{6,3}$ is a particular case, required the knowledge of certain properties of the adjoint representation of $SO(3) \times \cdots \times SO(3)$. Our proof is quite elementary in that it uses only basic properties of n-universal bundles. The appearance of Stiefel-Whitney classes at the end of the proof seems to be unavoidable. However calculations have been greately simplified.

(ii) Theorem 16.1 readily gives the solution for the stable paralelizability of $G_{n,k}$ for $2 \leq k \leq n-2$ and $(n,k) \neq (4,2), (6,3)$. In these two cases one sees easily that $w(G_{n,k}) \neq 1$. This proves the "real part" of Theorem 11.1.

(iii) Using Plücker co-ordinates one can prove that $\tilde{G}_{4,2} \cong S^2 \times S^2$, showing that $\tilde{G}_{4,2}$ is stably parallelizable but not parallelizable. See also [14], vol. 2, p.104.

§17. Stable Parallelizability of Flag$^+$ Manifolds.

In this section we consider the problem of determining which of the flag$^+$ manifolds are stably parallelizable. Theorem 17.1 summarizes the negative results and Theorem 17.3 the positive results. The further question of parallelizability is treated in the following section (see Theorem 18.1). These theorems solve the questions of parallelizability or stable parallelizability for most flag$^+$ manifolds, and an appendix summarizing the still unsolved cases appears at the end of §18.

Let $\mu = (n_1,\ldots,n_s)$ and let $1 \leq r \leq s$. Let $s \leq 3$ (when $s = 2$, $G(\mu;r)$ is just the oriented Grassmann manifold whose parallelizability was considered in the previous section.) Since $G(\mu;s-1) \cong G(\mu;s)$ as noted in Definition 15.1 it will be assumed that $r \neq s-1$. Further, in case $r = s$ we assume that at least two of the n_i's are different from 1, for, otherwise, $G(\mu;s) = V_{n,s-1}$ by Example 15.2 and hence are parallelizable by [26] or [48]. As in §10 we let $\mu_k = n_1,\ldots,n_k$, $|\mu_k| = n_1+\cdots+ n_k$ for $1 \leq k \leq s$, and let $n = |\mu_s|$.

We state the main results of this section in Theorems 17.1 and 17.3.

THEOREM 17.1

With the above notations, $G(\mu;r)$ is not stably parallelizable in the following cases:

(i) $r \leq s$, and $\{n_1,\ldots,n_s\} \not\subset \{2,1\}$ or $\{3,1\}$.

Now assume that $\{n_1,\ldots,n_s\} \subset \{2,1\},\{3,1\}$ or that $n_q \neq 1$ for precisely one value of q, $1 \leq q \leq s$.

(ii) Let $n_{r+1} = \cdots = n_s = 1$. (a): $1 \leq r \leq s-2$, $n_i \neq 1,2$, or 6 for some i, $1 \leq i \leq r$. (b): $r \leq s-4$, and $n_j \neq 1,2$ for some j, $1 \leq j \leq r$.

(iii) $n_j > 1$ for some j, $r+1 \leq j \leq s$, $1 \leq r \leq s-3$.

(iv) Let $r = s-2$, $n_i > 1$ for some $i > r$. The cases $\{n_{s-1},n_s\} \neq \{1,3\}, \{1,7\}$.

Examples 17.2

None of the following are stably parallelizable using (i), (ii)(a), (ii)(b), (iii) and (iv) respectively:

$$\tilde{G}(5,2,1), \qquad \tilde{G}(3,2,2,2)$$

$$G(10,1,1,1;3)$$

$$G(3,1,1,1,1;1)$$

$$G(5,4,2,2,1;2)$$

$$G(4,1,1,8,7;2), \qquad G(1,1,1,3,7;3)$$

THEOREM 17.3.

$G(\mu;r)$ is stably parallelizable in the following cases:

(i) $r = s$ and $\mu = (2,\ldots,2)$, $(2,\ldots,2,1)$.

(ii) See also Theorem 18.1 for parallelizable cases.

We now prove a general lemma on "inclusions" for flag$^+$ manifolds. The notation is somewhat awkward and will be dealt with first.

Let $I' = (k_1,\ldots,k_t)$ $t \geq 2$, be an increasing subsequence of $I = (1,\ldots,s)$. Let r' be the number of terms in I' which are $\leq r$. Let $\mu' = (m_1,\ldots,m_t)$ where $m_i = n_{k_i}$. Thus μ' is a subsequence of μ. When $r' = 0$, we convene that $G(\mu',r') = G(\mu')$. Let $i : G(\mu') \longrightarrow G(\mu)$ and $\tilde{i} : G(\mu';r') \longrightarrow G(\mu;r)$ be the inclusions induced by the inclusion ι of $\mathbb{R}^{|\mu'|} = \sum_{1\leq i\leq t} X_{k_i}$ into $\mathbb{R}^{|\mu|} = \sum_{1\leq j\leq s} X_j$ where $X_j = \{(a_1,\ldots,a_s) \in \bigoplus_{1\leq i\leq s} \mathbb{R}^{n_i} \mid a_i = 0 \text{ if } i \neq j\}$. More precisely, for $(A_1,\ldots,A_t) \in G(\mu')$,

$$i(A_1,\ldots,A_t) = \begin{cases} \iota(A_i) & \text{if } j = k_i \text{ for some } i \\ X_j & \text{otherwise.} \end{cases}$$

In the case of the flag$^+$ manifold $G(\mu';r')$ $\tilde{i}$ is defined similarly, with orientation on B_j for $1 \leq j \leq r$ being the same as on A_{k_i}, $j = k_i$ for some i, and the standard orientation on $X_j \cong \mathbb{R}^{n_j}$ otherwise. There is a commutative diagram

$$\begin{array}{ccc} G(\mu';r') & \xrightarrow{\tilde{i}} & G(\mu;r) \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ G(\mu') & \xrightarrow{i} & G(\mu) \end{array},$$

f and f' being the maps that forget the orientations. Note that f and f' are (finite) covering projections. Therefore

$$f^*(\tau(\mu)) \approx \tau(\mu;r)$$

$$f'^*(\tau(\mu')) \approx \tau(\mu';r')$$

as vector bundles. As in the proof of Theorem 11.1 one verifies that

$$i^*(\tau(\mu)) \sim \tau(\mu').$$

Hence, using the commutativity of the above diagram, (or directly), one obtains

$$\tilde{i}^*(\tau(\mu;r)) \sim \tau(\mu';r').$$

The following lemma is now immediate.

LEMMA 17.4

With the above notations, $\tilde{i}^*(\tau(\mu;r))$ is stably equivalent to $\tau(\mu';r')$ and $f^*(\tau(\mu))$ is isomorphic as a vector bundle to $\tau(\mu;r)$. Thus $G(\mu;r)$ is not stably parallelizable if $G(\mu';r')$ is not, and is (stably) parallelizable if $G(\mu)$ is (stably) parallelizable.

We are now ready to prove Theorem 17.1.

Proof of 17.1 (i):

By assumption, there exist two numbers i and j, $1 \leq i,j \leq s$ such that $n_i, n_j > 1$ and $(n_i, n_j) \neq (2,2),(3,3)$. Take $I' = (i,j)$ so that $\mu' = (n_i, n_j)$. By Lemma 17.4, and Theorem 16.1 it follows that $G(\mu;r)$ is not stably parallelizable.

Proof of 17.1 (iia):

When $1 \leq r \leq s-2$, take $I' = (i,s-1,s)$ so that $r' = 1$ and $\mu' = (n_i,1,1)$. Thus $G(\mu';r') \cong X_{n_i+2,2}$ by Example 15.2. The statement in this case follows from Theorem 14.5 and Lemma 17.4.

Proof of 17.1 (iib):

When $1 \leq r \leq s-4$, we may assume, from what has just been shown, that $n_i = 6$. Take $I' = (j,s-3,s-2,s-1,s)$, so that $G(\mu',r')$ $= G(6,1,1,1,1;1) = M$, say. Now, consider the covering projection

$$p : X_{10,4} \longrightarrow M$$

where $p([v_1,\ldots,v_4]) = (\{v_1,v_2,v_3,v_4\}^{\perp}, \mathbb{R}v_1,\mathbb{R}v_2,\mathbb{R}v_3,\mathbb{R}v_4)$ for $(v_1,v_2,v_3,v_4) \in V_{10,4}$. The orientation on $\{v_1,v_2,v_3,v_4\}^{\perp}$ is that defined by an ordered basis $u_1,\ldots,u_6$ of $\{v_1,\ldots,v_4\}^{\perp}$ such that $u_1,\ldots,u_6, v_1,\ldots,v_4$ (equivalently $u_1,\ldots,u_6, -v_1,\ldots,-v_4$) is in the standard orientation of $\mathbb{R}^{10}$. Now since $X_{10,4}$ is not stably parallelizable (see Theorem 14.5) M is not stably parallelizable. Again, by Lemma 17.4 it follows that $G(\mu;r)$ is not stably parallelizable.

Proof of 17.1 (iii):

When $1 \leq r \leq s-3$, take $I' = (r+1,\ldots,s)$ so that $\mu' = (n_{r+1},\ldots,n_s)$ and $r' = 0$. Thus $G(\mu',r') = G(\mu')$. The statement in this case follows from Theorem 11.1 and Lemma 17.4 again. When $r = s-2$, the same proof applies in case $G(n_{s-1},n_s)$ is not stably parallelizable.

Proof of 17.1 (iv):

We now make use of the assumption that for some i, $i > r = s-2$, $n_i \neq 1$. Let $I' = (s-1,s)$. Then $G(\mu')$ is not stably parallelizable when $\{n_{s-1},n_s\} \neq \{3,1\},\{7,1\}$ by 10.2. Consequently, $G(\mu;r)$ is not stably parallelizable in these cases. We now prove that $N = G(3,3,1;1)$ is not stably parallelizable. Let $\xi_p = \xi_p(3,3,1;1)$ and let $\tau = \tau(3,3,1;1)$. Since $\xi_1 \oplus \xi_2 \oplus \xi_3 \approx 7\epsilon$ and ξ_1 is orientable, $\Lambda^2(\xi_1) \approx \xi_1$ and $\xi_2 \oplus \xi_3$ is orientable. Since ξ_3 is a line bundle, we must have $\xi_3 \approx \Lambda^3(\xi_2)$. Thus (see Example 6.11) $\Lambda^2(\xi_2) \approx \xi_2 \otimes \xi_3$. Now from the bundle equivalences $\xi_1 \oplus \xi_2 \oplus \xi_3 \approx 7\varepsilon$ and

$$\begin{aligned}\binom{7}{2}\varepsilon &\approx \Lambda^2(\xi_1) \oplus \Lambda^2(\xi_2) \oplus \Lambda^2(\xi_3) \oplus \tau \\ &\approx \xi_1 \oplus \xi_2 \otimes \xi_3 \oplus \tau,\end{aligned}$$

we see that, to show that τ is not stably trivial it suffices to prove that $w(\xi_2 \otimes \xi_3) \neq w(\xi_2 \oplus \xi_3)$. Now $w_4(\xi_2 \otimes \xi_3) = 0$. By Example 15.7 (ii),

$$w_4(\xi_2 \oplus \xi_3) = w_3(\xi_2)\cdot w_1(\xi_3) \neq 0,$$

showing that $G(3,3,1;1)$ is not stably parallelizable. By Lemma 17.4 again, this proves (iv).

Proof of Theorem 17.3:

Refer to Proposition 16.2.

Remark 17.5.

The results of Theorem 17.1 for the cases $r = s-1$, s are known mostly from the work of I.D. Miatello and R.J. Miatello [30]. However, their proof that $\tilde{G}_{2^r,2}$ and $\tilde{G}_{2^r+2,3}$, $r > 2$, are not stably parallelizable seems to contain an error. Consequently, there appears to be some gap in their proof that $\tilde{G}(6,2,2)$, for example, is not stably parallelizable. Our proof of non-stable parallelizability of flag$^+$ manifolds in some other cases is quite similar to theirs.

§18. Parallelizability of Flag$^+$ Manifolds.

In this section we continue to use the notations of §17. The main results of this section are contained in the following.

