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Table of Contents

Computational Thinking and Experiences of Arithmetic Concepts

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Abstract: In this paper, we discuss how students experienced number while learning to program their robot to move. First, we will provide overview of the context and the research by describing a task used to develop conceptions of "number." Then, we will introduce two discourses from the cognitive sciences that orient the work: Conceptual Metaphor Theory (Lakoff & Nuñez, 2000) and Conceptual Blending Theory (Fauconnier & Turner, 2003). Finally, we will analyze an interaction among two students and their teacher as they tacitly negotiate meanings of number that are appropriate to the task of programming their robot to move forward 100 cm. Our analysis suggests that computational settings may afford rich settings for experiencing and blending distinct instantiations of a range of number concepts in manners that support flexible and transferable understandings.

Keywords: Programming robot, number, Grade 4, elementary mathematics

Learning Discourses

Learning Discourses in Education (Davis & Francis, 2020) analyzes, critiques and sorts/organizes over 850 discourses on learning according to their core foci and implicit metaphors. One of the strategies used to highlight confluences and disjunctions among discourses is a map, the horizontal axis of which distinguishes between correspondence discourses (which assume radical separations of internal from external, self from other, individual from collective, etc.) and coherence discourses (which reframe dichotomies as heuristic conveniences, while embracing evolutionary dynamics within and across nested systems). The map's vertical axis is used to locate discourses according to their relative emphases on the nature of learning ("interpreting learning," the lower region) and advice for teaching ("influencing learning," the upper region). We situate all our work among coherence discourses, and the research reported lands in the lower region of the map (on making sense of learning) with, we believe, strong implications for the upper region (on informing teaching).

A secondary organizational strategy used on the map is the clustering of similarly themed discourses. The work described here fits most strongly with the cluster that has been labeled "Association-Making Strategies," which collects a variety of currently popular research foci – such as spatial reasoning, conceptual metaphor, varied modes of reasoning, and ranges of cognitive bias.

Celia Hoyes and Richard Noss (2020) situated their work within constructionism, like many other researchers with interests in computational thinking (e.g. Abelson, 1981; Buteau, 2019; Papert, 1993). We see constructionism as profoundly complementary to our interests. On the Learning Discourses map, we have located it directly above the "Association-Making Strategies" cluster, meaning that we interpret it to share similar theoretical commitments and metaphoric frames, but with a stronger focus on implications for teaching.

With the Association-Making Strategies cluster, we find two discourses – Conceptual Metaphor Theory and Conceptual Blending Theory – to be especially useful for our current efforts to combine research interests in computational thinking and learning arithmetic. Reflecting a core insight from recent decades of cognitive sciences research, Conceptual Metaphor Theory (Lakoff & Johnson, 1999) is based on the principle that human thought is mainly analogical and associative rather than logical and deductive. Consequently, conceptual metaphor theory looks at metaphor as a core tool of human thinking. The theory examines how metaphor makes it possible to understand one conceptual domain – that is, idea, cluster of related experiences, set of interrelated interpretations – in terms in terms of another conceptual domain. It also examines how metaphoric associations among domains can orient perception, prompt action, and serve as uncritical justifications for further interpretations. Metaphor is core to human thinking and is especially important for bridging bodily experience to abstract constructs.

Conceptual Blending Theory (Fauconnier & Turner, 2003a, 2003b) extends conceptual metaphor theory in the suggestion that complex concepts and creative leaps typically involve blends of multiple metaphors. Such processes are seen as core aspects in human thought and language – pervasive, constant, and largely nonconscious. Once blends have been made, they become resilient and invisible for the knower. A ready example is the concept of "number" – which, for most adults, operates as a seamless blend of count, size, rank, distance, location, and value. Typically, adults find it difficult to see these interpretations of number as different. By contrast, young learners may initially experience them as distinct and incompatible.

Grounding Metaphors of Arithmetic

Lakoff and Núñez (2000) identified four grounding metaphors of number: *object collection*, *object construction*, *measurement*, and *object along a path.* The metaphor of *arithmetic as object collection* is based on a one-to-one correspondence of numbers to physical objects. With this metaphor, numbers are understood as counts, and they differ from one another in terms of *how many*. The metaphor of *arithmetic as object construction* frames number in terms of size that differ from one another in terms of *how large*. The *measuring stick* metaphor maps numbers onto distances, and so numbers differ from one another in terms of *how long*. The metaphor of arithmetic as *object along a path* is based on location, through which numbers are different by virtue of their locations. While the importance of these metaphors for mathematical understanding may not be immediately obvious, Lakoff and Núñez (2000) argued that the development of robust understandings of each and the capacity to move nimbly among them is critical for the emergence of mathematical understanding. Table 1 below translates Lakoff and Núñez's (2000) grounding metaphors of arithmetic into metaphors of number.

