Systematic Construction of Natural Deduction Systems for Many-valued Logics: Extended Report

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Systematic Construction of Natural Deduction Systems for Many-valued Logics: Extended Report

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Abstract

We exhibit a construction principle for natural deduction systems for arbitrary finitely-many-valued first order logics. These systems are systematically obtained from sequent calculi, which in turn can be extracted from the truth tables of the logics under consideration. Soundness and cut-free completeness of these sequent calculi translate into soundness, completeness and normal form theorems for the natural deduction systems.

Keywords: Many-valued logic, natural deduction, sequent calculus, normal form, cut-free derivation

1 Introduction

The study of natural deduction systems for many-valued logics can be motivated by the following two issues: (1) Many-valued logics provide a general framework for the investigation of properties of classical (two-valued) sytems. (2) A general construction of sound and complete natural deduction calculi leads to an adequate syntactical (proof-theoretic) characterization of many-valued logics for which one wants to emphasize the rôle of a particular truth value. (For standard logics, such as the families of Gödel und Łukasiewicz logics, one usually considers such distinguished truth values.)

We consider finitely-many-valued first order logics with arbitrary truth-functional connectives and distribution quantifiers (see Definition 2.3). A natural deduction derivation for a logic with the truth values $\{v_1, \ldots, v_m\}$ is defined as a derivation

$$\begin{array}{c} \Gamma_1 \mid \ldots \mid \Gamma_{m-1} \\ \vdots \\ \Gamma_m \end{array}$$

where Γ_l $(1 \leq l \leq m)$ are sets of formulas ($\Gamma_1 \mid \ldots \mid \Gamma_{m-1}$ represents the assumption, Γ_m is the conclusion.) Each position l corresponds to one of the truth values, v_m is the distinguished truth value. The intended meaning is as follows: Derive that at least one formula of Γ_m takes the value v_m under the assumption that no formula in Γ_i takes the value v_i $(1 \leq i \leq m-1)$.

Our starting point for the construction of natural deduction systems are sequent calculi. (A sequent is a tuple $\Gamma_1 \mid \ldots \mid \Gamma_m$, defined to be satisfied by an interpretation iff for some $i \in \{1, \ldots, m\}$ at least one formula in Γ_i takes the truth value v_i.) For each pair of an operator $\Box$ or quantifier Q and a truth value v_i we construct a rule introducing a formula of the form $\Box(A_1, \ldots, A_n)$ or $(\mathsf{Q}x)A(x)$, respectively, at position i of a sequent. The resulting calculi are shown to be sound and cut-free complete by Schütte's reduction tree method.

Every sequent rule introducing a formula at a non-distinguished position is converted into an elimination rule; the sequent rule introducing formulas at the distinguished position is transformed into an introduction rule (in the sense of natural deduction). We show that any natural deduction derivation can be translated into a derivation of the corresponding sequent calculus. On the other hand, any cut-free sequent calculus proof translates into a normal natural deduction derivation. (Here normal means that for no branch of the proof tree an elimination follows an introduction; this excludes maximal segments in the sense of PRAWITZ [1971].) Consequently, the natural deduction sytems are sound and complete and every derivation can be transformed into a normal derivation. Such derivations consist of "analytical" paths.

2 Preliminaries

2.1. DEFINITION A *language* $\mathcal{L}$ for a logic $\mathbf{L}$ consists of: (1) free variables, (2) bound variables, (3) predicate symbols, (4) propositional connectives, (5) quantifiers, and (6) auxiliary symbols: "(", ")", ","

We use a, b, c, … to denote free variables; x, y, z, … to denote bound variables; P, Q, R, … to denote predicate symbols; $\Box$ to denote connectives; and Q to denote quantifiers, all possibly indexed.

2.2. *Remark* Note that the languages as defined above are function-free. This is only a matter of convenience. All our results remain applicable to logics where the language contains function symbols.

2.3. DEFINITION A *matrix* $\mathbf{L}$ for a language $\mathcal{L}$ is given by:

(1) a nonempty set of *truth values* $V = \{v_1, \ldots, v_m\}$ of size m,

(2) an abstract algebra $\mathbf{V}$ with domain V of appropriate type: For every n-place connective $\Box$ of $\mathcal{L}$ there is an associated *truth function* $\widetilde{\Box}\colon V^n \to V$, and

(3) for every quantifier Q, an associated truth function $\widetilde{\mathsf{Q}}\colon \wp(V) \setminus \{\emptyset\} \to V$

A language and a matrix for it together fully determine a *logic* $\mathbf{L}$. $\mathbf{L}$ is said to be m*-valued*.

The intended meaning of a truth function for a propositional connective is obvious and perfectly analogous to the two-valued case. A truth function for quantifiers is a mapping from nonempty sets of truth values to truth values: given a quantified formula $(\mathsf{Q}x)F(x)$, such a set of truth values describes the situation where the instances of F take exactly the truth values in this set as values under a given interpretation.

