Young Children's Understanding of Quantity: Conceptualizing Quantities Beyond That Which Can Be Held In Their Hand

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Conceptualizing Quantities Beyond That Which Can Be Held In Their Hand

by

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Abstract

Taking a phenomenological approach, this study explores young children’s understanding of quantity beyond that which can be held in their hand. Descriptions of children’s mathematical experiences are held up against Pirie and Kieren’s theory of growth in mathematical understanding (1994), illustrating the fluid, dynamic nature of children’s ‘moving out’ and ‘folding back’ through various modes of understanding. A closer look at the Primitive Knowing, Image Making, Image Having and Property Noticing levels of the Pirie Kieren model considers the unique characteristics of each. Throughout, children’s actions and expressions, interactions with one another, gestures and facial expressions offer glimpses into the embodied, co-emergent nature of understanding. Specific to quantity, distinctions between discrete and analogue ways of knowing suggest understandings rooted in very different bodily experiences in the world.
Acknowledgements

My heartfelt appreciation to the children who allowed me to learn alongside them; to the philosophers, researchers and theorists whose shoulders I stood upon to try to make sense of it all; to my supervisor, Jo Towers, whose patience and gentle guidance has been unparalleled in my experience as a learner; and to my children, Aaron, Emily and Ben, who weren’t exactly delighted with this undertaking but also didn’t complain.
Table of Contents

Abstract ............................................................................................................................... ii
Acknowledgements ........................................................................................................ iii
Table of Contents ........................................................................................................... iv
Table of Figures .............................................................................................................. vi

CHAPTER ONE: INTRODUCTION .................................................................................. 1

CHAPTER TWO: THE PIRIE KIEREN DYNAMICAL THEORY FOR THE GROWTH OF
MATHEMATICAL UNDERSTANDING ............................................................................. 3

CHAPTER THREE: UNDERSTANDING ........................................................................... 8
3.1 A Dilemma .................................................................................................................. 8
3.2 Languaging Understanding ...................................................................................... 9
3.3 Conceptualizing Understanding: Ideas from a Community of Thinkers ................. 19
3.4 Constructivism ......................................................................................................... 20
3.5 Metaphor .................................................................................................................. 22
3.6 Embodiment ............................................................................................................. 23

CHAPTER FOUR: QUANTITY ...................................................................................... 27
4.1 Subitizing, Magnitude Appreciation, Discrete and Analogue Quantity, Counting,
Metaphor and Analogy ................................................................................................. 29
4.2 Unique Understandings of Quantity: Savantism, Gerstmann Syndrome and Other
Neurological Conditions ............................................................................................... 35
4.2.1 GT and other Individuals with Savant Syndrome ............................................... 36
4.2.2 HP and other Individuals with Gerstmann Syndrome or other forms of
Dyscalculia .................................................................................................................... 40
4.3 Observations and Wonderings ................................................................................ 42

CHAPTER FIVE: METHODS AND METHODOLOGY .................................................... 44
5.1 A Phenomenological Approach ................................................................................ 44
5.2 Site Selection and the Design of Learning Experiences .......................................... 49
5.3 Ethical Considerations in Research Design ............................................................ 53
5.4 Ethical Considerations in the Research Experience ................................................ 56
5.5 Data Collection ......................................................................................................... 57
5.6 Data Analysis and Interpretation ............................................................................. 59
5.7 Researcher Reflexivity .............................................................................................. 62

CHAPTER SIX: EMBODIMENT AND CO-UNDERSTANDING ...................................... 65
6.1 Embodied Understanding: JJ and Marlo .................................................................. 65
6.2 Gesture: Marlo and JJ, Manraj, and Sam ................................................................. 68
6.3 Understanding as Co-emergent: Marlo and JJ ......................................................... 71
6.4 Observations and Insights ....................................................................................... 76

CHAPTER SEVEN: EXPERIENCES WITH QUANTITY ............................................... 77
7.1 Children’s Distinctions Between ‘Small’ Numbers and ‘Big’ Numbers ................. 77
7.2 Rani’s Rice: Reasoning Through Very Large Numbers ........................................... 80
7.3 Harjit ....................................................................................................................... 82
7.4 Heidi ....................................................................................................................... 85
7.5 Clark ....................................................................................................................... 86
7.6 Liam ....................................................................................................................... 88
7.7 Gloria ..................................................................................................................... 90
7.8 Observations and Insights ..................................................................................... 93

CHAPTER EIGHT: MAGNITUDE APPRECIATION AND ANALOGUE QUANTITY95
8.1 Scanning .............................................................................................................. 95
8.2 Gestures for Discrete and Analogue Quantity ....................................................... 99
8.3 Observations and Insights .................................................................................... 100

CHAPTER NINE: INFINITY ..................................................................................... 101
9.1 Sam ...................................................................................................................... 101
9.2 Adarsh, Jyoti and Indulal .................................................................................... 106
9.3 Explorations of Infinity in Grade Two ................................................................. 109
9.4 Mona .................................................................................................................... 113
9.5 Jackson, Donovan and Bob .............................................................................. 114
9.6 Debriefing as a Class ......................................................................................... 117
9.7 Observations and Insights ................................................................................. 119

CHAPTER TEN: DISCUSSION .............................................................................. 121
10.1 Revisiting Pirie and Kieren ............................................................................... 121
  10.1.1 Primitive Knowing ....................................................................................... 123
  10.1.2 Image Making and Image Having ............................................................... 127
  10.1.3 Property Noticing ...................................................................................... 133
  10.1.4 Formalising, Observing, Structuring and Inventising ......................... 135
  10.1.5 Moving out and Folding Back in Dynamic, Nonlinear Patterns of Growth 137
10.2 Understanding ................................................................................................... 138
  10.2.1 Embodiment .............................................................................................. 139
  10.2.2 Co-Understanding .................................................................................. 142
10.3 Discrete and Analogue Quantity ..................................................................... 144
10.4 Implications ........................................................................................................ 147
  10.4.1 Mathematics Instruction in Alberta ......................................................... 147
  10.4.2 Wonderings .............................................................................................. 154

CHAPTER ELEVEN: REFLECTIONS ON THE RESEARCH EXPERIENCE ........... 156
References .............................................................................................................. 160
Addendum ............................................................................................................. 166

APPENDIX A: PARTICIPANTS ............................................................................. 171
Table of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure 1</td>
<td>Pirie and Kieren’s Dynamical Model of Growth of Mathematical Understanding</td>
<td>4</td>
</tr>
<tr>
<td>Figure 2</td>
<td>Marlo Pictures in her Head by Feeling or Maybe Seeing</td>
<td>66</td>
</tr>
<tr>
<td>Figure 3</td>
<td>JJ Feels What Numbers are Like</td>
<td>67</td>
</tr>
<tr>
<td>Figure 4</td>
<td>Nine is a Four and a Four and One More</td>
<td>68</td>
</tr>
<tr>
<td>Figure 5</td>
<td>Marlo Breaks Apart 23</td>
<td>69</td>
</tr>
<tr>
<td>Figure 6</td>
<td>JJ Experiences Her Own Understanding of ‘Breaking Apart’</td>
<td>70</td>
</tr>
<tr>
<td>Figure 7</td>
<td>Sam Picking Stuff Up and Moving it Around</td>
<td>70</td>
</tr>
<tr>
<td>Figure 8</td>
<td>Bracketing</td>
<td>71</td>
</tr>
<tr>
<td>Figure 9</td>
<td>Marlo and JJ Add 60 and 60</td>
<td>74</td>
</tr>
<tr>
<td>Figure 10</td>
<td>Marlo and JJ Add 60 to 120</td>
<td>75</td>
</tr>
<tr>
<td>Figure 11</td>
<td>Manraj Explains Subitizing</td>
<td>77</td>
</tr>
<tr>
<td>Figure 12</td>
<td>Jada’s Jar of Beans</td>
<td>79</td>
</tr>
<tr>
<td>Figure 13</td>
<td>Jada Guesses My Age</td>
<td>79</td>
</tr>
<tr>
<td>Figure 14</td>
<td>One Grain of Rice</td>
<td>80</td>
</tr>
<tr>
<td>Figure 15</td>
<td>Students’ Estimation of Quantity and Container Size</td>
<td>84</td>
</tr>
<tr>
<td>Figure 16</td>
<td>Heidi’s 200 Grains of Rice</td>
<td>85</td>
</tr>
<tr>
<td>Figure 17</td>
<td>Clark Thought 100 Would Take the Whole Bottle</td>
<td>86</td>
</tr>
<tr>
<td>Figure 18</td>
<td>Heidi Changed her Estimate</td>
<td>87</td>
</tr>
<tr>
<td>Figure 19</td>
<td>Liam Estimates About 6000</td>
<td>89</td>
</tr>
<tr>
<td>Figure 20</td>
<td>Liam Finds a Pattern</td>
<td>89</td>
</tr>
<tr>
<td>Figure 21</td>
<td>Gloria’s Calculations to Day 14</td>
<td>90</td>
</tr>
<tr>
<td>Figure 22</td>
<td>Gloria Begins her Graph</td>
<td>91</td>
</tr>
<tr>
<td>Figure 23</td>
<td>Gloria Adds Pages</td>
<td>91</td>
</tr>
<tr>
<td>Figure 24</td>
<td>Gloria’s Graph to Day 16</td>
<td>92</td>
</tr>
<tr>
<td>Figure 25</td>
<td>Levi Counted a Little Bit and then Guessed</td>
<td>96</td>
</tr>
<tr>
<td>Figure 26</td>
<td>Jackson Estimates</td>
<td>96</td>
</tr>
<tr>
<td>Figure 27</td>
<td>Amir ‘Scanning’</td>
<td>97</td>
</tr>
<tr>
<td>Figure 28</td>
<td>Hank Gestures One Hundred</td>
<td>98</td>
</tr>
</tbody>
</table>
Figure 29: Levi's Pinching Hand and Sweeping Arm.................................................................99
Figure 30: Adarsh Explains his Reasoning for Discrete and Analogue Quantity..................100
Figure 31: Sam Shows how Numbers go in a Loop ................................................................. 102
Figure 32: A Model for Sam.....................................................................................................103
Figure 33: Sam's Sweeping Gesture for Infinity................................................................. 104
Figure 34: Sam’s Model for Infinity.....................................................................................105
Figure 35: Adarsh, Jyoti and Indula's First Model for Infinity ..................................................108
Figure 36: Siobhan’s Black Hole.............................................................................................112
Figure 37: Alyssa and Amir's Spiral........................................................................................112
Figure 38: Mona’s Curved Line..............................................................................................113
Figure 39: Mona’s Sweeping Gesture.......................................................................................113
Figure 40: Bob’s Sweep of the Arm.......................................................................................115
Figure 41: Jackson’s Sweeping Arm Gesture.........................................................................115
Figure 42: Donovan Adds Numbers to His Circle....................................................................116
Figure 43: Jackson’s Notebook.............................................................................................117
Figure 44: Amir Shows how a Square Would Turn ...............................................................118
Figure 45: Richard Compares Discrete and Analogue Counting............................................119
Figure 46: Lara Gesturing Pitch and Volume...........................................................................145
Figure 47: Excerpt from the Alberta Mathematics Program of Studies .................................148
Figure 48: Sample Learning Progression in an Alberta Recommended Resource ...............150
Figure 49: Lesson 4..................................................................................................................151
Figure 50: Introducing Numbers to 10 000..........................................................................152
Figure 51: “Visualization” .....................................................................................................152
Chapter One: **Introduction**

The aim of this study is to describe young children’s mathematical understandings. It addresses the question of how children conceptualize quantity beyond that which can be held in their hand—how they experience and make sense of quantity larger than a tangible, familiar, countable amount. This required two research components:

- an investigation of what may be meant by ‘understanding’, within mathematics and beyond, and
- an exploration of the experiences of young children working with quantity.

Taking a phenomenological approach, the design of this study focused careful attention on the experiences of individual students: their actions and expressions; their gestures and facial expressions; the questions they asked and the insights they shared. Interpretations involve placing descriptions of these experiences alongside one another and against a grounding theoretical framework. It is hoped that this offering—this positioning of my experiences, the experiences of the students, and the ideas of those who have come before me—will create a space for meaningful intersection with the experiences and understandings of the reader.

The Pirie–Kieren Dynamical Theory for the Growth of Mathematical Understanding provides the primary framework against which experiences of understanding are characterized, while the work of Kim, Roth and Thom (2011), Roth and Thom (2009 & 2011), Thom and Pirie (2009); Towers and Davis (2002), Wagner and Davis (2010), Maher and Davis (2010a and 2010b) and Davis (1996); Lakoff and Johnson (1980 & 1999), Lakoff and Nuñez (2000), Nuñez, Edwards and Matos (1999) and others supported my interpretations of gesture, action and other forms of
embodied understanding. In addition to the above, Martin and Towers (2009) informed my readings of co-emergent understandings, a significant finding unanticipated at the outset of the study.

The mathematics of quantity: subitizing, counting, magnitude appreciation and the difference between discrete and analogue quantity, are explored alongside Davis (1996), Wagner and Davis (2010), Cantlon (2012), Cappelletti et al. (2009), Dehaene (1997 & 2003), Gallistel and Gelman (1992 & 2000) and others.

Throughout, I return again and again to the guiding wisdom of Merleau-Ponty (1945) in hopes that he can keep me from straying too far; keep my means and my ends aligned.

Despite the collective genius of all of these profound thinkers and prolific writers, the wisdom I delighted in most was that of the children with whom I studied. The brilliance of Adarsh’s words, Sam’s hula hoop, Gloria’s skipping and Marlo and JJ’s fingertips were the true source of insight and inspiration.
Pirie and Kieren (1994) have developed a model for describing growth of mathematical understanding that is “whole, dynamic, leveled but non-linear—a transcendentally recursive process” (p. 166). In this model, the process of building understanding involves moving back and forth between different ways of knowing, represented as a series of nested layers or levels. Although a hierarchy of sorts exists within this structure, Pirie and Kieren are clear that their intention is not to delineate between “low level/high level” mathematics, nor to impart a value differential whereby certain ways of knowing are “better” or “worse” (p. 172). In this model, it is the movement that occurs within the structure that reveals the highly integrated and coordinated ways that growth of mathematical understanding occurs.

Moving back and forth between levels in the Pirie Kieren model involves “folding back” and “moving out”, a process whereby the learner returns to an earlier, inner-level way of knowing before returning to a more complex, outer-level way of knowing. In this folding back process, the learner brings along the outer-level knowledge when returning to inner levels, thereby “thickening” them—understanding them again in a new, more comprehensive way. The pathway taken when moving between levels is unique to each individual and to the context of their learning.

An additional feature of the Pirie Kieren model, evident in Figure 1 below, is a distinct border between certain levels of mathematical understanding. Pirie and Kieren label these distinctions
“don’t need boundaries” and describe them this way: “Beyond these boundaries the learner is able to work with notions that are no longer obviously tied to previous forms of understanding, but these previous forms are embedded in the new level of understanding and readily accessible if needed” (p. 172).

Figure 1: Pirie and Kieren’s Dynamical Model of Growth of Mathematical Understanding
Moving from the innermost to the outermost layer, the levels of mathematical understanding are described this way:

- **Primitive Knowing**: the starting place from which growth of mathematical understanding begins;
- **Image Making**: making distinctions within previous ways knowing and beginning to use that knowledge in new ways;
- **Image Having**: using a mental construct about a topic without having to do the particular activities which brought it about;
- **Property Noticing**: manipulating or combining aspects of one’s images to construct context-specific, relevant properties;
- **Formalising**: abstracting a method or common quality from the noticed properties;
- **Observing**: reflecting on or coordinating noticed properties in order to put forward theorems;
- **Structuring**: attempting, through recognition of the interrelationship of a collection of theorems, to justify or verify statements through logical or meta-mathematical argument; thinking about one’s formal observations as theory; and
- **Inventising**: breaking away from preconceptions and creating new questions which might grow into a totally new concept.

(Pirie & Kieren, 1994, pp. 165-171)

It is important to note that within this model the term Primitive Knowing does not imply simplistic or unsophisticated understanding, but rather describes a starting point for new
learning. Primitive Knowing may carry with it any range of understandings, and any outer level of understanding may in turn become the Primitive Knowing in a new context. “The model has a fractal-like quality: inspection of any particular Primitive Knowing will reveal the layers of inner knowings” (p. 172).

One further characteristic of the Pirie Kieren model worth discussing here has to do with the experiences of understanding that occur within each level. A complementarity of actions and expressions, Pirie and Kieren assert, compose each layer of understanding; students move back and forth between the two. *Actions*, in this model, are understood to encompass mental as well as physical activities the learner experiences, including making observations, discussing ideas with others, manipulating materials, hypothesizing, illustrating and modeling, testing strategies, etc. *Expression*, the complementary (and equally necessary) component of the learning experience, involves looking at and articulating what was involved in one’s actions. Expression, here, may take the form of intentional communication with others, or it may simply involve noticing for oneself what is of significance in the learning context. The distinction between action and expression, when one is made, may be found in the function the experience serves in supporting growth in understanding rather than in what is outwardly visible in the learner’s actions.

The Pirie Kieren model offers a way to characterize various types of mathematical understandings and to observe the patterns and pathways that emerge as learners move between different levels or layers of understanding. Growth in mathematical understanding involves continuous restructuring—a process of returning to earlier, inner level understandings in order to
build new foundations for outer level understandings. This process is neither linear nor fixed—an individual’s growth of understanding may skip over levels, step back and forth between neighbouring levels, linger in a certain space, or leap, in a moment of profound insight, past several levels at once. Regardless of the pattern that characterizes an individual’s growth of mathematical understanding, the process of “folding back” and “moving out” is dynamic and nonlinear; it allows for the thickening, strengthening and reconstruction of increasingly sophisticated and complex understandings (p. 173).
Chapter Three: Understanding

3.1 A Dilemma

While the Pirie Kieren model provides the grounding against which growth in mathematical understanding can be observed, questions remain for me about what occurs (in a cognitive, conceptual or experiential sense) at any particular level of understanding. As discussed previously, Pirie and Kieren suggest a complementarity of acting and expressing within each layer, and hold that that understanding is “whole, dynamic, leveled but non-linear” (p. 166)–by nature indivisible into discrete component parts. Furthermore, they note, “it is important that the reader realise that we see understanding as a process and not as an acquisition or location” (Pirie & Kieren, 1994, p. 175). True. It is not my contention that understanding ought to be objectified–I don’t believe it can be dissected, compartmentalized and laid bare for our scrutiny like some sort of biological specimen. But how might one go about observing and describing that which exists perpetually in a state of flux? Is it possible to snap a meaningful picture of something that is forever in motion? Forever evolving? Not even a thing, per se, but as per Pirie and Kieren, a process? Is it even appropriate to attempt to do so?

The application of the Pirie Kieren model–the function it serves in the broader community of mathematics education–is to offer a common language and possible structure for observing and describing growth in mathematical understanding. Of interest to this study was a closer look at the first four levels of understanding identified in their model: Primitive Knowing, Image Making, Image Having and Property Noticing. Without denying the holistic and dynamic nature of understanding, the phenomena of interest to this study are individual students’ understandings within each of these levels. I am intensely curious about what it could mean to make an image,
or to have one; to bring to a novel learning experience some sort of Primitive Knowing; or to notice properties. When a student embarks on a new learning experience, what is present in their Primitive Knowing? What is absent? What does it mean to have, or not to have, that understanding? What changes for the learner in that process?

3.2 Languaging Understanding

In the chapters that follow, I explore these questions in much greater depth, beginning with some general histories and philosophies of the notion of understanding. Before I do so, however, a preliminary discussion of how to language understanding is in order. Each author who influenced this study used different terminology for understanding. So what terms to use? Clearly there is a difference between the terms understanding and meaning. What is it? When does it matter to delineate between the two? And what about idea? Knowledge? Concept? Construct? Representation? Image?

Pirie and Kieren (1994) use the term idea throughout their work when referring to the thoughts of an individual student (e.g., p. 170), as well as to their own thoughts as they describe the process of developing their model (e.g., p. 165). In the more formal process of selecting appropriate language with which to label the levels within their model, however, they comment that the decision to use the term image rather than the term idea had to do with idea’s relative ambiguity (p. 166). Pirie and Kieren write in some depth about the difficulties inherent in selecting appropriate terminology to describe and define elements of their model. Decisions about when to use existing terminology and when to use invented terms were complex, and each possibility carried its own inherent weaknesses.
In addition to the term *idea* and the labels given to the various levels of mathematical understanding, Pirie and Kieren generally use the term *meaning* to describe the process by which an individual ascribes significance to their ideas and experiences (e.g., p. 174) while the term *concept* is used to describe an understanding belonging to a specific discipline or body of common knowledge (e.g., p. 172). Other terms used include *mental object* (e.g., p. 166), *mental construct* (e.g., p. 170), *mental picture* (e.g., p. 173), *mental activities* (e.g., p. 175) *mental images* (e.g., p. 165) and even a *mental note* (e.g., p. 180) in reference to a range of thought activities that occur within the Image Making and Image Having levels.

In exploring the language used by Scardamalia and Bereiter (2010); Lakoff, Johnson and Nuñez (1990, 1999, 2000); Merleau-Ponty (1945); Davis and Maher (1990, 1996, 2010); von Glasersfeld (1983, 1989a, 1989b, 1990, 2006); Roth and Thom (2009 and 2010) and others who influenced this study, there appeared to be little consistency in the choice of terms used and little clarity about the relative meaning of each term. At times, many different terms are used within a single text. Throughout the literature, the term *idea* is perhaps the most ubiquitous, and certainly Scardamalia and Bereiter (2010) and others make extensive use of the term in reference to the generic, prototypic version of an individual’s thoughts. Ideas are generally decontextualized or are referenced within broad, nonspecific applications. When referring to understandings that are more clearly contextualized and specific to the individual, Scardamalia and Bereiter use the term *knowledge constructs*, and when referring to more formal ideas (generally shared understandings within a broader community), the term *intellectual constructions* is used. As noted by Roth and Thom (2009), the term *concept* is generally understood to refer to a publicly held idea, whereas
the terms conception and conceptualizations are commonly used to describe an individual’s personal understanding.

Merleau-Ponty (1945) uses the term idea as well, but makes no reference to any sort of mental object or knowledge construct. Although he allows for the possibility of a mental field (e.g., p. 188), mental space (e.g., p. 146) or mental panorama (e.g., p. 150), he is clear that “there can be no question of isolating, in my physiological representation of the phenomenon, the retinal images and their cerebral counterpart from the total field, actual and possible, in which they appear” (p. 408). The world, after all, is “not what I think but what I live through” (p. xvi). For Merleau-Ponty, perceptions (understood as the interplay of sensation and judgment) and actions are thought and thinking. They are consciousness and “logic lived through” (p. 57). Aligning with Kierkegaard’s critique of Descartes’, cogito ergo sum, Merleau-Ponty speaks of “…an ‘I think’ which is, by itself and without any adjunct an ‘I am’” (p. 433).

Maher and Davis (1990), on the other hand, use the term mental representations to refer to the ideas that are ‘built up’ as students work to construct images of a problem and posit possible solutions. In Teaching Mathematics: Toward a Sound Alternative (1996), Davis refers to various forms of understanding (formulated, unformulated, enacted, subjective, shared, conceptual, procedural etc.), and meaning (collective, formalized, formulated, linguistic, enacted etc.).
When addressing the nuanced distinction between meaning and understanding, Davis acknowledges the importance of the gray area between the two terms and offers these insights:

- …understanding seems more closely aligned with the interpersonal realm of ever-evolving conversation…and meaning seems more closely linked to the objectifying realm of definition-seeking discussions (p. 200).

- Understandings, however, seem to be more immediate, more fluid, more negotiable. Meanings, while still dynamic and still subject to bargaining processes, are more authoritative, as the term connotes stronger senses of unilateral fixing, of definitiveness, of stating (p. 200).

- Understandings are thus not merely dynamic, they are relationally, contextually and temporally specific (p. 201).

- Conventionally, “meaning” tends to be understood in terms of connective associations (p. 205); and

- The bulk of our meanings are…lived through or enacted (p. 205).