Drawing on Conceptual Blending Theory (Fauconnier & Turner, 2003a), Davis (2020) worked with teachers to identify three additional metaphors for number that arise in blends of Lakoff & Núñez's four grounding metaphors. The resulting total of seven core metaphors of number for elementary school mathematics is presented in Table 1. Between *number as count* and *number as size* are two blends: *Number as rank* blends notions of count and location, making it possible to answer questions of "Which?" by making available the ordinal numbers, *Number as amount* blends notions of count and size in order to render large numbers accessible. The third blend, *number as reification*, collects all the other instantiations. This consolidated instantiation is able to operate without a referent. That is, for example, five is simply 5 – not 5 things, the 5th thing, 5 large groups, size 5, an interval of 5, or location 5. Simply – but complexly – 5.

Table 1. *Four metaphors of number associated with Lakoff and Núñez's (2000) four grounding metaphors of arithmetic*

Lakoff and Núñez's Grounding metaphor	Associated metaphor of number	Matter addressed (situation modeled)	An instantiation of '5'
OBJECT COLLECTION	NUMBER AS COUNT	How many?	
OBJECT CONSTRUCTION	NUMBER AS SIZE	How big?	
USING A MEASURING STICK	NUMBER AS LENGTH	How long?	
MOTION ALONG A PATH	NUMBER AS LOCATION	Where?	

Designs for Complementing Mathematics Learning

An integrated, well-blended conception of number is not automatic. For young learners, it is likely that grounding metaphors of number are initially engaged individually, and the learner may at first experience some difficulty reconciling different metaphors. In contrast, an expert is likely to move seamlessly among metaphors, and experienced knowers many even find it difficult to parse well-consolidated conceptions. That is, the experienced knower might not be able to make distinctions among elements that the young learner must struggle to connect. To make such connections, learners must encounter different instantiations of number at the same time.

For the past decade, we have been designing robotics tasks that, we believe, do just that. While our original motivations in moving to robotics settings were to participate in the growing interest in computational thinking and to examine their potential contribution to the development of spatial reasoning, one of the immediate, consistent, and striking realizations in working with robotics was the manner in which even simple tasks supported learners' understandings of number, number systems, and computation, along with the development of computational fluency.

Early on, it became clear to us that the number line is a critical element in supporting learners' consolidations of the concept of number. In Lakoff & Núñez's (2000) terms, we see the number line as a sort of "linking metaphor" – a construct that enables extensive bridging across domains of experience, potentially yielding sophisticated, abstract ideas. Lakoff & Núñez (2000) suggested that linking metaphors require explicit teaching; for the most part, they are inventions, not part of one's natural world, that are designed for specific conceptual purposes. The task described below, along with the analyses of the engagements around the task, were developed with this in mind. We sought to design a task that deployed the number line as a site to bring together multiple interpretations of number in manners that compelled learners to integrate those interpretations by grappling with varied entailments.

Robotics (Coding Motion) Focus: A Hypothesertion

We speculate that coding/computational-thinking environments – **and working with robotic motion in particular** – are superb spaces to develop senses of number and number sense. This is largely because multiple instantiations of number are typically invoked, usually simultaneously – and might be anticipated, given that computational thinking is an offspring of mathematics.

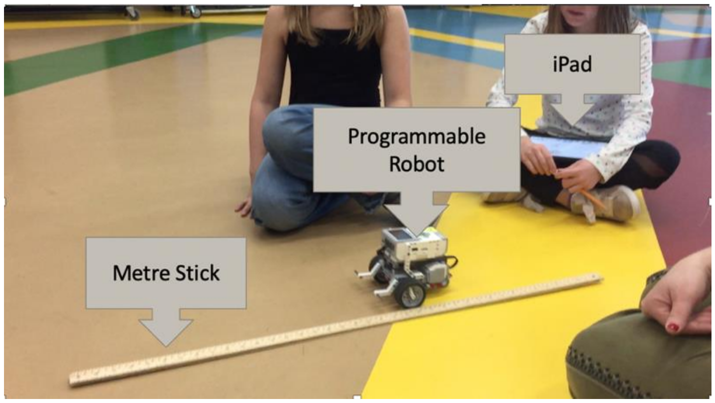
Research Setting

Context

The study took place in a local non-profit, independent K–12 school in Calgary, Alberta that specializes in working with students with learning differences. Weekly robotics classes were offered during regularly scheduled mathematics classes for all students in Grades 4, 5, and 6.