2.4. EXAMPLE The matrix for the three-valued Gödel logic $\mathbf{G}_3$ consists of:

(1) The set of truth values $V = \{f, *, t\}$

(2) The truth functions for the connectives:

$\neg$	
t	f
$*$	$*$
f	t

$\wedge$	t	$*$	f
t	t	$*$	f
$*$	$*$	$*$	f
f	f	f	f

$\vee$	t	$*$	f
t	t	t	t
$*$	t	$*$	$*$
f	t	$*$	f

$\supset$	f	$*$	t
f	t	t	t
$*$	f	t	t
t	f	$*$	t

(3) The truth functions for the quantifiers $\forall$ and $\exists$ (generalized $\wedge$ and $\vee$):

$$
\begin{array}{rclrcl}
\widetilde{\forall}(\{t\}) & = & t & \widetilde{\exists}(\{t\}) & = & t \\
\widetilde{\forall}(\{t,*\}) & = & * & \widetilde{\exists}(\{t,*\}) & = & t \\
\widetilde{\forall}(\{t,f\}) & = & f & \widetilde{\exists}(\{t,f\}) & = & t \\
\widetilde{\forall}(\{t,*,f\}) & = & f & \widetilde{\exists}(\{t,*,f\}) & = & t \\
\widetilde{\forall}(\{*\}) & = & * & \widetilde{\exists}(\{*\}) & = & * \\
\widetilde{\forall}(\{*,f\}) & = & f & \widetilde{\exists}(\{*,f\}) & = & * \\
\widetilde{\forall}(\{f\}) & = & f & \widetilde{\exists}(\{f\}) & = & f
\end{array}
$$

2.5. Definition An *interpretation* $\mathbf{M}$ is a mapping of all free variables to elements of a given domain D, and of predicate symbols to functions of type $D^n \to V$. We extend the language s.t. there is a constant symbol d for every $d \in D$ which is interpreted by itself.

A *valuation* $\mathrm{val}_{\mathbf{M}}$ is a mapping that extends the interpretation to formulas via the truth functions given in the matrix. We only give the precise definition of the valuation function for a quantified formula:

$$\mathrm{val}_{\mathbf{M}}\big((\mathsf{Q}x)G(x)\big) = \widetilde{\mathsf{Q}}\Big(\bigcup_{d\in D} \mathrm{val}_{\mathbf{M}} G[d/x]\Big).$$

3 Sequent calculi

Sequent calculi for classical logic were introduced by Gentzen [1934] and were later generalized to the many-valued case by Schröter [1955], Rousseau [1967], and others. More recently, equivalent formulations for tableaux calculi were given (see Surma [1977], Carnielli [1987] and Hähnle [1991]). The method used here can also be used to obtain calculi for transformation into clause form for many-valued resolution (see Baaz and Fermüller [1992]).

As a matter of convenience, we use *sets* instead of *sequences* of formulas in the definition of a sequent.

3.1. Definition An *(m-valued) sequent* is an m-tuple of finite sets Γ_i of formulas, denoted thus:

$$\Gamma_1 \mid \Gamma_2 \mid \ldots \mid \Gamma_m$$

For convenience, we will abbreviate $\Gamma \cup \Delta$ by Γ, Δ; $\Gamma \cup \{F\}$ by Γ, F; and sometimes $\Gamma_1 \mid \ldots \mid \Gamma_m$ by $\mid \Gamma_l \mid_{l=1}^{m}$. We say that F *stands (or occurs) at place* i, if $F \in \Gamma_i$; v_i then is the *truth value corresponding to place* i.

3.2. Definition An interpretation $\mathbf{M}$ is said to *satisfy* a sequent $\Gamma_1 \mid \ldots \mid \Gamma_m$, if there is an i ($1 \le i \le m$) and a formula $F \in \Gamma_i$, s.t. $\mathrm{val}_{\mathbf{M}}(F) = v_i$. A sequent is called *valid*, if it is satisfied under every interpretation.

3.3. Definition An *introduction rule for a connective* $\square$ *at place* i in the logic $\mathbf{L}$ is a schema of the form:

$$\frac{\Big\langle \Gamma_1^j, \Delta_1^j \mid \ldots \mid \Gamma_m^j, \Delta_m^j \Big\rangle_{j\in I}}{\Gamma_1 \mid \ldots \mid \Gamma_i, \square(A_1,\ldots,A_n) \mid \ldots \mid \Gamma_m}\ \square{:}i$$

where the arity of $\square$ is n, I is a finite set, $\Gamma_l = \bigcup_{j\in I} \Gamma_l^j$, $\Delta_l^j \subseteq \{A_1,\ldots,A_n\}$ and the following condition holds:

Let $\mathbf{M}$ be an interpretation. Then the following are equivalent:

(1) $\Box(A_1, \ldots, A_n)$ takes the truth value v_i under $\mathbf{M}$.

(2) For $j \in I$, $\mathbf{M}$ satisfies the sequents $\Delta_1^j \mid \ldots \mid \Delta_m^j$.

It should be stressed that the introduction rules for a connective at a given place are far from being unique: Let the expression A^{v_l} denote the statement "A takes the truth value v_l". Then every introduction rule for $\Box(A_1, \ldots, A_n)$ at place i corresponds to a conjunction of disjunctions of some A^{v_l} which is true iff $\Box(A_1, \ldots, A_n)$ takes the truth value v_i (namely, $\bigwedge_{j=1}^{k} \bigvee_{l=1}^{m} \bigvee_{A \in \Delta_l^j} A^{v_l}$). *Any* such conjunctive normal form for $\Box(A_1, \ldots, A_n)^{v_i}$ will do.

In particular, the truth table for $\Box$ immediately yields a *complete* conjunctive normal form, the corresponding rule is as in Definition 3.3, with: $I \subseteq V^n$ is the set of all n-tuples $j = (w_1, \ldots, w_n)$ of truth values such that $\tilde{\Box}(w_1, \ldots, w_n) \neq v_i$; and $\Delta_l^j = \{A_k \mid 1 \leq k \leq n, v_l \neq w_k\}$.