Throughout the literature, terminology associated with the word structure (e.g., construct, construction, structuring, instruction etc.), lends dimensionality to notions of understanding. The term conceptual structure, for example, is used by many authors (e.g., Arcavi, 2003; Murphy, 2006; von Glasersfeld, 1983) to refer to a specific network of interconnected ideas held by an individual, applicable within a particular context or contexts. Here, the term structure does not imply a neatly organized, easily recognized, generically assembled, right-angled construction. Rather, it refers to organic, dynamic, self-organized conceptual relationships. Although they do not use the term conceptual structure in this particular piece, in Structuring Occasions (2002),
Towers and Davis discuss the etymology of the term *structure* and, in an effort to push back against the “trivial interpretations of constructivist epistemology,” delineate differences between architectural and biological meanings:

…used in reference to modern building projects, structure and construction connote deliberate senses of planning and step-following. As taken up in formal discussions of teaching and learning, the allusion to architecture is caught up in a web of associations which includes such notions as basics, foundations, platforms, scaffolds, building blocks, hierarchies, frameworks, and so on—as well as, by implication, such ideals as order, rigidity, and permanence.

The biological meaning of structure is quite different. Used in such phrases as ‘the structure of an organism’ and ‘the structure of an ecosystem’, the word points to the complex history of organic forms. Structure in such cases is both caused and accidental, both familiar and unique, both dependent and autonomous.

The biologist’s use of the term is more faithful to its original meaning. Etymologically linked to ‘strew’ and ‘construe’, structure was first used to describe how collections of forms tend to spread out or pile up in ways that cannot be predetermined, but that are not completely random either.

(Towers & Davis, 2002, p. 315-16)
Although the primary focus for this particular publication was the co-opting of constructivist epistemologies as ‘constructivist teaching’, Towers and Davis describe *structures of personal understanding, an individual’s knowledge structures* or most simply *understanding* as complex and contingent (e.g., p. 318), emergent and evolving (e.g., p. 317), and continually revisited and reorganized (e.g., p. 318).

In *Facts And The Self From A Constructivist Point Of View* (1989a), and *Cognition, Construction of Knowledge*, von Glasersfeld (1989b) also uses the term conceptual structures to refer to thoughts or ideas deemed viable through one’s experience. He further discusses the delineation between *Vorstellung* (an autonomous internal construction) and *Darstellung*, (a structure that is considered the picture or illustration of something else) and discusses the relative inflexibility of English language in describing the various structures of one’s understanding. As with other constructivists, von Glasersfeld rejects the notion of an objective mental representation of an observer-independent world, but rather discusses the continual revision of understanding inherent to the organism’s eternal pursuit of equilibrium. As such, the term conceptual structure does not refer to a static representation of an external ‘truth’, but refers to a negotiated and continually evolving way of thinking one’s way forward.

In their writings on embodiment and metaphor, Lakoff and Johnson also make extensive use of the term conceptual structure. A linguist and a philosopher, respectively, they put forward a somewhat more positivist view of understanding. From their perspective, “conceptual structure arises from our sensorimotor experience and the neural structures that give rise to it” (Lakoff & Johnson, 1999, p. 77). Conceptual structures are “intrinsically meaningful by virtue of their
connection to our bodies and our embodied experience” (p. 77). In addition to the term conceptual structure, however, Lakoff and Johnson use a whole range of other terms as well, not entirely interchangeably but also without clearly defined boundaries of meaning:

- *neural architecture* or *neural networks* (referencing the broad structure or landscape of one’s cognition),
- *conceptual systems* (the organizational structure of thoughts or ideas—as per human conceptual systems or mathematical conceptual systems),
- *conceptual domains* (bounded collections of understandings held by a community),
- *mental structures* (akin to conceptual structures)
- *mental images* and *perceptual gestalts* (visual—cognitive representations of the world or of one’s ideas)
- *mental representations* (symbolic, although not necessarily visual, recreations of an external reality)
- *neural models* (schematic, functional prototypes that guide thought or action),
- and *conceptual frames* (thought structures that provide a referential structure or scaffold while transcending an individual idea).

(Lakoff & Johnson, 1999)

Lakoff and Núñez (2000) also use the term *cognitive mechanisms*, referring more specifically to thought processes or cognitive actions. One other term used extensively in the writings of Lakoff, Johnson and Núñez is *schema*, a label for the type of meaningful, patterned, often
subconscious thought that allows one to transfer the reasoning applicable to one context to an entirely different context.

Despite the wide range of language established thus far to describe understanding, there exists in the writings of Lakoff, Nuñez and Johnson a further breakdown or compartmentalization of terms: ideas are sub-categorized into mathematical ideas, human ideas, innate ideas, and so on; cognition into embodied cognition, disembodied cognition, metaphorical cognition; concepts into basic-level concepts, abstract concepts, aspectual concepts, formal concepts, spatial-relational concepts; and on, and on, and on...

For all its abundance and complexity, and in spite of the heroic and often poetic efforts of different authors to clarify what is meant by the various terms, most of the language still feels inadequate. Shallow. Reductionist. Perhaps what has not surfaced sufficiently in any of these terms is the notion of judgment—an acknowledgement of the contingent interplay of self and context. Merleau-Ponty (1945) establishes judgment as “what sensation lacks to make perception possible” (p. 37). Perception involves re-cognition, association, emotion; “any perception of relationship [is] a judgment” (p. 41). Certainly the constructivists impart active, intentional qualities to the act of building conceptual structures, and it follows that it is one’s judgment (at the level of conscious thought or not; often embodied) that allows for relationships to be recognized and for connections between ideas to be made, broken and rebuilt. In the Pirie Kieren model, each level of understanding, as well as one’s movement between them, is characterized by judgment. At each level of understanding, obstacles and incongruencies of experience prompt “folding back” to revise prior knowledge, and understanding at the levels of
Property Noticing and beyond involve a conscious and intentional analysis, synthesis or evaluation of ideas.

Still, the language available seems to limit, rather than illuminate, the topic.

The image of a student from long ago comes to mind—a six year old girl in pink-rimmed glasses and a blonde ponytail, describing how she would add two numbers in her head. Britta. Considering the question, she looked up and to the right, blinked several times, and then reached up with both hands to grasp an imaginary object just beside her left temple. *So I will put,* she said, *in my head will put 120 over here* [gives the air between her two hands a gentle squeeze] *and I will put 15 over here* [moves her hands to the right temple and squeezes again]. The thing she squeezed with her hands was more than an idea. It was something she could see and manipulate, pull apart and recombine, set aside and revisit. Analyze, evaluate and revise. As she carried out the task, she described what would happen when she put the numbers together—*first,* *I will take the 20 and I will put it with the 5* [moves her left hand, still grasping, over to the right], *and then I would put the hundred together with it and it would make one more...* As she worked through her reasoning, it was clear that understanding was something she was actively *doing;* her actions followed the structures and relationships that characterized her understanding. Moreover, in so doing, she was concurrently revisiting and revising the very ideas she was working with. Her doing was her understanding. Once she had arrived at her solution, she paused, gazing up and to the right once more, apparently reviewing her procedure and confirming its accuracy, gave a little nod and repeated her answer.
Britta understood. She made meaning; used judgment. Her understandings had spatial and relational qualities—they were *tangible*. They functioned within an organizational structure that provided them with internal coherence—an ecology of sorts. Furthermore, this student’s understanding was contingent on mathematical concepts that were not uniquely her own; they borrowed from and contributed to the broader structural-organizational system of mathematics. And judgment (her negotiation of fit) guided her thought process from beginning to end.

In the end, when she looked up and to the right, gave a little nod and repeated her answer, Britta had something in mind that hadn’t been there before. What was it? An idea? It was clearly more than an idea. An idea is too amorphous; inert. An idea can’t be squeezed. A mental object could be squeezed, but it would be closed to the possibility of ongoing revision. A mental object is too rigid. Static. Impermeable. Britta’s understanding involved strategy and creativity. So, was she using a conceptual structure? Perhaps. Her understanding could be imagined according to the etymological roots of the word understanding (to stand in the midst of), and one could think of her standing in the midst of a structure—in the ecological sense of the word—observing its relationships, moving its moveable pieces, unraveling and re-weaving the fabric of its reality. But then the conceptual structure would be something that *belongs to her*, or that she belongs to—it isn’t, *simply and without adjunct* as Merleau-Ponty might say, *her*.

Perhaps this thesis is my own thinking-as-being. My own becoming. As such the words I choose will come to mean what they will come to mean as they are written; and as they are read they will come to mean something else. In this text, I make use of the terms idea, understanding, meaning and concept. The term idea, as per Scardamalia and Bereiter (2010), will refer broadly
to an individual’s thoughts or thinking. I’ll borrow from Davis (1996) in creating a distinction between the terms understanding and meaning, where understanding is more dynamic, contextual and negotiated and meaning is somewhat more grounded in the significance one gives to ideas—the generalizability of one’s ideas to a broader context. For the term concept, I’ll defer to von Glasersfeld (1989a), who offers the suggestion that concepts (or facts or theories) “gain a higher degree of viability when successful predictions can be made by imputing the use of these facts to Others. This additional viability is the constructivist’s counterpart to ‘objectivity’” (p. 447). As such, concepts may be understood as ideas that have been deemed viable within a broader community of thinkers.

3.3 Conceptualizing Understanding: Ideas from a Community of Thinkers

For some 2500 years the Western world has manifested an overwhelming tendency to think of knowledge as a cognitive organism’s representation of an outside world, its structure, and how it works. The representation might not yet be quite perfect, but, in principle, it was thought to be perfectible. In any case, its goodness was supposed to depend on the degree of correspondence between it and the outside world called ‘reality’. Today, this way of thinking is no longer viable.

(von Glasersfeld, 1989a, p. 435)

What remains to be understood precisely is the way the world comes to belong to the subject and the subject to himself, which is that cogito which makes experience possible; our hold on things and on our ‘states of consciousness’.

(Merleau-Ponty, 1945, p. 435)
If we begin with what understanding is not—neither the pursuit of increasingly accurate mental representations of an observer-independent reality, nor the arbitrary individual creation of one’s unique idea of the world—then perhaps we can begin to look at understanding as *being*.

### 3.4 Constructivism

Constructivism, as a theory of knowing, takes a pragmatic approach in that it sets aside questions of a reality that may or may not exist separate from our experience of it, and focuses instead on questions of how concepts and experiences co-evolve. For von Glasersfeld, the pursuit of understanding is the pursuit of compatibility or viability. In his words, “compatibility is a matter of avoiding clash, passing between obstacles, fitting into space that is not encumbered by the conditions that have to be complied with” (von Glasersfeld, 1983, p. 209). Compatibility, here, has a fundamentally different nature than ‘matching’—our experience in building understanding is not one of creating an exact replica of something that exists objectively apart from our perception and our actions. Rather, it is one of pulling together previous experiences and testing their viability with the current circumstances of our experience.

For the constructivist, then, cognition is not about representing a real world, but rather is about organizing (and constantly reorganizing) one’s own subjective world of experience. In this frame, cognition (unlike the associated process of evolution) is seen to occur in fits and starts, involving the simultaneous revision, reorganization and reinterpretation of past, present, and projected actions and understandings.

(Davis, 1996, p. 184)
It is the negotiation of fit, here, that allows for new understanding to emerge—when existing ideas, or the particular combination of ideas that have been arranged in a new relationship, bump up against contextual constraints, it prompts a ‘folding back’ (Pirie & Kieren, 1994) in order to revise those ways of thinking. In this way, newer, more viable ideas can emerge. Similarly, the successful application of previous experiences within a new context may serve as a confirmation of viability, again prompting a ‘folding back’—this time adding depth and complexity to existing ideas and strengthening existing understandings.

In the sphere of cognition, though indirectly linked to survival, equilibrium refers to a state in which an epistemic agent’s cognitive structures have yielded and continue to yield expected results, without bringing to the surface conceptual conflicts or contradictions. In neither case is equilibrium necessarily a static affair, like the equilibrium of a balance beam, but it can be and often is dynamic, as the equilibrium maintained by a cyclist.

(von Glasersfeld, 1989a, p. 126)

This equilibrium-seeking—the endless negotiation of viable action—occurs not only on the level of the individual, but occurs as well in the dynamic relationships between one person and another, within communities, and at the scale of humanity. One of the fundamental ways we negotiate equilibrium, in fact, is through what Martin and Towers (2009) call *improvisational coactions*, a term that refers to the process of offering up ideas (or partial ideas) upon which the group may act, reflecting (informally and intuitively) on the efficacy of those ideas and actions, and then collectively revising the group’s approach. Although these ideas may be intentionally
and explicitly articulated, they are more often enacted or experienced—understanding emerges through action: as action.

3.5 Metaphor

In many ways, this view of understanding stands in stark contrast to the traditions of western thought. Our legacy, in terms of how we conceive of understanding, is rooted in a very different worldview. Lakoff and Johnson, in *Philosophy in the Flesh* (1999), trace the metaphoric and linguistic histories of the terms idea and understanding. The work of Plato, for example, was based on the foundational metaphors of ideas as objects and knowing as seeing (p. 366) which implied an observer-independent external reality that could be imprinted, as an object is imprinted in wax, on the mind of the seer. If ideas are objects, then the complex metaphors that arise about understanding are bound by the same rules that govern objects in the physical world:

- Thinking is object manipulation.
- Thoughts are objective. Hence they are the same for everyone; that is, they are universal.
- Communicating is sending.
- The structure of a thought is the structure of an object.
- Analyzing thoughts is taking apart objects.

(Lakoff & Johnson, 1999, p. 249)

The view of ideas as objects led to the common metaphorical expression, understanding is grasping. In Latin, the term *comprehend* originally meant both *hold tightly* and *understand* (p. 125). To this day, our inheritance in terms of how we conceptualize ideas and understandings is based on the primary metaphor of idea as object. The implications of this manner of thinking are
profound—and largely unnoticed, as the conceptual metaphors we use to think about understanding rarely surface to the level of conscious thought.

We might begin to reframe our thinking in this way: “we must not, therefore, wonder whether we really perceive a world, we must instead say: the world is what we perceive” (Merleau-Ponty, 1945, p. xvi). This notion essentially upends Plato’s notion of a mind upon which the world can be imprinted. “Normal functioning,” then, “must be understood as a process of integration in which the text of the external world is not so much copied as composed” (Merleau-Ponty, 1945, p. 10). Beginning with this worldview, and then considering theories of embodied understanding, the lines between perception, experience, action and cognition converge.

3.6 Embodiment

To understand is to experience the harmony between what we aim at and what is given, between the intention and the performance—and the body is our anchorage in a world.

(Merleau-Ponty, 1945, p. 167)

Our being in the world is the basis for understanding. It is understanding. Our experiences of movement and touch and sight, of hearing and taste and smell—our bodily interactions with that which is not our body—are our primary understandings. Our movements, unified by the body beyond the need for conscious thought, are brought together by virtue of their common meaning (Merleau-Ponty, 1945, p. 172). Through our bodily experience we come to understand relationships: associations, constancy and change, cause and effect, time and space—as well as the textures and contours and behaviours of all of those things that are not ourselves. The
physical body and its actions in the world are experiences of perception, reason, creation, judgment, emotion, inference, imagination…experiences of understanding. “Put differently, thought and actions are not precursors to or consequences of meaning: they are meaning” (Davis, 1996, p. 206).

Davis suggests that physical experiences are the basis for conceptualization and that our actions reveal the nature of our understanding: “all knowing is founded on bodily action or sensation; conversely, the evidence of one’s body of knowledge is found in one’s behaviour” (Davis, 1996, p. 185). Similarly, Nuñez et al. (1999) describe bodily ways of knowing as functioning beyond conscious thought, forming the basis for all understanding. “The tools provided by the embodied cognition approach allow one to characterize precisely how the inferential structure of everyday bodily experience, which underlies intuition, is mapped onto more abstract domains” (Nuñez, Edwards, & Matos, 1999, p. 61). Although the categorization of domains as concrete and abstract are considered an unnecessary or artificial distinction by other theorists (and certainly Pirie and Kieren (1994) note that the concrete may never be severed from the abstract), the premise of embodied understanding put forward by Nuñez and others suggests that bodily experience provides prototypical understandings that are flexible and familiar enough to be unconsciously repurposed in conceptualizing seemingly unrelated ideas.

Roth and Thom (2009) propose a radically embodied theory of mathematical understanding, suggesting that “conceptions do not exist as disembodied, decontextualized, and transcendental ideas but are imminent in each concrete realization of experience and in its relations to other experiences” (p. 175). In their analysis of gesture, bodily motion and other forms of physical
action and expression (both personal and interpersonal in nature), they contend that “cognition—at least in part—exists in bodily rather than in mental form” (2011, p. 209). Taking a phenomenological approach informed, in part, by Merleau-Ponty, drawing heavily on Vygostky, and incorporating elements of Lakoff and Nuñez’ work, Roth and Thom’s (2011) theory of mathematics in the flesh observes the “utter irreducibility of knowing and acting”, noting “it is not possible and does not provide insight into conceptual growth to decontextualize, disembody or part out what [a student] knows, how he knows, what he does, and how he works into discrete categories” (p. 281). Observing students at work, Roth and Thom describe children’s physical being not as a means by which activities or objects of the mind may be revealed, but simply as understanding.

Roth and Thom note that their theory of embodied mathematics represents a departure from a constructivist view of understanding. They reject the need for a distinction between concrete physical experience and abstract thought, “concerned as it is with the construction of mental entities and representations” (Kim, Roth, & Thom, 2011). The implied dis-integration of thought and action that may be read into the work of other theorists, for example, is reconciled in their approach.

As they build out the idea of knowledge as embodied action, Kim, Roth, and Thom make reference to Pirie and Kieren’s (1994) complementarity of acting and expressing, and echo their descriptions of recursive, dynamic, ongoing restructuring that characterize growth in mathematical understanding.
Consider their description of Alice, for example:

In the process of imaging on her own, her body orientation changed and her hands were more actively engaged in thinking. Her body was silent but active to process ideas analogically, in and through her body, and to give shape to concepts articulated in the situation. At this moment of her thinking and learning, her body was more centrally present than any verbal or written inter/ action.

(Kim, Roth & Thom, 2011, p. 211)

This description of Alice concurrently building and expressing her understandings of geometry appears to be congruent with Pirie and Kieren’s model for Image Making and Image Having:

By images the theory means any ideas the learner may have about the topic, any mental representations, not just visual or pictorial ones….It is the sense that is made of [physical representations], the personal meaning created by the learner for a concept, that is called an image. The theory posits that as a learner’s mathematical understanding of a particular concept grows, they will make, hold and extend particular images as they work on mathematical tasks.

(Martin & Towers, 2009, p. 2)

In the readings of the data that follow in chapter 10, I investigate the theories and ideas of Pirie and Kieren, Kim, Roth and Thom, Martin and Towers, and others in more detail. Increasingly, I am drawn to Davis’ (1996) simple statement: “understanding [is] thus not a goal to achieve, but a quality to enact” (p. 239).
Chapter Four: **Quantity**

In this study, seeking to describe children’s understanding of quantity meant sitting beside them, watching them work, and asking them to discuss their thinking. One question that was typically good for getting kids talking was an open-ended one: *What is it like?* Tailored somewhat to the context of the learning, the question might be, *What is the number 100 like for you?* Or, *you said that you went from 18 to 36. What was that like?* Or even, *you told me that numbers go on forever. What is it like for numbers to go on forever?* The question was intentionally biased toward analogic or metaphorical reasoning, and away from a positivist worldview. It wasn’t meant to uncover the degree to which the child was able to cognitively recreate an empirical truth, nor was it seeking out the degree of alignment between their ideas and those sanctioned by the broader mathematics community. Still, the discussions of student experience in the chapters that follow warrant an exploration of what has been written in the field of mathematics education about how quantity may be understood, as well as about the significance of quantity within the discipline of mathematics itself.

Questions of how many and how much, of more and less (and of how much more and how much less)—our experiences with making comparisons, constructing and deconstructing, combining and separating, constancy and change—are rooted in (and contribute to our ongoing negotiation of) understandings of quantity. Before I begin to explore these roots, however, it is worth noting that a discussion of quantity is not the same as a discussion of number; neither is it the same as a discussion of arithmetic. Although the boundaries between quantity, number and arithmetic overlap significantly and the understandings of each are interconnected, my focus here is specific to quantity.
The following topics are discussed:

- subitizing;
- proportionate reasoning and magnitude appreciation;
- discrete and analogue quantity;
- counting;
- metaphor and analogy; and
- unique or unusual understandings of quantity.

Understandings of quantity are complex and highly interconnected. Subitizing allows us to quickly recognize a small number of items at a glance, whereas magnitude appreciation allows us to make approximations of relative quantity. We might recognize and take action based on understandings of discrete, countable quantities or we may work with quantity as a continuous, non-numerical quality. Counting is one way of dealing with discrete stimuli; it requires not only an understanding of quantity, but also a whole host of additional thought processes and social conventions related to sequencing, labeling, and the allocation of counts. In certain circumstances, we might conceptualize quantity the way we would a number line or a simple bar graph, allowing us to judge relative numerical quantity or to reason arithmetically. Embodied understandings may treat quantity as collections, as constructions, as distance or as steps along a path.

We combine, connect and extend these experiences and analogies in ways that allow our reasoning to adjust and adapt to the context of each unique experience. It is the ecology of this understanding of quantity, in its nuanced interdependence, that allows it to be lived so flexibly and so organically in our day-to-day experience. It is the unique combination of ideas that
emerge within a given context that allows us to reason—to act—to understand. Acknowledging, first and foremost, the significance of this ecology and the limits inherent to attempting to separate the parts from the whole, what follows is a brief exploration of each topic.

4.1 Subitizing, Magnitude Appreciation, Discrete and Analogue Quantity, Counting, Metaphor and Analogy

Present in both humans and animals, subitizing is the process of recognizing, without counting, a small number of items (Davis, 1996; Feigenson et al., 2004). Typically limited to no more than four or five stimuli (objects, images, symbols or sounds), subitization is a quick, precise judgment of quantity (Coolidge & Overmann, 2012). Magnitude appreciation, on the other hand, is approximate and relative. It involves proportionate reasoning, and has been referred to as “an analogical representation of numerical quantities” (Langdon & Warrington, 1997). Non-numerical magnitude appreciation, such as the ability to judge relative brightness, pitch, length, weight, temperature and time, seems to function in much the same way as understandings of relative numerical quantity (Cantlon, 2012; Moyer & Landauer, 1967). We reason about which is bigger or brighter or louder in much the same way as we reason about which set contains more items.

According to Weber’s Law, the ability to discern the relative magnitude of two sets of stimuli is dependent on a minimum ratio between the two groups. When comparing the number of dots in a set, for example, a reliable judgment of quantity can be made between two relatively small sets even when the difference in the number of dots is marginal; comparisons of larger amounts, however, require a larger difference between the two groups (Cantlon, 2012; Dehaene, 2003). A
similar effect is noted with non-numerical comparisons: differences in pitch or luminance, for example, require a minimum ratio in order for a judgment of relative intensity to be made (Cappelletti et al., 2009). This “threshold of discrimination,” proposed by Weber, “increases linearly with stimulus intensity” (Dehaene, 2003, p. 145). The larger the quantities to be compared, the larger the difference must be between the two.

Depending on the context, understandings of quantity can be described as being either discrete or analogue. In some situations these understandings are enacted separately from one another; other situations call for an integrated approach. Put simply, discrete quantity is countable; analogue quantity is not (Henik et al., 2012). If one were to imagine a pile of sticks, for example, understandings of discrete quantity would relate to the number of individual sticks in the pile, whereas understandings of analogue quantity could apply to the length, weight or diameter of an individual stick or to the relative length, weight or diameter of the pile. Although measurement systems have been developed that allow us to superimpose countability onto uncountable quantities by way of arbitrary but universally agreed-to units and symbols, the basic process of noticing and being able to roughly compare non-numerical characteristics is the essence of analogue quantity. Analogue and discrete quantity are also distinct from one another in terms of their characteristics of change—whereas discrete quantity increases or decreases in clearly delineated units, increases or decreases in analogue quantity are continuous and non-defined (Cappelleti et al., 2009).

Counting deals with discrete quantity. It is a learned process. Whereas subitizing and magnitude appreciation may develop without targeted instruction, the incorporation of words and symbols
in the counting process, as well as the other cultural norms that counting adheres to, requires that the process be taught. Gallistel and Gelman (1992) suggest that counting involves “mapping” verbal and symbolic conventions onto a preverbal understanding of number—that our “innate” quantity sense is used in service of the more acculturated process of counting. In addition to the skill of connecting words and symbols with their relative quantities, Lakoff and Nuñez (2000) describe a complex network of knowledge and skills that must develop in order for one to be able to count, including specific understandings related to grouping, ordering, pairing, memory, exhaustion-detection, cardinal-number assignment, independent-order, combinatorial-grouping and symbolizing (pp. 51-52). In other words, counting involves recognizing that items belong to a group, ordering or sequencing those items, recalling the language or symbols used to label each item, pairing each item with one (and only) one count, and then recognizing that the items counted belong to a set having the quantity identified by the final count, regardless of their particular arrangement. Despite its complexity, however, counting is but one small component of our even more complex understandings of number.