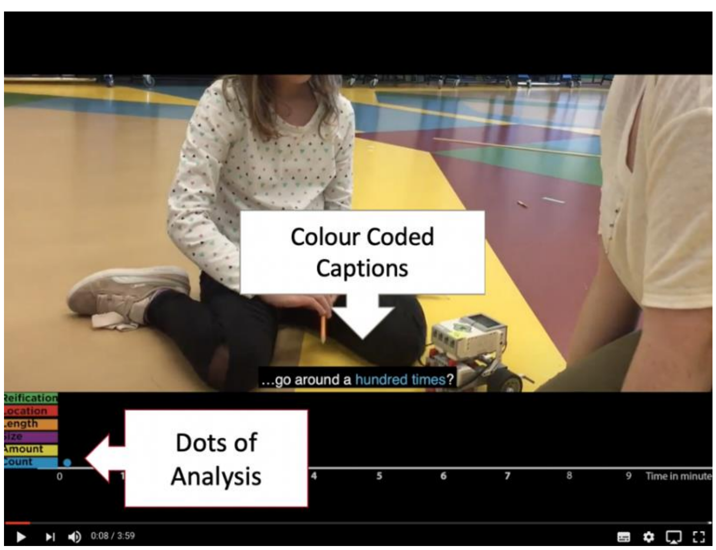
The video data highlighted in this paper were taken in a Grade 4 classroom at the beginning of the year. The students had not yet formally encountered decimal numbers, and they were just starting to program the movement of their robots. The question that oriented the activity captured in the episode was, “How many wheel rotations are needed for the robot to travel 100 cm?” With regard to equipment, each pair of students as a metre stick and used their iPad with EV3 Mindstorms software to program their EV3 Mindstorms robot.

Figure 1. *Two Grade 4 Students and Their Teacher Programming Their Robot to Move 100 cm.*



This 10-minute episode was selected through an exhaustive interpretive selection process (Knoblauch, 2013) of more than 240 hours of video recordings, based on the quality and focus of the action. Transcriptions of the video were imported into NVivo and we coded the videos together based on the instantiation of Table 1. The videos were then edited to include color coded captions and in-time analysis of the video. As you view the videos, watch for these color-coded captions and dots of analyses (see Figure 2).

Figure 2. *Color-Coded Captions and Dots of Analysis in the videos*



Minutes 00:00–03:59 – How far does the robot travel? https://vimeo.com/313928391

The recording begins with the students thinking that 100wheel rotations will be needed for the robot to travel 100 cm. The teacher shows them how far the robot travels with one-wheel rotation by aligning the pointed wheel cap vertically down and moving the robot forward by pushing it along with her hand, and she asks them to guess again. The students try 15-wheel rotations. Their next try is 7, and it is much closer to the desired distance. The students appear to be quickly gaining a sense of what a *count* of 1-wheel rotation means as a *length*. In this 4-minute portion of the episode, five different metaphors of number are invoked: in order of frequency, they are: *count (14), size (4), length (6), location (7), and reification (1).* With regard to the tracking of metaphors at the bottom of the screen, it can be observed that *count*, although dominant at the opening, was quickly abandoned – suggesting that the participants realized on some level that it was relevant, but not especially useful for solving the task.

Minutes 04:00–05:32 – How far does the robot travel with 7? https://vimeo.com/317354442

The second clip starts with the trio observing where the robot stops (*location*) with seven wheel rotations. The girls *count* each rotation as the robot travels alongside the ruler. The teacher asks, "Is seven too much …" (*amount*) "… or too little?" (*size*). Gabby responds with, "too much" (*amount*). They then find that six stops even closer, but still moves past the desired end point (*location*). The girls reason that they "needed to go back 1" (*location*) to five. In this video, the use of the instantiation of location becomes amplified. While references to *length* disappear, our suspicion is that those are conflated with *location*, en route to a more consolidated notion. In the 1.5 minutes of this clip, there are 14 number references across five different metaphors: *location* (7), *count* (1), *amount* (3), size (1), and *reification* (2).