A rule constructed this way can have up to $m^n - 1$ premises (if $\Box$ takes the value v_i only once in the truth table), but standard methods for minimizing combinational function, such as the Quine-McCluskey procedure, can be used to find minimal (in the number of premises and the number of formulas per premise) rules.

If a connective never takes a particular truth value v_i, then the introduction rule at place i is actually a weakening rule (see below).

3.4. Example Consider the implication in three-valued Gödel logic $\mathbf{G}_3$ given in Example 2.4. The conjunctive forms

$$
\begin{aligned}
(A \supset B)^f &= (A^* \vee A^t) \wedge B^f \\
(A \supset B)^* &= A^t \wedge B^* \\
(A \supset B)^t &= (A^f \vee A^* \vee B^t) \wedge (A^f \vee B^* \vee B^t)
\end{aligned}
$$

yield the following introduction rules:

$$
\frac{\Gamma \mid \Delta, A \mid \Pi, A \quad \Gamma', B \mid \Delta' \mid \Pi'}{\Gamma, \Gamma', A \supset B \mid \Delta, \Delta' \mid \Pi, \Pi'} \supset{:}f \quad \frac{\Gamma \mid \Delta \mid \Pi, A \quad \Gamma' \mid \Delta', B \mid \Pi'}{\Gamma, \Gamma' \mid \Delta, \Delta', A \supset B \mid \Pi, \Pi'} \supset{:}*
$$

$$
\frac{\Gamma, A \mid \Delta, A \mid \Pi, B \quad \Gamma', A \mid \Delta', B \mid \Pi', B}{\Gamma, \Gamma' \mid \Delta, \Delta' \mid \Pi, \Pi', A \supset B} \supset{:}t
$$

3.5. Definition An *introduction rule for a quantifier* Q *at place* i in the logic $\mathbf{L}$ is a schema of the form:

$$
\frac{\left\langle \Gamma_1^j, \Delta_1^j \mid \ldots \mid \Gamma_m^j, \Delta_m^j \right\rangle_{j \in I}}{\Gamma_1 \mid \ldots \mid \Gamma_i, (\mathsf{Q}x)A(x) \mid \ldots \mid \Gamma_m} \; \mathsf{Q}{:}i
$$

where I is a finite set, $\Gamma_l = \bigcup_{j \in I} \Gamma_l^j$, $\Delta_l^j \subseteq \{A(a_1), \ldots, A(a_p)\} \cup \{A(t_1), \ldots, A(t_q)\}$, the a_l are metavariables for free variables (the *eigenvariables* of the rule) satisfying the condition that they do not occur in the lower sequent, the t_k are metavariables for terms, and the following condition holds:

Let $\mathbf{M}$ be an interpretation. Then the following are equivalent:

(1) $(\mathsf{Q}x)A(x)$ takes the truth value v_i under $\mathbf{M}$.

(2) For all $d_1, \ldots, d_p \in D$, there are $d'_1, \ldots, d'_q \in D$ s.t. for all $j \in I$, $\mathbf{M}$ satisfies $\Delta'^j_1 \mid \ldots \mid \Delta'^j_m$ where Δ'^j_l is obtained from Δ_l^j by replacing the eigenvariable a_k with d_k $(1 \leq k \leq p)$ and the term variable t_k with d'_k $(1 \leq k \leq q)$.

The truth function for a quantifier Q immediately yields introduction rules for place i in a way similar to the method described above for connectives: Let $I = \{j \subseteq \{v_1, \ldots, v_m\} \mid \widetilde{\mathsf{Q}}(j) \neq v_i\}$, then the rule is given as in Definition 3.5, with $\Delta_l^j = \{A(a_w^j) \mid w \in j, w \neq v_l\} \cup \{A(t^j) \mid v_l \in V \setminus j\}$. Again, it should be stressed that in general this is not the only possible rule.

3.6. Example Consider the universal quantifier $\forall$ in three-valued Gödel logic $\mathbf{G}_3$ given in Example 2.4. Intuitively, $(\forall x)A(x)$ takes the value f, if $A(t)$ is false for *some* t; t, if $A(a)$ is true for *all* a; and $*$, if $A(t)$ takes the value $*$ for some t and $A(a)$ never takes the value f. We obtain the following rules:

$$\frac{\Gamma, A(t) \mid \Delta \mid \Pi, A}{\Gamma, (\forall x)A(x) \mid \Delta \mid \Pi}\ \forall{:}f \qquad \frac{\Gamma \mid \Delta \mid \Pi, A(a)}{\Gamma \mid \Delta \mid \Pi, (\forall x)A(x)}\ \forall{:}t$$
$$\frac{\Gamma \mid \Delta, A(a) \mid \Pi, A(a) \quad \Gamma' \mid \Delta', A(t) \mid \Pi'}{\Gamma, \Gamma' \mid \Delta, \Delta', (\forall x)A(x) \mid \Pi, \Pi'}\ \forall{:}*$$

3.7. Definition A *sequent calculus* for a logic $\mathbf{L}$ is given by:

(1) Axioms of the form: $A \mid \ldots \mid A$, where A is any formula,

(2) For every connective $\Box$ and every truth value v_i an introduction rule $\Box{:}i$,

(3) For every quantifier Q and every truth value v_i an introduction rule $\mathsf{Q}{:}i$,

