2016

Inference for Dependent Generalized Extreme Values

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Inference for Dependent Generalized Extreme Values

by

Jialin He

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF MASTER OF SCIENCE

GRADUATE PROGRAM IN MATHEMATICS AND STATISTICS

CALGARY, ALBERTA

August, 2016

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Abstract

The Generalized Extreme Value (GEV) distribution is the most commonly used distribution for analyzing extreme values. However, the existing GEV models are based on the assumption that the extreme values are independent, which is sometimes not the case in real data analysis. This thesis aims to overcome this issue by bringing forward a new GEV model that considers the correlation between two successive extreme values. The proposed model can be applied to both independent and dependent extreme values. The point estimation and interval estimation methods for the model parameters are introduced. Simulation studies describe the estimation performance under different combinations of parameters and show that the proposed methods have better performance than the traditional GEV model. Moreover, a study of the Average Run Length (ARL) for the GEV model is conducted through simulation. In the end, two real data analyses are included to illustrate the application of our methodology.
Acknowledgements

First, I would like to express my sincere appreciation to my supervisor Dr. Gemai Chen. He gave me this precious opportunity to be his student. I still remember the warm welcome and the helpful suggestions he gave me when I first came to Canada. During the two years study, he not only taught me the theoretical knowledge about statistics, but also shared his valuable experience and guided me on applying these knowledge to solve real world problems. I really appreciate him for his guidance, patience, enthusiasm and immense knowledge.

Besides my supervisor, I would like to thank Dr. Xuewen Lu for spending time on discussing bootstrap method with me. His rigorous attitude and generous help inspired me for my future work. I also would like to thank Zixiang Guan for sharing his opinions towards extreme values with me and helping clarify some theorems.

I would like to thank my committee members, Dr. Xuewen Lu and Dr. Bingrui(Cindy) Sun for spending precious time on reading my thesis and giving valuable suggestions.

I am also grateful to Yanmei Fei for helping me schedule the oral defense and related matters.

Special thanks are given to the Department of Mathematics and Statistics at the University of Calgary for offering the financial support during my graduate studies.

Moreover, I have to give my thanks to my dear friends, Hongyan Wang, Yuan Dong, Kunlin Hao and Jixian Li, for their love and support, for all the colorful memories they gave me.

Last but not the least, I would like to thank my parents and my boyfriend. Distance makes me realize how much you love me and how much I love you.
# Table of Contents

**Abstract** ........................................................................................................... ii

**Acknowledgements** ........................................................................................ iii

**Table of Contents** ............................................................................................ iv

**List of Tables** .................................................................................................... v

**List of Figures** .................................................................................................. vi

**List of Symbols** ................................................................................................. vii

1 Introduction ........................................................................................................ 1

1.1 Extreme Events and Extreme Values ......................................................... 1

1.2 Introduction to the GEV Model ................................................................. 3

1.3 Introduction to the GEV AR(1) Model ....................................................... 4

1.4 Review of Previous Studies ....................................................................... 4

1.5 Thesis Plan .................................................................................................... 5

2 Methodology ..................................................................................................... 7

2.1 Asymptotic Models of Extreme Values .................................................... 7

2.1.1 Model Formulation ........................................................................... 7

2.1.2 Extremal Types Theorem ................................................................. 8

2.1.3 The Generalized Extreme Value Distribution ................................ 9

2.1.4 Maximum Likelihood Estimation for GEV Model ......................... 9

2.1.5 Confidence Interval for GEV Model ............................................... 10

2.1.6 Return Level and Return Period ...................................................... 11

2.2 GEV Autoregressive(1) Model ................................................................. 12

2.2.1 Model Formulation ........................................................................... 12

2.2.2 Maximum Likelihood Estimation for GEV AR(1) Model ................. 13

2.2.3 Confidence Interval for GEV AR(1) Model ..................................... 14

2.2.4 Return Level and Run Length for Gumbel AR(1) Model ................ 15

3 Simulation Study ............................................................................................... 16

3.1 Parameters Estimation .............................................................................. 23

3.2 Confidence Interval .................................................................................. 30

3.3 Return Level and Average Run Length .................................................... 34

3.4 Comparison ............................................................................................... 35

4 Real Case Study ............................................................................................... 40

4.1 Annual Maximum Water Flow of Bow River ......................................... 40

4.2 Weekly and Daily Maximum Sea Level at Tofina Station ..................... 42

4.2.1 Weekly Maximum Sea Level ............................................................ 42

4.2.2 Daily Maximum Sea Level ............................................................... 44

5 Conclusions and Future Work ..................................................................... 46

**Bibliography** ................................................................................................... 48

**Appendix A** Calculation of the likelihood function ....................................... 50

**Appendix B** Useful R Code ............................................................................. 55
List of Tables

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>The mean bias and the MSE (in the brackets) of $\hat{\rho}$ and $\hat{\mu}$ with sample size $n = 30$.</td>
<td>17</td>
</tr>
<tr>
<td>3.2</td>
<td>The mean bias and the MSE (in the brackets) of $\hat{\sigma}$ and $\hat{\xi}$ with sample size $n = 30$.</td>
<td>18</td>
</tr>
<tr>
<td>3.3</td>
<td>The mean bias and the MSE (in the brackets) of $\hat{\rho}$ and $\hat{\mu}$ with sample size $n = 200$.</td>
<td>19</td>
</tr>
<tr>
<td>3.4</td>
<td>The mean bias and the MSE (in the brackets) of $\hat{\sigma}$ and $\hat{\xi}$ with sample size $n = 200$.</td>
<td>20</td>
</tr>
<tr>
<td>3.5</td>
<td>The mean bias and the MSE (in the brackets) of $\hat{\rho}$ and $\hat{\mu}$ with sample size $n = 1000$.</td>
<td>21</td>
</tr>
<tr>
<td>3.6</td>
<td>The mean bias and the MSE (in the brackets) of $\hat{\sigma}$ and $\hat{\xi}$ with sample size $n = 1000$.</td>
<td>22</td>
</tr>
<tr>
<td>3.7</td>
<td>Summarized iteration times for different combinations of parameters for finding useful confidence intervals for the GEV AR(1) model parameters ($n=30$).</td>
<td>31</td>
</tr>
<tr>
<td>3.8</td>
<td>Summarized iteration times for different combinations of parameters for finding useful confidence intervals for the GEV AR(1) model parameters ($n=200$).</td>
<td>32</td>
</tr>
<tr>
<td>3.9</td>
<td>Summarized iteration times for different combinations of parameters for finding useful confidence intervals for the GEV AR(1) model parameters ($n=1000$).</td>
<td>33</td>
</tr>
<tr>
<td>3.10</td>
<td>Average Run Length of $p = 0.1$ and $p = 0.05$.</td>
<td>36</td>
</tr>
<tr>
<td>3.11</td>
<td>Average Run Length of $p = 0.02$ and $p = 0.01$.</td>
<td>37</td>
</tr>
</tbody>
</table>
# List of Figures and Illustrations

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Two different ways of identifying extreme values</td>
<td>2</td>
</tr>
<tr>
<td>3.1</td>
<td>The mean bias and MSE of 4 parameter estimators with sample size n=30</td>
<td>24</td>
</tr>
<tr>
<td>3.2</td>
<td>The mean bias and MSE of 4 parameter estimators with sample size n=200</td>
<td>25</td>
</tr>
<tr>
<td>3.3</td>
<td>The mean bias and MSE of 4 parameter estimators with sample size n=1000</td>
<td>26</td>
</tr>
<tr>
<td>3.4</td>
<td>The mean bias of $\hat{\rho}$ for different n and $\xi$</td>
<td>28</td>
</tr>
<tr>
<td>3.5</td>
<td>The mean bias $\hat{\mu}$ for different n and $\xi$</td>
<td>28</td>
</tr>
<tr>
<td>3.6</td>
<td>The mean bias of $\hat{\sigma}$ for different n and $\xi$</td>
<td>29</td>
</tr>
<tr>
<td>3.7</td>
<td>The mean bias of $\hat{\xi}$ for different n and $\xi$</td>
<td>29</td>
</tr>
<tr>
<td>3.8</td>
<td>Average run length of the GEV AR(1) model</td>
<td>35</td>
</tr>
<tr>
<td>3.9</td>
<td>Comparison between GEV (solid curves) and GEV AR(1) (dash curves) regarding MSE and the mean bias of $\hat{\xi}$</td>
<td>39</td>
</tr>
<tr>
<td>4.1</td>
<td>The plot of river flow of the Bow River</td>
<td>40</td>
</tr>
<tr>
<td>4.2</td>
<td>Plot of weekly maximum sea level</td>
<td>43</td>
</tr>
<tr>
<td>4.3</td>
<td>Consecutive plot of weekly maximum sea level</td>
<td>44</td>
</tr>
<tr>
<td>4.4</td>
<td>Consecutive plot of daily maximum sea level</td>
<td>45</td>
</tr>
</tbody>
</table>
# List of Symbols, Abbreviations and Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>ARL</td>
<td>Average run length</td>
</tr>
<tr>
<td>CI</td>
<td>Confidence interval</td>
</tr>
<tr>
<td>EVT</td>
<td>Extreme value theory</td>
</tr>
<tr>
<td>GEV</td>
<td>Generalized extreme value</td>
</tr>
<tr>
<td>GEV AR(1)</td>
<td>Generalized extreme value first order autoregressive</td>
</tr>
<tr>
<td>i.i.d.</td>
<td>Independent and identically distributed</td>
</tr>
<tr>
<td>MLE</td>
<td>Maximum likelihood estimator</td>
</tr>
<tr>
<td>MSE</td>
<td>Mean square error</td>
</tr>
</tbody>
</table>
In some cases, rare phenomena are at the centre of researchers’ interest. These phenomena, which are named extreme events as well, have a relatively small probability of occurring while will have a huge impact once happen. Therefore, it is crucial to study and try to predict the extreme events.

The Generalized Extreme Value (GEV) model is a well-founded statistical model that analyzes extreme events. It is built on Extreme Value Theory (EVT) which was developed by R.A. Fisher and L.H.C. Tippett in 1928. The GEV model has become a well established model now and has been applied in many fields such as engineering, meteorology, insurance and finance.

One of the problems in using the GEV model lies in the assumption that extreme values should be independent and identically distributed (i.i.d.). However, in real life, this assumption is not always satisfied. Therefore, this dissertation offers a modified GEV model that addresses this issue by analysing dependent extreme values.