If it is possible to ascribe physical or spatial characteristics to understandings of quantity, it has been suggested that the process would involve placing relative magnitude along a continuum with a left-to-right orientation (Cantlon, 2012; de Hevia & Spelke, 2008; Moeller et al., 2009). This “mental number line” is thought to develop over time in most cultures. It may co-develop along with the learned left-to-rightness of reading and writing, the use of measurement tools such as rulers, and the culturally modeled practice of physically ordering or touching items left to right when counting (de Hevia & Spelke, 2008). Some researchers have further documented that in cultures where reading and writing are oriented right-to-left, the mental number line is
reversed (Dehaene, 1992). Other researchers, however, posit that a spatially structured understanding of quantity develops independently of cultural norms; that the nature of quantity favours this type of cognitive modeling.

In describing the structure and function of the mental number line, researchers make the distinction between logarithmic representations of number, typically found in young children and in adults with limited formal schooling, and linear representations of number, which correlate with increasing maturity and education (Feigenson et al., 2004; Portugal & Svaiter, 2009). Logarithmic representations of number compress the scale of growth as magnitudes increase, resulting in disproportionately spaced approximations for small vs large quantities. Linear representations, on the other hand, maintain a steady rate of change with proportionate intervals between numbers (Booth & Seigler, 2008; Feigenson et al., 2004; Moeller et al., 2009). It has been observed that young children will position single digit numbers at equal intervals along a number line, but when they work with multidigit numbers, the relative position of each numeral is no longer proportionate—the space between ten and twenty, for example, is not equivalent to ten times the distance between one and two. In adults, however, numbers are generally positioned along an evenly distributed, proportionate, scale—the space between zero and sixty will be approximately six times as far as the space between zero and ten.

Moeller, Pixner, Kauffman and Nuerk (2009), recognize and have confirmed these findings with children and adults. They have, however, questioned the validity of previous researchers’ interpretations of the data. They suggest that an additional process may be at play whereby multidigit numbers are decomposed in order to reason about the tens and the units on separate
but parallel mental number lines. Essentially, Moeller et al. hypothesize that rather than evolving from a logarithmic mental number line to a linear one, the model we use for reasoning about number moves from a double number line to a single. At this point, cognitive behavioural scientists, cognitive neuroscientists and psychologists concede that either model may elicit the same observable results. As Dehaene (2003) notes, “it is hard to see how behavioral observations alone could ever disentangle the linear and logarithmic hypotheses” (p. 145).

As alluring as the idea of a mental number line may be, it would be unreasonable to conclude that all understandings of quantity could simply be reduced to a conceptualized plotting of numbers along a line. Could embodied understanding reasonably be imagined to function as actions along a mental number line? Perhaps in some ways—in certain circumstances—but the mental number line cannot be conceived of as the beginning and the end of our reasoning about quantity. In considering the possibilities inherent to embodied understandings of quantity, Lakoff and Nuñez (2000) provide an interesting place to begin. They suggest four grounding metaphors for embodied arithmetic that arise directly through physical experience. Although these metaphors are offered up for consideration in terms of arithmetic thinking, they are fundamentally rooted in our lived experiences with quantity, both discrete and analogue.

Consider these grounding metaphors:

1. **Object collection:** making groups, subitizing, counting physical objects, adding to or taking away from a collection

2. **Object construction:** a specific type of object collection involving making wholes from parts or deconstructing wholes into their component parts
3. *Measuring stick:* using physical segments (ie. sticks, hands, feet, string) placed end to end to measure length; both unidirectional and continuous

4. *Motion along a path:* moving in a straight line from one place to another

*(Lakoff & Nuñez, 2000)*

In my embodied understanding, then, quantity may be something I have, something I make, somewhere I go, or the steps I take to get there. When quantity is something I have or something I make, my collection of things may have an overall organizational structure or it may not. The structure, if it exists, may take any number of forms—there may be elements of balance or equivalence; it may have iterative qualities; it may follow some logic of proportionate distribution. Or it may have no visible structure at all—a collection of objects piled helter skelter into a heap or woven together so seamlessly that its shape takes on a single form. I may think of quantity as the space between where I am and where I might be; or perhaps I may think of quantity as the destination itself. Moving from here to there may occur as one smooth, continuous motion (and the significance of the journey may lie solely in the sweeping gesture of start to finish), or it may be a journey of discrete steps, jumps or leaps whereby the path I take is segmented or demarcated into each of the recognizable parts that make the whole.

These embodied understandings of quantity emerge and evolve through lived experience. More complex situations cause us to build more complex metaphors, but the grounding metaphors remain those that are most closely tied to our physical experience.
4.2 Unique Understandings of Quantity: Savantism, Gerstmann Syndrome and Other Neurological Conditions

"Nature is nowhere accustomed more openly to display her secret mysteries," wrote William Harvey, in the seventeenth century, "than in cases where she shows traces of her workings apart from the beaten path." (in Sacks, 1995, p. 109)

It would be naïve and shortsighted to suggest that each one of us reasons about quantity in the same way, using the same grounding metaphors, similar mental number lines, equivalent processes for subitizing and magnitude appreciation, and so on. While it is interesting to consider some of the patterns that emerge when examining population data—norms—there is a risk in becoming too comfortable in discussing these patterns as “the way things are”. The experiences of individuals whose understandings of quantity lie somewhere outside these norms—those whose reasoning is marked by unique associations, relationships, processes and skills, and who may be identified as having exceptional gifts or deficits (although I really hate those terms)—are equally important to consider. The following section is by no means representative of the vast and varied ways that understandings of quantity may be lived, however it touches on the stories of a few individuals whose experiences add richness and texture to the concepts of quantity that have been explored thus far.
4.2.1 GT and other Individuals with Savant Syndrome

In their investigation into savant abilities, Soulières et al. (2010) discuss a young boy they call GT. He is introduced this way:

Before he turned three, behavioural markers of the restricted interests and repetitive behaviours area were noticed in the form of visual fixation of rotating objects (tops, fans), prolonged visual inspection of objects under various angles, and an interest in letters and calculation. However, a functional use of objects was also observed. When the boy was three, his parents noticed that he had absolute pitch. He could read time at three and a half and has been focusing on clocks ever since. He was considered clearly atypical by his milieu in first grade, due to invasion of discourse and social routines by repetitive questions on restricted topics of interest (e.g., buses). He could also address somebody by giving his height and weight. At nine, he used to compare small differences in time indicated by the various clocks in his home and to systematically locate activities and interests in relation to time. Discourse and leisure time were filled with special interests, mental calculation, estimation of surfaces of rectangles and circles, and determining time in relation to the size of projected shadows. G.T. was conscious of both his interests and capacities. He reported he had a kilometre indicator in his head and used [it] to compute street distances by adding up the number of pedal revolutions he had to do when riding a bicycle. He considered himself as having “a clock in his head” and being able to estimate elapsed time. He explained his ability to estimate weights under 10 kg by summing a gold standard of 35 g, which is the weight of a cereal bar.

(Soulières et al., 2010, p. 264)
In their investigation, Soulières et al. explored the estimation abilities of GT and another child with savant syndrome. They tested skills in estimating numerosity, time, weight, length, surface and distance, comparing the results with those of children of comparable age and IQ who did not have characteristics of autism or savantism. Although GT’s performance on tests of subitization was comparable to those of his peers, the skills he displayed in estimating numerosity, surface and distance were “outstanding” (p. 273). In each of these subtests, the underlying skill involved “mapping a code (e.g., measurement units such as grams or centimetres) to absolute values of a continuous noncoded dimension of a physical aspect of the world (e.g., weight or length)” (p. 273). It was a process of applying discrete, rule-bound units of measure to items or experiences that were analogue in nature. In discussing the nature of his thinking with the researchers, GT described a system of relating the overall distance, surface or numerosity to personally meaningful referents. The process, he explained, was patterned—it involved using known measures (e.g. the weight of a cereal bar is 35g) iteratively in order to make a reasonable estimate of the whole. Although it was not tested, the researchers suggest that a similar process may underlie GT’s perfect pitch.

A unique opportunity presented itself in the research with GT in that he could confidently and articulately describe his own thinking. Often, individuals with savant syndrome (and particularly those with autism spectrum disorders) are unable to describe, analyze and discuss the processes that underlie their exceptional skills (Snyder, 2009). Although savant syndrome itself is exceedingly rare, occurring in about 10% of individuals with autism and approximately 0.1% of the general population, the types of skills demonstrated by people with savant syndrome typically fall into five general categories: music, art, calendar calculations, mathematics and...
mechanical or spatial skills (Treffert, 2009). While each individual may demonstrate special skills in one or more of these areas, all individuals with savant syndrome exhibit prodigious memory in their area(s) of specialization; a skill that has been described as “automatic, mechanical, concrete and habit-like… characteristically very deep but exceedingly narrow” (Treffert, 2009, p. 1353).

It has been suggested that this ability to store and retrieve isolated units of data may be what underlies the uncanny ability to recall seemingly trivial bits of information—what the weather was like on any day of one’s life, for example, or what was served for lunch, or the prize amount for the winning player on Jeopardy. This automated storage and retrieval of personally meaningful information may be what allows some individuals with savant syndrome to remember and recite thousands of digits of pi or to recite encyclopedia pages, verbatim, from memory.

Other skills, such as determining which day of the week falls on any date, past or future, for example, or picking out the primes from random lists of multidigit numbers, or instantaneously performing complex computations, relies not only on memory, but also on the use of coding patterns. It is suggested that in individuals with savant syndrome, such as GT, the ability to recognize and apply coding patterns between more than one element is sophisticated and highly developed (Soulières et al., 2010). Even the stunning musical or artistic abilities of some savants may be based on this type of complex coding system. The combination of heightened perception and the ability to recognize or develop patterned relationships for what they have heard or seen
may be what allows them to reproduce a piece of jazz, note for note, or draw a city skyline, to scale, in minute detail.

The fact that GT can accurately estimate the time an object was photographed to within a 20-minute range, simply by looking at the object’s shadow; or the fact that he can estimate the heights of adults to within 10cm when observed at a distance of 3m; or even his ability to name or recreate any pitch he hears has been attributed to a process of “memorizing the relation between an element of an ordered code and a range of anchored values on a physical continuum, based on an isomorphism between the scale and the measured or labelled magnitude” (Soulières et al., 2010, p. 274). The process involves heightened perceptual awareness, an exceptional understanding of structure and relationships, and the ability to correlate the two.

Savant skills have also been observed in otherwise neurotypical individuals following brain injury or disease (Treffert, 2009), leading researchers to speculate that the patterns of thought typical of those with savant syndrome exist in all of us and may be ‘released’ through certain forms of damage to the brain. Snyder (2009) has conducted experiments where savant-type skills (including savant-like numerosity) are induced in otherwise normal individuals through a process of artificially inhibiting selected neurological functions. In these trials, he attempted to replicate the savant experience of unfiltered access to detailed, unprocessed information. Snyder suggests that although each of us likely perceives and stores a vast amount of data through our daily experiences, we typically cannot retrieve it or put it to conscious use without ascribing some meaning to it—without some manner of conceptualization. By removing the filter of
meaning-making, he is able to ‘release’ the ability to accurately perceive and recall information that subjects would otherwise not be able to access.

While GT’s gifts for understanding analogue quantity are exceptional, elements of the ways he makes sense of time and space, sound, weight and distance are likely to be present to a certain degree in others. Equally likely is that elements of deficit in understandings of quantity are present as well, such as those that were experienced by HP as described in the narrative that follows.

4.2.2 HP and other Individuals with Gerstmann Syndrome or other forms of Dyscalculia

On February 28, 1994, HP suffered a stroke and quite suddenly lost the ability to write, calculate or dial telephone numbers. Upon examination, it was found that although he had no other visual, sensory, motor or neurological deficits, HP displayed all four characteristic symptoms of Gerstmann Syndrome: finger agnosia, confusion of left-to-right, agraphia and acalculia. Mayer et al. (2009), present the case of HP and discuss the common roots of these four apparently interconnected aspects of thought.

As a result of his stroke, HP was found to have “a focal ischaemic lesion, situated subcortically in the inferior part of the left angular gyrus and reaching the superior posterior region of T1” (Mayer et al., 2009, p. 1107). One effect of this lesion was finger agnosia—he was unable to recognize or differentiate between his fingers, on either hand, other than his thumb and pinkie finger (although success was marginally improved when he was permitted to look at his hands when completing the tasks). The stroke also impaired his ability to distinguish left from right—
when asked to point to parts of his body labeled as left or right (for example, when asked to point to his right eye or his left leg) HP had a rate of error far above that of the control group in both physical tests and in tests involving various diagrams of a human body. HP’s stroke led to agraphia as well. When writing, HP was frustrated by his penmanship, which he described as resembling that of a child, and he was noted to flip the letters b, d, q and p along the horizontal plane (e.g. dame was written as qame). It was further noted that even this very limited ability to write was only possible when he visually mediated the task. One final impact of the stroke, and the fourth in the characteristic symptoms of Gerstmann Syndrome, was aaculzia. HP was noted to count slowly, and with great difficulty, sometimes pausing as long as five seconds when moving from one number to the next and often skipping numbers as he went. HP was unable to even begin the task of counting by threes. According to the researchers, every task requiring numbers was severely impaired.

The patient could write down an operation with no difficulty but was unable to solve it. Moreover, he could not make any estimation about the results of the operation. He also failed in quantity comparisons using Arabic and alphabetic codes. Moreover, when asked about the size or the weight of drawings of objects, his perceptual estimation of quantities was wrong.

(Mayer et al., 2009, p. 1115)

This raises some fascinating questions about understandings of quantity. The notion of finger agnosia is almost unfathomable. What is it like to lose understanding of one’s own fingers? And how does that particular loss correlate with a sudden loss in the ability to estimate? And to
count? What is the relationship between understanding one’s own bodily orientation and the ability to make reasonable approximations of magnitude? Certainly Mayer et al. support Dehaene’s (1997) proposed mental number line and the idea of a left-to-right orientation of both discrete and analogue quantity. And certainly there are a whole range of neurological conditions that lead to different variants of acalculia and dyscalculia with different associated impairments in language or reasoning or memory (Ardila & Rosselli, 2002). Gerstmann syndrome, however, is unique both in its particular constellation of impairments and its relative lack of other associated deficits. Individuals with Gerstmann syndrome don’t suffer losses in language skills, motor skills or memory; they don’t have impaired social interactions or reasoning abilities. So why finger agnosia, agraphia, acalculia and left to right confusion? The link between the four symptoms has previously been hypothesized to relate to defects in visual integration or visuospatial-language integration, or even in the so-called “body schema” (Arbuse, 1947), however Mayer et al. suggest a different hypothesis. They propose that the fundamental link in the constellation of symptoms that characterize Gerstmann syndrome is the ability to rotate, transform or otherwise manipulate mental images (Mayer et al., 2009); it is an impairment of embodied understanding.

4.3 Observations and Wonderings

It may be the case that our understandings of quantity, in some circumstances, are very much like that of GT—that we perceive, retrieve and code data according to a personally meaningful structure in such a way that the information is essentially unaltered in our experience of it. Perhaps we, too, “have” clocks or scales that we use for reference when making sense of quantity. We may, like the blind mathematician Bernard Morin, think about quantity by “feeling
its weight and pondering it” (Jackson, 2002, p. 1248). Like HP, our understanding of quantity may rest in our fingertips, deeply connected to our sense of directionality. It may very well be the case that our understanding of quantity is all of these things. And more. And that as we live and learn and grow, we continually fold back to incorporate each new experience into our embodied understandings.
5.1 A Phenomenological Approach

Empiricism cannot see that we need to know what we are looking for, otherwise we would not be looking for it, and intellectualism fails to see that we need to be ignorant of what we are looking for, or equally again we should not be searching.

(Merleau-Ponty, 1945, p. 33)

The aim of this study was to describe young children’s understanding of quantity beyond that which could be held in their hand. It required two research components:

- an investigation of what may be meant by ‘understanding’, within mathematics and beyond, and
- an exploration of the experiences of young children working with quantity.

Taking a phenomenological approach, and acknowledging that understanding is complex, unpredictable and often ambiguous, the design of this study focused careful attention on the experiences of individual students and the insights they shared. Interpretations involved placing descriptions of these experiences alongside one another and against a grounding theoretical framework. Although the use of such a framework is not necessarily typical of a phenomenological approach, the decision to do so was made in response to an ongoing struggle within the community of mathematics researchers about the validity and reliability of qualitative methods. “To undertake even theory-building research, we must have specific, justifiable aims that are based to some degree within a theoretical reasoning,” writes Pirie (1997, pp. 158-9).
describes coherence within the “theoretical framework, research questions, data-collection techniques and interpretive techniques” (p. 159) as the standard by which the credibility of one’s research might be judged. The grounding theoretical framework helped me to articulate the nature of my investigation, shaped my research approach, and offered a point of reference against which to interpret my experiences.

It is hoped that this offering—this positioning of my experiences, the experiences of the students, and the ideas of those who have come before me—will create a space for meaningful intersection with the experiences and ideas of the reader.

In this study, my intent was to be with students as they engaged in mathematically rich learning experiences so that I could observe, discuss and document their insights. When working with a student, I attempted to create a space where we might explore understandings of quantity together—a space where preconceived notions about how one ought to go about doing so were, inasmuch as it is possible to do so, set aside. In this approach, the relationship between subject, researcher and experience is acknowledged as being inseparable; the phenomenon and being cannot exist in isolation of one another (Adorno & Salada, 2002). My interactions with students and with the ideas they were exploring were both rigorous and fun—there was a spirit of intense curiosity, openness and wonder that seemed to characterize our shared experiences. Van Manen and Adams (2010) describe wonder as stepping back to allow things to speak to us, “a radical passive receptivity to let the things of the world present themselves in their own terms” (p. 452). Acknowledging that it may be considered strange to speak of wonder as a method, they suggest
that “if we understand method as *methodos*, as path or way, then we may indeed consider wonder an important motive in human science inquiry” (p. 452).

It is a phenomenological approach—a *descriptive* methodology (van Manen, 1990). In documenting and interpreting the phenomena, my aim was to keep intact, as much as possible, the nuances of the experiences, the ways that students expressed their understandings, and the context in which these understandings emerged. As a teacher approaching 20 years of experience, I found myself needing to continually readjust my stance—checking the urge to repeatedly interrupt their thinking with a barrage of questions and prompts, noticing my tendency to paraphrase everything they said, and reminding myself to just stop to watch and listen rather than leading them along the direction I thought they appeared to be heading. We are patterned, as teachers, to watch with an evaluative eye; staying in a descriptive space was a profoundly important pedagogical learning experience for me.

In documenting and interpreting my experiences with students, I offer descriptions of their interactions with one another, with materials and with me, creating images of their understanding at a given point in time, as well as creating a sense of how their understandings shifted and evolved over time. Dimensions of these images are built out through students’ words, gestures and facial expressions; their written work, models and drawings; as well as the pauses, questions, frustrations and epiphanies that marked their exploration.
One must ask: how is this topic actually experienced? What are examples of possible incidents or events? Phenomenological inquiry is continually oriented to the beginning, to the concrete, to experience as lived.

(van Manen & Adams, 2010, p. 453)

In the field of mathematics education, qualitative research methods are quickly gaining ground on their more traditional, positivist counterparts; it is increasingly accepted that the investigation of understanding cannot be decontextualized, as all understanding is situated. As Pirie (1997) observes, “it is the paradigmatic denial of an absolutist epistemology that allows for the human element, allied to the specific goals of the research, to be influential” (p. 158).

The theoretical framework that grounds the readings of the phenomena, and that shapes the interpretations that follow, centers on theories of mathematical understanding put forward by Pirie and Kieren (1994, 1998) and Davis (1990, 1996, 2010); insights about embodiment and metaphor shared by Lakoff, Nuñez and others (1980, 1999, 2000); and ideas about gesture and embodiment developed by Kim, Thom and Ross (2010, 2011). Pirie (1997) offers this wisdom:

Possibly the most crucial consideration, if not always the most dominant in the presentations, is that of the role of theory in qualitative research. There are in fact two issues here: first, the intention of the research itself in regard to its contribution to the development of general theory within mathematics education and second, the theoretical standpoint of the researcher and the theoretical basis for the methods and instruments used.

(Pirie, 1997, p. 157)
Within this study, the grounding theoretical framework serves two purposes: first, it helps to illuminate the unique characteristics of the phenomena, and second, it pulls together the common threads between each of the varied experiences. The challenge was to allow the theoretical framework to function in these ways without allowing it to unilaterally define the phenomenon. While the grounding theoretical framework did very much shape my standpoint as a researcher and was the basis for the methods used, the intent of this study was not necessarily to provide a direct contribution to the development of general theory within mathematics education. I didn’t seek out images of a generic, generalizable, universal phenomenon, and the intent was not to objectively categorize and theorize. However, in the spirit of Einstein’s notion of theory as a ‘self-sharpening tool’, perhaps bringing the work of the theorists alongside my own research will serve to sharpen those tools.

A phenomenological reading of the research experience based within a theoretical framework has allowed for patterns and relationships between individual phenomena to emerge. Of interest in this study was the particular—I sought out, documented and interpreted descriptions of the understandings of this child, at this time, in this context. When held up alongside one another, networks of ideas and patterns of experience emerged that offer possible ways of thinking about how young children understand quantity beyond that which can be held in their hand. It is hoped that the experiences of the reader find a place of intersection with these ideas as well. As Merleau-Ponty (1945) offers, “the phenomenological world is not pure being, but the sense which is revealed where the paths of my various experiences intersect, and also where my own and other people’s intersect and engage each other like gears” (p. xxi).
5.2 Site Selection and the Design of Learning Experiences

This investigation was designed to be flexible enough to adapt to the context of the learner and the learning rather than attempting to rigidly structure experiences that could be consistently replicated from one learning context to another. Learning tasks used for this study were aligned with students’ day-to-day experiences in the math classroom. In order to find coherence between the conditions suitable for the study and the student’s daily learning experiences, sites were selected where math tasks would typically involve an exploratory, investigative approach, and where students would normally be actively involved in developing, refining and discussing personal understandings. Having found this coherence in the sites selected, students were on familiar ground during the study.

In all, seven classrooms were selected within a total of five large suburban schools. Students in these classrooms were typical of the demographics of our large urban school division—they came from a wide range of household incomes and parents’ education levels; a mixture of those brand new to Canada, first-generation Canadians, multi-generational Canadians and First Nations, Métis and Inuit learners; and students who came to their learning with a range of abilities and experiences.

Working together, the classroom teacher and I designed learning tasks that enabled students to explore and develop understandings of quantity. These investigations, conducted with students in grades 1, 2 and 3, were aligned with the learning outcomes and mathematical processes outlined in the Alberta Mathematics Program of Studies (2007) and were designed to integrate meaningfully into a broader instructional sequence. The Math Program of Studies recommends
instructional approaches that allow students to construct understanding and develop number sense through problem solving (pp. 2 to 11), so we created learning tasks that encouraged students to explore, ask questions, make connections, pose and test theories, consider ambiguity and inconsistencies, make judgments and draw conclusions.

While the specific topics of the learning experiences varied (including investigations using nonstandard units of measurement, estimation challenges, patterning and making predictions) the nature of the learning tasks had much in common. Each offered a problem-based context for students to explore, examine and refine their own understandings of quantity. Below is a brief description of the learning tasks; more detailed information accompanies the data in chapters six through nine.

- **Classroom AM01**: Students in this grade one class started their math lessons with several glass jars of assorted shapes and sizes. Each contained a collection of familiar objects. One was filled with erasers, another with marshmallows, one with pipe cleaners, another with Lego pieces, and so on. Their investigation had to do with the number of items in each jar.

- **Classroom AS01**: Students in this grade one class used pan-balance scales and a variety of classroom and household objects (toy dinosaurs, plastic straws, dinky cars, canned goods, potatoes, etc.) to make predictions and explore quantity through balance.

- **Classroom CH02**: In this grade two class, students were investigating the distance along mapped routes from the point of departure to a given destination. They estimated and made predictions about distance using nonstandard units of measurement (paper clips, unifix cubes, popsicle sticks, etc).
• **Classroom CH03**: Students in this third grade class were given a problem where quantity doubled each day for thirty days. They explored patterns and made predictions based on growth over time.

• **Classroom FC01**: In this first grade class, students mapped out spaces in an imaginary zoo, exploring how much space might be needed to house different numbers of animals of different sizes.