(4) Weakening rules for every place i:

$$\frac{\Gamma_1 \mid \ldots \mid \Gamma_i \mid \ldots \mid \Gamma_m}{\Gamma_1 \mid \ldots \mid \Gamma_i, A \mid \ldots \mid \Gamma_m}\ \mathrm{w}{:}i$$

(5) Cut rules for every two truth values $v_i \neq v_j$:

$$\frac{\Gamma_1 \mid \ldots \mid \Gamma_i, A \mid \ldots \mid \Gamma_m \quad \Delta_1 \mid \ldots \mid \Delta_j, A \mid \ldots \mid \Delta_m}{\Gamma_1, \Delta_1 \mid \ldots \mid \Gamma_m, \Delta_m}\ \mathrm{cut}{:}ij$$

A sequent is *provable* in a given sequent calculus, if there is an upward tree of sequents s.t. every topmost sequent is an axiom and every other sequent is obtained from the ones standing immediately above it by an application of one of the rules.

3.8. Theorem *(Soundness) For every sequent calculus in the sense of Definition 3.7 the following holds: If a sequent is provable, then it is valid.*

Proof. By induction on the length of proofs: Axioms are obviously valid, since every formula has to take *some* truth value. Introduction rules preserve validity by definition. The weakening rules are obviously sound. The cut rules are sound, since no formula can take two different truth values. □

3.9. Theorem *(Completeness) For every sequent calculus in the sense of Definition 3.7 the following holds: If a sequent is valid, then it is provable without cuts from atomic axioms.*

Proof. We use the method of *reduction trees*, due to Schütte [1956] (see Takeuti [1987], Ch. 1, § 8). We show that every sequent S is either provable in the sequent calculus or has a counter-model.

Let E be an enumeration of all tuples of free variables. A free variable is *available* at stage k, if it occurs in the tree constructed before stage k (if there is no such variable, pick any and call it available). A tuple of variables is *available* at stage k, if all the variables occuring in it are available; it is *new* if none of the variables occuring in it are available.

A reduction tree is a tree of sequents constructed from S in stages as follows:

Stage 0: Write S at the root of the tree.

Stage k: If the topmost sequent $S' = \Gamma'_1 \mid \ldots \mid \Gamma'_m$ of a branch contains an atomic formula A s.t. $A \in \bigcap_{j=1}^{m} \Gamma'_j$ then stop the reduction for this branch. Call a branch *open* if it does not have this property.

Apply the following reduction steps for every formula F occuring at place i in the topmost sequent $S' = \Gamma'_1 \mid \ldots \mid \Gamma'_m$ of an open branch, which has neither already been reduced at place i on this branch in this stage, nor is the result of a reduction at this stage:

(1) $F \equiv \Box(A'_1, \ldots, A'_n)$: Replace S' in the reduction tree by:

$$\frac{\left\langle \Gamma'_1, \Delta'^j_1 \mid \ldots \mid \Gamma'_i, \Delta'^j_i \mid \ldots \mid \Gamma'_m, \Delta'^j_m \right\rangle_{j \in I}}{\Gamma'_1 \mid \ldots \mid \Gamma'_i \mid \ldots \mid \Gamma'_m}$$

where Δ'^j_r is an instance of Δ^j_r in the rule $\Box{:}i$ introducing F as in Definition 3.3, obtained by instantiating $A_1, \ldots, A_n$ with $A'_1, \ldots, A'_n$, respectively.

(2) $F \equiv (\mathsf{Q}x)A'(x)$: Let $a_1, \ldots, a_p$ be all eigenvariables and $t_1, \ldots, t_q$ be all term variables in the premises of the rule schema $\mathsf{Q}{:}i$. Replace S' in the reduction tree by:

$$\frac{\left\langle \Gamma'_1, \Delta'^j_1 \mid \ldots \mid \Gamma'_i, \Delta'^j_i \mid \ldots \mid \Gamma'_m, \Delta'^j_m \right\rangle_{j \in I}}{\Gamma'_1 \mid \ldots \mid \Gamma'_i \mid \ldots \mid \Gamma'_m}$$

where Δ'^j_r is an instance of Δ^j_r in $\mathsf{Q}{:}i$ introducing F as in Definition 3.5, obtained by instantiating A with A', the eigenvariables $a_1, \ldots, a_p$ with the first new p-tuple of variables in the enumeration E, and the term variables $t_1, \ldots, t_q$ with the first available q-tuple of variables in E which has not been used in a reduction of F at place i on this branch in a stage before k. Observe that $F \in \Gamma'_i$ and thus occurs in all upper sequents.

Now let T_S be the reduction tree constructed in this manner. If T_S is finite, then every topmost sequent contains an atomic formula that occurs at each place in that sequent. A cut-free proof of S from axioms containing these formulas is easily constructed.

If T_S is infinite it has an infinite branch B by König's Lemma. For every atomic formula $P(a_1, \ldots, a_n)$ in B, there is a position l where it never occurs in the sequents in this branch. We construct an interpretation $\mathbf{M}$ as follows: the domain is the set of all free variables, $\mathbf{M}a = a$ (a a free variable) and $\mathbf{M}P(a_1, \ldots, a_n) = v_l$, where v_l is the truth value corresponding to the place l.

If F is a formula occuring in B, and F occurs at place i anywhere in B, then $\mathbf{M}F \neq v_i$. This is easily seen by induction on the complexity of F:

(1) F is atomic: $\mathbf{M}F \neq v_i$ by the construction of $\mathbf{M}$.