1.1 Extreme Events and Extreme Values

Extreme events are events with a relatively small probability of occurring. They have relatively quite large or small value for a random variable $X$. For example, an extreme event would be a person who is two meters tall, or the temperature of Calgary reaching $42^\circ$C. If we draw a probability distribution plot or probability mass function for these events, they are represented as lying in the tail of the distribution.

Extreme events become a research focus because they usually have a significant influence and can lead to serious consequences. In nature, extreme weather such as tornadoes, strong hurricanes and extreme temperatures can cause billions of economic damage and lead to the death of a large number of people. In financial markets, the extreme low price of stocks can cause crashes for the
whole financial market around the world and the collapse of many companies. In insurance, it is possible that individual claims are so high as to exceed the capacity of an insurance company. Therefore, it is crucial to build a statistical model for extreme events to study the trend and provide effective protection.

There are two main ways of defining extremes in real data. One is the block-maxima method, the other is the peak over threshold (POT) method. The block-maxima method selects the maximum of the process over $n$ time units of observation, for example 4 months or 10 years. If $n$ is the number of observations in a year, then the extreme values correspond to the annual maximum. The POT method considers the observations exceeding a given (high or low) threshold as extreme values and the choice of threshold is crucial when using POT method. The difference for these two methods is shown in Figure 1.1. In the left panel, the observations $X_1, X_5, X_9$ and $X_{11}$ represent the block maxima for four periods of three observations each while $X_1, X_2, X_7, X_9, X_{11}$ are the POT extreme values. In this thesis, we use the first method, the block-maxima method, to identify extreme values.
1.2 Introduction to the GEV Model

The Extreme Value Theory (EVT) is a foundational theory for extreme values regarding asymptotic distribution of them. It plays a central role for extreme values as the central limit theory for averages. The EVT states that, for properly centered and scaled maxima \( M_n = \max\{X_1, X_2, \ldots, X_n\} \), where \( X_n \) is a sequence of iid random variables, the nondegenerated limit distribution of \( M_n \) can only be the Generalized Extreme Value (GEV) distribution. The GEV distribution is the united form of three distribution families, Gumbel, Frechet and Weibull. In other words, if the rescaled sample maxima converges in distribution to a variable, then the distribution of this variable can only be one of these three extreme value distributions. From this point, the distribution of extreme values can be represented and thus the likelihood function of sample data and the following statistical inference can be made. The detailed explanation will be covered in Section 2.1.

The assumption of the GEV model is that the original data \( X_n \) are i.i.d. However, the real data always have dependence in some way since they are time series data. Leadbetter (1983) studied the extreme values for stationary sequence and concluded that if \( X_n \) are stationary sequence, then the block maxima \( M_n \) still follow the EVT and can be modeled by GEV model. Based on this conclusion, the GEV model can be used to analyze extreme values as long as the extreme values are iid.

In application, the correlation between extreme values is usually avoided by extracting maxima with proper block interval and discarding the remaining regular data. For example, the daily temperature is correlated, if today has a high temperature, then it is very likely that tomorrow will be hot. If we choose 365 (number of days in one year) as the block interval, then the block maxima, annual highest temperature is reasonable to be regarded as iid. However, in some cases, the correlation still exists between block maxima for some data, especially high-frequency data. For example, if we have a minute-by-minute sampled data and our interest parameter is the daily maxima, then this maxima are correlated. The dependence can be removed if we only extract annual maxima while about 5 million data need to be discard and only one remain. It is not advisable to
waste precious data. Meanwhile, the independent maxima may not be the one we are interested in. Therefore, we consider to improve the classic GEV model to adapt dependent extreme values.

1.3 Introduction to the GEV AR(1) Model

As Toulemonde et al. (2010) mentioned, linear autoregressive (AR) processes offer a simple and elegant framework for capturing temporal dependencies. The idea of our new model is to combine AR processes with GEV model, expand the GEV model to analyze the dependent extreme values. In this thesis, we choose the simple AR process, the first order autoregressive process (AR(1)), to form the new model. We name the new model as Generalized Extreme Value First Order Autoregressive Model and use GEV AR(1) to denote it. The GEV AR(1) model can analyze extreme values that relate to their previous ones. Note that we don’t assume extreme values follow a AR(1) process since it may not be the case in real data analysis, the only thing we know is they have short term dependency.

The AR(1) process can be represented as,

\[ x_t = \rho x_{t-1} + \epsilon_t, \]  

where \(|\rho| < 1\), \(\epsilon_t\) is a white noise process with zero mean and constant variance \(\sigma_\epsilon^2\). Then the dependent extreme values \(y_t, t = 1, 2, \cdots, n\), are functions of \(x_t\),

\[ y_t = h(x_t). \]  

The function \(h(\cdot)\) is the link between the AR(1) process and the GEV AR(1) model. The method to find this function will be introduced in Section 2.2.1.

1.4 Review of Previous Studies

Fisher and Tippett (1928) pointed out the Generalized Extreme Value Theory saying that the limit distribution of block-maxima extreme values (largest or smallest variable) of i.i.d. variables can
only be one of the three families, Gumbel, Frechet and Weibull. Then Gnedenko (1943) and Smith (1992) provided mathematical proof of the theorem and studied the characterization of these three families. In the following decades, the significant benefit for using the GEV model has been recognized and the model has been applied in many disciplines. In financial markets, the model has been used to study the risk measures of financial portfolios (Bensalah et al. (2000), Diebold et al. (2000), Gencay and Selcuk (2004), Embrechts et al. (2013)). Meteorologists and hydrologists applied the GEV model to analyze wind speeds, sea-surges, climate change and so on (Katz et al. (2002), Martins et al. (2000)).

Researchers have worked on dealing with dependent data when applying the GEV model. Tawn (1988) proposed a new method to filter dependent data so that the filtered extreme values satisfy the i.i.d. assumption. Rust et al. (2011) brought forward a modified bootstrap inference method to predict the confidence interval for dependent data. Toulemonde et al. (2010) combined linear autoregressive (AR) processes with Gumbel distribution to analyze Gumbel distributed dependent maxima. Zhang (2015) improved this approach by using the combination of a least squares method estimation and a new method of moments estimation.

1.5 Thesis Plan

The purpose of this thesis is to bring out a new estimation method for analyzing temporally dependent extreme values and study the performance of the estimates of the GEV AR(1) model. The thesis is organized as follows. In Chapter 2, we will introduce the foundation of EVT and the GEV model. The point estimation method and interval estimation method will be included. Then, we will expand the GEV model to adopt dependent extreme values, the idea of our GEV AR(1) model will be explained and the corresponding parameter inference methods are going to be introduced. In Chapter 3, simulations will be carried out to study the performance of the GEV AR(1) model. The performance will be evaluated in terms of two measures. One is the bias and mean squared error (MSE) for estimators, the other is the coverage probability for confidence interval. In Chapter
4, two real case studies will be conducted to show how to use our model to do data analysis on dependent extreme values. In Chapter 5, some conclusions and future work will be discussed. The detailed procedure for calculating maximum likelihood estimations is given in the appendix.
Chapter 2

Methodology

In this chapter, we are going to introduce the GEV model and the new proposed GEV AR(1) model. In the first section, the foundational knowledge about GEV model and its inference methods will be introduced. Then, in the second section, we will talk about GEV AR(1) model and its inference methods.

2.1 Asymptotic Models of Extreme Values

2.1.1 Model Formulation

Based on the block-maxima method, we develop the statistical model that concerns extreme values

$$M_n = \max\{X_1, X_2, \ldots, X_n\},$$

where $X_1, \ldots, X_n$ is a sequence of independent random variables having a common distribution function $F$. By statistical theory, the cumulative probability function of $M_n$ can be calculated by

$$P(M_n \leq x) = P(X_1 \leq x, X_2 \leq x, \ldots, X_n \leq x)$$

$$= P(X_1 \leq x)P(X_2 \leq x) \ldots P(X_n \leq x)$$

$$= [F(x)]^n. \quad (2.1)$$

The distribution of $M_n$ can be derived as long as $F$ is known. Unfortunately, $F$ is usually unknown in real life. It is possible that we estimate the $F$ by sample data and then get the estimated distribution for $M_n$ by Equation 2.1. However, as [Coles et al. (2001)] mentioned, even small discrepancies in the estimation of $F$ can lead to substantial discrepancies for $F^n$. This disadvantage can be explained by the following simple example. 0.95 is quite close to 1 while $0.95^{10} = 0.5987369$ is far away from 1. Therefore, we should regard $M_n$ as a random variable and look for its distribution.
function $F^n$ directly. Notice that for any $x$ smaller than the upper end-point of $F$, the distribution of $M_n$ degenerates to a point mass as $n$ goes to infinity, $F^n(x) \to 0$ as $n \to \infty$. To avoid this degeneration problem, we can apply a linear renormalization of the variable $M_n$

\[ M^*_n = \frac{M_n - \mu_n}{\sigma_n}, \]

where $\{\mu_n\}$ and $\{\sigma_n > 0\}$ are sequences of constants. This transformation could rescale $M_n$ and stabilize the location and scale of $M^*_n$ as $n$ increases, avoiding the degeneration problem that arise with $M_n$. We therefore analyze the limit distributions for $M^*_n$ rather than $M_n$.

### 2.1.2 Extremal Types Theorem

The limit law for the normalized maxima $M^*_n$ is given by the following theorem.

**Theorem 1 (Extremal Types Theorem)** If there exist sequences of constants $\{\sigma_n > 0\}$ and $\{\mu_n\}$ such that

\[ \Pr\{(M_n - \mu_n)/\sigma_n \leq x\} \to G(x) \text{ as } n \to \infty, \]  

(2.2)

where $G$ is a non-degenerate distribution function, then $G$ belongs to one of the following families:

\[ I : G(x) = \exp\left\{-\exp\left[-\left(\frac{x - \mu}{\sigma}\right)\right]\right\}, -\infty < x < \infty; \]

\[ II : G(x) = \exp\left\{-\left(\frac{x - \mu}{\sigma}\right)^{-\xi}\right\}, x > \mu; \]

\[ III : G(x) = \exp\left\{-\left[-\left(\frac{x - \mu}{\sigma}\right)^{\xi}\right]\right\}, x < \mu \]

(2.3)

for parameters $\mu \in \mathbb{R}, \sigma > 0$, and $\xi > 0$.