THEOREM 18.1

Let $\mu = (\underset{k}{a,\ldots,a},\ \underset{m}{1,\ldots,1})$.

(i) $G(\mu;r)$ is parallelizable if $a = 1$; or if $a = 2$, $k \le r \le s$, $m > 1$; or if $a = 3$, $r = s$.

(ii) $G(6,1,1;1)$ and $G(6,1,1,1;2)$ are parallelizable.

(iii) $\tilde{G}(2,\ldots,2)$ and $\tilde{G}(2,\ldots,2,1)$ are not parallelizable.

In order to prove Theorem 18.1 we convert the parallelizability problem to a lifting problem in homotopy theory. For this purpose we now turn to the Λ^2 construction, a key tool in our work. It seems to have been first utilized by K.Y. Lam [26] to obtain immersion results for flag manifolds.

For $\mu = (n_1,\ldots,n_s)$, let $\nu = \left[\begin{bmatrix} n_1' \\ 2 \end{bmatrix},\ldots,\begin{bmatrix} n_k' \\ 2 \end{bmatrix},d\right]$ where $(n_1',\ldots,n_k')$ is the subsequence of μ obtained by omitting the 1's in the sequence μ, and $d = \begin{bmatrix} n \\ 2 \end{bmatrix} - \sum_{1\le i\le s} \begin{bmatrix} n_i \\ 2 \end{bmatrix} = \dim G(\mu)$. Let $1 \le r \le s$ and let $\ell = |\{i_p|\ 1 \le p \le r\}|$. Thus ℓ is the number of oriented components in each flag$^+$ of $G(\mu;r)$ of dimension greater than one. We define a map

$$g : G(\mu;r) \longrightarrow G(\nu;\ell) \text{ as follows.}$$

Regard $\mathbb{R}^{|\nu|} = \mathbb{R}^{\binom{n}{2}}$ as $\Lambda^2(\mathbb{R}^n)$. For $\underline{A} = (\tilde{A}_1,\ldots,\tilde{A}_r,A_{r+1},\ldots,A_s) \in G(\mu;r)$

$$g(\underline{A}) = (\Lambda^2(A_{i_1}),\ldots,\Lambda^2(A_{i_k}),T_{\underline{A}})$$

where $T_{\underline{A}}$ is the orthogonal complement of $\Lambda^2(A_{i_1}) + \cdots + \Lambda^2(A_{i_k})$ in $\mathbb{R}^{|\nu|}$ with respect to the induced inner product on $\mathbb{R}^{\binom{n}{2}} \cong \Lambda^2(\mathbb{R}^n)$. Also note that the orientation on $\Lambda^2(A_{i_p})$ for $1 \le p \le \ell$ is that defined by the ordered basis $a_1 \wedge a_2,\ldots,a_1 \wedge a_m,\ldots,a_{m-1} \wedge a_m$ of $\Lambda^2(A_{i_p})$, where $m = n_p'$ and $a_1,\ldots,a_m$ is any positively oriented basis of $\tilde{A}_{i_p}$. It is clear that g is a well defined, continuous map of $G(\mu;r)$ into $G(\nu;\ell)$. We now prove the following lemma.

LEMMA 18.2.

With the above notations, $g^*(\xi_{k+1}(\nu;\ell))$ is bundle isomorphic to $\tau(\mu;r)$.

Proof:

Recall that $\tau(\mu;r) \approx \bigoplus_{1\le i<j\le s} \xi_i(\mu;r) \otimes \xi_j(\mu;r)$ as vector bundles. Let $T(\mu;r)$ denote the total space of the tangent bundle $\tau(\mu;r)$. Define $\hat{g} : T(\mu;r) \longrightarrow E(\xi_{k+1}(\nu;\ell))$ as follows: For $\underline{A} = (\tilde{A}_1,\ldots,\tilde{A}_r,A_{r+1},\ldots,A_s) \in G(\mu;r)$, $\hat{g}$ is the map defined by $\hat{g}(\underline{A},a_i \otimes a_j) = (g(\underline{A}),\ a_i \wedge a_j)$ for $a_i \otimes a_j \in A_i \otimes A_j \subset \tau(\mu;r)|\underline{A}$. Note that $a_i \wedge a_j$ is in the image of $\xi_{k+1}(\nu;\ell)$ over $g(\underline{A})$, for clearly if $i \neq j$ then $A_i \wedge A_j$ is orthogonal to all $\Lambda^2(A_{i_m}) = A_{i_m} \wedge A_{i_m}$, $1 \le m \le k$. Since exterior product is linear, $\hat{g}$ is well-defined. $\hat{g}$ is continuous, fibre preserving, and restricted to each fibre it is a vector space isomorphism. Therefore $\hat{g}$ is a bundle map, which covers g. Hence $g^*(\xi_{k+1}(\nu;\ell) \approx \tau(\mu;r)$ as vector bundles.

COROLLARY 18.3

$G(\mu;r)$ is parallelizable if and only if g can be lifted to a map $\bar{g}$ making the following diagram commutative

$$\begin{tikzcd}
 & G\left[\underbrace{1,\ldots,1}_{d},\begin{bmatrix} n_1' \\ 2 \end{bmatrix},\ldots,\begin{bmatrix} n_k' \\ 2 \end{bmatrix};\ \ell+d\right] \arrow[d, "\pi"] \\
G(\mu;r) \arrow[r, "g"] \arrow[ur, dashed, "\bar{g}"] & G(\nu;\ell)
\end{tikzcd}$$

where
$$\pi(t_1,\ldots,t_d,\tilde{B}_1,\ldots,\tilde{B}_{\ell+1},\ldots,B_k)$$
$$= (\tilde{B}_1,\ldots,\tilde{B}_\ell,B_{\ell+1},\ldots,B_k,\mathrm{span}\{t_1,\ldots,t_d\}).$$

Proof:

If g can be lifted to a map $\bar{g}$ as in the above diagram, then

$$\begin{aligned}\tau(\mu;r) \approx g^*(\xi_{k+1}(\nu;\ell)) &= (\pi \circ \bar{g})^*(\xi_{k+1}(\nu;\ell)) \\ &= \bar{g}^*(d\varepsilon) \\ &= d\varepsilon\end{aligned}$$

as vector bundles, since $\pi^*(\xi_{k+1}(\nu;\ell))$, being a Whitney sum of d *oriented* line bundles over a compact manifold, is a trivial bundle of rank d.

If $\tau(\mu;r)$ is trivial, then there exist d orthonormal sections $s_1,\dots,s_d$ of $g^*(\xi_{k+1}(\nu;\ell)) \approx \tau(\mu;r)$. $\bar{g}$ is defined by

$$\bar{g}(\underline{A}) = (s_1(\underline{A}),\dots,s_d(\underline{A}),g_1(\underline{A}),\dots,g_k(\underline{A}))$$

where $\qquad g(\underline{A}) = (g_1(\underline{A}),\dots,g_{k+d}(\underline{A}))$.

This completes the proof.

Proof of Theorem 18.1. (i):

The case a = 1 follows from the fact that $G(1,\dots,1)$ is parallelizable. See Theorem 11.1 and Lemma 17.4.

When a = 2, $k \le r \le s-1$, we use Corollary 18.3. Note that $\nu = (\underbrace{1,\dots,1}_{k},d)$ and $\ell = k$. Hence $G(\nu;\ell)$ is the Stiefel manifold $V_{k+d,k}$.

Let $m \ge 2$. Consider the following diagram:

$$\begin{tikzcd}
& V_{k+d,k+d} \arrow[d, "\pi"] & \\
G(\mu;r) \arrow[ur, dashed, "\bar{g}"] \arrow[r, "g"] \arrow[drr, "p"'] & V_{k+d,k} & \\
& & G(\mu';r) \arrow[ul, "g'"'] \arrow[uul, dashed, "\bar{g}'"']
\end{tikzcd}$$

where $\mu' = (\underbrace{2,\ldots,2}_{k}, \underbrace{1,\ldots,1}_{m-2}, 2)$ and p is the map that sends the flag^{+} $\underline{A} = (\tilde{A}_1,\ldots,\tilde{A}_r,A_{r+1},\ldots,A_s)$ to $(\tilde{A}_1,\ldots,\tilde{A}_r,A_{r+1},\ldots,A_{s-2},A_{s-1}+ A_s)$. Since $g(\underline{A})$ depends only on $\tilde{A}_1,\ldots,\tilde{A}_k$ for each $\underline{A} \in G(\mu;r)$, g defines a map $g'\colon G(\mu';r) \longrightarrow V_{k+d,k}$ such that $g' \circ p = g$. Now $\dim G(\mu';r) = \dim G(\mu;r) - 1 = d - 1$ equals the connectivity of $V_{k+d,k}$. Therefore there exists a map $\bar{g}'\colon G(\mu';r)$ so that $\pi \circ \bar{g}' = g'$ in the diagram above. Taking $\bar{g} = \bar{g}' \circ p$, we see that

$$\pi \circ \bar{g} = \pi \circ (\bar{g}' \circ p) = (\pi \circ \bar{g}') \circ p = g' \circ p = g$$

completing the proof of parallelizability in this case.

It will be shown in §23 that span $G(\mu;r) >$ span S^d in case $a = 3$, $m \geq 2$, $r = s-1$ or s. Since by Propositiion 16.2, $G(\mu;r)$ is stably parallelizable, it follows by the Bredon-Kosinski Theorem 3.2 that $G(\mu;r)$ is parallelizable.The cases $m = 0,1$ are due to Stong [42].

Proof of 18.1 (ii):

Parallelizability of $G(6,1,1;1) \cong X_{8,2}$ is a known result due to Zvengrowski [48] (cf. Theorem 14.5.).

Let $\xi_i = \xi_i(6,1,1,1;2)$ and let τ denote the tangent bundle of $M = G(6,1,1,1;2)$. Thus $\xi_2 \approx \varepsilon$. Since $\bigoplus_{1\leq i\leq 4} \xi_i \approx 9\varepsilon$, and since ξ_1 and ξ_2 are oriented, it follows that $w_1(\xi_3) = w_1(\xi_4)$. Since ξ_3 and ξ_4 are line bundles, it follows that $\xi_3 \approx \xi_4$ as vector bundles. Let $\zeta \approx \xi_3 \approx \xi_4$. We now have the following bundle equivalences:

$$\tau \approx \bigoplus_{1\le i<j\le 4} \xi_i \otimes \xi_j$$
$$\approx \xi_1 \oplus \xi_3 \oplus \xi_4 \oplus \xi_1 \otimes (\xi_3 \oplus \xi_4) \oplus \xi_3 \otimes \xi_4$$

since $\xi \otimes \xi_2 \approx \xi_2 \otimes \xi \approx \xi$, as $\xi_2 \approx \varepsilon$. Now $\xi_3 \otimes \xi_4 \approx \zeta \otimes \zeta \approx \varepsilon \approx \xi_2$ as vector bundles. Therefore

$$\tau \approx \xi_1 \oplus \xi_2 \oplus \xi_3 \oplus \xi_4 \oplus \xi_1 \otimes 2\zeta$$
$$\approx 9\varepsilon \oplus \xi_1 \otimes 2\zeta.$$

Hence span $M \ge 9 >$ span S^{21}.

Since dim $M = 21$, by the Bredon-Kosinski Theorem 3.2 we need only show that $\tau \sim 0$. In $KO(M)$, $\xi_1 \approx 9\varepsilon - \xi_2 - \xi_3 - \xi_4 \approx 8\varepsilon - 2\zeta$. Therefore

$$\tau \sim \xi_1 \otimes 2\zeta \approx (8\varepsilon - 2\zeta) \otimes 2\zeta \sim 16\zeta.$$

Consider the map $q : M \longrightarrow \mathbb{R}P^8$ which is the projection onto the last co-ordinate. Denoting the canonical line bundle over $\mathbb{R}P^8$ by Υ, we see that $q^*(\Upsilon) \approx \xi_4 = \zeta$. Since the order of Υ is 16 by [1], it follows that $16\zeta \sim 0$. Hence $M = G(6,1,1,1;2)$ is parallelizable.