(2) $F \equiv \Box(A_1, \ldots, A_n)$: F is reduced somewhere in B. By Definition 3.3, F takes the truth value v_i under $\mathbf{M}$ iff for every $j \in I$, $\mathbf{M}$ there is an $A_l \in \Delta^j_k$ that evaluates to v_k ($1 \leq l \leq n$, $1 \leq k \leq m$). By induction hypothesis none of the A_l in the premise that belongs to B evaluates to the truth value corresponding to its place in the premise. Hence, $\mathbf{M}F \neq v_i$.

(3) $F \equiv (\mathsf{Q}x)A(x)$: By Definition 3.5, F takes the truth value v_i under $\mathbf{M}$ iff for every premise j of the introduction rule (a) all substitution instances of $A(a)$ (where a is some eigenvariable s.t. $A(a)$ occurs in some Δ'^j_k) evaluate to the truth values corresponding to the place at which it stands in the premise, or (b) there is a substitution instance of $A(t)$ (where t is a term variable s.t. $A(t)$ occurs in some Δ'^j_k) evaluates to the truth value at which it stands. By the construction of T_S and the induction hypothesis, the following holds for the premise belonging to B: There is at least one $A(a)$ s.t. $A(a)$ evaluates to a truth value *not* corresponding to the place at which it stands, and *all* substitution instances of all $A(t)$ evaluate to a truth value *not* corresponding to the place at which they stand. Hence, $\mathbf{M}F \neq v_i$.

In particular, no formula in S evaluates to the truth value corresponding to the position at which it stands. Hence $\mathbf{M}$ does not satisfy S. □

4 Natural deduction systems

Gentzen [1934] formulated natural deduction for intuitionistic logic as the system **NJ**. In correspondence with the intuitionistic sequent calculus **LJ**, where the right side of a sequent is restricted to at most one formula, **NJ** deals with inference patterns ("Schlußweisen") of *one* conclusion from a set of assumptions. At the application of rules, assumptions of a certain form can be *cancelled* in parts of the proof. A proof of a formula is a deduction tree where all assuptions have been cancelled.

In **NJ**, the symbol "⊥" is used to denote falsehood, or equivalently, an empty conclusion. Gentzen gives introduction and elimination rules for the connectives and quantifiers, as well as the weakening rule for **NJ**:

$$\frac{\bot}{A}$$

Natural deduction for classical logic **NK** is then obtained by adding *tertium non datur* to **NJ**. Alternatively, one can drop the restriction to one formula in the conclusion and allow sets of formulas. We generalize the classical multi-conclusion system of natural deduction to the m-valued case.

4.1. Definition Let the □-introduction rules at place i be given as in Definition 3.3. The (natural deduction) *introduction rule* □:I *for* □ is given by:

$$\frac{\left\langle \begin{array}{c} \Gamma^j_1, [\Delta^j_1] \mid \ldots \mid \Gamma^j_{m-1}, [\Delta^j_{m-1}] \\ \Gamma^j_m, \Delta^j_m \end{array} \right\rangle_{j \in I}}{\Gamma_m, \Box(A_1, \ldots, A_n)}$$

The *elimination rule* □:E_i *for* □ *at place* $i < m$ is given by:

$$\frac{\begin{array}{c} \Gamma'_1, [\Box(A_1, \ldots, A_n)] \mid \ldots \mid \Gamma'_i \mid \ldots \mid \Gamma'_{m-1}, [\Box(A_1, \ldots, A_n)] \\ \Gamma'_m, \Box(A_1, \ldots, A_n) \end{array} \quad \left\langle \begin{array}{c} \mid \Gamma^j_l, [\Delta^j_l] \mid^{m-1}_{l=1} \\ \Gamma^j_m, \Delta^j_m \end{array} \right\rangle_{j \in I}}{\Gamma_m, \Gamma'_m}$$

The formulas in square brackets are those which can be cancelled at this inference.

4.2. Example The introduction rule for ⊃ in the logic $\mathbf{G}_3$ is:

$$\frac{\begin{array}{cc} \Gamma, [A] \mid \Delta, [A] & \Gamma', [A] \mid \Delta', [B] \\ \Pi, B & \Pi', B \end{array}}{\Pi, \Pi', A \supset B}$$

The elimination rule at place $*$ is:

$$\frac{\begin{array}{c}\Gamma, [A \supset B] \mid \Delta \\ \Pi, A \supset B\end{array} \quad \begin{array}{c}\Gamma'' \mid \Delta'' \\ \Pi', A\end{array} \quad \begin{array}{c}\Gamma'' \mid \Delta'', [B] \\ \Pi''\end{array}}{\Pi, \Pi', \Pi''}$$

The elimination rule at place f is:

$$\frac{\begin{array}{c}\Gamma \mid \Delta, [A \supset B] \\ \Pi, A \supset B\end{array} \quad \begin{array}{c}\Gamma'' \mid \Delta'', [A] \\ \Pi', A\end{array} \quad \begin{array}{c}\Gamma'', [B] \mid \Delta'' \\ \Pi''\end{array}}{\Pi, \Pi', \Pi''}$$

4.3. Definition Let the Q-introduction rules at place i be given as in Definition 3.5. The (natural deduction) *introduction rule* Q:I *for* Q is given by:

$$\frac{\left\langle \begin{array}{c}\Gamma_1^j, [\Delta_1^j] \mid \ldots \mid \Gamma_{m-1}^j, [\Delta_{m-1}^j] \\ \Gamma_m^j, \Delta_m^j\end{array} \right\rangle_{j \in I}}{\Gamma_m, (\mathsf{Q}x)A(x)}$$