• **Classroom FC03**: Students in this third grade class had recently spent a week at zoo school. Back in their math class they used the known mass of the various animals they had studied at the zoo to investigate problems of equivalence. The explored questions about the number of animals of one species that would be required to balance a number of animals of a different species on an imaginary scale.

• **Classroom SR03**: In this third grade class, students were working through textbook problems on data analysis. Rather than work with the whole class on a given problem, I brought snippets of my experiences with other schools back to a small group of students in this class. These children investigated some ideas around estimation, some notions of balance and space, and some ideas about growth over time.

As discussed in Chapter Two, foundational to this study is the dynamic theory of growth of mathematical understanding developed by Pirie and Kieren (1994). Therefore, in designing learning experiences, consideration was given to three general instructional approaches or methods of questioning: “provocative, which have the effect of moving the student outwards, *invocative*, which have the effect of causing the student to fold back in order to enlarge or alter his image, and *validating*, which allow the teacher to view, or the student to confirm, existing
understandings” (Pirie & Kieren, 1994, p. 188). Learning experiences were provocative in that they were designed to spark curiosity, to initiate exploration and to get the students thinking. Small group work, investigations with materials, discussions with other students, and dialogue with their classroom teacher and me were both provocative and invocative, as students developed, articulated and revised preliminary understandings. The slightly more formalized interviews I held one on one with students were primarily validating, as clarity was sought about what they understood and why they understood it to be so.

As this study did not focus on growth over time per se, but rather centered on understanding within a relatively short timeframe, the instructional interventions used by the teacher were not a focus for investigation. As well, less emphasis was placed on learning (the changes evident in student understanding over time) while a greater emphasis was placed on understanding. That being said, understanding is like quicksilver. Given its nature to perpetually change, I found it exceedingly difficult to pinpoint student understanding as any sort of static phenomenon—it simply wouldn’t hold still long enough to be viewed as anything other than the process of becoming something else.

During these investigations, field notes, photographs and videos were taken to document student insights and understandings. As they worked, I occasionally drew together small groups of students to dig a little deeper; here students refined and articulated their ideas in response to questions of a probing, clarifying or reflecting nature. At times, digging deeper took the form of a semistructured interview. Some students agreed to have these interviews videorecorded; others did not. In situations where the student declined the use of videorecording, field notes, audio
recordings and/or photographs were used in its place. In all circumstances, parent consent was obtained prior to working with students.

5.3 Ethical Considerations in Research Design

In designing this investigation, I was particularly interested in seeking out experiences with students whose understanding of quantity was unique or unusual in some way. As such, a critical ethical concern arose in regards to the potential of this approach to compromise the dignity of the students involved. An additional concern, given the study’s use of images, video and audio recordings, was the amount of detail to be shared about each student. Although steps were taken to ensure the identities of the children were protected, their individual characteristics, unique insights and personal idiosyncrasies were captured in much greater detail than would have been the case for anonymous, amalgamated data. As Pirie (1997) notes, in general, qualitative research methods accord less anonymity to the people involved than do most quantitative approaches.

I agonized over ethical questions before the study began: How does one protect the dignity and autonomy of the individual if the story that is told centers on what is unique and distinct? In this study, where the research subjects were too young to realistically be considered a potential audience of the study, where their identities were concealed, and where the subject matter was relatively benign, why did it still feel so invasive? Like I was taking something from them that wasn’t mine to take?
Perhaps it had to do with the fact that during the work in the field, students would be asked to tell their own story—to describe their own understanding—but the full body of evidence would include work samples and field notes, the gestures and facial expressions captured on video, etc. Ultimately, the research would tell quite a different story than any of these children would have consciously and intentionally told themselves. The ethical dilemma, perhaps, was about whose voice would be heard.

It also had something to do with the notion of belonging. The research was to be conducted in classrooms where individual differences in ways of thinking and being were celebrated—where the uniqueness of ideas were understood as important learning opportunities for everyone in the learning community. Looking back over the last hundred years of public education, though, that certainly is not the nature of the dominant culture of elementary schools. In a culture of compliance and conformity, being identified as an outlier is a big risk. In the classroom communities where this study took place, students who shared unique insights did so with a sense of belonging and acceptance. When their stories would eventually be told to a much broader audience, however, questions arose about how valued and celebrated their idiosyncrasies would be.

Couser (2001) speaks to the role of the author in directing the gaze of the reader. Offering a modicum of relief to some of the ethical considerations outlined above, he describes critical elements of the author’s approach to mediating the reader’s experience. The descriptions of the individual, the experience, the evidence and the insights must be woven together in such a way that the gaze of the reader is neither gawking nor scrutinizing, but rather is a gaze of compassion,
respect and generosity. With an approach that honours the lived experience of the individual who is the subject of the study, the author can guard against distancing and objectification and can help to build empathy and understanding. What I hope is evident in my descriptions and interpretations of the research experience is the sheer joy I took in my encounters with the children. In writing, I felt once again (and attempted to convey) my own delight, wonder and admiration for each individual.

Yet another ethical consideration was the question of student assent. Regardless of research methodology, working with children involves a certain duty of care. Their autonomy is of no less value than that of an adult, and yet the power difference is potentially even more significant than would be the case with adult research subjects. While a formalized approach to informed consent was required on the part of the parents, what constituted appropriate autonomous assent for five to eight year olds varied from one child to the next. In all cases, assent was complicated by the complexity of truly understanding the research process—how would they understand the role of the researcher as different from, but similar to, that of the classroom teacher? How could they understand the nature of a research study? Or what it might mean to have their story told in scholarly publications or professional learning sessions?

Clearly within the scope of student assent was if, when, and in what way they would choose to share their understandings with me. As an outside researcher with no prior relationship with these students, that meant checking in often (e.g. That looks interesting; would you like to talk about what you are doing? Shall we keep working or would you like to be done now? What do you think about videotaping our work together? And so on…). It often also meant providing
possible scripts to help children negotiate their participation (e.g. *When you decide you’ve had enough, you can just tell me you’d like to go back to your desk and work on your own...*) and posing questions in such a way that power imbalances were somewhat mitigated through a sense of collaboration (e.g. *Shall we grab some blocks for this work? Should we take some pictures of this?*).

One final ethical consideration for conducting research within the course of regular classroom instruction was the integrity of student learning experiences. Classrooms were selected where exploration, building ideas and sharing understandings were the norm. As well, the tasks that were designed for this investigation were based on the learning outcomes and processes laid out in provincial curriculum. Finally, topics of study that were familiar and of interest to the students were chosen so that the research experience fit seamlessly into the course of their regular day-to-day learning.

### 5.4 Ethical Considerations in the Research Experience

As the study got underway, many of the concerns that had arisen during the design phase simply took care of themselves. The process of identifying students whose understanding of quantity was unique or unusual in some way didn’t actually feel stigmatizing or ‘othering’ at all. Students expressed a range of understandings, and the understandings of the outliers were not so strange or unusual as to preclude belonging to the majority.

In the assent process, students quite comfortably asserted their own autonomy—perhaps even to a greater extent than adults might have done. It was quite common for students to decline an
interview when they were busy with a learning task, and time and time again students told me that they would let me know when they were ready to work together. Many students declined the use of video, and those that approved seemed quite comfortable in their decision. It was interesting to note that the younger children were (and perhaps the less schooled), the more willing they were to define their own boundaries for participation. Ruby, for example, a six-year-old student in grade one, was drawing some pictures and talking about them with me when the rest of her classmates left the room for gym. She looked up and took note of them leaving, then continued to draw and to talk about her thinking with great animation. About ten minutes later, Ruby quite suddenly stopped drawing and announced that she was going to gym now. No question of permissions—it was clearly and happily her decision.

The final consideration, the integrity of the students’ math learning, was also a non-issue. The classroom teachers and I met prior to the classroom investigations and found a natural fit for our work together, and then at the end of each lesson we met briefly to discuss what students were understanding, where they were stuck and what might make sense in terms of next steps. It was exactly the same process as we would have taken in team teaching, and the scope and sequence of the students’ math learning didn’t seem to be compromised in any way.

5.5 Data Collection

In this study, data consisted of field notes, photographs and videorecordings. As the data were highly contextualized, and as the study sought out complex and nuanced expression through words, gesture and facial expression, videorecording was determined to be the ideal method for data collection. Powell et al. (2003) describe the potential of video as a flexible instrument for
capturing nuanced communication and complex interactions in a way that is both inclusive and non-intrusive. The use of video data in the study of mathematics education is understood to provide opportunities unparalleled in the history of research in the field. More than ever before, it is possible to capture large amounts of data that preserve, in detail, many of the verbal, nonverbal and social elements of student learning as it occurs.

The decision not to transcribe the video reflects the highly integrated and largely nonverbal nature of the evidence. Working exclusively with videorecordings rather than working with transcriptions is an approach supported by many investigators (Powell et al., 2003, p. 411). When children were asked to explain their thinking, what often emerged was a combination of words and gestures, pauses, triumphant insights and points of confusion. The act of translating this complex, nuanced communication into a written record would irrevocably alter the data.

Decisions made about what to capture on video, of course, greatly influenced the study. Some decisions were made in the design stage and others made in the moment, as events unfolded. In both circumstances, these decisions required that some data were included and other data were omitted; a complete picture is never possible, and a ‘true’ or ‘objective’ picture does not exist. Looking back on the data collected, the same students in the same context could have revealed untold variations of the phenomena depending on the decisions made about what to capture and what to exclude.

Evidence was triangulated as much as possible—field notes of classroom observations were held up against videos of students at work, samples of written work, models, illustrations, and videos
of interviews between the student and me. Since “inquiry relies on multiple sources of evidence, with data needing to converge in a triangulating fashion” (Yin, 2009, p. 18), comparisons and connections were made between various pieces of evidence throughout the study. Dimensions of student understanding were brought into focus through points of alignment within the evidence, as well as through the tensions or asynchronies that emerged between data sources. Together, these dynamic forces within the triangulation process gave shape and form to the data.

5.6 Data Analysis and Interpretation

As indicated earlier, the data gathered for this study weren’t selected based on the extent to which it was perceived to be normative or representative of the “typical” ways that children might think about quantity. Neither was the collection of data categorical—taken together, this data set is not illustrative of each of all possible ways children might think about quantity. Participants were chosen because their ways of thinking offered something unique and interesting to consider about how understandings of quantity might possibly exist.
Analysis was a multistep process. The work of Powell, Francisco, and Maher (2003) influenced the design of this process significantly.

1. Following each session with students, video data was reviewed with the most basic of filters—*Is this evidence likely to be significant in the context of this study?* Video that showed nothing of apparent significance was set aside for later review.

   It is ultimately a research issue to determine the nature and contours of what constitutes mathematical ideas and reasoning. Our position invites researchers to decide in the context of their research the important aspects of ideas and reasoning to focus upon. (Powell et al., 2003, pp. 413–414)

2. The remaining video data was then reviewed several times in order to develop familiarity with the content, and brief descriptions were made alongside time counts for ease of review.

3. Next, critical events were identified.

   Significant contrasting moments may be events that either confirm or disaffirm research hypotheses; they may be instances of cognitive victories, conflicting schemes, or naïve generalizations; they may represent correct leaps in logic or erroneous application of logic; they may be any event that is somehow significant to a study’s research agenda.

   (Powell et al., 2003 p. 417)

4. At this point, the antecedent and consequent events were explored and questions were identified that prompted one of three next steps:

   a. a return to the grounding theoretical framework to explore the significance of the event;
b. a triangulation of data with other evidence gleaned through field notes, photographs, or samples of student work; and/or

c. additional interactions with the student to build a clearer picture of his or her understanding.

5. A coding system was developed that reflected the grounding theoretical framework as well as important themes that were emerging within the data.

6. Next, patterns were identified within the evidence that appeared to fit together in what Powell et al call a ‘pivotal strand’—an intersection of data that illustrates something of significance related to the research question.

   By connecting sequences of critical events and further analyzing them, for example, using constant comparisons (Glaser & Strauss, 1967), researchers build narratives that initially are amalgams of hypotheses and interpretations and that in turn influence subsequent identification and analyses of critical events. In this sense, critical events and narratives co-emerge.

   (Powell et al., 2003 p. 417)

7. Throughout this process, a storyline was built—an intricate narrative that wove together grounding theories, research data, emerging insights and new questions.

   Constructing a storyline requires the researcher to come up with insightful and coherent organizations of the critical events.

   (Powell et al., 2003, p. 430)
5.7 Researcher Reflexivity

The study was designed to be flexible and responsive, allowing for an iterative process of reflection and adjustment throughout the investigation. Decisions about task design, interview techniques, data gathering methods and approaches to analysis were monitored and modified on an ongoing basis in order to refine the approach and meaningfully explore the questions under investigation. This approach also allowed for careful monitoring of emerging ethical issues—reflections on experiences with students were critical to ensuring that the methods used were as appropriate and unobtrusive as possible.

The question, then, is—on what grounds were these modifications made? What was the lens through which events were monitored? What underlying beliefs about the world, about understanding and about mathematics gave shape to the research methods and informed the decisions and adjustments that were made? My introduction provides a description of my research stance, as do the grounding theories. What follows are statements of my beliefs as a researcher and how they influenced the research process.

Children are autonomous individuals. As such, despite having parental consent for student participation, decisions about whether or not to participate, how, and for how long, were always brought back to the child. As well, I strived to build a relationship of collaborative investigation with each student, where we explored important ideas together. Rather than interrogating student understanding as an outside observer, we sat on the same side of the table, so to speak. “The more open and reciprocal the interaction, the more ethically sound the representation” (Couser, 2001, p. 5).
Understanding is highly contextual. As such, the topic of the exploration, the materials available, the questions asked, the nature of the learning environment, and the relationships between the students, their teacher and the researcher all impacted the learning experience and the data that this learning made available. “Qualitative research, in seeking to describe and make sense of the world, does not require researchers to strive for objectivity and distance themselves from research participants” (Meyer, 2001, p. 344).

I believe that children are capable of constructing and expressing complex ideas about the world. As such, students were invited explore in ways that made sense to them and that allowed them to make sense of the mathematical ideas under investigation. “Normal functioning must be understood as a process of integration in which the text of the external world is not so much copied as composed” (Merleau-Ponty, 1945, p. 10). A further implication of this belief was that in interactions with students, what they said and what they showed about their thinking was taken as valid evidence of their understanding.

I believe that children understand quantity in embodied ways and that their ways of expressing understanding are inherently metaphorical. As such, the data I gathered was first and foremost what I believed I would see—much of the data deemed relevant to the study were evidence of these embodied, metaphorical ways of thinking and knowing.

I believe, as per Merleau-Ponty, that “the world is not what I think but what I live through” (1945, p. xvi). Thus, the primary aim of the research and the foremost consideration in working with children was to describe the phenomenon in context.
The phenomenological world is not the bringing to explicit expression of a pre-existing being, but the laying down of being. Philosophy is not the reflection of a pre-existing truth, but, like art, the act of bringing truth into being.

(Merleau-Ponty, 1945, p. xxii-xxiii)

In the chapters that follow, I’ve captured a small number of the research experiences the students and I shared. All together, 25 students are included in the stories—some individually and some clustered together with other students. I refer to students by pseudonym, with images (photographs and image captures from video data) that illustrate their experiences and understandings. For ease of reference, Appendix A provides the pseudonym and an image of each participant, with the exception of Jyoti, Indulal and Gloria, each of whom declined to be photographed.
Chapter Six: **Embodiment and Co-Understanding**

Students in this grade three class had recently finished a week-long animal study at the local zoo. Back in class, their teachers used the experience as a stepping off point for an early exploration of proportion and relative quantity, a study that required students to estimate, approximate, combine and compare. Using the approximate weights of animals they had studied (meerkats, penguins, gorillas, giraffes and elephants) and the approximate weight of a grade three student as referents, they were invited to try a series of balancing challenges using an imaginary simple beam scale. For example:

- *How many meerkats would it take to balance a grade three student?*

- *How many to balance an elephant?*

- *How would that compare to the number of penguins it would take to balance the elephant?*

- *What if we tried to balance all three grade three classes against a group of animals? What could that look like?*

- *And so on...*

### 6.1 Embodied Understanding: JJ and Marlo

During their exploration, JJ (short sleeves) and Marlo (long sleeves) talked to me for a while about what it was like to estimate: to make comparisons using these large and rather intangible quantities. They explained how important it was to become familiar with a small quantity first before making an estimation of the whole. Marlo took an aside from the task they were currently working on and offered an example of estimating the number of gumballs in a jar: “If you took a small number of gumballs, like ten or maybe twenty, and held them in your hand, you could picture in your head how many gumballs were in your hand by feeling or maybe seeing.” Notice
her hands as she spoke (Figure 2). As she described feeling the gumballs, her hands moved as though holding and manipulating them; and although she didn’t actually look down at her hands at all, she described how this process would allow her to *picture in her head by feeling or maybe seeing*.

![Figure 2: Marlo Pictures in her Head by Feeling or Maybe Seeing](image)

JJ nodded in agreement (Figure 3), adding “you have to feel the number…to feel what it is like.” And here, JJ’s whole body leaned inward as she grasped the “number”, and caressed it near her cheek.
Carrying on with their descriptions of quantification scenarios, Marlo and JJ talked about estimating and counting other things—polka dots, marshmallows, meerkats, germs…their imaginations took them down several paths. Marlo noted “if it was so small, or if it was dangerous so you couldn’t hold it, it’s kind of hard how to picture it in your hand and how it feels and how big it is…how much space it would take up.”

Kim, Roth and Thom, whose research into understanding in mathematics deals extensively with gesture and embodiment, suggest that “knowing cannot be the affair of a pure, independent, abstract, and disembodied mind,” rather, “cognition—at least in part—exists in bodily rather
than in mental form” (Kim, Roth, & Thom, 2011, p. 209). It would seem that understandings of quantity reside in Marlo’s fingertips or alongside JJ’s cheek as genuinely and as meaningfully as they do in the mind.

6.2 Gesture: Marlo and JJ, Manraj, and Sam

While Marlo and JJ spoke directly to the importance of ‘feeling’ numbers, embodied understandings were evident in some of the ways other students worked with objects, or in the gestures they used when talking through their thinking. As Roth and Thom (2009) show, “gestures, bodily experiences, and words are but different, one-sided expressions of the mathematical concepts that they metonymically denote” (p. 211). When students were working with or describing small numbers or known quantities, the gestures they used frequently mimicked grasping and manipulating physical objects.

For example, in a first-grade classroom, Manraj and I were talking about the number nine. He explained that, “for a nine you take a four and a four.” As he spoke, his hand patted the tops of two imaginary piles of equal height (Figure 4). “But if you take one more,” he said (and here he pointed to the table), “and put it here,” he explained (lifting up...
and moving the one he ‘took’ and setting it on the top of one of the piles), “then you have four and five”. To illustrate his point, Manraj patted the piles again, this time with one hand landing slightly higher than the other. Within Manraj’s image of nine as two groups of four and one more, he showed an embodied understanding of the equivalence of the two fours. Although nothing that he said indicated he held this understanding, it was nevertheless expressed in his actions. His embodied understanding of five being one more than four was evident in his gesture of picking ‘one’ up and placing it on the pile.

Embodied understandings of quantity were evident, too, as Marlo and JJ worked through their challenges with zoo animals. Strategizing for how to determine the total number of students in two classes of 23 and one class of 24, Marlo devised a method for ‘breaking’ the 23s into 20s and 3s. As she talked, her hands grasped and separated the space in front of her into a larger piece and a smaller one (Figure 5). JJ, following Marlo’s reasoning, made her own gesture of separating 23 (Figure 6). It was interesting to notice that JJ wasn’t mirroring Marlo’s gestures precisely, but was physically experiencing and understanding the breaking apart of 23 for herself.

![Figure 5: Marlo Breaks Apart 23](image-url)
Another day, Sam, a first-grade student in a class down the hall, was showing me his mental math. I asked if he could subtract 24 from 51. He thought for a moment, then responded, “thirty…three”. Although his answer was incorrect, his reasoning was what was of interest to the study. Together we figured out that he had taken twenty away from fifty and then had taken one away from four. Using the language of “taking away” that Sam had used, I asked him what it was like to take 24 away from 51, and he gestured to show grasping and pulling (Figure 7), saying, “it’s like picking stuff up and moving it around.”
Pinching, grasping and clustering gestures were observed in several students as they worked with easily countable quantities. Another gesture that students used frequently when working with ‘small’, defined amounts was a gesture I have been calling bracketing, where children used fingers or flat hands to show imaginary boundaries (brackets) around the quantity they were working with (Figure 8).

Like grasping, breaking or moving objects, feeling for the outer boundaries of an object is a physical experience that is not uniquely mathematical, nor exclusively related to a particular problem-solving context. Each of these children understood quantity in and through the physical action of manipulating objects, even though no objects were physically present.

6.3 Understanding as Co-emergent: Marlo and JJ

The research findings of Kim, Roth and Thom (2011) suggest that this bodily understanding is not limited to any individual. Rather, it recognizes the co-emergence of understanding in and
through interactions with others. Consider the encounter between Marlo and JJ described below (Figure 9) as they try to figure out the combined weight of three grade three students, each of whom weighs approximately 60 pounds. In this shared experience, although they never discuss a strategy overtly, they work fluidly with one another in such a way that understanding never resides solely with one child or the other. Their actions and expressions are so completely intertwined that they not only finish one another’s sentences, they finish one another’s gestures as well.

Martin and Towers (2009) describe this phenomenon in depth, and have coined the term *improvisational coactions* to refer to the particular forms of interaction that lead to the collective growth in mathematical understanding within and amongst a group of learners. These coactions are characterized by an interweaving of fragments of ideas through a sophisticated (though unscripted) pattern of interaction in the learning context. Through their actions, learners put forward innovations—ideas that challenge or affirm images held by the group. Others accept or reject these innovations based on their perceived value in furthering the actions of the group. As opposed to a collaborative process where learners might “act on each others’ ideas in a reciprocal, complementary way,” improvisational coactions “can be said to act *with* the contributions of other group members in a mutual, joint way” (p. 16). In this way, understanding is not something to be observed at the level of the individual, but at the level of the group.

Sitting side by side with their notebooks open, Marlo and JJ started by devising a strategy for adding 60 and 60.
Here is a brief play-by-play of their one-minute encounter:

- they begin by counting by ones on their fingers (image A);
- without conferring with one another, they simultaneously reject that approach and move to counting by tens, both switching to an open-handed gesture (image B);
- they count 10...20..., then both pause again (image C), without discussing their change in tactics, and begin again with 60 as their starting point;
- as they count 60...70...80...90...100..., each child (in sync) lays out an open hand to mark off each group of ten (image D);

(Figure continues on the following page)
• when they pass 100, they both say 101…102… (instead of 110…120…);
• again, neither Marlo nor JJ says anything about the error, yet they both pause (image E);

Recognize that something isn’t right (image F);

Think for a moment (image G);

Then ‘self’-correct, saying 110…120… (images H, I and J);

Reaching the answer of 120, they again move, synchronously, from counting to writing down the answer in their notebooks (images K, L and M);

Next, JJ notices an error in how she wrote the answer in her notebook and erases it while Marlo looks on (image N);

Fluidly and without question reaches over, erases and corrects the same error on Marlo’s page (image O).

The experience of Marlo and JJ’s understanding was fluid, dynamic and nonlinear. They moved through a complementarity of acting and expressing that caused them to extend their understanding, to fold back and revise their thinking and then push out again to new understandings. What was so powerful in this experience, though, was the dynamic of understanding that existed between the two girls. There was no planning, no debate about strategy, no discussion of errors—no dialogue at all. And yet, understanding was very much a
shared experience. There was meaning to be found in the move from a single finger count to the open-palmed gesture for counting by tens that didn’t need to be named. There was understanding in the pause after their initial miscount: look at Marlo and JJ’s faces in the series of images labeled E, F, G and H. Throughout the entire process of realizing there was something wrong with their approach, pausing and revising their strategy, not a single word was exchanged between the two of them. In fact, from start to finish the only words either of them uttered were numbers! And yet rapid cycles of moving out and folding back were involved in the co-emergence of their understanding.

As they moved on to the next step of the problem, adding another 60 to the 120 they had already found, a new shared action emerged between the two girls (Figure 10). This time they spontaneously decided to start at 120 and count on by tens. Again, what follows is a brief play-by-play of an encounter that lasted less than a minute:

- Starting at 120, JJ pulls up Marlo’s fingers one at a time (images Q and R), tapping each one on the fingertip to keep track as they both count by tens …130…140…150…;
• when they reach their solution (180), Marlo holds up six fingers, showing the count of sixty, while JJ shows one thumb to indicate one group of 60 (image S);
• again, the agreement as to the correctness of their answer is unspoken and they both record their findings in their notebooks (image T).