Proof of 18.1 (iii):

The non-parallelizability of $\tilde{G}(2,\ldots,2)$ and $\tilde{G}(2,\ldots,2,1)$ follow from the fact that their span are zero by Corollary 15.4.

Remarks 18.4.

The proof of parallelizability in the case $\tilde{G}(2,\ldots,2,1,\ldots,1)$, $k \le r$, given here is more direct than the one found in [30]. Further, it covers a much wider class of manifolds. For example, parallelizability of $X_{2n,2n-2}$ follows from that of $M = G(2,\underbrace{1,\ldots,1}_{2n-2};1)$ by considering the covering map $f : X_{2n,2n-2} \longrightarrow M$ which sends $[v_1,\dots,v_{2n-2}]$ to $(\tilde{A},\mathbb{R}v_1,\dots,\mathbb{R}v_{2n-2})$ with $\tilde{A}$ the oriented 2-plane $\mathbb{R}u_1 + \mathbb{R}u_2$ (the orientation being given by the ordered basis u_1,u_2 where $u_1,u_2,v_1,\dots,v_{2n-2}$ is an ordered basis in the standard orientation on $\mathbb{R}^{2n}$). The proof of parallelizability of $\tilde{G}(3,\dots,3,1,\dots,1)$ is the same as that given in [30].

(ii) Theorems 17.1, 17.3, and 18.1 do not settle the question of parallelizability and stable parallelizability of all the flag$^+$ manifolds. The following table indicates the types of flag$^+$ manifolds, up to diffeomorphisms arising from the permutations of $n_1,\dots,n_r$ and $n_{r+1},\dots,n_s$, <u>not</u> known to be parallelizable. By a proof similar to that of Theorem 18.1 (ii), one shows easily that $\tau(\mu;s-2) \sim 16\xi_s$ in case (a), and in case (b) $\tau(\mu;s-3) \sim 8(\xi_{s-2}\oplus \xi_{s-1}\oplus \xi_s)$. Similarly, in case (c) $\tau(\mu;s-2) \sim (a+1)\xi_{s-1}$.

<u>Table 18.5</u>

The following flag$^+$manifolds are not known to be stably parallelizable.

(a) $G(6,1,\dots,1;s-2)$ $\quad n \geq 10.$

(b) $G(6,\dots,1;s-3)$ $\quad n \geq 9.$

(c) $G(1,\dots,1,a;s-2)$ $\quad a = 3$ or 7, $s \geq 3.$

CHAPTER FIVE

$H^*(Y_{n,k};\mathbb{Z}_2)$ - A COMPUTATIONAL AID

§19 Introduction.

The $\mathbb{Z}_2$-cohomology of real flag manifolds has been computed by Borel [6]. In this chapter we study the relations among a set of algebra-generators of the cohomology algebra $H^*(Y_{n,k};\mathbb{Z}_2)$, where $Y_{n,k}$ is the flag manifold $G(\underbrace{1,\dots,1}_{k}, n-k)$.

The space $Y_{n,k}$ is related to the Grassmann manifold $G_{n,k}$ as follows: The map $q : V_{n,k} \longrightarrow G_{n,k}$ can be factored as

$$V_{n,k} \xrightarrow{p} Y_{n,k} \xrightarrow{f} G_{n,k}$$

where $p(v_1,\dots,v_k) = (\mathbb{R}v_1,\dots,\mathbb{R}v_k, \{v_1,\dots,v_k\}^{\perp}) \in Y_{n,k}$, and $f(A_1,\dots,A_{k+1}) = (A_{k+1}^{\perp} , A_{k+1}) \in G_{n,k}$.

It is shown here that f induces a monomorphism in $\mathbb{Z}_2$-cohomology and that the relations among the generators of $H^*(Y_{n,k};\mathbb{Z}_2)$ have a very simple description. See Theorem 20.8. Thus $f^*: H^*(G_{n,k};\mathbb{Z}_2) \longrightarrow H^*(Y_{n,k};\mathbb{Z}_2)$ can be used to decide the vanishing (or not) of Stiefel-Whitney classes of $G_{n,k}$. Applications of the results of this chapter are postponed to the next.

We remark that Hiller [15], Stong [40] and Hiller-Stong [17] have made use of the space $Y_{n,n}$ to study the relations among the generators $w_i(\gamma_{n,k})$, $1 \le i \le k$, of $H^*(G_{n,k})$, by considering the map in cohomology induced by the map $Y_{n,n} \longrightarrow G_{n,k}$ that takes the flag $(A_1,\dots,A_n)$ in

$Y_{n,n}$ to $(A_1 + \cdots + A_k, A_{k+1} + \cdots + A_n)$ in $G_{n,k}$. Using the relations so obtained, Hiller and Stong [17] derive some lower bounds for the immersions of $G_{n,k}$ in Euclidean spaces. Hiller [15] also obtains lower bounds for the Lusternik-Schnirelmann category for real Grassmann manifolds.

The observation that the use of $Y_{n,k}$ rather than $Y_{n,n}$ makes computations comparatively much easier in the calculation of Stiefel-Whitney classes of Grassmann manifolds is due to Prof. P. Zvengrowski.

§20. The Canonical Generators for $H^*(Y_{n,k};\mathbb{Z}_2)$

Let $A^* = \sum_{i \geq o} A^i$ be a graded commutative algebra over $\mathbb{Z}_2$ with unit 1. For a subset $S \subset A^*$, $\langle S \rangle$ denotes the ideal of A^* generated by S. Let A^+ denote the subalgebra $\sum_{i \geq 1} A^i$ of A^*. For $a \in A^*$, $a^{(m)}$ denotes the m^{th}-degree term of a. Thus,

$$a = a^{(o)} + \cdots + a^{(m)} + \cdots$$

with $a^{(j)} = 0$ for $j > |a|$, the degree of a.

Let A^{**} denote the commutative $\mathbb{Z}_2$-algebra of formal sums $a^{(o)} + \cdots + a^{(m)} + \cdots$ with $a^{(m)} \in A^m$, $m \geq 0$, where for $a,b \in A^{**}$

$$(ab)^{(m)} = \sum_{i+j=m} a^{(i)} b^{(j)} \qquad (*)$$

Note that if $a \in A^{**}$ with $a^{(o)} \in U(A^o)$, the group of units of A^o, then a has an inverse in A^{**}. We consider A^* as a subalgebra of A^{**} in the obvious way.

Let $S[X_1,\ldots,X_r]$ denote the subalgebra of symmetric polynomials in $X_1,\ldots,X_r$ of the polynomial algebra $\mathbb{Z}_2[X_1,\ldots,X_r]$ freely generated by elements $X_1,\ldots,X_r$, with $\deg X_i = 1$, $1 \le i \le r$. $\sigma_i(X_1,\ldots,X_r)$ denotes the i^{th}-elementary symmetric polynomial in $X_1,\ldots,X_r$ for $1 \le i \le r$. If r is clear from the context we simply write σ_i instead of $\sigma_i(X_1,\ldots,X_r)$. Note that $(1 + \sigma_1 + \cdots + \sigma_r) = \prod_{1\le i\le r} (1 + X_i)$ gives

$$(1 + \sigma_1 + \cdots + \sigma_r)^{-1} = \prod_{1\le i\le r} (1 + X_i)^{-1}$$
$$= \prod_{1\le i\le r} (1 + X_i + X_i^2 + \cdots).$$

This can be written as $1 + t_1 + t_2 + \cdots$, where $t_m = \Sigma\, X_1^{m_1}\cdots X_r^{m_r}$, the sum being taken over all sequences $(m_1,\ldots,m_r)$ with $m_i \ge 0$, $m_1 + \cdots + m_r = m$. We call $t_m = t_m(X_1,\ldots,X_r)$ the total symmetric polynomial of degree m; by convention $X_i^o = 1 = \sigma_o = t_o$. For example $t_2(X_1,X_2,X_3) = X_1^2 + X_2^2 + X_3^2 + X_1X_2 + X_1X_3 + X_2X_3$.

Let $V_{\infty,n}$ denote the infinite Stiefel "manifold" of orthonormal n-frames in $\mathbb{R}^\infty$. We have the universal principal O(n)-bundle $\gamma_{\infty,n} = (V_{\infty,n}, BO(n), O(n))$. Note that $O(n_1) \times \cdots \times O(n_s) \subset O(n)$ acts on $V_{\infty,n}$ for $n_1 + \cdots + n_s = n$ by restriction of the O(n)-action on it. Consider the following commutative diagram:

$$\begin{tikzcd}
 & O(n_1) \times \cdots \times O(n_s) \arrow[r] \arrow[d] & O(n) \arrow[r] \arrow[d] & G(\mu) \\
 & V_{\infty,n} \arrow[r, equal] \arrow[d] & V_{\infty,n} \arrow[d] & \\
G(\mu) \arrow[r, "j"] & B(O(n_1) \times \cdots \times O(n_s)) \arrow[r, "p"] & BO(n) &
\end{tikzcd}$$

Here $B(O(n_1) \times \cdots \times O(n_s)) = V_{\infty,n}/(O(n_1) \times \cdots \times O(n_s))$ is a classifying space for the group $O(n_1) \times \cdots \times O(n_s)$. A point in $B(O(n_1) \times \cdots \times O(n_s))$ is a sequence $\underline{A} = (A_1,\ldots,A_s)$ of pairwise orthogonal subspaces $A_1,\ldots,A_s$ of $\mathbb{R}^\infty$ with $\dim A_i = n_i$, and $p(\underline{A}) = A_1 + \cdots + A_s \in BO(n) = G_{\infty,n}$.

Taking $\mathbb{R}^n \in BO(n)$ as the base point, j is the inclusion of the fibre, which is the (real) flag manifold $G(\mu)$, into $B(O(n_1) \times \cdots \times O(n_s))$ $(\mu = (n_1,\ldots,n_s))$.

Let γ_i denote the i^{th}-canonical n_i-plane bundle over $B(O(n_1) \times \cdots \times O(n_s))$ whose fibre over $\underline{A}$ is the vector space A_i. Then $j^*(\gamma_r) = \xi_r$ for $1 \le r \le s$, and $p^*(\gamma_{\infty,n}) = \gamma_1 \oplus \cdots \oplus \gamma_{n_s}$. Let $w_i(r) = w_i(\xi_r)$ for $1 \le i \le n_r$, $1 \le r \le s$.

We state, without proof, the following theorem due to Borel [6].

Recall that $H^*(BO(m);\mathbb{Z}_2) = \mathbb{Z}_2[w_1(\gamma_{\infty,m}),\ldots,w_m(\gamma_{\infty,m})] \cong S[X_1,\ldots,X_m]$ under the algebra isomorphism that take $w_i(\gamma_{\infty,n})$ to $\sigma_i(X_1,\ldots,X_m)$. (cf. [32]).

THEOREM 20.1

p^* is a monomorphism and $H^*(G(\mu);\mathbb{Z}_2)$ is isomorphic as an algebra to the quotient of $H^*(B(O(n_1) \times \cdots \times O(n_s));\mathbb{Z}_2)$ by the ideal generated by elements of positive degree in the image of p^*. Thus

$$H^*(G(\mu);\mathbf{Z}_2) \cong \frac{S[X_1,\dots,X_{n_1}]\otimes\cdots\otimes S[X_{n-n_s+1},\dots,X_n]}{\langle S^+[X_1,\dots,X_n]\rangle}.$$

The $\mathbf{Z}_2$-Poincaré polynomial of $G(\mu)$ is

$$P(G(\mu);t) = \frac{(1-t)(1-t^2)\cdot\cdots\cdot(1-t^{n-1})(1-t^n)}{\prod_{1\le r\le s} (1-t)(1-t^2)\cdot\cdots\cdot(1-t^{n_r})}.$$

We now prove the following.

PROPOSITION 20.2.