This theorem implies that, no matter what distribution $X_i$ follows, the rescaled sample maxima $M^*_n$, which is $(M_n - \mu_n)/\sigma_n$, has a limiting distribution that must be one of these three types of families. The three families, labelled I, II and III, are known as the Gumbel, Fréchet and Weibull families respectively. They all have a location parameter $\mu$ and a scale parameter $\sigma$. Fréchet and Weibull have one more, the shape parameter $\xi$. 

8
2.1.3 The Generalized Extreme Value Distribution

These three types of families have distinct forms of behavior, corresponding to the different forms of tail behavior for the distribution function \( F \) of \( X_i \). In applications, the three families give different representations of extreme value behaviors. If we choose one of the three families to build the model, it could be a wrong model and the subsequent inferences could be wrong. The unification of these three families of extreme value distribution into a combined family can solve this problem.

**Theorem 2** If there exist sequences of constants \( \{\mu_n > 0\} \) and \( \{\sigma_n\} \) such that

\[
\Pr \left\{ \left( M_n - b_n \right) / a_n \to G(x) \right\}, \quad \text{as} \quad n \to \infty
\]

for a non-degenerate distribution function \( G \), then \( G \) is a member of the GEV family

\[
G(x) = \exp \left\{ -\left[ 1 + \xi \left( \frac{x - \mu}{\sigma} \right) \right]^{-1/\xi} \right\}
\]

defined on \( \{x : 1 + \xi (x - \mu)/\sigma > 0\} \), where \(-\infty < \mu < \infty, \sigma > 0 \) and \(-\infty < \xi < \infty\).

This is the generalized extreme value (GEV) family of distributions. When \( \xi > 0 \), the GEV model corresponds to the Fréchet (Type II) family; when \( \xi < 0 \), the GEV model corresponds to the Weibull (Type III) family; when \( \xi = 0 \), the GEV distribution leads to the Gumbel family (Type I) and is interpreted as the limit as \( \xi \to 0 \), which is,

\[
G(x) = \exp \left\{ -\exp \left\{ -\left( \frac{x - \mu}{\sigma} \right) \right\} \right\}, \quad -\infty < x < \infty.
\]

Then, the probability density function (pdf) of the GEV family is

\[
g(x) = \begin{cases} 
\exp\{ -\left[ 1 + \xi \left( \frac{x - \mu}{\sigma} \right) \right]^{-1/\xi} \} \cdot \left[ 1 + \xi \left( \frac{x - \mu}{\sigma} \right) \right]^{-1/\xi - 1} \cdot \frac{1}{\sigma}, & \xi \neq 0, \\
\exp\{ -\exp\{ -\left( \frac{x - \mu}{\sigma} \right) \} \} \cdot \exp\{ -\left( \frac{x - \mu}{\sigma} \right) \} \cdot \frac{1}{\sigma}, & \xi = 0,
\end{cases}
\]

2.1.4 Maximum Likelihood Estimation for GEV Model

The Maximum Likelihood Estimator (MLE) is the estimated parameter that maximizes the likelihood function. It can be calculated by maximizing the log-likelihood function as well since the
logarithm function is monotonic and achieves maximum at the same value. For the classical GEV model, which assumes the block maxima \( X_1, X_2, \ldots, X_m \) are independent variables, the likelihood function can be calculated by multiplying individual probability density functions,

\[
L(\mu, \sigma, \xi; x_1, \ldots, x_n) = \prod_{i=1}^{m} g(x_i).
\]

(2.8)

Thus, the log-likelihood function for GEV model is

\[
l(\mu, \sigma, \xi) = \begin{cases} 
-m \log \sigma - (1 + 1/\xi) \sum_{i=1}^{m} \log[1 + \xi \left( \frac{x_i - \mu}{\sigma} \right)] - \sum_{i=1}^{m} \left[ 1 + \xi \left( \frac{x_i - \mu}{\sigma} \right) \right]^{-1/\xi} & \xi \neq 0, \\
-m \log \sigma - \sum_{i=1}^{m} \left( \frac{x_i - \mu}{\sigma} \right) - \sum_{i=1}^{m} \exp \left( -\frac{x_i - \mu}{\sigma} \right) & \xi = 0,
\end{cases}
\]

(2.9)

provided that

\[
1 + \xi \left( \frac{x_i - \mu}{\sigma} \right) > 0, \quad \text{for} \quad i = 1, \ldots, m.
\]

(2.10)

Thus, under the GEV model, the MLEs for the four parameters are calculated by maximizing the log-likelihood function (2.9). MLE has consistency properties under the regularity conditions. Such conditions are not always satisfied by the GEV model. Smith (1985) pointed out that the information matrix is finite over the range \(-\infty < \xi < 0.5\) and thus MLEs are consistent when \(\xi < 0.5\). The case \(\xi > 0.5\) is not likely to happen in applications so that this limitation is usually not an obstacle in practice. Therefore, we study the performance of the model for \(\xi < 0.5\) in the following part of this paper.

2.1.5 Confidence Interval for GEV Model

**Theorem 3** Let \( x_1, \ldots, x_n \) be independent realizations from a distribution within a parametric family \( \mathcal{F} \), and let \( \ell(\cdot) \) and \( \hat{\theta}_0 \) denote respectively the log-likelihood function and the maximum likelihood estimator of the \( d \)-dimensional model parameter \( \theta_0 \). Then, under suitable regularity conditions, for large \( n \),

\[
\hat{\theta}_0 \sim \text{MVN}_d(\theta_0, I_E(\theta_0)^{-1}),
\]

(2.11)
where $I_E(\theta)$ is the information matrix,

$$I_E(\theta) = \begin{bmatrix}
  a_{1,1}(\theta) & a_{1,2}(\theta) & \ldots & a_{1,d}(\theta) \\
  a_{2,1}(\theta) & a_{2,2}(\theta) & \ldots & a_{2,d}(\theta) \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{d,1}(\theta) & a_{d,2}(\theta) & \ldots & a_{d,d}(\theta)
\end{bmatrix}$$

with

$$a_{i,j}(\theta) = E\left\{-\frac{\partial^2}{\partial \theta_i \partial \theta_j} \ell(\theta)\right\}.$$

This theorem can be used to obtain approximate confidence intervals for individual components of $\theta_0 = (\theta_1, \cdots, \theta_d)$. For large $n$, $\hat{\theta}_i \sim N(\theta_i, I_E(\theta_0)^{-1}_{i,i})$. Hence, an approximate $(1-\alpha)$ confidence interval for $\theta_i$ would be

$$\hat{\theta}_i \pm z_{\alpha/2} \sqrt{I_E(\theta_0)^{-1}_{i,i}/n}. \quad (2.12)$$

For the GEV model, the parameters are $\theta_0 = (\mu, \sigma, \xi)$. Then the approximate $(1-\alpha)$ confidence interval for each parameter could be calculated by Equation (2.12).

### 2.1.6 Return Level and Return Period

In application, researchers such as meteorologists and finance experts need to deal with real problems and make decisions. They are more interested in the return levels rather than the parameters themselves. Return levels are upper quantiles of the estimated GEV distribution. Under i.i.d. assumption, they can be calculated by inverting the GEV distribution function (Equation (2.5)).

$$x_p = G^{-1}(1-p)$$

$$= \begin{cases}
  \mu + \frac{\sigma[1 - (-\log(1-p))^{\xi}]}{\xi}, & \text{for } \xi \neq 0, \\
  \mu - \sigma \log(-\log(1-p)), & \text{for } \xi = 0,
\end{cases} \quad (2.13)$$

The return level $x_p$ is thought to be the value that is expected to be exceeded, on average, once every $1/p$ years. In other words, in any given fixed time period, the maximum value is expected
to be greater than \( x_p \) with probability \( p \). A value of \( x_{0.01} \) of 7 means that the maximum observed value during a period of time will exceed 7 once in 100 years on average. The estimated return levels can be calculated by substituting the parameters \( \mu, \sigma, \xi \) by their MLEs \( \hat{\mu}, \hat{\sigma}, \hat{\xi} \) into Equation (2.13).

2.2 GEV Autoregressive(1) Model

2.2.1 Model Formulation

We want to build a model for extreme values that are dependent on the previous value. First, we define a series of block maximum values \( y_t, t = 1, \ldots, n \), then each \( y_t \) follows Generalized Extreme Distribution and has cumulative distribution function (CDF) as

\[
G(y_t) = \begin{cases} 
\exp\{-\exp[-\frac{y_t - \mu}{\sigma}]\}, & \text{if } \xi = 0 \\
\exp\{-[1 + \xi (\frac{y_t - \mu}{\sigma})^{-\frac{1}{\xi}}]\}, & \text{if } \xi \neq 0
\end{cases}
\]  

(2.14)

Since \( y_t \) are dependent on \( y_{t-1} \), they can be represented as a function of a AR(1) process. For simplicity, we define the AR(1) time series \( \{x_t\} \) by the following equations,

\[
\begin{cases} 
x_1 = e_1 \\
x_t = \rho x_{t-1} + e_t, & t = 2, \ldots, n,
\end{cases}
\]  

(2.15)

where \( e_1 \sim N(0,1), e_i \sim N(0,1-\rho^2), -1 < \rho < 1, i = 2, 3, \ldots, n \), \( e_1 \) and \( \{e_i\} \) are all independent. Then \( \{x_t\}_{t=1}^n \) is an AR(1) process and \( x_t \sim N(0,1), t = 1, \ldots, n \).

The link between extreme values \( \{y_t\}_{t=1}^n \) and AR(1) process \( \{x_t\}_{t=1}^n \) exists through their CDFs.

**Theorem 4 (Probability Integral Transform)** Suppose that a random variable \( X \) has a continuous distribution for which the cumulative distribution function (CDF) is \( F_X \). Then the random variable \( Y \) defined as \( Y = F_X(X) \) has a uniform distribution \( U(0,1) \).

The Probability Integral Transform theorem reveals that random variables from any given continuous distribution can be converted to random variables having a uniform distribution \( U(0,1) \).
Thus, both $y_t$ and $x_t$ can be converted to $U(0,1)$. This means $G(y_t)$ and $F(x_t)$ have the same distribution, where $G(\cdot)$ is the CDF for $y_t$ and $F(\cdot)$ is the CDF for $x_t$. In this way, these two variables are connected with each other. Our model is based on this connection.