6.4 Observations and Insights
Understanding, for Marlo and JJ, was embodied, enacted and co-emergent. It was not something that either one of them ‘had’, but rather something that both girls, together, lived through. Davis (1996) might have described it this way:

Played out in the complex choreography of existence, knowing cannot be separated from action, nor can it be extricated from inter-action. Cognition/knowledge—the parallel, co-implicated, co-emergent process—is what knits us together, and it is where both individual and collective identities (selves) come to form.

(Davis, 1996, p. 192)
Chapter Seven: *Experiences with Quantity*

It is worth noting that gestures such as pinching and moving, breaking apart, patting and bracketing (Figures 4 through 8), as well as JJ and Marlo’s ‘feeling’ for numbers (Figures 2 and 3), each emerged as students worked with and talked about relatively small quantities—quantities that could be held in their hand. As we talked about quantities beyond that which could be held in one’s hand, however, children’s gestures suggested embodied understandings of a different nature. Their own distinctions between ‘small’ numbers and ‘big’ numbers signified an important shift in thinking.

7.1 Children’s Distinctions Between ‘Small’ Numbers and ‘Big’ Numbers

As I sat and worked with students, they frequently made reference to ‘small’ numbers and ‘big’ numbers. The distinction between what constituted a ‘small’ or a ‘big’ number varied according to the student and to the learning context, but their distinction between the two seemed to be significant to how they went about engaging with a task. I talked to several children about how they determined if a number was big or small.

![Figure 11: Manraj Explains Subitizing](image)

When I talked with first-grader Manraj about small numbers and big numbers (Figure 11), he told me that the most important way you could understand the number you were working with was through subitizing. I asked him to tell me more about subitizing. He explained it this way: “Subitizing is
like, oooh, yup! That’s a five! It’s like, you see a number and you’ve seen it before so you know what number it…is”. As we talked, Manraj described quantities that could be subitized as familiar—known in ways that other quantities were not. Together we determined that Manraj used subitizing to make the distinction between small and large numbers.

Marlo, describing her process for estimating quantity, made her own distinction between small and large numbers. She defined a small amount as “ten, or maybe twenty”. Later, when she was working on solving the problem involving the number of students in each classroom, she commented that 23 was kind of a ‘big’ number, “but not really a big number, just kind of big”. Bailey, in first grade, told me that small numbers have one digit and big numbers have more than one. Meg, another grade one student, agreed—saying that big numbers “are like ten or a hundred”.

Another day, in another school, I asked third-grade students Adarsh, Jyoti and Indulal to tell me about what made numbers ‘big’ or ‘small’. In the conversation that followed, they explained that it depended on what was being counted, offering examples that illustrated that if the objects to be counted were small, a larger number of items would still be considered a small amount, and vice versa. I asked them about the number 37; they agreed that it was “kind of big and kind of small” —it was big for the age of an animal, moderate for the age of a person and small for a number of grains of rice.
Jada had quite a complex working definition of big and small numbers. We had been working on a balance problem in her first-grade class when I spotted a large glass jar filled with dried beans over on the side counter (Figure 12). Wandering over together, I asked her about how many beans she thought there might be in the jar. Reaching in to grab a handful of beans, she chirped “One hundred!” Jada then started to sort the beans into two piles on the counter: large beans like kidney beans and limas, and small beans like peas and black beans. “The big ones,” she told me, “are big numbers like in the teens and the small ones are small numbers like 1, 2, 3, 4, 5, 6, 7, 8 and 9.” I asked her to tell me about big numbers and small numbers. “It’s like, it’s like…my baby brother was first one,” she said, “and then he grewed. So that means…so that means one is the smallest age. So right now, one turns into two, and two turns into three…” I asked Jada how old she was. “Six!” she grinned. Then I asked her how old she thought I was. She flashed a quick smile to see if I was joking, then looked up to think. “Sssssss…eventeen?” She asked (Figure 13). “Yeah?” I asked, “That’s a good number; tell me about that number.” Talking together, we figured out that teens are big numbers, and seven is older than six (her age), hence my age could reasonably be estimated at 17.
When Jada estimated my age, she reasoned through to determine an appropriate quantity. When she guessed a hundred beans in the jar, though, it was not so much about reasoning through to find a suitable estimate as it was simply describing its gestalt—expressing its ‘big’ness. Many of the students I worked with used a known numeral as a placeholder of sorts when they judged the quantity at hand to be a ‘big number’. Students would use the word hundred, or million, or billion as an archetype—a name for a very large quantity that didn’t necessarily relate to an image of a known amount. When Meg said it would take ‘a hundred’ straws to balance her scale, for example, a hundred simply meant ‘a lot’. Often, in classroom experiences, I noticed that the relationships between students’ understandings of the language of large numbers, the quantity signified by those numbers and images of what those quantities might be like were complex and multifaceted.

7.2 Rani’s Rice: Reasoning Through Very Large Numbers

In a grade three class, we ventured into explorations of very large numbers—quantities well beyond that which could be held in one’s hand. The teachers and I planned a series of lessons around Demi’s (1997) picture book, *One Grain of Rice*. This folktale tells the story of a young girl, Rani, who outsmarts a greedy rajah who had been withholding rice from the villagers during a time of famine. When the rajah offers her a reward for her good deed, Rani simply asks for one grain of rice, doubled each day for 30 days.
In the first class, the teacher read the story to her students, stopping at the part when the rajah accepts Rani’s seemingly innocuous request. Students worked together to record the important information and the questions the problem was asking (Figure 14) and were then asked to consider a reasonable estimate for the quantity of rice Rani would have been given all together after 30 days. Students were asked to show how they came up with their estimate and explain why it was a reasonable amount.

The students set out to work, some forming and reforming informal partnerships and groups, others working on their own. Many students made initial estimates of about 60 grains of rice, relying on their Primitive Knowing of doubling to simply double the 30-day period. Other students tried doubling the number of grains of rice each day for a few days and noticed that the growth pattern led to relatively large numbers quite quickly. This recognition generally led them to make a guess based on what they considered to a ‘big’ number—generally somewhere between 100 and 200. A few students (three of twenty-five) brought more complex images of doubling and of growth patterns to their first explorations of this problem; these students made initial estimates of 1000 grains of rice or more.

Over the next couple of class periods, I sat with several students to talk about what these quantities meant to them—how they understood the relationships between numerals and quantity, and how they understood relationships between quantity, size and space.
7.3 Harjit

I sat with Harjit during her initial exploration of the problem. She told me that her mother made rice every night and that it took “a lot” of rice to feed her whole family, especially when the extended family came for dinner. Looking to her first estimate for the doubling problem, I saw that Harjit had predicted that Rani would have 22 grains of rice after the first five days. I asked her if that amount was a lot of rice. “Yes,” she answered. “Enough to feed the whole village?” I wondered. She wasn’t sure about that. I asked her if she could show me about how much rice it would be; she gazed around the room, thinking. “About a bagful?” she speculated, looking towards the coatrack where her classmates’ backpacks hung in neat rows. I asked again to confirm, “22 grains of rice would fill a bag?” Harjit nodded, confidently, in agreement.

Although the prior knowledge Harjit was relying on as she approached this problem was grounded in experience with feeding her family, the image she held of 22 as a large number (or a lot of something) had not yet been meaningfully integrated with her experience with rice.

I wondered if other students had begun their investigation with similarly disconnected or conflicting experiences with number, quantity, and spatial estimation. Conferring with the teacher, we decided to add two additional questions to the task: first, we asked students to think about the size of container that would be required to hold their estimated quantity of rice; second, we asked them to determine whether or not the amount of rice they had estimated would be sufficient to feed all of the hungry villagers.
The results were fascinating. Most students I worked with decided that yes, Rani would be victorious and the villagers would have enough to eat, but that inference seemed to be based in past experiences with folktales rather than in any sort of mathematical reasoning. When asked to represent the quantity of rice in terms of filling containers, most students’ numerical and spatial representations were remarkably divergent (Figure 15):

- Mason thought that 95 grains of rice would fit in a big box.
- Clark thought that 100 grains of rice would fill a garbage can as big as the chair.
- Leah agreed that a garbage can would suffice for 100 grains of rice, but suggested a ‘big, big, big box’ would be needed to hold 148.
- Kevin, who had estimated 169 grains of rice, thought that he would need four recycling bins to hold them all.
Harjit suggests that a bag would hold 22 grains of rice.

Mason thinks it would take a big box to hold 95 grains.

Clark estimates that 100 grains of rice would fill a garbage can as big as the chair.

Leah proposes a garbage can for 100 grains of rice and a big, big, big box for 148 grains.

Kevin thinks that 169 grains of rice will fill four recycling bins.

Heidi found that 200 grains of rice would fit in her hand, but was not yet sure of an estimate for day 30.

Figure 15: Students’ Estimation of Quantity and Container Size
Consider the fact that these students were in third grade. In our province, that means that instructional time dedicated to hands-on work with quantity would be coming to an end very soon for these young learners, and the rest of their work with number and computation would focus almost exclusively on numerals. How many children, I wonder, find themselves working through higher level mathematics without ever developing an embodied sense of number beyond that which can be held in their hand?

7.4 Heidi

Looking over Heidi’s written work, it appeared that she had taken a different approach from that of many of her classmates. While Kevin, Clark, Leah, Mason and Harjit had made their preliminary estimates based solely on work with numbers, Heidi had started working with rice very early in the problem solving process. By the time she was asked to consider the relationship between quantity and container size, she had already done some experimentation. She knew that 200 grains of rice would fit in her hand (Figure 16).

Heidi explained her experience this way: at first, she had estimated 60 grains of rice, a conclusion reached by doubling the number of days. Next, she used her chart to calculate the quantity after five days. Looking at the pattern of growth over

Figure 16: Heidi’s 200 Grains of Rice
a few days, she realized that doubling each day resulted in a rapid growth rate; her 30-day estimate, therefore, would need to be a ‘big number’—“maybe even 100”, she said. She revised her written estimate to 100, and then started counting grains of rice into her hand to get a sense of how much space that would take. She ended up counting 200 grains of rice into her hand before deciding that her hand was full. Taking a look at the meager handful, she explained, “I thought 200 but if they [the villagers] tried to live on that, they wouldn’t do too good”. Next, Heidi returned to the doubling chart, telling me that she needed to calculate the doubling pattern for a few more days before she could make a reasonable estimation for the 30 days.

7.5 Clark

Clark’s understanding followed a path very similar to Heidi’s. Looking to his written work from the first class, I saw that he had made an initial estimate of 100 grains of rice. On his page (Figure 15), Clark had suggested that 100 grains of rice would fill “a garbage can as big as a chair”. I met with him the next day to talk about his thinking.

When I sat down to chat with Clark on the second day, he had counted 160 grains of rice into the bottom of a 30ml container; it came up to just below the 5ml mark. “Was that surprising to you?” I asked. “Yeah!” he answered, wide-eyed and grinning, “I thought that 100 would take the whole bottle,” he exclaimed, and
he capped his hand over the top of the container, “that’s why I brought two!” (Figure 17). We talked about his thinking—like Heidi, he had initially thought that doubling grains of rice for 30 days would give a total of about 60, and then he had revised his estimate to a ‘big’ number. At some point between his image of filling a garbage can with 100 grains of rice and his decision to use medicine cups for 200 grains, his thinking about size and quantity must have changed, but in the moment, I didn’t think to ask how or why it did so. “Now how many grains of rice do you think it will take to fill this container?” I asked. “Maybe one million”, he answered. I wanted to explore the idea of one million with him a bit further, but he told me that first he needed to figure out what 248 would look like. Looking to his paper, I saw that he done some calculations that took him to day nine. He kept counting.

In the next lesson, I sat with Heidi again. She had done some more exploring, and had recorded some of her thinking on her page. I noticed that she had carried on with calculations of the doubling pattern for several more days, and then had written “I changed my estimate to 1 billion because 10 days is 1082” (Figure 18). I asked Heidi to talk to me about what a billion means. She responded “I think a billion is ten one thousands?” She made questioning face, then nodded, “yeah, ten one thousands”. I asked her how much space a billion grains of rice would take and she said, “from what I learned so far, maybe a
bowl?” and she cupped her hands in front of her, “yeah, like a little bowl? Because 700 only took that kind of [medicine] cup”.

In this experience, Heidi’s understanding involved moving out and folding back through embodied understandings of quantity, emerging understanding of growth patterns and prior knowledge about the structure and function of the base ten number system. Her estimate of one billion was not an archetype for a very large quantity (as was likely the case with Jada’s ‘hundred’ beans), but was rather a reasoned and embodied prediction of a quantity she understood.

7.6 Liam

Liam approached the problem with rather sophisticated understandings in several key areas: patterns of growth, orders of magnitude, and the relationship between quantity and capacity. When he returned from a doctor’s appointment mid-math class, Liam’s teacher quickly caught him up on the nature of the investigation. He listened attentively, and then still standing next to her with his coat and backpack on, Liam quickly calculated the first five days out loud. He shared a passing observation about how quickly the numbers grew, and suggested that thirty days would take Rani past a thousand grains of rice. I asked what kind of space would be needed to hold that quantity; he glanced around the room and then gestured towards a jar that measured about 20cm by 20cm.

Liam settled in to work. Carrying on his verbal doubling to day ten, he revised his original estimate, saying that 30 days would take Rani to at least 32 000 grains of rice. I handed him a
plastic cup, half-filled with rice (Figure 19). “How many grains of rice do you think there are in this cup right now?” I asked him. He picked it up, shook it, placed his index finger at the approximate fill line, looked up briefly to think, and answered, “maybe about 6000?”

![Figure 19: Liam Estimates About 6000](image)

After working a little longer, Liam waved me over to show me a pattern he had noticed in his chart (Figure 20). The first five days, he pointed out, “went 1, 2, 4, 8, 16,” and days 11 to 15 “went 1000”…he paused… “well,” he clarified, “about 1000…2000, 4000, 8000, 16000”.

![Figure 20: Liam Finds a Pattern](image)

Very quickly and comfortably, he shared his insight that “once it gets to the 32 000s and the 64 000s then it will get to the hundred thousands and then five hundred thousands and the millions”.

I asked him how much space millions of grains of rice would take. “I don’t know,” he said,
gesturing towards the basin of rice the class was using for their explorations, “I wonder if we even have enough in the tub?”

Pointing to the third row on his page, days 26 to 30, I asked him “if these are going to be in the millions, then what kind of number will go down here?” His response? A quick shrug and a flat statement: “I dunno”. My sense, in this exchange, was that what Liam didn’t know about the last five days had little to do with pattern recognition or a judgment of relative magnitude, but as was the case with Heidi, not having the language to name those numbers.

7.7 Gloria

During the initial whole-class presentation of the problem, Gloria had pattered off numbers just under her breath, eyes wide with excitement about the growth pattern she recognized immediately: “1, 2, 4, 8, 16, 32, 64, 128, 256…” When students were dismissed from the carpet, Gloria had left the group discussion with one hand clasped firmly over her mouth to prevent herself from spilling out her discovery before others had had a chance to explore the problem.

When I went to sit with her and talk about her work, Gloria showed me how she had used mental math to double the amount each day up to day 14 (Figure 21: Gloria’s Calculations to Day 14)
When she read the number for that day, however, she said, “eight million one hundred ninety-two” for the written numeral 8192. When I asked her what size container she was likely to need for that amount of rice, she predicted a small cup would be sufficient. Like Heidi, Gloria had a reasonable sense of the quantity 8192 (at least as it related to grains of rice in this problem) but had simply misnamed the numeral.

As Gloria ventured into next steps, I suggested she try graphing the pattern of growth so that she could see what repeated doubling looked like. She had never created a graph before, but grasped the concept quite quickly and started plotting the data she had generated (Figure 22). Shortly after beginning her graph, Gloria came to tell me she had run out of space on the chart paper she was using; we set her up in the hall with markers and a giant stack of paper (Figure 23). Although she did the calculations herself, and determined on her own where to place each day’s mark, she was repeatedly astonished by her own reasoning. It was absolutely fascinating to watch her reaction to her own findings. Like Clark’s reaction to seeing how little space was taken up by 100 grains of rice (Figure 17), Gloria’s response was one of fascination, wonder and delight.
When Gloria went from day 12 to day 13, for example, (having run out of room on her first page) she backtracked several times, thinking she had made a mistake. Her brow furrowed in intense concentration as she calculated and recalculated her findings. Finally, she reconciled herself with the fact that the next step would take almost an additional full piece of paper and set about making space for her chart to reflect the giant leap that would be required.

It was in the shift from day 13 to day 14, though, that her demeanour changed from one of watchfulness—attentive curiosity—to one of joy and amazement. Realizing she would need another two full pages to hit her mark, Gloria walked back and forth along the length of the graph—skipping, really—delighting in her discovery. It was a dramatic folding back that fundamentally changed the image she had previously held of doubling, and likely her images of relative magnitude as well. She carried on, bringing her classmates alongside to share in her discovery, adding sheets and sheets of paper to her graph as she worked on representing days 14 and 15.

When Gloria reached day 16, though, she decided she could no longer continue with her graph. Having explained what would be required to work through all 30 days, she decided that what she had already drawn was sufficient to illustrate her key findings. We brought her classmates out to talk about what she had discovered. With 25 eager faces looking on, Gloria explained, pragmatically, the structure and function of a bar graph, and then talked through the calculations
to day five that they had already discussed as a class. Then she got into the wonder of it all.
“Let’s say this was five inches,” she said, pointing to day 12, “then the next day would be ten inches because it’s doubling how tall it is.” She answered some questions from her teacher and her classmates, and then I asked her to explain why she didn’t continue her graph any farther.
“Because,” she said, “if I went all the way to day 30 it would take all of the paper in the whole school.” Her classmates chimed in with suggestions that the paper would go all the way around the school or perhaps all the way home, and Gloria took the opportunity to show where day 17 would land by walking down the hallway a distance equivalent to sheets of paper on the existing graph. The class was delighted and joined her down the hallway. I asked where day 18 would go and they scampered down the hallway and around the corner. Together, in a manner both physical and joyful, they experienced a new, embodied understanding of doubling.

7.8 Observations and Insights

The understandings of quantity that most children brought to their first Image Making experiences—their understanding of what a hundred grains of rice would look and feel like, for example—was vastly different from what they discovered in the early stages of exploration. It was so interesting to watch how thoroughly mind-blowing the experiences of large numbers were to some of these children. When Clark saw how little space 160 grains of rice took up in the little medicine cup (Figure 17), he was completely awe-struck. When Gloria realized that the next bar on her graph would require an additional full sheet of chart paper (Figure 23), and that doubling that number again would require four full sheets of chart paper, she was nothing short of astonished. Astonished! In both cases, there was a dramatic folding back to revise the images of number they were bringing to the task.
Liam, on the other hand, was one student who began the investigation with complex understandings of quantity that incorporated well-developed images of number, size and space, proportion and orders of magnitude. One might describe Liam’s number sense as intuitive—the images he held were robust enough, and his Primitive Knowing ‘thick’ enough (as Pirie and Kieren might say) to be used with ease and flexibility in this novel context. This ‘feel’ for numbers has been identified in the Alberta K to 9 Mathematics Program of Studies (2007) as “the most important foundation of a numerate child” (p. 11). Liam’s understanding of large numbers was certainly not typical of those in his class, however, which led me to wonder about a central question about understanding large numbers posed by Wagner and Davis (2010), “Is it at all possible to develop a feel for such immense quantities, or are we condemned to dealing with only the numbers and a vague sense that they are larger than we can imagine?” (p. 45).
Chapter Eight: **Magnitude Appreciation and Analogue Quantity**

Embodied understanding occurs as the physical expression of numerosity, with or without the written or verbal notation of a numeral or number words. A young child playing pat-a-cake, for example, might follow the pattern of clapping her hands twice before clapping her mother’s hands, and would recognize the difference between a double-clap and a single without any need to name it ‘two’. In my research experiences, this embodied understanding of two, or the images of number held by Marlo and JJ that allowed them to “break” 23 into 20 and 3, or that enabled Sam to “pick stuff up and move it around” as he subtracted, appeared to be characteristic of understandings of discrete quantity. The gestures and metaphors that evidenced children’s understanding of number as embodied, tangible amounts generally emerged in the context of small, countable quantities.

### 8.1 Scanning

As discussed in Chapter 6, Marlo and JJ had developed a generalizable estimation strategy that involved getting a feel for a small quantity and then using that embodied understanding to make a reasonable estimate of a larger amount. I asked them if this strategy was like skip counting, and I modeled the thought process out loud, saying “Okay, I know this is ten,” and I gestured as if to grab a small group, “so I would count ten, twenty, thirty…” gesturing at similar groups in the set. “Or,” I asked them, “was it more like scanning?” This time I modeled a different approach: “Okay I know this is ten,” and again I gestured at a small handful, “so –“ and I swept my hand across the larger group in one smooth gesture “– this is about a hundred”. Both JJ and Marlo agreed that their approach was not like the first example, but very much like the second.
This idea of ‘scanning’ had initially been brought to my attention through an experience with Levi, a first-grade student from a different school. Levi had estimated that there were 20 books on the bookshelf because, he explained, “it kind of looks like a 20”. I asked how he came up with that number and he responded, “I counted a little bit and then I guessed” (Figure 25). With Levi as well, I had asked if this process was like skip counting; he had responded that it wasn’t.

![Figure 25: Levi Counted a Little Bit and then Guessed](image)

Jackson, a second-grade student, described a similar process. He was the one to coin the term ‘scanning’. He had a large pile of centimeter cubes in front of him. We talked a bit about how many blocks there might be in the pile, and when I asked him about how he would go about making a good estimate, he explained that he would count a small quantity and would then use that number to figure out the whole amount. First he took four blocks out of the pile, and then he spent a few seconds looking at the large group of blocks, his hand resting gently on the group of four.

![Figure 26: Jackson Estimates](image)
I asked Jackson what he was thinking about when he was looking at the blocks. He responded, “I kind of…I kind of did this in my head,” he explained, and then, using an overhand grasp with slightly clawed fingers, he pulled a group of about four blocks out of the pile and into the empty space in front of him (Figure 26). Alternating hands in a quick, iterative, uninterrupted gesture much like pulling a rope, he grabbed and pulled approximate piles, saying, “I just keep doing this and this and the number goes up and up”. I asked him if he was skip counting when he did that or if it was more like comparing. He said it was “like comparing…comparing a little number to a big number and that makes it…make sense”. He elaborated, explaining, “I count, like, four…and then I scan it.”

Across the room, Amir, Jackson’s classmate, was estimating the length of a line using unifix cubes. He counted ten blocks, lining them up along the beginning of the path. Next, he let his gaze slowly pass across the set of ten. Looking along the rest of the path (unmarked by unifix cubes), he swiped his hand along its full length and then made an estimate (Figure 27). He, too, referred to this process as ‘scanning’.

Back in Jada’s first-grade class, Hank used a similar gesture to show his estimate of a hundred. He had been watching me talk with Jada (7.1) about the number of beans in a large glass jar.

“About how many beans do you think are in this jar?” I had asked Jada. “One hundred!” was her
delighted response. Hank wrinkled his nose in disagreement. “Naw,” he said, “it would be way more than a hundred!” I asked why he thought it would be more than a hundred. “Because look at it!” he exclaimed, reaching forward and dragging a finger up the side of the jar (Figure 28), “ooone uuuup! And this row would be up to a hundred already!”

Figure 28: Hank Gestures One Hundred
8.2 Gestures for Discrete and Analogue Quantity

This way of ‘feeling’ number is not the same type of embodied understanding that Marlo experienced in breaking apart 23, or that Manraj expressed by patting piles of four and five, or that JJ caressed near her cheek. Levi pinched the air with his fingers when talking about a couple, but drew his arm across its full reach to show a long time until lunch (Figure 29).

Figure 29: Levi's Pinching Hand and Sweeping Arm

Liam (7.6), too, used a wide sweeping arm when talking about thousands of grains of rice, describing his estimate as ‘flowing’, while Clark (7.5) raised his hand slowly up from the table as he talked about quantities beyond what might possibly fit in his cup. While discrete quantity was understood through pinching, grasping, moving or bracketing, the gestures and metaphors for large quantities, it would seem, were analogue.
Adarsh, a grade three student, explained it quite clearly (Figure 30).