$H^*(G(\mu);\mathbf{Z}_2)$ is generated by the Stiefel-Whitney classes $w_i(r)$, $1 \le i \le n_r$, $1 \le r \le s$, as a polynomial algebra subject only to the relations which arise from the relation

$$\prod_{1\le r\le s} (1 + w_1(r) + \cdots + w_{n_r}(r)) = 1.$$

Proof:

Note that $w_i(r) = w_i(\xi_r) = j^*(w_i(\gamma_r))$. Since $\{w_i(\gamma_r) \mid 1 \le i \le n_r,\ 1 \le r \le s\}$ generates the $\mathbf{Z}_2$-cohomology of $B(O(n_1) \times \cdots \times O(n_s))$ it follows from the above theorem that $\{w_i(r) \mid 1\le i\le n_r,\ 1\le r\le s\}$ generates the cohomology algebra $H^*(G(\mu);\mathbf{Z}_2)$. Also $\xi_1 \oplus\cdots\oplus \xi_s = n\varepsilon$ gives $w(\xi_1 \oplus\cdots\oplus \xi_s) = 1$. Hence

$$\prod_{1\le r\le s} (1 + w_1(r) + \cdots + w_{n_r}(r)) = 1. \tag{*}$$

To show that there exist no further relations among $w_i(r)$'s other than those that arise from (*) we proceed as follows:

Let $S[\mu]$ denote the subalgebra of $\mathbf{Z}_2[X_1,\dots,X_n]$ generated by the elements $\sigma_i(r) = \sigma_i(X_{|\mu_{r-1}|+1}, \ldots, X_{|\mu_r|})$, where $|\mu_r| = n_1 + \cdots + n_r$, $1 \le r \le s$, $|\mu_o| = 0$. Note that letting

$$\sigma_i = \sigma_i(X_1, \ldots, X_n)$$

$$1 + \sigma_i + \cdots + \sigma_n = \prod_{1\le i\le n} (1+X_i) = \prod_{1\le r\le s} \left[\prod_{|\mu_{r-1}|<i\le|\mu_r|} (1+X_i) \right]$$

$$= \prod_{1\le r\le s} (1 + \sigma_1(r) + \cdots + \sigma_{n_r}(r)).$$

Thus the ideal I_μ generated by the relations

$$\prod_{1\le r\le s} (1 + \sigma_1(r) + \cdots + \sigma_{n_r}(r)) = 1$$

is the same as that generated by $\{\sigma_i \mid 1\le i\le n\}$. Therefore the Poincarè polynomial of $S[\mu]/I_\mu$ is

$$= \frac{\text{Poincarè polynomial of } S[\mu]}{\text{Poincarè polynomial of } \mathbf{Z}_2[\sigma_1, \ldots, \sigma_n]}$$

$$= \frac{\prod_{1\le r\le s} (1-t)^{-1} \cdot \cdots \cdot (1-t^{n_r})^{-1}}{(1-t)^{-1} \cdot \cdots \cdot (1-t^n)^{-1}}$$

$$= P(G(\mu); t).$$

Since there exists an algebra homomorphism $\eta : \frac{S[\mu]}{I_\mu} \xrightarrow{\text{onto}} H^*(G(\mu);\mathbf{Z}_2)$ with $\eta(\sigma_i(r)) = w_i(r)$ and since their Poincarè polynomials are equal, it follows that η is an isomorphism.

Hence the only relations among $w_i(r)$ are those that arise from the relation (*).

This completes the proof.

In case $\mu = (1, \ldots, 1, n-k)$, $G(\mu) = Y_{n,k}$. Write $x_i = w_1(\xi_i)$ for $1 \le i \le k$. Using the above proposition one can consider the

Stiefel-Whitney classes of ξ_{k+1} as being defined in terms of x_i's. Indeed, writing $\sigma_i = \sigma_i(x_1,\dots,x_k)$ we have

$$1 = (1 + w_1(\xi_{k+1}) + \cdots + w_{n-k}(\xi_{k+1})) \cdot \prod_{1\le i\le k}(1 + x_i)$$
$$= (1 + w_1(\xi_{k+1}) + \cdots + w_{n-k}(\xi_{k+1})(1 + \sigma_1 + \cdots + \sigma_k).$$

Hence $1 + w_1(\xi_{k+1}) + \cdots + w_{n-k}(\xi_{k+1}) = (1 + \sigma_1 + \cdots + \sigma_k)^{-1}$

$$= 1 + t_1 + \cdots + t_{n-k} + t_{n-k+1} + \cdots.$$

Therefore we obtain the following corollary.

COROLLARY 20.3

$H^*(Y_{n,k})$ is generated by 1-dimensional classes $x_i = w_1(\xi_i)$, $1 \le i \le k$ subject to the conditions that $(1 + \sigma_1 + \cdots + \sigma_k)\cdot(1 + t_1 + \cdots + t_{n-k}) = 1$ (i.e. that $t_j = 0$ for $j > n-k$.)

Proof:

From the above computation, $w_i(\xi_{k+1}) = t_i$ for $1 \le i \le n-k$ and $t_j = 0$ for $j > n-k$. The corollary now follows from Proposition 20.2.

COROLLARY 20.4

$f : Y_{n,k} \longrightarrow G_{n,k}$ induces a monomorphism of $\mathbf{Z}_2$-cohomology algebras. In fact Im f^* is generated by $\{\sigma_i \mid 1 \le i \le k\}$ as a subalgebra.

Proof:

Note that $f^*(\gamma_{n,k}) = \xi_1 \oplus \cdots \oplus \xi_k$. Hence

$$f^*(w_i(\gamma_{n,k})) = \sigma_i \text{ and } f^*(w_i(\beta_{n,k})) = t_j.$$

Therefore, by Corollary 20.3, Ker f^* is generated as an algebra by $w_j(\beta_{n,k})$ for $j > n-k$. Since $\beta_{n,k}$ is an $(n-k)$-plane bundle, it follows that $w_j(\beta_{n,k}) = 0$ for $j > n-k$. The corollary follows.

Remark 20.5.

The above corollary also follows from the Splitting Principle once we note that $f : Y_{n,k} \longrightarrow G_{n,k}$ is a splitting bundle of $\Upsilon_{n,k}$. Compare [§2, 40].

Since the total symmetric polynomials are difficult to handle we need to describe the relations among the generators x_i's in $H^*(Y_{n,k};\mathbf{Z}_2)$ in terms of monomials. For this purpose we need the following lemmas.

Let $1 \le k \le n$, $n \ge 2$. For $1 \le r \le k$ define $I_{n,r}$ to be the ideal of $\mathbf{Z}_2[X_1,\dots,X_k]$ generated by $\{t_j(r) \mid j > n-r\}$. Here $t_j(r)$ denotes the total symmetric polynomial $t_j(X_1,\dots,X_r)$. We follow the convention that whenever we write a monomial $X_{i_1}^{m_1} \cdots X_{i_r}^{m_r}$ the subscripts $i_1,\dots,i_r$ are understood to be distinct.

LEMMA 20.6.

$I_{n,r} \subset I_{n,k}$ for $1 \le r \le k$.

Proof:

$t_m(r)$ is the coefficient of z^m in the formal expansion of

$$\prod_{1\le i\le r} (1 + X_j z)^{-1} = \prod_{i\le i\le k} (1 + X_i z)^{-1} \cdot \prod_{r+1\le j\le k} (1 + X_j z)$$
$$= (1 + t_1 z + t_2 z^2 + \cdots) \cdot (1 + Q_1 z + \cdots + Q_{k-r} z^{k-r})$$

where $t_j = t_j(k)$ and Q_i are some elements of $\mathbf{Z}_2[X_{r+1},\dots,X_k]$.

Now the coefficient of z^m of the expression on the right hand side is

$$t_m + t_{m-1}Q_1 + \cdots + t_{m-k+r}Q_{k-r}.$$

Note that if $m > n-r$ and $r \le k$, $m - k + r > n - k$. Therefore, it follows that $t_m + t_{m-1}Q + \cdots + t_{m-k+r}\, Q_{k-r} \in I_{n,k}$ for $m > n-r$. Hence $t_m(r) \in I_{n,k}$ for $m > n-r$. The lemma follows.

LEMMA 20.7

The ideal $I_{n,k}$ contains the monomials $X_{i_1}^n$; $X_{i_1}^{n-1} X_{i_2}^{n-1}$; $X_{i_1}^{n-1} X_{i_2}^{n-2} X_{i_3}^{n-2};\dots;X_{i_1}^{n-1} X_{i_2}^{n-2} \cdots X_{i_{r-1}}^{n-r+1} X_{i_r}^{n-r+1}$ for $\{i_1,\dots,i_r\} \subset \{1,\dots,k\}$, $2 \le r \le k$.

Proof:

Proof is by induction on r.

$$X_1^n = t_n(1) \in I_{n,1} \subset I_{n,k}, \text{ by Lemma 20.6}$$

Now assume that $X_{i_1}^{n-1} \cdots X_{i_{r-1}}^{n-r+2} X_{i_{r-1}}^{n-r+2} \in I_{n,k}$ for any $(r-1)$ distinct numbers i_m between 1 and k, for some $r \le k$.

By Lemma 20.6 again, $t_{n-r+1}(r) \in I_{n,k}$. But

$$X_1^{n-1} \cdots X_{r-1}^{n-r+1}\, t_{n-r+1}(r)$$
$$= X_1^{n-1} \cdots X_{r-1}^{n-r+1} X_r^{n-r+1} + \text{terms in } I_{n,k}$$

(due to the induction hypothesis).

Hence $X_1^{n-1} \cdots X_{r-1}^{n-r+1} \cdot X_r^{n-r+1} \in I_{n,k}$, completing the proof.

We are now ready to prove the main theorem of this section.

THEOREM 20.8.

Let $m_1 \geq \cdots \geq m_r$ be a decreasing sequence of non-negative integers.

(i) The monomial $x_{i_1}^{m_1} \cdots x_{i_r}^{m_r} \in H^*(Y_{n,k};\mathbf{Z}_2)$ is non-zero if and only if $m_j \leq n-j$.

(ii) $x_1^{n-1} \cdots x_k^{n-k} = x_{i_1}^{n-1} \cdots x_{i_k}^{n-k} \neq 0$.

Proof:

The "only if" part of (i) follows from Corollary 20.3 and Lemma 20.7.

Note that $d = \dim Y_{n,k} = \binom{n}{2} - \binom{n-k}{2} = \sum_{1\leq i\leq k} (n-i)$. Since $H^d(Y_{n,k};\mathbf{Z}_2) \approx \mathbf{Z}_2$, it follows that there must be a monomial of degree d that is non-zero. By the "only if" part of (i), it follows that

$$x_{i_1}^{n-1} \cdots x_{i_k}^{n-k} \neq 0$$

for some sequence of distinct integers $i_1,\ldots,i_k$ between 1 and k.

Now let $\{j_1,\ldots,j_k\} = \{1,\ldots,k\}$. The map induced in cohomology by the homeomorphism $Y_{n,k} \longrightarrow Y_{n,k}$ that takes $(A_1,\ldots,A_{k+1})$ to $(B_1,\ldots,B_k,A_{k+1})$ where $B_{i_r} = A_{j_r}$, $1 \leq r \leq k$, maps $x_{i_1},\ldots,x_{i_k}$ onto $x_{j_1},\ldots,x_{j_k}$ respectively. It follows that $x_{j_1}^{n-1}\cdots x_{j_k}^{n-k} \neq 0$. Since there is only one non-vanishing class in $H^d(Y_{n,k};\mathbf{Z}_2)$, it follows that $x_{j_1}^{n-1} \cdots x_{j_k}^{n-k} = x_{i_1}^{n-1} \cdots x_{i_k}^{n-k} \neq 0$. This proves (ii). The "if part" of (i) now follows from (ii).

§21. Computations in $H^*(Y_{n,k};\mathbf{Z}_2)$.

We continue to use the notations of §20.

Recall that $f : Y_{n,k} \longrightarrow G_{n,k}$ denotes the map that sends the flag $(A_1,\ldots,A_{k+1}) \in Y_{n,k}$ to $(A_{k+1}^{\perp},A_{k+1})$ in $G_{n,k}$. In view of Corollary 20.4 $f^*(H^*(G_{n,k};\mathbf{Z}_2))$ is the subalgebra generated by the elementary symmetric polynomimals $\sigma_1,\ldots,\sigma_k$. In this section we derive formulae for multiplying two symmetric monomials in $H^*(Y_{n,k};\mathbf{Z}_2)$ for $k = 3$, the formula for an arbitrary $k \le n$ being very difficult.