Denote $u_t = G(y_t)$ and regard it as a new variable. $u_t$ is a function of $y_t$ and this function is a one-to-one non-decreasing function. For each $u_t$, there is a corresponding $y_t$. In this way, we can get the inverse function

$$y_t = G^{-1}(u_t) = \begin{cases} 
\mu + \sigma \left[-\ln(-\ln u_t)\right], & \text{if } \xi = 0 \\
\mu + \frac{\sigma}{\xi} \left[-\ln u_t\right]^{-\frac{\xi}{\xi - 1}}, & \text{if } \xi \neq 0
\end{cases} \quad (2.16)$$

Since $x_t \sim N(0,1)$, $\Phi(x_t)$ should follow a uniform distribution $U(0,1)$ where $\Phi(x_t)$ is the CDF for standard normal distribution, $u_t = \Phi(x_t) \sim U(0,1)$. Then $\Phi(x_t)$, $G(y_t)$ and $u_t$ have the same distribution function and the relationship between $x_t$ and $y_t$ can be represented as a function of each other.

$$y_t = G^{-1}(\Phi(x_t)) \quad (2.17)$$

$$x_t = \Phi^{-1}(G(y_t)) \quad (2.18)$$

Therefore, the GEV AR(1) distribution has cumulative distribution function

$$G(y_t) = \begin{cases} 
\exp\{-\exp\left[-\frac{y_t - \mu}{\sigma}\right]\}, & \text{if } \xi = 0 \\
\exp\left[-1 + \xi \left(\frac{y_t - \mu}{\sigma}\right)^{-\frac{1}{\xi}}\right], & \text{if } \xi \neq 0
\end{cases}$$

where $y_t = G^{-1}(\Phi(x_t))$ and $x_t$ is an AR(1) process defined by Equation (2.15).

### 2.2.2 Maximum Likelihood Estimation for GEV AR(1) Model

In our GEV AR(1) model, extreme values are not independent, the MLEs for parameters cannot be calculated by the method mentioned in Section 2.1.4 since the likelihood function is different. Notice that the function $y_t = G^{-1}(\Phi(x_t))$ has the following three properties,

1. $y_t = G^{-1}(\Phi(x_t))$ is an increasing function for all $x$ in the support of $f_X(x)$;
2. the function is a one-to-one function;
3. there exists an inverse function $\Phi^{-1}(G(\cdot))$.

Then, by applying the Jacobian transformation, the joint density function for $y_t$ is given by

$$
\begin{align*}
\tilde{f}_Y(y_1, y_2, \ldots, y_n) &= f_X(x_1, x_2, \ldots, x_n) \cdot |J_1| \\
&= f_E(e_1, e_2, \ldots, e_n) \cdot |J_2| \cdot |J_1|,
\end{align*}
$$

(2.19)

where $|J_1|, |J_2|$ are the absolute values of the Jacobians. After some calculation (see Appendix), we can get the likelihood function of $\{y_t\}_{t=1}^n$ as below,

$$
\begin{align*}
f_Y(y_1, y_2, \ldots, y_n) &= \begin{cases} 
A \cdot \prod_{t=1}^{n} [-\ln(G(y_t))], & \xi = 0, \\
A \cdot \prod_{t=1}^{n} [-\ln(G(y_t))]^{\xi+1}, & \xi \neq 0,
\end{cases}
\end{align*}
$$

(2.20)

where

$$
A = \left( \frac{1}{\sqrt{2\pi}} \right)^n \left( \frac{1}{\sqrt{1 - \rho^2}} \right)^{n-1} \cdot \frac{\prod_{t=1}^{n} G(y_t)}{\sigma^n \cdot \prod_{t=1}^{n} \Phi(\Phi^{-1}(G(y_t)))} 
$$

$$
\cdot \exp \left\{ -\frac{1}{2} (\Phi^{-1}(G(y_1)))^2 - \frac{1}{2 \sqrt{1 - \rho^2}} \sum_{t=2}^{n} (\Phi^{-1}(G(y_t)) - \rho \Phi^{-1}(G(y_{t-1})))^2 \right\}
$$

The maximum likelihood estimators are the values that maximize $f_Y(y_1, y_2, \ldots, y_n)$. There is no analytical solution for this estimation problem, but estimators can be obtained by using numerical optimization algorithms. There are some packages that can help us get the maximization done in a straightforward way, the one we use here is the "optim" in R.

2.2.3 Confidence Interval for GEV AR(1) Model

Under suitable assumptions, the consistency of MLE for dependent samples still holds. However, in the GEV AR(1) model, the information matrix doesn’t have a closed form since the expectation cannot be solved analytically. Instead, we use parametric bootstrap method to estimate confidence intervals (CI) for the parameters. The idea of parametric bootstrap method is treating the estimated parametric distribution as if it was the real distribution and estimate the distribution by resampling data from the observed data. The detailed procedures are as follows.
Step 1. Use observed values $y_1, y_2, \cdots, y_n$ to calculate the MLE $\hat{\theta}$ of $\theta$, plug $\hat{\theta}$ into the likelihood function $f(\theta)$, use $f(\hat{\theta})$ to estimate $f(\theta)$.

Step 2. Treat $\hat{\theta}$ as the real parameter and $f(\hat{\theta})$ as the real likelihood function. Generate a data sample with size $n$, $x^*_1, \cdots, x^*_n$, $n$ is usually equal to the original sample size.

Step 3. Based on the sample data $x^*_1, \cdots, x^*_n$, calculate the MLE $\hat{\theta}^*$.

Step 4. Repeat Step 2 and Step 3 many times, denote the iteration number as $n_{\text{boot}}$, then we can get $\hat{\theta}_{i}^*$, $i = 1, 2, \cdots, n_{\text{boot}}$.

Step 5. Find the corresponding quantiles for $\hat{\theta}^*$ to construct bootstrap CI for $\theta$. For example, if we want to find a 95% CI, the corresponding quantiles can be $\hat{\theta}_{0.025}^*$ and $\hat{\theta}_{0.975}^*$.

2.2.4 Return Level and Run Length for Gumbel AR(1) Model

In section 2.1.5, we introduce the definition of return level $x_p$ for GEV under i.i.d. assumption. When the extreme values are not independent, the $x_p$ defined by Equation (2.13) cannot be used to represent the return level for GEV AR(1) model any more and the corresponding return period $1/p$ will be different as well. Instead of return period, we study the average run length (ARL) to describe the expected waiting time (Zhang (2015)). ARL is defined as the average waiting time of the first observation that passes the specific return level. Let $L$ represent the waiting time, then the ARL is defined as follows,

$$\text{ARL} = E(L) = \sum_{k=1}^{\infty} k \cdot P(L = k).$$

The probability of a specific value $L = k$ is,

$$P(L = k) = \int_{y_p}^{\infty} \int_{-\infty}^{y_p} \cdots \int_{-\infty}^{y_p} f_{Y_1, Y_2, \ldots, Y_k}(y_1, y_2, \ldots, y_k) \, dy_1 \cdots dy_{k-1} \, dy_k,$$

where $f_{Y_1, Y_2, \ldots, Y_k}(y_1, y_2, \ldots, y_k)$ is the joint probability density function of the first $k$ components of $\{Y_t\}_{t=1}^{\infty}$.

There is no closed form for Equation (2.22). Thus, we use numerical method to get the probability and the ARL.
Chapter 3

Simulation Study

In this chapter, we are going to conduct simulations to capture the characters of the parameters and study the performance of our new model. For a fixed combination of \( \rho, \mu, \sigma, \xi \) and sample size \( n \), the simulated data are generated by the following steps:

Step 1. Generate one random value \( e_1 \) from \( N(0, 1) \) and \( (n-1) \) random values \( e_i \) from \( N(0, 1-\rho^2) \), \( i = 2, \ldots, n \).

Step 2. Generate \( \{x_t\}_{t=1}^n \) by setting \( x_1 = e_1, x_t = \rho x_{t-1} + e_t, t = 2, \ldots, n \). Then \( \{x_t\}_{t=1}^n \) follows AR(1) and \( x_t \sim N(0,1), t = 1, \ldots, n \).

Step 3. Get the simulated extreme values \( y_t \) through the transformation function \( y_t = G^{-1}(\Phi(x_t)) \), then \( \{y_t\}_{t=1}^n \) are the generated data set from GEV AR(1) model with the specified parameters.

For each candidate \( \rho \) values in \( \{-0.9, -0.8, \ldots, 0.8, 0.9\} \), we run simulations for \( \xi \) in \( \{-0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4\} \). Without losing generality, the \( \mu \) and \( \sigma \) are set to be 0 and 1, respectively. We choose the sample size \( n = 30, 200, 1000 \). For each combination of \( \rho, \xi \) and \( n \), \( N = 10000 \) samples are generated from the GEV AR(1) model. For each sample, the MLE of the four parameters and the corresponding bias and squared error are calculated. The mean bias and MSE for the \( N = 10000 \) samples are computed by the following formula,

\[
\text{Bias}(\hat{\theta}) = \frac{1}{N} \sum_{i=1}^{N} (\hat{\theta}_i - \theta),
\]

\[
\text{MSE}(\hat{\theta}) = \frac{1}{N} \sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2,
\]

where \( \theta \) is the true parameter (could be \( \rho, \mu, \sigma, \xi \)), and \( \hat{\theta}_i \) is the MLE of \( \theta \) based on the \( i \)th sampling dataset. These two statistics are used as measurements to compare the model performance.

The partial results of the mean bias and the MSE of the four parameters with different sample size \( n = 30, 200, 1000 \) are shown in Table 3.1-Table 3.6. Numbers in brackets are the MSE and
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Table 3.1: The mean bias and the MSE (in the brackets) of \( \hat{\rho} \) and \( \hat{\mu} \) with sample size \( n = 30 \).
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Table 3.2: The mean bias and the MSE (in the brackets) of $\hat{\sigma}$ and $\hat{\xi}$ with sample size $n = 30$. 
Table 3.3: The mean bias and the MSE (in the brackets) of $\hat{\rho}$ and $\hat{\mu}$ with sample size $n = 200$. 

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Table 3.4: The mean bias and the MSE (in the brackets) of $\hat{\sigma}$ and $\hat{\xi}$ with sample size $n = 200$. 