When I asked him how he thought about small numbers and big numbers, he responded, “bigger numbers I think about tall and number line, smaller numbers I think about size.”

Figure 30: Adarsh Explains his Reasoning for Discrete and Analogue Quantity

8.3 Observations and Insights

In addition to that which may be expressed through oral language, writing and drawing, children are abundantly expressive in their gestures, facial expressions and use of objects. The gestures children used when talking about quantity offered a glimpse into the kinds of embodied understandings they held and the ways those understandings were re-experienced as their understandings grew. Davis (1996) offers this insight: “they [bodily understandings] are part of our acting in the world—an acting that “understands” the difference between a single or a pair of raised fingers before it can count, an acting that “understands” that a sequence of two perpendicular cuts produces four pieces before it realizes the process is multiplicative. These are understandings that are aspects of the body’s doing, and are thus conditioned by that which is encountered in moving through the world” (p. 201). Davis references thinkers who contend that “the learner’s actions, and not socially-sanctioned preestablished truths, are the most important source of one’s mathematical understandings” (p. 202).
Chapter Nine: **Infinity**

My conversations with young children about big and small numbers always (and very quickly) seemed to find their way to the notion of infinity. Acknowledging that the focus of this study was *quantity* and that I had no intention to delve into mathematical constructs of infinity, what occurred was that infinity arrived in our conversations anyway. As we made estimates of large numbers, students often used the term ‘infinity’ to refer to the largest conceivable amount. A colloquialism that is common amongst young children, their use of the term infinity did not necessarily take us away from an exploration of quantity. The other point of entry for the notion of infinity was a certain fascination that children have for the idea that numbers ‘go on forever’. While this, in and of itself, has little to do with how quantity is understood, what was of interest to this study were the parallels in the language and gestures used by children who were discussing large quantities (quantities greater than that which could be held in their hand) and the language and gestures used by children as they explored the notion of infinity.

**9.1 Sam**

When I arrived in Sam’s grade one class, the math lesson had yet to begin. I hadn’t begun my research yet, and was visiting the class to get to know the students and to explain the purpose of my study. I found an empty chair next to Sam. He was reading a book, his brow furrowed in concentration. I sat next to him, and when he was finished reading we exchanged hellos and introductions. I asked what they would be learning about in the upcoming math lesson. “I don’t know,” he answered. “What were you learning about yesterday?” I asked. “Numbers,” he replied. When I asked what he knew about numbers he sat up taller, raising his little blonde
eyebrows in enthusiasm, “I know that numbers never end,” he declared. I acknowledged that that was, indeed, amazing. “What is it like,” I asked, “for numbers to never end?”

Sam told me that you could count forever and you would never come to the end of numbers. I asked him how that could be? The elephant he had drawn on the page, I pointed out, has a beginning and an end. So did the book I held in my hand. And the pencil on his desk. “What is it like for numbers to go on forever and never end?” I asked. “It’s kind of like a number thing,” he said, “going in a circle.” In the air, with a single finger, he drew a line that went away from him and then looped back before going out again for a second round (Figure 31).

Figure 31: Sam Shows how Numbers go in a Loop
The next part of what he said was muffled, so I asked him to clarify: “say that again—it goes in a loop?” “Yeah,” he said, “that would make it stop every time if you went in a loop and a loop and a loop,” and here he looped his finger out, stopped in front of himself and then looped out again and again. “Every time it goes higher and higher and it would have different numbers.” “How do you know when you come to the end of a loop?” I asked, and he answered, as though it were obvious, “because it starts back at one.” “What happens after you’ve done a hundred loops…or a thousand loops…or a million loops,” I asked him, “are there still more loops?” “Yeah,” he said, hesitating a little, “you could draw more.” I asked him if he could think of anything else that went on forever. After thinking for a moment, he perked up and said, “dates never stop! You know, dates of the year? Or videos? A video on replay?”

![Figure 32: A Model for Sam](image)

The class ended and we went our separate ways, but the next day I brought in a model that I had made based on what he had told me the day before (Figure 32). Before I shared it with him, I talked to him about our prior conversation, saying, “last time you told me that numbers go on forever” (“and never stop,” Sam interjected) “and never stop, and I asked you what it was like and you said it was like a circle” (and here I looped a finger in the air as he had done). “Yeah,” he said, “it would stop if it was going in a circle, but it never stops so it would be like going in a straight line forever.” Here he reached back over his shoulder
and swiped a finger across the full span of his reach (Figure 33). “Aaah,” I said, “like a wheel going over a line forever?” And he nodded. I had misunderstood. Or perhaps his understanding had already changed?

![Figure 33: Sam's Sweeping Gesture for Infinity](image)

I told Sam that I had made a model based on what I had pictured in my mind when he was talking the day before; I brought it out and asked him what he thought about it. “Oooh, cool!” he exclaimed, taking it apart to look more closely. Then he picked up each circle in turn, counting out loud as he touched each numeral with the tip of his finger. “Cooo-ool,” he declared. “Does that look anything like what you had in your head?” I asked. “Yeah,” he answered, but his face said not really. I asked if we could make something that looked like what he had in mind, and he agreed, suggesting we, “get a piece of paper and, like, cut it out in circles and make it, like, bigger and bigger.”

Using red construction paper, Sam made a couple of first attempts to cut out a circle, but was clearly frustrated by his attempts. We talked through what he was trying to make, establishing that when he said ‘circle’, he was trying to make a sphere. We weren’t having much luck making a sphere out of paper either, but he quickly readjusted and decided that a cylinder would
work just as well. We made a paper cylinder about 5cm in diameter and 25cm in circumference, but it just wasn't quite right; he explained that it needed to get bigger every time it went around. I asked him if we were going to put numbers on the cylinder and he explained that, no, the cylinder would go along the numbers, getting bigger each time it passed a one.

Through a two-day process of building and revising Sam’s model, we eventually ended up with about 50 little pieces of paper lined up down the hallway, each with a numeral that Sam had printed on it. Rolling along the numbers was a hula hoop with a paper cylinder taped around the outside (Figure 34). When the time came to finally try out his model, the lunch bell had already sounded, greatly diminishing the joy Sam was taking in our experiment. As he tried rolling the hoop along the number line, I asked him if he thought it was starting to work the way he had imagined. “Yeah,” he said, noting that the cylinder still wasn’t getting bigger and that the hula hoop wouldn’t stay on the track when he pushed it and let go. He shrugged, “can I go get my lunch now?” I was disappointed. Here was this tiny little person who held such a sophisticated image of how numbers worked, and our experiment was trumped by lunch.
9.2 Adarsh, Jyoti and Indulal

Following my experience with Sam, I was curious about how other students might think about
the notion of infinity, so I brought the question to Adarsh, Jyoti and Indulal. These students
attended grade three at a large new school at the outskirts of the city where 98% of the students
were identified as English Language Learners; most of Indian or Pakistani heritage. The lesson
students were working on in their math class was textbook work: reading bar graphs. Since they
seemed to have a good grasp of the concepts they were working with in class, I asked them if
they wouldn’t mind working with me on a different task for a while.

Following a preliminary exploration of the problem of Rani’s Rice (7.2) and a short discussion of
what they might consider small and large quantities (7.1), I asked the students what kind of
number would be too big to count. At first they tossed around ideas of thousands and millions,
and then Jyoti suggested infinity, using the colloquial use of the term meaning the largest
countable amount. Indulal commented that numbers never end. As I had done with Sam, I asked
what it was like for numbers to never end. I held up a book and a pencil and noted that they both
had beginning and ends, and commented that most things don’t go on forever. Jyoti piped up
first: “a circle doesn’t have a beginning and an end…same thing with a line!” “Yes!” Indulal
interjected, “it extends to infinity!” “Same with the world!” said Adarsh, “anything round keeps
on going forever…like a…sphere or…like an ordinary circle.”

I stopped the discussion to ask them a question. “But when you’re doing math in the classroom,”
I noted, “I haven’t seen any circles in any of your work.” I gave some examples of the kinds of
materials and models they had at hand in their math class—base ten blocks and hundreds charts
and whatnot—and then asked, “What would it be like to make a model that showed how numbers go on forever?” They debated a few ideas: Adarsh suggested a line that students would “watch and watch and watch forever.” Indulal suggested circles. He thought perhaps if we began with a circle of a given circumference, the outer edge could then be divided and subdivided to make a progressively larger circle as numbers increased. Then he revised his idea and suggested that “the circle is rolling and leaves the numbers behind from one to ten in one roll…one whole roll it leaves a ten behind it…and the circle would go on and on and on and on….numbers are endless.”

I must pause here to recap. Indulal had suggested essentially the same model for infinity as Sam had imagined, despite the fact that the two boys were attending school two grade levels and fifty kilometers apart. Despite the fact that Sam’s grade one math program was largely exploratory and experiential while the Indulal’s grade three math class was largely textbook-driven. Despite the fact that Sam came from an English speaking family and Indulal was learning English as a second language. And despite the fact that while Sam was sighted, Indulal was legally blind.

I asked Jyoti, Adarsh and Indulal if they would be interested in building a model that could show how numbers work, and they all jumped at the opportunity. We went hunting for supplies. Interestingly, the only thing they thought they might work with in the math storage room was a bin of little geometric solids. In the phys. ed. storage room, though, they pulled out skipping ropes, balls of assorted sizes, pinnies and pylons, and the coup de grace—a red nylon rope about a hundred feet long. Back in the hallway outside of their classroom, they quickly rejected the geometric solids and went to work with the gym supplies.
After half an hour or so of exploration and invention, the three students were locked in heated debate. The red rope lay stretched down the length of the hallway. Strung along one end like beads on a necklace were yellow whiffle balls (Figure 35). Jyoti and Indulal were negotiating the space between the whiffle balls—they agreed that the balls represented numbers, but while Jyoti thought they should be spaced apart, Indulal disagreed. “What is between the numbers?” he demanded indignantly, “there cannot be nothing between the numbers…it would be more numbers!” Jyoti looked unsure about how to respond. Adarsh weighed in: “What about this? There are 20 holes on each ball so each of these could be 20?” He slid the whiffle balls together so they touched, and counted: “20, 40, 60? Like this?”

Figure 35: Adarsh, Jyoti and Indulal's First Model for Infinity

Neither Jyoti or Indulal really engaged with Adarsh’s idea, so he took the opportunity to return to a point he had been trying to make early in the exploration process. “If these are numbers,” he demanded, gesturing toward the whiffle balls, “what is the rope?” Clearly, in Adarsh’s mind this was a rhetorical question. “We don’t need these things,” he said, and pulled the whiffle balls off the rope, straightening it out again. “The rope is the numbers. Why do we need the balls? The
rope is the numbers going on forever.” Again, here was a nine year old child whose understanding of number far outreached anything he would have encountered in his four years of elementary school mathematics.

9.3 Explorations of Infinity in Grade Two

My curiosity was piqued. On my next visit to one of the grade two classes I particularly enjoyed, I asked the teacher to just start a conversation about numbers and then follow it where it naturally led. Gathering her students on the carpet, the teacher asked her students an open-ended question: “Tell me what you know about numbers.”

Donovan offered the idea that numbers were for counting stuff to see how much you have. Richard suggested you could count with them, but qualified his response as different from Donovan’s idea by adding, “like you could count…up?” and he gestured with an upsweep of his hand. It would appear that there was distinction to be made between counting ‘things’ and counting ‘up’—between discrete and analogue quantity. The teacher asked Richard how high he could count, and as he pondered the question, some of his classmates tossed out suggestions: “one hundred!” “one-finity!” Others started counting, “1, 2, 3, 4…,” as though the question was a challenge to be met. Amir interjected: “it’s unlimited! It never stops.”

We were maybe three minutes into the lesson, tops. The teacher asked Amir to hold on a minute and explain. “What never stops?” she asked. The class erupted in enthusiastic responses and the teacher settled them down again. Amir then explained that numbers never stop, they just keep going and going. With a bit of a ‘gotcha’ look, he suggested, “If I say this is the highest number,
and then, well, then, why don’t you just add one more? Then it’s not the highest number.” So there.

I talked to Amir, and to the class, about what he had just suggested, explaining that most things that we are familiar with (this book, this pencil), have a beginning and an end. As I had with Sam, and with Jyoti, Indulal and Adarsh, I asked them what it was like for numbers to go on and on forever. Mohammed stepped in to disagree, explaining that numbers had to end, otherwise “people would be making numbers until…until the end of the world…until they die and until this world ends.” Another student speculated that numbers would outlast people, but not the universe, explaining that when the sun burns out, the universe would end and that would be the end of numbers.

Mohammed wasn’t ready to consider the idea of numbers going on forever, but he did want to talk about very big numbers. He told his classmates that “you could fill a whole chapter book with numbers the size of a dot and it would still be a number.” Richard elaborated, suggesting that you could just copy that number and copy that number and keep copying to make a number that never ends. Amir pointed out that every number is just made up of ten digits: “1, 2, 3, 4, 5, 6, 7, 8, 9…” he paused, “not ten…0.” He explained to his classmates that you could just write down any sequence of digits and it would be a real number. Again, Richard elaborated, talking about how people might have invented names for numbers and how they might have decided that “once you get to 99 you change to 100 by using a ten and then putting a zero and then inventing a name for that number.” Donovan commented that people often get confused about the names of numbers when they are counting.
As the conversation wound its way through the properties of numerals, I stopped to clarify a bit about digits, positions of digits and place value—not introducing new ideas, per se, but overlaying a little bit of language and structure to what they were discussing. I also talked a bit about the relative magnitude of numbers, telling them that it would take 11 days to count to a million (at a rate of one count per second), 32 years to count to a billion and back to the time of the cavemen to count to a trillion. They were duly impressed.

Danielle asked us to wait a minute. “How do they know numbers never end?” she demanded indignantly, “they couldn’t start counting back when there were cavemen because cavemen didn’t know what numbers even were!” We talked a bit about how those calculations were made, and then Donovan wondered aloud, “now I’m starting to wonder what number even means…does it mean never ending, or does it mean it ends…” Richard interjected: “Does it mean all the numbers in the universe or what?” and then both Richard and Amir asked in unison (a tone of pleading anxiety in their little voices) “What does it mean?” The class was experiencing a collective existential crisis of sorts. I responded by repositioning the question: “You began the conversation by saying that numbers were for counting, and now I think you’re wondering if numbers are about something more than just counting.”

Jackson stopped us to bring the conversation to the language of numbers and how people invented the names for numbers. “You’re implying that people invented numbers,” I observed, “do you think humans invented numbers, or do you think numbers would exist whether people invented them or not?” Richard speculated that maybe the first numbers were dinosaur
footprints, and that the first people saw them and decided to name them. The dinosaur footprints were numbers, he decided, whether we named them or not.

Next, the students set off to work. They were asked to think about how they could show what it was like for numbers to go on forever. Working in pairs or small groups, occasionally stepping aside to work on their own, the students talked and drew and thought and talked some more; they made conjectures, negotiated meaning and worked through new understandings. What follows are some glimpses into our shared experience.

Working together, Alyssa and Amir designed a model that spiraled outward (Figure 37), with the small numbers in the middle and the larger numbers towards the outside. Siobhan started by drawing a black hole (Figure 36), explaining that the more numbers there were, the darker the black hole would be. Richard played out several variations on the theme of generations, starting with cavemen and working forward in time.
Mona began by drawing a curved line that traveled up and across her notebook page from the bottom left hand corner to the top right hand corner. Written along the line was a string of digits (Figure 38).

“Tell me why this line goes up like this,” I asked, tracing the line with my finger, “instead of straight up like this, or straight across like this, or in rows…?” Mona explained that she drew it to resemble a lifecycle.

She talked to me about caterpillars, describing the stages of their development, and noting how one generation always followed the next. With sweeping loops of her arm (Figure 39), she explained that “a whole bunch of them could keep doing it forever and ever and it would never stop.” Her gesture was identical to Sam’s (Figure 33); one I saw repeated several times under similar circumstances.

After working with this idea a while longer, Mona revised her analogy, deciding that it was really more like the earth orbiting around the sun, “it goes around the circle and the time goes more and more—like using more and more time.”
9.5 Jackson, Donovan and Bob

Jackson started off working on his own, then exchanged ideas with a few different groups of classmates before settling in to work with Donovan and Bob. Within the space of this 45-minute lesson, his understanding shifted significantly. Initially, he speculated about infinity as being analogous to a “really long rectangle that goes on forever” and here he reached both arms up as high as he could to show a long vertical column. Next he had a go at playing with the patterns found in base ten. “You know how one thousand has three zeroes and one hundred has two?” he asked me, holding up two fingers for emphasis. “Well…if a trillion has twelve zeroes, then maybe infinity would have thirteen?” he said, turning the end of the sentence upwards into a question. As Heidi (7.4) and Gloria (7.7) had experienced, it can be challenging to integrate the language of number, base ten rules, numerical representation and the quantity signified.

As if in answer to his own question, Jackson shrugged, looking unsure. “But then,” I asked him, pushing back against his tentative conjecture, “would it be possible to have a number with fourteen zeroes? Or would that be impossible?” He twisted his mouth to the side and squinted his eyes in concentration. “It would be possible…,” he said, “but I don’t know what you’d call it.” Having answered his own question about reaching infinity by increasing powers of ten, Jackson set about trying a new approach while I worked with other students.

Later, I checked in with Jackson when was he working with Donovan and Bob. The group had decided that numbers would need to loop in some way in order to be able go on forever.
At first, they decided that each loop would be a ‘count’ of infinity. Bob explained it with a broad, curved sweep of his arm followed by a straight path back to the start (Figure 40), illustrating how they might count up until they reached infinity and then they would go back to zero and start again. Jackson mirrored Bob’s gesture, then drew loops in his math notebook agreeing that yes, once they reached infinity they would have to loop back.

I asked them to clarify, “so, every time you go around you get to infinity?” Jackson drew a mark on the circle in his notebook to indicate the end/beginning of a cycle, “you go around, and when you hit infinity, you have to go back to zero,” he explained. “No, no,” Bob interjected, excited by a sudden insight, “you have to go back to the negatives!”

“Yeah,” agreed Donovan, “you have to do the negatives and then zero, and then one and then up to infinity.” As they spoke, each of the three
boys in turn drew circles in the air with their hands (Figure 41), Bob punctuating the circle with taps of his hand at the point in the cycle where he showed counting one, two, three...

As they continued the dynamic negotiation of meaning, Jackson introduced the idea that perhaps it was like a ball rolling. He began tracing with a finger around and around the circle he had drawn in his notebook, saying that numbers never stop. He paused to think. Then, with one hand drawing circles in the air, he said, “it’s like a ball rolling”. In a flurry of gestures and words and excitement, he moved from one hand drawing circles in the air to two hands cycling over and under one another. Then he adjusted again, using one hand to represent the flat space along which the ball was rolling, saying that the ball would keep rolling along the line forever. I asked him to tell me more about the line, and in a spark of insight, Jackson decided that the line was like a number line and that the ball would keep rolling along it to infinity. Here was a third child, in a different classroom, essentially inventing the same model for infinity that Sam (9.1) and Indulal (9.2) had each created.
As we discussed Jackson’s ideas, Donovan sat beside us, busily drawing away in his notebook. Looking at his work, I saw that he was writing numbers around the inside edge of the circle he had drawn. Looking more closely, I noticed that his model actually depicted a smaller circle traveling within the bigger, numbered circle. Jackson, conversely, drew a straight line across the width of his notebook, made the infinity symbol at the end of the line, paused a moment, and drew a looping line back to the beginning (Figure 43). The ball, he explained, would travel along the line and loop back.

9.6 Debriefing as a Class

Later in the lesson (which all told was less than an hour), the students gathered on the carpet to share their ideas. Jackson, Donovan and Bob had moved along with their thinking some more, describing the cycles they had created as behaving like the earth orbiting the sun, similar to the approach Mona had eventually taken. After a few students had shared images and metaphors with their classmates, and we had talked about the similarities between their ideas, I made an observation: “So far today, I have seen circles, life cycles, solar systems, black holes, rolling balls…and all of these things are based on a circle. I’ve been in a lot of math classes and I’ve never seen a circle in a math class [used this way]. I see lots of blocks, squares, grids, cubes, hundreds charts—all in rows and all in lines. Would it be possible to use circles in a math class when you work with numbers?”
A chorus of ‘yes’s and ‘no’s erupted. Danielle offered the suggestion that you could “take circle stuff” and she gestured placing something on the ground in front of her “and put it out and that is still counting.” Amir noted that you could take shapes of any sort and set them out to count them. Interestingly, the simple suggestion of the math class context was enough to bring them sharply back to ideas of discrete quantity. The teacher probed a little deeper into Amir’s thinking: “but why did you choose a circle to show how numbers work?” Amir explained that he had chosen a circle because it “never stops,” tracing circles in the air with his hand as he spoke. Then he observed that you could take any shape and roll it. If it was a square, he noted, it would go “dunh-dunh-dunh-dunh.” With this, he drew a square moving in a series of turns the air (Figure 44), pausing to articulate each corner that would hit the ground as it rolled.

Figure 44: Amir Shows how a Square Would Turn
As he listened to Amir, Richard suddenly had an epiphany (which he explained with delight and great animation even through the sounding of the lunchtime dismissal bell): “’cause a circle, it’s like, it’s kind of like…a circle…the number keeps going around and around” (Figure 45). He traced smooth circles in the air to show the motion, “but it doesn’t go, like, 1, 2, 3, 4, 1, 2, 3, 4.” And here he emphasized the punctuation of each count, “it doesn’t go like that. It just goes and goes like that but the numbers go higher.” Essentially, what Richard explained was the difference between discrete and analogue quantity.

9.7 Observations and Insights

Interestingly, the kinds of images that children generated related to infinity (at separate schools, on separate occasions) evolved along similar trajectories. Students settled quite quickly into the idea of circles, loops or cycles, then went on to negotiate the rules or patterns that would define their motion or iterations. The question of how to indicate growth (through size, distance or numerals) was another key factor in determining how one’s model would work. For Sam, this meant that the circle would need to get larger with each iteration and for Indulal, this meant that the space between numbers would need to be infinitely subdivided. Adarsh, Sam, Jackson and Donovan all experimented with a number line of sorts, while Adarsh was the only one to come to the conclusion that the analogy of infinite motion could be sufficiently represented with the line and that no additional components were necessary.
The idea of infinity is intangible; it cannot be experienced bodily and thus is understood in relationship to those experiences that can be understood in an embodied way. Lakoff and Nuñez (2000) suggest that, “outside mathematics, a process is seen as infinite if it continues (or iterates) indefinitely without stopping. That is, it has imperfective aspect (it continues indefinitely) without an endpoint. This is the literal concept of infinity outside mathematics. It is used whenever one thinks of perpetual motion—motion that goes on and on forever” (p. 156). Since it is difficult to conceptualize motion as never ending (or continuing for extended periods of time), the images we make are iterative—repeated cycles of finite motion. These images may be based in our experiences of motion that is sustained by iterative actions—walking, for example (Lakoff & Nuñez, 2000). When Mona, Jackson or Sam, for example, looped a wide arm out and then back in broad circular gestures, they were expressing the iterative motion required to sustain an infinite process. Adarsh’s model of infinity as “just the rope”, then, is remarkable.
Chapter Ten: Discussion

The purpose of this study was two-fold. Broadly, it explored the notion of understanding, with a particular focus on the Primitive Knowing, Image Making, Image Having and Property Noticing modes described in the Pirie and Kieren model of growth in mathematical understanding (1994). Specifically, it examined young children’s understanding of quantity beyond that which could be held in their hand. This discussion begins with a return to Pirie and Kieren, moves into an exploration of understanding, examines some particulars about quantity, and then positions the findings in relationship to possible implications for teaching and learning.

10.1 Revisiting Pirie and Kieren

Pirie and Kieren (1994) describe growth in mathematical understanding as “whole, dynamic, leveled but non-linear—a transcendentally recursive process” (p. 166). The model they developed for characterizing and representing growth in mathematical understanding describes a process of moving back and forth between different ways of knowing, represented in their model as a series of nested layers or levels. Of particular interest to me when designing this study were the characteristics of understanding at the first four levels of the Pirie Kieren model—Primitive Knowing, Image Making, Image Having and Property Noticing. Through the research process, however, what surfaced as being even more significant was the dynamic fluidity between modes of understanding—the “moving out” and “folding back” that characterized not just growth and learning in mathematics, but the nature of understanding itself.