Notation: 21.1

Let $m_1,\ldots,m_k$ be a sequence of non-negative integers with $m_1 + \cdots + m_k > 0$. Denote by $[m_1,\ldots,m_k]$ the symmetric polynomial $\Sigma\, X_{i_1}^{m_1}\cdots X_{i_k}^{m_k}$ in $\mathbf{Z}_2[X_1,\ldots,X_k]$ where the sum is taken over all possible sequences $i_1,\ldots,i_k$, $1 \le i_1,\ldots,i_k \le k$, which yield distinct monomials. $[m_1,\ldots,m_k]$ will be called the symmetric monomial of type $m_1,\ldots,m_k$. We caution the reader that symmetric monomials are are not monomials, unless $k = 1$. If k is clear from the context we write $[m_1,\ldots,m_r]$ to denote $[m_1,\ldots,m_r,\underbrace{0,\ldots,0}_{k-r}]$. It is clear from the definition that all symmetric monomials (of a given degree m) form a $\mathbf{Z}_2$-basis for the vector subspace of symmetric polynomials (of degree m). By a little abuse of notation we denote the image of $[m_1,\ldots,m_k]$ in $H^*(Y_{n,k};\mathbf{Z}_2) \cong \mathbf{Z}_2[x_1,\ldots,x_k]$ under the algebra homomorphism that sends X_i to x_i for $1 \le i \le k$, by the same symbol $[m_1,\ldots,m_k]$.

Examples 21.2

(i) $[\underbrace{1,\ldots,1}_{r}] = \sigma_r$ for $1 \le r \le k$.

(ii) $[m] = \sum_{i=1}^{k} X_i^m$

(iii) $t_m = \sum_{m_1+\cdots+m_k=m} [m_1,\ldots,m_k]$.

(iv) Assume that $m_1 \ge \cdots \ge m_k$. By Theorem 20.8 $[m_1,\ldots,m_k] \in H^*(Y_{n,k};\mathbf{Z}_2)$ is zero if for some r, $m_j \ge n-j$ for $1 \le j \le r-1$ and $m_r > n-r$.

We now prove the following lemma, which is found to be very useful in all the computations in $H^*(Y_{n,3};\mathbf{Z}_2)$ that we will need to carry out.

LEMMA 21.3

Let $m_1 > m_2$. The coefficient of a symmetric monomial $[a,a,b]$ is zero in the expression of the product $[m_1,m_2]\cdot[n_1,n_2,n_3]$ in $\mathbf{Z}_2[X_1,X_2,X_3]$ as a $\mathbf{Z}_2$-linear combination of the symmetric monomials, unless $m_2 = 0$, $n_i = n_j$ for some $i \ne j$, and $b = n_k + m_1$, where $\{i,j,k\} = \{1,2,3\}$.

Proof:

In the expansion of $[m_1,m_2]\cdot[n_1,n_2,n_3]$ the monomial $X_1^a X_2^a X_3^b$ occurs only if $a = n_k$ or $b = n_k$ for some k.

Case (i): Let $b = n_k$. Then $a = m_1 + n_i = m_2 + n_j$ for some i,j, $\{i,j,k\} = \{1,2,3\}$. Since $m_1 \ne m_2$, $n_i \ne n_j$. Therefore $X_1^a X_2^a X_3^b$ occurs an even number of times: for, $X_1^{m_1} X_2^{m_2} \cdot X_1^{n_i} X_2^{n_j} X_3^{n_k}$ can be paired with $X_1^{m_2} X_1^{m_1} \cdot X_1^{n_j} X_2^{n_i} X_3^{n_k}$. Hence the coefficient of $X_1^a X_2^a X_3^b$ is zero. By symmetry the required coefficient of $[a,a,b]$ is zero.

Case (ii): Let $a = n_i$. In view of case (i) we assume $b \neq a$. Then $a = m_2 + n_j$ and $b = m_1 + n_k$ with j,k such that $\{i,j,k\} = \{1,2,3\}$. The monomial $X_1^a X_2^a X_3^b$ occurs in the multiplication $[m_1,m_2] \cdot [n_1,n_2,n_3]$ first as $X_2^{m_2} X_3^{m_1} \cdot X_1^{n_i} X_2^{n_j} X_3^{n_k}$. If $n_i \neq n_j$, then $X_1^{n_i} X_2^{n_j} X_3^{n_k} \neq X_1^{n_j} X_2^{n_i} X_3^{n_k}$. Therefore $X_1^a X_2^a X_3^b$ appears again as $X_1^{m_2} X_3^{m_1} \cdot X_1^{n_j} X_2^{n_i} X_3^{n_k}$.

As before, we conclude that in case $n_i \neq n_j$ the required coefficient of $[a,a,b]$ is zero. If, however, $n_i = n_j$, then $m_2 = 0$, completing the proof.

The following example will be used to obtain certain Stiefel-Whitney classes of the Grassmann manifolds $G_{n,3}$.

Example 21.4.

Assume that $n = q(2^r + 1)$, $r \geq 2$, and $k = 3$. For $1 \leq j \leq r$ one has

$$\prod_{j \leq p \leq r} [2^p q, 2^{p-1} q] = [2^r q, 2^r q - 2^{j-1} q, 2^r q - 2^j q]$$

in $H^*(Y_{n,3};\mathbf{Z}_2)$.

Proof:

When $j = r$ the statement is trivially valid.

Assume that the formula is valid for some $j + 1$ with $1 \leq j < r$.

Then $2^r q + 2^{j-1} q \geq n$. Therefore, using Example 21.2 (iv) and Lemma 21.3, we obtain

$$\prod_{j \leq p \leq r} [2^p q, 2^{p-1} q] = [2^j q, 2^{j-1} q] \cdot [2^r q, 2^r q - 2^j q, 2^r q - 2^{j-1} q]$$
$$= [2^r q, 2^r q - 2^j q + 2^{j-1} q, 2^r q - 2^{j+1} q + 2^j q]$$
$$= [2^r q, 2^r q - 2^{j-1} q, 2^r q - 2^j q],$$

completing the proof.

The reader is referred to Chapter I, [29] for an exposition of symmetric functions that deals with the relationships among the symmetric monomials, total symmetric polynomials, elementary symmetric polynomials, etc. in a more general setting.

CHAPTER SIX

SPAN OF FLAG MANIFOLDS.

§22. Introduction.

The problem of determining the span of a Grassmann manifold $G_{n,k}$ appears to be very difficult in general. Some progress has been made in obtaining lower bounds by Leite-Miatello [27] and more recently by Korbaš [24], who considers the wider class of flag manifolds. Bartík and Korbaš [5] have also obtained upper bounds for span $G_{n,k}$ by computations of the Stiefel-Whitney classes $w_i(G_{n,k})$ for $1 \leq i \leq 9$.

In the present chapter, we consider the question of span of flag manifolds. Our methods are well-known. We use a Stiefel-Whitney class argument to obtain upper bounds, and the results of Thomas [44] to obtain better lower bounds than those known from the work of Leite and Miatello. The span of $G_{n,k}$ has been determined in infinitely many non-trivial cases. The computational techniques of Chapter 5 are used throughout to calculate Stiefel-Whitney classes. Korbaš has used completely different methods to calculate the Stiefel-Whitney classes.

The present chapter is organized as follows: In §23, a lower bound for the span of flag manifolds is given in terms of the Radon-Hurwitz function. §24 deals with upper bounds for the span of Grassmann manifolds. In §25 we obtain better lower bounds for span $G_{n,k}$ using mainly the results of Thomas [44].

§23. The Radon-Hurwitz Function.

Let n be a positive integer. Write n as $n = s \cdot 2^{4a+b}$ where s is odd and $0 \leq b \leq 3$, $a \geq 0$. Define $\rho(n) = 8a + 2^b$. ρ is called the Radon-Hurwitz function.

The following well-known result is due to Radon [34] and Hurwitz [21]. See also [11].

Let n be even, so that $\rho(n) \geq 2$.

THEOREM 23.1

There exist $\rho(n) - 1$ antisymmetric orthogonal linear transformations $\varphi_i \in O(n)$, $1 \leq i \leq \rho(n) - 1$ satisfying the conditions: $\varphi_i^2 = -I$ and $\varphi_i\varphi_j + \varphi_j\varphi_i = 0$ for $1 \leq i < j \leq \rho(n)-1$, where I stands for the identity transformation.

For a construction of such linear transformations cf. [47].

LEMMA 23.2

Let $a_i \in \mathbb{R}$, $1 \leq i \leq \rho(n)-1$ with $\sum_{1 \leq i \leq \rho(n)-1} a_i^2 = 1$. Let $\varphi_i \in O(n)$ be as in Theorem 23.1 above. Then for the antisymmetric transformation $\varphi = \sum_{1 \leq i \leq \rho(n)-1} a_i\varphi_i$ one has $\varphi^2 = -I$. Equivalently $\varphi \in O(n)$.

Proof:

$$\begin{aligned}
\varphi^2 &= \sum_{1 \leq i \leq n} a_i^2\varphi_i^2 + \sum_{1 \leq i < j \leq n} a_ia_j(\varphi_i\varphi_j + \varphi_j\varphi_i) \\
&= \left[\sum_{1 \leq i \leq n} a_i^2\right](-I) + 0 \\
&= -I.
\end{aligned}$$

Since φ is antisymmetric, equivalently $\varphi \in O(n)$.

The following lemma gives some useful properties of the Radon-Hurwitz function ρ.

LEMMA 23.3

Let k, m be non-negative integers. Then,

(i) if $m = r\cdot 2^k$, r odd, then $\rho(m) = \rho(2^k)$.

(ii) $\rho(2^k) < \rho(2^{k+1})$.

(iii) $2k \le \rho(2^k) \le 2k+2$.

Proof:

(i) is obvious from the definition of ρ. Writing $k = 4a + b$ with $0 \le b \le 3$, $a \ge 0$, $\rho(2^k) = 8a + 2^{b+1}$. If $0 \le b \le 2$, then it is clear that $\rho(2^{k+1}) = 8a + 2^{b+1}$, hence $\rho(2^k) < \rho(2^{k+1})$. When $b = 3$, $k + 1 = 4(a+1)$. Therefore $\rho(2^{k+1}) = 8(a+1) + 2^0 > 8a + 2^3 = \rho(2^k)$. This proves (ii).

(iii) is immediate from the following inequalities: For $0 \le b \le 3$, $2b \le 2^b \le 2b + 2$. Hence, for $a \ge 0$,

$$2(4a+b) \le 8a + 2^b \le 2(4a+b) + 2.$$

The following theorem is a first step towards obtaining the span of flag manifolds. We use the notations of Chapter 3 and 4. Thus $\mu = (n_1,\ldots,n_s)$ is a sequence of positive integers with $s \ge 2$, $n = n_1 + \cdots + n_s = |\mu|$.

THEOREM 23.4 (Zvengrowski).

If n_i is odd for some i, $1 \le i \le s$, then span $G(\mu) \ge \rho(n) - 1$.

Proof:

Since $\rho(n) = 1$ when n is odd, obviously span $G(\mu) \geq \rho(n) - 1$ in this case.