The *elimination rule* Q:E$_i$ *for* Q *at place* $i < m$ is given by:

$$\frac{\begin{array}{c}\Gamma_1', [(\mathsf{Q}x)A(x)] \mid \ldots \mid \Gamma_i' \mid \ldots \mid \Gamma_{m-1}', [(\mathsf{Q}x)A(x)] \\ \Gamma_m, (\mathsf{Q}x)A(x)\end{array} \quad \left\langle \begin{array}{c}\mid \Gamma_l^j, [\Delta_l^j] \mid_{l=1}^{m-1} \\ \Gamma_m^j, \Delta_m^j\end{array} \right\rangle_{j \in I}}{\Gamma_m, \Gamma_m'}$$

The eigenvariables in Δ_l^j must not occur in $\Gamma_1, \Gamma_1', \ldots, \Gamma_m, \Gamma_m'$ nor in $(\mathsf{Q}x)A(x)$.

4.4. *Remark* Note that some of Gentzen's original rules for **LK** are different from the those as obtained by Definition 3.3, in that a rule for a given place is split into two, which *together* give a complete characterization of the connective. These rules can also be translated into natural deduction rules as above.

4.5. Definition A *natural deduction system* for a logic **L** is given by:

(1) For every connective $\Box$ and every truth value v_i an introduction rule $\Box$:I and an elimination rule $\Box$:E$_i$;

(2) For every quantifier Q and every truth value v_i an introduction rule Q:I and an elimination rule Q:E$_i$;

(3) The weakening rule:

$$\begin{array}{c}\Gamma_1 \mid \ldots \mid \Gamma_{m-1} \\ \vdots \\ \dfrac{\Gamma_m}{\Gamma_m, A}\ \text{w}\end{array}$$

Weakenings are considered as introductions.

In the classical case, a derivation of a formula F from an assumption A has the intuitive meaning of exactly that: assuming A, we can deduce F. Viewed truth-functionally, this means: assuming that A is true, i.e., not false, then F is true as well. The generalization to the many-valued case is as follows: Given a derivation of F from the assumption $A_1 \mid \ldots \mid A_{m-1}$: if A_i does not take the truth value v_i $(1 \leq i \leq m-1)$, then F takes the truth value v_m.

4.6. Definition A *natural deduction derivation* is defined inductively as follows:

(1) Let A be any formula. Then

$$\frac{\mid A \mid_{l=1}^{m-1}}{A}$$

is a derivation of A from the *assumption* $\mid A \mid_{l=1}^{m-1}$ (an *initial derivation*).

(2) If D_j are derivations of Γ_m^j, Δ_m^j from the assumptions $\Gamma_1^j, \Delta_1^j \mid \ldots \mid \Gamma_{m-1}^j, \Delta_{m-1}^j$, and

$$\frac{\left\langle \begin{array}{c} \Gamma_1^j, [\Delta_1^j] \mid \ldots \mid \Gamma_{m-1}^j, [\Delta_{m-1}^j] \\ \Gamma_m^j, \Delta_m^j \end{array} \right\rangle_{j\in I}}{\Gamma_m}$$

is an instance of a deduction rule (the Δ_i^j may be empty) satisfying the eigenvariable conditions, then

$$\frac{\langle D_j \rangle_{j\in I}}{\Gamma_m}$$

is a derivation of Γ_m from the assumptions $\bigcup_{j\in I} \Gamma_1^j \mid \ldots \mid \bigcup_{j\in I} \Gamma_{m-1}^j$. The formulas in Δ_i^j which do not occur in $\bigcup_{j\in I} \Gamma_i^j$ are said to be *cancelled* at this inference.

4.7. Definition In an elimination, the premises (sets of formulas) containing the formula to be eliminated are called *major premises*, the other premises are called *minor premises*.

We call a formula occurence A

(1) the *conclusion formula* of an introduction, if it is the formula being introduced,

(2) a *premise formula* of an introduction, if it is one of the formulas in Δ_m^j in that introduction,

(3) the *major premise formula* of an elimination, if it is the formula being eliminated,

(4) a *minor premise formula* of an elimination, if it is among the formulas in Δ_m^j in that elimination,

(5) a *cancelled assumption formula* of an elimination, if it stands immediatley below an assumption which contains the formulas in Δ_l^j $(1 \leq j \leq m-1)$ being cancelled at that elimination.

A formula occurence A is said to *follow* A', if both are of the same form and A' stands immediately above A.

4.8. Theorem *(Soundness) If a set of formulas Γ_m can be derived from the assumptions $\Gamma_1 \mid \ldots \mid \Gamma_{m-1}$, then the following holds for every interpretation* $\mathbf{M}$*: If no formula in Γ_i $(i < m)$ evaluates to the truth value v_i, then there is a formula in Γ_m that evaluates to v_m.*

Proof. The statement of the theorem is obviously equivalent to: If Γ_m can be derived from assumptions $\Gamma_1 \mid \ldots \mid \Gamma_{m-1}$, then $\Gamma_1 \mid \ldots \mid \Gamma_m$ is valid. We prove this by inductively translating every derivation D of Γ_m from $\Gamma_1 \mid \ldots \mid \Gamma_{m-1}$ to a sequent calculus proof of $\Gamma_1 \mid \ldots \mid \Gamma_m$:

(1) D is only an initial derivation: The translated proof is an axiom.

