Examining Fraction Development Through Analysis of Symbolic and Non-Symbolic Magnitude Mental Representations of Whole Numbers and Fractions

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Examining Fraction Development Through Analysis of Symbolic and Non-Symbolic Magnitude Mental Representations of Whole Numbers and Fractions

by

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ABSTRACT

A plethora of previous research has identified important contributions of early magnitude mental representations of whole numbers to later achievement with arithmetic and other whole number skills (De Smedt et al., 2013). Of relatively little research are the contributions of magnitude mental representations of fractions, especially non-symbolic, to fractions knowledge (Fazio et al., 2014). The current study aimed to fill this gap in knowledge by examining the relationship between symbolic and non-symbolic magnitude mental representations and procedural and conceptual fraction knowledge of eighth-grade Canadian students. Results indicate that both symbolic and non-symbolic magnitude mental representations of fractions are related to achievement in procedural and conceptual fraction knowledge. The current study also identifies five distinct learner profiles based on students’ levels of procedural and conceptual fraction knowledge (e.g., lower procedural, higher conceptual), with initial indications that proficiency in magnitude mental representations may account, in part, for differences between these groups.
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CHAPTER 1: INTRODUCTION

Fractions are one of the most difficult mathematical concepts for students to learn (Gabriel et al., 2013a; Kouba, Zawojewski, & Strutchens, 1997; National Mathematics Advisory Panel [NMAP], 2008). Even adults who require fractions in their daily activities struggle with correctly calculating problems with fractions (Geary et al., 2008). However, fraction knowledge is essential for learning more advanced mathematical concepts, such as algebra and physics (Bailey, Hoard, Nugent, & Geary, 2012; NMAP, 2008; Siegler et al., 2012). In addition, the understanding of fractions is essential for careers in many different areas, including medicine, business, and technology fields (Orpwood, Schollen, Leek, Marinelli-Henriques, & Assiri, 2012). It is essential for educators and researchers to identify the various struggles that students have with fractions, and how these difficulties can be remedied.

Several theories exist that provide explanations for the origin of why students struggle with fractions. Many of these theories describe the effect known as the whole number bias, which states that whole number understanding interferes with gaining fraction knowledge (Ni & Zhou, 2005; Vamvakoussi & Vosniadou, 2010). The whole number bias may be an artifact of how the brain processes numerosity (i.e. a favoring of the magnitude mental representations of whole numbers instead of fractions; Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013) or originate from a focus on whole numbers in the students’ formative mathematical education years (Iuculano & Butterworth, 2011; Post, Cramer, Behr, Lesh, & Harel, 1993). Consistent with the whole number bias theory, research has indicated that students often rely on symbolic magnitude mental representations of whole numbers when solving fraction magnitude problems, which may be the cause of many errors and increased reaction times (Meert, Grégoire, & Noël, 2010). This result suggests that the whole number bias may occur because students’ rely on a
whole number magnitude mental representational framework to solve fraction problems, which interferes with understanding components of fractions together as a whole. In contrast, the *integrated theory of numerical development* suggests that the development of fractions should not be negatively impacted by whole number understanding (Siegler, Thompson, & Schneider, 2011). The *integrated theory of numerical development* posits that the magnitude basis of all real numbers, including both whole numbers and fractions, gives an initial framework for students to understand all numbers. In this vein, magnitude of fractions can be understood through previous understanding of magnitude of whole numbers, and then new rules can be learned and applied to fractions. Central to the *integrated theory of numerical development* are the notions that the understanding of numbers, for both whole numbers and fractions, is a developmental process rooted in the common magnitude basis among all real numbers, including whole numbers and fractions (Siegler et al., 2011).

While fractions and whole numbers may be rooted in a common understanding of magnitude, it is not currently known whether they are supported by a common or distinct magnitude systems (or mechanisms), however several researchers have suggested that both fractions and whole numbers are processed by similar systems (Liu, 2014; Matthews, Chesney, & McNeil, 2014; Siegler et al., 2010). Siegler and colleagues (2010) have argued that both whole numbers and fractions, as well as other types of numbers, are represented on the same mental number line. Similarly, arguments have been made for both whole numbers and fractions as being processed by the Approximate Number System (ANS; Liu, 2014). Many researchers have concluded that fraction magnitudes can be mentally represented both by their parts and as a whole. The magnitude mental representations used to represent fractions appears to depend on the task demands (Gabriel Szucs, & Content, 2013b; Hurst & Cordes, 2016; Kallai & Tzelgov,
magnitude mental representations of fractions are distinct from magnitude mental representations of whole numbers because the magnitude mental representations of fractions either represent a whole ratio or two separate integers, whereas magnitude mental representations of whole numbers represent one integer. In addition, the mental representations of fractions on a mental number line is much more densely populated than the mental representations of whole numbers because there are an infinite number of possible fractions between two fractions, whereas there is only a set number of plausible whole numbers between two whole numbers. Furthermore, there is an infinite amount of equivalent fractions that have the same magnitude as a single fraction, whereas there is only one whole number that has the same magnitude as a whole number (Gabriel et al., 2013a). Although it is not yet known if fractions and whole numbers are processed by similar or distinct systems, initial studies have suggested that both types of numbers are processed by the same systems (Liu, 2014; Matthews et al., 2014; Siegler et al., 2010), even though fractions and whole numbers can be represented in distinct ways (Gabriel et al., 2013a).

While further study of the exact systems that support magnitude mental representations of fractions is needed, there is accumulating evidence to support early magnitude mental representations of fractions as an important precursor to the development of fraction procedural knowledge (Siegler & Pyke, 2013) and overall mathematical knowledge (Fazio, Bailey, Thompson, & Siegler, 2014; Siegler & Pyke, 2013). This work builds upon fairly well-established research that supports the role of magnitude mental representations in the development of whole number understanding. In particular, it has been identified that both
symbolic (e.g., Arabic numerals) and non-symbolic (e.g., dots) magnitude mental representations of whole numbers are related to mathematical skills (De Smedt, Nöel, Gilmore, & Ansari, 2013). Specific to mental representations of fractions, research has indicated that an accurate symbolic representation is linked to achievement in arithmetic with fractions (Fazio, Bailey, Thompson, & Siegler, 2014; Siegler & Pyke, 2013). There is also emerging evidence to support the impact of non-symbolic magnitude mental representations of fractions on fraction knowledge (Fazio et al., 2014; Hansen et al., 2015), however, further replication and study in this area is warranted. At this juncture, however, it is not yet clear how these magnitude mental representations (both symbolic and non-symbolic) impact different types of fraction knowledge. Fraction knowledge is a large term that encompasses many types of fraction knowledge. One way to divide fraction knowledge is by separating it into procedural and conceptual fraction knowledge. Procedural knowledge of numbers involves the ability to perform actions and manipulations on numbers (Baroody, Feil, & Johnson, 2007). Conceptual knowledge of numbers involves understanding of numerical principles and concepts (Baroody et al., 2007). Although there are many theories about the developmental relationship between procedural and conceptual knowledge, the most supported theory is the iterative model, which suggests that there is a bidirectional relationship between procedural and conceptual knowledge (Rittle-Johnson & Schneider, 2015; Rittle-Johnson, Siegler, & Alibali, 2001). Thereby, increases in one type of knowledge supports increases in the other type of knowledge. One way to examine the developmental relationship between procedural and conceptual fraction knowledge is through examining students’ individual differences in these two types of fraction knowledge (Hallett, Nunes, & Bryant, 2010; Hallett Nunes, Bryant, & Thorpe, 2012; Hecht & Vagi, 2012). Current studies suggest that there may not be one developmental model of procedural and conceptual fraction knowledge, but
instead each student may develop in dissimilar ways (Hecht & Vagi, 2012). Examining fraction knowledge through the components of procedural and conceptual fraction knowledge may allow for greater precision in the research of how students learn fractions.

To extend the literature on early mathematical skills that support fraction knowledge, the current study examines the relationship between symbolic and non-symbolic magnitude mental representations of numbers (whole numbers and fractions) and fraction knowledge. As further expanded upon in Chapter two, the current study is built on a theoretical and empirical framework of mathematical development that examines the process of acquiring number sense (with a focus on magnitude mental representations of symbolic and non-symbolic quantities), to learning about whole numbers and rational numbers and the procedural and conceptual knowledge related to these numbers. Chapter three details the methods of the current research study, including participants, measures, procedures, and analyses. Chapter four reports on the results of the current research study. Chapter five discusses the results of the current research study and its implications, limitations, as well as exploring future directions of the study of early mathematical skills that support fraction knowledge. Understanding an optimal progression of mathematical learning could allow for curriculum and intervention advances that may be of great benefit to mathematics students and educators. Specifically, additional insight pertaining to the developmental progression of fraction knowledge may allow educators to adapt teaching methods to the individual differences of their students.

Definitions of Key Terms

Key terms used in the current study are defined here to provide clarity for the reader. These are by no means the only definitions, but they serve as working definitions in the context of this thesis.
Mathematics: “a group of related sciences, including algebra, geometry, and calculus, concerned with the study of number, quantity, shape, and space and their interrelationships by using a specialized notation” (Mathematics, n.d.).

Number: “a property of sets of discrete entities” (Brannon, 2006, pp. 223).

Number sense: “..fluidity and flexibility with numbers, the sense of what numbers mean, and an ability to perform mental mathematics and to look at the world and make comparisons” (Gersten & Chard, 1999, pp. 19-20).

Numerosity: a cardinal value (Hyde, 2011) or number of objects (Wagner & Davis, 2010) in a set.

Numerical magnitude: “the absolute value of a number” (National Joint Apprenticeship & Training Committee, 2005, pp. 564)

Expressed representations: expression of a quantity that can be shared between people through the senses (i.e. vision, auditory, etc.; Holloway & Ansari, 2015).

Mental representations of numbers: a person’s “semantic knowledge of a given quantity” in the mind (Holloway & Ansari, 2015, pp. 531).

Symbolic: expressed representations of quantities that indirectly provide meaning to a quantity through the use of numerals (Holloway & Ansari, 2015).


Symbolic magnitude mental representations: expression of numerical symbols and their quantity or value in the mind (Holloway & Ansari, 2015).
Non-symbolic magnitude mental representations: semantic knowledge of the “expressed representation of quantities that [are] encode[d] in a direct 1-to-1 fashion” (Holloway & Ansari, 2015, pp. 532), for example understanding the presence and magnitude of 17 circles in a visual display.

Procedural knowledge of numbers: ability to perform actions and manipulations on numbers (Baroody, Feil, & Johnson, 2007), actions/manipulations can include shrinking, expanding, adding, subtracting, dividing, multiplying, balancing, and many other operations.

Conceptual knowledge of numbers: understanding about numerical principles and ideas (Baroody et al., 2007).
CHAPTER 2: LITERATURE REVIEW

Introduction

The following literature review examines the growth of mathematical skills and understanding by first situating the context of the study within a model that examines broad factors that promote mathematical skills and understanding. The breadth of the review narrows to the development of number sense, followed more specifically by symbolic and non-symbolic magnitude mental representations. Subsequently, the transition from understanding whole numbers to fractions and North American students’ difficulty with fraction is explored. Lastly, the development of procedural and conceptual understanding of fractions is investigated.

From Number Sense to Whole Numbers to Fractions: The Development of Mathematical Skills

Pathways to mathematics understanding. Several factors are involved in the development of early mathematics understanding, which is a precursor to gaining fraction knowledge. A recently developed model, the Pathways to Mathematics Model (LeFevre et al., 2010; further refined in Sowinski et al., 2015), posits there are three cognitive domains (quantitative, linguistic, and working memory) that each contribute to unique aspects of the development of early mathematical understanding. Initial results indicated that subitizing latency (the time to accurately recognize a small (3-4) quantity of objects without needing to count them) in preschool(kindergarten predicted preschool performance in non-symbolic arithmetic and together, these quantitative variables predicted performance in numeration, calculation, and number line estimation two years later (Lefevre et al., 2010). In addition to subitizing and non-symbolic arithmetic, two other number sense skills of counting and symbolic magnitude comparison were found to be important components of the quantitative pathway of
mathematical development in second and third grade students (Sowinski et al., 2015). More specifically, subitizing, counting, and symbolic magnitude comparison accounted for a significant amount of variance in counting backwards, arithmetic fluency, calculation, and number system knowledge (number line task, digit recognition task, and next number task). It is noted that some of the components of the quantitative pathway may only be predictive of mathematical skills during specific times of development. For example, subitizing may only be predictive of arithmetic while counting skills are being developed (Sowinski et al., 2015). The linguistic pathway (as measured by receptive vocabulary and phonological awareness) significantly predicted all measures of number development (i.e. backward counting, addition and subtraction arithmetic fluency, addition and subtraction calculation, number system knowledge, and word reading), especially number system knowledge. The working memory pathways significantly predicted backward counting and arithmetic fluency (Sowinski et al., 2015).

While a number of competing models and theories have been posited to account for early number development (Geary, 2004; Spelke & Kinzler, 2007; Treacy & Willis, 2003), the above detailed Pathways to Mathematics model highlights a widely accepted viewpoint within the field of numerical cognition that several broad (e.g., language, working memory) and domain-specific (e.g., number sense skills) abilities and skills are involved in supporting future mathematical skill development. Not only are these factors important to the development of whole number understanding, but also some of these factors (i.e. those in the quantitative pathway) are likely important to the development of fraction knowledge (Siegler et al., 2011). The following section further details some of the early precursor mathematical skills (e.g. number sense and magnitude mental representations) that support later mathematics understanding, specifically those
associated with the quantitative pathway, and discusses their role in supporting growth in fraction knowledge.

**Number sense.** Early in the development of mathematical knowledge is the growth of number sense, a skill consisting of proficiency in basic numerical concepts including counting, comparing quantities, mental calculations, estimation, and number patterns (Jordan, Kaplan, Locuniak, & Ramineni, 2007; Jordan, Kaplan, Nabors Oláh, & Locuniak, 2006). Number sense skills develop early on in life and one such number sense skill, the recognition of quantities and magnitudes of objects, begins in early infancy (Brannon, 2006; Cantrell & Smith, 2013). Together, the *Approximate Number System* (ANS), which provides a rough estimate of numerical magnitude and begins developing in early infancy without explicit instruction (Izard, Sann, Spelke, & Streri, 2009), and the *parallel individuation system* (a.k.a. *object tracking system*; Trick & Pylyshyn, 1994), which provides more precise estimates of small sets of objects (about 3-4), work together to encode numerosity of sets of objects or events (Hyde, 2011). With formal education, individuals develop a symbolic magnitude mental representations of numbers that appears to reference the ANS through the Intraparietal Sulcus (Pinel, Piazza, Le Bihan, & Dehaene, 2004; Walsh, 2003). Number sense skills, including comparison of numerical magnitude mental representations (De Smedt et al., 2013; Desoete, Ceulemans, De Weerdt, & Pieters, 2010; Sasanguie, Göbel, Moll, Smets, & Reynvoet, 2012), subitizing (Gray & Reeve, 2014), and counting (Desoete, Stock, Schepens, Baeyens, & Roeyers, 2009), support the development of later arithmetic skills (Bartelet et al., 2014; Price & Fuchs, 2016) and mathematical skills (Jordan, Glutting & Ramineni, 2010).

**Symbolic/non-symbolic magnitude mental representations of numbers.** To understand how basic numerical competencies are mentally represented, many researchers have examined
the symbolic and non-symbolic magnitude mental representations of numbers (Wood, Willmes, Nuerk, & Fischer, 2008). It is important to define two uses of the word ‘representation’, firstly, expressed representations refer to what people can see, feel, taste, smell, or hear; whereas mental representations refer to how people understand these expressed representations in their mind (Holloway & Ansari, 2015). Symbolic magnitude mental representations are the referents of numerals and their value in the mind, whereas non-symbolic magnitude mental representations are the referents of pictures/objects, as well as tones (Agrillo et al., 2013), food, and other objects/events (Feigenson et al., 2002), and their value.

In terms of the development of mental representations, non-symbolic magnitude mental representations of numerosity may begin to be evident as early as immediately after birth (Izard et al., 2009). It is suggested that numerical quantity is accessed from the ANS and mapped onto a mental number line (Park & Starns, 2015), however, this may not be a direct mapping (Holloway & Ansari, 2015). It is proposed that non-symbolic magnitude mental representations of numbers are mapped onto a logarithmic, not linear, mental number line from birth, such that smaller numbers are given greater space on the left side of the number line than larger numbers on the right side of the number, instead of equal space for each number as given by the linear mental number line (Booth & Siegler, 2006; Dehaene, Izard, Spelke, & Pica, 2008). As determined by habituation studies, it is suggested that infants recognize differences between sets of dots with a larger ratio (e.g. 7 and 14, not 7 and 10) as early as six-months old (Xu & Spelke, 2000). The precision of these magnitude mental representations of non-symbolic quantities increase over time, as exhibited by the ability for six-month old infants to distinguish between items at a 2:3 ratio (van Oeffelen & Vos, 1982), which increases to around a 7:8 ratio in adulthood (Jordan, Suanda, & Brannon, 2008).
As children mature, they develop symbolic magnitude mental representations of numbers through counting and arithmetic, which are precise representations, by indicating a specific quantity (i.e. number). As opposed to non-symbolic magnitude mental representations of numbers, the symbolic magnitude mental representations of numbers are more precise because the difference between 34 and 35 is much more clear with numerals than pictures or sounds. With formal education, these symbolic magnitude mental representations of numbers are initially mapped onto a logarithmic mental number line (Booth & Siegler, 2006; Dehaene et al., 2008; Siegler & Opfer, 2003), and gradually changes to a linear mental number line (Dehaene et al., 2008; Siegler & Opfer, 2003). Thus, by around second grade North American students represent symbolic quantities along a linear mental number line (Geary, Hoard, Byrd-Craven, Nugent, & Numtee, 2007). Not only do magnitude mental representations tend to differ with age, but also it is suggested that males’ symbolic magnitude mental representations of numbers are more automatic and accurate than females’ symbolic magnitude mental representations of numbers (Bull, Cleland, & Mitchell, 2013), possibly due to differences in surface area of the parietal lobe (Koscik, O’Leary, Moser, Andreasen, & Nopoulos, 2009).