Minutes 05:33–08:40 – Are there numbers between 5 and 6? https://vimeo.com/319520044

Recall that this episode represents the students' first formal encounter with decimal numbers. Notice their body language, starting at about 28 seconds into the clip, when the teacher asks them if there are numbers "between 5 and 6"? The students nod their heads "no," and to our observation, their faces suggest questioning and lack of understanding. At this point, the teacher cycles through multiple instantiations, as though searching for something that resonates. She starts by using money (*amount*) to talk about decimal and common fractions. She then moves to *location*. In the three minutes of the clip, there are 38 utterances of number across four different metaphors: *count* (3), *amount* (9), *location* (11), and *reification* (15). Notable in this clip are, firstly, the pronounced shift to *reification* – which signals to us both a further consolidation of multiple instantiations and an enhanced interpersonal accord on that emerging consolidation – and, secondly, a shift from discrete to continuous notions of number. On these matters, it seems appropriate that the teacher invokes the blend *amount*, which sits across discrete and continuous metaphors.

Minutes 08:41–09:43 – Video 4: Homing in on an appropriate metaphor
https://vimeo.com/325933850

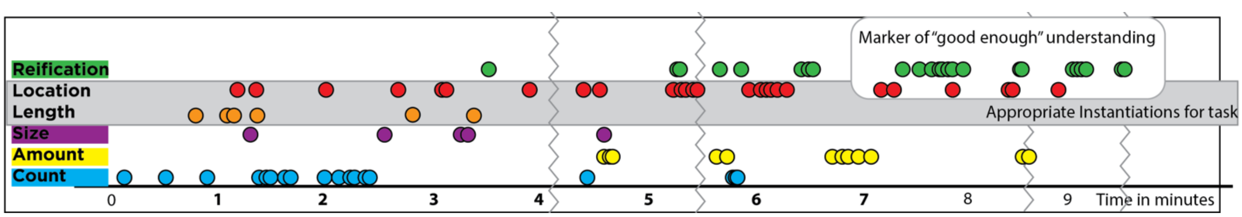
In final clip, the pair of students use 5.7 as their final try. The teacher finishes the discussion with an instantiation of *location*. In the minute-long clip, there are seven utterances of number across two different metaphors: *location* (1), and *reification* (6).

Notably, in the week after this episode, this same pair of students used decimal numbers in another context – fluently, appropriately, and without prompting.

Summary

Across this episode, there were 91 instantiations of number (see Figure 3). Textbooks have very few instantiations. As illustrated in Figure 3, the metaphors that proved to be appropriate for the task are *location* and *length*, and the strong presence of *reification* was an indicator of adequately shared understandings. We suspect this sort of pattern – that is, the shift to a single metaphor, coupled to significant usage of *reification* – is a common and important marker of good-enough common understanding … or, in more fraught encounters, total bafflement.

Figure 3. *Use of Metaphors in Conversation about Robot-Measurement Task*



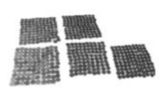
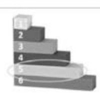
Final Thoughts

We have used this video episode in several situations. Of note, when viewers are not alerted to attend to metaphors of number, few observers notice the somewhat incoherent barrage of interpretations that fly around in the first five minutes. By contrast, when asked to attend to metaphor (even when not provided with the analysis presented in Figure 3), viewers tend to notice that barrage without much difficulty.

With regard to task design, our suspicion is that the number line is a critical feature for blending multiple instantiations. With the number line, one can simultaneously *count* spaces, compare *sizes*, determine *lengths*, and identify *locations*. As we attempt to illustrate with Figure 4, this simultaneity of instantiations offers more than elaborated spaces for interpretation. They also afford access to new types of number and number systems. Rational numbers, for example, are readily discussed in terms of only *counts*, but are readily accessible with blends of *counts* and *sizes* – and even more readily accessed when a well-parsed number line affords blends of *counts*, *sizes*, *lengths*, and *locations*.