Prior to conducting this study, I believed that I might find students ‘at’ a particular level or ‘in’ a particular mode of understanding. Now I see these modes of understanding more as verbs than
as nouns—Primitive Knowing, Image Making, Image Having and Property Noticing are not places in which ways of knowing might dwell, but rather are ways of being that are forever in flux. Understanding, as I have come to know it through this study, is always on the way. As such, the dilemma I identified in 3.1 about what is present and what is absent within each level of understanding really isn’t a dilemma at all—I am no longer searching for a static version of understanding.

Pirie and Kieren stress that their model for growth in mathematical understanding “is not meant to be used as a tool with which one can categorize, level, or sequence forms of mathematical knowledge,” but rather offers a way to describe “the complexities inherent in mathematical understanding” (Thom & Pirie, 2006, p. 187). My experiences with children in this study surfaced certain characteristics of these complexities. Primitive Knowing was made visible through conjecturing, planning, and predicting, for example, while metaphorizing, analogizing and gesturing offered glimpses into Image Making experiences. Image Having looked like the calm after the storm, while the ‘aha moments’, sudden u-turns and spontaneous new trajectories were often characteristic of experiences of Property Noticing. A sole reliance on spoken language or written work would have done little to bring these characteristics to light. Meaning was found in the combination of actions, expressions and interactions undertaken in any mode of understanding—children’s bodily movements and facial expressions, their use of materials, the fragments of ideas shared between members of a group, and their co-emergent gestures all worked in concert to make their understanding visible.
In the sections that follow, I describe characteristics of children’s understanding as experienced in the various modes of the Pirie-Kieren model. In doing so, my aim is not to compartmentalize or artificially deconstruct what is understood to be a holistic, integrated phenomenon, but rather to look carefully at the whole from different perspectives. The experiences described below are “complex collectives” (Towers & Davis, 2002), where “the phenomena are understood to be enfolded in and to unfold from one another. As such, the divisions are merely heuristic conveniences” (p. 335).

As well, my intention in focusing this analysis on the first four levels of the Pirie Kieren model was not to disregard or diminish the significance of the other levels (Formalising, Observing, Structuring and Inventising). While I do take note of certain Formalising, Observing and Structuring actions and expressions, of particular interest to this study were the characteristics of understanding that lead to, emerge from, occur as Image Making and Image Having experiences.

**10.1.1 Primitive Knowing**

Primitive Knowing, the starting point for any particular growth in mathematical understanding, is the particular set of ideas and understandings that the individual or the group brings to the learning experience. Primitive knowing is highly contextualized; it encompasses what students think, know and can do in relationship to a particular set of circumstances.
The kinds of experiences and understandings students used as their starting point for Image Making actions and expressions were varied, and were not exclusively mathematical. In the grade two classroom where we discussed the idea of infinity, for example, Jackson (9.5) drew on the mathematical knowledge he brought with him about the rules of the base ten number system, suggesting that infinity could be ‘reached’ by adding a zero to the largest known number. Mona, on the other hand (9.4), began with her knowledge of the patterns found in the lifecycle of a butterfly, using it as the basis for her model of a curved number line, while Sam (9.1) and Jyoti (9.2) each brought to mind the properties of a circle. Sam used his experience with circles in motion to posit an iterative pattern for growth in number over time, and Jyoti suggested that the circumference of a circle could be infinitely subdivided, allowing for infinite growth.

In another context, when we began our exploration of One Grain of Rice for example, Harjit (7.3) reflected on prior experiences with her mother cooking dinner for the extended family, using it as a point of reference for ‘a lot’ of rice, and Heidi (7.4) and Clark (7.5) each used their prior experience with doubling to make an estimate of 60 for the total amount after 30 days. Leah, Mason, Kevin and others (Figure 15) came with Primitive Knowing of a hundred or hundreds as very large quantities, leading them to suggest that very large containers would be required for what they later discovered were very small quantities of rice.

In each circumstance, Primitive Knowing was the vantage point from which conjectures could be made about a viable course of action; a stepping off point for mathematical exploration. The understandings and experiences that children brought with them to a novel learning context set the trajectory for the actions and expressions involved in Image Making. Typically, the two
blended seamlessly together—a student’s Primitive Knowing might be observed in their actions or recounted in the expressions of their Image Making experiences. Often, a child’s Primitive Knowing was not fully realized until the process of folding back brought new dimensions of understanding to bear on their original starting point. With Heidi, for example, looking at the tiny mound of rice that she held in the palm of her hand was a point of reflection on her unchecked assumption that a hundred would be a large amount.

Primitive Knowing, of course, wasn’t limited to those experiences that occurred at the outset of a learning task, as was evidenced by the dynamics of Jackson’s mathematical understanding during a single class period (9.5). In his preliminary investigations, Jackson drew on his Primitive Knowing of the structure and function of the base-ten number system to suggest that infinity could be reached by adding a zero to the end of the largest known numeral. When he reached the outer limits of that possible course of action, he started a new trajectory based on his experiences with looping motions, then with a ball rolling, then with a ball rolling along a flat surface, and finally incorporating his prior knowledge of number lines. Each incorporation of additional Primitive Knowing was a readjustment—a folding back in order to move outward with a new approach to Image Making.

Similarly, when Marlo and JJ were adding 60 and 60 (6.3), they began with a Primitive Knowing of counting by ones, quickly folded back and began again with a strategy rooted in their understanding of counting by tens, revised their approach again to incorporate their previous experience with counting on from a known amount, and finally incorporated their concepts of place value as it applied to passing the hundred mark when counting by tens. Within the space
of just a few seconds, at least four distinct areas of mathematical understanding were called upon to inform their approach to Image Making.

In my experiences with children in this study, Primitive Knowing might be characterized as familiar—personally meaningful to the student or students involved, and a comfortable fit with the unique circumstances of the learning task. At the same time, Primitive Knowing was typically acknowledged as tentative—ideas were offered up as conjectures or hypotheses, rationales or wonderings, leaving space for the possibility that through Image Making things might be otherwise. Often, this mode of understanding was clearly recognizable as embodied—children would return to physical experiences, either by overtly naming them or by living them out through actions, gestures, body positions and facial expressions. When JJ and Marlo (6.1) were developing a strategy for estimating weights, for example, they began by recalling experiences with estimating countable objects; they physically re-experienced touching and feeling items to get a sense of their count before they set about developing an approach that would be appropriate for the new learning context.
10.1.2 Image Making and Image Having

The first layer of coming to understand is when one performs actions — mental or physical—in order to create some “idea” of the new topic. Here, understanding grows from making distinctions through mathematical actions, on a base in Primitive Knowings. The intent of working at this level is to give rise to the creation of new mathematical “images,” which may exist in mental, verbal, written, or physical forms.

(Pirie & Thom, 2006, p. 189)

The actions and expressions of Image Making are the experiences that bring mathematical understanding to form; they are the ‘mucking about’ with mathematics that enable one to test the viability of their prior understandings in a new context, feel for the outer limits of their viability within a familiar context, integrate previously disparate understandings, and explore new ways of knowing. Often embodied or enacted, Image Making occurs through a wide range of verbal and nonverbal actions and expressions.

By images the theory means any ideas the learner may have about the topic, any mental representations, not just visual or pictorial ones….It is the sense that is made of [physical representations], the personal meaning created by the learner for a concept, that is called an image. The theory posits that as a learner’s mathematical understanding of a particular concept grows, they will make, hold and extend particular images as they work on mathematical tasks.

(Martin & Towers, 2009, p. 2)
When Gloria (7.7) was exploring the problem of Rani’s Rice, she engaged in several Image Making experiences. At the carpet, when the teacher first introduced the problem, Gloria quickly began doubling the quantity, saying each progressive increment just under her breath. In this first Image Making action, she experimented with the process of doubling, playing it out through the first fourteen days. Once she reached the point of Image Having—when she had developed a sense of the kind of quantity that might be achieved after 30 days of doubling—Gloria stopped and adopted a different approach. It wasn’t necessary for this first experience of Image Making to bring her all the way to finding a solution to the problem; rather, she engaged in the process until she was satisfied with the Image she had made about the impact of this pattern of growth over time. The purpose of her first Image Making experience, therefore, was to bring clarity to (or to add depth and complexity to) her Primitive Knowing of doubling in order to inform her next Image Making experience.

Through her next course of action, plotting her results on a bar graph, Gloria explored the pattern of growth in a new way, allowing her to incorporate a more tangible sense of relative proportion. Again, Gloria worked through Image Making actions and expressions up to the point where a new Image of the growth pattern was realized. A solution to the problem was not the destination yet—she had her ‘aha’ moment once again at day 14. At this stage, Image Making was a back and forth process of performing calculations, marking her graph, double-checking her work, looking at the graph and pondering her findings. It was characterized by deep concentration and periods of reflection. When Gloria went from day 12 to day 13, for example, (having run out of room on her first page) she backtracked several times, thinking she had made a mistake. Her brow furrowed in intense concentration as she calculated and recalculated her findings. Finally,
she reconciled herself with the fact that the next step would take almost an additional full piece of paper and she set about making space for her chart to reflect the giant leap that would be required. It was in the shift from day 13 to day 14, though, that her demeanour changed from one of watchfulness—attentive curiosity—to one of joy and amazement. Realizing she would need another two full pages to hit her mark, Gloria skipped back and forth along the length of the graph, delighting in her discovery. For Gloria, this marked the shift from Image Making to Image Having; the graphing process had given her a substantial enough idea (a robust enough Image) of doubling that she could begin to think about the problem in new ways.

Clark (7.5) and Heidi (7.4) had Image Making experiences that were very similar to one another. They entered into the problem with Primitive Knowing of doubling that informed their first Image Making experience: doubling the 30 day period to make a first estimate of 60 grains of rice. Their next Image Making actions involved step-by-step doubling for the first five days, an action that led them to fold back and incorporate an element of ‘big’ness into their next estimate. Beginning again with an estimate of 100, both Heidi and Clark then undertook an Image Making experience of counting grains of rice (Heidi into her hand and Clark into a medicine cup) that integrated understandings of quantity as ‘count’ with ideas about quantity as ‘space’. For these two, as with other students in the class, this Image Making action sought to reconcile previous Images of a hundred or hundreds as being a very large amount with their current experience working with grains of rice. As with Gloria’s experience, the Image Making process undertaken through the counting of individual grains of rice was not intended to bring them to a solution to the problem, but rather to ‘thicken’ or refine previous understandings.
Clark’s shift from Image Making to Image Having, like Gloria’s, was marked by expressions of wonder and delight. His finding that 160 grains of rice scarcely passed the 5ml mark on his medicine cup fundamentally changed the Image he held of 100, and his response was to marvel, joyfully, at his own discovery.

As Davis and Towers (2002) observe, in Image Making “the learner is being asked to make distinctions in previous knowings and use them in new circumstances or to new ends” (p. 324). For Clark and Heidi, counting grains of rice allowed for such distinctions to be made, as did their next Image Making actions of completing pencil-and-paper computations, drawing their findings out on paper and talking through the problem with other students. Towers and Davis (2002) note, “it is these repeated Image Making activities which are significant in providing the ground for the deeper understandings” (p. 331).

Often, Image Making actions and expressions were observed in the gestures students made as they worked through new understandings. The scanning gestures used by Levi, Jackson and Amir as they estimated quantity (8.1), for example, were likely to have been Image Making experiences, as were the bracketing gestures used by Nina, Lisa and others as they worked with discrete, countable amounts (6.2). Image Making also took the form of literally drawing images out on paper. The process of drawing out a spiral model of infinity (Figure 37), for example, was a means for Alyssa and Amir to try out their ideas about how such a concept might work, while the physical act of drawing the black hole (Figure 36) allowed Siobhan to make her theory more complex and robust.
One characteristic of Image Making that became evident in the course of the study was the fact that the process couldn’t be rushed or interrupted. Students who were engaged in Image Making experiences frequently told me they couldn’t stop to discuss their work at the moment, or if they did pause to talk about their thinking, it was only to explain the actions they needed to complete before they would be able to discuss their understanding further. Ruby (5.4), for example, was so thoroughly engrossed in her Image Making experience that when the rest of the class left for gym she took note but kept on working. It wasn’t until she reached the Image Having level that she felt comfortable enough to leave her work and join her classmates. Similarly, both Heidi (7.4) and Clark (7.5) politely declined to answer my questions until they had finished counting out grains of rice.

Students involved in Image Making were highly engaged—thoroughly engrossed in their experiences, both physically and intellectually. When the Image Making actions and expressions were shared understandings, as was the case with Donovan, Jackson and Bob (9.5), or with Marlo and JJ (6.3), the learning space was filled with fragments of dialogue that needn’t be completed, co-emergent gestures and facial expressions, and writings and drawings that mirrored, assimilated and elaborated on one another’s offerings. These shared Image Making experiences had a dynamic energy that was almost tangible; observing them I couldn’t help but feel that I was in the presence of something of profound importance. At the time of this publication, new research from Martin and Towers (2015) investigated this particular phenomenon, describing and defining new constructs they refer to as Collective Image Making, Collective Image Having and Collective Property Noticing (p. 4). Although I did not make use
of their research in the course of this study, a further investigation of this phenomenon would most certainly require consideration of their construct.

Another characteristic of Image Making experiences was that the actions and expressions were generally in service of, and ended with, Image Having, Property Noticing or some other dimension of understanding—children rarely carried through with a single approach to Image Making until they reached a solution to the problem. Heidi, for example, only counted grains of rice into her hand until she had a new Image of hundreds, and then she went back to pencil and paper computation (7.4). Liam completed computations in his chart until he reached a Property Noticing stage of understanding triggered by the recognition of a pattern in his results. At this point he switched to an Image Making action of making predictions orally based on the pattern he had identified.

Returning to Pirie and Kieren’s original descriptions (1994), we see that at the Image Making stage, children make distinctions within previous ways knowing and begin to use that knowledge in new ways, and at the Image Having stage they use a mental construct about a topic without having to do the particular activities which brought it about (p. 165).

Whereas Image Making experiences were noteworthy for the intensity of students’ physical and intellectual engagement, Image Having experiences had the feel of students returning to the present. Often this ‘return’ was marked with delight and awe. When Gloria’s experience with graphing led to a new Image of the relative proportion of the numbers she was working with (7.7), her expression was one of pure joy. In the Image Making stage, she had been so deeply
engaged—so completely focused—that she appeared to have been in her own world. At her ‘aha’ moment—the point in which the experience of drawing a line twice as long as the line before became her own mathematical Image—her face simply lit up with pleasure. Similarly with Clark (7.5), the realization that a hundred grains of rice scarcely covered the bottom of the medicine container elicited a huge grin and delighted laughter. Again, this seemed to be a natural emotional response to the experience of Having a new Image of a hundred.

10.1.3 Property Noticing

Once a person has made certain images, one is in a position to look at those images and make connections and distinctions among them. This is the layer of Property Noticing. It is a form of “standing back” and reflecting on one’s existing understanding in order to further that understanding.

(Pirie & Thom, 2006, p. 190)

Liam (Figure 20) approached the doubling problem by working within the chart his teacher had provided. When he recognized a pattern, he was able to move into a Property Noticing mode of understanding, using his recognition of mathematical properties to make predictions about the next steps. “The first five days, he pointed out, “went 1, 2, 4, 8, 16,” and days 11 to 15 “went 1000”…he paused… “well,” he clarified, “about 1000…2000, 4000, 8000, 16000”. This Property Noticing led to the insight that “once it gets to the 32 000s and the 64 000s then it will get to the hundred thousands and then five hundred thousands and the millions”. For Liam, Property Noticing allowed him to bypass the repeated computation and allow the mathematics to take care of itself.
Indulal and Adarsh, as they debated how to go about modeling infinity, were observed to work through Property Noticing actions and expressions as well. At the point where they had strung the red rope out along the floor with yellow whiffle balls strung out at intervals along its length, they both stood back to consider its viability. Indulal started by questioning the properties that would be illustrated if they presumed the balls represented numbers. “What is between the numbers?” he demanded indignantly, “there cannot be nothing between the numbers…it would be more numbers!” Adarsh weighed in: “What about this? There are 20 holes on each ball so each of these could be 20?” He slid the whiffle balls together so they touched, and counted: “20, 40, 60? Like this?” With both of them still unsatisfied, Adarsh moved on to Structuring, as described below.

In my research experience, Property Noticing was characterized by the metacognitive exclamations that students expressed, either physically or verbally. The ‘aha’ moments or the moments of ‘what the…?’ could be read on their faces, seen in their gestures and heard in their tone of voice. Property Noticing experiences often seemed to mark the moments where the outer limits of one’s Primitive Knowing suddenly came into view. Coming to a Property Noticing level of understanding, the children I worked with would recognize the strengths and the shortcomings of their prior understandings and would use that information to change tack. Immersed in a problem that piqued their interest, they effortfully, and typically agreeably, went back to Image-Making experiences that helped them to thicken, revise or move forward in a new direction from their prior understandings.
10.1.4 Formalising, Observing, Structuring and Inventising

In and amongst the multitude of Primitive Knowing, Image Making, Image Having and Property Noticing experiences encountered through the course of my research, I caught several glimpses of the four “outermost” layers of the Pirie-Kieren model as well. Amir’s response to Danielle, for example, during the large group debriefing of their exploration of infinity (9.6), could be understood an instance of Formalizing. In this exchange, Danielle had suggested that circles could be used as counters in the math classroom. Amir directed our attention to the fact that although we were investigating properties that were unique to circles, the characteristic Danielle was attending to (its potential for discrete count) was not specific to circles but rather was generalizable to a countable object of any shape. “You could take any shape,” he argued, “and use it as a counter. You could take a circle, a square, a rectangle…but a circle, it *turns*… it keeps on going and doesn’t stop.”

Later in the same conversation, Amir’s understanding showed characteristics of Observing. In response to the teacher’s probing questions, Amir shared his Formalized understanding that “you could take any shape and roll it,” but then he elaborated, observing that if the shape were a square, it would go “dunh-dunh-dunh-dunh.” With this, he drew a square moving in a series of turns the air (Figure 44), pausing to articulate each corner that would hit the ground as it rolled. Richard, too, stepped back to compare the properties of a circle and a square, Observing that a square “goes 1, 2, 3, 4, 1, 2, 3, 4, but a circle doesn’t go like that. It just goes and goes like that (tracing circles in the air) but the numbers go higher.”
Adarsh, in his negotiations with his Jyoti and Indulal about the model they were building for infinity (9.2), used Structuring as a way of inviting shared understanding of the mathematical properties that were relevant to the task at hand. He explained his formal observations in terms of a logical structure, challenging the others with a rhetorical question: “If these are numbers,” he demanded, gesturing toward the whiffle balls, “what is the rope? We don’t need these things,” he said, and pulled the whiffle balls off the rope, straightening it out again. “The rope is the numbers. Why do we need the balls? The rope is the numbers going on forever.”

Inventising was observed as well, as students were seen to break away from preconceptions and ask new questions that might give rise to a totally different topic (Pirie & Thom, 2006). When the grade two class was invited to share what they knew about numbers during their first class discussion, a tipping point was reached when they moved beyond their familiar use of number as a count of discrete quantity (9.3). When Donovan wondered aloud, “now I’m starting to wonder what number even means…does it mean never ending, or does it mean it ends…” and Richard interjected: “Does it mean all the numbers in the universe or what?” and then both Richard and Amir asked in unison “What does it mean?”; they were (in their completely appropriate six-year-old way), breaking free from preconceptions about number such that space was created where new ideas might emerge.
10.1.5 Moving out and Folding Back in Dynamic, Nonlinear Patterns of Growth

Depth and breadth of mathematical understanding then, increases as one moves outwards and inwards within the model; moving outwards opens possibilities for more generalized mathematical knowledge to emerge and movements inwards allow one to return to previous realms of knowing… Folding Back is defined, not as just the recollection of a mathematical experience or piece of information, but as providing a means by which a learner or group of learners can reconstruct, reintegrate, or re-evaluate known mathematics so that they may function in the outer layers with a “thicker” understanding.

(Pirie & Thom, 2006, p. 187)

According to the Pirie Kieren model of growth of mathematical understanding, the process of ‘moving out and folding back’ is described as experience that brings us to the edges of viable understanding before returning to a previous understanding in order to thicken or restructure it. Kim, Roth, and Thom (2011) describe “an ongoing cycle of embodied learning [that] takes place in the dynamic balance of different forms of knowing” (p. 226). Entering into this study, I had dramatically underestimated the speed and fluidity with which this cycling between different forms of knowing would occur. What I came to understand from my work with children was that the kinetic energy of understanding was, in fact, its defining feature.

My experience with Heidi (7.4) was typical of the type of dynamic and nonlinear growth of mathematical understanding observed throughout my research. The process of unfolding and folding back was so fluid and continuous that the points in time where I could ‘pin down’ and
describe her understanding were perhaps less significant than the growth process itself. Complementarities of acting and expressing that characterized Heidi’s interactions with the problem, with her own thinking, and with me, were lived through as a constant back-and-forthness: doing, noticing, wondering, rethinking, and doing differently. As Pirie and Kieren (1994) explain, understanding is “a whole, dynamic, leveled but non-linear, transcedently recursive process” (p. 166). In Heidi’s ongoing negotiation of fit, her understanding never paused long enough to become a ‘thing’; understanding was always a verb—a process—a way of being.

For Marlo and JJ as well (6.3), the co-emergence of their understanding was like quicksilver—they moved from Primitive Knowing through Image Making to Property Noticing, back to Image Making and on to Image Having in a matter of seconds. Like watching dancers in motion, there were observable elements that could be noticed and characterized separately from one another, but the meaning was found in the whole, integrated interaction.

10.2 Understanding

Our bodies are shaped by the world they participate in shaping; they render the mind-and-world; subject-and-object; individual-and-collective; mental-and-physical inseparable. These phenomena are co-emergent: fluidly defined against one another.

(Merleau-Ponty, 1945, p. 78)
Davis (1996) suggests ways to “conceive of ‘understanding’ and ‘meaning’ in ways that will help us to avoid the conventional tendency of locating them in the heads of ostensibly autonomous agents” (p. 196). It speaks to Maturana and Varela’s (1987) widely cited axiom (e.g., Davis, 1996; Lakoff & Johnson, 1999), “all doing is knowing, and all knowing is doing” or to Pirie and Thom (2006): “mathematical understanding is viewed as being brought forth by interrelated and fluid processes, evolving in a fractal-like manner from existing knowing actions” (p. 186).

Understanding is continually structured and restructured through a negotiation between self and that which is not oneself, and in the actions and experiences of the body are the living through of understanding. As Roth and Thom (2011) propose, “conceptions do not exist as disembodied, decontextualized, and transcendental ideas but are imminent in each concrete realization of experience and in its relations to other experiences” (p. 176). As such, understanding is not something we gain and hold onto, but that which we experience; what we do—our acting in the world is understanding.

10.2.1 Embodiment

Merleau-Ponty suggests that the body is that which renders the mind and the world inseparable. Far from representing a discrete demarcation between subject and object, one’s body is simultaneously of oneself and of the world.

(Davis, 1996, p. 9)
Marlo and JJ, literally, in an embodied way, understood quantity by getting a ‘feel’ for number (6.1). When JJ rubbed her hands together and talked about what it was like to ‘feel’ a number, the Image she held wasn’t simply an abstract, objective representation of quantity, it was a tangible, bodily way of knowing. Similarly, Marlo’s statement that by holding something in your hand you could picture it in your head “by feeling or maybe seeing” reflects something profoundly important about understanding, generally, and about the Image Making/Image Having processes in particular.

Núñez, Edwards and Matos (1999), in *Philosophy in the Flesh*, speak to the connection between metaphorical thought and embodied understanding: “The mind is inherently embodied. Thought is mostly unconscious. Abstract concepts are largely metaphorical…Unlike traditional studies of metaphor, contemporary embodied views don’t see conceptual metaphors as residing in words, but in thought” (pp. 3 & 52). Extending beyond the distinction between whether these ideas reside in words or thought, Roth and Thom (2009) might suggest that contemporary embodied views see conceptual metaphors as “immanent in each concrete realization of experience” (p. 175). When Adarsh, Jyoti and Indulal (9.2) were scavenging through the school to find materials to build their model of infinity, it was not insignificant that they found their treasure trove in the gym storage room rather than in the math supply closet or the art room. The embodied metaphors that made the idea of infinity even remotely accessible to them were rooted in their bodily experiences in the world.

Reviewing video data gathered through the course of this study allowed me to stop and notice gesture in a way that may otherwise have simply slipped by in the course of our classroom
interactions; studying these gestures offered a unique perspective on the nature of children’s understanding. Studying children’s bodily orientation, movements, interactions with objects, physical mimicry and gesture was often as important as analyzing their dialogue or written work. The difference between the clawed hand that Jackson used to show how he thought through an estimation for a large but discrete quantity (Figure 26) versus the sweeping loops of the arm he made when discussing the concept of infinity (Figure 42) made visible the embodied understandings that were driving his Image Making experiences. Similarly, the difference between Levi’s bracketing gesture (Figure 8) for groups of 24 and his use of a wide outswept arm to show the amount of time until lunch (Figure 29) offered glimpses into how he understands discrete and analogue quantity.