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Table 3.5: The mean bias and the MSE (in the brackets) of $\hat{\rho}$ and $\hat{\mu}$ with sample size $n = 1000$. 
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Table 3.6: The mean bias and the MSE (in the brackets) of $\hat{\sigma}$ and $\hat{\xi}$ with sample size $n = 1000$. 
the ones without brackets are the mean bias. In order to see the pattern of the performance, three corresponding figures are also displayed here (Figure 3.1-Figure 3.3). Colors and linetypes are applied here to represent curves of different $\xi$ since there are seven curves drawn in one graph. Green lines, yellow lines and red lines represent curves of $\xi = -0.2, -0.1$ and 0 respectively and dashed lines, dotted lines, dotdashed lines and longdash lines represent $\xi = 0.1, 0.2, 0.3, 0.4$ respectively.

3.1 Parameters Estimation

As we can see from Table 3.1 and Figure 3.1, with sample size $n = 30$, the estimation of $\rho$ has a generally ascending MSE as $\rho$ increases from -0.9 to 0.9. The absolute value of the mean bias of $\hat{\rho}$ has a decreasing pattern from a positive number to a negative number. It is almost equal to 0 when the real $\rho = -0.4$. The seven curves are almost overlapped indicating that the impact of $\xi$ is little for estimating $\rho$. In general, the estimation for $\rho$ has a better performance when the correlation between two consecutive extreme values are not positively strong.

In estimation of $\mu$, both the mean bias and MSE grow exponentially as $\rho$ increases. They have small value that are close to 0 when $\rho < 0$ and go up rapidly when $\rho > 0.5$. This means the proposed method has good performance for estimating $\mu$ in terms of the mean bias and MSE when extreme values are negatively dependent or have weak positive dependence with the previous one. Moreover, the performance of $\hat{\mu}$ with different $\xi$ are similar.

When estimating $\sigma$, according to Table 3.2 and Figure 3.1, the absolute value of the mean bias has the same pattern with the one for $\hat{\mu}$. $\hat{\sigma}$ has a small absolute value of the mean bias when $\rho < 0$ and it increases exponentially when $\rho > 0.5$. The difference between $\hat{\mu}$ and $\hat{\sigma}$ is that $\hat{\mu}$ is always greater than $\mu$ while $\hat{\sigma}$ is smaller than $\sigma$. In terms of MSE, the estimation of $\sigma$ has a quite small MSE except when extreme values have strong relationship (both positive and negative).

The estimation of $\xi$ performs well under the mean bias and MSE measurements when the correlation is not strong, i.e. $-0.5 < \rho < 0.5$. When extreme values have strong positive relationship
Figure 3.1: The mean bias and MSE of 4 parameter estimators with sample size n=30

Green line: $\xi = -0.2$, yellow line: $\xi = -0.1$, red line: $\xi = 0$, dashed line: $\xi = 0.1$, dotted line: $\xi = 0.2$, dotdashed line: $\xi = 0.3$, longdash line: $\xi = 0.4$
Figure 3.2: The mean bias and MSE of 4 parameter estimators with sample size n=200

Green line: $\xi = -0.2$, yellow line: $\xi = -0.1$, red line: $\xi = 0$, dashed line: $\xi = 0.1$, dotted line: $\xi = 0.2$, dotdashed line: $\xi = 0.3$, longdash line: $\xi = 0.4$
Figure 3.3: The mean bias and MSE of 4 parameter estimators with sample size n=1000

Green line: $\xi = -0.2$, yellow line: $\xi = -0.1$, red line: $\xi = 0$, dashed line: $\xi = 0.1$, dotted line: $\xi = 0.2$, dotdashed line: $\xi = 0.3$, longdash line: $\xi = 0.4$
(ρ > 0.5), the estimator has relatively high absolute value of the mean bias. When the relationship is strongly negative (ρ < −0.5), the model with larger ξ will have higher MSE.

If we focus on lines for ξ = −0.2, −0.1, 0, 0.1, 0.2 and 0.3, as the sample size becomes larger, n = 200, n = 1000, the graphs of the mean bias and MSEs of the four estimators keep the similar patterns while the absolute values shrink to small values, therefore the consistency property of the parameter MLEs is verified. When ξ = 0.4, both the mean bias and MSE of ˆµ, ˆσ and ˆξ have significantly different patterns and have relatively high values. Some curves for ξ = 0.4 do not show completely in Figure 3.2 and Figure 3.3 due to the huge difference. We do not adjust the scales of graphs for the reason that we want to show the patterns of other ξ values clearly. The curves for ξ = 0.4 will be shown in the following figures.

For better visual comparison, we combine the figures of biases with n = 30, 200, 1000 together and put them under the same unit (Figure 3.4- Figure 3.7). Seven different linetypes (lines with squares, lines with circle, solid, dashed, dotted, dotdash and longdash) are used to denote the lines for ξ = −0.2, −0.1, 0, 0.1, 0.2, 0.3, 0.4 respectively. Three colors, black, blue and orange are applied to represent n = 30, 200, 1000. Figure 3.4 represents the mean bias of ˆρ for different sample sizes and ξ values. The curves with the same sample size are almost the same, which indicates the effect of ξ is not significant. All the three groups show decreasing patterns. As the sample size increases from 30 to 1000, the mean biases of ˆρ become closer to zero and the slopes of the curves become more gentle. When sample size achieves 1000, the mean bias of ˆρ is almost equal to zero no matter what the real ρ is. As we can see from Figure 3.5 and Figure 3.7, the biases for ˆµ and ˆξ diminish to zero when sample size becomes large enough. Figure 3.6 shows that the mean bias of ˆσ also tends to zero when ξ = −0.2, −0.1, 0, 0.1, 0.2, 0.3 while does not seem to have the same property when ξ = 0.4.
Figure 3.4: The mean bias of $\hat{\rho}$ for different $n$ and $\xi$

Figure 3.5: The mean bias $\hat{\mu}$ for different $n$ and $\xi$
Figure 3.6: The mean bias of $\hat{\sigma}$ for different $n$ and $\xi$

Figure 3.7: The mean bias of $\hat{\xi}$ for different $n$ and $\xi$
3.2 Confidence Interval

For the GEV AR(1) model, we apply the parametric bootstrap method mentioned in Section 2.2.3 to get confidence intervals for parameters. The iteration time $n_{\text{boot}}$ is an important parameter in the bootstrap method. The increase of $n_{\text{boot}}$ will lead to a more accurate interval and longer computation time as well. We would like to find an appropriate $n_{\text{boot}}$ such that the interval is relatively accurate while the computation time is not too long. The accuracy is measured by the coverage rate which is defined as the percentage of simulated intervals that cover the real parameter.

Simulations are conducted to find the appropriate $n_{\text{boot}}$ for different combinations of $\rho \in \{-0.9, -0.8, \ldots, 0.8, 0.9\}$ and $\xi \in \{-0.2, -0.1, 0, 0.1, 0.2, 0.3\}$. We do not consider $\xi = 0.4$ here since the mean bias and MSE of $\hat{\xi}$ when $\xi = 0.4$ do not show consistency property based on our simulations. The confidence level is set to be 95%. For each combination of $\rho$ and $\xi$, we calculate 1000 confidence intervals by using the bootstrap method and count the number of intervals that contain the real parameters. The size of drawn samples in each iteration is the same as the original sample size. The iteration time $n_{\text{boot}}$ is set to be $\{100, 500, 1000, 1500, 2000\}$, we increase the iteration time until the coverage rate reaches 95%, or equivalently, there are no less than 950 confidence intervals containing the real parameter. Among the four parameters, $\rho$ is the one that we care most. It decides whether the GEV AR(1) should be used. When $\rho$ is significantly different from 0, the GEV AR(1) model is meaningful and should be used to describe the correlation between extreme values. Therefore, we focus on the coverage rate of $\rho$.

For example, we generate $N = 1000$ data sets with real parameter $\rho = 0.1, \mu = 0, \sigma = 1, \xi = 0.1$, each data set has sample size $n = 30$. By using bootstrap method, set iteration times $n_{\text{boot}} = 100$ and the number of drawn samples in each iteration is 30. Then 95% confidence intervals for each data set are calculated and there are $N = 1000$ CIs for each parameter. After that, we count the number of intervals which contain the true parameter. In these $N = 1000$ trials, there are 930 CIs for $\rho$ that contain 0.1 and 887, 883, 893 CIs for $\mu, \sigma, \xi$ that contain the true values respectively. The correct rates for $\rho$ doesn’t reach the confidence level 95%, therefore, we increase the iteration
Table 3.7: Summarized iteration times for different combinations of parameters for finding useful confidence intervals for the GEV AR(1) model parameters (n=30).

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The iteration times $n_{boot}$ from 100 to 500 and repeat this process. The correct numbers of confidence intervals of the four parameters $\rho, \mu, \sigma, \xi$ are 952, 906, 910, 911 respectively. Since the coverage rate of $\rho$ is 95.2%, higher than 95%, then we get the conclusion that when the estimated $\hat{\rho} = 0.1$ and $\hat{\xi} = 0.1$, the iteration time $n_{boot}$ should be set to 500 to get an appropriate bootstrap confidence interval of $\rho$.

The iteration times $n_{boot}$ for different combinations of $\rho$ and $\xi$ are summarized in Table 3.7-Table 3.9. In real application, these are the suggested $n_{boot}$ to use when finding the bootstrap confidence intervals.
Table 3.8: Summarized iteration times for different combinations of parameters for finding useful confidence intervals for the GEV AR(1) model parameters (n=200).

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Table 3.9: Summarized iteration times for different combinations of parameters for finding useful confidence intervals for the GEV AR(1) model parameters (n=1000).
3.3 Return Level and Average Run Length

In the classical GEV model, under the i.i.d. assumption, the $p$ represents the probability that the extreme value will exceed the return level $x_p$. It also means the GEV return level $x_p$ is expected to be achieved or exceeded once in a $1/p$ time period. However, in the GEV AR(1) model, since the extreme values could be dependent, the return period $1/p$ is not as meaningful as it is in the GEV model. Instead, we use the ARL to represent the expected waiting time.

Since there is no analytical solution, we use simulation method to study the ARL of the GEV AR(1) model. For each $\rho \in \{-0.9, -0.8, -0.7, \ldots, 0.8, 0.9\}$ and $\xi \in \{-0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4\}$, the ARLs are calculated respectively with return probability $p \in \{0.1, 0.05, 0.02, 0.01\}$. These four numbers correspond to the most commonly used return periods of 10, 20, 50 and 100 time units.