The non-symbolic magnitude mental representational system provides a basis for the symbolic magnitude mental representational system to build upon and allow for more sophisticated mathematical skills to develop (Feigenson, Dehaene, & Spelke, 2004; Jordan et al., 2010). However, the symbolic magnitude mental representations may not replace the non-symbolic magnitude mental representational system, but instead it is suggested that these two systems work together to provide a deeper understanding of numerosity (Mundy & Gilmore, 2009). In an analysis of the symbolic and non-symbolic magnitude mental representations of whole numbers in children aged 6-9, researchers measured mathematical achievement (as
measured by questions on magnitude; ordering; number lines; and calculations in addition, subtraction, multiplication, and division) and used a mapping task that required students to choose an Arabic numeral that corresponded to a set of dots or a set of dots that corresponded to an Arabic numeral (Mundy & Gilmore, 2009). The results identified that students aged 6-9 exhibited a bidirectional mapping process of quantity, such that quantity is accurately determined from dots to Arabic numerals and from Arabic numerals to dots. Furthermore, both symbolic and non-symbolic magnitude mental representations of whole numbers were positively correlated with mathematics achievement, such that greater accuracy of mapping between symbolic and non-symbolic magnitude mental representations is associated with higher mathematics achievement scores. The bidirectional mapping and its association with number skills suggests that both symbolic and non-symbolic magnitude mental representations are important for children’s development of numeration and arithmetic skills and each representation may have differential contributions to this development (Mundy & Gilmore, 2009).

In contrast to the bidirectional mapping hypothesis, initial research suggests a full mediation hypothesis (Price & Fuchs, 2016). In an analysis of the relationship between symbolic and non-symbolic magnitude mental representations of whole numbers and mathematical competence in third-grade students (mean age of 9.01 years), researchers measured mathematical competence (i.e. arithmetic and numeration) and numerical magnitude task with sets of dots and Arabic digits (Price & Fuchs, 2016). The results indicated that automaticity and accuracy in symbolic magnitude mental representations fully mediated the relationship between automaticity and accuracy in non-symbolic magnitude mental representations and mathematical competence. These results suggest that non-symbolic magnitude mental representations support the development of symbolic magnitude mental representations, which supports the development of
mathematical competencies in arithmetic and numeration (Price & Fuchs, 2016). Alternatively, other researchers have argued that symbolic and non-symbolic magnitude mental representations of numbers are independent of each other. Examining this relationship through mixing modalities has lead researchers to suggest that symbolic and non-symbolic magnitude mental representations of numbers utilize separate systems because adults performed faster and more accurately when given stimuli with the same modality (i.e. dot-dot or symbol-symbol) than mixed modalities (i.e. dot-symbol or symbol-dot; Lyons, Ansari, & Beilock, 2012).

Others have researched the connection between each type of magnitude mental representation and mathematics achievement to determine the relationship between symbolic and non-symbolic magnitude mental representations of numbers. Fairly consistently in the literature, an automatic and precise symbolic magnitude mental representations of numbers has been positively correlated with understanding and achievement with whole numbers with various age groups, both through age comparative and longitudinal studies (De Smedt et al., 2013; Desoete et al., 2010; Sasanguie et al., 2012). However, the relation between non-symbolic magnitude mental representations of numbers and mathematics achievement has shown inconsistent results in the literature, with correlations obtained between .13 (Mussolin, Nys, Leybaert, & Content, 2012) and .52 (Libertus, Feigenson, & Halberda, 2013).

A recent meta-analysis was conducted to examine the inconsistent research results between studies examining non-symbolic magnitude mental representations of numbers (measured with the magnitude comparison task with dots) and mathematics achievement (Fazio et al., 2014). As expanded upon below, the results of the meta-analysis identified several factors that differed between the studies and may contribute to the inconsistent finding including sample size, age, and achievement level, as well as measures used. The correlation between non-
symbolic magnitude mental representations of numbers and mathematics achievement is
determined to have a small effect size, therefore, many of the samples in previous research
studies may have been too small to detect an effect between non-symbolic magnitude mental
representations of numbers and mathematics achievement. The distribution of various
performance levels within the sample may impact the results because the relation between non-
symbolic magnitude mental representations of numbers and mathematics achievement is stronger
in students who are lower achieving in mathematics (Bonny & Lourenco, 2013; Mazzocco,
Feigenson, Halberda, 2011). Furthermore, the age of the sample may impact the results, such
that the relation between non-symbolic magnitude mental representations of numbers and
mathematics achievement is stronger for younger age groups (Fazio et al., 2014). In addition,
there are various measurements used in the magnitude comparison task (Price, Palmer, Battista,
& Ansari, 2012), such as the use of Weber’s fraction as opposed to reaction time slope
measurements. Lastly, the use of different measures may yield different results in the relation
between non-symbolic magnitude mental representations of numbers and mathematics
achievement. For example, measures that include advanced arithmetic and geometry have
stronger relations (Lourenco, Bonny, Fernandez, & Rao, 2012). Certain areas of mathematics,
such as geometry, may better map onto non-symbolic expressed representations than symbolic
expressed representations, suggesting that different representations may have differential
contributions to different areas of mathematics. In their meta-analysis, Fazio and colleagues
(2014) concluded that, despite discrepant research findings, there is a small and reliable
correlation between non-symbolic magnitude mental representations and mathematics
achievement.
In sum, the current literature strongly suggests that whole number symbolic magnitude mental representations supports the development of later mathematical skills, including, mathematical skills involved in statewide assessments (Fazio et al., 2014), fraction arithmetic skills (Siegler and Pyke, 2013), whole number arithmetic skills (Desoete et al., 2010), and skills involved in a timed arithmetic-task and those used in mathematics curriculum tasks (Sasanguie et al., 2012). It is suggested that symbolic numerical magnitude processing supports the development of arithmetic skills similar to how phonological awareness supports the developments of reading skills (Vanbinst, Ansari, Ghesquière, & De Smedt, 2016). In addition, there is a correlation between symbolic and non-symbolic magnitude mental representations of whole numbers (Fazio et al., 2014; Mundy & Gilmore, 2009). However, the relationship between mathematics achievement and non-symbolic magnitude mental representations of whole numbers has varied between research studies, a relationship that is impacted by many methodological factors, including participant’s age, sample size, achievement level, and measurement tools.

Related to this research focus on symbolic and non-symbolic magnitude mental representations of whole numbers, recent research has begun to examine symbolic and non-symbolic magnitude mental representations of fractions. This represents a departure from what has been the more traditional and largely singular emphasis on whole numbers. This area is an interesting line of investigation with some work being undertaken on the relationship between mathematics achievement and symbolic magnitude mental representations of fractions (Fazio et al., 2014; Siegler and Pyke, 2013) with very little examining non-symbolic magnitude mental representations of fractions (Fazio et al., 2014).
Symbolic/non-symbolic magnitude mental representations of fractions. It is important to explore the developmental process of symbolic and non-symbolic magnitude mental representations of fractions. Similar to whole numbers, it is generally believed that magnitude mental representations of fractions exhibit a developmental process in which an understanding of the non-symbolic mental representations precedes that of symbolic mental representations. An understanding of non-symbolic magnitude mental representations of fractions begins early in childhood (Siegler, Fazio, Bailey, & Zhou, 2013). At six months of age, infants can identify differences between two non-symbolic fractions expressed representations (dot sets as ratios) if one of the fractions is larger than the other by a factor of two (McCrink & Wynn, 2007). By the age of three, children are able to match equivalence of fractions between different shapes (i.e. half of a circle is the same as half of a square; Goswami, 1995; Singer-Freeman & Goswami, 2001). As early as the age of four, children begin to be able to calculate with non-symbolic magnitude mental representations of fractions, as measured by a task where students see two pieces of a circle (one at a time) and are asked to combine or subtract these two pieces and then choose one of four options that equals the calculation (Mix, Levine, & Huttenlocher, 1999). This task requires students to use spatial transformation and memory skills to solve the fraction problems. It has been suggested that children are able to calculate with non-symbolic magnitude mental representations of whole numbers and fractions because they are able to construct mental images of each piece of the problem and combine or subtract the mental images (Huttenlocher, Jordan, & Levine, 1994; Mix et al., 1999).

In comparison, symbolic magnitude mental representations of fractions develops later than non-symbolic magnitude mental representations of fractions. By the age of seven, children begin to understand proportional reasoning, such that they understand that sharing a set of
objects with more people results in less objects for each person (Sophian, Garyantes, & Chang, 1997). With experience, children begin to be able to understand the magnitude of written fractions, however, this skill is a challenge for many students (Hecht & Vagi, 2010; Siegler et al., 2013).

A number of researchers have explored how fractions are understood and processed mentally (Bonato, Fabbri, Umiltà, & Zorzi, 2007; Gabriel et al., 2013b; Hurst & Cordes, 2016; Kallai & Tzelgov, 2009; Meert et al., 2009, Meert et al., 2010; Schneider & Siegler, 2010; Zhang et al., 2014). Before discussing some of the findings in this area, it is helpful to first discuss commonly used symbolic and non-symbolic expressed representations of fractions. As depicted in Figure 1, similar representations have been used in the study of fractions as have been used for whole numbers, with focus commonly given to both magnitude and the number line. The magnitude expressed representations of whole numbers and fractions uses both numerals (symbolic) and various arrays, such as dots (non-symbolic) to represent quantity.

The number line expressed representations of whole numbers and fractions is typically depicted as a line with equal partitions to signify the distance of a quantity, expressed as either a numeral (symbolic) or as an array (non-symbolic), from zero. With non-symbolic fraction representations, the focus is on the ratio of the display. For example, in Figure 1 (below) the illustration of the non-symbolic number line for fractions is anchored on the left-hand side with ten blue dots and on the right-hand side by ten red dots. As one progresses along the line, the amount of blue dots decreases and the amount of red dots increases. Similarly, the magnitude expressed representations of fractions focuses on the ratio of the display, such that individuals are instructed to compare the ratio of blue dots to red dots. To some extent, how these expressed representations are perceived is individual-dependent. For example, a set of three dots could be
perceived as a whole, thus representing the quantity of three, or could be perceived as one set
where each dot represents a third of the total display. Given the similarities between such
stimuli, it is important that researchers cue participants to the particular mathematical
properties/features of interest (e.g., whole number versus fractions). For a non-symbolic fraction
magnitude comparison task, for example, participants might be shown two sets of red and blue
dots (each set consisting of differing numbers of red and blue dots), instructed that the blue dots
are worth $100 and the red dots are worth $1, and asked to select the set that would give them
the best chance of getting a blue dot if only one dot could be randomly drawn.

<table>
<thead>
<tr>
<th></th>
<th>Symbolic</th>
<th>Non-symbolic</th>
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<tbody>
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<tr>
<td><strong>Rational</strong></td>
<td></td>
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</tr>
<tr>
<td><strong>Magnitude</strong></td>
<td>$\frac{1}{10}$ or 1:9 $(part\text{-}whole)$ or $(part\text{-}part)$</td>
<td></td>
</tr>
<tr>
<td><strong>Number Line</strong></td>
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<tr>
<td><strong>Whole</strong></td>
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<tr>
<td><strong>Magnitude</strong></td>
<td>3</td>
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<td><strong>Number Line</strong></td>
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Figure 1. *Examples of symbolic and non-symbolic expressed representations of numbers.*

Note. The pictures and numerals displayed in Figure 1 depict some typical expressed
representations of whole and rational numbers in both symbolic and non-symbolic form.
The understanding of the expressed representations of fractions requires mental processing of the components of the fraction separately or together. Symbolic magnitude mental representations of whole numbers represent a single integer, whereas the symbolic magnitude mental representations of fractions may represent two whole numerals (numerator numeral/denominator numeral) or a single rational number. Understanding symbolic fraction magnitude as a single rational number is called holistic processing (Zhang et al., 2014; also called global magnitude of fractions in Gabriel et al., 2013b), whereas understanding symbolic fraction magnitude through two whole numbers is called componential processing (Zhang et al., 2014). A study examining symbolic magnitude mental representations of fractions with adults found that the participants used componential processing to understand symbolic fraction magnitude (Bonato et al., 2007). In addition, a study examining symbolic magnitude mental representations of fractions of sixth- and eighth-grade students found that students tend to focus on either the numerator or denominator, instead of the fraction as one quantity expressed by the relationship between the numerator and the denominator (Siegler & Pyke, 2013). In contrast, another study identified that some students use holistic processing to understand symbolic fraction magnitude, however, they were still strongly influenced by the magnitude of the denominator (Meert et al., 2010). Overall, the majority (as posited by Hurst & Cordes, 2016) of the research propose that individuals (both children and adults) prioritize using componential processing, but will use holistic processing if demanded by the task (Gabriel et al., 2013b; Hurst & Cordes, 2016; Kallai & Tzelgov, 2009; Meert et al., 2009, Meert et al., 2010; Meert et al., 2012; Schneider & Siegler, 2010; Zhang et al., 2014). For example, when the denominators are not alike between two fractions (i.e. 5/7 and 2/9), a holistic processing strategy is more likely to
be utilized because comparing the components of the fractions is less meaningful than comparing the fraction quantities (Hurst & Cordes, 2016). In support of the differential fractions processing strategies, a neural study with adults identified that the use of the componential processing and holistic processing of fraction magnitude resulted in different brain activity in the parietal lobe depending upon the processing strategy used (Barraza, Gómez, Oyarzún, & Dartnell, 2014). Together, these studies suggest that fractions may be mentally represented by both their component parts (i.e. considering the numerals of the numerator and the numerals of the denominator somewhat independently) and as a quantity relationship (i.e. considering the numerals of the numerator in relationship to the numerals of the denominator). Therefore, magnitude mental representations of fractions may be understood more than one way; magnitude mental representations of fractions may occur by understanding the magnitude of two individual parts or by understanding the magnitude of the whole fraction. The componential processing of fractions is sufficient for comparing fractions in the forms of 1/x because only the denominator needs to be processed to compare fractions, however, the holistic processing of fractions is more beneficial for comparing other types of fractions, such as those with different denominators or improper fractions (Gabriel et al., 2013a). One difference between the magnitude mental representations of whole numbers and fractions is that whole numbers can only be processed holistically, whereas fractions can be understood through holistic or componential processing.

Of interest are the similarities and differences between how magnitudes of whole numbers and fractions are mentally represented. One study found that middle school and college students’ individual differences in ANS acuity (as measured by a non-symbolic whole number magnitude comparison task) and fraction knowledge (as measured by a symbolic fraction magnitude comparison task) were correlated, which suggests that fraction magnitude mental
representations of fractions are processed through the same ANS that processes magnitude mental representations of whole numbers (Liu, 2014). In addition, it is suggested that both fractions and whole number magnitude can be represented along the same mental number line (Schneider & Siegler, 2010; Siegler et al., 2011). In contrast to whole numbers, accessing the magnitude mental representations of fractions is slower and less accurate than mental magnitude representations of whole numbers (Schneider & Siegler, 2010), which suggests there is greater noise, less precision, in the magnitude mental representations of fractions on the mental number line. In addition, there are greater individual differences in magnitude mental representations of fractions than magnitude mental representations of whole numbers, such that individuals take much longer to compare two-digit fractions than one-digit fractions and adults at different education levels have significantly different reaction times. However these differences are not significant with whole numbers (Schneider & Siegler, 2010). In an analysis of 8 to 11 year-old students’ discussions of fractions, Pitta-Pantazi, Gray, and Christou (2004) found individual differences in the way students represented fractions. Students who performed higher in arithmetic were more likely to comment on the procedural or conceptual nature of fractions (i.e. saying “half” or “part-whole”) or responses about general concepts related to fractions (i.e. saying “part of a shape” or “sharing”) than those who performed lower in arithmetic (Pitta-Pantazi et al., 2004). These results suggest that there are individual differences in the way students represent fractions (Pitta-Pantazi et al., 2004). Although magnitude mental representations of fractions and whole numbers are similar, such as both being represented along a mental number line, these magnitude mental representations denote distinctly different types of numbers and differ by precision and size of individual differences. Therefore, it is important to assess the magnitude mental representations of both whole numbers and fractions.
One of the important reasons to study the magnitude representations of fractions is because of its relation to mathematics achievement (Siegler & Pyke, 2013). To examine the relationship between fraction and overall mathematics achievement and symbolic fraction magnitude knowledge, as well as whole number division, executive functioning, and problem solving, sixth and eighth graders were tested on their performance in the aforementioned areas (Siegler & Pyke, 2013). A standardized statewide achievement test was used to divide 120 participating students into a low achieving group (35th percentile and below) and a high achieving group (above 35th percentile). Symbolic fraction magnitude representations were measured with a number line estimation task (indicate where a fraction is located on a number line ranging from 0-1 or 0-5) and magnitude comparison task (indicate if a presented fraction is larger than 3/5). Results indicated that students performed better in fraction arithmetic in eighth-grade than sixth-grade; however, the low achieving groups in sixth- and eighth-grade had similar low fraction arithmetic scores. These results could be due to the finding that students in the low achieving group used whole number strategies to solve fraction problems in both sixth- and eighth-grade, whereas students in the sixth-grade high achieving group used whole number strategies but the eighth-grade high achieving students used fraction strategies to solve fraction problems. After controlling for executive functioning and reading achievement, there was a significant correlation between fraction arithmetic performance and symbolic magnitude representations of fractions. In addition, there was a significant correlation between overall mathematics achievement and symbolic magnitude representations of fractions, which was found to be a stronger relationship in eighth grade, than in sixth grade. Furthermore, these results suggest that symbolic fraction magnitude mental representations are likely an important
foundation for fraction knowledge, as well as many other areas of mathematics (Siegler & Pyke, 2013).