(2) D ends in an introduction rule:

$$\dfrac{\left\langle \begin{array}{c} \Gamma_1^j, [\Delta_1^j] \mid \dots \mid \Gamma_{m-1}^j, [\Delta_{m-1}^j] \\ \vdots\, D_j \\ \Gamma_m^j, \Delta_m^j \end{array} \right\rangle_{j\in I}}{\bigcup_{j\in I} \Gamma_m^j, A}$$

The corresponding sequent calculus proof $\sigma(D)$ is:

$$\dfrac{\left\langle \begin{array}{c} \vdots\, \sigma(D_j) \\ \Gamma_1^j, \Delta_1^j \mid \dots \mid \Gamma_m^j, \Delta_m^j \end{array} \right\rangle_{j\in I}}{\bigcup_{j\in I} \Gamma_1^j \mid \dots \mid \bigcup_{j\in I} \Gamma_m^j, A}$$

(3) D ends in an elimination rule:

$$\dfrac{\begin{array}{c} \Gamma_1', [A] \mid \dots \mid \Gamma_i' \mid \dots \mid \Gamma_{m-1}', [A] \\ \vdots\, D' \\ \Gamma_m', A \end{array} \qquad \left\langle \begin{array}{c} \mid \Gamma_l^j, [\Delta_l^j] \mid_{l=1}^{m-1} \\ \vdots\, D_j \\ \Gamma_m^j, \Delta_m^j \end{array} \right\rangle_{j\in I}}{\bigcup_{j\in I} \Gamma_m^j \cup \Gamma_m'}$$

The corresponding sequent calculus proof $\sigma(D)$ is:

$$\begin{array}{c} \dfrac{\begin{array}{c} \vdots\, \sigma(D') \\ \Gamma_1', A \mid \dots \mid \Gamma_i' \mid \dots \mid \Gamma_m', A \end{array} \quad \dfrac{\left\langle \begin{array}{c} \vdots\, \sigma(D_j) \\ \Gamma_1^j, \Delta_1^j \mid \dots \mid \Gamma_m^j, \Delta_m^j \end{array} \right\rangle_{j\in I}}{\bigcup_{j\in I} \Gamma_1^j \mid \dots \mid \bigcup_{j\in I} \Gamma_i^j, A \mid \dots \mid \bigcup_{j\in I} \Gamma_m^j}}{\vdots\ \text{cuts}} \\ \bigcup_{j\in I} \Gamma_1^j \cup \Gamma_1' \mid \dots \mid \bigcup_{j\in I} \Gamma_m^j \cup \Gamma_m' \end{array}$$

(4) D ends in a weakening: Add a weakening at place m to the translated sequent calculus proof.

Note that eigenvariable conditions remain satisfied. □

4.9. *Remark* Translating sequent rules for two-valued logic yield natural deduction elimination rules which differ from those given by Gentzen. However, Gentzen's rules can be obtained in a systematic way by a simplification of the constructed rules. The resulting schema falls outside of our definition of natural deduction rules. We demonstrate this simplification *pars pro toto* for the ∀-elimination rule. The classical version as given by Parigot [1992] is:

$$\dfrac{\begin{array}{c} \Gamma \\ \Delta, (\forall x)A(x) \end{array}}{\Delta, A(t)}$$

The constructed rule is:

$$\dfrac{\begin{array}{c} \Gamma \\ \Delta, (\forall x)A(x) \end{array} \qquad \begin{array}{c} \Gamma', [A(t)] \\ \Delta' \end{array}}{\Delta, \Delta'}$$

Taking $\{A(t)\}$ for Δ' and $\emptyset$ for Γ', we obtain Parigot's rule by disregarding the redundant right premise.

5 Normal derivations

A maximum segment in the intuitionistic natural deduction calculus **NJ** is a sequence of formulas in a derivation that starts with an introduction and end with an elimination. In the classical, multi-conclusion system, it is a sequence starting with an introduction of a formula and ending in an elimination acting on the same formula. A maximum segment constitutes a redundancy in the proof. In **NJ**, and also in multi-valued natural deduction, there are always proofs without such redundancies (see PRAWITZ [1971]).

5.1. DEFINITION A sequence $A_1, \ldots, A_r$ of occurrences of *one and the same* formula is called a *maximum segment*, if A_1 is the conclusion formula of an introduction, A_{j+1} stands immediately below A_j, and A_r is the the major premise formula in an elimination.

5.2. DEFINITION A *normal derivation* is a natural deduction derivation where no major premise of an elimination stands below an introduction.

5.3. PROPOSITION *A normal derivation contains no maximum segments.*

5.4. THEOREM *Every cut-free sequent calculus proof of $S = \Gamma_1 \mid \ldots \mid \Gamma_m$ can be translated into a normal natural deduction derivation of Γ_m from the assumptions $\Gamma'_1 \mid \ldots \mid \Gamma'_{m-1}$, where $\Gamma'_l \subseteq \Gamma_l$ $(1 \leq l \leq m-1)$.*

Proof. By induction on the height h of the proof P of S.