For a specified probability $p$, the GEV return level $x_p$ can be calculate by Equation (2.13). With assigned $\rho$ and $\xi$, random variables are generated from the GEV AR(1) model until one random variable is not less than the return level $x_p$. Then the total number of simulated random variables $t_i$ is the run length. We repeat this procedure for $N = 10000$ times. The average run length $\bar{t} = \sum_{i=1}^{N} t_i / N$ is regarded as the simulated average waiting time to see the first extreme value that exceeds the GEV return level $x_p$. The summarized ARLs for selected $\rho$, $\xi$ and $p$ are shown in Table 3.10, Table 3.11 and Figure 3.8.

There are four groups of curves shown in Figure 3.8. They correspond to $p = 0.01, 0.02, 0.05, 0.1$ respectively from top to bottom. Each group represents the ARL for the same probability $p$. Within each group, the seven curves show the ARL patterns for different $\xi \in \{-0.2, -0.1, 0, 0.1, 0.2, 0.3, 0.4\}$. The overlapping indicates the shape parameter $\xi$ has little influence on the ARL. The more important fact we need to notice is the relationship between the return period $1/p$ and the ARL. All these four groups of curves illustrate a clear pattern. When the correlation is weak, the ARL is around the return period $1/p$, while it increases dramatically as the correlation becomes strong, i.e. $|\rho| > 0.7$. Moreover, the increase becomes more and more significant as the probability $p$
Figure 3.8: Average run length of the GEV AR(1) model decreases from 0.1 to 0.01. This can be explained by the fact that when the dependency is strong, it is less likely to see an extreme value achieve or exceed the GEV return level $x_p$, and the more extreme the $x_p$ is, the longer waiting time is expected to be seen.

3.4 Comparison

A simulation is conducted to compare the performances of the two models, GEV and GEV AR(1). For each $\rho$ in $\{-0.9, -0.8, -0.7, \ldots, 0.8, 0.9\}$ and $\xi$ in $\{-0.2, -0.1, 0, 0.1, 0.2, 0.3\}$, data samples are generated from GEV AR(1) with sample size $n = 30$. Then MLE of unknown parameters are calculated. Repeat this procedure for $N = 1000$ times and the mean bias and MSE are computed by Equation 3.1 and Equation 3.2. Since there is no $\rho$ parameter in the GEV model, we compare the mean biases and MSEs of $\hat{\xi}$ for the two models, the result is shown in Figure 3.9. The solid curves represent the GEV model while the dash curves represent the GEV AR(1) model. The plots from top to bottom represent $\xi = \{-0.2, -0.1, 0, 0.1, 0.2, 0.3\}$ successively.
Table 3.10: Average Run Length of $p = 0.1$ and $p = 0.05$
| $\xi$ | $p=0.02$ |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |�|p=0.01|       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |       |�|Table 3.11: Average Run Length of $p = 0.02$ and $p = 0.01$
As we can see from Figure 3.9, GEV AR(1) model has a smaller MSE than GEV model when $\xi$ is less than 0.2 or when the autocorrelation coefficient $\rho$ is greater than -0.4. GEV AR(1) model has a smaller absolute value of bias in general as well. When $\rho$ is around 0, which means there is no correlation or quite weak correlation between extreme values, the two models have almost the same MSE and the mean bias. This simulation result verifies the fact that these two models are equivalent when the extreme values are independent.

Based on the simulation study, GEV AR(1) model has better performance when the extreme values don’t have a strong negative correlation.
Figure 3.9: Comparison between GEV (solid curves) and GEV AR(1) (dash curves) regarding MSE and the mean bias of $\hat{\xi}$.
Chapter 4

Real Case Study

In this chapter, we will use two examples of real data to show the application of the GEV AR(1) model.

4.1 Annual Maximum Water Flow of Bow River

The first example is about the annual maximum water flow of the Bow river in Calgary from 1913 to 2014. The historical data is downloaded from the website of Wateroffice. There are 102 records in the data set and the unit of data is $m^3/s$. The plot of the annual maximum water flow is shown in Figure 4.1.

![Figure 4.1: The plot of river flow of the Bow River](image)

If we assume the extreme values are i.i.d, then the GEV model can be applied here. According
to the method mentioned in Section 2.1.4, the estimated parameters are

$$(\hat{\mu}, \hat{\sigma}, \hat{\xi}) = (286.8781, 103.2107, 0.2228085).$$

Then, the estimated $1/p$ years return level can be calculated by Equation (2.13),

$$\hat{x}_p = \hat{\mu} + \frac{\hat{\sigma}[1 - (-\log(1 - p))\hat{\xi}]}{\hat{\xi}}$$

When $p = 0.1, 0.05, 0.02, 0.01$, the estimated results correspond to 10-years, 20-years, 50-years and 100-years return levels, namely, $\hat{x}_{0.1} = 469.5383$, $\hat{x}_{0.05} = 511.1146$, $\hat{x}_{0.02} = 555.9193$, $\hat{x}_{0.01} = 583.8972$.

However, the extreme values could have correlation in some degree. If we assume such dependence and use the GEV AR(1) model, the estimated parameters are calculated by the method mentioned in Section 2.2.2,

$$(\hat{\rho}, \hat{\mu}, \hat{\sigma}, \hat{\xi}) = (0.3424, 287.4407, 103.3160, 0.2272)$$

Since $\hat{\rho} = 0.3424, \hat{\xi} = 0.2272$ and sample size is 102, according to the summarized simulation in Table 3.7, the iteration time of bootstrap method $n_{\text{boot}}$ is set to be 1500. Then, the 95% confidence interval for $\rho$ and $\xi$ are $(0.1279, 0.5531)$ and $(0.0515, 0.4633)$ respectively. The confidence interval for $\rho$ is above 0, which means the true $\rho$ is likely to be larger than 0. Thus, the GEV AR(1) model is meaningful for this case.

Under the GEV AR(1) model, according to the summarized average run length table (Table 3.10, Table 3.11), the above four return levels have corresponding average run lengths equal to 12.72, 23.41, 54.97 and 107.42 years. For example, when $p = 0.01$, the estimated return level is 583.8972. If the dependency between extreme values is ignored, then 583.8972 is expected to be exceeded once every 100 years. However, based on our analysis, the dependency is significant and cannot be ignored. If we consider the dependency, then the average run length of extreme values exceeding 583.8972 will be 107.42 years.
4.2 Weekly and Daily Maximum Sea Level at Tofina Station

The second example is about the sea level in Tofina, B.C., Canada. The original data set is the historical hourly value of sea level measured at Tofina Station (49.15,-125.91666) from 00:00 am, January 1st, 2011 to 11:50 pm, December 31st, 2014. The data is published on the website of Global Sea Level Observing System (GLOSS).

There are 35064 records in this data set. Among them, 912 records are missing data. After checking their locations in the data set, we know that there are 19 days that do not have sea level records. When we do analysis on the daily maximum data, these 19 days are deleted. When our interest variable is the weekly maximum data, we use the remaining days records to find the weekly maxima.

4.2.1 Weekly Maximum Sea Level

Weekly maximum sea level is extracted from the original data set. Here we define a week as from Sunday to Saturday. The start date, January 1st, 2011, is Saturday. The records for the start date are ignored and first record of weekly maximum sea level starts from the first Sunday, January 2nd, 2011. The last day of the original record is Wednesday. Thus, the last record of weekly maximum sea level is the maximum record of the last four days. The plot of the weekly maximum sea level at Tofina Station is shown in Figure 4.2.

If we assume the weekly extreme values are i.i.d, then the GEV model can be applied here. According to the method mentioned in Section 2.1.4, the estimated parameters are

$$(\hat{\mu}, \hat{\sigma}, \hat{\xi}) = (3463.6094, 238.5364, 0.1423).$$

Then, the estimated $1/p$ years return level can be calculated by Equation (2.13).

When $p = 0.1, 0.05, 0.02, 0.01$, the estimated results correspond to 10-years, 20-years, 50-years and 100-years return levels, or $\hat{x}_{0.1} = 3922.94$, $\hat{x}_{0.05} = 4041.424$, $\hat{x}_{0.02} = 4177.825$, $\hat{x}_{0.01} = 4268.818$. 

42
If we assume dependence and use the GEV AR(1) model, the estimated parameters are calculated by the method mentioned in Section 2.2.2,

\[(\hat{\rho}, \hat{\mu}, \hat{\sigma}, \hat{\xi}) = (0.0794, 3484.1251, 231.2302, -0.1112).\]

Since \(\hat{\rho} = 0.0794, \hat{\xi} = -0.1112\) and sample size is 206, according to the summarized simulation in Table 3.8, the iteration time of bootstrap method \(n_{\text{boot}}\) is set to be 500. Then, the 95% confidence interval for \(\rho\) is \((-0.0563, 0.2091)\). This interval contains 0, which means the correlation between two consecutive weekly maxima is not significantly different from 0. The dependency can be checked by the consecutive plot (Figure 4.3) as well. Since there is no clear pattern in the consecutive plot, we cannot say the correlation is significant. In this case, the analysis results from the GEV model and the GEV AR(1) model should be quite similar.

Under the GEV AR(1) model, according to the summarized average run length table (Table 3.10, Table 3.11), these four return levels have corresponding average run lengths equal to 10.76,
Figure 4.3: Consecutive plot of weekly maximum sea level

20.73, 50.93 and 101.24 weeks. They are quite similar with the return periods of GEV model, 10, 20, 50 and 100 weeks.

4.2.2 Daily Maximum Sea Level

Based on the previous analysis, the weekly maximum sea level can be regarded as independent. It may not be the case when we analyze daily maximum sea level. From the common sense, daily hydrological data and daily meteorological data are usually correlated. The consecutive plot of daily maximum sea level (Figure 4.4) shows that there is strong relationship between the consecutive two daily maximum values.

The parametric estimations for the GEV model for daily maximum values are

\( (\hat{\mu}, \hat{\sigma}, \hat{\xi}) = (3237.6110, 277.8583, -0.1794) \).

Then, the estimated 10-years, 20-years, 50-years and 100-years return levels are \( \hat{x}_{0.1} = 4007.961 \), \( \hat{x}_{0.05} = 4327.649 \), \( \hat{x}_{0.02} = 4807.756 \), and \( \hat{x}_{0.01} = 5223.956 \) respectively.
However, it is clear that the correlation between daily maximum values are too significant to be ignored. Under the GEV AR(1) model, the estimated parameters are

\[(\hat{\rho}, \hat{\mu}, \hat{\sigma}, \hat{\xi}) = (0.8983, 3222.9576, 287.8060, -0.1515).\]

Since \(\hat{\rho} = 0.8983\), \(\hat{\xi} = -0.1515\) and sample size is 1456, according to the summarized simulation in Table 3.9, the iteration time of bootstrap method \(n_{boot}\) is set to be 2000. Then, the 95% confidence interval for \(\rho\) is (0.8680, 0.9167). It is significantly above 0 meaning that the true \(\rho\) is larger than 0. Thus, the GEV AR(1) model is meaningful for this case.