In contrast to symbolic magnitude mental representations of fractions, very limited research has examined non-symbolic magnitude mental representations of fractions. It is possible that processing non-symbolic fraction magnitudes uses the ANS, parallel individuation system, and specific spatial abilities to determine fraction magnitude by comparing the ratios of two sets (Hansen et al., 2015). In an analysis of adults’ magnitude mental representations of numerical ratios with a magnitude-estimation task (i.e. participants adjust a sliding bar to match a dot ratio), it was suggested that ratio magnitude was understood through componential processing, instead of the whole ratio (Meert et al., 2012). In addition, the results suggest that the non-symbolic estimate of either the components or of the whole ratio may be converted into a symbolic magnitude mental representations before providing a response (Meert et al., 2012). In addition, it was noted that performance on the non-symbolic proportional reasoning task was less accurate than the symbolic fraction magnitude fraction task, which suggests that the non-symbolic magnitude mental representations of rational numbers may be less precise and/or that there is a greater focus on the components in the non-symbolic proportional reasoning task than the symbolic fraction magnitude task (Meert et al., 2012). These results suggest that non-symbolic magnitude mental representations of rational numbers are distinct from symbolic magnitude mental representations of fractions; however, similar systems may be utilized to understand the magnitude of fractions represented either symbolically or non-symbolically.

While limited research has examined non-symbolic magnitude mental representations of rational numbers, initial results suggest that these magnitude mental representations provide significant contributions to the understanding of fractions. Fazio and colleagues (2014) were one
of the first to examine magnitude comparison of non-symbolic fractions. The relationship
between the magnitude mental representations of symbolic/non-symbolic fractions/whole
numbers and overall mathematics achievement was measured in fifth grade students using
symbolic and non-symbolic magnitude comparison and number line estimation tasks with whole
numbers and fractions (Fazio et al., 2014). Across two sessions, fifth grade students’ overall
cognitive intelligence and performance on eight magnitude comparison and number line
estimation tasks (symbolic/non-symbolic, whole number/fraction) was assessed. Students’
overall mathematics achievement was measured by their score on a state-wide assessment of
achievement (Pennsylvania System of School Assessment). The performance on the eight
magnitude tasks was analyzed in three different models to assess which model best correlated to
overall mathematics achievement.

Aligned with the integrated theory of numerical development (Siegler et al., 2011), Fazio
and colleagues found a positive correlation between the symbolic magnitude mental
representations of whole numbers and fractions (Fazio et al., 2014). In contrast to the full
mediation hypothesis of symbolic and non-symbolic magnitude mental representations and
mathematical competence (Price & Fuchs, 2016), Fazio and colleagues found that both symbolic
and non-symbolic magnitude mental representations of numerical quantities independently
contribute to overall mathematics achievement, and the relationship was not better explained by
a full or partial mediation by the opposite type of magnitude mental representations (symbolic
magnitude mental representations did not mediate the relationship between non-symbolic
magnitude mental representations and mathematics achievement, nor did non-symbolic
magnitude mental representations mediate the relationship between symbolic magnitude mental
representations and mathematics achievement). When broken down by task, performances on
each of the tasks were positively related to mathematics achievement, except for the non-symbolic fraction magnitude comparison task. The symbolic numerical tasks (both whole numbers and fractions) and the non-symbolic whole number magnitude comparison tasks had the strongest correlations to mathematics achievement. In addition, the results indicated that there is a correlation between symbolic and non-symbolic magnitude mental representations of whole numbers and fractions. These results confirm that an understanding of both the magnitudes of visual displays and Arabic numerals is related to mathematical achievement (Fazio et al., 2014).

Important to highlight is that the non-symbolic fraction magnitude comparison task (i.e. comparing two dot ratios and indicating which set had a larger ratio of blue dots) used in this study was the first of its kind and allows future research to also examine the contributions of non-symbolic magnitude mental representations of fractions to various areas of mathematics achievement. Due to the novelty of the task, Fazio et al. (2014) conducted more trials (186) with the non-symbolic whole number and fraction magnitude comparison tasks than is typically used in studies with non-symbolic whole number magnitude comparison tasks and an elementary student sample (between 30-40 trials as used in: Bonny & Lourenco, 2013; Price & Fuchs, 2016), however, not significantly more trials than many other studies (between 120-160 trials as used in: Jordan et al., 2013; Meert et al., 2010; Yang et al., 2014).

Beyond measuring overall mathematics achievement, it is important to assess the contributions of symbolic and non-symbolic magnitude mental representations of fractions to specific mathematical competencies, such as fraction concepts. In an examination of math-specific and general competencies of sixth-grade students, symbolic magnitude mental representations of whole numbers were assessed through a number line estimation task and non-symbolic proportional reasoning (i.e. computation with pictures that depict ratios) was assessed
through an equivalence task (Hansen et al., 2015). Researchers identified that whole number line estimation and non-symbolic proportional reasoning uniquely contribute to the understanding of fraction concepts, and just whole number line estimation contributed to the understanding of fraction procedures (Hansen et al., 2015). These results indicate that non-symbolic magnitude mental representations of rational numbers (i.e. understanding and manipulation of pictures that depict ratios) independently support the development of fraction knowledge, specifically conceptual knowledge.

In sum, the development of magnitude mental representations of both whole numbers and fractions begins early in children’s development. However, there are differences between the magnitude mental representations of whole numbers and fractions, including how fractions can be understood through both componential and holistic processing. Of particular importance, current results suggest that both symbolic and non-symbolic magnitude mental representations of fractions independently contribute to fraction skills and overall mathematics achievement, albeit the contributions of non-symbolic magnitude mental representations of fractions has had limited examination to date. As further study is given to this field of research, one area, in particular, that requires greater attention is determination of the specific components of fraction skills and mathematics achievement that are uniquely influenced by the symbolic and non-symbolic magnitude mental representations of both fractions and whole numbers. Fazio and colleagues (2014) examined the contribution of symbolic and non-symbolic magnitude mental representations of both whole numbers and fractions to overall mathematics achievement; however, they did not explore the contributions of these magnitude mental representations to fraction skills. In contrast, Hansen and colleagues (2015) examined the contributions of mental representations to procedural and conceptual fraction knowledge, however, they only examined
symbolic whole number mental representations and non-symbolic fraction reasoning. Therefore, it is important to combine these research efforts and assess the contributions of symbolic and non-symbolic magnitude mental representations of fractions and whole numbers to procedural and conceptual fraction knowledge. In order to understand the influence of magnitude mental representations of fractions and whole numbers on fraction skills, it is important to examine the development of specific fraction skills.

Theories about the Development of Fractions. Once individuals have developed their basic number sense skills (including comparison of magnitude mental representations of numerosity), there are several other concepts (e.g. operations with whole numbers, fractions, decimals, etc.) they must learn to develop mathematical skills. One concept that many students find exceptionally difficult is operations with fractions (Gabriel et al., 2013a; Kouba et al., 1997). One reason that fraction operations are more difficult than whole numbers is that there are multiple perspectives of fractions: part-whole, quotient, ratio, measure, and operator (Moseley & Okamoto, 2008). The part-whole perspective involves a comparison of a part to the whole (one of the most commonly used constructs in North American education; Thompson & Saldanha, 2003), ratio/part-part perspective involves comparing different quantities (such as 1 part concentrate to 3 parts water), the quotient perspective involves the process of division (such as divide 6 cups of juice into 18 glasses), a measure perspective involves identifying the distance from zero to the fraction (such as placing ¼ on a number line), and operator perspective involves identifying the impact (i.e. shrinking/expanding) of the fraction on other numbers (such as halving a recipe, or having only 1 cup of concentrate instead of 2, how much less water should be used; Moseley & Okamoto, 2008). It is possible that the ratio/part-part perspective may favor componential processing of fraction magnitude, where as the part-whole perspective may favor
the holistic processing of fraction magnitude. In addition, having a precise magnitude mental representations of fractions on a mental number line would likely aid in the measure perspective of fractions. The multiple ways that fractions can be perceived is one complication that makes fractions a more difficult mathematical concept to understand for teachers and students, particularly when it is unclear about which type of fractions construct is being referred to explicitly in mathematics discussions.

Many researchers and educators hypothesize that the discrete representations of a whole number may interfere with understanding ordinality and continuity of fractions and rational numbers, an effect referred to previously as the whole number bias (Ni & Zhou, 2005). The whole number bias may manifest in several ways, including using incorrect procedures (e.g. adding across numerators and denominators e.g. $1/8 + 3/8 = 4/16$) and not accepting that there are numbers between 0 and 1 (Ni & Zhou, 2005). The origin of the whole number bias is debated among researchers, some of whom suggest that this bias is innate (Mix et al., 2002; Obersteiner et al., 2013), while others suggest that this bias is an artifact of the students’ education and a lack of conceptual framework (Post et al., 1993). For example, it has been suggested that some mathematical language used in early grades may hinder development of fraction understanding, such as ‘one over four’ or ‘one out of four’ may imply two separate whole numbers instead of a single fraction number (Mack, 1995). In addition, students may have difficulty with understanding fractions because they process individual components of fractions separately as two whole numbers instead of one fraction (Gabriel et al., 2013a, 2013b). Further, it is suggested that teaching students symbolic representations of fractions before they understand non-symbolic representations of fraction quantities hinders their later understanding of fractions and related procedures (Petit, Laird, & Marsden, 2010).
Alternatively, the Undifferentiated Amount Hypothesis suggests that early magnitude mental representations of quantities are based upon the total amount (spatial dimension), not just numbers, which represents both discrete and continuous quantities. Following the Undifferentiated Amount Hypothesis, the whole number bias lies in the difficulty with the symbolic magnitude mental representations of fractions, which suggests that only the system using fraction numerals would be impaired by previous experience with whole numbers but the system using fraction pictorials would not be impaired by previous experience with whole numbers (Ni & Zhou, 2005). As identified by Siegler and colleagues (2011), most theories of numerical development (e.g. evolutionary and conceptual change theories) suggest that there is an early understanding of whole numbers and much later, and often flawed, is the development of fraction concepts and procedures.

As an alternative to the previous theories of numerical development, Siegler and colleagues (2011) suggest an integrated theory of numerical development, which posits that there is a developmental continuum of all types of real numbers, such that individuals progressively acquire new types of real numbers as they understand their magnitudes and how to use these numbers in operations. For example, students will learn to represent whole numbers on a mental number line, eventually this framework can be applied to rational numbers, such as fractions. The theory of numerical development asserts that individuals must learn that not all the properties of whole numbers apply to all real numbers, but unlike many other theories the integrated theory of numerical development suggests that this challenge is just one component of numerical development. This assertion places fraction knowledge as one critical point of numerical development, instead of as a secondary development (Siegler et al., 2011). In support of the integrated theory of numerical development is that training in fraction magnitude
knowledge has been found to support the development of rational numbers (including decimals, percentages, and fractions) in fourth-grade students (Kalchman, Moss, & Case, 2001) and those who were low achieving in mathematics (Fuchs et al., 2013), which suggests that growth in one area of mathematics can support growth in other areas of mathematics. Furthermore, the development of fraction knowledge is important for later mathematics achievement (Bailey et al., 2012; Siegler et al., 2011).

**Procedural and Conceptual: Types of Fraction Knowledge.** In addition to the previously detailed multiple perspectives of fractions, fraction knowledge can be viewed as composed of at least two types: procedural and conceptual fraction knowledge (Hecht & Vagi, 2012). There is a lack of consensus in the literature regarding the definitions, but procedural knowledge is typically defined as actions and manipulations on numbers, and conceptual knowledge as understanding of numerical principles and constructs/ideas (Baroody et al., 2007, pg. 123). Conceptual knowledge includes the understanding of links between facts, procedures, and principles and is commonly measured by explanations of key mathematical ideas or procedures, judging depth and quality of an example, or relating pictorial representations and conceptions to symbolic numbers. Procedural knowledge includes the understanding of how to conduct a series of mathematical steps to accomplish a given goal and is commonly measured by accuracy in executing those steps. Flexibility with procedures is suggested to indicate deep procedural knowledge, such that a full understanding of procedures allows an individual to choose between strategies and/or create new strategies to solve problems. Although the different types of knowledge can be separately conceptualized, most mathematical problems require the use of both procedural and conceptual knowledge (Rittle-Johnson & Schneider, 2015).
Several theories have hypothesized the developmental relationship between conceptual and procedural knowledge. These theories include the concepts-first (Geary, 1994; Gelman & Williams, 1998) and procedures-first (Siegler & Stern, 1998; Sun, Merrill, & Peterson, 2001) views, which suggest one type of knowledge develops first and supports the development of the other type of knowledge. Another theory is the inactivation view, which suggests that conceptual and procedural knowledge develop independent of each other (Resnick & Omanson, 1987). The most supported theory is the iterative model that posits a bidirectional relationship between conceptual and procedural knowledge, such that increases in one type of knowledge leads to increases in the other type of knowledge (Rittle-Johnson & Koedinger, 2009; Rittle-Johnson & Schneider, 2015; Rittle-Johnson et al., 2001). The positive correlations that have been found between procedural and conceptual knowledge, as well as predictive relationships between both procedural and conceptual knowledge, suggest that there is a bidirectional relationship between these two types of numerical knowledge. Although there is likely a bidirectional relationship between procedural and conceptual knowledge, it is not a directly proportional relationship, such that an increase in conceptual knowledge does not necessarily result in growth of the same magnitude in procedural knowledge (Rittle-Johnson & Schneider, 2015).

To evaluate the various theories of the developmental relationship between procedural and conceptual fraction knowledge, researchers have suggested that an investigation of individual differences in fraction procedural and conceptual knowledge would provide evidence to support one or more of these theories (Hallett et al., 2010; Hecht & Vagi, 2012). Hecht and Vagi (2012) examined the individual strengths and weaknesses of fourth and fifth graders’ understanding of common fractions. Classroom testing was conducted with 181 students in fourth grade and again in fifth grade on measures of procedural knowledge, conceptual
knowledge, and fraction skills, as well as a standardized mathematics test in fourth grade (Woodcock Johnson Test of Achievement, 3rd Edition [WJ III]; Woodcock, McGrew, & Mather, 2001). In fourth grade, the results indicated four distinct learning profiles based on grade level expectations: (1) above expected conceptual performance and below expected procedural performance (34% of sample), (2) below expected conceptual performance and above expected procedural performance, (3) below expected conceptual performance and expected procedural performance (those in categories 2 and 3 made up 27% of the sample), and (4) above expected conceptual performance and above expected procedural performance (39% of the sample) (Hecht & Vagi, 2012). Similar clusters were found in an earlier study by Hallett and colleagues (2010) with fourth and fifth grade students in the United Kingdom. In fifth grade, the same clusters were prominent with the addition of a cluster with expected conceptual performance and below expected procedural performance (Hecht & Vagi, 2012). From fourth to fifth grade, almost three-fourths of the students were identified in a different cluster than the cluster they were identified in fourth grade, which suggests that there is little stability in these groupings over time.

Hecht and Vagi (2012) also examined the fraction skills and mathematics achievement of students in the different fraction knowledge clusters. Fraction skills were highest for those high in both procedural and conceptual understanding, followed by the cluster with high conceptual and low procedural knowledge, and lowest for those with lower conceptual and higher procedural knowledge. Many of those with math difficulties, as identified by the WJ III, were in the cluster group with lower conceptual knowledge. In addition, students with higher than expected conceptual and procedural knowledge had greater gains from fourth to fifth grade in fraction word problem and estimation skills than students with above expected procedural knowledge and below expected conceptual knowledge. These results provide support for the
iterative model by indicating that both procedural and conceptual knowledge are important for solving fraction word and estimation problems (Hecht and Vagi, 2012). Furthermore, these results indicate that the theories about development of procedural and conceptual knowledge are not necessarily oppositional to each other because not all students develop in the same manner (e.g. the development of students in the higher procedural and lower conceptual knowledge may better support the procedures-first theoretical model, but students exhibiting higher procedural and conceptual knowledge may better support the bidirectional model of development). One developmental model may not encompass all students’ learning experiences with procedural and conceptual fraction knowledge (Hallett et al., 2010; Hecht & Vagi, 2012). Overall, the results of these studies (Hallett et al., 2010; Hecht & Vagi, 2012) indicate that students individually differ on their procedural and conceptual fraction knowledge and these differences are related to their ability to solve mathematical fraction questions.