Now assume that n is even. Without loss of generality, let n_1 be odd. Recall that

$$\begin{aligned}\tau(\mu) &\approx \bigoplus_{1\leq i<j\leq s} \xi_i(\mu) \otimes \xi_j(\mu) \\ &\approx \xi_1(\mu) \otimes \Big(\bigoplus_{2\leq j\leq s} \xi_j(\mu)\Big) \oplus \bigoplus_{2\leq i<j\leq n} \xi_i(\mu) \otimes \xi_j(\mu) \\ &\approx \mathrm{Hom}\Big(\xi_1(\mu), \bigoplus_{2\leq j\leq s} \xi_j(\mu)\Big) \\ &\qquad \oplus \bigoplus_{2\leq i<j\leq s} \mathrm{Hom}(\xi_i(\mu), \xi_{j(\mu)}) \qquad (*)\end{aligned}$$

Hence, to show that span $G(\mu) \geq \rho(n) - 1$, it suffices to prove that $\mathrm{Hom}(\xi_1(\mu), \bigoplus_{2\leq j\leq s} \xi_j(\mu))$, which is a subbundle of $\tau(\mu)$ by (*), admits $\rho(n) - 1$ linearly independent cross sections. To simplify notations we let $\xi = \xi_1(\mu)$ and $\eta = \bigoplus_{2\leq j\leq s} \xi_j(\mu)$. Note that the fibre of ξ (respectively η) over an arbitrary flag $\underline{A} = (A_1,\ldots,A_s)$ in $G(\mu)$ is A_1 (respectively $A_1^{\perp}$). We construct $(\rho(n)-1)$ cross sections $\Psi_i\colon G(\mu) \longrightarrow \mathrm{Hom}(\xi,\eta)$ as follows:

Choose linear transformations $\varphi_i\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$, $1 \leq i \leq \rho(n)-1$,satisfying the conditions of Theorem 23.1. For a subspace X of $\mathbb{R}^n$, let q_X denote the orthogonal projection of $\mathbb{R}^n$ onto $X^{\perp}$ with respect to the standard inner product on $\mathbb{R}^n$. Define $\Psi_i(\underline{A}) \in \mathrm{Hom}(A_1,A_1^{\perp})$ as $\Psi_i(\underline{A}) = q_{A_1} \circ (\varphi_i|A_1)$.

CLAIM:

$\varphi_i(\underline{A})$, $1 \le i \le \rho(n)-1$, are linearly independent for each $\underline{A}$ in $G(\mu)$. For, otherwise, there would exist $a_i \in \mathbb{R}$, $1 \le i \le \rho(n)-1$ with $\Sigma\, a_i^2 = 1$ and $\Sigma\, a_i\varphi_i(\underline{A}) = 0$ for some $\underline{A} \in G(\mu)$. For such an $\underline{A}$, we therefore have

$$0 = \Sigma\, a_i\varphi_i(\underline{A}) = \Sigma\, a_i q_{A_1} \circ (\varphi_i|A_1) = q_{A_1} \circ (\Sigma\, a_i\varphi_i)|A_1$$

$$= q_{A_1} \circ \varphi|A_1 \text{ where } \varphi = \Sigma\, a_i\varphi_i.$$ This means that $\varphi(A_1) \subset A_1$, in other words A_1 is an invariant subspace of φ. Since φ is real antisymmetric and orthogonal by Lemma 23.2 and since A_1 is of odd dimension, this is impossible. This establishes our claim that $\varphi_i(\underline{A})$, $1 \le i \le \rho(n)-1$, are linearly independent for each $\underline{A} \in G(\mu)$.

To complete the proof of the theorem, we need only check for continuity of the everywhere linearly independent cross sections φ_i, $1 \le i \le \rho(n)-1$. To see that φ_i are continuous, note that as $\underline{A} \in G(\mu)$ varies continuously in $G(\mu)$ one obtains a <u>continuous</u> splitting, $\mathbb{R}^n = A_1 \oplus A_1^{\perp}$, of $\mathbb{R}^n$. This shows that $\phi_i\ G(\mu) \longrightarrow \mathrm{Hom}(\xi, n\varepsilon)$, $q : G(\mu) \longrightarrow \mathrm{Hom}(n\varepsilon, \eta)$ defined respectively as $\phi_i(\underline{A}) = \varphi_i|A_i$, $1 \le i \le \rho(n)-1$, and $q(\underline{A}) = q_{A_1}$ for all $\underline{A} \in G(\mu)$ are continuous. Now φ_i is the composition $G(\mu) \xrightarrow{(\phi_i, q)} \mathrm{Hom}(\xi, n\varepsilon) \oplus \mathrm{Hom}(n\varepsilon, \eta) \xrightarrow{\text{"composition"}} \mathrm{Hom}(\xi, \eta)$. Hence φ_i is continuous for $1 \le i \le \rho(n)-1$, completing the proof.

COROLLARY 23.5

Span $G(\mu; r) \ge \rho(n)-1$ providing at least one of the n_i's is odd.

Proof:

Since $f : G(\mu;r) \longrightarrow G(\mu)$, the map that forgets the orientations, is a covering projection span $G(\mu;r) \geq$ span $G(\mu)$. The corollary is now immediate from the theorem.

Remark 23.6

Theorem 23.4 was obtained also by Korbaš [24] using a different description of the tangent bundle $\tilde{G}(\mu)$, generalizing the results of Leite-Miatello [27] for oriented Grassmann manifolds.

By Corollary 15.4, all complex and quaternionic flag manifolds have span zero. For this reason only flag$^+$ or real flag manifolds $G(\mu)$ where at least two of the n_i's are odd will be considered in what follows. Hence in the case of Grassmann manifolds $G_{n,k}$ n is assumed to be even and k odd.

THEOREM 23.7.

Let $n_1 = \cdots = n_k = 1$ and let $2 \leq k \leq r \leq s$. Then

$$\text{span } G(\mu) \geq \frac{1}{2}\, k(2n-k-1)$$

Proof:

When $k \geq s-1$, $G(\mu;r) \cong V_{n,k}$ by Example 15.2, and in this case the theorem follows from the parallelizability of the Stiefel manifold $V_{n,k}$ for $k \geq 2$.

Assume that $r \le s-1$. Let $\mu' = (1,\ldots,1,n_{k+1}+\cdots+n_s)$. Now consider the map $q : G(\mu;r) \longrightarrow G(\mu';k)$ defined as

$$q(\tilde{A}_1,\ldots,\tilde{A}_r,A_{r+1},\ldots,A_s) = (\tilde{A}_1,\ldots,\tilde{A}_k,\ A_{k+1}+\cdots+A_s).$$

Clearly, $q^*(\xi_i(\mu';k)) = \begin{cases} \xi_i(\mu;r) \text{ for } 1 \le i \le k \\ \bigoplus_{k+1\le j\le s} \xi_j(\mu;r) \end{cases}$

Now an easy computation shows that $q^*(\tau(\mu';k))$ is a subbundle of $\tau(\mu;r)$. Since by Example 15.2, $G(\mu';k) \cong V_{n,k}$, $\tau(\mu';k)$ is a trivial bundle of rank $\dim V_{n,k} = \frac{k}{2}(2n-k-1)$. Consequently $\tau(\mu;r)$ has a trivial subbundle of rank at least $\frac{1}{2}k(2n-k-1)$. The theorem follows.

THEOREM 23.8. (Miatello-Miatello.)

Let $k \le s$ and let $n_{k+1} = \cdots = n_s = 3$. Then $\tilde{G}(\mu)$ is parallelizable in the following cases:

(i) $n_1 = \cdots = n_k = 1$, $k \ge 2$, or $k = 1 = n_1$ and $n \equiv 0,1,2,5,6 \bmod 8$.

(ii) $k = 0$, $n \equiv 0, 3 \bmod 4$.

Proof:

It was shown in §16 that the manifolds $\tilde{G}(\mu)$ of cases (i) and (ii) are stably parallelizable (see Proposition 16.2). Recall that a stably parallelizable manifold of dimension d is either parallelizable or has span $\rho(d+1)$, where $d = \dim \tilde{G}(\mu)$ for the cases under consideration.

When $n_1 = \cdots = n_k = 1$, $k \ge 2$, span $\tilde{G}(\mu) \ge \frac{1}{2}k(2n-k-1) \ge 2n-3$, by Theorem 23.6. Now $d = \dim \tilde{G}(\mu) = \binom{n}{2} - \sum_{1\le i\le s} \binom{n_i}{2} < \binom{n}{2} - 1$. Therefore

$d+1 \le \frac{1}{2} n^2$. Let ℓ be a positive integer such that $2^{\ell-1} \le d+1 < 2^\ell$. Then $2^{\ell-1} < \left[\frac{n}{2}\right]$. Now using Lemma 23.3,

$$\rho(d+1) \le \rho(2^{\ell-1}) \le 2\ell-2.$$

Since $2^{\ell-1} < \left[\frac{n}{2}\right]$, and since $n \ge 5$, we have $2^\ell-2 < 2n-3$. Hence $\rho(d+1) < 2n-3$. Therefore $\tilde{G}(\mu)$ is parallelizable in this case.

In case $k = 1 = n_1$, $n \equiv 1,2 \bmod 4$, or $k = 0$, $n \equiv 0,3 \bmod 4$, $\dim \tilde{G}(\mu) = d = \left[\frac{n}{2}\right] - (n-k)$ is even. Therefore $\rho(d+1) = 0$. Since span $\tilde{G}(\mu) \ge 1$ by Corollary 15.4, parallelizability of $\tilde{G}(\mu)$ follows.

When $k = 1 = n_1$, and $n \equiv 0 \bmod 8$, $\dim \tilde{G}(\mu) = d = \left[\frac{n}{2}\right] - n + 1$. Thus $d + 1 = \frac{n}{2} \cdot (n-3) + 2 \equiv 2 \bmod 4$. Therefore $\rho(d+1) = 2 < 7 \le \rho(n) - 1 \le \text{span } \tilde{G}(\mu)$ by Theorem 23.4.

§24. Upper Bounds.

In this section we adopt the notations of Chapters 2 and 5. Thus $Y_{n,k}$ denotes the (real) flag manifold $G(1,\dots,1,n-k)$ and $f : Y_{n,k} \longrightarrow G_{n,k}$ the map that sends the flag $(A_1,\dots,A_{k+1}) \in Y_{n,k}$ to $A_{k+1}^\perp \in G_{n,k}$. $\tau_{n,k}$ denotes the tangent bundle of $G_{n,k}$. We write $w_{n,k}$ to denote the total Stiefel-Whitney class $w(G_{n,k})$ of $G_{n,k}$, Thus $w_{n,k}^{(i)} = w_i(G_{n,k})$.

PROPOSITION 24.1

(i) $w_{n,k} = (w(\gamma_{n,k}))^n (w(\gamma_{n,k} \otimes \gamma_{n,k}))^{-1}$.

(ii) $w_{n,k}^{(i)} = 0$ if and only if $f^*(w_{n,k}^{(i)}) = 0$.

Proof:

By Theorem 10.1, $\tau_{n,k} \approx \Upsilon_{n,k} \otimes \beta_{n,k}$. Hence
$\tau_{n,k} \oplus (\Upsilon_{n,k} \otimes \Upsilon_{n,k}) \approx (\Upsilon_{n,k} \otimes \beta_{n,k}) \oplus (\Upsilon_{n,k} \otimes \Upsilon_{n,k}) \approx$
$\Upsilon_{n,k} \otimes (\beta_{n,k} \oplus \Upsilon_{n,k}) \approx n\Upsilon_{n,k}$ since $\Upsilon_{n,k} \oplus \beta_{n,k} \approx n\varepsilon$. (i) follows from this.

(ii) is an immediate consequence of the fact that
$f^* : H^*(G_{n,k};\mathbf{Z}_2) \longrightarrow H^*(Y_{n,k};\mathbf{Z}_2)$ is a monomorphism.

Recall from Corollary 20.4 that $f^*(w_i) = \sigma_i$, the i^{th} elementary symmetric polynomial in the canonical generators $x_1,\ldots,x_k$ of $H^*(Y_{n,k};\mathbf{Z}_2)$. Therefore, from Theorem 8.3 (i) and Example 8.2 (iv)
$f^*(w(\Upsilon_{n,k} \otimes \Upsilon_{n,k})) = \phi_k = (q_{k,2})^2$ in $H^*(Y_{n,k};\mathbf{Z}_2)$, where
$\phi_k = \phi_k(\sigma_1,\ldots,\sigma_k)$ etc. Also since $f^*w(\Upsilon_{n,k}) = \prod_{1\le i\le k} (1+x_i)$, we have the following.

COROLLARY 24.2.

$$f^*(w_{n,k}) = \Big(\prod_{1\le i\le k} (1+x_i)^n\Big)\, \phi_k^{-1}.$$

Proof:

Apply f^* to 24.2 (i).

We now prove the following main theorem of this section:

THEOREM 24.3.

Let $m = 0$ or 2^r, $r \ge 1$.

(i) If $n = 4(m+1)$, then span $G_{n,3} = 3$.

(ii) If $n = 8(m+1)$, then span $G_{n,3} = 7$.