Looking to Kim, Roth and Thom, whose research into student understanding in mathematics deals extensively with gesture and embodiment, we see that gesture is not merely a communicative act, but a process by which understanding is developed and enacted (Kim, Roth, & Thom, 2011). They suggest that “knowing cannot be the affair of a pure, independent, abstract, and disembodied mind,” rather, “cognition—at least in part—exists in bodily rather than in mental form” (Kim, Roth, & Thom, 2011, p. 209).
10.2.2 Co-Understanding

Collective knowledge and individual understandings are dynamically co-emergent phenomena. In response to the question, Where is the mathematics?, one might thus say that it is located in the activity—or, perhaps more descriptively, in the interactivity—of learners.

(Davis, 1996, p.113)

When working together to solve a problem, such as was the case with Jackson, Donovan and Bob (9.5); Adarsh, Jyoti and Indulal (9.2); or Marlo and JJ (6.2), physical actions, gestures, facial expressions, and fragments of written or verbal dialogue brought forth emergent understandings that became the understandings of the group. Through a reciprocal process of sharing, accepting, rejecting, modifying and improving upon their own and one another’s offerings, a group of learners developed understandings that could not be said to ‘reside’ with any individual member of the group.

Kim, Roth and Thom (2011) observe, “intersubjective knowing emerges whenever individual learners reciprocally assimilate the actions of the other in socially interactive situations” (p. 225).

When Marlo acted through an understanding of ‘breaking apart’ 23, for example, (Figure 5), JJ, who was engaged in the same stream of thought, assimilated Marlo’s ideas into her own gestures (Figure 6). This was not a matter of mimicry, as the gestures were not the same, but was more a matter of joining in with her own unique interpretation—if their learning had been musical, JJ would be harmonizing.
Similarly, as Jackson, Donovan and Bob (9.5) were navigating the rules that might govern the iterative motion of their looping model for infinity, they played around with various end points and points of return. As they did so, the gestures that evolved within the group ebbed and flowed along a similar course: the gesture for the looping action was originally smooth and uninterrupted for all three children, but when they determined that they needed to demarcate an endpoint for each cycle, they all adopted a gesture of tapping the side of one hand against the open palm of the other. As they negotiated the difference between a starting point of zero and their vague understanding of how negative numbers might come into play, all three used a gesture of pulling their ‘driving’ hand back behind their side body before moving up to the unspoken but agreed upon ‘zero’ at their left hip. Another distinction in gesture that emerged, collectively, was the difference between a gesture for discrete count (tapping the side of one hand against the palm of the other, as with the gesture that marked the end of a cycle) and a gesture for analogue counting, or the count going up continually (a sweeping loop of the arm). Again, these shared gestures were more than mimicry—they both denoted and brought about shared understandings.

While gesture was certainly a significant component of students’ shared understandings, other actions and expressions (verbal and nonverbal) were noteworthy as well. Alyssa and Amir, for example, lying side by side on the carpet drawing their spiral together (Figure 37), created something that only came into being through their shared actions. Richard’s ‘aha’ moment regarding the properties of a circle as it relates to iterative motion (9.6), was woven in and through Amir’s tracing of the paths in the air that might be taken by a circle and or a square. When JJ pulled up Marlo’s fingers one at a time (Figure 10), their shared understanding went
unspoken, but emerged, fingertip by fingertip, through their interaction. As Davis (1996) explains, “the space of collective action is not merely a device in promoting the individual sense-making; it is a location for (shared) meaning and understanding” (p. 197).

10.3 Discrete and Analogue Quantity

Young children treat counting as analogue before they count discrete quantity—they prattle off strings of numbers the way they might recite the alphabet or sing a song. Ask a three-year-old her age (one of the most socially significant applications of number at that stage of development), and her answer is likely to be one-two-three!—a continuous stream of number. In this context, quantity (inasmuch as the child’s response refers to quantity) is experienced as analogue. It is akin to the answer to the question how big are you? (to which the answer is, of course, sooooo big!, with the arms stretched way up high). As Jada explained (7.1) when talking about her brother’s age: “So right now, one turns into two, and two turns into three…” One turning into two is a distinctly different experience from adding one more. It has the same feel to it as the increasing light at sunrise or the volume being turned up on the radio. It is the gesture of Levi’s arm (Figure 29) sweeping across the full length of its reach to show how long until lunch. Or of Jackson ‘scanning’ the pile of blocks with his swift hand over hand motion (8.1). It’s analogue.

Music is analogue. A video made the rounds in social media recently that showed Lara, a toddler, ‘conducting’ a choir in Kyrgyzstan (https://www.youtube.com/watch?v=gE9r1LkRCV0). While there is almost certainly some element of imitation at play in her gestures (Figure 46), the way her arms reach out, rise and fall in response to the music, and the way she leans in with her
body and then stretches back—even the movement of her eyebrows—*embodies* the music. Acknowledging there is more at play here than a physical response to the sound, I nevertheless recognize similarities between the upsweep of Lara’s arm as the pitch and volume rise and the sweeping gestures I observed in students working with analogue quantity.

![Figure 46: Lara Gesturing Pitch and Volume](image)

Gallistel and Gelman (1992), in studying the verbal and preverbal counting and arithmetic of both human and non-human animals, found “strong evidence that the preverbal representatives of numerosity are magnitudes” (p. 53). They suggest that understandings of quantity are
analogous in structure and function to histograms, whereby numerosities may be compared or combined according to their height.

In a later study, Gallistel and Gelman (2000) describe the difference between discrete and analogue quantity this way: “The distinction between the integers and the reals, between the discrete and the continuous, lies precisely here: integers are discretely ordered and countably infinite, like the levels you get when you fill an (infinitely tall) beaker one cupful at a time. By contrast, the reals are continuously ordered and uncountably infinite, like the levels you get when you fill the beaker with a hose that is ‘on’ for different, random amounts of time” (p. 62).

In my research experience, students were noted to make personally significant distinctions between small numbers and large numbers, essentially according to what they could fit in their hand. Discrete quantities, where the defining characteristic of the number in a particular context was its precise count, were understood through actions and expressions indicative of an embodied understanding of individual, countable objects. The bracketed hand gesture, flat hand tap and pinched or cupped hands, for example, were only observed when students were talking about quantity they could hold in their hand. Conversely, the actions and expressions through which children built understandings of large quantities were consistent with analogue experiences—smooth, continuous, often iterative motions. While both understandings were embodied, they were rooted in very different actions and expressions.
10.4 Implications

Wagner and Davis (2010) argue that there is “a moral imperative to connect number sense with such a quantity sense that allows students to feel the weight of numbers” (p. 39). This view requires that we reconsider what it might mean to create conditions for learning that are conducive to growth in mathematical understanding. As Kim, Roth and Thom (2011) observe, “the recognition of the body in one’s thinking and knowing of concepts challenges the received views of knowledge in learning and teaching” (p. 233). Yet a growing body of research would suggest that embodied understanding is at the heart of all learning.

10.4.1 Mathematics Instruction in Alberta

If young children tend to make sense of large quantities by treating them as analogue, and if they develop these embodied understandings through experience, what are the implications for mathematics instruction? Consider the approach recommended by the Alberta Mathematics Program of Studies (2007) for grades 4 to 6 (Figure 47). The learning outcomes could be interpreted generously: representing and describing whole numbers, for example, could refer to any properties of number, including both analogue and discrete quantity. The achievement indicators, however, make it quite clear that the kinds of understandings that students and teachers ought to concern themselves with is not an embodied sense of quantity at all, but rather the functional application of the conventions for number.
<table>
<thead>
<tr>
<th>Learning Outcomes</th>
<th>Achievement Indicators</th>
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| **4.1** Represent and describe whole numbers to 10 000, pictorially and symbolically. [C, CN, V] | • Read a given four-digit numeral without using the word *and*, e.g., 5321 is five thousand three hundred twenty-one, NOT five thousand three hundred *AND* twenty-one.  
• Write a given numeral, using proper spacing without commas; e.g., 4567 or 4 567, 10 000. Write a given numeral 0–10 000 in words.  
Represent a given numeral, using a place value chart or diagrams.  
Express a given numeral in expanded notation; e.g., $321 = 300 + 20 + 1$.  
• Write the numeral represented by a given expanded notation.  
• Explain the meaning of each digit in a given 4-digit numeral, including numerals with all digits the same; e.g., for the numeral 2222, the first digit represents two thousands, the second digit two hundreds, the third digit two tens and the fourth digit two ones.  
• Create and order three different 4-digit numerals.  
Identify the missing numbers in an ordered sequence or on a number line. Identify incorrectly placed numbers in an ordered sequence or on a number line.  
• Order a given set of numbers in ascending or descending order, and explain the order by making references to place value. |
| **4.2** Compare and order numbers to 10 000. [C, CN, V] |  |
| **5.1** Represent and describe whole numbers to 1 000 000. [C, CN, V, T] | • Write a given numeral, using proper spacing without commas; e.g., 934 567. Describe the pattern of adjacent place positions moving from right to left.  
Describe the meaning of each digit in a given numeral.  
Provide examples of large numbers used in print or electronic media.  
• Express a given numeral in expanded notation; e.g., $45 321 = (4 \times 10 000) + (5 \times 1000) + (3 \times 100) + (2 \times 10) + (1 \times 1)$ or $40 000 + 5000 + 300 + 20 + 1$.  
• Write the numeral represented by a given expanded notation. |
| **6.1** Demonstrate an understanding of place value, including numbers that are:  
• greater than one million  
• less than one thousandth. [C, CN, R, T] | • Explain how the pattern of the place value system, i.e., the repetition of ones, tens and hundreds within each period, makes it possible to read and write numerals for numbers of any magnitude.  
• Provide examples of where large and small numbers are used; e.g., media, science, medicine, technology. |
Looking more closely at the outcomes, we might also consider the mathematical processes students are expected to apply. Most learning outcomes related to working with large numbers include CN (connections), C (communication), and V (visualization). R (reasoning), on the other hand, is not required until sixth grade. Furthermore, it would appear that the application of these processes has nothing to do with developing a sense of quantity; students are expected to visualize, communicate about and make connections with the numerals themselves.

Wagner and Davis (2010) caution strongly against such an approach: “Our worry as educators is that number sense is often divorced from quantity sense and that curricular and pedagogical emphases in school mathematics actually contribute to their separation…. Something is lost in the decoupling of number and concrete experience, as there are conceptual and cultural implications of coupling and decoupling” (pp. 42-43).

Although there are a great many teachers who do very interesting, engaging, hands-on, quantity-based work with number, the textbook examples in Figures 48 to 50 are typical of the instructional approaches taken in many of our grade four to six classrooms. For example, the instructional sequence recommended in one of the approved teaching resources for fourth grade math (Figure 48), suggests eight lessons over a span of 13 days. Activities include modeling numbers with base ten blocks, putting numbers in order, writing numbers in expanded form and expressing numbers in words.
Figure 48: Sample Learning Progression in an Alberta Recommended Resource

Looking to the only activity that appears to address ideas of quantification, Lesson 4 (Figure 49), we see that the lesson nonetheless begins with the teacher reviewing conventions of writing large numbers, explaining when to leave a space between three digits in a numeral. In the second part
of the lesson (allocated 20-25 minutes), students are invited to consider what *Ethan* thinks about 10 000 (the boy on the front page who, through some miraculous feat of physical and cognitive endurance, is juggling a soccer ball past 5424 hits), and to think about how they might show someone else what 10 000 means. The lessons that follow Lesson 4 return to the conventions of reading and writing large numbers, suggesting that out of the entire school year, as little as 20 minutes may be dedicated to developing any sort of ‘feel’ for numbers.

Another fourth grade resource introduces students to whole numbers to 10 000 with an amazing fact: *The largest marching band ever assembled had 4526 members. There were students from 52 different school bands* (Figure 50). The very first investigation of this amazing fact has nothing whatsoever to do with marching bands, though—students are asked to explore ways to show the *number* 4526. The next task, as described in the teacher’s guide, involves visualization—the textbook interpretation of that mathematical process, however, is simply drawing base ten blocks (Figure 51).
I could carry on, showing a fifth-grade example that asks students to investigate world record holding feats of strength and endurance solely through the use of tables and charts, or the teacher’s guide for third-grade mathematics that would allow for an entire unit on number to rely solely on base ten blocks for hands-on work with number, but teaching practice was not a focus for this study.

Prior to undertaking this investigation, I wondered about a possible disconnection between the understandings of quantity that students bring to new learning experiences and the school-sanctioned models and representations of number used in most classrooms. I wondered if the models, metaphors and associations we use in the math classroom build on or enhance the conceptual structures students find intuitive, or if they actually prune away potentially better models. I wondered if our reliance on certain images of quantity and numerosity served some aspects of mathematical thought better than others.

Figure 50: Introducing Numbers to 10 000

Figure 51: "Visualization"
In Alberta, understandings of quantity are not required learning outcomes in grades four to six. *Numerals* are modeled in our classrooms, not quantity. There are few opportunities to explore and enhance the models of, metaphors for and associations with quantity that students may bring with them to the math classroom because quantity is simply not a topic of investigation.

As a discipline, mathematics education is concerned not only with creating effective means and methods of instruction, but with understanding why certain methods are effective and others are not, and with larger questions about the nature and development of mathematical knowledge. Our answers to these questions, and even the ways we choose to investigate them, are strongly influenced by our implicit or explicit conceptualization about the nature of human thought, and about mathematics itself. When mathematics is conceived of as an external realm of objective truths, to be ‘discovered’ through the application of rational thinking, then the investigation of mathematics learning focuses on accurate mappings, models, and internal representations of mathematical entities and relationships. If, on the other hand, mathematics is conceived as a product of adaptive human activity in the world, shared and made meaningful through language, but based ultimately on biological and bodily experiences unique to our species, then mathematics education must take a different approach. New practices in mathematics education, from classroom teaching to scientific research and curriculum design, should emerge that present mathematics as a genuine mind-based activity with all its embodied peculiarities and beauty.

(Nuñez et al, 1999, p. 60)
10.4.2 Wonderings

Although the aim of this study was not to investigate instructional techniques, learning resources, pedagogy or curriculum, when I return to teaching elementary school students, I wonder what I might do differently as a result of my research experience.

- How could I more consciously attend to the kinds of Images of quantity children were actually making, rather than the ones we, as adults, presume will make sense to them?

- If, as Davis and Towers (2002) suggest, “it is these repeated Image Making activities which are significant in providing the ground for the deeper understandings” (p. 331), how should the patterns of teaching and learning shift in my classroom in order to maximize opportunities for Image Making?

- If understanding is co-emergent, how might I design opportunities for students to bump up against others who will help to dynamically grow their understanding?

- And if understanding can be seen to occur not ‘within’ one child or another but rather in the intersubjective space, what is required in terms of how I think about learning outcomes and assessment?

- What might it look like to incorporate opportunities for developing and refining embodied understandings of quantity across grade levels?

- How might I cultivate the development and refinement of mathematical understandings based in Images of motion—along a horizontal or vertical number line, through cyclical, iterative motions, as expanding or contracting shapes, or as physical movement through the learning environment? Would these images feel familiar and meaningful to even the youngest of our learners?
As well, the study opened up a number of possibilities for further research. There is so much worth investigating; for example, I would be interested to inquire into:

- the tensions that exist between our histories and inheritances as educators and our capacity to consider understanding as dynamic and recursive;
- the patterns and characteristics of intersubjective or co-emergent understanding;
- the conditions under which embodied understanding might live out in Alberta math classrooms;
- the kinds of manipulatives, models and visual representations of mathematical concepts used in elementary math classrooms versus the kinds of models and manipulatives children themselves might design or invent;
- how older children or adults understand quantity beyond that which can be held in their hand (including the kinds of figures used to describe national spending, global epidemics, natural disasters or other events of common concern);
- the role of classroom assessment and standardized achievement testing in perpetuating an outdated view of what constitutes understanding in mathematics;
- the role of teacher as learner and the impact of such a stance on the learning of the children in the classroom;
- and so much more.
Chapter Eleven: **Reflections on the Research Experience**

I have been privileged, over the course of my teaching career, to countless experiences such as that of Britta (3.2) squeezing the air beside her temple while she explained her thinking that I absolutely treasure; they are gifts that I can re-experience and understand anew time and time again. The fieldwork I conducted for this study and the time I spent reviewing the videos of students at work offered more such gifts. The moment when Sam (Figure 33) reached his little arm out in a loop to show how numbers go on forever. The moment when JJ (Figure 3) closed her eyes and rubbed her fingers together while she recalled the blissful sensory experience of “feeling numbers”. The blunt and good-natured answer to my research question that Adarsh (Figure 30) laid out for me in his beautiful Indian accent: “Bigger numbers I think about tall and number line, smaller numbers I think about size.”

While I had approached the research experience as an opportunity to come alongside children in their work and to experience their understandings of quantity as they emerged in the learning context, I did come into the study with certain ideas about understanding and about quantity that shaped my approach. I had expected children’s understandings of quantity to be highly visual. I had thought I would find an organized, iterative approach to estimation, where building blocks of known amounts were used to make sense of larger quantities. In the design of the study I hadn’t really considered the possibility that students would treat quantity as analogue. If I knew then what I know now, I would have designed the learning tasks differently.

In designing this study, I intentionally sought out a variety of learning contexts: a range of students, at different grade levels, engaged in all sorts of math tasks. By working with a variety
of math tasks rather than attempting to replicate a similar learning experience in each classroom, I hoped to be able to investigate a range of ways young children understand quantity. Diverse learning tasks, I reasoned, would elicit diverse perspectives. In practice, the nature of the learning tasks led students toward certain types of understandings, using certain types of metaphorical thinking that were strongly influenced by the type of problem to be solved and the types of questions they were asked.

Tasks that involved smaller quantities were all structured around questions of how many and how much—students estimated and determined the length of a line using nonstandard units of measurement, for example, or estimated the number of objects in a jar, or considered how many of one item it would take to balance a different number of another item. These tasks elicited reasoning about discrete quantity. Tasks that involved very large quantities, though, invited students to make predictions of quantity based on growth over time and thus had an element of motion inherent in the problem. So, too, did our discussions about infinity. These learning experiences drew upon students’ understandings of analogue quantity. In the end, none of the learning tasks were intentionally focused on growth or motion limited to very small quantities, nor were they necessarily targeted towards very large amounts as discrete quantities. These were present, at times (the first several days of doubling in the grains of rice problem, for example, or estimating quantities in jars to hundreds of items), but students tended not to linger in those places for long.

A more significant bias I brought to the design of the study was the way I previously understood understanding. Although I had read and superficially understood Pirie and Kieren’s descriptions
of understanding as dynamic and recursive, at the outset of the study I still held some vestigial notion of understanding as being something one acquires. In designing the study to look for evidence of understanding at each of the first four levels of the Pirie Kieren model, some part of me still imagined that understanding would hold still long enough for me to document what it was like. Looking back to the way I originally positioned the purpose of this study, the unchecked assumptions now seem obvious:

Without denying the holistic and dynamic nature of understanding, in this study I endeavor to provide images that illuminate individual students’ understandings within each of these levels. I am intensely curious about what it could mean to make an image, or to have one; to bring to a novel learning experience some sort of Primitive Knowing; to notice properties. When a student embarks on a new learning experience, what is present in their Primitive Knowing? What is absent? What does it mean to have, or not to have, that understanding? What changes for the learner in that process? (3.1)

Perhaps the most profound insight I’ve gained through the course of this study has to do with understanding as being. Years ago, when Britta (3.2) had reached up beside her temple and squeezed her hands together, I took her gestures to be referencing an idea that lived, like a physical object, in her mind; now I interpret her actions more as the expression of understanding that lives, like physical experience, in her body. “Our mathematical knowledge…is neither “out there” nor “in here,” but exists and consists in our acting. As such the character of our knowledge changes with every action” (Davis, 1996, p. 79).
It was Merleau-Ponty who first tipped the balance for me between an unchecked positivist mindset and an emerging phenomenological perspective, so it seems fitting to end with a return to Phenomenology of Perception (1945):

In the proposition: ‘I think, I am’, the two assertions are to be equated with each other, otherwise there would be no *cogito*. Nevertheless we must be clear about the meaning of this equivalence: it is not the ‘I am’ which is pre-eminently contained in the ‘I think,’ not my existence which is brought down to the consciousness which I have of it, but conversely the ‘I think,’ which is re-integrated into the transcending process of the ‘I am’, and consciousness into existence. (Merleau-Ponty, 1945, p. 446)


de Hevia, M.D. & Spelke, E.S. (2008). Spontaneous mapping of number and space in adults and young children. *Cognition*, 110, 198-207.


4Russians. 2014. *Mini Maestro, little girl conducting a church choir, Kyrgyzstan* [Video]. Retrieved from https://www.youtube.com/watch?v=gE9r1LkRCV0
Addendum

Here’s something. I don’t know what it is, or why it might be, but it stood out from the other images as I reviewed the data from my fieldwork as something worth attending to. I didn’t include it in my analysis, but decided to offer it here so that if it means something to you, you may share in my experience.

As I worked with children, we talked about things that caused them to think. They would pause, quietly thinking, and as they thought they looked up and off into the distance. This was not observed with all students, and was not noted every time they paused to think. Interestingly, Indulal (the student with significant visual impairments) did not gaze off into the distance, but he did tip his head up and I watched his white eyelashes fluttering as he thought. He is not included in these images as he chose not to be photographed.

When I noticed it during the interviews, I asked the students about what was happening in their minds when they were gazing off (What were you thinking about? What were you doing in your head? What was it like in your mind?), and sometimes I asked what they were looking at. Most often, they responded to the question of what was going on in their minds with responses about strategies and solutions. The answer to the question of what they were looking at? I dunno.

Having considered the characteristics of the various layers or levels of the Pirie Kieren model of growth in mathematical understanding, I wonder if this particular experience is evidence of Image Making. One of the characteristics noted during Image Making was that of deep immersion, where students were observed to be deeply engrossed in the experience, both
physically and intellectually; Image Having, on the other hand, was often characterized by a sense of a ‘return’. Given the embodied nature of understanding, perhaps this gaze was indicative of a process of re-experiencing (or experiencing for the first time) an Image Making action. The return, then, when the child re-established eye contact and offered a solution, might be a cue that they were experiencing Image Having.
Me: Can you add 25 and 25 and 25 and 9 in your head?
[Adarsh looks up]
Adarsh: 84?

Natalie: I can equal 7. It's 3 and 4.
Me: Ok. Can you equal 14?
[Natalie looks up]
Natalie: 10 and 4?

Me: Can you picture what a pike of 100 candies would look like?
[Fernando looks up]
Fernando: Yes

Me: Can you make 26 into two or more parts?
[Pete looks up]
Pete: Just two.
Me: How long do you think it would take you to count 32 000 grains of rice?
Liam looks up
Liam: I don’t really want to be unreasonable, but probably, like, at least a couple weeks.

Me: How old do you think I am?
Jada looks up
Jada: Ssss…eventeen?

Rhys: So that would be 11 000 and 6 637…
Rhys looks up
Rhys: 17 637

Victoria: I just used 50s because 50 is close to 60.
Me: What is that like in your head, for 50 to be close to 60?
Victoria looks up, then writes the equation 60-10=50
Victoria: If you go from 60 to 50, you only take away ten things [holds up ten fingers].
Me: Can you imagine (on a scale) how many meerkats it would take to balance a gorilla?

[Cameron looks up]

Cameron: No

Me: Are there other ways you can break down 26?

Manraj: I can get, like, ten, and then I can get, like, 20...

[Manraj looks up]

Manraj: ...and then I can put a six there.

[Cameron looks up]

Cameron: No

Me: How many straws would you need to put in to balance Natalie’s?

Meg: A hundred.

[Meg looks up]

Fred: Yup, it’s 20.

Me: What were you doing in your head?

Fred: I was counting back from 24. I was hmming and hmming and I was going like this [mocks looking up to think].
APPENDIX A: PARTICIPANTS

Adarsh  Amir  Bob  Clark  Danielle

Donovan  Gloria  Harjit  Heidi  Indulal

Jackson  Jada  JJ  Jyoti  Levi

Liam  Manraj  Marlo  Meg  Mona

Richard  Sam  Siobhan