Breaking down fraction knowledge to procedural and conceptual knowledge may allow for greater precision in research. Related to the previously detailed research into the contributions of symbolic and non-symbolic mental representations of whole numbers and fractions, it is suggested that these representations positively influence overall mathematics achievement and whole number and fraction arithmetic (Fazio et al., 2014; Siegler & Pyke, 2013), however, the mechanisms of this influence are not yet known. As such, the study of symbolic and non-symbolic magnitude mental representations of whole numbers and fractions to fraction knowledge may benefit from study of their influence specific to fraction knowledge subtypes, one possibility being procedural and conceptual fraction knowledge. Specifically, different types of magnitude mental representations may differentially contribute to specific types of fraction knowledge. This research will aid in the theoretical understanding of
procedural and conceptual fraction development. In addition, this research could be of benefit to educators to aid in the identification of students’ strengths and areas of need (Hecht & Vagi, 2012) and lay groundwork for future research to evaluate effective interventions for students struggling with the procedural and/or conceptual aspects of mathematics fraction education.

The Current Study

Building upon previous theoretical and empirical knowledge, the current study seeks to expand the existing body of knowledge in the area of fraction knowledge of elementary students by examining the individual profiles of students’ magnitude mental representations of whole numbers and fractions, as well as their procedural and conceptual knowledge. The current study seeks to replicate the American findings of Fazio and colleagues (2014) by examining the connection between symbolic and non-symbolic magnitude mental representations of fractions and whole numbers to each other and to overall mathematics achievement with a population of eighth grade Canadian students. Due to the novelty of the non-symbolic fraction magnitude comparison task, a primary goal of this study was to replicate the methods of Fazio and colleagues (2014) on an additional sample. Therefore, the current study examines the contributions of symbolic and non-symbolic magnitude mental representations of fractions and whole numbers to procedural and conceptual fraction knowledge. In addition, the current study builds upon previous research (Hallett et al., 2010; Hecht & Vagi, 2012) by replicating the cluster identification process of procedural and conceptual fraction knowledge and further identifying possible characteristics of these groupings by examining correlations to symbolic and non-symbolic magnitude task performance.
Research Questions

The overall goal of the current study is to determine the relationship between Canadian eighth grade students’ magnitude mental representations of whole numbers and fractions and fraction knowledge. This will be addressed through five specific research questions: (1) Is there a relationship between students’ symbolic magnitude mental representations of whole numbers and fractions? (2) Is there a relationship between student’s non-symbolic magnitude mental representations of whole numbers and fractions? (3) What is the relationship between symbolic and non-symbolic magnitude mental representations of whole numbers and fractions and procedural and conceptual fraction knowledge? (4) What are the clusters of individual differences of procedural and conceptual fraction knowledge of the eighth-grade students? and (5) Do these clusters of students, based on fraction knowledge, differ by performance on symbolic and non-symbolic magnitude mental representations of whole numbers and fractions?

Development of the magnitude mental representation of numbers. Following Fazio and colleagues’ (2014) research, the following two questions seek to replicate their findings and evaluate theoretical models of symbolic and non-symbolic magnitude mental representations development. The first question is: “Is there a relationship between students’ symbolic magnitude mental representations of whole numbers and fractions?” Results for this research question will provide information about the relationship between whole numbers and fractions. Supporters of the whole number bias theory suggest that understanding of whole numbers interferes with acquisition of fraction knowledge (Ni & Zhou, 2005). Alternatively, the integrated theory of numerical development (Siegler et al., 2011) suggests that these two types of mathematical understanding may support each other because of their underlying foundation in magnitude mental representations. Fazio and colleagues (2014) found a positive correlation
between symbolic magnitude mental representations of whole numbers and fractions, such that fifth grade students who performed faster on symbolic whole number magnitude tasks were also faster at symbolic fraction number magnitude tasks, which suggests that there is a relationship between whole number and fraction knowledge. In support of the null hypothesis, results would indicate that there is no correlation between symbolic magnitude mental representations of whole numbers and fractions. In support of the integrated theory of numerical development, the results would indicate that there is a strong correlation between symbolic whole number and fraction magnitude mental representations.

The second question is: “Is there a relationship between student’s non-symbolic magnitude mental representations of whole numbers and fractions?” There is limited research examining the relationship between both types of non-symbolic magnitude mental representations. Although Fazio and colleagues (2014) assess non-symbolic magnitude mental representations of both whole numbers and fractions, correlational analyses were not provided for the relationship between these two types of magnitude mental representations on all the measures used (including reaction time and accuracy). In support of the null hypothesis, results would indicate that there is no correlation between non-symbolic magnitude mental representations of whole numbers and fractions. In support of the integrated theory of numerical development, the results would indicate that there is a strong correlation between non-symbolic magnitude mental representations of whole numbers and fractions.

Relationship Between Magnitude Mental Representations of Numbers and Fraction Skills.

The third research question is: “What is the relationship between symbolic and non-symbolic magnitude mental representations of whole numbers and fractions and procedural and conceptual fraction knowledge?” The third research question will provide information regarding
the relationship between magnitude mental representations and fraction knowledge. Previous research has identified that both symbolic and non-symbolic magnitude mental representations relate to mathematical achievement (Siegler et al., 2011). Fazio and colleagues (2014) found that each symbolic and non-symbolic magnitude mental representations were significantly related to each other and each uniquely predicted variance in mathematical achievement, which suggests that both numerical magnitude mental representations support mathematical development and may support different areas of this development. Examining procedural and conceptual fraction knowledge may help differentiate the areas of development that each magnitude mental representations primarily supports. Using a theory of the bidirectional relationship between symbolic and non-symbolic magnitude mental representations, it is hypothesized that performance on symbolic and non-symbolic magnitude comparison tasks will be related. Further, it is predicted that each magnitude mental representational system will uniquely predict outcomes in the two mathematics achievement measures (procedural and conceptual fraction knowledge), because a bidirectional relationship would suggest that each magnitude mental representational system independently supports areas of mathematics, as well as conjointly working to support mathematical skills. In addition, support for the whole number bias would be identified if results indicate a greater relationship between magnitude mental representations of whole numbers and fraction knowledge (of either type), as opposed to magnitude mental representations of fractions. Alternatively, support for the integrated theory of numerical development would be garnered if results indicate that both magnitude mental representations of whole numbers and fractions are predictive of both conceptual and procedural fraction knowledge. Finally, support for the null hypothesis would be identified if the results
indicate that there is no relationship between magnitude mental representations of whole
numbers and fractions and procedural and conceptual knowledge.

**Individual differences in procedural and conceptual knowledge.** The fourth question is “What are the clusters of individual differences of procedural and conceptual fraction knowledge of the eighth-grade students?” Of particular interest is identifying if the clusters found in the current sample are similar to or different than the clusters found by previous research. As previously noted, research identifying clusters of differences in procedural and conceptual fraction knowledge of elementary grade students have found the following clusters: high conceptual and low procedural knowledge, low conceptual and high procedural knowledge, low conceptual and expected levels of procedural knowledge, and high conceptual and high procedural knowledge (Hallett et al., 2010; Hecht & Vagi, 2012). It is hypothesized that similar clusters will be found in the current research study.

The fifth question asks whether these obtained clusters of students, based on fraction knowledge, differ by performance on symbolic and non-symbolic magnitude mental representations of whole numbers and fractions tasks? It is possible that certain types of magnitude mental representations (symbolic or non-symbolic) are more associated with a type of fraction knowledge (procedural or conceptual). It has been hypothesized that numerical magnitude mental representations are highly linked to conceptual knowledge, because the magnitude mental representations provide a basis for understanding numbers and quantity (Hecht & Vagi, 2012). However, research has identified that the link between numerical magnitude mental representations and conceptual or procedural knowledge is enmeshed and difficult to separate (Siegler et al., 2011). The current study seeks to untangle these associations by examining the individual differences of procedural and conceptual knowledge and association of
these clusters to symbolic and non-symbolic magnitude mental representations of whole numbers and fractions, as well as examining the direct connection between both numerical magnitude mental representations and procedural and conceptual fraction knowledge. Following a null hypothesis, the results would indicate that clusters of fraction knowledge types do not differ by symbolic/non-symbolic magnitude mental representations of whole numbers and fractions. It is hypothesized that clusters high in conceptual knowledge will perform better than clusters low in conceptual knowledge on measures of non-symbolic magnitude mental representations of whole numbers and fractions because of the common relation to basic magnitude knowledge necessary for conceptual fraction knowledge. In addition, it is hypothesized that clusters high in procedural knowledge and low in conceptual knowledge will be more accurate on measures of symbolic, as compared to non-symbolic, magnitude mental representations of whole numbers and fractions, because procedures focus on the use of symbolic numbers, as opposed to understanding of quantity. Support for the whole number bias would be garnered if those low in both types of fraction knowledge perform better on whole number magnitude mental representational tasks than fraction magnitude mental representational tasks. In addition, support for the whole number bias would be garnered if those high in both types of fraction knowledge perform better on fraction magnitude mental representational tasks than those low in both types of fraction knowledge because this would suggest that those low in fraction knowledge may be using whole number magnitude mental representations and knowledge to solve fraction problems. Alternatively, support for the integrated theory of numerical development would be garnered if results indicated that those high in both types of fraction knowledge performed better on both fraction and whole number magnitude mental representational tasks than those low in both types of fraction knowledge.
CHAPTER 3: METHODS

Participants

Thirty schools in the Southern Alberta area were contacted to request consent for participation in the current research study. Seven principals of schools in this district consented to allow their teachers and eighth grade students to participate in the current research study. Teachers were given the option to participate in the research study; at which point nine teachers from six schools consented to participate in the research study. Teachers were given consent forms and information letters to send home with their students. Seventy students (average age = 13.97 years) participated in the research study (32 males and 38 females).

Measures

Fraction test. Students’ conceptual and procedural knowledge of fractions was assessed in a classroom test. The fraction test (see Appendix C for the fraction test as it was given to the students and Appendix D for details about how the questions are conceptualized and citations about where the questions were derived from in the literature) was developed through a literature review and three expert review sessions. Experts consisted of three university professors in the field of mathematics education. Experts were consulted regarding the level of difficulty of the questions, type of fraction knowledge (i.e. conceptual and procedural knowledge) that the questions assess, inclusion and exclusion of questions, wording of the questions, scoring of the questions, and how to present the questions (i.e. order and number of parts). Students’ conceptual fraction knowledge was assessed by asking them to colour in a specific fraction of given shapes (3 questions), choose images that show a specified fraction (2 questions), identify the fraction displayed in an image (1 question), order given fractions from smallest to largest (1 question), make fractions equivalent by filling in the missing number (3 questions), estimate (1 question),
and solve word problems (3 multiplication questions). Procedural knowledge was assessed with several computation questions: (1) proper fractions with same denominators (1 addition, 1 subtraction, and 2 multiplication); (2) proper fractions with different denominators (1 addition, 1 subtraction, and 1 multiplication); (3) one mixed fraction and one proper fraction with same denominators (1 addition); (4) one mixed fraction and one proper fraction with different denominators (1 subtraction, and 1 multiplication); and (5) one word problem (1 subtraction). The fraction test covers many of the multiple perspectives of fractions, including part-whole, ratio, operator, and measure perspectives (Moseley & Okamoto, 2008). It is important to note that attention was given in the task design to best delineate conceptual and procedural fraction knowledge (e.g., expert review). That said, as was outlined in Chapter 2, it is acknowledged that most mathematical problems require the use of both procedural and conceptual knowledge (Rittle-Johnson & Schneider, 2015). As this paper-and-pencil task did was not accompanied by student interviews, it is possible that students may have used an approach other than intended to correctly respond to a particular question (e.g., procedural approach used to solve a question targeted conceptual knowledge).

**Magnitude mental representational tasks.** Symbolic and non-symbolic magnitude mental representations of whole numbers and fractions were assessed with magnitude comparison tasks on a computer. The symbolic magnitude comparison tasks (tasks and procedures replicated from Fazio et al., 2014) presents the student with two numbers, ranging from 5-21 for the whole number task and having denominators that range from 5-21 for the fraction task. The numbers remain on the screen until the student presses ‘z’ on the computer keyboard if the number on the left is larger and ‘?’ if the number of the right is larger. Forty trials for each of the whole number and fraction conditions were presented. For both the whole
number and fraction tasks, 10 sets of numbers were generated “from each of the 4 ratio bins: 1.15-1.28, 1.28-1.43, 1.48-1.65, and 2.46-2.71,” with ratios calculated by dividing the larger number by the smaller number of the set (Fazio et al., 2014, pg 60). The quantities in the non-symbolic magnitude comparison tasks were matched to the numbers used in the symbolic magnitude comparison conditions (Fazio et al.).

The whole number non-symbolic magnitude comparison task (downloaded from www.panamath.org, procedures replicated from Fazio et al., 2014) consisted of 186 trials (2 practice and 46 trials from each type of ratio bin), half of these trials had equivalent dot size for each set and the other half had the dots cover an equivalent overall area. For each trial, participants were presented with a set of blue dots on the left side and a set of yellow dots on the right side for 1382 ms, followed by 500ms of a backward mask of yellow and blue dots, which then fades to a blank grey screen until the student responds. At any point after the dots are presented (including after the backward mask has disappear) students may indicate their response using the keyboard by pressing ‘d’ for the left and ‘k’ for the right side.

For the fraction non-symbolic magnitude comparison task (task and procedures replicated from Fazio et al., 2014), the students were instructed that the blue candies taste the best and they are going to reach into a bag and pull out one candy; therefore, they should pick the side “that gives them the best chance of getting a blue candy.” In addition, they were told that the size of the candy does not matter. There were 182 trials in the fraction task (2 practice trials and 45 of each of the four ratio types), half of these trials had equivalent dot size for each set and the other half had the dots cover an equivalent overall area. A trial presents two sets of blue and yellow dots (as seen in Figure 2) for 2 seconds, followed by a mask that covers the entire screen in small yellow and blue dots until the study responded. Students could respond at any point during the
trial, using the keyboard by pressing ‘d’ for the left and ‘k’ for the right side. It is important to note that this task assesses an individual’s understanding of fractions as ratio.

![Figure 2](image-url)

*Figure 2.* Example of one trial of the non-symbolic fraction magnitude comparison task.

**Data Collection Procedure**

After parental consent was obtained, the students were taken individually or in pairs to a separate testing room, provided an explanation of the study, and their assent was obtained. Testing began with the magnitude comparison tasks, with students tested individually or in pairs on separate computers. The students completed two of the magnitude comparison tasks (one symbolic and one non-symbolic of either whole number or fraction tasks) and completed the other two magnitude comparison tasks on a second session (approximately 10 to 20 minutes per session). Tasks were presented to the students in one of the four following orders: (1) symbolic whole number, non-symbolic whole number, symbolic fraction, and non-symbolic fraction magnitude comparison tasks (N=18); (2) non-symbolic whole number, symbolic whole number, non-symbolic fraction, and symbolic fraction magnitude comparison tasks (N=18); (3) symbolic fraction, non-symbolic fraction, symbolic whole number, and non-symbolic whole number magnitude comparison tasks (N=17); (4) non-symbolic fraction, symbolic fraction, non-symbolic whole number, and symbolic whole number magnitude comparison tasks (N=17). The presentation of whole number tasks and fraction tasks was randomized to ensure 50% of participants received the whole number tasks first and fraction tasks second, as well as,
randomized for the presentation of symbolic and non-symbolic tasks to ensure 50% of participants received the symbolic tasks first and non-symbolic tasks second.

After the magnitude comparison task data was collected and on a different day, students were given a 20-40 minute fraction test. Teachers were given the option to administer the test themselves with all of their students in the classroom or to have the researcher pull out the students into a separate classroom and complete the fraction test. For the teachers that chose to administer the test themselves (3 teachers chose this option, which involved 25 participating students), all of their students (i.e. participating and non-participating) completed the fraction test; the researcher photocopied the fraction tests of the participating students’ fraction tests and all of the fraction tests were given to the teacher.

**Analyses Procedures**

For the magnitude comparison tasks, preliminary analyses were conducted on students’ reaction times, such that each students’ reaction times were separately converted to z-scores, and reaction times greater than 2.5 standard deviations from the student’s mean reaction time were removed (used by Fazio et al., 2014). Accuracy measures for the magnitude comparison tasks were calculated as the average of correct verses incorrect responses. To remove a degree of shared variance, conceptual and procedural fraction knowledge were broken down into residuals though linear regression, such that conceptual knowledge residuals were obtained by regressing conceptual knowledge on procedural knowledge and vice versa for procedural knowledge (Hallett et al., 2010; Hecht & Vagi, 2012). The residuals indicate the level that one type of fraction knowledge is at based on what is predicted based on their performance on the other type of fraction knowledge. For example, a student with a lower procedural residual has lower procedural knowledge than what is predicted by their conceptual
knowledge. Qualitative descriptors assigned to the residual scores were influenced by those used in previous studies (Hallett et al., 2010, 2012; Hecht and Vagi, 2012), with a residual of 0 indicating that the score of one type of knowledge is the same as the predicted value of one type of fraction knowledge based on the score on the other type of fraction knowledge. Residuals at or lower than -1.5 are classified as much lower, residuals between -0.5 and -1.5 are classified as lower, residuals between -0.5 and +0.5 are classified as expected, residuals between +0.5 and +1.5 are classified as higher, and residuals at or higher than +1.5 are classified as much higher.