Under the GEV AR(1) model, according to the summarized ARL table (Table 3.10, Table 3.11), these four return levels have corresponding average run lengths equal to 44.04, 80.80, 171.54 and 310.73 days. This means, the average run length of daily maximum sea level exceeding 5223.956 will be 310 days. The result is much different from the 100 days under the GEV model.
Chapter 5

Conclusions and Future Work

In this thesis, we improved the GEV model by taking into account the autocorrelation in the block-maxima series and proposed a new parametric model (GEV AR(1)) to analyze the dependent block-maxima. In the GEV model based on the i.i.d. assumption, the autocorrelations between extreme values are ignored and the independence of data is sometimes achieved by discarding a large volume of precious data. This method is a waste of information especially when the data is high-frequency. Therefore, we bring forward a new model called the GEV AR(1) model.

By using GEV AR(1) model, we can analyze autocorrelated extreme values and get the following estimations and inferences.

1. Parameter Estimation (MLE)

We introduced a new method to calculate the MLEs of the four parameters \( \rho, \mu, \sigma, \xi \) in GEV AR(1) model. The idea of this method is to use Jacobian transformation to get the joint probability density function, which is the likelihood function for GEV AR(1) model.

2. Interval Estimation

The interval estimations for the four parameters is calculated by bootstrap method. The suggested iteration time for different combinations of parameters are summarized through simulations.

3. Estimated ARL for return level

Return level of GEV model is one of the most concerned results for researchers. However, under the GEV AR(1) model, return levels are not statistically meaningful since the assumption of independence may not be satisfied. Instead, the ARL is applied in the GEV AR(1) model to represent the average length of time period that the extreme values exceed the return level. ARL is calculated by simulations and the summarized ARL table is shown in Table 3.5. The ARLs are always greater than the corresponding return period \( 1/p \) and the difference increases exponentially.
as the dependence increase. Actually, the return period $1/p$ which is defined under the classic GEV model is a special case of the ARL when the correlation coefficient $\rho = 0$. The ARL is almost equal to $1/p$ when $\rho = 0$.

Besides offering the estimation method for application, simulations have been run to test the performances of parameter estimations and the coverage rate of confidence intervals. The performance is measured by the mean bias and MSE. Simulation results reveal that parameter estimations perform well in general when the correlation between two consecutive extreme values are weak and moderate. Strong positive dependence will lead to increased mean bias and MSE of $\hat{\mu}, \hat{\sigma}, \hat{\xi}$. However, this shortage can be diminished by large sample size.

In addition, we conducted a comparison between the GEV model and the new-proposed GEV AR(1) model with multiple combinations of $\rho$ and $\xi$. The new model has smaller absolute value of the mean bias than the traditional GEV model and the MSEs are lower when $\xi < 0.2$. The GEV AR(1) model with high $\xi$ (when $\xi = 0.2, 0.3$) performs better than GEV when the dependence of extreme values are not strongly negative.

This thesis focuses on the special case of dependent extreme values which are first order correlated. The model overcomes the shortage of the requirement of independent extreme values, however, it still has some limitations. When data are correlated with more previous data, they are not first order autoregressive. In this case, a GEV AR(p) model with larger $p$ order would be a better choice. For future work, it is valuable to work on the following areas.

1. Extend the GEV AR(1) model to GEV AR(p) models for $p > 1$ and consider more general model to fit more complicated cases.

2. Develop a method to test whether GEV AR(1) or GEV AR(p) model can fit the sample data.
Bibliography


Appendix A

Calculation of the likelihood function

The detailed calculation procedure is shown as follows. In order to get the MLE for the four parameters, the likelihood function should be represented at first. For calculating the likelihood for \(\{X_t\}_{t=1}^n\), we can consider the following transformation. Since \(X_1 = e_1\), \(X_t = \rho X_{t-1} + e_t, \ t = 2, \ldots, n\),

\[
\begin{align*}
  x_1 &= e_1 \\
  x_2 &= \rho x_1 + e_2 = \rho e_1 + e_2 \\
  x_3 &= \rho x_2 + e_3 = \rho^2 e_1 + \rho e_2 + e_3 \\
  & \quad \vdots \\
  x_n &= \rho x_{n-1} + e_n = \rho^{n-1} e_1 + \rho^{n-2} e_2 + \cdots + \rho e_{n-1} + e_n,
\end{align*}
\]

(A.1)

which can be written as

\[
\begin{align*}
  e_1 &= x_1 \\
  e_2 &= x_2 - \rho x_1 \\
  e_3 &= x_3 - \rho x_2 \\
  & \quad \vdots \\
  e_n &= x_n - \rho x_{n-1}
\end{align*}
\]

(A.2)

where \(e_1 \sim N(0, 1), e_i \sim N(0, 1 - \rho^2), \ -1 < \rho < 1, i = 2, 3, \ldots, n\), \(e_1\) and \(\{e_i\}\) are all independent. Notice that,

\[
f_X(x_1, x_2, \ldots, x_n) = f(e_1, e_2, \ldots, e_n) \cdot |J_1|,
\]

(A.3)

where \(f\) is the density function of \(e_1, \ldots, e_n\), and \(|J_1|\) is the absolute value of Jacobian and it can
be calculated by
\[
|J_1| = \begin{vmatrix} \frac{\partial x_1}{\partial e_1} & \frac{\partial x_1}{\partial e_2} & \cdots & \frac{\partial x_1}{\partial e_n} \\ \frac{\partial x_2}{\partial e_1} & \frac{\partial x_2}{\partial e_2} & \cdots & \frac{\partial x_2}{\partial e_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial e_1} & \frac{\partial x_n}{\partial e_2} & \cdots & \frac{\partial x_n}{\partial e_n} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ \rho & 1 & 0 & \cdots & 0 \\ \rho^2 & \rho & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{n-1} & \rho^{n-2} & \rho^{n-3} & \cdots & 1 \end{vmatrix} = 1. \quad (A.4)
\]

Since \(\{e_i\}\) are independent, the likelihood for \(\{e_i\}\) is the joint probability density
\[
\left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\sqrt{1-\rho^2}}\right)^{n-1} \exp\left\{-\frac{1}{2} e_1^2 - \frac{1}{2} \sum_{t=2}^{n} e_t^2 \right\}. \quad (A.5)
\]

Then the likelihood for \(\{x_t\}\) is
\[
\begin{align*}
  f_X(x_1, x_2, \ldots, x_n) &= f(e_1, e_2, \ldots, e_n) |J_1| \\
  &= \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\frac{1}{\sqrt{1-\rho^2}}\right)^{n-1} \exp\left\{-\frac{1}{2} x_1^2 - \frac{1}{2} \sum_{t=2}^{n} (x_t - \rho x_{t-1})^2 \right\}.
\end{align*}
\]

Then we can try to calculate the likelihood for \(\{y_t\}_{t=1}^n\) by using the same method, the Jacobian transformation formula is
\[
\begin{align*}
  f_Y(y_1, y_2, \ldots, y_n) &= f_X(x_1, x_2, \ldots, x_n) \cdot |J_2| \\
  |J_2| \text{ is calculated by the following steps:}
\end{align*}
\]

\(a. \) When \(\xi = 0\)
\[
\begin{align*}
y_t &= f(u_t) = f(g(x_t)) = \mu - \sigma \cdot \ln(-\ln(u_t)) \\
\frac{dy}{dx} &= \frac{\sigma \cdot \phi(x_t)}{-\ln(\Phi(x_t)) \cdot \Phi(x_t)} \\
x_t &= \Phi^{-1}(G(y_t)) \\
\frac{dx}{dy} &= \frac{1}{\frac{dy}{dx}} = \frac{-\ln(\Phi(x_t)) \cdot \Phi(x_t)}{\sigma \cdot \phi(x_t)} = \frac{-\ln(\Phi^{-1}(G(y_t))) \cdot \Phi(\Phi^{-1}(G(y_t)))}{\sigma \cdot \phi(\Phi^{-1}(G(y_t)))} \\
  &= \frac{-\ln(G(y_t)) \cdot G(y_t)}{\sigma \cdot \phi(\Phi^{-1}(G(y_t)))} \\
\end{align*}
\]
Therefore, the Jacobian of the transform is

\[
|J_2| = \left| \begin{array}{cccc}
\frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\
\frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n}
\end{array} \right| = \left| \begin{array}{cccc}
\frac{\partial x_1}{\partial y_1} & 0 & \cdots & 0 \\
0 & \frac{\partial x_2}{\partial y_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{\partial x_n}{\partial y_n}
\end{array} \right|
\]

\[= \prod_{t=1}^{n} \frac{\Phi^{-1}(G(y_t))}{\sigma \cdot \prod_{t=1}^{n} \phi(\Phi^{-1}(G(y_t)))} \cdot \prod_{t=1}^{n} d\text{gev}(y_t) \]

\[= \prod_{t=1}^{n} \frac{\Phi^{-1}(G(y_t))}{\sigma \cdot \prod_{t=1}^{n} \phi(\Phi^{-1}(G(y_t)))} \cdot \prod_{t=1}^{n} d\text{gev}(y_t) \]

\[(A.9)\]