Proof:

We know by Theorem 23.4 that span $G_{n,k} \geq \rho(n) - 1$. Denoting $\dim G_{n,k} - \rho(n) + 1$ by s, we see that to prove the theorem it suffices to show in each case that $f^*(w_s(G_{n,k})) \neq 0$.

Case (i).

When $m = 0$, $G_{4,3} \cong G_{4,1} \cong \mathbb{R}P^3$ and the statement is true in this case. When $m = 2$, a direct computation shows that $f^*(w_{12,3}^{(24)}) = [8,8,8]$, which, by Theorem 20.8, is non-zero (see also Remark 24.5).

Now assume that $m \geq 4$. By Corollary 24.2

$$\begin{aligned} f^*(w_{n,3}) &= (1 + [1] + [1,1] + [1,1,1])^{4m+4} \cdot \phi_3^{-1}. \\ &= (1 + [4] + [4,4] + [4,4,4]) \cdot (1 + [4m] + \\ &\qquad [4m,4m] + [4m,4m,4m]) \cdot \phi_3^{-1}. \qquad (*) \end{aligned}$$

since m is a power of 2. This leaves the computation of ϕ_3^{-1}.

Now by Theorem 20.8, $x_i^{4m+4} = 0$. Hence $x_i^{8m} = 0$. Therefore, by Example 21.2 (iv)

$$\phi_3^{4m} = (1 + [4] + [2,2] + [4,2])^{4m} = 1. \quad \text{Consequently,}$$

$$\phi_3^{-1} = (1 + [4] + [2,2] + [4,2])^{4m-1}.$$

Since $4m - 1 = \sum_{p=1}^{r+2} 2^{p-1}$, we have

$$\begin{aligned} \phi_3^{-1} &= \prod_{1 \leq p \leq r+2} (1 + [2^{p+1}] + [2^p,2^p] + [2^{p+1},2^p]) \\ &= (1 + [4m,4m]) \cdot \alpha \cdot (1 + [4] + [2,2] + [4,2]) \end{aligned}$$

where $\quad \alpha = \prod_{2 \leq p \leq r+2} (1 + [2^{p+1}] + [2^p,2^p] + [2^{p+1},2^p])$.

Note that $(1 + [4m] + [4m,4m] + [4m,4m,4m])(1 + [4m,4m]) = 1 + [4m]$

since $x_i^{8m} = 0$ and thus $[4m]\cdot[4m,4m] = [4m,4m,4m]$, $[4m,4m][4m,4m] = [4m,4m]\cdot[4m.4m,4m] = 0$. Then substituting in (*) we obtain

$$f^*(w_{n,3}) = (1 + [4m])\cdot\alpha\cdot(1 + [4] + [4,4] + [4,4,4]) \cdot (1 + [4] + [2,2] + [4,2]). \qquad (**)$$

Note that $\alpha = z^4$ for some $z \in H^*(Y_{n,3};\mathbf{Z}_2)$ and that $s = 3(n-3) - 3 = 12m$. Before we proceed further, we now look at $\mathrm{Ann}[4m]$, the annihilator of $[4m]$ in $H^*(Y_{n,3};\mathbf{Z}_2)$. Since $x_i^n = 0$, $[a,b,c] \in \mathrm{Ann}[4m]$ if $a,b,c \geq 4$. Thus the symmetric monomials of $H^*(Y_{n,3};\mathbf{Z}_2)$ which are of the form $[4a,4b,4c]$ and possibly not in $\mathrm{Ann}[4m]$ in degree $8m - 4i$, $i = 0,1,2,3$ are listed in the table below.

Table 24.4

Degree	$\mathbf{Z}_2$-basis
8m	[4m,4m]
8m-4	[4m,4m-4]
8m-8	[4m,4m-8], [4m-4, 4m-4]
8m-12	[4m,4m-12], [4m-4, 4m-8]

Note that $m \geq 4$ is used in the above table so that $8m-12 \geq 4m+4$. Using Lemma 21.3 one sees that $[4m,4m-4j] \in \mathrm{Ann}[4m]$ for $j = 1,2,3$. Also $[4]\cdot[4m-4, 4m-4]$, $[4,4]\cdot[4m-4, 4m-8] \in \mathrm{Ann}[4m]$.

Now degree $\alpha = 12m - 12$. Hence

$$\alpha^{(12m)} = \alpha^{(12m-4)} = \alpha^{(12m-8)} = 0.$$

Also $\alpha^{(12m-12)} = \prod_{2\leq p\leq r+1} [2^{p+1},2^p] = [4m,4m-4,4m-8]$ by Example 21.4.

Consider $\alpha^{(12m-16)} = [4m, 2m] \cdots [16,8]([8] + [4,4])$. (Again $m \geq 4$ is crucial here.) $[4,4,4] \cdot [4m,2m] = 0$ gives $[4,4,4] \cdot \alpha^{(12m-16)} = 0$. We are now ready to calculate $f^*(w_{n,3}^{(12m)})$.

From (**),

$$
\begin{aligned}
f^*(w_{n,3}^{(12m)}) = [4m] \cdot \{\alpha^{8m)} &+ \alpha^{(8m-4)}(\cdots\cdots) \\
&+ \alpha^{(8m-8)}([4,4] + [4])[4] + [2,2])) \\
&+ \alpha^{(8m-12)}([4,4,4] + \{4,4]\ (\cdots\cdots)) \\
&+ \alpha^{(8m-16)} \cdot [4,4,4] \cdot (\cdots\cdots)\} \\
&+ \alpha^{(12m-12)} \cdot \{[4,4,4] + [4,4] \cdot (\cdots\cdots)\} \\
&\alpha^{(12m-16)} \cdot [4,4,4] \cdot (\cdots\cdots),
\end{aligned}
$$

where the precise expressions inside $(\cdots\cdots)$ will be found to be irrelevant.

From the above observations, $[4m] \cdot \alpha^{(8m-4)} = 0$, $[4m] \cdot \alpha^{(8m-8)} \cdot [4] = 0$, $\alpha^{(8m-12)} \cdot [4,4,4][4m] = 0$ and $[4m]\alpha^{(8m-12)} \cdot [4,4] = 0$. Also $\alpha^{(12m-12)} \cdot [4,4,4] = 0 = \alpha^{(12m-16)}[4,4,4] = \alpha^{(12m-12)} \cdot [4,4] = 0$.

Hence $\qquad f^*(w_{n,3}^{(12m)}) = [4m] \cdot \alpha^{(8m)} + [4m] \cdot \alpha^{(8m-8)}[4,4]$.

Now

$$
\begin{aligned}
\alpha^{(8m-8)} &= ([4m] + [2m,2m]) \cdots ([8] + [4,4]) \\
&\qquad + \text{terms having } [2^i,2^{i-1}] \text{ as a factor for some } i \geq 3 \\
&= [2m,2m] \cdots [4,4] + \text{terms having} \\
&\qquad [2^i,2^{i-1}] \text{ or } [2^i] \text{ as a factor} \\
&= [4m-4,\ 4m-4] + \text{terms of the form } [4a,\ 4b,\ 4c] \text{ where}
\end{aligned}
$$

$a,b,c \geq 1$ or terms having $[2^i]$ or $[2^i,2^{i-2}]$ as a factor.

Applying Lemma (21.3), we find that terms having $[2^i]$ or $[2^i,2^{i-1}]$ as a factor, in the above expression must be a $\mathbf{Z}_2$-linear combination of symmetric monomials of the form $[4a,4b,4c]$ with $a \neq b$ if $c = 0$. Hence such terms are in Ann$[4m]$. We see therefore that

$$[4m]\cdot[4,4]\cdot\alpha^{(8m-8)} = [4m,4m,4m] \neq 0.$$

(See Theorem 20.8)

By a similar argument

$$[4m] \cdot \alpha^{(8m)} = 0.$$

Hence $f^*(w_{n,3}^{(12m)}) = [4m,4m,4m] \neq 0.$

Proof of (ii) is similar and will therefore be omitted. In this case one obtains

$$f^*(w_{n,3}^{(24m+8)}) = [8m+4,\ 8m+4,\ 8m]$$
$$\neq 0.$$

As in (i), one computes $w_{24,3}^{(56)}$ and $w_{8,3}^{(8)}$ directly, and lets α equal

$$\prod_{4\leq p\leq r+4}(1 + [2^p] + [2^{p-1},2^{p-1}] + [2^p,2^{p-1}]).$$

Remarks 24.5

(i) To simplify calculations of $w_{12,3}^{(24)}$, $w_{24,3}^{(56)}$ and $w_{8,3}^{(8)}$ the reader may follow the method of proof used above.

(ii) Note that $[4m,4m,4m] = [4]\cdot[4m,4m,4m-4]$ in $H^*(Y_{n,3};\mathbf{Z}_2)$, n as in 24.3(i). Since $[4] = f^*(w_1^4)$, $w_{n,3}^{(12m)} \in \mathrm{Im}(\cup w_1)$. Consequently $w_{12m}(\tilde{G}_{n,3}) = p^*(w_{n,3}^{(12)}) = 0$ where $p : \tilde{G}_{n,k} \longrightarrow G_{n,k}$ is the double covering map that forgets the orientation. Similarly $w_{24m+8}(\tilde{G}_{n,3}) = 0$ since $[8m+4,,\ 8m+4,\ 8m] = [4]\cdot[8m+4,\ 8m+4,\ 8m-4]$ in $H^*(Y_{n,3};\mathbf{Z}_2)$, where

n is as in 24.3 (ii). Thus a Stiefel-Whitney class argument does not yield the same upper bound in the corresponding oriented Grassmann manifolds.

(iii) When $n = 2^{4a}(2c+1)$, $\rho(n) - 1$ is even. In this case $s = \dim G_{n,k} - \rho(n) + 1$ is odd when k is odd. Since the total Stiefel-Whitney class of $G_{n,k}$ is a square element when n is even (see Corollary 24.2 and Example 8.2(iv)), it follows that $w_s(G_{n,k}) = 0$. Thus the upper bound obtainable by a Stiefel-Whitney class argument does not equal the lower bound given by Theorem 23.4.

The proof of the following theorem is similar to the that of Theorem 24.3 above and will therefore be omitted. Note that the fact that $f^*\colon H^*(G_{n,k};\mathbf{Z}_2) \longrightarrow H^*(Y_{n,k};\mathbf{Z}_2)$ is a monomorphism is crucial here.

THEOREM 24.6

Let $n = 2(m+1)$, $m = 2^r$, $r \geq 1$. Then $w_s(G_{n,3}) = 0$ where $s = \dim G_{n,3} - \rho(n) + 1 = 3n - 10$. $w(G_{6,3}) = 1 + (w_1(\gamma_{6,3}))^2$.

The above theorem will be helpful in obtaining *lower* bounds for the span of $G_{n,3}$ (where n is as in the theorem above).

§25. Lower Bounds.

We begin this section with an application of the following theorem due to Pontrjagin and Thom. Its proof can be found in [41].

THEOREM 25.1

A compact smooth manifold M is cobordant to zero if and only if all its Stiefel-Whitney numbers are zero.

We state also the following lemma, proof of which can be found in the book of Husemoller [22].

LEMMA 25.2.

Let M be a compact smooth manifold on which $\mathbb{Z}_2$ acts effectively. Then M is cobordant to zero.

PROPOSITION 25.3

(i) $\tilde{G}_{n,k}$ is cobordant to zero for any n,k.

(ii) $G_{n,k}$ is cobordant to zero if n is even and k odd, or if $n = 2k$.

Proof:

Note that the map $\theta : \tilde{G}_{n,k} \longrightarrow \tilde{G}_{n,k}$ that sends $\tilde{A} \in \tilde{G}_{n,k}$ to the vector space A with the opposite orientation defines a smooth effective $\mathbb{Z}_2$-action on $\tilde{G}_{n,k}$. Hence $\tilde{G}_{n,k}$ is cobordant to zero.

If $n = 2k$, $\varphi : G_{n,k} \longrightarrow G_{n,k}$ that sends $A \in G_{n,k}$ to $A^{\perp} \in G_{n,k}$ defines a smooth effective $\mathbb{Z}_2$-action. As before, it follows that $G_{n,k}$ is cobordant to zero.