A correlational analysis was used to assess if there is a relationship between the magnitude mental representations of symbolic whole numbers and fractions and non-symbolic whole numbers and fractions, which addresses the first and second research questions. Regression analyses were used to determine the unique contributions of symbolic and non-symbolic magnitude mental representations of whole numbers and fractions to conceptual and procedural fraction knowledge and overall mathematics achievement, which will address the third research question.

Using the two residualized variables (conceptual and procedural), a hierarchical cluster analysis (Ward’s method) was conducted to identify individual differences between the students on these measures (Hallett et al., 2010; Hecht & Vagi, 2012; Tan, Steinbach, & Kumar, 2005), which answers the fourth research question. The clusters were compared on magnitude mental representations of symbolic and non-symbolic whole numbers and fractions. Significant omnibus tests were followed-up with Mann-Whitney U tests (Hecht & Vagi, 2012), which answers the fifth research question.
CHAPTER 4: RESULTS

Chapter four details the results of the current study. First, the preliminary analyses are described, then the descriptive statistics are presented for each task, followed by the results of analyses to answer each research question in order.

Preliminary Analyses

For the tasks with reaction time (RT) data, trials that were greater than or equal to 2.5 standard deviations from the students’ mean RT were deleted (0-18% of trials were deleted for each student) and a mean RT was calculated for each student (as used by Fazio et al., 2014). Analyses of normality were conducted and appropriate non-parametric analyses were used to accommodate for non-normal distributions of variables. Similar to Fazio and colleagues (2014), all of the magnitude comparison tasks have high reliabilities (as measured by Cronbach’s alpha, symbolic fraction magnitude comparison task, alpha = .94; symbolic whole number magnitude comparison task, alpha = .93; non-symbolic fraction magnitude comparison task, alpha = .99; non-symbolic whole number magnitude comparison task, alpha = .99). In addition, the fraction test measures had reliabilities in the acceptable range (as measured by Cronbach’s alpha, overall fraction test, alpha = .85; conceptual fraction knowledge, alpha = .81; procedural fraction knowledge, alpha = .70).

Descriptive Statistics

Table 1 provides the means, standard deviations, and ranges for the accuracy and RT variables of the magnitude comparison tasks, as well as the procedural, conceptual, and overall fraction test scores. Accuracy data is an average of correct (i.e. 1) and incorrect (i.e. 0) responses, such that a score of .95 is equivalent to answering 95% of the trials correctly. Reaction time data is recorded in milliseconds. Fraction test scores (overall, procedural, and
conceptual) are calculated by an average of correct (i.e. 1), partially correct (i.e. .5, used on two questions on the conceptual fraction knowledge measure), and incorrect (i.e. 0), such that a score of .74 is equivalent to scoring 74% on the questions. The fraction test has a total of 24 questions: 10 questions are predominantly procedural and 14 questions are predominantly conceptual.
Table 1

*Descriptive Statistics for Magnitude Comparison Tasks and Fraction Test Variables.*

<table>
<thead>
<tr>
<th>Variables</th>
<th>Mean</th>
<th>SD</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>SWNMC accuracy</td>
<td>.96</td>
<td>.05</td>
<td>.76 - 1.00</td>
</tr>
<tr>
<td>NSWNMC accuracy</td>
<td>.90</td>
<td>.08</td>
<td>.49 - .98</td>
</tr>
<tr>
<td>SFMC accuracy</td>
<td>.84</td>
<td>.16</td>
<td>.36 – 1.00</td>
</tr>
<tr>
<td>NSFMC accuracy</td>
<td>.80</td>
<td>.13</td>
<td>.13 - .96</td>
</tr>
<tr>
<td>SWNMC RT</td>
<td>696.45 ms</td>
<td>157.31 ms</td>
<td>399.77 – 1230.37 ms</td>
</tr>
<tr>
<td>NSWNMC RT</td>
<td>813.01 ms</td>
<td>332.58 ms</td>
<td>307.15 – 1943.84 ms</td>
</tr>
<tr>
<td>SFMC RT</td>
<td>2392.34 ms</td>
<td>1050.91 ms</td>
<td>986.34 – 6578.69 ms</td>
</tr>
<tr>
<td>NSFMC RT</td>
<td>1463.52 ms</td>
<td>539.67 ms</td>
<td>689.64 – 3191.26 ms</td>
</tr>
<tr>
<td>Procedural</td>
<td>.60</td>
<td>.24</td>
<td>.10 – 1.00</td>
</tr>
<tr>
<td>Conceptual</td>
<td>.71</td>
<td>.21</td>
<td>.14 – 1.00</td>
</tr>
<tr>
<td>Fraction Test</td>
<td>.66</td>
<td>.20</td>
<td>.17 – 1.00</td>
</tr>
</tbody>
</table>

*Notes.* Fraction Test is the overall score on the fraction test, Procedural refers to the procedural knowledge questions on the fraction test, Conceptual refers to the conceptual knowledge questions on the fraction test, SWNMC accuracy is the accuracy measure of symbolic whole number magnitude comparison task, NSWNMC accuracy is the accuracy measure of non-symbolic whole number magnitude comparison task, SFMC accuracy is the accuracy measure of symbolic fraction magnitude comparison task, NSFMC accuracy is the accuracy measure of non-symbolic fraction magnitude comparison task, SWNMC RT is the RT measure of symbolic whole number magnitude comparison task, NSWNMC RT is the RT measure of non-symbolic whole number magnitude comparison task, SFMC RT is the RT measure of symbolic fraction magnitude comparison task, NSFMC RT is the RT measure of non-symbolic fraction magnitude comparison task.
Order. Students were randomly assigned one of four orderings of the magnitude comparison tasks to balance effects of task performance changes due to ordering. The differences on the magnitude comparison tasks by ordering groups were analyzed with Kruskal-Wallis H test due to non-normal distributions of the variables. Results indicated that students’ accuracy on the symbolic fraction magnitude comparison task ($\chi^2(3) = 12.03, p = 0.007$) differed by the order of the tasks, as well as the RT performance on the symbolic fraction magnitude comparison task ($\chi^2(3) = 16.62, p = 0.001$) and the non-symbolic whole number magnitude comparison task ($\chi^2(3) = 14.51, p = 0.002$). For the accuracy performance on the symbolic fraction magnitude comparison task, students were statistically significantly less accurate when presented with this task third (order 1) instead of first (order 3; $U = 56.50, p = .001$), second (order 4; $U = 76.50, p = .010$), or fourth (order 2; $U = 80.50, p = .009$). For the RT performance of symbolic fraction magnitude comparison task, students were statistically significantly faster when presented with this task third (order 1) instead of first (order 3; $U = 57.00, p = .001$) or second (order 4; $U = 68.00, p = .004$), as well as statistically significantly faster when presented with this task fourth (order 2) instead of first (order 3; $U = 72.00, p = .007$) or second (order 4; $U = 82.00, p = .019$). For the RT performance of non-symbolic whole number magnitude comparison task, students were statistically significantly slower when presented with this task first (order 2) instead of second (order 1; $U = 96.00, p = .037$), third (order 4; $U = 65.00, p = .003$) or fourth (order 3; $U = 53.00, p = .001$). These results indicate that there was an order effect and the randomization of the presentation of tasks was important to the current study.

Development of the Magnitude Mental Representations of Numbers.

The first and second research questions ask, “is there a relationship between students’ symbolic magnitude mental representations of whole numbers and fractions?” and “is there a
relationship between student’s non-symbolic magnitude mental representations of whole numbers and fractions?” To answer the first and second research questions, t tests and correlational analyses were performed on the symbolic and non-symbolic whole number and fraction magnitude comparison tasks’ accuracy and RT. Table 2 and 3 depict the correlation tables between the magnitude comparison tasks and fraction test for both accuracy data and RT data for the magnitude comparison tasks.

**Accuracy.** Students were significantly more accurate on the whole number tasks (symbolic: \( M=0.96, SD= 0.05 \); non-symbolic: \( M=0.90, SD=0.08 \)) than the fraction tasks (symbolic: \( M=0.84, SD= 0.16 \); non-symbolic: \( M=0.80, SD=0.13 \)) for both symbolic and non-symbolic tasks (symbolic: \( t(69)= -6.12, p < .001 \); non-symbolic: \( t(69)= -7.33, p < .001 \)).

Students’ accuracy on non-symbolic fraction and symbolic fraction tasks was positively correlated (\( r=.57, p < .001 \)), as well as accuracy on non-symbolic fraction and whole number magnitude comparison tasks (\( r=.47, p < .001 \)), and accuracy on the non-symbolic whole number task and symbolic fraction task (\( r=.39, p = .001 \)). However, students’ accuracy on the symbolic whole number task was not correlated with accuracy on any of the other tasks, this result could be due to the ceiling effect of the symbolic whole number task.

**Reaction time.** Students were significantly faster on the whole number tasks (symbolic: \( M=696.45, SD= 157.31 \); non-symbolic: \( M=813.01, SD=332.58 \)) than the fraction tasks (symbolic: \( M=2392.34, SD= 1050.91 \); non-symbolic: \( M=1463.52, SD=539.67 \)) for both symbolic and non-symbolic tasks (symbolic: \( t(69)= 19.04, p < .001 \); non-symbolic: \( t(69)= 11.13, p < .001 \)). Students’ RT on non-symbolic fraction and symbolic fraction was positively correlated (\( r=.51, p < .001 \)), as well on RT of non-symbolic whole number and symbolic whole number tasks (\( r=.47, p < .001 \), non-symbolic fraction and non-symbolic whole number tasks (\( r=.45, p <
.001), and RT on the symbolic whole number and non-symbolic fraction tasks ($r=.24, p = .042$). However, students’ RT on the symbolic fraction task was not correlated with RT of the symbolic whole number ($r=.13, p = .276$) and non-symbolic whole number ($r=-0.12, p = .920$) tasks.
Table 2

Correlations Among the Fraction Test Variables and Accuracy Measures of the Magnitude Comparison Tasks.

<table>
<thead>
<tr>
<th>Variables</th>
<th>Fraction Test</th>
<th>Procedural Knowledge</th>
<th>Conceptual Knowledge</th>
<th>SW</th>
<th>NSW</th>
<th>SF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fraction Test</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Procedural Knowledge</td>
<td>.87**</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Conceptual Knowledge</td>
<td>.93**</td>
<td>.65**</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SW</td>
<td>-.00</td>
<td>.13</td>
<td>-.11</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>NSW</td>
<td>.38*</td>
<td>.42**</td>
<td>.29*</td>
<td>.15</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SF</td>
<td>.61**</td>
<td>.53**</td>
<td>.57**</td>
<td>-.02</td>
<td>.39*</td>
<td></td>
</tr>
<tr>
<td>NSF</td>
<td>.49**</td>
<td>.41**</td>
<td>.48**</td>
<td>.10</td>
<td>.47**</td>
<td>.57**</td>
</tr>
</tbody>
</table>

Notes. SW is the accuracy measure of symbolic whole number magnitude comparison task, NSW is the accuracy measure of non-symbolic whole number magnitude comparison task, SF is the accuracy measure of symbolic fraction magnitude comparison task, and NSF is the accuracy measure of non-symbolic fraction magnitude comparison task.

* $p < .05$.

** $p < .001$.
Table 3

*Correlations Among the Fraction Test Variables and RT Measures of the Magnitude Comparison Tasks.*

<table>
<thead>
<tr>
<th>Variables</th>
<th>Fraction Test</th>
<th>Procedural Knowledge</th>
<th>Conceptual Knowledge</th>
<th>SW</th>
<th>NSW</th>
<th>SF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fraction Test</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Procedural Knowledge</td>
<td>.89**</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Conceptual Knowledge</td>
<td>.93**</td>
<td>.65**</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SW</td>
<td>-.23</td>
<td>-.13</td>
<td>-.28*</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>NSW</td>
<td>.01</td>
<td>.02</td>
<td>.00</td>
<td>.47**</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SF</td>
<td>.14</td>
<td>.16</td>
<td>.09</td>
<td>.13</td>
<td>-.01</td>
<td></td>
</tr>
<tr>
<td>NSF</td>
<td>.44**</td>
<td>.44**</td>
<td>.38**</td>
<td>.24*</td>
<td>.45**</td>
<td>.51**</td>
</tr>
</tbody>
</table>

*Notes.* SW is the RT measure of symbolic whole number magnitude comparison task, NSW is the RT measure of non-symbolic whole number magnitude comparison task, SF is the RT measure of symbolic fraction magnitude comparison task, and NSF is the RT measure of non-symbolic fraction magnitude comparison task.

* *p < .05.
** *p < .001
Relationship Between Magnitude Mental Representations of Numbers and Fraction Skills.

The following research questions examine procedural and conceptual fraction knowledge. The number of correct and partially correct questions were added together and divided by the number of questions associated with the category (i.e. procedural, conceptual, or total). As displayed in Table 1, there was a large range in scores on the fraction test (procedural: range = 0.10 – 1.00, $M = 0.60, SD = 0.24$; conceptual: range = 0.14 – 1.00, $M = 0.71, SD = 0.21$; overall fraction test: range = 0.17 – 1.00, $M = 0.66, SD = 0.20$). As displayed in Table 2 and 3, there were statistically significant positive strong correlations between procedural and conceptual fraction knowledge ($r = .65, p < .001$). The correlation between procedural and conceptual fraction knowledge likely represents the difficulty in separating and individually assessing each type of knowledge (Rittle-Johnson & Schneider, 2015).

The third research question asks, “What is the relationship between symbolic and non-symbolic magnitude mental representations of whole numbers and fractions and procedural and conceptual fraction knowledge?” To answer the third research question, four linear multiple regression analyses were run, two with procedural fraction knowledge as the dependent variable and two with conceptual fraction knowledge as the dependent variable, one of each of these sets was run with the four predictor variables of accuracy (for each of the symbolic and non-symbolic whole number magnitude comparison tasks and the symbolic and non-symbolic fraction magnitude comparison tasks) and the other was run with the four predictor variables of RT (for each the symbolic and non-symbolic whole number magnitude comparison tasks and the symbolic and non-symbolic fraction magnitude comparison tasks).

**Procedural knowledge.** All together, the accuracy on the symbolic and non-symbolic magnitude comparison tasks (for both fractions and whole numbers) accounted for 35% of the
variance in procedural knowledge ($R^2 = .345, F(4, 65) = 8.56, p < .001$), however when examining each task separately, only the accuracy on the symbolic fraction magnitude comparison task significantly predicted performance on the procedural knowledge questions ($\beta = .41, t(69)=3.31, p = .002$).

All together, the RT on the symbolic and non-symbolic magnitude comparison tasks (for both fractions and whole numbers) accounted for 27% of the variance in procedural knowledge ($R^2 = .273, F(4, 65) = 6.10, p < .001$), however when examining each task separately, only the RT on the non-symbolic fraction magnitude comparison task significantly predicted performance on the procedural knowledge questions ($\beta = .64, t(69)=4.39, p < .001$).

**Conceptual knowledge.** All together, the accuracy on the symbolic and non-symbolic magnitude comparison tasks (for both fractions and whole numbers) accounted for 38% of the variance in conceptual knowledge ($R^2 = .376, F(4, 65) = 9.79, p < .001$), however when examining each task separately, only the accuracy on the symbolic fraction magnitude comparison task significantly predicted performance on the conceptual knowledge questions ($\beta = .42, t(69)=3.43, p = .001$). All together, the RT on the symbolic and non-symbolic magnitude comparison tasks (for both fractions and whole numbers) accounted for 30% of the variance in conceptual knowledge ($R^2 = .301, F(4, 65) = 6.99, p < .001$), however when examining each task separately, only the RT on the non-symbolic fraction magnitude comparison task ($\beta = .59, t(69)=4.09, p < .001$) and symbolic whole number magnitude comparison task ($\beta = -.35, t(69)=-2.93, p = .005$) significantly predicted performance on the conceptual knowledge questions.
Individual Differences in Procedural and Conceptual Knowledge.