The likelihood for \(\{y_t\}\) is

\[f_Y(y_1, y_2, \ldots, y_n) = f(x_1, x_2, \ldots, x_n) | J_2 |\]

\[= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\sqrt{1 - \rho^2}}\right)^n \exp\left\{ -\frac{1}{2} \left[\Phi^{-1}(G(y_1))\right]^2 - \frac{1}{2 \sqrt{1 - \rho^2}} \sum_{t=2}^{n} [\Phi^{-1}(G(y_t)) - \rho \Phi^{-1}(G(y_{t-1}))]^2 \right\} \cdot \frac{\prod_{t=1}^{n} \left[-\ln(G(y_t))\right] \cdot \prod_{t=1}^{n} G(y_t)}{\sigma \cdot \prod_{t=1}^{n} \phi(\Phi^{-1}(G(y_t)))} \cdot \prod_{t=1}^{n} d\text{gev}(y_t) \]

\[= \frac{1}{\sqrt{2\pi}} \left(\frac{1}{\sqrt{1 - \rho^2}}\right)^n \exp\left\{ -\frac{1}{2} \left[\Phi^{-1}(G(y_1))\right]^2 - \frac{1}{2 \sqrt{1 - \rho^2}} \sum_{t=2}^{n} [\Phi^{-1}(G(y_t)) - \rho \Phi^{-1}(G(y_{t-1}))]^2 \right\} \cdot \prod_{t=1}^{n} \frac{\Phi^{-1}(G(y_t))}{\sigma \cdot \prod_{t=1}^{n} \phi(\Phi^{-1}(G(y_t)))} \cdot \prod_{t=1}^{n} d\text{gev}(y_t) \]

\[(A.10)\]

where \(\Phi(\cdot)\) is cumulative distribution function (CDF) of standard Normal distribution \(N(0, 1)\) and \(\phi(\cdot)\) is the probability distribution function. Specifically,

\[
\Phi(x_n) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x_n} \exp\left\{ -\frac{x_n^2}{2} \right\} dt
\]

\[
\phi(x_n) = \frac{1}{\sqrt{2\pi}} \exp\left\{ -\frac{x_n^2}{2} \right\}
\]
The likelihood function is,

$$\ln f(y_1, y_2, \ldots, y_n)$$

$$= -\frac{n}{2} \ln(2\pi) - \frac{n - 1}{2} \ln(1 - \rho^2) - \frac{1}{2} \Phi(G(y_1))^2 - \frac{1}{2(1 - \rho^2)} \sum_{t=2}^{n} [\Phi^{-1}(G(y_t)) - \rho \Phi^{-1}(G(y_{t-1})]$$

$$+ \sum_{t=1}^{n} \ln[-\ln(G(y_t))] + \sum_{t=1}^{n} \ln(G(y_t)) - n \ln \sigma - \sum_{t=1}^{n} \ln[\phi(\Phi^{-1}(G(y_t)))]$$.

(A.11)

\[b. \text{ When } \xi \neq 0\]

$$y_t = f(u_t) = \frac{\sigma}{\xi}((-\ln(u_t))^{-\xi} - 1)$$

$$f'(u_t) = \frac{\sigma}{\xi}((-\xi)(-\ln u_t)^{-\xi - 1}(-1/u_{t})$$

$$f'(x_t) = \frac{\sigma}{\xi}((-\xi)(-\ln u_t)^{-\xi - 1}(-1/u_{t})\phi(x_t) = \frac{\sigma}{\Phi(x_t)}[-\ln(\Phi(x_t))]^{-\xi - 1}$$

$$x_t = \Phi^{-1}(G(y_t))$$

$$\frac{\partial x}{\partial y} = \frac{1}{\frac{\partial y}{\partial x}} = \frac{\Phi(x_t)}{\sigma \phi(x_t)}[-\ln(\Phi(x_t))]^{\xi + 1}$$

$$= \frac{\Phi(\Phi^{-1}(G(y_t)))}{\sigma \phi(\Phi^{-1}(G(y_t)))}[-\ln(\Phi(\Phi^{-1}(G(y_t))))]^{\xi + 1}$$

$$= \frac{G(y_t)}{\sigma \phi(\Phi^{-1}(G(y_t)))}[-\ln(G(y_t))]^{\xi + 1}$$

Therefore, the Jacobian is,

$$|J_2| = \frac{\prod_{t=1}^{n} G(y_t)}{\sigma^n \prod_{t=1}^{n} \phi(\Phi^{-1}(G(y_t)))} \prod_{t=1}^{n} [-\ln(G(y_t))]^{\xi + 1}$$

(A.13)

The likelihood function is,

$$f_y(y_1, y_2, \ldots, y_n)$$

$$= (\frac{1}{\sqrt{2\pi}})^n \cdot (\frac{1}{\sqrt{1 - \rho^2}})^n \cdot \exp\{-\frac{1}{2} \Phi^{-1}(G(y_1))^2 - \frac{1}{2(1 - \rho^2)} \sum_{t=2}^{n} [\Phi^{-1}(G(y_t)) - \rho \Phi^{-1}(G(y_{t-1})]]^2\} \cdot \frac{\prod_{t=1}^{n} G(y_t)}{\sigma^n \prod_{t=1}^{n} \phi(\Phi^{-1}(G(y_t)))} \cdot \prod_{t=1}^{n} [-\ln(G(y_t))]^{\xi + 1}$$

(A.14)
So the Jacobian value is calculated by:

$$|J_2| = \begin{vmatrix}
\frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots & \frac{\partial x_1}{\partial y_n} \\
\frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots & \frac{\partial x_2}{\partial y_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial x_n}{\partial y_1} & \frac{\partial x_n}{\partial y_2} & \cdots & \frac{\partial x_n}{\partial y_n}
\end{vmatrix} = \begin{vmatrix}
\frac{\partial x_1}{\partial y_1} & 0 & \cdots & 0 \\
0 & \frac{\partial x_2}{\partial y_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \frac{\partial x_n}{\partial y_n}
\end{vmatrix}$$

(A.15)

So the likelihood for $y_t$ is

$$f_y(y_1, y_2, \ldots, y_n) = f_x(x_1, x_2, \ldots, x_n) |J_2|$$

$$= \begin{cases} 
A \cdot \prod_{t=1}^{n} [-\ln(G(y_t))] \cdot \prod_{t=1}^{n} G(y_t), & \xi = 0, \\
A \cdot \prod_{t=1}^{n} [-\ln(G(y_t))]^{\xi+1} \cdot \prod_{t=1}^{n} G(y_t), & \xi \neq 0
\end{cases}$$

(A.16)

where

$$A = \left( \frac{1}{\sqrt{2\pi}} \right)^n \left( \frac{1}{\sqrt{1-\rho^2}} \right)^{n-1} \exp \left\{ -\frac{1}{2} \Phi^{-1}(G(y_1))^2 - \frac{1}{2\sqrt{1-\rho^2}} \sum_{t=2}^{n} \left( \Phi^{-1}(G(y_t)) - \rho \Phi^{-1}(G(y_{t-1})) \right)^2 \right\}$$

and

$$\frac{\prod_{t=1}^{n} G(y_t)}{\sigma^n \cdot \prod_{t=1}^{n} \phi(\Phi^{-1}(G(y_t)))}$$

Then the maximum likelihood estimator of parameters can be calculated by maximizing the likelihood function, or equivalent with maximizing the log likelihood function.
Appendix B

Useful R Code

-------------------------------------------------------------

Define the cdf of the GEV distribution

-------------------------------------------------------------

d gev=function(y,mu,sig,xi){
  v=(y-mu)/sig
  if(xi==0){
    u=exp(-v)
  }else{
    u=(1+xi*v)^(-1/xi)
  }
  (1/sig)*u^(xi+1)*exp(-u)
}

-------------------------------------------------------------

Define the pdf of the GEV distribution

-------------------------------------------------------------

p gev=function(y,mu,sig,xi){
  v=(y-mu)/sig
  if(xi==0){
    u=exp(-v)
  }

else{
    u=(1+xi*v)^(-1/xi)
}
exp(-u)

Define the log-likelihood function of the GEV AR(1) distribution

loglikelihoodgev=function(theta,y){
    n=length(y)
    rho=theta[1]
    mu=theta[2]
    sig=theta[3]
    xi=theta[4]
    a=1+xi*(y-mu)/sig
    if(any(a<=0)||sig<=0||abs(rho)>=1) return(10^8)
    x=qnorm(pgev(y,mu,sig,xi))
    return(-sum(log(dnorm(x[2:n]-rho*x[1:(n-1)],0,sqrt(1-rho^2)) /
        dnorm(x[2:n]))) - sum(log(dgev(y,mu,sig,xi))))
}

Calculate the mle of the GEV AR(1) distribution

mle=function(data){

ybar=mean(data)
s=sqrt(var(data))
inil=c(0,ybar-0.57721*sqrt(6)*s/pi,sqrt(6)*s/pi,0.1)
out1=optim(inil,loglikelyhoodgev,hessian = F,method="Nelder-Mead",
          control=list(maxit=2000),y=data)
est=out1$par
return(est)

Generate data from the GEV AR(1) distribution

simudata=function(n,rho,mu,sig,xi){
x=rep(0,n)
e=rnorm(n,0,sqrt(1-rho^2))
x[1]=rnorm(1)
for(i in 2:(n)){
x[i]=rho*x[i-1]+e[i]
}
u=pnorm(x)
if(xi==0){
y=mu+sig*(-log(-log(u)))
}else{
y=mu+sig/xi*((-log(u))^(-xi)-1)
}
return(y)
Calculate the bootstrap CI

```
bootstrapci=function(data,n,nboot){
  theta.hat=mle(data)
  rho.hat=mle(data)[1]
  mu.hat=mle(data)[2]
  sig.hat=mle(data)[3]
  xi.hat=mle(data)[4]
  simuy=matrix(0,n,nboot)
  theta.star=matrix(0,4,nboot)
  ncounterr=0
  j=1
  while(j<=nboot){
    simuy[,j]=simudata(n=n,rho=rho.hat,mu=mu.hat,sig=sig.hat,
                      xi=xi.hat)
    #theta.star[,j]=mle(simuy[,j])
    inibstrp=c(0,mean(simuy[,j])-0.57721*sqrt(6)*sqrt(var(simuy[,j]))/pi,
               sqrt(6)*sqrt(var(simuy[,j]))/pi,0.0)
    outbstrp=try(optim(inibstrp,loglikelyhoodgev,hessian=F,method
                      ="Nelder-Mead",control=list(maxit=2000),y=simuy[,j])
                      ,silent=T)
    if(’try-error’%in% class(outbstrp))
      {
        
```
ncounterr=ncounterr+1
next
}else
{
    theta.star[,j]=outbstrap$par
    j<-j+1
}
}
d1=quantile(theta.star[1,],0.025)
d2=quantile(theta.star[1,],0.975)
d3=quantile(theta.star[2,],0.025)
d4=quantile(theta.star[2,],0.975)
d5=quantile(theta.star[3,],0.025)
d6=quantile(theta.star[3,],0.975)
d7=quantile(theta.star[4,],0.025)
d8=quantile(theta.star[4,],0.975)
return(c(d1,d2,d3,d4,d5,d6,d7,d8,ncounterr))}
}