In case n is even and k odd, $\dim G_{n,k} = d = k(n-k)$ is odd. We know from 24.1(i) and 8.2(iv) that in this case $w(G_{n,k})$ is square of a certain element of $H^*(G_{n,k};\mathbb{Z}_2)$. Consequently all the monomials

$(w_1(G_{n,k}))^{m_1}\cdots(w_d(G_{n,k}))^{m_d}$ are either zero or of even degree. It follows since dim $G_{n,k}$ is odd that all the Stiefel-Whitney numbers are zero. By Theorem 25.2 $G_{n,k}$ is cobordant to zero. This completes the proof of the proposition.

Similarly Lemma 25.2 can be applied to prove the following. It will simplify notations to let $G(\mu;0) = G(\mu)$.

PROPOSITION 25.4

A flag$^+$ manifold $G(\mu;r)$ is cobordant to zero if $r \geq 1$ or if $n_i = n_j$ for some $i \neq j$, where $\mu = (n_1,\ldots,n_s)$.

We are now ready to prove the following.

THEOREM 25.5.

(i) Span $G_{n,k} \geq 2$ only if $\begin{bmatrix} n \\ k \end{bmatrix} \equiv 0 \bmod 4$.

(ii) Span $G_{n,3} \geq 2$ if $n = 2(2^r+1)$, $r \geq 1$.

Proof of (i):

Let Span $G_{n,k} \geq 2$. Then n is even and k odd. If $n = 4m$, $m \geq 1$, then since k is odd $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{n}{k} \cdot \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = 4 \cdot \left[\frac{m}{k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}\right] \equiv 0 \bmod 4$. It remains to consider $n \equiv 2 \bmod 4$. In this case dim $G_{n,k} \equiv 1 \bmod 4$. Therefore by Theorem (3.3), $k(M) = 0$. Since $G_{n,k}$ is a boundary manifold by Proposition 25.3, we see that by Theorem 3.4 $k(M) = x^*(G_{n,k})$. Proceeding as in the proof of Example (3.9),

$$x^*(G_{n,k}) \equiv \left[\tfrac{1}{2}\, P(G_{n,k};1)\right] \bmod 2,$$

where $P(M;t)$ denotes the $\mathbf{Z}_2$-Poincaré polynomial of M. By Theorem 20.1,

$$P(G_{n,k};t) = \frac{(1-t^n)\ \cdots\ (1-t^{n-k+1})}{(1-t)\ \cdots\ (1-t^k)}$$
$$= \frac{(1+t+\cdots+t^{n-1})\ \cdots\ (1+\cdots+t^{n-k})}{1\cdot(1+t)\ \cdots\ (1+t+\cdots+t^{k-1})}$$

Hence

$$P(G_{n,k};1) = \frac{n\cdot(n-1)\ \cdots\ (n-k+1)}{1\cdot 2\cdots\ k} = \begin{bmatrix} n \\ k \end{bmatrix}.$$

Thus $0 = (k)(G_{n,k}) = \chi^*(M) \equiv \left[\frac{1}{2}P(G_{n,k};1)\right]$ mod 2 gives.
$P(G_{n,k};1) = \begin{bmatrix} n \\ k \end{bmatrix} \equiv 0 \bmod 4$.

Proof of (ii):

Let $n = 2\cdot(2^r+1)$, $r \geq 1$. Then $\begin{bmatrix} n \\ 3 \end{bmatrix} \equiv 0 \bmod 4$. Therefore $k(G_{n,3}) = \chi^*(G_{n,3}) \equiv \left[\frac{1}{2}P(G_{n,3};1)\right] = \left[\frac{1}{2}\begin{bmatrix} n \\ 3 \end{bmatrix}\right] \equiv 0 \bmod 2$. By Theorem 24.6, $w_{d-1}(G_{n,3}) = 0$. It follows from Theorem 3.3, that span $G_{n,3} \geq 2$.

Remark 25.6.

Leite and Miatello have shown in [27] that when $n = 2(r+s)+2$, $k = 2r+1$ where r and s have same parity, span $\tilde{G}_{n,k} > 1$ if and only if $\begin{bmatrix} r+s \\ r \end{bmatrix} \equiv 0 \bmod 2$. Theorem 25.5 (i) also follows from this (since span $(G_{n,k}) \geq 2$ implies span $(\tilde{G}_{n,k}) \geq 2$).

We now state, without proof the following theorem due to U. Koschorke. The author is grateful to Professor J. Korbaš of Katedra Matematiky VŠDS, Czechoslovakia, for bringing this reference, and Theorem 25.8, to his notice.

THEOREM 25.7.

Let M^d be a compact orientable manifold such that $d \geq 9$, $d \equiv 1 \bmod 4$. Assume that $Sq^1 : H^1(M;\mathbf{Z}_2) \longrightarrow H^2(M;\mathbf{Z}_2)$ is injective. Then span $M \geq 3$ if and only if $w_{d-1}(M) = 0$, $W_{d-2}(M) = 0$ in $H^{d-2}(M;\mathbf{Z})$, and $k(M) = 0$. Here the "integral" Stiefel-Whitney class $W_{d-2}(M)$ is equal to $\beta\, w_{d-3}(M)$, β being the Bockstein homomorphism associated to the coefficient exact sequence $0 \longrightarrow \mathbf{Z} \xrightarrow{x2} \mathbf{Z} \xrightarrow{r} \mathbf{Z}_2 \longrightarrow 0$.

Proof:

Refer to p.166 of [25].

Let us apply the above theorem to the case $G_{6,3}$. Recall from 24.6 that $w(G_{6,3}) = 1 + w_1^2$, $w_i = w_i(\gamma_{6,3})$. Since $w_1(G_{6,3}) = 0$, $G_{6,3}$ is orientable. The only element of $H^1(G_{6,3};\mathbf{Z}_2)$ is w_1 and $Sq^1(w_1) = w_1^2 \neq 0$. $w_6(G_{6,3}) = 0$ implies that $W_7(G_{6,3}) = 0$. Also since $w_8(G_{6,3}) = 0$ and $k(G_{6,3}) = \begin{bmatrix}6\\3\end{bmatrix} \equiv 0 \bmod 4$, it follows that all the conditions of Theorem 25.7 are satisfied when $M = G_{6,3}$. Thus we have shown that span $G_{6,3} \geq 3$. Since $w_2(g_{6,3}) = w_1^2 \neq 0$, and dim $G_{6,3} = 9$, we obtain span $G_{6,3} \leq 7$. We record these facts as a

THEOREM 25.8 (Korbaš).

$3 \leq \text{span } G_{6,3} \leq 7$.

As another application of the computational techniques developed in Chapter 5, we determine the height of $w_1 \in H^1(G_{n,k};\mathbf{Z}_2)$ for $k = 2$ or $n = 2^s + 1$, $k = 3$. Complete results for height of w_1 have been obtained by Stong [40].

Example 25.9

Let $2^s < n \le 2^{s+1}$. Height of $w_1 = 2^{s+1} - 2$ for $w_1 \in H^1(G_{n,k};\mathbb{Z}_2)$ if (i) $k = 2$ or (ii) $n = 2^s + 1$, $k = 3$.

Proof:

(i) In view of Corollary 19.3, it suffices to show that $\sigma_1 = x_1 + x_2 = f^*(w_1)$ has height $2^{s+1} - 2$ where $f : Y_{n,2} \longrightarrow G_{n,2}$.

$$\text{Now } \sigma_1^{2^{s+1}-2} = (x_1^{2^s} + x_2^{2^s}) \cdots (x_1^2 + x_2^2)$$
$$= x_1^{2^s} \cdot x_2^{2^s-2} + \sum_{(i,j)\neq(2^s,2^{s-2})} x_1^i x_2^j$$

Multiplying both sides by $x_1^{n-2^s-1} x_2^{n-2^s}$ we obtain

$$x_1^{n-2^s-1} x_2^{n-2^s} \sigma_1^{2^{s+1}-2} = x_1^{n-1} x_2^{n-2} + \sum_{(i,j)\neq(2^s,2^s-2)} x_1^{n-2^s-1+i} x_2^{n-2^s+j}$$

Since i and j are even and $i+j = 2^{s+1} - 2$, using Theorem 20.8 one sees readily that each term in the summation Σ must be zero (either x_1 or x_2 will have exponent $\ge n$). $x_1^{n-1} x_2^{n-2}$ being non-zero, it follows that

$$\sigma_1^{2^{s+1}-2} \neq 0.$$

Consider

$$\sigma_1^{2^{s+1}-1} = (x_1^{2^s} + x_2^{2^s}) \cdots (x_1^2 + x_2^2)(x_1 + x_2)$$
$$= t_{2^{s+1}-1} = 0 \text{ by Corollary 20.3.}$$

(ii) As in (i), we show that height of $\sigma_1 = x_1 + x_2 + x_3 = f^*(w_1)$ is $2^{s+1} - 2$ where $f : Y_{2^s+1,3} \longrightarrow G_{2^s+1,3}$.

$$\sigma_1^{2^{s+1}} = (x_1^{2^s} + x_2^{2^s} + x_3^{2^s}) \cdots\cdots (x_1^2 + x_2^2 + x_3^2)$$

$$= x_1^{2^s} \cdot x_2^{2^s-2} + \sum_{(i_1,i_2,i_3)\neq(2^s,2^s-2,0)} x_1^{i_1}x_2^{i_2}x_3^{i_3}$$

$$= x_1^{n-1} \cdot x_2^{n-3} + \sum_{(i_1,i_2,i_3)\neq(n-1,n-3,0)} x_1^{i_1}x_2^{i_2}x_3^{i_3}$$

Multiplying both sides by $x_2 \cdot x_3^{n-3}$ we obtain

$$x_2 \cdot x_3^{n-3} \cdot \sigma_1^{2^{s+1}-2}$$

$$= x_1^{n-1}x_2^{n-2}x_3^{n-3} + \sum_{(i_1,i_2,i_3)\neq(n-1,n-3,0)} x_1^{i_1}x_2^{i_2+1}x_3^{i_3+n-3}.$$

The only (necessarily even) values for i_3 which give possibly non-zero monomials in the sum are 0 and 2. In case $i_3 = 0$, one must have $\{i,i_2+1\} = \{n-1,n-2\}$ to obtain a non-zero monomial inside Σ. This forces $i_1 = n-2 = i_2$. But $i_1 \neq i_2$ for monomials appearing in the sum due to uniqueness of binary expansion. In case $i_3 = 2$, binary expansions again show $|i_1-i_2| \geq 4$, so $\{i_1,i_2+1\} \neq \{n-2,n-3\}$. Therefore, we see by Theorem 20.8 again that the only non-zero term on the right hand side is $x_1^{n-1}x_2^{n-2}x_3^{n-3}$. Hence $\sigma_1^{2^{s+1}-2} \neq 0$.

Now

$$\begin{aligned}\sigma_1^{2^{s+1}-1} &= (x_1^{n-1}+ x_2^{n-1}+ x_3^{n-1})\cdot(x_1 + x_2 + x_3)^{2^s-1}\\ &= x_1^{n-1}(x_2+ x_3)^{2^s-1}+ x_2^{n-1}(x_3+x_1)^{2^s-1}+ x_3(x_1+ x_2)^{2^s-1}\\ &= x_1^{n-1} t_{2^s-1}(x_2,x_3) + x_2^{n-1}t_{2^s-1}(x_3,x_1)\\ &\qquad + x_3^{n-1}t_{2^s-1}(x_1,x_2).\end{aligned}$$

We will now prove that $x_1^{n-1}t_{2^s-1}(x_2,x_3) = 0$. Using symmetry, it follows that $\sigma_1^{2^{s+1}-1} = 0$. Since by Corollary 20.3, $t_{n-2}(x_1,x_2,x_3) = 0$, we have $t_{n-2}(x_2,x_3) = x_1 \cdot P(x_1,x_2,x_2)$ for some polynomial P. Therefore

$$\begin{aligned} x_1^{n-1} t_{2^s-1}(x_2,x_3) &= x_1^{n-1} \cdot x_1 \, P(x_1,x_2,x_3) \\ &= 0. \end{aligned}$$

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