The fourth question asks “What are the clusters of individual differences of procedural and conceptual fraction knowledge of the eighth-grade students?” Standardized residuals of procedural and conceptual fraction knowledge were calculated (standardized residuals were calculated with a linear regression of each type of fraction knowledge against the other type of fraction knowledge) before conducting cluster analysis. To answer the fourth research question, Ward’s method of cluster analysis, a hierarchical cluster analysis, was used on standardized residualized procedural and conceptual fraction knowledge variables (as used by Hallett et al., 2010 and Hecht & Vagi, 2012) because Ward’s method limits the amount of clusters with few members (Everitt, Landau, & Leese, 2001). Cluster solutions and F-statistic ratio (displayed in Table 4) were determined for two clusters up to seven clusters based on the clustering of residualized scores of both procedural and conceptual fraction knowledge variables. An ideal clustering is one that has large variance between clusters and small variance within the clusters, which can be calculated through a MANOVA by using the cluster groupings as the independent variables and the procedural and conceptual fraction knowledge standardized residualized scores as dependent variables (method created by Calinski & Harabasz, 1974; method also used by Hecht & Vagi, 2012).
Table 4

\textit{F-statistic Ratio for Different Cluster Solutions.}

<table>
<thead>
<tr>
<th>Clusters</th>
<th>$F$</th>
<th>$p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 Clusters</td>
<td>54.23</td>
<td>&lt; .001</td>
</tr>
<tr>
<td>6 Clusters</td>
<td>54.40</td>
<td>&lt; .001</td>
</tr>
<tr>
<td>5 Clusters</td>
<td>55.42*</td>
<td>&lt; .001</td>
</tr>
<tr>
<td>4 Clusters</td>
<td>52.06</td>
<td>&lt; .001</td>
</tr>
<tr>
<td>3 Clusters</td>
<td>53.16</td>
<td>&lt; .001</td>
</tr>
<tr>
<td>2 Clusters</td>
<td>46.05</td>
<td>&lt; .001</td>
</tr>
</tbody>
</table>

Note: Clusters are based on the grouping of students’ scores in both procedural and conceptual fraction knowledge. The * indicates the cluster solution with the highest $F$-statistic ratio, thus suggesting that it has the least variance within the clusters and the most variance between the clusters, as compared to other cluster solutions.

The five-cluster solution was identified as being a valid clustering of procedural and conceptual knowledge (see Table 4), which aligns with the number of clusters identified by previous researchers (Hallett et al., 2010, 2012; Hecht and Vagi, 2012). The clusters in the five-cluster solution (pictured in Figure 3) can be classified as: (1) higher procedural knowledge and expected conceptual knowledge (N=23), (2) lower procedural knowledge and higher conceptual knowledge (N=18), (3) expected procedural knowledge and expected conceptual knowledge (N=13), (4) much higher procedural knowledge and much lower conceptual knowledge (N=7), (5) expected procedural knowledge and lower conceptual knowledge (N=9).
The residuals indicate the level that one type of fraction knowledge is at based on what is predicted based on their performance on the other type of fraction knowledge. For example, a student with a lower procedural residual has lower procedural knowledge than what is predicted by their conceptual knowledge.

![Figure 3. Five-cluster solution with average procedural and conceptual knowledge standardized residual scores for each cluster. As outlined earlier, the assignment of qualitative descriptors to the various ranges of residual scores was influenced by classifications used in previous studies (Hallett et al., 2010, 2012; Hecht and Vagi, 2012), such that residuals at or lower than -1.5 are classified as much lower, residuals between -0.5 and -1.5 are classified as lower, residuals between -0.5 and +0.5 are classified as expected, residuals between +0.5 and +1.5 are classified as higher, and residuals at or higher than +1.5 are classified as much higher.](image)

The fifth research question asks, “do these clusters of students, based on fraction knowledge, differ by performance on symbolic and non-symbolic magnitude mental
representations of whole numbers and fractions?” To answer the fifth research question, Kruskall-Wallis analyses (due to the non-normal distributions of the magnitude comparison task variables) were run to determine if cluster membership (for the five-cluster solution) has an effect on symbolic and non-symbolic whole number and fraction magnitude comparison tasks performance (both accuracy and RT). The Kruskall-Wallis tests indicated that there was a statistically significant difference between the five clusters on their accuracy on the non-symbolic fraction magnitude comparison task ($\chi^2(4) = 15.75, p = 0.003$), the symbolic fraction magnitude comparison task ($\chi^2(4) = 14.87, p = 0.005$), and the non-symbolic whole number magnitude comparison task ($\chi^2(4) = 11.38, p = 0.023$); as well as their speed on the non-symbolic fraction magnitude comparison task ($\chi^2(4) = 9.93, p = 0.042$).

For the non-symbolic fraction magnitude comparison task, cluster 1 ($M = 0.85, SD = 0.08$) is statistically significantly more accurate than cluster 2 ($M = 0.76, SD = 0.10; U = 90.00, p = .002$) and cluster 5 ($M = 0.71, SD = 0.22; U = 23.00, p = .001$), such that those higher in procedural fraction knowledge (i.e. cluster 1, which has expected conceptual knowledge) performed more accurately on the non-symbolic fraction magnitude comparison task than those lower in procedural knowledge (i.e. cluster 2, which has higher conceptual knowledge) or at expected levels of procedural knowledge (i.e. cluster 5, which has lower conceptual knowledge). In addition, for the non-symbolic fraction magnitude comparison task, cluster 3 ($M = 0.84, SD = 0.09$) is statistically significantly more accurate than cluster 2 ($M = 0.76, SD = 0.10; U = 56.50, p = .015$) and cluster 5 ($M = 0.71, SD = 0.22; U = 23.00, p = .018$), such that those with expected levels of both procedural and conceptual knowledge (i.e. cluster 3) performed more accurately on the non-symbolic fraction magnitude comparison task than those lower in procedural knowledge.
knowledge and higher in conceptual knowledge (i.e. cluster 2) or at expected levels of procedural knowledge and lower in conceptual knowledge (i.e. cluster 5).

With respect to speed on the non-symbolic fraction magnitude comparison task, cluster 1 ($M = 1594.27\text{ms}, SD = 514.26\text{ms}$) is statistically significantly slower than cluster 2 ($M = 1236.55\text{ms}, SD = 405.02\text{ms}; U = 122.00, p = .026$) and cluster 5 ($M = 1166.05\text{ms}, SD = 239.81\text{ms}; U = 46.00, p = .016$), such that those higher in procedural fraction knowledge (i.e. cluster 1, which has expected conceptual knowledge) performed slower on the non-symbolic fraction magnitude comparison task than those lower in procedural knowledge (i.e. cluster 2, which has higher conceptual knowledge) or at expected levels of procedural knowledge (i.e. cluster 5, which has lower conceptual knowledge). In addition, for the non-symbolic fraction magnitude comparison task, cluster 3 ($M = 1677.52\text{ms}, SD = 649.55\text{ms}$) is statistically significantly slower than cluster 2 ($M = 1236.55\text{ms}, SD = 405.02\text{ms}; U = 67.00, p = .045$) and cluster 5 ($M = 1166.05\text{ms}, SD = 239.81\text{ms}; U = 22.00, p = .015$), such that those with expected levels of both procedural and conceptual knowledge (i.e. cluster 3) performed slower on the non-symbolic fraction magnitude comparison task than those lower in procedural knowledge and higher in conceptual knowledge (i.e. cluster 2) or at expected levels of procedural knowledge and lower in conceptual knowledge (i.e. cluster 5).

For the symbolic fraction magnitude comparison task, cluster 1 ($M = 0.89, SD = 0.11$) is statistically significantly more accurate than cluster 2 ($M = 0.80, SD = 0.16; U = 122.00, p = .025$) and cluster 5 ($M = 0.68, SD = 0.22; U = 36.00, p = .005$), such that those higher in procedural fraction knowledge (i.e. cluster 1, which has expected conceptual knowledge) performed more accurately on the non-symbolic fraction magnitude comparison task than those lower in procedural knowledge (i.e. cluster 2, which has higher conceptual knowledge) or at
expected levels of procedural knowledge (i.e. cluster 5, which has lower conceptual knowledge). In addition, for the non-symbolic fraction magnitude comparison task, cluster 3 \((M = 0.92, SD = 0.07)\) is statistically significantly more accurate than cluster 2 \((M = 0.80, SD = 0.16; U = 52.50, p = .010)\) and cluster 5 \((M = 0.68, SD = 0.22; U = 15.50, p = .004)\), such that those with expected levels of both procedural and conceptual knowledge (i.e. cluster 3) performed more accurately on the non-symbolic fraction magnitude comparison task than those lower in procedural knowledge and higher in conceptual knowledge (i.e. cluster 2) or at expected levels of procedural knowledge and lower in conceptual knowledge (i.e. cluster 5).

For the non-symbolic whole number magnitude comparison task, cluster 1 \((M = 0.93, SD = 0.04)\) is statistically significantly more accurate than cluster 2 \((M = 0.86, SD = 0.13; U = 107.00, p = .008)\) and cluster 5 \((M = 0.88, SD = 0.05; U = 42.00, p = .010)\), such that those higher in procedural fraction knowledge (i.e. cluster 1, which has expected conceptual knowledge) performed more accurately on the non-symbolic fraction magnitude comparison task than those lower in procedural knowledge (i.e. cluster 2, which has higher conceptual knowledge) or at expected levels of procedural knowledge (i.e. cluster 5, which has lower conceptual knowledge). The reaction time and accuracy performance on the other symbolic and non-symbolic magnitude comparison tasks did not statistically significantly differ between the five clusters.
Chapter 5: Discussion

The aim of the current study was to examine the relationship between magnitude mental representations and procedural and conceptual fraction knowledge. This chapter details the findings of the current study, implications of the results, future directions for this area of research, and finally a discussion of the strengths and limitations of the current study.

Development of the Magnitude Mental Representations of Numbers.

The first and second areas of investigation in the current study are about the relationship between Canadian eighth-grade students’ performance on symbolic and non-symbolic magnitude comparison tasks for whole numbers and fractions. Similar to Fazio and colleagues’ (2014) results, the current study identified that students were faster and more accurate on the whole number magnitude comparison tasks than the fraction magnitude comparison tasks. These results suggest that magnitude mental representations of whole numbers on the mental number line are more precise and easier to access than magnitude mental representations of fractions. It is likely that students are faster and more accurate on magnitude comparison tasks of whole numbers than fractions because students’ have more experience and practice with whole numbers. Alternatively, the slower RT on fraction tasks could suggest that students’ access fraction understanding through magnitude mental representations of whole numbers.

When looking more closely at the relationship between accuracy of magnitude mental representations of fractions and whole numbers, the results indicated a strong correlation between students’ accuracy on both the symbolic and non-symbolic fraction magnitude comparison tasks (symbolic and non-symbolic). In addition, students’ accuracy on the non-symbolic whole number magnitude comparison task was moderately correlated to both the symbolic and non-symbolic fraction magnitude comparison tasks. However, students’ accuracy
on the symbolic whole number magnitude comparison task was not significantly correlated with any of the other tasks. This lack of statistically significant correlations to the accuracy on symbolic whole number magnitude comparison task may be due to ceiling effects on the symbolic whole number magnitude comparison task, such that many students had near perfect performance on this task.

When looking more closely at the relationship between speed of accessing magnitude mental representations of fractions and whole numbers, speed on the non-symbolic fraction magnitude comparison task was strongly correlated to the symbolic fraction magnitude comparison task, moderately correlated to non-symbolic whole number magnitude comparison task, and weakly correlated to symbolic whole number magnitude comparison task. In addition, symbolic whole number and non-symbolic whole number magnitude comparison tasks were moderately correlated. However, students’ speed on the symbolic fraction magnitude comparison task was not significantly correlated to either of the whole number magnitude comparison tasks (symbolic and non-symbolic). The accuracy and speed results suggest that students’ understanding of both symbolic and non-symbolic magnitude mental representations of fractions are strongly related and these may also be related to magnitude mental representations of whole numbers.

These accuracy and RT findings of the current study are slightly different than the findings of a previous study. Fazio and colleagues (2014) found that speed and accuracy on the symbolic magnitude comparison tasks (whole number and fraction) were positively correlated to each other, however, the non-symbolic magnitude comparison tasks (whole number and fraction) had weak or non-significant correlations with each other and with the symbolic magnitude comparison tasks. In contrast, the results of this study identified moderate correlations between
accuracy and RT performance on the non-symbolic fraction magnitude comparison task and the non-symbolic whole number and symbolic fraction magnitude comparison tasks, but weak (for the RT measure) or non-significant (for the accuracy measure) correlations with performance on the symbolic whole number magnitude comparison task. The differences between Fazio and colleagues’ (2014) results compared to the current study could be due to differences in education in different countries (i.e. America vs. Canada) or differences in age (i.e. Grade 5 vs. Grade 8). In addition, these differences support the need for further replication and study in the area of non-symbolic magnitude mental representations of fractions and whole numbers. Although slightly different, the overall results of Fazio and colleagues (2014) and of the current study suggest that magnitude mental representations of whole numbers and fractions are related and in support of the integrated theory of numerical development (Siegler et al., 2011) because these correlational results suggest that the magnitude mental representations whole numbers may support the development of the magnitude mental representations of fractions. It is possible that magnitude mental representations of whole numbers and fractions support the development of each other and they may be represented along the same mental number line.

**Relationship Between Magnitude Mental Representations of Numbers and Fraction Skills.**

The third area of investigation of the current study is about the contribution of Canadian eighth-grade students’ symbolic and non-symbolic magnitude mental representations of whole numbers and fractions to their procedural and conceptual knowledge of fractions. Contrary to previous studies (Hansen et al., 2015; Fazio et al., 2014; Siegler & Pyke, 2013), the current study examined the symbolic and non-symbolic magnitude mental representations of both whole numbers and fractions, as well as assessing specific fraction skills (procedural and conceptual knowledge). The accuracy with which students were able to compare magnitudes of symbolic
fractions was predictive of their level of conceptual and procedural fraction knowledge. The speed with which students were able to compare magnitude of non-symbolic fractions and symbolic whole numbers was predictive of their level of conceptual and procedural fraction knowledge. These results suggest that magnitude mental representations of fractions are related to fraction knowledge and may be more important predictors of both procedural and conceptual fraction knowledge than magnitude mental representations of whole numbers (expect for possibly speed on the symbolic whole number magnitude comparison task, which significantly predicted conceptual fraction knowledge). Contrary to the whole number bias (Ni & Zhou, 2005), these results suggest that fraction knowledge may rely more on magnitude mental representations of fraction than whole numbers, however these results are correlational and the current study cannot measure the magnitude mental representations that students’ draw upon while solving fraction problems.

This study was the first to examine the contributions of symbolic and non-symbolic magnitude mental representations of fractions and whole numbers to procedural and conceptual fraction knowledge. Contrary to the hypothesis that different types of magnitude mental representations will support different types of fraction knowledge, the results indicate that both types of fraction magnitude mental representations (symbolic and non-symbolic) predict performance on both types of fraction knowledge (procedural and conceptual). However, accuracy measures suggest that symbolic magnitude mental representations of fractions contribute to both types of fraction knowledge, where as speed measures suggest that non-symbolic magnitude mental representations of fraction contribute to both types of fraction knowledge. These results could be a product of the magnitude mental representations because symbolic mental representations are more precise than non-symbolic mental representations.
Therefore, accuracy measures may more precisely measure the magnitude mental representations of symbolic quantities. In contrast, speed measures may better access magnitude mental representations of non-symbolic quantities because speed measures the automaticity of the magnitude mental representations and non-symbolic quantities may not be mentally represented as precisely as symbolic quantities (as suggested by Meert et al., 2012). Overall, these results indicate that fraction magnitude mental representations are likely important to students’ knowledge of fractions, both conceptually and procedurally.

**Individual Differences in Procedural and Conceptual Knowledge.**

The fourth area of investigation in the current study is about the individual differences among Canadian eighth-grade students’ procedural and conceptual knowledge of fractions. The five-cluster solution best depicts the groupings of individual differences in students’ conceptual and procedural fraction knowledge: students who are higher in procedural knowledge and expected conceptual knowledge, those lower in procedural knowledge and higher in conceptual knowledge, those expected in procedural knowledge and expected in conceptual knowledge, those much higher in procedural knowledge and much lower in conceptual knowledge, and those expected in procedural knowledge and lower in conceptual knowledge. Clusters found by the current study were similar to the clusters of fraction knowledge of fourth- and fifth-grade American students previously identified both in terms of types of clusters and number of clusters, such that previous studies have identified clusters with higher knowledge in one area with lower knowledge in the other area, as well as clusters that are high in one type of knowledge with expected levels in the other type of knowledge (Hallett et al., 2010; Hecht & Vagi, 2012). Similar to the current study, Hallett and colleagues (2010) identified five clusters of students’ individual differences in fraction knowledge. However, Hecht and Vagi (2012)
identified four clusters (for fourth grade) and seven clusters (for fifth grade) for students’ individual differences in fraction knowledge.

Of long-standing debate is the way that procedural and conceptual number understanding develops. The results of the current study may provide support for the iterative model of procedural and conceptual numerical development (Rittle-Johnson & Koedinger, 2009; Rittle-Johnson & Schneider, 2015; Rittle-Johnson et al., 2001). The iterative model suggests that there is a bidirectional relationship between conceptual and procedural knowledge, in other words, increases in one type of knowledge leads to increases in the other type of knowledge (Rittle-Johnson & Koedinger, 2009; Rittle-Johnson & Schneider, 2015; Rittle-Johnson et al., 2001). The results of the current study indicate that students’ performance on conceptual and procedural fraction knowledge is correlated, therefore suggesting that these two types of knowledge are related. In addition, the results indicate that students’ level of procedural and conceptual fraction knowledge is different between individuals, which suggests that students may develop different procedural and conceptual fraction knowledge through different pathways.