$h = 1$: Then S is an axiom of the form $\mid A \mid_{l=1}^{m}$. The corresponding natural deduction proof $\pi(P)$ is the initial derivation

$$\frac{\mid A \mid_{l=1}^{m-1}}{A}$$

$h > 1$: We distinguish cases according to the last rule in P:

(1) P ends in an introduction rule at place m:

$$\frac{\left\langle \begin{array}{c} \vdots\ P_j \\ \Gamma_1^j, \Delta_1^j \mid \ldots \mid \Gamma_m^j, \Delta_m^j \end{array} \right\rangle_{j \in I}}{\Gamma_1 \mid \ldots \mid \Gamma_m, A}$$

Construct a natural deduction proof $\pi(P)$ of Γ_m, A from the assumptions $\Gamma_1 \mid \ldots \mid \Gamma_{m-1}$ as follows:

$$\frac{\left\langle \begin{array}{c} \Gamma_1^j, [\Delta_1^j] \mid \ldots \mid \Gamma_{m-1}^j, [\Delta_{m-1}^j] \\ \vdots\ \pi(P_j) \\ \Gamma_m^j, \Delta_m^j \end{array} \right\rangle_{j \in I}}{\Gamma_m, A}$$

This only adds an introduction at the end of the derivation, hence $\pi(P)$ is normal.

(2) P ends in an introduction rule at place $i \neq m$:

$$\frac{\left\langle \begin{array}{c} \vdots\ P_j \\ \Gamma_1^j, \Delta_1^j \mid \ldots \mid \Gamma_m^j, \Delta_m^j \end{array} \right\rangle_{j \in I}}{\Gamma_1 \mid \ldots \mid \Gamma_i, A \mid \ldots \mid \Gamma_m} \ \Box{:}i$$

Construct a natural deduction proof $\pi(P)$ of Γ_m from the assumptions $\Gamma_1 \mid \dots \mid \Gamma_i, A \mid \dots \mid \Gamma_{m-1}$ as follows:

$$\dfrac{\begin{array}{c}[A] \mid \dots \mid A \mid \dots \mid [A] \\ A\end{array} \qquad \left\langle \begin{array}{c} \mid \Gamma_l^j, [\Delta_l^j] \mid_{l=1}^{m-1} \\ \vdots\ \pi(P_j) \\ \Gamma_m^j, \Delta_m^j \end{array} \right\rangle_{j\in I}}{\Gamma_m}$$

This only adds an elimination at the beginning of a normal derivation, hence $\pi(P)$ is normal.

(3) P ends in a weakening at place m: Append a weakening to the natural deduction proof.

(4) P ends in a weakening at place $i < m$: Do nothing.

Note that the eigenvariable conditions remain satisfied in the translated proof. □

5.5. Corollary *(Completeness)* *Natural deduction systems are complete.*

Proof. By Theorem 3.9, cut-free sequent calculus is complete, hence every valid sequent $\Gamma_1 \mid \dots \mid \Gamma_m$ has a cut-free sequent calculus proof. The translation of this proof yields a natural deduction derivation of Γ_m from the assumptions $\Gamma_1 \mid \dots \mid \Gamma_{m-1}$. □

5.6. Corollary *(Normal Form Property)* *For every natural deduction derivation, there exists a normal natural deduction derivation of the same set of formulas from the same assumptions.*

Proof. If there is a derivation of Γ_m from $\Gamma_1 \mid \dots \mid \Gamma_{m-1}$, then by Theorem 4.8 there is a cut-free sequent calculus proof of $\Gamma_1 \mid \dots \mid \Gamma_m$, whose translation yields a normal derivation. □

5.7. Definition A *path* in a natural deduction derivation is a sequence of occurences of formulas $A_1, \dots, A_r$ s.t.

(1) A_1 is either
   (a) a formula standing immediately below an assumption or
   (b) is the conclusion formula of an introduction *without premise formulas* (e.g., weakenings);

(2) A_r is either
   (a) an end formula of the derivation or
   (b) a minor premise formula of an elimination or
   (c) a major premise formula of an elimination *without cancelled assumption formulas*, and

(3) A_{j+1} $(1 \leq j \leq r-1)$ is either
   (a) a cancelled assumption formula of an elimination rule, if A_j is the major premise formula of that elimination, or
   (b) the conclusion formula of an introduction if A_j is a premise formula of that rule, or

(c) follows A_j.

5.8. Proposition *A path in a normal derivation can be divided into three (possibly empty) parts:*

(1) The analytical part $A_1, \ldots, A_p$*, where each formula is the major premise formula of an elimination and stands immediately below an assumption;* A_j *is a subformula of* A_{j-1} $(2 \leq j \leq p)$*.*

(2) The minimum part $A_{p+1}, \ldots, A_q$*;* A_j *is equal to* A_{j+1} $(p \leq j \leq q)$*.*

(3) The synthetical part $A_{q+1}, \ldots, A_r$*;* A_{q+1} *is the conclusion formula of an introduction with premise formula* A_q*;* A_{j-1} *is a subformula of* A_j $(q+1 \leq j \leq r)$*.*

5.9. *Remark* If a cut-free sequent calculus proof *with atomic axiom sequents* is translated as in the proof of Theorem 5.4, the minimum segment in a path with non-empty analytical and synthetical part is atomic.

6 Conclusion

We emphasize the fact that the construction of the logical calculi as well as the translations given are purely systematic and can in principle be automatised. Moreover, soundness, completeness and normal form theorems for the systems considered are derived in a uniform way.

It remains to be investigated for which collections of operators one can achieve strong normalisation (i.e. normal form transformations with Church-Rosser property) according to some reasonable definition. (See Parigot [1992] for positive and Zucker [1974] for negative results in the two-valued case.)

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