It thus goes without saying that coding robot motion is likely to be a powerful learning space, with regard to number concepts. In our experience, it has been particularly powerful for rational numbers, and especially decimal fractions, for students in upper elementary. We routinely encounter learners whose understandings of number are clearly fragmented and whose abilities to manipulate decimal fractions are highly procedural. Yet, consistently, even preliminary encounters with programming robots to move have proven to be powerful sense-making spaces, as learners emerge with demonstrably greater fluency with varied applications of number. To re-emphasize, our strong suspicion is that the built-in number lines of such encounters are core to their effectiveness – and, thus, an important focus for task designers interested in supporting arithmetic learning while promoting computational thinking.

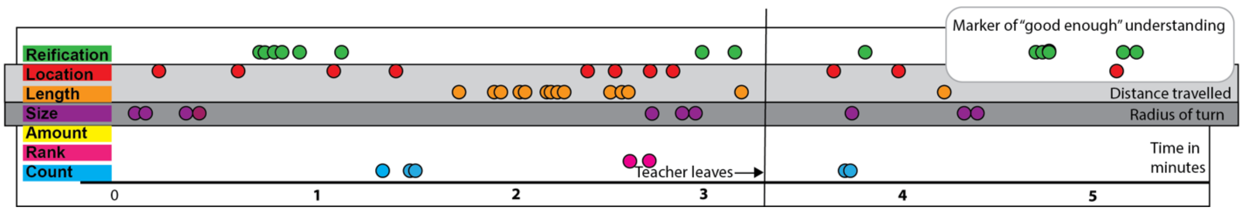
Figure 4. Seven metaphors of number, along with some illustrative entailments (from Davis, 2020)

Metaphor, Metonym, or Metaform of Number		Matter addressed (situation modeled)	Associated Grounding Metaphor(s) of Arithmetic	An instantiation of '5'	How 'less' and 'greater' tend to be expressed		How addition tends to be seen	Some encounters/ contexts/ uses	Numbers made available
					less	greater			
COUNT	quantity	*How many?* (discrete)	OBJECT COLLECTION		fewer	more	combining sets	counting; sorting; clustering	Whole N^os; Natural N^os; cardinals
RANK	quantity	*Which?* (discrete)	OBJECT COLLECTION		ahead	behind	changing rank	sequencing; ranking; grading	ordinals
AMOUNT	quantity	*How much?* (discrete, but experienced as continuous)	OBJECT COLLECTION & CONSTRUCTION		less	more	pooling amounts; amassing	pricing; accounting; apportioning	large numbers; discrete fractions
SIZE	magnitude	*How big?* (continuous object)	OBJECT CONSTRUCTION		smaller	larger	growing; joining pieces	assembling; sharing; ratios	continuous fractions
LENGTH	magnitude	*How long?* (continuous dimension)	USING A MEASURING STICK		shorter	longer	extending; moving farther	scale-based measuring; traveling	Rational N^os; Irrational N^os; Integers
LOCATION	entity	*Where?* (discrete site in continuous space)	MOTION ALONG A PATH		left of (or, lower)	right of (or, higher)	shift in location	locating; scheduling; reading time	Real N^os; Imaginary N^os; Complex N^os
REIFICATION	entity	*What?* (disentangled from physical instantiations)	All of the above	5	<	>	binary operation	symbolic manipulation; computing	Any/all of the above

To repeat an earlier point, none of this should be surprising. Coding is an offspring of mathematics; it always already involves powerful and sophisticated conceptual blends of concepts. When coding is combined with motion, orienting attention towards the number line can provide insights into number. The number line is perhaps the most powerful instantiation for number, and coding motion supports rapid familiarization, robust understanding, and flexible usage.

Effective pedagogy in enabled by nuanced pre-understanding of which instantiations to invoke when. In another recent study, a teacher was briefly informed of the instantiations of number invoked with coding motion (link here). His awareness of knowing when to invoke which instantiation prompted him to be more deliberate in his conversations, contributing to clearer communication about and quicker resolution to a more complex task. While the full analysis is not yet complete, we are able to offer a summary chart (see Figure 5). We leave it here as a provocation, and we invite the reader both to use it to follow the linked video and to contrast it with the trace presented in Figure 3. We believe it serves as further confirmation of our hypotheasertion on the potential contributions of coding motion to learning arithmetic.

Figure 5. *Use of Metaphors in Conversation about Robot-Steering Task*



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Additional Resources

Links to all resources can be found on the series website (http://mkn-rcm.ca/online-seminar-series-on-programming-in-mathematics-education/).

1. Seminar presentation slides
2. Rich mathematical robotic tasks (Grades 4 – 9) for teachers
3. Research paper elaborating further the ideas presented in the seminar