Lastly, the fifth area of investigation in the current study is about the contributions of magnitude mental representations to the different clusters. Results indicated that students who did not have lower than expected procedural or conceptual knowledge performed more accurately on the non-symbolic fraction, symbolic fraction, and non-symbolic whole number magnitude comparison tasks than those who were lower (but not much lower) in either procedural or conceptual knowledge. These results suggest that students who do not have a weakness in either type of fraction knowledge may have a more accurate magnitude mental representation of symbolic and non-symbolic fractions and non-symbolic whole numbers than students who are slightly lower in one type of fraction knowledge. In addition, the results
indicated that students who were lower than expected procedural or conceptual knowledge performed faster on the non-symbolic fraction magnitude comparison task than those who were not lower in either procedural or conceptual knowledge. These results suggest that students who were lower in one type of knowledge may have used whole number strategies to solve the non-symbolic fraction magnitude comparison task. For example, these students may have only compared the number of blue dots instead of the ratio of blue to yellow dots, therefore decreasing their accuracy and decreasing their reaction time because most students are faster to compare whole numbers than fractions/ratios. Alternatively, students who are lower in one type of knowledge may have quickly indicated their answers on the non-symbolic fraction magnitude comparison task without thinking about their response. However, the exact nature of this requires further study as one of the cluster groups with the much lower procedural fraction knowledge was not found to significantly differ in terms of accuracy or speed of mental representations as compared to the other clusters.

Overall, these results suggest that students who display lower fraction knowledge also perform lower on some magnitude comparison tasks (i.e. symbolic fraction and non-symbolic whole number and fraction) than those who do not display fraction knowledge that is lower than expected. These findings indicate that students who struggle with fractions may be missing the early mathematical skills of magnitude knowledge of fractions and whole numbers. Furthermore, these results suggest magnitude mental representations (specifically of symbolic fraction and non-symbolic whole number and fraction) support procedural and conceptual fraction knowledge.
Strengths and Limitations

One strength of the current study is that it examined both magnitude mental representations and fraction knowledge through analysis of patterns of individual differences and the relationship between these skills. This is a novel approach to assessing students’ fraction knowledge and serves to expand on past research that separately looked at magnitude mental representations (Fazio et al., 2014) and individual differences of procedural and conceptual fraction knowledge (Hallett et al., 2010; Hallett et al., 2012; Hecht & Vagi, 2012). The benefit of this approach is that participants were grouped by their strengths and weaknesses in procedural and conceptual fraction knowledge instead of grouping all of the students together, which could prevent important differences from being averaged out of the data. Attending to individual differences in fraction knowledge through this method should provide a more precise understanding of most students’ experience in acquiring fraction mathematical skills. The previous work of Hecht and Vagi (2012) found that there are patterns of individual differences in students’ fraction knowledge, thus it is important to consider variable performance between different groups in their fraction knowledge. Further, because students may develop fraction knowledge through different pathways, this method will help reconcile contradictory findings in the procedural and conceptual fraction knowledge development literature by aiding in the identification of all possible pathways of development (Hallett et al, 2010; Hecht & Vagi, 2012).

One possible limitation of the current study is that classroom instruction was not controlled. Some students may have more training in one area of fraction knowledge (procedural or conceptual). In an examination of the benefits of concepts-first verses iterative learning (alternating procedural and conceptual lessons), researchers evaluated the two different types of sixth grade instruction sequencing of decimal place-value concepts and arithmetic procedures
(Rittle-Johnson, & Koedinger, 2009). Results indicated that students performed higher on arithmetic procedures when taught with the iterative sequencing as opposed to the concepts-first sequencing. The knowledge of decimal place-value concepts was similar in both sequencing conditions. In addition, higher pre-test knowledge predicted higher post-test knowledge in the other domain (e.g. higher conceptual pre-knowledge predicted higher procedural post-training knowledge). These results suggest that the instruction method can influence the students’ fraction knowledge. Assessing fraction knowledge of students from different classes when instructional method has not been controlled for is problematic because it is difficult to determine if differences in fraction knowledge are due to study variables or the instructional method. However, the current study analyzed the individual differences of students’ fraction knowledge and magnitude mental representations of whole numbers and fractions, which allows for analysis of differences between students’ fraction knowledge.

The results of the order effects of the current study indicated that students performed differently, in terms of speed and accuracy, on the symbolic fraction and non-symbolic whole number magnitude comparison tasks dependent upon the order that the tasks were presented. In addition, students may have become fatigued by the magnitude comparison tasks and performed slower. For this reason, the tasks were randomized to balance fatigue effects approximately evenly across all tasks. Furthermore, the tasks were structured to include breaks for students to ensure that they do not become more fatigued than a typical school day.

In terms of limitations of the measures, while both of the symbolic magnitude comparison tasks had the same procedure, the two non-symbolic magnitude comparison tasks had slightly different procedures. In particular, there were different time intervals for the presentation of stimuli, such that the dots for the fraction magnitude comparison task were
displayed for 2 seconds and the dots for the whole number magnitude comparison task were displayed for 1382 milliseconds. Both the fraction and whole number non-symbolic magnitude comparison tasks had backward masks of blue and yellow dots, however, the whole number task’s mask filled half of the screen and was only viewed for half a second and the fraction tasks’ mask filled the entire screen and was viewed until the student responded. These differences between the non-symbolic magnitude comparison tasks may have slightly impacted the results simply due to variances in presentation. Furthermore, there was a ceiling effect of accuracy on the symbolic whole number magnitude comparison task, which may have interfered with identification of relations between performance on the other magnitude comparison tasks, as well as procedural and conceptual fraction knowledge.

To best ensure the test developed for this study accurately measured procedural and conceptual fraction knowledge, the questions were developed and reviewed through literature review and expert review and questions were classified as procedural or conceptual based on researcher deliberations and expert reviews. The decision was made because there is not an available test in the literature known by the researcher that separately assesses procedural and conceptual fraction knowledge. However, piloting and reliability analyses on the fraction test were not completed before being used in the current research study, thus it is difficult to determine the reliability and validity of the fraction test. In addition, because interviews were not conducted to assess the type of knowledge that the students were using when solving the fraction questions, it is possible that students were using procedural knowledge to solve predominantly conceptual questions or vice versa. Given the extensive review of the literature and by experts, it is expected that the fraction test sufficiently measures procedural and
conceptual fraction knowledge, however further testing with a psychometrically sound research fraction task is needed to further examine these fraction skills.

Another possible limitation of the current study is the small sample size. Although a power analysis indicated that 64 participants (the current study had 70 participants) is a sufficient sample size to evaluate effects with a regression analysis, the sample size may have not been sufficient to conduct a cluster analysis. There is no general guideline of sample size for cluster analysis (Dolnicar, 2002), however previous studies examining clusters of students’ procedural and conceptual fraction knowledge had greater than 100 participants in their samples (Hallett et al., 2010; Hallett et al., 2012; Hecht & Vagi, 2012). It is possible that differences in clusters, as compared to previous studies, likely resulted from a smaller sample size for analysis.

**Implications and Future Directions**

In general, the results of the current study indicate that the magnitude mental representations of fractions and whole numbers (both symbolic and non-symbolic) are related. This relationship suggests that the magnitude mental representations of whole numbers and fractions support the development of each other, as proposed by the *integrated theory of numerical development* (Siegler et al., 2011). Similar to Siegler and colleagues’ (2011) suggestion, this relationship could also imply that magnitude mental representations of whole numbers and fractions are represented along the same mental number line. However, magnitude mental representations of fractions may be more difficult to access because of the limited practice in accessing these representations or limited development of these representations, as compared to magnitude mental representations of whole numbers.

In addition, the results of the current study indicate that magnitude mental representations of fractions are more predictive of performance on a fraction test (measuring both procedural and
conceptual knowledge of fractions) than magnitude mental representations of whole numbers. Although these results do not directly measure the magnitude mental representations that are accessed when solving fraction questions, these results do suggest that students may rely more on magnitude mental representations of fractions than whole numbers when solving fraction questions. In addition, these results could suggest that magnitude mental representations of fractions better support the development of procedural and conceptual fraction knowledge than magnitude mental representations of whole numbers. These results could suggest that even though fractions are mapped onto the same mental number line as whole numbers, fractions may require more time to develop precise and automatic magnitude mental representations of fractions (Siegler et al., 2011). Unlike the whole number bias (Ni & Zhou, 2005), these results suggest that students’ performance on a fraction test is more related to their magnitude mental representations of fractions than whole numbers. However, few students solved all of the fraction problems correctly and may have been using whole number strategies to solve the fraction problems, which supports the whole number bias (Ni & Zhou, 2005). Future studies could examine students’ use of whole number rules to solve fraction problems and assess whether these incorrect procedures are linked to poorer magnitude mental representations of fractions.

Much of the emphasis of magnitude mental representations is research with whole numbers, which has identified that training in number sense areas (including magnitude mental representations of whole numbers) improves number sense skills and mathematics calculation skills (Dyson, Jordan, & Glutting, 2013) and early mathematics achievement (Chard, Baker, Clarke, Jungjohann, Davis, & Smolkowski, 2008). Implications of the current study, along with a few other research studies (Fazio et al., 2014; Hansen et al., 2015; Siegler & Pyke, 2013),
provide initial support for the importance of magnitude mental representations of fractions for the development of fraction knowledge. These findings will likely result in increased attention given to the study of specific training in magnitude mental representations of fractions to support the students’ development of fraction knowledge.

The current study highlights the importance of researching students’ knowledge of fractions because many students struggled with answering the questions on the fraction test, some students scoring as low as 10%. Therefore, it is important to identify targeted interventions that will best help students struggling in fractions to improve their fraction understanding. Future studies should continue to examine the contribution of magnitude mental representations of fractions to different types of fraction knowledge. The definition of procedural and conceptual knowledge should be further refined and improve fraction tests according to these definitions and through piloting the fraction test. Other types of fraction knowledge than procedural and conceptual knowledge should be examined directly, such as the multiple perspectives of fractions: part-whole, quotient, ratio, measure, and operator (Moseley & Okamoto, 2008). In addition, future studies should conduct longitudinal research to identify the best times to implement number sense interventions related to fractions. Together with the current research, research in this area would provide further information regarding how best to adapt or maintain curriculum to improve students’ skills with fractions.

In addition, future studies should consider examining the growth of fraction knowledge longitudinally. Hecht and Vagi (2012) identified that individual differences in fraction knowledge may not be stable over time, simply, students differentially improve in procedural and conceptual fraction knowledge. Therefore, it would be important to examine factors that contribute to growth in students’ fraction knowledge across several years of schooling. Results
from a longitudinal study would allow for more of a causal interpretation of the factors to fraction knowledge than a single time-point study (longitudinal method used to examine the characteristics that support reading and mathematics achievement by Claessens, Duncan, & Engel, 2009; and Hecht and Vagi, 2012, measured procedural and conceptual knowledge when students where in fourth grade and again in fifth grade).

**Conclusion**

In order to provide appropriate support to students who struggle to grasp fraction concepts and procedures, it is important to understand the underlying magnitude mental representations that support the various fraction skills. The current study expands the present literature by examining both symbolic and non-symbolic magnitude mental representations of fractions and whole numbers, as well as procedural and conceptual knowledge. Examining magnitude mental representations of fractions is important for several theoretical reasons, including adding to the literature regarding how these representations develop and resolves conflict in theories about the impact of whole numbers of the development of fractions (whole number bias versus integrated theory of numerical development). Similar to prior research (Fazio et al., 2014), the current study identified a relationship between magnitude mental representations of whole numbers and fractions. In addition, this area of study is important to identify possible skills that may help students understand fractions and allow for the development of targeted interventions. The current research study identified that both symbolic and non-symbolic magnitude mental representations of fractions are related to procedural and conceptual fraction knowledge. The relationship that was observed was complex but does provide initial support of the important contributions of early magnitude mental representations of fractions to both procedural and conceptual fraction knowledge. As the current study
represents only initial research into this area, future research is required to further explore the manner in which magnitude mental representations of fractions support the acquisition of fraction knowledge.

In addition to pointing to the importance of research on magnitude mental representations of fractions, the current study also highlights individual differences in fraction knowledge acquisition. The current study identified that students can be grouped based on their procedural and conceptual fraction knowledge, with initial indications that proficiency in certain magnitude mental representations may account, in part, for differences between these groups. In particular, several groups characterized with lower performance in one type of fraction knowledge were less accurate on non-symbolic fraction, symbolic fraction, and non-symbolic whole number magnitude comparison tasks than those who were not lower in one type of fraction knowledge. However, the relationship is complex as not all groups that were lower in one type of fraction knowledge had accompanying lower performance in these magnitude comparison tasks. As work in this research area continues, it is important to look at differences among subgroups because students may differentially develop fraction skills, thus interventions must be tailored to the manner of students’ developmental pathway to fraction understanding. All together, the results suggest there is likely a relationship among magnitude mental representations of fractions and whole numbers. In addition, magnitude mental representations of fractions and whole numbers may be related to procedural and conceptual fraction knowledge. Future research should seek to replicate these findings, as well as, examine the impact of specific training in magnitude mental representations of fractions on the development of students’ fraction knowledge.
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APPENDIX A: PARENT/GUARDIAN CONSENT FORM

Name of Researcher, Faculty, Department, Telephone & Email:
Kaleigh Pelletier, School and Applied Child Psychology, Werklund School of Education, 587-438-0475, kaleigh.pelletier@ucalgary.ca

Supervisor:
Michelle Drefs, School and Applied Child Psychology, Werklund School of Education, 403-220-3675, madrefs@ucalgary.ca

Title of Project:
Examining Fraction Development Through Analysis of Symbolic/Nonsymbolic Representation of Whole Number and Fractions

Sponsor:
N/A

This consent form, a copy of which has been given to you, is only part of the process of informed consent. If you want more details about something mentioned here, or information not included here, you should feel free to ask. Please take the time to read this carefully and to understand any accompanying information.

This study has been approved by the University of Calgary Conjoint Faculties Research Ethics Board.

Purpose of the Study
The purpose of the current study is to gain an understanding of Grade 8 students’ knowledge of fractions. The study will examine how students think about numbers and current fraction knowledge.

What Will Your Student Be Asked To Do?
If you and your student consent to participation in the study, your student will be asked to complete a series of math-related activities during two 20 to 30-minute individualized testing sessions. Activities will include: viewing a series of dots and numbers on a computer screen and identifying which set is larger. In addition, there will be one 30-minute fraction test that will be administered to the whole classroom. The researcher will score all the tests but only collect the scores from those who choose to participate in the research study. The fraction test will examine their ability to add/subtract/multiply fractions, as well as, the ability to understand basic principles of fractions. Participation in this research study is completely voluntary, and the student and/or their parents may refuse to participate altogether, may refuse to participate in parts of the study, may decline to answer any and all questions, and may withdraw from the study at anytime without penalty (i.e. will not impact their grades and will still be entered into the draw to win a $20 gift card).

What Type of Personal Information Will Be Collected?
Should you agree to participate, your student will be asked to provide their gender, birth date, and name.

Are there Risks or Benefits if I Participate?
There are not considered to be any risks to this study above what students will experience in a typical school day, although it is possible that students may experience some boredom, fatigue, or eye strain from participating in the computer tasks. Should any of these latter occur, the student will be given a break and reminded that they may withdraw from the study.

Participation in the study is voluntary and participation or non-participation will have no effect on the student’s relationship with their class teacher or their school. Students who choose to participate will be entered into a draw
to win one of three $20 gift cards to a location of their choice. In addition, their participation will contribute to the growing body of scientific knowledge about the development of fraction skills. This information can be used for the future planning of mathematics curriculum and interventions.

**What Happens to the Information I Provide**

Participation is completely voluntary and data will be kept confidential and anonymous. You are free to discontinue participation at any time during the study and up to one month after data collection; any data collected will be deleted from the data file if consent is withdrawn. No one except the research team will have access to the collected information. Students will be assigned a participant number that will be used to identify them, so that their scores on the fraction test and tasks are not stored with their name. There will be a document that matches the participant numbers to the student names, but this document will be stored in a separate file and will be encrypted and will be deleted one month after data collection. Only group information will be summarized for any presentation or publication of results. The information will be stored on an encrypted computer and photocopies of the participating students’ fraction tests will be stored in a locked cabinet in a locked office at the University of Calgary. The anonymous data will be stored for five years on a computer disk, at which time, it will be permanently erased.

**SIGNATURES**

Your signature on this form indicates that 1) you understand to your satisfaction the information provided to you about your participation in this research project, and 2) you agree to participate in the research project.

In no way does this waive your legal rights nor release the investigators, sponsors, or involved institutions from their legal and professional responsibilities. You are free to withdraw from this research project at any time. You should feel free to ask for clarification or new information throughout your participation.

Parent/Custodian’s Name: (please print) ____________________________________________

Parent/Custodian’s Signature: __________________________ Date: ______________

Researcher’s Name: (please print) ______________________________________________

Researcher’s Signature: __________________________ Date: ______________

Questions/Concerns

If you have any further questions or want clarification regarding this research and/or your participation, please contact:

Kaleigh Pelletier, Werklund School of Education/ Faculty of School and Applied Child Psychology (SACP) 587-438-0475, kaleigh.pelletier@ucalgary.ca and Dr. Michelle Drefs, Werklund School of Education/ Faculty of SACP, 403-220-3675, madrefs@ucalgary.ca

If you have any concerns about the way you’ve been treated as a participant, please contact the Research Ethics Analyst, Research Services Office, University of Calgary at (403) 210-9863; email cfreb@ucalgary.ca.

A copy of this consent form has been given to you to keep for your records and reference. The investigator has kept a copy of the consent form.
APPENDIX B: STUDENT CONSENT FORM

**Project Title:** Examining Fraction Development Through Analysis of Symbolic/Nonsymbolic Representation of Whole Number and Fractions

**Principal Investigator:** Dr. Michelle Drefs and Kaleigh Pelletier

What is a research study?

A research study is a way to find out new information about something. Children do not need to participate in a research study if they don’t want to participate.

Why are you being asked to be part of this research study?

You are being asked to take part in this research study because we are trying to learn more about how eighth-grade students understand fractions. About 60 children will be in this study.

If you join the study what will happen to you?

We want to tell you about some things that will happen to you if you are in this study.

- You will be in the study two times for 20 to 30 each time. Each time you will answer simple math questions about how big numbers and pictures are.
- Also, we will take your scores from one test on fractions that your teacher will give you.

Will any part of the study hurt?

This study may make you tired or your eyes may hurt from staring at a computer screen.

Will the study help you?

If you join the study, you will have the chance to win a $20 gift card to a place of your choice.

Will the study help others?

This study might find out things that will help other children with learning fractions some day.

Do your parents know about this study?

We will talk to your parents about your participation in this study as well. You can talk this over with them before you decide.

Who will see the information collected about you?

The information collected about you during this study will be kept safely locked up. Nobody will know it except the people doing the research. The study information about
you will not be given to your parents or teachers. The researchers will not tell your friends or anyone else

**What do you get for being in the study?**

You be entered to win a $20 gift card to a place of your choice.

**Do you have to be in the study?**

You do not have to be in the study. No one will be upset if you don’t want to do this study. If you don’t want to be in this study, you just have to tell us. It’s up to you. You can also take more time to think about being in the study.

**What if you have any questions?**

You can ask any questions that you may have about the study. If you have a question later that you didn’t think of now, either you can email or have your parents email (kaleigh.pelletier@ucalgary.ca). You can also take more time to think about being in the study and also talk some more with your parents about being in the study.

**What choices do you have if you say no to this study?**

This study is extra, so if you don’t want to do it nothing else will change.

**Other information about the study.**

If you decide to be in the study, then please write your name below. You can change your mind and stop being part of it at any time. All you have to do is tell the person in charge. It’s okay. The researchers and your parents won’t be upset.

You will be given a copy of this paper to keep

Would you like to take part in this study?

_____ Yes, I will be in this research study.      _____ No, I don’t want to do this.

__________________________        ___________________

Child’s name        Signature of the child        Date

__________________________        ___________________

Person who received assent        Signature        Date
APPENDIX C: FRACTION TEST

Each question is scored out of 1, only question 10 in part A and question 8 in part B could be awarded a half point, the rest of the questions were marked either 1 (correct) or 0 (incorrect).

Fractions Part A

Name (First name, Last initial): _________________ Date: ________________

Please show all your work.

1. What fraction of the hexagon is shaded red?

2. Add. Show your work.

\[
\frac{4}{13} + \frac{2}{13} =
\]

3. Circle the image that shows 1/7

4. Put an X on the image that shows 1/3 of 1/7
5. Subtract. Show your work.

\[
\frac{2}{3} - \frac{2}{9} =
\]

6. Write down these fractions in order from smallest to largest:

\[
\frac{1}{4}, \frac{1}{2}, \frac{1}{100}, \frac{1}{3}
\]

7. Subtract. Show your work.

\[
5 \frac{1}{5} - \frac{3}{4} =
\]

8. Harriet’s horse eats \(\frac{1}{4}\) of a barrel of hay each day, how much hay does Harriet need to feed her horse for a week? Show your thinking.

9. Multiply. Show your work.

\[
\frac{3}{5} \times \frac{1}{5} =
\]
10. Luke and Janna are asked to estimate the answer to $\frac{1}{4} \times \frac{2}{3}$

Luke thinks the answer is less than $\frac{1}{4}$.

Janna thinks the answer is greater than $\frac{1}{4}$.

Who is correct? How do you know?

11. Jessica walks $\frac{7}{8}$ of a kilometer to school and Joe walks $\frac{1}{4}$ of a kilometer to school. Who walks more and by how much? Show your work.
Fractions Part B

Name (First Name, Last Initial): _____________________ Date: ______________

Please show all your work.

1. Subtract. Show your work.

\[
\frac{5}{8} - \frac{1}{8} =
\]

2. Shade in two-thirds of each of these shapes:

(a) ![Shape A](image1)

(b) ![Shape B](image2)

(c) ![Shape C](image3)

3. Add. Show your work.

\[
\frac{4}{5} + \frac{3}{10} =
\]

4. Fill in the missing numbers: Show your thinking.

(a) \( \frac{1}{3} = \frac{2}{ \quad } \)

(b) \( \frac{5}{10} = \frac{30}{ \quad } \)

(c) \( \frac{4}{12} = \frac{1}{ \quad } \)
5. Add. Show your work.

\[
\frac{4}{10} + \frac{1}{10} = \]

6. Sue Ling made too much spaghetti for dinner. She ate \( \frac{1}{5} \) of the pot of spaghetti for dinner and packed \( \frac{1}{10} \) of the pot of spaghetti for lunch. What portion of the spaghetti is still in the pot? Show your thinking.

7. Multiply. Show your work.

\[
\frac{1}{3} \times \frac{2}{7} = \]

8. What is one-quarter of one-fifth? Use a drawing to show your thinking.

9. Multiply. Show your work.

\[
2\frac{1}{4} \times \frac{3}{8} = \]
APPENDIX D: Fraction Test Details

Appendix D details the questions used in the fraction test by the conceptual or procedural nature of the question, fraction perspective that may be used, and literature that supports the use of the question to measure conceptual or procedural fraction knowledge. The procedural and conceptual nature of the question was determined through literature review of studies defining and assessing procedural and conceptual fraction knowledge, as well as through an expert review of the questions. Although the type of rational number perspective (Moseley & Okamoto, 2008) used depends on how the student understands and processes the question, the perspectives listed is the type of meaning that is suspected to be used by the students.

<table>
<thead>
<tr>
<th>Part</th>
<th>Question</th>
<th>Conceptual or Procedural</th>
<th>Perspective</th>
<th>Literature</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1. What fraction of the hexagon is shaded red?</td>
<td>Conceptual</td>
<td>Part-whole</td>
<td>type of question used in Hecht and Vagi, 2012</td>
</tr>
<tr>
<td>A</td>
<td>2. Add. Show your work. 4/13 + 2/13 =</td>
<td>Procedural</td>
<td>Part-whole (addition)</td>
<td>type of question used in Hecht and Vagi, 2012</td>
</tr>
<tr>
<td>A</td>
<td>3. Circle the image that shows 1/7</td>
<td>Conceptual</td>
<td>Part-whole</td>
<td>Mason, 2015</td>
</tr>
<tr>
<td>A</td>
<td>4. Put an X on the image that shows 1/3 of 1/7</td>
<td>Conceptual</td>
<td>Part-whole</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>5. Subtract. Show your work. 2/3 - 2/9 =</td>
<td>Procedural</td>
<td>Part-whole (subtraction)</td>
<td>Type of question used in Hecht and Vagi, 2012</td>
</tr>
<tr>
<td>A</td>
<td>6. Write down these fractions in order from smallest to largest: 1/4, 1/2, 1/100, 1/3</td>
<td>Conceptual</td>
<td>Measure (comparisons)</td>
<td>Question from Brown, Hart, &amp; Kucherman (1984) and used in Hallett, Nunes, &amp; Bryant (2010)</td>
</tr>
<tr>
<td>A</td>
<td>7. Subtract. Show your work. 5 1/5 - 3/4 =</td>
<td>Procedural</td>
<td>Part-whole (subtraction)</td>
<td>Type of question used in Hecht and Vagi, 2012</td>
</tr>
<tr>
<td>A</td>
<td>8. Harriet’s horse eats 1/4 of a barrel of hay each day, how much hay does Harriet need to feed her horse for a week? Show your thinking.</td>
<td>Conceptual</td>
<td>Part-whole and operator</td>
<td>Word problems use conceptual knowledge (Quintero, 1983)</td>
</tr>
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<td></td>
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<td></td>
</tr>
<tr>
<td><strong>A</strong></td>
<td>9. Multiply. Show your work. 3/5 x 1/5 =</td>
<td>Procedural</td>
<td>Part-whole</td>
<td>Type of question used in Hecht and Vagi, 2012</td>
</tr>
<tr>
<td><strong>A</strong></td>
<td><strong>10. Luke and Janna are asked to estimate the answer to 2 1/4 x 2/3. Luke thinks the answer is less than 2 1/4. Janna thinks the answer is greater than 2 1/4. Who is correct? How do you know?</strong></td>
<td>Conceptual</td>
<td>Operator</td>
<td>Type of question used in Rittle-Johnson et al., 2001</td>
</tr>
<tr>
<td><strong>A</strong></td>
<td><strong>11. Jessica walks 7/8 of a kilometer to school and Joe walks 1/4 of a kilometer to school. Who walks more and by how much? Show your work.</strong></td>
<td>Procedural</td>
<td>Part-whole (comparison)</td>
<td>Developed from expert review</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td><strong>1. Subtract. Show your work. 5/8 - 1/8 =</strong></td>
<td>Procedural</td>
<td>Part-whole (subtraction)</td>
<td>Type of question used in Hecht and Vagi, 2012</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td><strong>2a. Shade in two-thirds of each of these shapes:</strong></td>
<td>Conceptual</td>
<td>Part-whole</td>
<td>Questions from Brown, Hart, &amp; Kucherman (1984) and used in Hallett, Nunes, &amp; Bryant (2010).</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td><strong>2b. Shade in two-thirds of each of these shapes:</strong></td>
<td>Conceptual</td>
<td>Part-whole</td>
<td></td>
</tr>
<tr>
<td><strong>B</strong></td>
<td><strong>2c. Shade in two-thirds of each of these shapes:</strong></td>
<td>Conceptual</td>
<td>Part-whole</td>
<td></td>
</tr>
<tr>
<td><strong>B</strong></td>
<td><strong>3. Add. Show your work. 4/5 + 3/10 =</strong></td>
<td>Procedural</td>
<td>Part-whole (addition)</td>
<td>Type of question used in Hecht and Vagi, 2012</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td><strong>4. Fill in the missing numbers: Show your thinking. 1/3 = 2/x</strong></td>
<td>Conceptual</td>
<td>Part-Part (Ratio)</td>
<td>Question from Brown, Hart, &amp; Kucherman (1984) and used in Hallett, Nunes, &amp; Bryant (2010)</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td><strong>4. Fill in the missing numbers: Show your thinking. 5/10 = x/30</strong></td>
<td>Conceptual</td>
<td>Part-Part (Ratio)</td>
<td></td>
</tr>
<tr>
<td><strong>B</strong></td>
<td><strong>4. Fill in the missing numbers: Show your thinking. 4/12 = 1/x</strong></td>
<td>Conceptual</td>
<td>Part-whole</td>
<td></td>
</tr>
<tr>
<td><strong>B</strong></td>
<td><strong>5. Add. Show your work. 3 4/10 + 1/10 =</strong></td>
<td>Procedural</td>
<td>Part-whole (addition)</td>
<td>Type of question used in Hecht and Vagi, 2012</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td><strong>6. Sue Ling made too much spaghetti for dinner. She ate 1/5th of the pot of spaghetti for dinner and packed 1/10th of the pot of spaghetti for lunch.</strong></td>
<td>Conceptual</td>
<td>Part-whole</td>
<td>Word problems use conceptual knowledge (Quintero, 1983)</td>
</tr>
<tr>
<td></td>
<td>What portion of the spaghetti is still in the pot? Show your thinking.</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>7. Multiply. Show your work. $\frac{1}{3} \times \frac{2}{7} =$</td>
<td>Procedural</td>
<td>Operator (multiplication)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Type of question used in Hecht and Vagi, 2012</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>8. What is one-quarter of one-fifth? Use a drawing to show your thinking</td>
<td>Conceptual</td>
<td>Part-whole and Quotient (division)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Word problems use conceptual knowledge (Quintero, 1983)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>9. Multiply. Show your work. $2 \frac{1}{4} \times \frac{3}{8} =$</td>
<td>Procedural</td>
<td>Part-whole and Operator (multiplication)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Type of question used in Hecht and Vagi, 2012</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX E: Type of Solutions Used to Solve Fraction Questions

Appendix E details students’ performance and solutions for each of the questions on the fraction test. Each box indicates the number of students (total = 70) that either got the correct answer or used a particular solution. For question 10 in Part A and question 8 in Part B, a number of students who got partial credit (i.e. 0.5) for their answer and number of students who received full credit (i.e. 1) for their answer is provided. None of the other questions allowed for partial credit. The students’ answer was not tallied under a particular solution, only the work the student showed to arrive at the answer. Both correct and incorrect solutions were tallied. A symbolic solution is one where a student used only numerals to solve the problem. A rectangular, circle, or other shape model is one where a student draws a shape to solve the problem. A set model is one where a student draws several items where each item represents a part and all the items consist of the whole, and the student may use this drawing to solve the problem. A number line model is one where a student uses a line with several ticks to depict the distance between two numbers, and the student may use this drawing to solve the problem.

<table>
<thead>
<tr>
<th>Question</th>
<th>Pro or Con</th>
<th>Number Correct</th>
<th>Symbolic</th>
<th>Rectangular Model</th>
<th>Circle Model</th>
<th>Number Line</th>
<th>Set Model</th>
<th>Other Shape Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Part A</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1. What fraction of the hexagon is shaded red?</td>
<td>C</td>
<td>58</td>
<td>19</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2. Add. Show your work. 4/13 + 2/13 =</td>
<td>P</td>
<td>58</td>
<td>30</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3. Circle the image that shows 1/7</td>
<td>C</td>
<td>50</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4. Put an X on the image that shows 1/3 of 1/7</td>
<td>C</td>
<td>33</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
5. Subtract. Show your work. \( \frac{2}{3} - \frac{2}{9} = \)

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>47</th>
<th>54</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
</table>

6. Write down these fractions in order from smallest to largest: \( \frac{1}{4}, \frac{1}{2}, \frac{1}{100}, \frac{1}{3} \)

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>55</th>
<th>4</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
</table>

7. Subtract. Show your work. \( 5 \frac{1}{5} - \frac{3}{4} = \)

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>29</th>
<th>60</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
</table>

8. Harriet’s horse eats \( \frac{1}{4} \) of a barrel of hay each day, how much hay does Harriet need to feed her horse for a week? Show your thinking.

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>51</th>
<th>60</th>
<th>5</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
</table>

9. Multiply. Show your work. \( \frac{3}{5} \times \frac{1}{5} = \)

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>43</th>
<th>26</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
</table>

10. Luke and Janna are asked to estimate the answer to \( 2 \frac{1}{4} \times \frac{2}{3} \) Luke thinks the answer is less than \( 2 \frac{1}{4} \). Janna thinks the answer is greater than \( 2 \frac{1}{4} \). Who is correct? How do you know?

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>0.5 - 26</th>
<th>40</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
</tr>
</thead>
</table>

11. Jessica walks \( \frac{7}{8} \) of a kilometer to school and Joe
walks 1/4 of a kilometer to school. Who walks more and by how much? Show your work.

Part B

1. Subtract. Show your work. 5/8 - 1/8 =
   - P: 61 50 0 0 0 0 0 0

2a. Shade in two-thirds of each of these shapes:
   - C: 57 6 0 0 0 0 0 0

2b. Shade in two-thirds of each of these shapes:
   - C: 55 5 0 0 0 0 0 0

2c. Shade in two-thirds of each of these shapes:
   - C: 56 5 0 0 0 0 0 0

3. Add. Show your work. 4/5 + 3/10 =
   - P: 51 56 0 0 0 0 0 0

4. Fill in the missing numbers: Show your thinking. 1/3 = 2/x
   - C: 66 46 0 3 0 0 0 0

4. Fill in the missing numbers: Show your thinking. 5/10 = x/30
   - C: 66 45 0 0 0 0 0 0

4. Fill in the missing numbers: Show your thinking. 4/12 = 1/x
   - C: 60 46 0 0 0 0 0 0

5. Add. Show your work. 3 4/10 + 1/10 =
   - P: 27 51 0 0 0 0 0 0

6. Sue Ling
   - C: 38 50 4 6 0 0 0 0
made too much spaghetti for dinner. She ate 1/5th of the pot of spaghetti for dinner and packed 1/10th of the pot of spaghetti for lunch. What portion of the spaghetti is still in the pot? Show your thinking.

<p>| | | | | | | | |</p>
<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>7. Multiply. Show your work. $\frac{1}{3} \times \frac{2}{7} = $</td>
<td>P</td>
<td>47</td>
<td>36</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8. What is one-quarter of one-fifth? Use a drawing to show your thinking</td>
<td>C</td>
<td>0.5 - 21</td>
<td>27</td>
<td>30</td>
<td>24</td>
<td>0.5</td>
<td>1</td>
</tr>
<tr>
<td>9. Multiply. Show your work. $2 \frac{1}{4} \times \frac{3}{8} = $</td>
<td>P</td>
<td>27</td>
<td>60</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>